DOCUMENT RESUME

ED 476 065 SE 067 806

AUTHOR	Cockburn, Anne D., Ed.; Nardi, Elena, Ed.
TITLE	Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education (26th, Norwich, England, July 21-26, 2002). Volumes 1-4.
INSTITUTION	International Group for the Psychology of Mathematics Education.
ISBN	ISBN-0-9539983-6-3
PUB DATE	2002-07-00
NOTE	1770p.; For select individual papers, see SE 067 779-805.
PUB TYPE	Collected Works - Proceedings (021) -- Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE	EDRS Price MF15/PC71 Plus Postage.
DESCRIPTORS	*Computer Uses in Education; Concept Formation; Curriculum Development; Ethnomathematics; *Graphing Calculators; Group Activities; Higher Education; Instructional Development; Interdisciplinary Approach; Learning Activities; *Mathematics Instruction; *Mathematics Teachers; *Professional Development; Teacher Education Programs; Teacher Effectiveness

ABSTRACT

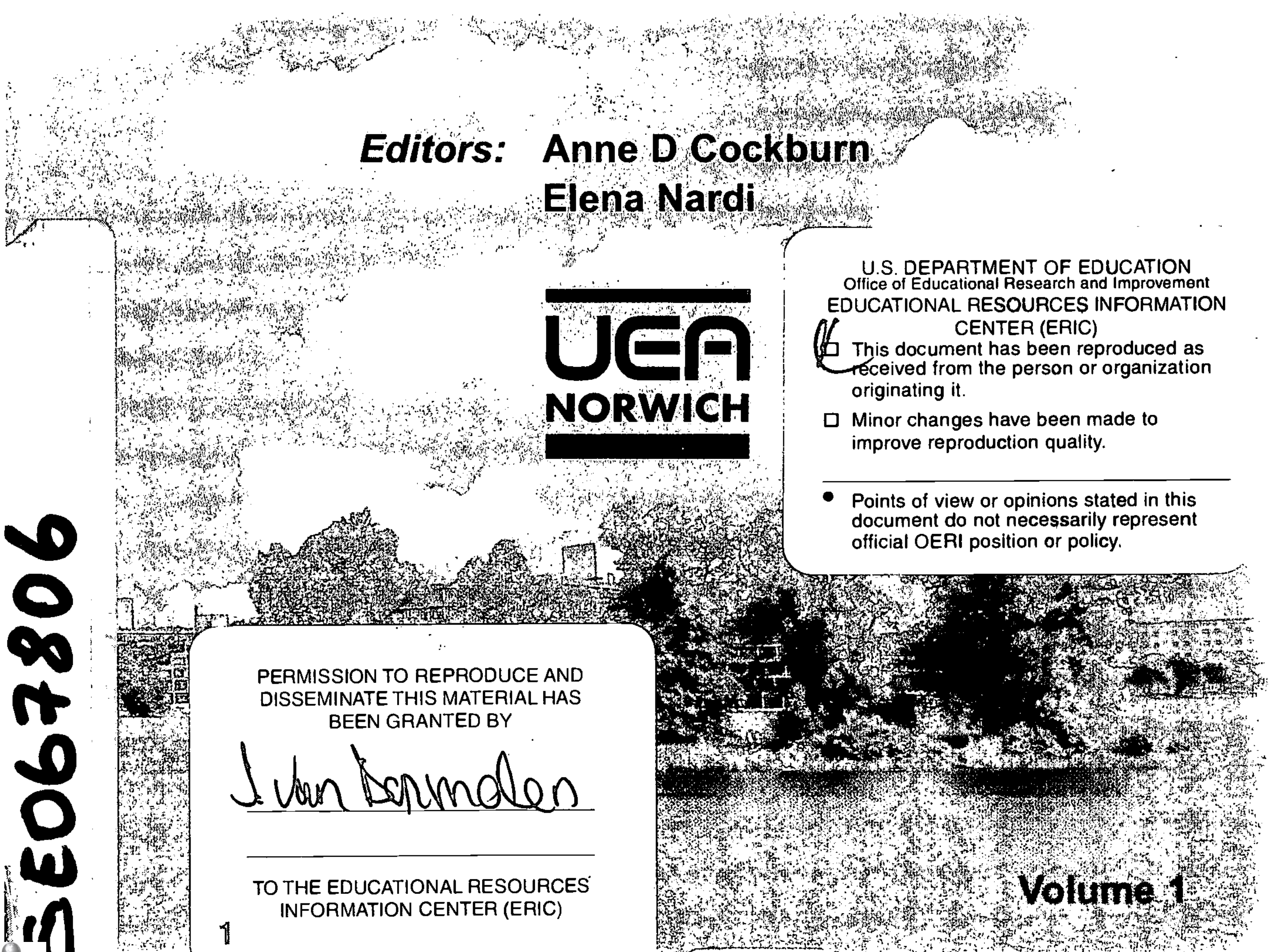
This document contains the first volume of the proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education. Conference presentations are centered around the theme "Learning from Learners". In total, 290 presentations were given ranging from the psychology of mathematics in the very early years to discussions on advanced mathematical concepts. This volume features nine plenary papers, papers from four groups of research forums, discussion groups, working sessions, and short oral communications. Plenary papers include: (1) "Little Children, Big Mathematics: Learning and Teaching in the Pre-School" (Herbert P. Ginsburg); (2) "Researching Primary Numeracy" (Margaret Brown); (3) "How Students Structure Their Own Investigations and Educate Us: What We've Learned from a 14 Year Study" (Carolyn A. Maher); (4) "Mathematical Epistemologies at Work" (Richard Noss); (5) "Taking the Modeling Perspective Seriously at the Elementary School Level: Promises and Pitfalls" (Lieven Verschaffel); (6) "Learning from Learners" (Joao Filipe Matos); (7) "Episode 1: How Karin Disturbed Her Teacher's Line of Reasoning" (Joop Van Dormolen); (8) "Episode 2: Marco and Anna--Each Following Their Own Path" (Rosetta Zan); and (9) "Episode 3: Calculators--The First Day" (Susie Groves). (KHR)

INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

PME26

UNIVERSITY OF EAST ANGLIA ▪ UK

Proceedings of the 26th Annual Conference



Editors: **Anne D Cockburn**
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UEA NORWICH

Volume 1

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INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

PME26
UNIVERSITY OF EAST ANGLIA • UK

21 - 26 July 2002
Norwich, UK

PROCEEDINGS
Volume 1

Editors
Anne D Cockburn
Elena Nardi

UEA
NORWICH

School of Education and Professional Development
University of East Anglia

Proceedings of the 26th Conference of the International group for the Psychology of Mathematics Education

Volume 1

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Website: www.uea.ac.uk/edu
The Proceedings are also available on cd-rom

ISBN number 0-9539983-6-3

Design of the PME26 logo: Steve Giles

Printed by the University of East Anglia, Norwich, UK

Welcome to PME 26

Welcome to PME 26! The School of Education and Professional Development at the University of East Anglia is delighted to be hosting the twenty-sixth annual meeting of the International Group for the Psychology of Mathematics Education. The theme for the conference this year is 'Learning from Learners' which, we are pleased to report, has captured the imagination of many of this year's contributors. In total we have in the region of 290 presentations, ranging from the psychology of mathematics in the very early years to discussions on advanced mathematical concepts. We will also be offering you a variety of entertainment that will restore your energy levels amid the demands of such a busy scientific programme.

We very much hope you enjoy your stay in Norwich and that you will feel welcome to return in years to come. The city dates back to Anglo-Saxon times and offers a wealth of sights, including a Norman castle (1067), two magnificent cathedrals (1096 and 1894) and what the eighteenth century historian, Francis Bloomfield, described as the 'grandest market in all England'. We recommend that you take the opportunity to wander around the city and discover its delights for yourself.

We hope that the conference will run smoothly with, apparently, very little effort on our part. If it doesn't, I can only reassure you that we have done, and will do, our very best! If it does all go well it will be due to many months of preparation and commitment from numerous people. We thank them all, especially members of the Local and Programme Committees. Particular attention must go to Gilah Leder for her encouragement in the early days of our work and to Rina Hershkowitz for continuing such support. Thanks too to Janet Ainley who proved to be an excellent Chair for the Programme Committee. The invaluable help of Lucila Recart and Bella Radford was also much appreciated. Angela Hook's and Claire Butler's hard work should not go unmentioned either. Joop van Dormolen's ceaseless labours in both creating and refining the computer programme we relied on and his endless patience and enthusiasm also deserve particular recognition. And finally, a special mention to Elena Nardi, without whom this venture would have been impossible.

More formally, we would like to thank the following sponsors:

- The Lord Mayor of Norwich for his most generous contribution on our arrival at the dinner dance;
- Jarrolds Department Store for heavily subsidising the Norwich guidebooks in your conference pack;
- Falmer Press for providing conference pens;
- Paul Chapman Publishers for donating conference notebooks;
- Colmans of Norwich for the savoury packs in your conference bag;
- The University of East Anglia for providing us with both time and space to host this conference.

Finally, should you have any questions/concerns during your stay, please do not hesitate to ask members of the conference team. We very much hope that you enjoy PME 26 and that you return home feeling refreshed and invigorated.

With very best wishes,

Anne Cockburn
on behalf of the PME 26 Conference Committees

Summer 2002

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INTRODUCTION

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

History and aims of PME

PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany in 1976. Its past presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gerard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), Carolyn Kieran (Canada), Stephen Lerman (UK), and Gilah Leder (Australia).

The major goals of the Group are:

- To promote international contacts and the exchange of scientific information in psychology of mathematics education
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators
- To further a deeper understanding into the psychological aspects of teaching and learning mathematics and the implications thereof.

PME membership and other information

Membership is open to people involved in active research consistent with the Group's goals, or professionally interested in the results of such research. Membership is on an annual basis and requires payment of the membership fees (US$40 or the equivalent in local currency) per year (January to December). For participants of PME26 Conference, the membership fee is included in the Conference Deposit. Others are requested to contact their Regional Contact, or the Executive Secretary. More information about PME as an association can be obtained through its home page at: http://igpme.tripod.com or contact the Executive Secretary.

Honorary Members of PME

Hans Freudenthal (The Netherlands, deceased)
Efraim Fischbein (Israel, deceased)

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Members of the Communication, Conference and Catering Services at UEA

Secretarial staff: Lucila Recart, Bella Radford and other very welcome occasional helpers.

IT staff at UEA: Rob Barnes, Peter Saward, Elena Nardi, Angela Hook, Dave Knock, and members of the Audio Visual Services.

PROCEEDINGS OF PREVIOUS PME CONFERENCES

Copies of some previous PME Conference Proceedings are still available for sale. For information, see the PME home page at http://igpme.tripod.com, or contact the Executive Secretary Dr. Joop van Dormolen, Rehov Harofeh 48A, Haifa 34367, Israel;
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Al proceedings, except PME1, are included in ERIC. Below is a list of the proceedings with their corresponding ERIC code.

PME International

No.	Year	Place	ERIC number
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerpen, Belgium	ED226943
7	1983	Shoresh, Israel	ED241295
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, Netherlands	ED411130(vol.1), ED411131(vol.2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128(vol.1), ED411129(vol.2)
13	1989	Paris, France	ED411140(vol.1), ED411141(vol.2), ED411142(vol.3)
14	1990	Oaxtepex, Mexico	ED411137(vol.1), ED411138(vol.2), ED411139(vol.3)
15	1991	Assisi, Italy	ED413162(vol.1), ED413163(vol.2), ED413 164(vol.3)
16	1992	Durham, USA	ED383 538
17	1993	Tsukuba, Japan	ED383 536
18	1994	Lisbon, Portugal	ED383 537
19	1995	Recife, Brazil	ED411134(vol.1), ED411135(vol.2), ED411136(vol.3)
20	1996	Valencia, Spain	Being processed
21	1997	Lahti,Finland	ED416082(vol.1), ED416083(vol.2), ED416084(vol.3), ED416085(vol.4)
22	1998	Stellenbosch, South Africa	ED427969(vol.1), ED427970(vol.2), ED427971(vol.3),

			ED427972(vol.4)
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	Being processed
25	2001	Utrecht, Netherlands	
26	2002	Norwich, United Kingdom	

PME North American Chapter

No.	Year	Place	ERIC number
2	1980	Berkeley, California (with PME2)	ED250186
3	1981	Minnesota	ED223449
4	1982	Georgia	ED226957
5	1983	Montréal, Canada	ED289688
6	1984	Wisconsin	ED253432
7	1985	Ohio	ED411127
8	1986	Michigan	ED301443
9	1987	Montréal, Canada (with PME11)	ED383532
10	1988	Illinois	ED411126
11	1989	New Jersey	ED411132(vol.1), ED411 133(vol.2)
12	1990	Oaxtepex, Mexico (with PME14)	ED411137(vol.1), ED411138(vol.2), ED411139(vol.3)
13	1991	Virginia	ED352274
14	1992	Durham, New Hampshire (with PME16)	ED383538
15	1993	California	ED372917
16	1994	Louisiana	ED383533(vol.1), ED383534(vol.2)
17	1995	Ohio	ED389534
18	1996	Panama City, Florida	ED400178
19	1997	Illinois	ED420494(vol.1), ED420495(vol.2)
20	1998	Raleigh, North Carolina	ED430775(vol.1), ED430776(vol.2)
21	1999	Mexico	ED433998
22	2000	Tucson, Arizona	ED446945
23	2001	Snowbird, Utah	
24	2002	Athens, Georgia	

Abstracts from articles can be inspected on the ERIC web site (http://www.accesseric.org) and on the web site of ZDM/MATHDI (http://www.emis.de/MATH/DI.html).

Micro fiches with the content of the proceedings might be available for inspection at university libraries or can be bought from: ERIC/CSMEE, 1929 Kenny Road, Columbus, OH 43210-1080,
Tel: (614) 292-6717, Fax: (614) 292-0263, e-mail:ericse@osu.edu
MATHDI is the web version of the Zentralblatt für Didaktik der Mathematik (ZDM, English subtitle: International Reviews on Mathematical Education). For more information on ZDM/MATHDI and its prices or assistance regarding consortia contact Gerhard König, managing director, fax: +49 7247 808 461, email: gk@fiz-karlsruhe.de.

THE REVIEW PROCESS OF PME26

Research Forums

Four Research Forums were suggested by the International Committee and the Programme Committee for PME26: **RF1** Abstraction: Theories about the Emergence of Knowledge Structures; **RF2** The Nature of Mathematics as Viewed From Mathematics Education Research; **RF3** Measuring Mathematics Learning and Describing Goals for Systemic Reform; **RF4** From Number Patterns to Number Theory: Issues in Research and Pedagogy. For each Research Forum the proposed structure, the contents, the contributors and their role were reviewed and agreed by the Programme Committee.

Discussion Groups and Working Sessions

Eleven DG proposals, of which all but two were accepted, and four WS proposals were received, of which all were accepted. After consultation with the PC one DG agreed to become a WS. Overall there are 8 DG and 5 WS in the programme.

Research Reports

The Programme Committee received 237 Research Report proposals. Each proposal was sent for blind review to three reviewers. As a rule, proposals with at least two recommendations for acceptance were accepted. The reviews of proposals with only one recommendation for acceptance were carefully read by at least two members of the Programme Committee. When necessary, the Programme Committee members read the full proposal and formally reviewed it. Proposals with three recommendations for rejection were not considered for presentation as research reports. Altogether, 165 research report proposals were accepted. When appropriate, authors of proposals that were not accepted as Research Reports were invited to re-submit their work — some in the form of a Short Oral Communication and some as a Poster Presentation.

Short Oral Communications and Poster Presentations

The Programme Committee received 91 Short Oral Communication proposals and 18 Poster Presentation proposals. Each proposal was reviewed by at least two Programme Committee members. From these proposals 59 Short Oral Communications and all Poster Presentations were accepted. Of the rejected Short Oral Communication proposals, 12 were recommended as Posters.

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The PME26 Programme Committee thanks the following people for their help in the review process:

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INDEX OF PRESENTATIONS BY RESEARCH DOMAIN

The papers in the Proceedings are indexed below by research domain, as indicated by the authors on their proposal form. The papers are indicated by their presenting author and page number.

1 Advanced mathematical thinking

2 Affective factors

3 Algebra

4 Assessment

5 Beliefs

15 Mathematical modeling

16 Measurement

17 Mental models

18 Metacognition

19 Proof

20 Probability

21 Problem solving

22 Rational numbers

23 Sociocultural studies

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PLENARY LECTURES

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LITTLE CHILDREN, BIG MATHEMATICS:
LEARNING AND TEACHING IN THE PRE-SCHOOL

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Teachers College, Columbia University

Three-, 4-, and 5-year-old children (in the U.S., we call them pre-schoolers) are little. They are short. They do not know as much as adults. As Piaget suggested, their thought may be different from adults'. Mathematics is big. It is one of the crowning intellectual achievements of humanity. It is abstract, symbolic, and difficult. Many believe that mathematics is too big to teach to little children because they cannot understand it a meaningful way.

I strongly disagree, and in this talk will present a different view of early mathematics education. I will show that little children are more competent than we usually think they are. They have a great interest in mathematical ideas and even in mathematical symbolism. They can learn mathematics and can benefit from teaching. I will also describe how research can both stimulate and be stimulated by the development of a rich program of early childhood mathematics education.

COUNTING AND EVERYDAY MATHEMATICS

In this talk, I will focus on one small area of young children's mathematical knowledge, namely their acquisition of counting <u>words</u>, the language of counting—not enumeration, not understanding of cardinality, not any of the other important topics on which researchers have focused.

Why place so much emphasis on counting words? At first, the topic appears to be relatively uninteresting. Counting is often seen as a mechanical process; it is not considered to entail genuine mathematical thinking. Children seem to acquire counting through a process of rote learning. In this view, although children clearly need to learn to count, mathematics educators and researchers should devote attention to deeper mathematical topics.

I think these views are wrong. First, counting is not simply a matter of rote memory; it is an interesting cognitive activity. And second, counting can present young children with opportunities for rich mathematical learning. It is worth thinking about how children learn to count and how it can be taught. Moreover, the example of counting suggests some general lessons about research on the psychology of mathematics education and on curriculum development.

We know several things about counting language. First, young children like to count, even to high numbers. Walkerdine (1988) reports that "... children initiate a counting sequence and often count spontaneously" (p. 105). Irwin and Burgham (1992) describe how children at 5 years of age enjoy counting up to relatively large numbers, like 100. Similarly, Gelman (1980) observes that some children exhibit a spontaneous interest in large numbers. "I remember a five-year-old who told me that she had been trying to count to a million, had been at it for two days and said it would still take her a long time to reach her goal!" (p. 65).

I have observed similar phenomena.

> One day during "free play" Kasheef, a 5-year-old African-American boy, was sitting at a table alone. He poured a large number of small beads from his hand onto the table. "Oh, man! I got one hundred," he proclaimed. Then, he began to count the beads as he picked them up one by one, although grasping them was difficult because they were so small. When he reached "ten," Barbara joined the activity. She said the numbers but did not actually count out the beads. As he progressed through the twenties, Kasheef made a mistake. He dropped the bead as he said "twenty-six," but ignored it and continued counting. Barbara kept pace with him. When he got to "thirty," Kasheef dropped the bead again. He paused for a second and said, "Wait! I made a mistake."

So here is a child who spontaneously decided to enumerate a large collection of objects, estimated their number, said the numbers correctly through 30, and performed a 1-1 matching of the spoken numbers to the objects. No one told him to undertake these activities. He made two mistakes, but recognised the second. He was highly motivated, persisting over a period of time despite the physical difficulty of the task. And he was joined by a peer who also found the activity of some interest.

> Kasheef began to count the beads again. "One, two, three...." When he counted "three," Barbara picked out one bead, showed it to him, and said, "I have one." But Kasheef ignored her and continued to count. Barbara joined him at "five." When he counted "ten," Barbara again showed her beads to Kasheef, "I got, look...." Kasheef again ignored her and continued counting. Barbara kept pace with him, uttering the same number words. When he counted "twelve," Barbara shouted in his ear, as if she wanted to distract his attention. Kasheef ignored her again. When Kasheef reached "forty-seven," another girl came over and asked, "What do you count?" Again, he ignored the distraction. After the "forty-nine", Kasheef paused, apparently not knowing what came next. Barbara said, "fifty" and Kasheef followed her with "fifty-one, fifty-two...."

Kasheef had clearly decided to count the whole collection of beads and was not about to be distracted from the task. He persisted despite several interruptions. He counted very well, making almost no mistakes until he reached "forty-nine." At this point, he needed and benefited from Barbara's help. But once given "fifty," he knew what to do with it: he knew the rule for constructing numbers from decades and units.

> When they reached "fifty-two," Ruthie came to the table, picked out one bead, and joined their counting, "fifty-two, fifty-three..." Madonna also joined the counting, "fifty-six, fifty-seven..." The female chorus said the counting words but sometimes picked out

several beads at once or sometimes didn't pick out a single bead. After "seventy-nine," Kasheef again paused. After the girls said, "eighty," Kasheef continued, "eighty-one, eighty-two..." When they reached "eighty-five," the girls fought with one another to grab beads, which then scattered on the table, dropped on the floor and rolled in every direction. Despite the disturbance, Kasheef continued, "eighty-six, eighty-seven... ninety-four."

Kasheef's counting drew a crowd of girls who found the activity of interest. They were competent in saying the counting numbers, but were not careful in enumerating the objects. They were disruptive. Kasheef largely ignored them, except when they provided him with the numbers he did not know. Kasheef showed remarkable persistence in his attempt to count each bead and to say the numbers correctly. He was almost always accurate, getting help on only two occasions.

After he picked out the ninety-fourth bead, there were no more on the table. He picked up some beads from those on the floor, and continued, "ninety-five, ninety-six, ninety-seven..." What he said next was inaudible. Madonna shouted, "One hundred!" raising her arm. Kasheef, Barbara, and Ruthie said, "One-hundred!" right after her. And Kasheef concluded, "We all got one hundred." In the course of the celebration, the girls got down under the table and began to squabble over picking up the beads. Kasheef joined the fight. At this point the teacher came over and yelled at the children for their misbehaviour.

We see then that Kasheef continued to count very well, at least to "ninety-seven," and like the other children was very proud to reach the goal, "one-hundred." What the teacher observed was a group of children squabbling under the table.

Many features of this example stand out. Kasheef and the other children were able to count, making only a few errors, to 100. They showed great interest both in the counting numbers themselves and in counting as a means to determine a quantity. Also, Kasheef persisted, despite frequent distraction; he was excited, interested, enthusiastic, hard-working.

Kasheef's behaviour was not unique. In a large naturalistic study—which I will not describe in detail here—we found that during free play, 4- and 5-year old American children from various social class and ethnic groups spontaneously and frequently engaged in "everyday mathematics" (Ginsburg, Pappas, & Seo, 2001). This refers to the mathematical skills and competencies children employ in the ordinary environment. Young children's everyday mathematics is usually intuitive (like the idea that adding makes more), (Brush, 1978); it is informal (not acquired through formal instruction); and it is free of written symbolism. In our study, children exhibited everyday mathematics in a little more than 40 percent of the minutes during which we observed them. Further, we failed to uncover statistically significant social class differences in the overall frequency of everyday mathematics. For example, the average percentages of minutes in which lower-, middle-, and upper-SES children engaged in everyday mathematical behaviour were 44, 43, and 40 respectively.

We also found that three types of everyday mathematics occurred most frequently in children's free play: enumeration, magnitude, and pattern. Enumeration included saying the counting words, enumerating objects (like Kasheef), and observing the quantity of a set of objects (e. g., "There are five here"). Magnitude involved such activities as judging that one set had more than another or that a quantity was very large. And pattern was shown when, for example, children devised elaborate symmetries, in three dimensions, during block play. Again, we failed to uncover statistically significant social class differences in the frequency or complexity of these three types of everyday mathematics.

In brief, our research clearly reveals that children frequently and spontaneously engage in an everyday mathematics that includes not only counting but also work with magnitudes and patterns. The research also shows that teachers were almost never involved in these mathematical activities.

We also know something about the cognitive processes underlying counting (Miller & Parades, 1996). To count successfully, learners (children or adults) must first memorise a relatively small set of meaningless sounds in an arbitrary sequence. In English, these sounds—a kind of meaningless "number song" (Ginsburg, 1989)—include "one, two... ten." Second, learners must induce and then use rules for generating subsequent number words. In English, the rules for generating 21 through 99 are reasonably clear: take the decade word (twenty, thirty, and the like) and add to it the memorised sequence "one... nine" and then go to the next decade word. Use of rules like these was evident in the fact that once given a decade word he did not know, Kasheef easily attached to it the unit numbers.

Unfortunately, English presents learners with several difficulties. One is that a few number words above "ten," namely "eleven" and "twelve," appear to be arbitrary. A second is that generating the decade words is not so simple. "Seventy" bears a clear relation to "seven" whereas the relation between "twenty" and "two" is not entirely transparent. A third difficulty is really serious. The number words from "thirteen" to "nineteen" put the units (e. g., "thir") first and the tens ("teen") second, whereas all number words after 20 do the reverse, as in the case of "twenty-three."

Counting need not be as odd as occurs in English. East Asian languages based on the Chinese are perfectly regular. To be sure, their learners must memorise the equivalent of "one" through "ten," but after that they can benefit from a consistent set of rules. After "ten," comes "ten-one... ten-nine," and after that, with perfect regularity, come "two-ten, two-ten-one, ... nine-ten-nine." Perhaps the coherent design of Chinese counting contributes to its speakers' mathematical understanding.

In brief, learning the language of counting is a key aspect of everyday mathematics. Children enjoy counting and often try to count to large numbers. Counting requires more than memorisation; like other aspects of language, it is based on a system of rules.

Consider next how counting can be taught and what the teaching of mathematics teaches us.

A MATHEMATICS CURRICULUM

For many years, the message taken from Piagetian research was that young children are egocentric and cannot understand important mathematical ideas. Certainly young children's thought suffers from various limitations. At the same time, the recent research literature on everyday mathematics paints a different portrait, namely that young children possess greater competence and interest in mathematics than we ordinarily recognise.

Given this view of young children's abilities, I decided to develop a comprehensive mathematics curriculum for 4- and 5-year-olds. Why is it useful to do so? One might argue that a better policy would be let young children enjoy and grow their everyday mathematics. In this view, young children learn from play and cannot profit from formal instruction; indeed, early lessons will only have the effect of turning them against mathematics at an earlier age than usual.

My answer to this argument is that play is not enough for several reasons. First, in the American context at least, many children, particularly poor children, are at risk of school failure and failing schools. They need a kind of mathematical head start to succeed in school. Second, mastering challenging tasks fosters children's feelings of confidence and competence (Stipek, 1997). Third, many teachers are not aware of young children's mathematical competence. Pre-schoolers' successful learning of formal mathematics may help teachers appreciate the children's competence and raise expectations of how much they should be learning in school in the later grades. And most importantly, our observations show that in their everyday play, the children already engage in and enjoy significant mathematical learning. It is great fun for them and of great intellectual significance.

As John Dewey (1976) advised: "Abandon the notion of subject-matter [like mathematics] as something fixed and ready-made in itself, outside the child's experience..." (p. 278). As we have seen, the child's experience includes an everyday mathematics that is more substantial and enjoyable than much of what is typically taught in school. Given this fact, as Dewey puts it, "Guidance is not external imposition. <u>It is freeing the life-process [that is, children's everyday mathematics] for its own most adequate fulfilment</u>" (p. 281). We therefore choose to engage in early mathematics education to help children achieve the fulfilment and enjoyment of their intellectual interests.

In collaboration with my colleagues Carole Greenes and Robert Balfanz, and with the generous support of the National Science Foundation, I have been engaged in the development of a comprehensive mathematics curriculum for little children, Big Math for Little Kids (Ginsburg, Greenes, & Balfanz, 2003). Our goal is to help children to think mathematically and to explore mathematical ideas in depth, over a lengthy period of time and through extended activities. Building on children's everyday mathematics, the curriculum consists of what we think are exciting and enjoyable activities, games, and stories, organised into six major strands, namely number, shape, measurement, patterns and logic, operations, and spatial relations. Our program presents the study of mathematics both as a separate "subject" and as an integrated part of other pre-school activities. Sometimes, the curriculum presents "math activities" like counting or studying shapes. Sometimes, it blends the mathematics into such activities as stories, songs, and block building. The program employs large group activity, small groups, and individual exploration. We believe that young children need to learn how to behave and learn in large groups. But they also profit from the greater degree of teacher attention possible in small groups. And they need time for individual learning and exploration. We characterise the program as offering playful but purposeful learning.

TEACHING COUNTING WITH PERSONALITY AND PIZZAZZ

I would like to describe what we learned from one small but I think important part of the program, namely learning the counting words. Our first counting activities were in good measure based on the research already described. We knew that children enjoy counting and like to count to high numbers. We also knew that the first 12 numbers in English are meaningless sounds that need to be memorised, that the numbers from 13 to 19 are "backwards," and that the numbers thereafter are very regular. In most curricula, 4-year-olds are generally taught to count to about 20, and 5-year-olds to 31 (the highest number of days on the calendar). Just think what this means. We ask young children to learn the harder part of counting before the easier part. They must learn the memorise the first 12 meaningless sounds, then learn the backwards numbers that violate the rules that later become evident, and finally are not allowed to count far enough into the larger numbers so that they could detect the only rational part of the English counting system. It is as if we wanted to make learning to count as hard as possible for young children.

Our solution was to make counting easier by helping young children to count to one hundred rather than 20 or 31. Counting from 21 to 99 makes sense, and then it's fun to shout "one hundred" at the end. Counting from 1 to 20 does not make sense and is hard to do. So we devised several activities to promote counting. In "Numbers with personality," children assign each "decade" a distinctive personality. They decide that the numbers from "one" to "nine" will be timid, and say them in a quiet, mousy voice. Later they might describe the twenties as "roaring like a mouse" or the forties as whining like a baby. We almost always refer to the numbers between "ten" and "nineteen" as the "yucky teens," because they suffer from ugly design features.

In another variant of this activity, "Counting with Pizzazz," the children choose different body movements for the decades. They may jump the twenties and hop the thirties. And when they get to the end of a decade, they always say, "cut," or "think," to indicate that they have reached the end of a decade and that they must decide what new decade term comes next.

Teachers using our program generally choose to do these activities every day throughout the school year. In New York, our wonderful pre-school teacher, Suzanne Mir, does counting at the beginning of every day during "circle time" (which in her room morphs throughout the year into square, rectangle, and pentagon time). The children enjoy the activities, they seem to be learning some interesting things about counting that I will soon describe, and we all get a good workout.

INTRODUCING NUMERALS

For some time, we taught counting with the activities described above. But then we developed another aspect of the counting curriculum. The story begins with an observation of a 3-year-old's behaviour. One of our activities is a game in which children step from one written numeral, printed on a large cardboard card, to another. They may start on 1, then have to locate 2, step on it, and then proceed in order to 5 or 10. This obviously helps them recognise the sequence of written numbers. For our present purpose, the game itself is not important. What is interesting is that one day, 3-year-old Arlo went to the place where the numerals were stored, took them out, and during free play tried to arrange them in order. He was successful in arranging the first several numerals, but skipped one of them. He then put a few numerals in the wrong order. After receiving some help from one of the graduate students working on the project, Angelika Yiassemides, he quickly learned to order the numerals to 10. Then he continued, and within a few days, got up to about 30 with few mistakes. He kept asking for more numerals to arrange in order, and we had to keep making them. We eventually made numerals up to 100 and he was well on his way to mastering the entire sequence.

I was surprised at his intense interest in these symbols. But then it occurred to us that use of numerals could aid counting. Numerals after all are perfectly regular with respect to place value and the base ten system and hence may be easier to deal with than English counting. The tens are always on the left and the units on the right. There are no irregularities like "thirteen." We write "thirteen" as 13, not 31, (which is similar to the way we say it). Of course, knowledge of counting can help children learn to read and give meaning to the numerals. In brief, if children are interested in counting and they are also interested in numerals, then the two activities might reinforce each other. Further, from the child's point of view, reading numerals is even more "grown-up" than counting them.

So we devised a number chart in which starts with numerals ending in 0 on the left and 9 on the right. We developed activities in which the teacher pointed to the numerals as children counted out loud. "Cut" or "think" was always said when children reached a numeral ending in 9. The new decade could be found by going down a row and to the end at the left. At first the teacher said nothing about the numerals. Gradually she pointed out various features, for example that the 3 in 34 referred to thirty; that 20, 30... are just like 2, 3...; and that as you go straight down a column all numbers end with the same numeral. She sometimes left out a numeral in the chart and asked the children to tell what was missing. In short, we felt that linking written numerals to spoken numbers could make underlying patterns relatively transparent and thereby facilitate their discovery.

One of our reasons for doing this is that our previous research showed that discovering and using patterns is one of children's most frequent everyday activities. Children are pattern detectors—in language, perception, and mathematics. This is not really a constructivist idea; it is a "realist" notion. Patterns exist in the world—sometimes the constructed world of mathematics—and can be detected. The detection may provide the necessary conditions for subsequent construction.

Of course the discovery of mathematical patterns in charts is not new. For example, Trivett (1980) offers a wonderful description of the mathematics that can be discovered in the multiplication table. But what is new, I think, is using this method at the pre-school level to teach counting, and potentially even related notions revolving around place value and the base ten system.

WHAT CAN WE LEARN FROM THE LEARNING OF COUNTING?

These then are some of the ways we teach counting. What can we learn from our experiences? How successful are the activities and what do children learn from them? Although we have not yet done traditional evaluations of Big Math, we have had the opportunity to conduct, over a long period of time, many informal observations of children's learning in our schools in New York, Baltimore and Chelsea, Massachusetts. In my experience, these "anecdotal observations" often point to important phenomena, can provide useful insights, and lead to interesting speculations and topics for further research.

First, almost all children thoroughly enjoy counting activities (and other aspects of mathematics). At the outset, young children seem highly motivated to learn school mathematics—at least in our Big Math curriculum. They are not turned off by early mathematics and do not feel that learning it is an imposition. There is no reason for early childhood educators to fear that little children will find mathematics instruction to be unpleasant. The issue is rather how and why the initial motivation gets undermined over the course of the first few years in school.

Second, almost all children are proud of their success. Little children see counting prowess as a sign of being a "big kid" and being "smart." Probably their teachers and their parents share this belief. It is important for little children to achieve big results. Real achievement makes for self-confidence and heightened motivation (Stipek, 1997). It is important to present children with challenging work so that they can master it. We need to allow children to succeed.

Third, it appears that by the end of the year many of the 4-year-olds do indeed succeed. Many can count to at least 100 and can count by tens to 100 as well. (They can also achieve at a high level in many other areas of our curriculum.) Yet I do not want to exaggerate their success. It is clear that not all children succeed at this level. For example, one child I recently observed (after about 7 months of counting activities) claimed that "twenty" comes not only after "nineteen" but also after "thirty-nine" and any other number she was asked about. Clearly, careful evaluations with pre- and post-tests and control groups are necessary to make firm statements about children's achievement.

Fourth, we have observed that some 3-year-olds in the classroom participate in the counting activities for many months without saying anything, but after a period of time may suddenly start counting to a reasonable level. They seem to be absorbing a great deal without exhibiting their learning in external behaviour. It will be important to investigate the kinds of learning in which 3-year-olds engage during systematic mathematics instruction.

Fifth, we have observed that children's learning seems to be very inconsistent. Some children seem to "get it" on one day but not another. Others, like the 3-year-olds, seem not to "get it" at all for long periods of time but then seem to catch on very quickly. Most often we assume that that learning is more or less orderly. But writers like Siegler (1996) have pointed to the inconsistencies and general messiness of learning, and researchers like Dowker (1998) have highlighted the important role of individual differences in mathematical abilities. Clearly, greater insight into these matters will help us calibrate instruction to young children's needs. Our teachers sometimes worry that the children do not seem to be learning immediately. (Some parents are even more worried about this than the teachers.) We have learned to ask teachers (and parents) to be patient and not to worry about whether the children "get it" right away. I think our advice is generally sound; but it would be useful to have a deeper understanding of these issues and some firmer evidence upon which to base the recommendations.

Sixth, what mathematical ideas, if any, do children learn when they are learning to count in the Big Math program? Our observations suggest that they learn not only to say the counting words, but to discover the rules that underlie counting and to explore patterns that can be discovered in the counting numbers. Even though it is a highly symbolic and abstract activity, learning to count can be an exciting pattern detection and generation activity. Exploring patterns in mathematics need not involve manipulating concrete things. Instead, counting involves the manipulation of words and of ideas about words. The children also seem to learn that counting and numerals are related and have a similar structure, which, of course, involves ideas about place value and the base ten system. At the same time, the children probably do not learn from our counting activities very much about cardinality. But that does not necessarily imply that the children are failing to do interesting mathematical work when they count. Cardinality is not the only important thing there is for young children to learn about mathematics. (Of course, other parts of our program deal directly with cardinality). In brief, even learning a counting language and relating it to abstract written symbolism can present little children with important intellectual challenges and enable them to do some interesting mathematics. There's a conjecture for further research. And of course, studying many other topics at the pre-school level can also lead to genuine mathematical learning.

Seventh, although I have been stressing young children's competence, it is also clear that there are many things they cannot do and learn. We have not had great success in teaching ordinality, for example, and I can once again confirm that ideas of equivalence are very difficult for young children. Little children cannot learn everything. But the central message is that they can learn a great deal and we don't yet have a clear idea of the limits on their learning when they are engaged in an exciting curriculum with a good teacher.

Eighth, group instruction appears to be very effective in teaching counting and almost all aspects of our program. Most of our Big Math activities are introduced in the group setting, later extended to small groups, and then to individual work. I was never a believer in group instruction for little children. In fact, I think I have written some nasty things about this practice. Yet I have been surprised at how effective it can be. But I don't understand how it works and how effective teachers operate. Little is known about group and individual pedagogy in the early childhood setting; learning more about it would no doubt lead to many practical benefits, including effective teacher education.

And finally, our experiences teach us something about the processes of research and curriculum development. We began with some research on young children's abilities. Drawing upon this research, along with other resources, like our informal experiences with young children, we developed a curriculum. In the process of trying it out, we observed that children were engaged in some surprising learning. Indeed, it is fair to say that the children would not likely have exhibited this learning and the competencies it implies had we not developed the curriculum. The observation of the new learning then led to the revision of our educational goals and the creation of new curriculum, which in turn led to new and perhaps unexpected behaviour that needs to be studied and explained. So research leads to curriculum and curriculum leads to research. Exciting mathematical environments create new expressions of competence, which is not static. As Vygotsky (1978) maintained, we need to study the "...dynamic mental state, allowing not only for what has been achieved developmentally but also for what is in the course of maturing" with adult assistance (p. 87). Observing children within a challenging curriculum allows us to do this.

CONCLUSIONS

Yes, mathematics is big, but little children are bigger than you might think. Early mathematics education is a great opportunity for children, teachers and researchers alike.

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RESEARCHING PRIMARY NUMERACY

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Abstract: I describe a major research programme on primary mathematics in the UK, the Leverhulme Numeracy Research Programme, and give a sample of some of the preliminary results. The Programme combines large-scale longitudinal survey and case studies, quantitative and qualitative data, and observation and intervention studies in order to try to ascertain how and in what way different factors affect pupils' progress. The case-studies have focused on factors relating to home cultures, pupil behaviours, curriculum and teaching styles, teacher subject knowledge, and school policies/leadership. The results quoted relate to these areas and also to effects of the implementation of a national reform which occurred during the research programme.

BACKGROUND

As in other countries, there has been continual concern over the standards of calculation in primary schools (ages 4-11) in the UK (Brown, 1999). This has become more urgent and more political with the publication of international comparisons first at secondary and more recently at primary level (e.g. Lapointe, Mead & Askew, 1992; Mullis *et al.*, 1997), and the realisation that countries in the developed world will need a highly skilled workforce to maintain their economic competitiveness.

In English-speaking countries this has led to an increasing desire by governments to control primary mathematics. In England this first led from complete freedom for teachers over both curriculum and teaching methods in the 1970s to a legally imposed broadly defined national curriculum in 1989/90 (Johnson & Millett (Eds.) 1996) together with related national tests to be at the end of each stage (2 to 4 years). More recently continuing political concern over standards of basic skills in primary schools led to a National Numeracy Strategy, introduced in 1999/2000, which has incorporated a much tighter prescription of content, teaching sequence and teaching methods.

The key features of this National Strategy are:

- *an increased emphasis on number and on calculation, especially mental calculation,* including estimation, and selection from a repertoire of strategies;
- *a three-part template for daily mathematics lessons,* starting with 10-15 minutes of oral/mental arithmetic practice, then direct interactive teaching of whole classes and groups, and finally 10 minutes of plenary review;

• *detailed planning using a suggested week-by-week framework of objectives,* specified for each year group, which introduces many skills at an earlier stage than previously, and covers areas of mathematics other than number;

• *a systematic standardised national training programme,* run by consultants locally and by school mathematics co-ordinators in all schools, using videos to demonstrate 'best practice', with in-school support for low-performing schools.

Although not legally imposed, the Numeracy Strategy has been almost universally implemented, and is being extended in a slightly modified form to secondary schools.

The meaning of *numeracy* reflects the social context of its use (Brown et al., 1998). It was accepted by educational policymakers in the UK that numeracy was to be defined broadly, as in other countries, as the competence and inclination to use number concepts and skills to solve problems in everyday life and employment. Nevertheless it was felt necessary, for political and educational reasons, that the aspects of numeracy to be newly emphasised at primary level should focus on proficiency (DfEE, 1998), regarding numeracy as a culturally neutral and value-free set of autonomous skills, underpinned by visual models (e.g. the number line). In contrast to 1980s developments, there are few references to problem-solving and those which occur are mainly traditional 'word-problems', with artificial contexts. In the remainder of this paper 'numeracy' is to be interpreted in this narrow way, although I would espouse a much broader interpretation relating to social practices (Baker & Street, 1993).

The national concern about numeracy also led the Trustees of the Leverhulme Trust, a charity, to fund a £1 million 5-year study, the *Leverhulme Numeracy Research Programme*, to run from 1997 to 2002. The aim of this programme is

• *to take forward understanding of the nature and causes of low achievement in numeracy and provide insight into effective strategies for remedying the situation.*

The design of the research programme is reported in the next section. We wanted to examine the contribution of many different factors to low attainment, in individual children, classes, schools or population groups, by studying, on both a large and small-scale, cases in which these factors varied. Intervention studies were also planned.

When the proposal was written it could not have been anticipated that a new government would quickly implement the National Numeracy Strategy. Clearly this has affected the Leverhulme Programme as the implementation occurred in the middle year (1999/2000) of the 5-year programme. For example it has meant that curriculum objectives, teaching sequence and aspects of teaching methods no longer vary between classes, and thus the effects of differences in these can only be perceived in data from the early years of the project. An intervention project concerning teacher professional development has also had to be modified to fit with the Strategy training courses. The Leverhulme work addresses fundamental issues in primary numeracy and will not merely act as an evaluation of the implementation of the Numeracy Strategy, which is being done, with the parallel Literacy Strategy, by a Canadian team (Earl *et al.*, 2000, 2001). But inevitably our data can be used to inform some aspects of the evaluation.

RESEARCH DESIGN

The Leverhulme Numeracy Research Programme is a longitudinal study that combines large-scale survey in a 'core project' with case-study data in five 'focus projects'. Two of the focus projects take the form of intervention projects.

The Core Project: Tracking numeracy *(Margaret Brown, Mike Askew, Valerie Rhodes, Hazel Denvir, Esther Ranson, Helen Lucey, & Dylan Wiliam; 1997-2002)*

Aim: To obtain large-scale longitudinal value-added data on numeracy to:

• inform knowledge about the progression in pupils' learning of numeracy throughout the primary years, and

• to assess relative contributions to gains in numeracy of the different factors to be investigated in the programme.

Methods: Data on pupil attainment has been gathered twice a year for 4 years, on two longitudinal cohorts of about 1600 pupils, one moving from Year 1 (age 5/6) to Year 4 (age 8/9) and the other from Year 4 to Year 7. Each cohort includes all children of the appropriate age in 10 primary schools in each of 4 varied local education authorities (about 75 classes). Detailed data is collected on pupils, teachers and schools including lesson observations, teacher interviews and questionnaires. Many instruments are modifications of those designed for our 'Effective Teachers of Numeracy' project (Askew *et al.,* 1997). This data forms the basis for both statistical and qualitative analysis to investigate the relative contributions of different factors. (The methodology of the testing in this project is discussed in a paper (Brown *et al.*, 2002) in the PME 26 Research Forum: Measuring Mathematics Learning and Goals for Systemic Reform.)

The core study provides a base for the case-study investigations in the focus projects, and has both generated hypotheses to be explored in the focus projects and allowed hypotheses arising from those to be checked on a larger sample.

Focus 1) Case-studies of pupil progress *(Mike Askew, Valerie Rhodes, Hazel Denvir, Margaret Brown & Helen Lucey; 1997-2002)*

Aim: To obtain a clear and detailed longitudinal picture of the numeracy development of a range of pupils taught in a varied set of schools and to examine this in the light of their classroom experiences, to ascertain what works, what goes wrong, and why.

Methods: This project is exploring the classroom practice factors influencing pupil attainment, including school, teacher, teaching, curriculum and individual pupil factors. From the longitudinal core sample we selected 5 schools which presented interesting contrasts. In each of these schools we selected children of varied attainment, six from a Reception (age 4/5) and six from a Year 4 class to provide longitudinal case study data, plotting progression in learning over 4 or 5 years. Children are observed and informally interviewed in two blocks of five lessons each year, and their written work collected. Longer interviews concerning perceptions of progress, attitudes and home support, and involving assessment questions, occurred at the end of Years 3 & 6.

Focus 2) Teachers' conceptions and practices and pupils' learning *(Mike Askew, Alison Millett & Shirley Simon; 1999-2002)*

Aim: To investigate the relationships between teachers' beliefs about, knowledge of and practices in teaching numeracy and whether changes in beliefs, knowledge and/or practices raise standards.

Methods: The project is following twelve teachers before, during and after their experience of a short course of professional development as part of the National Numeracy Strategy. We are adapting the methods of eliciting teachers' subject knowledge and beliefs from our earlier work (Askew *et al.*, 1997) in order to construct teacher profiles. Changes in teachers' practices are monitored using video recording of lessons, and changes in pupils' attainment by using the tests developed for the core project. The teachers' profiles, their classroom practices and their pupils' attainment will be monitored over three years.

Focus 3) Whole school action on numeracy *(Alison Millett & David Johnson; 1997-2001)*

Aim: To identify whole-school and teacher factors which appear to facilitate or inhibit the development of strategies for raising attainment in numeracy.

Methods: This research focuses on six schools as they each experienced an inspection and then implemented the National Numeracy Strategy. Each school had identified the need for improvements in their teaching of numeracy and we have collected data both on the strategies schools used to develop the teaching of numeracy and the effect of these strategies on pupils' attainment. The research is investigating the complex interplay of school factors (policies, leadership) and teacher factors involved in the implementation of change over four years. The research uses documentary analysis, observation in classrooms and at meetings, and interviews with a range of informants (headteachers, maths co-ordinators, classroom teachers, governors and parents).

Focus 4) School and community numeracies *(Brian Street, Alison Tomlin & Dave Baker; 1998 - 2002)*

Aim: To refine and establish the meanings and uses of numeracy in home and school contexts; to establish differences between practices in the two environments and to draw inferences for pedagogy.

Methods: This project is investigating the influence of social factors on attainment, in particular differences between numeracy practices, and the linguistic practices associated with them, in the pupils' home and school contexts. Three schools were selected to provide a range of home cultures. Case-study pupils were then chosen from Reception classes (age 4/5) and followed through Year 1 and into Year 2. We have been using ethnographic methods including participant observation of classrooms and of informal situations in and out of school, and interviews with teachers, parents and pupils. The study extends previous work on literacy practices (Street, 1996) into numeracy, but retains a comparative element between the two.

Focus 5) Primary CAME (Cognitive Acceleration in Mathematics Education) *(David Johnson, Mundher Adhami, Michael Shayer, Rosemary Hafeez, Sally Dubben, Ann Longfield & Jeremy Hodgen; 1997-2000)*

Aim: To investigate the effect on the development of numeracy of managed cognitive challenge/conflict designed to encourage verbal interactions and metacognitive activity in whole-class and various small group arrangements of children in Year 5 and Year 6

Methods: An experimental design is used to investigate whether intervention in classroom practices aimed at promoting intellectual development can be effective. It extends our earlier work on CAME (Cognitive Acceleration in Mathematics Education) in secondary schools (Adhami *et al.*, 1998) which uses Piagetian and Vygotskian paradigms. The research team, including teacher-researchers in each of two laboratory schools, first devised and trialled a sequence of mathematical problem situations designed to challenge children, and promote teacher-child and child-child discussion in cooperative small group work and whole-class discussion. This led to the main fieldwork involving research with teachers in a further 8 schools, with the teacher-researchers as tutors. We have used systematic observation of lessons and professional development sessions, and pre- and post- intervention pupil assessments of cognitive development and mathematical attainment. A linked study is demonstrating how this intervention acts as a basis for teachers' continuous professional development.

Coherence of themes

Although the structure of the Leverhulme Programme has been described as six projects, there has been great added benefit in the projects being part of the larger programme. The results of the programme will be published under a sequence of four common linked themes, to each of which several projects contribute.

- children's learning and progression
- teachers and teaching
- home, culture and school
- professional development of teachers.

In the next sections it is not possible to report the complete findings of the programme under each theme, but I will provide some sample results. This not only for lack of space (we are contracted to write a series of four books for Kluwer), but because at the time of writing we have not analysed all the data (we have another six months which will be devoted to data analysis). The results presented here must be regarded as provisional since neither all the data nor all the analysis has been finally checked.

SOME RESULTS: CHILDREN'S LEARNING AND PROGRESSION

1a) Lessons aimed at accelerating the cognitive level of children's mathematical thinking appear to show some generic results, but the results in terms of national assessment levels are more ambivalent.

The fifth focus project *Primary CAME* generated a series of 'Thinking Maths' lessons which teachers could use occasionally alongside the National Numeracy Strategy. The results in the group of experimental schools showed a significant difference compared with control schools in the rises in children's generic cognitive level as measured on a well-validated test. There was a problem administering a final mathematical problem-solving test because it was close in time to national tests at the end of primary school, so the raw national test scores were used instead as a post-test. Although there was some indication of a higher performance for experimental schools, there was no significant overall effect because of unexplained unexpectedly high results in the national tests from two control schools. This raises questions about using high stakes test results as a research measure as they may not be sufficiently reliable.

1b) The proportion of pupils who can answer a specific question increases with age approximately following a cumulative normal distribution with variations from this relating to the curriculum and testing regime. Changes in the curriculum as a result of the National Numeracy Strategy have had a significant effect on attainment in some areas.

Where children are exposed at an early stage to a fact, skill or idea the improvement in facility (the proportion of pupils who are successful) follows roughly a cumulative normal curve, as one might expect. The items that occur over several years in our tests which most closely match this model are those relating to multiplication facts shown in Table 1.

Item	*Year 1* Oct Jun	*Year 2* Oct Jun	*Year 3* Oct Jun	Year 4 Oct Jun	Year 5 Oct Jun	Year 6 Oct Jun
4x5	*10 38*	*37 65*	*72 87*	85 91		
7x8		*3 14*	*12 32*	31 47	56 74	77 83
9x9			*15 41*	49 65	75 86	89 92

Table 1: Facilities of multiplication items: younger cohort results in italic from October 1998 to June 2001; older cohort results from October 1997 to June 2000 (n>1600)

Interestingly there is no evidence of significant improvement on multiplication facts between the earlier results for the start of Year 4 in 1997 for the older cohort and the end of Year 3 in 2001 for the younger cohort; yet improvement of knowledge of multiplication facts was what the Education Minister promised the public as the effect of the Numeracy Strategy.

However there is evidence of the effect of the Strategy on other items. (This can be seen also in Table 1 in the other paper on the Programme in the PME26 Proceedings (Brown *et al.*, 2002)). On average, performance at the end of Year 3 is 10% higher than expected for the younger cohort who have experienced the National Numeracy Strategy; this is the equivalent of about 4 months' learning.

This effect seems to reflect a curriculum which is both more ambitious in that pupils are taught some material earlier than previously, and more focused on mental strategies which are a focus of the tests we use. (This comparison before and after the Numeracy Strategy will become clearer when we have the 2001/2002 results analysed for the younger cohort in Year 4 to compare with the previous cohort 4 years earlier.)

It might be claimed that the improved performance was due to changes in generic pedagogy rather than curriculum content but an examination of differences in performance on individual items shows that this is unlikely.

We found that for six items out of 65, the younger cohort at the end of Year 3 in 2001 had percentage success rate greater than 10% higher than that for the older cohort at the start of Year 4 in 1997 (and for only one item was the facility more than 10% higher at the start of Year 4). In the case of all these items improvements can be explained by references to curriculum change, in particular to increased early emphasis on the number line, inverse operations and horizontal recording. This seems to provide evidence of effectiveness of curriculum change in enhancing achievement, but early introduction may not necessarily result in long term benefit.

These results depend on data from the large-scale survey, although they can be understood by reference to case-studies. For example it was clear that at the start of the project in 1997/8 most schools were following broadly similar curricula related to one or more published schemes. However St. Luke's, a case-study school in the second focus project *Case-studies of pupil progress,* was following a significantly more ambitious curriculum than all the others. Most children in the class right from Reception year (age 4-5 years) were using textbooks intended for children one year older. The results from that school in both our tests and the national tests were exceptionally high. (The school had a somewhat above average intake, sufficient to make this policy viable but not to fully account for the results.) By the end of the project the school had purchased a new set of texts which were matched to the National Numeracy Framework, but this time most children were following the books intended for the correct year group, as the school was satisfied that the Numeracy Strategy curriculum was sufficiently ambitious. Correspondingly the results of the school although still above average are now somewhat lower than previously in relation to those of other schools.

Strong curriculum effects have been suggested by international studies at secondary level such as SIMS (the IEA Second International Mathematics Study) (Burstein, 1992). Nevertheless the between-countries, within-topic correlations between coverage and attainment across countries in SIMS were quite low (Robitaille & Garden, 1989). It is also true that when curriculum variation was taken into account in TIMSS - the Third International Mathematics and Science Study, only very small changes in the international rankings resulted at both primary and secondary levels (Mullis *et al.*, 1997, Beaton *et al.*, 1996). Other TIMSS studies suggest more subtle relations between curriculum and attainment (Schmidt *et al.*, 1996; Stigler & Hiebert, 1997).

1c) Over several years most children remain at roughly the same percentile of attainment, although with some oscillation. A few however gradually change their relative positions. The progress of some children appears to be held up because of some fundamental conceptual gaps.

We have been observing classroom behaviour in mathematics lessons of 30 children in each cohort (6 from each of 5 classes in different schools) over 4/5 years, and related these to their test results. For most pupils their percentile in the sample remains roughly constant with minor oscillations; however in some cases there seem to be longer-term trends.

Debbie, in the older cohort at Pinedene school, is a child whose test results oscillate considerably around the median, with no obvious long term trend. She started at about the median in Year 4, moving up to the 65th percentile at the end of the year but her performance dropped gradually through Year 5 to about the 35th percentile at the start of Year 6. By the end of Year 6 and again at the end of Year 7 she was back at about the 60th percentile. When we talked to Debbie she felt that she had learned a lot in Year 4 but had found the teacher and the work in Year 5 difficult to understand, and had recovered in Year 6 with a more supportive and more relaxed teacher. Her perceptions of the quality of the teaching and of her reaction to it correspond to our classroom observation data. This, and the fact that the changes in her performance are similar to, but more extreme than, the changes in the class performance suggests that in her case the quality of teaching is a key factor and that Debbie was particularly sensitive to it. (Some case-study children in the class did not follow the trend of class performance.) It also seems likely that her parents splitting up in Year 5 might have exacerbated the problems that year.

However it is also instructive to look at Debbie's mathematical profile. Although we saw her working with her friend excitedly learning about equivalent fractions in a pictorial context in Year 4, more abstract fractions, and more especially decimal fractions, remained a problem; in our tests she made no progress in this area between the end of Year 4 and the start of Year 6. Debbie volunteered to us in Year 5 that she did not understand these ideas and always got wrong answers in tests. The teaching we observed in that year was not addressing her problems. However during Year 6 the ideas fell into place and she scored quite well on those items at the end of that year and in Year 7.

Another child whose progress was held up by fundamental conceptual problems was Joseph. Joseph, at St. Luke's School and like Debbie in the older cohort, had a performance which gradually declined from near the 80th percentile at the start of Year 4 to about the 60th at the end of Year 6 and Year 7. Our classroom observation suggests that Joseph tried hard to remember standard algorithms but he often became confused and had a fragile understanding of place value to fall back on (his parents had arranged for a private tutor the previous year but Joseph said he did not find this useful). This apparent gap in understanding led to some continuing quite basic errors in place value, although at the end of Year 6 in some areas like fractions he showed quite sophisticated understanding. After a year in a high set at a prestigious school, Joseph's performance in the test deteriorated, both on fractions and on place value. It is interesting to speculate whether, in contrast to many successful children in his class, Joseph's poor progress was part of the 'collateral damage' of the decision by St. Luke's referred to earlier to accelerate the curriculum by a year.

Thus the progress of individual children shows many variations and appears to depend on many factors, relating to the child's ability, personality and inclinations, the home circumstances, and especially to whether the teaching addresses the mathematical ideas causing problems. For different children the balance of importance of these factors also changes. In spite of general trends, it is impossible to predict the future progress of any specific child from one or two test results.

SOME RESULTS: HOME, CULTURE AND SCHOOL

In this area we probably have as yet fewer conclusions than elsewhere since we are still writing up case studies and are awaiting the full data set on the tests before doing the analysis for gender, ethnicity, postcode, parental pattern, etc.. However we still have a few results, both on a micro and a micro-level, which suggest the likely final pattern.

2a) While the numeracy attainment of a school is quite closely related to a 'poverty indicator' of its intake, the gains made in numeracy are independent of this indicator.

In the UK as in some other countries the only readily available indicator of the background of the children in a school is the proportion of children claiming an entitlement to free school meals. As is generally the case with educational data (Levin, 1999) we found a negative correlation between this and average attainment for the school. For example in Year 5 the correlation was -0.63.

However when we looked at the relation between the gains children made over a year and the proportion of children claiming free school meals this correlation was reduced to almost zero (-0.06 for Year 5). This suggests that it is the fact that children from less advantaged homes start behind others which causes continued weak performance; there is no evidence that they make slower progress (see also Burstein, 1992). This is however an average result and the case studies point to many individual variations.

2b) The relationship between numeracy attainment and home circumstances for any individual child is very complex.

In the fourth focus project *School and community numeracies*, we have been studying social factors which may affect children's progress, with case study children moving from Reception to Year 2 in three schools, one an inner city multi-racial school, one a school in a prosperous suburb, and one on a long-established social housing estate with many social problems but with mainly white pupils.

This has thrown up some counter-examples to the general relationships referred to above. For example one of the most materially deprived families is that of Aaysha, whose parents are recent immigrants from Pakistan, as yet unemployed, speak little English (although her father attends classes) and live in temporary accommodation where facilities are basic and shared with many other families. Some of their home numeracy practices (e.g. methods of finger counting) differ from those taught at school. Nevertheless Aaysha is doing very well at school, becoming fluent in English and good at mathematics. An explanation for this is that both parents are well-educated and numerate, and worked in the insurance business in Pakistan; indeed her father has an

MA in statistics. Thus they have intellectual and educational, if not economic, capital, and a determination to succeed which is associated with recent immigrants.

On the other hand some second generation immigrants are less well placed. For example Kim lives with his mother and grandmother. His mother works long hours in a reasonably well-paid full-time administrative job which involves some numeracy, and he spends a lot of time with his grandmother, a nurse working shifts who many years previously had run a small 'home school' in Jamaica. Like many other children, Kim is expected to practise his numeracy skills at home. The problem in the Reception class (age 4/5) was that the pedagogy and underlying epistemology used by Kim's grandmother seemed to be out of line with those of the school. She expected Kim to be able to recall quite sophisticated number facts which he could not do, while asserting that the homework set by the school was too difficult. Kim therefore refused to engage with much of the numeracy work either at home or at school, and his frequent condoned absences led the school to believe that he was overindulged. There was improvement in home-school understanding and in Kim's attainment in Year 1 when his mother arranged for a discussion with his male teacher, also of Jamaican origin.

The experiences of Kim illustrate a general point that many families of our case-study children expect children to regularly do school mathematics at home, in addition to the homework set (normally an hour per week). Thus the predominant numeracy practices at home may not be domestic but those the parents perceive as 'school practices'. Some parents say they do this to support the work at school; for others it is to compensate for what they see as a lack of challenging numeracy teaching in the school. These practices occur across all social groups.

Yet in some cases, like Aaysha, this additional home support seems to be informal, sensitive and successful and in other cases, like Kim, to be limited to recall of facts with an unproductive outcome that adds to the child's sense of failure and discourages them from engaging further. It may be that families with greater intellectual and social capital can be more supportive as they are more sensitive both to the child's needs and to the nature of school numeracy practices (Galbraith & Chant,1990).

SOME RESULTS: TEACHERS AND TEACHING

3a) It is difficult when comparing across many classes of the same age to find a clear relationship between factors relating to teachers or teaching and the learning which occurs.

We looked in two different ways for relationships between different factors and the average gains for each class in a particular year group between October and June. First we correlated against the gains in attainment for Years 4 and 5 all our data from teacher questionnaires (at least 60 were returned from the 75 classes in each year group; many of the omissions were due to changes of teachers) which related to personal qualifications and experience and also to self-reported practices and beliefs. Several items were, with permission, taken from TIMSS (Mullis et al., 1997).

There were no factors which had significant effect sizes for both year groups, and only a very small number which were significant even for one year group. The factors included the age, length of teaching experience, training and qualifications of teachers, as well as frequency of whole class teaching, type and frequency of homework, and use of calculators and computers. These results are consistent with those for TIMSS both internationally (Beaton *et al.*, 1996; Mullis *et al.*, 1997) and when the English results were analysed independently for the Numeracy Strategy Task group.

In a previous study (Askew *et al.*, 1997) we found that teachers who were effective in teaching numeracy, defining effectiveness in terms of gains achieved by their classes over the year, tended to have a particular orientation to the teaching of numeracy which we characterised as 'connectionist'. In the Leverhulme study we wondered whether in the same way we could characterise 'teaching' as more or less effective. There have recently been several 'evidence-based' claims in the UK for effective styles of teaching (e.g. DfEE, 1998; Hay McBer, 2000; Muijs & Reynolds, 2001) and it seemed useful to see whether we arrived at similar results.

Having analysed what seemed to differentiate the lessons observed of Year 4 teachers whose classes made high and low average gains in 1997/8, we compiled a list of characteristics which were organised using an adapted version of a framework from Saxe (1991). We used this to devise a scale and rated the lessons of teachers in Years 1 and 5 the following year and Years 2 and 6 in 1999/2000. However the correlations with average gains were very low ($r<0.2$, except in Year 6, where $r= 0.39$). (It is not clear why the correlation was higher in Year 6 but it might be that the curriculum variation is reduced that year because of the high-stakes national tests). We tried another way of rating the lessons in terms of the connections between the mathematics, the teacher and the children. However again the correlations were low ($r<0.2$).

This suggests either that our observations, our ranking systems or our tests are unreliable, or that compared with pupil factors, teaching and teachers have a rather small effect on pupils' gains. Again this would not be out of line with other findings (e.g. Mortimore *et al.*, 1988; Creemers, 1997; Burstein, 1992), which suggest that even after the effect of the pupil variables have been removed the teaching accounts for at most 10% of the variance in attainment.

3b) For a particular class across different years, the relationship between teaching and learning seems clearer than across different classes.

We have not yet analysed the data from the full sample in terms of progress made in different years by the same class, but examination of case study data suggest that controlling the pupil factors more closely in this way means that the factors relating to teaching become more salient. Classes appear to make slower progress in years where the teaching ranking is low on our adapted Saxe scale. Those case study children who can review their progress over different years are generally able to identify accurately the years when least and most progress was made, and attribute this to the teaching.

SOME RESULTS: PROFESSIONAL DEVELOPMENT OF TEACHERS

4a) A short programme of professional development supported by in-school action can be almost universally effective in implementing change in curriculum and in aspects of teaching.

Teachers and headteachers in our sample have universally welcomed the National Numeracy Strategy and the support it has given them. They report improved grasp of the priorities for teaching and have changed their methods and curriculum to meet these (see also Earl *et al.*, 2001, 2002). They have praise for the training delivered in local centres, and in schools by the mathematics co-ordinators who are fellow-teachers. It is clear from our interviews and observations that all teachers have changed their practices and their curriculum. In some cases we have observed the same teacher with a Year 4 class in 1997/8 and 2001/2, and found a noticeable change in confidence.

Data from the six case study schools in the third focus project, *Whole school action on numeracy,* suggested that there were six factors relating to the effectiveness of the co-ordinator in improving the standards in the school:

- clarity of vision about priorities of action and ways of working;
- enthusiasm about the role;
- balance between headteacher and co-ordinator, each valuing each others' role;
- high priority to resourcing the co-ordination role, for example to enable co-ordinators to work with other teachers in their classrooms;
- coherence and consistency within the school community;
- regular external support available and used.

In the three schools where most of these factors were not present, a change of co-ordinator occurred during the implementation of the Numeracy Strategy. By the end of that year all three schools had improved their position with respect to these factors and the results in national tests correspondingly improved (Millett & Johnson, 2000).

4b) Professional development which changes and links together teachers' subject knowledge, beliefs, and practice is a much longer term and more difficult enterprise.

In the second focus project *Teachers' conceptions and practices and pupils' learning* we have been examining the changes taking place in relation to a 5-day Numeracy Strategy training course focusing on teacher subject knowledge. Although some of the teachers reported that it had increased their confidence, and some superficial changes in practice and curriculum occurred due to new aspects of pedagogical subject knowledge (e.g. the teaching of different methods of calculation), the course did not appear to have been sufficiently sustained or to have involved enough informed collaborative reflection (Cobb *et al*., 1997) to have had a strong effect in terms of subject knowledge, beliefs or practices.

In contrast the professional development practices developed in the fifth focus project *Primary CAME* involved teacher researchers modelling lessons in which teachers acted as pupils, then teachers trying lessons out in their own schools and finally meeting to discuss the outcomes. The cycle has similarities to the Japanese model (Stigler & Stevenson, 1991). These meetings continued over two years which was judged to be generally necessary for teachers to develop their beliefs, teaching practices and subject knowledge. The essential ingredients were *cycles of practice and (collective) reflection*, informed *by clear theoretical perspectives*.

CONCLUSION

The opportunity to view primary numeracy from many different perspectives, using different research methods, has allowed a holistic view of teaching and learning and avoided simplistic conclusions. Factors relating to individual pupils have strong effects. Curriculum seems a more salient factor than generic pedagogy. Both these can be changed quite rapidly at a systemic level, but development in teachers' deeper understanding is a long term process.

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HOW STUDENTS STRUCTURE THEIR OWN INVESTIGATIONS AND EDUCATE US: WHAT WE'VE LEARNED FROM A FOURTEEN YEAR STUDY

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This talk reports on a portion of a fourteen-year study of the mathematical thinking of a cohort group of students that is based on an identifiable perspective on how mathematical ideas are built. Three videotape episodes are presented so that we can view together groups of students working together at three points in time: as fourth graders (ages 9-10), and in grades 10 and 11 (ages 15-16). Excerpts from interviews are offered to provide student perspectives on their own learning over the years.

INTRODUCTION

Choosing a theme for this paper has been difficult; fourteen years of research have provided many possibilities[1]. After much deliberation, I decided to focus on how students, working in small groups, structure their investigations and what we have learned so far by studying how their ideas develop. This talk will introduce some of the students who participated in elementary and secondary school, and later as college students. During the last few months, several of these students have talked with us about their participation in the long-term study.[2] These data along with interview data from their upper-high school years are included here.[3]

Videotape and written data come from students' early investigations of counting problems in elementary school, through their investigations of combinatorics and probability in middle and high school, and their investigations of ideas related to calculus in high school time (Kiczek, 2000; Kiczek, Maher & Speiser, 2001; Maher & Martino, 1996a, 1996b, 2001; Maher & Kiczek, 2000; Maher & Speiser, 1997; Martino, 1992; Muter, 1999; Speiser, 1997; Speiser, Walter, Maher, 2001). Observations and analyses of the students' early explorations provide foundations for later thinking about particular ideas in mathematics. Videotape data make it possible

[1] We would like to thank the Kenilworth students for their continued, invaluable contributions to our work. Thanks, also, to the research team for their dedicated work on the project.

[2] Jeff, Romina, Michael and Brian, now second year college students, joined graduate seminars in which video segments of problem solving were viewed together. The seminar sessions were audio or videotaped. In these sessions, the students talked about how they learned by working together.

[3] Fred Rica, Principal of the Harding Elementary School, Kenilworth invited me, in 1984, to visit his school and to observe mathematics lessons. These earlier classes, for the most part, emphasized drill and memorization. Inspired by the belief that the children's mathematics learning could be significantly improved, a three-year teacher development project was launched in 1984 between the Kenilworth Public Schools and our group at Rutgers (See O'Brien (1995) for a ten-year analysis of the teacher-development project). The longitudinal study was an outgrowth of the partnership.

to view these sessions together. The three episodes -The Gang of 4 (ages 9-10); Romina's Proof (ages 16-17), and the Night Session (ages 17-18) - taken together, provide illustrations over time of how the students worked together in small groups, and of how the teacher(s)/researcher(s) interacted with them. Finally, video clips of student interviews (ages 17, 18, and 19) provide further commentary by participants about their own learning. Jeff (March 2002), reflecting on his participation in the research since grade one, focuses on the depth of their investigations and the impression this left on him.

You didn't come in and say, "this is what we were learning today and this is how you're going to figure out the problem." We were figuring out how we were going to figure out the problem. We weren't attaching names to that but we could see the commonness between what we were working on there and maybe what we had done in school at some point in time and been able to put those things together and come up with stuff and to do these problems to come up with, what would be our own formulas because we didn't know that other people had done them before. We were just kind of doing our own thing trying to come up with an answer that was legitimate and that no matter how you tried to attack it, we could still answer it. It was a solid formula that works no matter how you tried to do it.

BACKGROUND, PURPOSES, RATIONALE

The study was initiated in 1989 with a class of 18 first-grade children at a public school in a working-class community (Martino, 1992; Maher and Martino, 1996b)[4]. The work reported here is a component of the longitudinal study of the development of students' mathematical ideas. Attention has been given to studying how learners build mathematical ideas, create models, invent notation, and justify, reorganize, extend, and generalize their ideas. Data come from a cohort group of students whose mathematical activity has been followed by the research team for over 14 years[5].

The main objective of our research has been to gain a deeper understanding of mathematical learning when particular conditions are in place. We have been interested in creating conditions whereby we can give children an opportunity to show us how they think about mathematics. These conditions are essential to the context of the study and may be helpful in understanding issues of commitment, motivation, and value to participating students. In the early years of the project,

[4] The class was one of three first grades in the elementary school. The students in each class remained together for their first three years of elementary school. In grade four, new classes were formed. The study continued with a smaller subset of the original class and several other students who joined. The group that was followed for fourteen years consists of seven students; others (seven) participated for approximately eight years.

[5] Earlier work for this study was supported, in part, by National Science Foundation grants MDR9053597 (directed by R.B.Davis and C.A.Maher) and REC-9814846 (directed by C.A. Maher) The opinions expressed are not necessarily of the sponsoring agency and no endorsement should be inferred.

when the research was conducted in classrooms, these conditions, negotiated with the participating school district administrators and classroom teachers, guided the establishment of the context for our research. What came to be called "Rutgers mathematics" occurred in the early elementary years four to six times a year, for three days duration. For two of the days, their math period was extended approximately an hour to an hour and a half. The third day was the regularly scheduled math time of about 45 minutes. Students continued conversations through follow-up individual or small group interviews the same or the following week. When feasible, the classroom teacher observed the interview.

Interview design was motivated by our observations of the children doing mathematics in the classroom and of our study of videotapes and researcher notes. For example, we might notice that there would be an interesting idea that was being pursued by a child or a group, and we would ask the children to tell us about what they were doing. In the interview setting, we invited the reconstruction and extension of ideas put forward. To a large extent, the direction we took in the presentation of investigations was inspired by what the children showed us in their talking, drawings inscriptions, and building of physical models.

Students were invited to think about mathematical situations, often over long periods of time. They were asked to present their ideas with suitable justifications that were convincing to them (and to us) and to consider generalizations and extensions. They were not graded for their work; they revisited problems over months and years; they offered arguments for the validity of their solutions. Ideas were listened to and treated with respect by the teacher/researcher(s). In the early years, we called the students' attention to the variety of ways they represented their ideas, with the intent of making public both similarities and differences in their thinking (Maher, 1998a). We invited reflection and discussion among students about such differences (Maher, 1998b; Maher, Martino & Pantozzi, 1995; Maher, Davis & Alston, 1992). The children, in justifying their ideas, provided arguments that exemplified several important types of mathematical proof, for example, proof by contradiction, proof by cases, and reasoning by induction. In the later years, we observe students using all these forms of reasoning as natural parts of their discourse.

During the first eight years, the study was classroom based. Since high school, cohorts of students participated in small group after school sessions. This came about, at least in part, because the county school district formed a regionalization of the high schools. This resulted in the closing of the Kenilworth high school for several years. Most of the children, upon completing 8 years of elementary school, continued to a regional high school in another town. Others attended parochial schools in the area. Public pressure from Kenilworth citizens (and from some neighboring communities) to regain their local high school led to a public referendum that resulted in a vote to de-regionalize the schools, an historical event for the state. Some local high schools, including Kenilworth, re-opened in 1997 and most of the original students in the study were reunited. In that interim year, we continued our research with several students after school hours in private homes. The interest of students to continue once their high school was re-opened sparked the continuation of what we refer to as "after-school" mathematics. In 1996, classroom research was replaced largely by small group research.

THEORETICAL PERSPECTIVE ON LEARNING

We investigate the development of mathematical ideas by examining, from moment to moment, the development of students' thinking as it is indicated by their conversation and their inscriptions as they work on well-designed investigations.[6] The guiding framework for this level of analysis comes from research on representations (Davis, 1984; Davis & Maher, 1990; Davis & Maher, 1997; Goldin 2002, 2000; Kiczek & Maher, 1998; Kiczek, Maher & Speiser, 2001; Maher & Martino, 1997; Maher & Davis, 1995). In doing mathematics, mental images can be formed by individuals, to be used in building representations of mathematical ideas. These representations can be carried forth and used, and revisited and modified, in the light of new experiences. Although the internal, cognitive representations are not available to us and perhaps the individual, certain features of them can be made public and open to discussion. This can occur as ideas are explained, justified and shared with others. To represent an idea, an individual may create a structure or present a notation[7]. In this way, the ideas are made public in their discourse in the form of explanations, actions, writings, and notations. Records make possible later re-examination of the relationships between ideas. In this way, the ideas can be discussed and reflected upon (Dörfler, 2000).

RESEARCH METHODS

We regard events as connected sequences of utterances and actions by the learners. An event is called "critical" when it demonstrates a significant advance from previous understanding, or a conceptual leap in earlier understanding, or the identification of a cognitive obstacle (Maher & Martino, 1996a; Kiczek, 2000; Steencken, 2001). These episodes are obvious and striking in that they can be connected to prior events. Identification of critical events makes it possible to examine their influence on later understanding and to trace the development of ideas.

Data Source. Our main sources of data are: (1) behavior of students as they work on mathematical investigations recorded on videotape; (2) written work of students; (3) follow-up interviews of individual or small groups of students, often taking the form

[6] The mathematical investigations posed are central to the research. To view the tasks, visit the Robert B. Davis Institute for Learning website, http://www.rbdil.gse.rutgers.edu

[7] Willi Dörfler (2000) makes an important distinction between the visible representations and the internal, cognitive (re)presentations that are not visible to us.

of teaching experiments; (4) individual student interviews; and (5) researcher notes. Groups of us observed the children, took notes, noticed behaviors, and developed interests in what they produced individually, together, and through sharing with others. At various times, students revisited tasks and talked about their ideas. It was not uncommon for ideas to re-emerge for discussion over long periods.

Videotaping. Videotapes are made of nearly all sessions or task-based interviews. These have three main forms: (1) videotapes of task-based (or "clinical") interviews, where there would usually be one interviewer, one student, and two camera operators (one to record work; the other to record the conversation.) Also, two or more observers take notes, but not in view of the student; (2) videotapes (made outside the classroom in an office or quiet setting) similar to those just described, with 2-5 students working together on a task. During student investigations, there is usually no interviewer present. There are two cameras and a sound technician; (3) videotapes made in actual classroom settings, but otherwise similar to the second category with small groups of students working together. There are three cameras and two sound technicians. Our research results emerge through systematic study of extensive, archived videotape data, often from tape segments, which now, because of recent data, have been re-analysed from new directions, with newly developed tools and a more detailed framework.[8]

FRAMEWORK FOR ANALYSIS

A framework is offered that takes into account how ideas develop and travel within the group and how the teacher/researcher interacts in the process. The analysis begins with the identification of critical events. The mathematical content of each critical event is identified and described, taking into account the context in which the event appears, the identifiable student strategies and/or heuristics employed, earlier evidence for the origin of the idea, and subsequent mathematical developments that follow its emergence. Together, these components provide a "trace", the data tracking the development of the idea(s) (Maher & Martino, 1996a; Maher, Pantozzi, Martino, Steencken & Deming, 1996; Kiczek, 2000; Steencken, 2001).[9] We identify and code their traces in the form of diagrams. Concurrently, transcripts are verified, and explicitly co-ordinated to diagrammed events. Our interpretations evolve from all of these.

Event diagrams and codes.[10] Each critical event defines a timeline, consisting of a past, a present and a future.

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[8] See Davis, Maher & Martino (1992) for a discussion of using videotapes to study the construction of knowledge.

[9] The set of connected critical events with past, present and future defines a "pivotal strand". See Kiczek (2000) and Steencken (2001) for a discussion of pivotal strand.

[10] The framework was developed for the National Science Foundation grant MDR9053597 and further elaborated and extended in the project work. See Speiser (1998) for a prototype-coding

past————————present——————— future——

(critical event)

The critical event itself defines the present. Prior images to which the critical event folds back[11] define the past (both recent and more distant), while later events which help us understand (or fold back to) the present critical event define its future. The timeline is followed in strands of analysis, all of which are coded.

Constructing a storyline. Coded nodes denote events along the timeline, and descriptive codes are used to mark strands of events which we call the "flow of ideas". The construction of a storyline begins with the flow of ideas. We examine and identify codes and their respective critical events in an attempt to trace an emerging and evolving story about the data. A storyline is constructed from a coherent organization of the critical events, and often involves complex flowcharting. Hence, the process of producing a trace involves identifying a collection of events, coding those events, and then interpreting them, to provide insight into a student's cognitive development. The trace contributes to the narrative of a student's personal intellectual history as well as to the collective history of a group of students who collaborate.

Constructing narrative. In our model, a narrative phase enables researchers to view the recorded material from the data set holistically. Although they appear last, interpretative actions actually begin from the inception of research; they are originally formulated through theoretical perspectives and research questions of interest (Powell, Francisco & Maher, 2001).

TASKS AND VIDEO SEGMENTS[12]

Three tasks and video episodes are considered here. The tasks are selected from the counting/combinatorics strand and the episodes span an eight year period.

Grade 4 Task: Building Towers. Convince each other and the researchers that you found the number of towers that could be made 3-cubes tall, selecting from two colors.

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scheme for critical events and Powell, Francisco & Maher (2001) for further discussion of the framework and methodology.

[11] See Pirie & Kieren (1994).

[12] See the Private Universe Project in Mathematics (PUP-Math), 2001 for video episodes and accompanying workshop materials at http://www.learner.org/channel/workshops/math and Maher, Alston, Dann & Steencken (2000).

Grade 10 Task: Ankur's[13] Challenge. Find as many towers as possible that are 4-cubes tall if you can select from three colors and there must be at least one of each color in each tower. Show that you have found all the possibilities.
Grade 11 Task: Pascal's Triangle. For Pascal's triangle, how does the addition rule work?

In a May 1999 interview, seventeen-year old Jeff describes the way the students work. He reports that members of the group would put forth their ideas, review them, and select the most salient. He points out that the ideas of others are to be valued.

Well, we break up into groups...like five groups of three, say, and everyone in their own groups would have their own ideas, and you'd argue within your own group, about what you knew, what I thought the answer was, what you thought the answer was and then from there, we'd all get together and present our ideas, and then this group would argue with this group about who was right with this...

Grade 4 Task: Building Towers. In grade 3, the students worked on building towers, 4-tall and 5-tall, selecting from two colors. Sixteen months later, in the fourth grade, they investigated towers 5-tall. About one month later, a group interview with fourth graders: Jeff, Michelle, Milin, and Stephanie was conducted. In this session, we were especially interested in what made the students' reasoning convincing. For about half an hour, the students shared their different approaches.

Grade 4 Video Episode: Gang of Four.[14] In grade four, Jeff, Michelle, Milin and Stephanie discovered the idea of mathematical proof. For at least a year before this, the children had been building arguments in their block tower investigations in which they controlled for variables, argued by cases, and used inductive reasoning and argument by contradiction. For example, in the case of towers, students noticed that, as they built from towers of height n to height n+1, thy could choose one of two colors, thereby doubling the number of towers. So they investigated a doubling idea and come up with a doubling rule and posited this as a generalization. Other students, in looking for patterns, recognized certain organizations that accounted for all possibilities of a given height, and suggested an argument by "cases".

In an interview, Mike (May 1999) talks about exploring and gaining understanding:

In our class, all we did was just explore. We took days at a time, and I have a good understanding of it...like, if you were going to, I guess, a normal class, you'd have to be, like, only selected kids might understand it. But in a class where everybody's working

[13] For a full discussion, see Muter (1999).
[14] See Maher & Martino (1996a, b) for a transcript of the videotape session and an analysis of the data.

together, everybody's a part of the teaching, and everybody, or at least the majority of kids will understand it.

<u>**Grade 10 Task: Ankur's Challenge**</u>. In the 10th grade, five students (Ankur, Brian, Jeff, Romina, and Michael) met again and considered variations of the tower problems. First, they were asked to find all 5-tall towers, choosing from colors, red and yellow, such that each tower contains 2 red cubes, and to justify their solution. Mike and Ankur quickly solved the problem. While they were waiting for the rest of the group, Ankur poses another problem:

How many combinations can you make with towers four tall, selecting from a choice of 3 colors, and using at least one of each color in every tower?

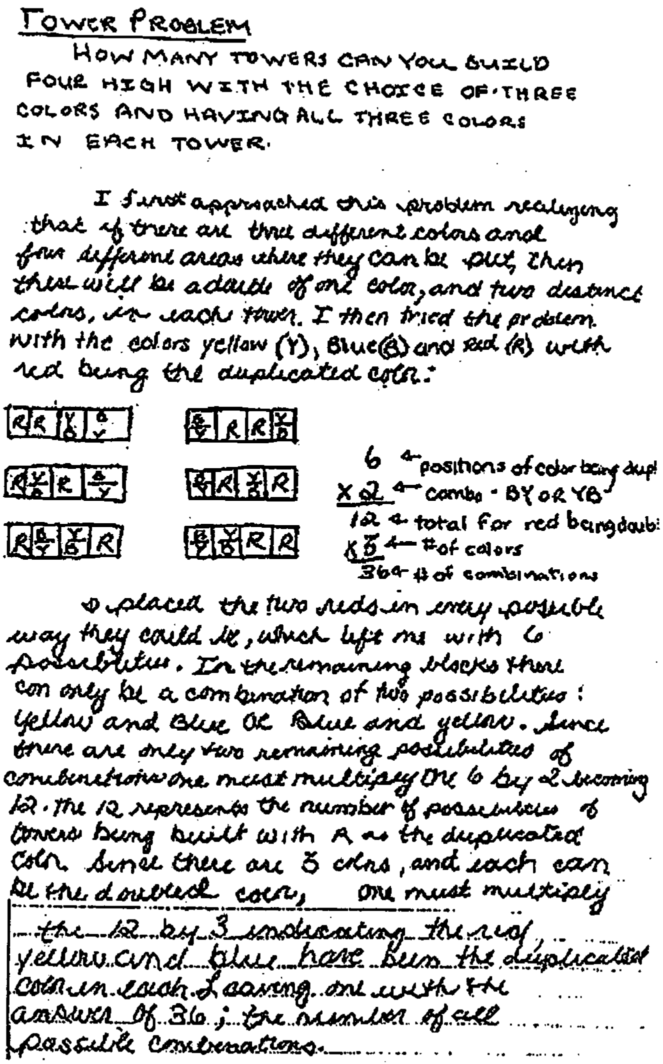
They worked on Ankur's new problem for approximately 15 minutes. Mike and Ankur, after calculating that there were 81 total towers when selecting from 3 colors, returned to the conditions of the problem and, by subtraction, came up with 39 towers. Romina, working with Jeff and Brian, said that there were 36 towers.
Michael continued to work on the problem by himself. Unaware of the work of Romina and her group, he asked to hear their solution. Romina went to the chalkboard and presented her justification.[15] She indicated that the set of all possible towers could be partitioned into six groups. Since every tower would have two of one color, Romina focused on the placement of the duplicate color, using x and o. She indicated that for each placement of the first or duplicate color, there would be two possible combinations for the second and third colors. She also indicated that these combinations would have two opposite arrangements for the second and third colors. She then tripled the 12 possibilities to represent every color, concluding that there should be a total of 36. Romina was asked to write her solutions. Figure 1 shows the refinements in her written work presented at the next after-school session.

<u>**Grade 10 Video Episode: Romina's Proof.**</u> In an earlier session, Michael introduced the idea of using binary notation to count towers. Other students soon integrated binary numbers in their coding. When Michael indicated that he was "ready to listen", Romina shared her solution with the group.

Romina (March 2002) comments on the way they worked:

If I didn't understand the problem, or if I didn't work enough to it, by myself to understand, and I guess if Michael didn't know where I was heading with what I was doing, and if I didn't understand where the other person was heading I would like to work on it before I came up with a couple of options myself to see which one we take.

[15] See Muter (1999) and Muter & Maher (1998) for a more complete discussion


TOWER PROBLEM

HOW MANY TOWERS CAN YOU BUILD FOUR HIGH WITH THE CHOICE OF THREE COLORS AND HAVING ALL THREE COLORS IN EACH TOWER.

I first approached this problem realizing that if there are three different colors and four different areas where they can be put, then there will be a double of one color, and two distinct colors, in each tower. I then tried the problem with the colors yellow (Y), Blue (B) and red (R) with red being the duplicated color:

I placed the two reds in every possible way they could be, which left me with 6 possibilities. In the remaining blocks there can only be a combination of two possibilities: yellow and blue or blue and yellow. Since there are only two remaining possibilities of combinations one must multiply the 6 by 2 becoming 12. The 12 represents the number of possibilities of towers being built with R as the duplicated color. Since there are 3 colors, and each can be the doubled color, one must multiply the 12 by 3 indicating the red, yellow and blue have been the duplicated color in each. Leaving one with the answer of 36; the number of all possible combinations.

Figure 1. Romina's written work of her solution to Ankur's Challenge.

Grade 11 Task: Pascal's Triangle. In May of their junior year, the high school students returned to school one evening around 7:30 pm for a research session. The researcher began by asking the students to review what they had discussed in their pre-calculus class earlier that day. They reported that the class had learned to use a calculator to find the coefficient of any term in the binomial expansion without having to write out rows in Pascal's Triangle. The students were also asked why the addition rule for Pascal's Triangle worked. In response, they showed a 1-1 correspondence between terms in Pascal's Triangle and choices in particular pizza and tower problems. During their discussion of these problems, they gave meaning to the addition rule. [16]

Video Episode: The Night Session. The students were asked to write the general n^{th} row in Pascal's triangle, using the bracket notation for nCx. They responded to the researcher's request to formulate the addition rule with this notation. They explained the correctness of the notation by referring first to particular cases of the pizza problem,[17] and then to the meaning and structure of the addition rule as additional toppings are added.[18] Jeff, assisted by Michael, wrote the following equation on the chalkboard:

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Challenged by the researcher to express their result in factorial notation, the students worked together to produce the following equation:

$$\left(\frac{n!}{(n-x)!x!}\right) + \left(\frac{n!}{(n-x+1)!(x+1)!}\right) = \left(\frac{(n+1)!}{(n-x)!(x+1)!}\right)$$

After succeeding to write the addition rule for nCx, Jeff remarks:

Do you know, like, how intimidating this equation must be like if you just picked up a book and looked at that.

[16] For a full discussion, see Kiczek (2000); Kiczek & Maher (1998); Kiczek, Maher, & Speiser (2001).

[17] The metaphoric reference is to the general Pizza Problem. Since grade 5, the students worked on variations of Pizza investigations. The reference here is to a general problem of finding how many different pizzas that could be made using any number out of, say n, different topping choices and of considering how they could account for organizations of pizzas as additional toppings are made available.

[18] A detailed analysis of the session is described in Dörfler & Maher (in progress).

Mike (April 2002), in an interview 3 years later, was again asked how he might explain the addition rule. Mike recalled that the group had given a general rule in the 1999 after-school session; he immediately began to reconstruct it. This time, Mike called a row, r [to denote the number of toppings], and a "spot" in the row, n. He used the notation to show that "r choose n" plus "r choose n+1" equals "r+1 choose n+1", referring again to adding pizza toppings to explain the rule.

In the same interview, Mike talked about how he looks for relationships while he works on problems:

The process while I am doing the problem...I just start understanding more that this is related to that...how this is related to just a triangle that's made up of numbers. At first when they showed us a triangle, we didn't know that has anything to do with...once you start understanding things have a relation to each other you just start convincing yourself...and then you come to a point where you know it's right, or you think it's right.

STUDENTS REFLECT ON THEIR LEARNING

Through a series of individual and small group interviews, we present student views on how they structured their learning, thereby gaining insight into their views of the process.

Just giving the answer was never enough. Jeff (May 1999) indicates that the students themselves took on the expectation for presenting a careful argument. Consequently, they reviewed their own argument, focused on meaning, and anticipated questions and "holes". They questioned each other and put ideas together before offering their solution to the researchers. Listening to and asking questions of each other were essential components of the process of working together.

Just giving the answer was never enough, in order to do it. You'd have to have a good, like, structural record. It's almost like doing, like a proof...like you need to show every step from point A to point B...you couldn't just, like, skip some things and jump around. You had to go straight, and everything had to be written out and good, and ... understanding, and if you had a problem with somebody, to ask another question about it, so you ended up doing whole types of things, just to get from the beginning to the end, and through it, that's how you really understand what you were doing, that's why we'd learn, like what we were doing without actually calling anything a certain thing...

We would come to it ourselves. Jeff (May 1999) talks about not being told how to do things.

And then, like, later now, we would be doing things, like, "Oh, that's what we were learning." because Rutgers never really told us what the answers were, or what we were actually doing...like, 'This is what we're going to do today; it's called the... theorem,' or anything like that. We would come to it ourselves, and then later, we would realize that that's what we were doing this whole time.

Jeff (March 2002) indicates a building process.

If we tried to just present a final thing and really didn't know it from the beginning we couldn't explain it in a way that that you would accept from us. So in order to explain it in a way that you would accept we'd really have to start from bare bones, from the beginning.... We didn't start talking about what we were doing with you until very late in what we were doing. There was not a lot of communication back from them to us about the work we were doing.

We got so in-depth. Jeff (May 1999) reviews what they accomplished.

Well, even though we didn't spend much time together, and they [researchers] only came a few times a year, we did so much, we covered so much, we got so in-depth on topics, that it leaves an impression. I mean, we could talk about doing the blocks in first grade, and we can almost go through problems: We did shirts and pants in second grade. I mean, how many other people can tell you the math that they were doing in second grade...like a word problem, you know? Because you go in deep, you work on it so much, and you go so far into it, that it just sticks with you...That's why it leaves such an impression, because of the depth you get into it...

We just sat and thought for hours a day. Romina (July 1999) talks about the confidence she gained. She indicates that they spent days thinking about problems and that presenting them to others was a valuable undertaking.

We did a lot of problem solving. We did a lot of thinking. We just sat and thought for hours a day, and we came up with a lot of interesting things. We were able to go in front of a large audience and talk about our ideas and argue our points, and prove our points. I think it was a very good experience.

I think a lot of what we were doing was working together. Jeff (February and March 2002, respectively) talks about the benefits of collaboration and the frustration of working alone.
Well that's...how we got to wherever we were going...we were like four different people with four ideas and we all thought we knew something on how to do a problem but...you

just cover so much more when everyone is discussing what you're doing, I mean that's what it was really all about...that's really how we got anywhere was kinda work, doing our thing together, you know, and using what we each knew, to work something out.

I think it would have been very different if it was all of us producing our own solutions....I think a lot of what we were doing was working together. I think when you are working alone, when you reach a part where you don't know anymore it is very easy to just be frustrated and say I don't know anymore. I'm not going to do this. I can't think about this. Like forget it. I think that by working with everybody when you got to that point, you can kind of peak over a little bit and it was all right...it was encouraged. That allowed everybody to really we could all move forward.

Everything has to make sense. Romina (March 2002) talks about understanding.

Everything has to make sense in my terms. Someone else may have done it already in a book, but I just don't understand it unless I try it myself and put it in my own term.

CONCLUSIONS AND IMPLICATIONS

We have engaged in a research program, extended over 14 years, within which sense making has become a cultural necessity. An aspect of this culture has been the emergence, beginning in the elementary grades, of argumentation, justification, proof making, and generalization. Such processes have developed in the context of coherent strands of mathematics. The reflections by students about their learning over the project years gives further insight into the process of how they worked together and structured their learning. They reported that giving the answer was never enough. They understood that they would be expected to provide a written account to support their reasoning and that details in arguments were important. They accepted that they were not to be told the answers or how to solve a problem and took on early the expectation that they would produce the result and offer appropriate support for it. The support came from convincing first themselves and then each other. Student expectations guided how far they were willing to go in solving problems. These expectations came from interactions with researchers, who challenged them to be attentive to details, provide evidence for their results, and consider extensions and generalizations to investigations posed. The students reported that they were aware of what would be asked of them and took upon themselves on the responsibility of developing satisfactory solutions beforehand. Their work and conversation, backed by extensive interviews, indicate that they maintained high expectations for themselves and for each other, and these expectations help explain the way they worked together, over months and years. As evidenced by their comments, they took on, progressively, responsibility for their own learning and for the maintenance of communication and collaboration in their working groups. They reported increased confidence over the long run, which may explain, at least in part, their evident commitment and responsibility for helping the project to continue and evolve through time.

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MATHEMATICAL EPISTEMOLOGIES AT WORK

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In this paper, I draw together a corpus of findings derived from two sources: studies of students using computers to learn mathematics, and research into the use of mathematics in professional practice. Using this as a basis, I map some elements of a theoretical framework for understanding the nature of mathematical knowledge in use, and how it is conceptualised by practitioners. I then draw some provisional implications for a set of design principles for activity systems aimed at fostering mathematical learning. I propose a central role for digital technologies, both in assisting understanding of the genetic construction of knowledge, and in developing learning environments that instantiate the design principles.

In mathematical terms, there is a celebrated tension between forms of discourse and cognition that are delicately tuned to cultural practices, and those that are focused explicitly on mathematics *per se,* recognisable by its symbolic forms and epistemological structures. This tension parallels (and is perhaps derived from) the epistemological duality of mathematical thought as both tool and object, simultaneously as a component of pragmatic activity and theoretical endeavour.

The preparation of this paper has afforded an opportunity to reflect retrospectively on this duality, and on a corpus of research in which I and my colleagues have been involved, spanning a variety of sub-fields and a couple of decades. I hope it is not too fanciful to impose upon this work a narrative that was not necessarily evident to any of us while we were engaged upon it. Here is a first outline of that narrative.

I will begin with a pervasive finding that arises from investigations with (mainly young) people expressing mathematical ideas with computers. These studies led to a series of thoughts concerning the generation of mathematical meanings that nagged away until the early nineteen nineties, when Celia Hoyles and myself began to formulate a theoretical framework for describing the phenomena we encountered. Shortly after this, we had the opportunity to work in a variety of settings with the broad common aim of elaborating the mathematics used in working practices. I will illustrate how these studies began to throw light on some fundamental questions, particularly concerning the nature of mathematical practices, and encouraged us to investigate further the problem of mathematical meaning from both cognitive and sociocultural perspectives. This effort has led to some general principles about the design of mathematical activity systems for learning, and in particular, the rather special role that digital technologies may play within them. Thus, perhaps fittingly but probably over-ambitiously, I will conclude where I began, with the assertion that digital technologies can play an unusually powerful role in helping us to understand and reshape the nature of mathematical sense-making.

Before I start, I would like to make two general observations. The first concerns my wish to consider both cognitive and social dimensions. To steer a course between these two approaches is not easy, not least because proponents of each often ignore the work of the other, or denounce as mere eclecticism any attempt at synthesis (there are important exceptions to this: see for example, Cobb and Bowers 1999; Kieran, Foreman & Sfard, 2002). One organising idea for thinking about this apparent dichotomy has been suggested by Andy diSessa (personal communication) who distinguishes between phenomena that are *distally* and *proximally* social. Much of what I have to say comes from a recognition that many phenomena concerned with mathematical meaning are proximally social, in that they manifestly involve social and cultural relations between people and within communities. But I also recognise that many facets of human thought are only distally social; while it is true that what I think, and the techniques I use for thinking and communicating are shaped both socially and culturally, I think in ways that are structured by my personal cognitive history at least as strongly as by the socio-cultural relationships in which I find myself embedded. No attempt to understand how mathematics is learned by human beings can afford to ignore this, essentially cognitive element, any more than it can afford to ignore the social and cultural relations in which cognitive activity is embedded. Thus, in what follows I hope to illustrate, not only that such a perspective need not necessarily lapse into eclecticism, but rather that the coordination of the two approaches provides a possible and even necessary methodological stance.

The second observation concerns the title of this paper. I recognise that it is bad form to tell a joke and then explain it. Forgive me then, if I explain the *double entendre* in the title. I want to talk about mathematical epistemology as it is found *in work*, to understand how mathematics is used, and how it is conceived by participants in their cultural practices. But I also want to talk about mathematical epistemology as a crucial element *at work* in learning situations; how we, as mathematics education researchers can develop, not just new approaches to teaching, but new mathematical epistemologies, that is more learnable and, at least for all but the few, more expressive.

INSIGHTS FROM OBSERVATIONS OF ACTIVITIES WITH COMPUTERS

Over some two decades, Celia Hoyles and I have engaged in studies of children and adults interacting with computational systems designed to afford mathematical expression. Throughout this time, we have noticed an interesting phenomenon, which we can simplistically characterise as follows: learners are often able to express themselves in terms that might be considered abstract, yet which seem to be bound tightly into the tools and symbols of the computational world. Learners can, in other words, say and do things with suitably-designed systems that they may be unable to say or do without them, and they can often do so in ways that are interestingly different from conventional means.

I would like to elaborate two points. The first centres on the ways that learners use technology to shape their mathematical expression, how some elements of the invariant relationships between the given objects are identified and related within the symbolic discourse of the environment. In the sense that these invariant relationships remain articulated only within activity and interaction using the notational system of the virtual world, they might not be said to constitute a formal abstraction. But to the extent that they become transformed into something coherent, reusable and general, it does make sense to consider such activity as involving an abstraction of some kind. For further elaboration of this argument, in the context of stochastic thinking, see Pratt, 1998; Pratt & Noss, in press: for a study in relation to students' conceptualisations of non-euclidean geometry from a similar perspective, see Stevenson, 1996; Stevenson & Noss, 1999: and for a recent study on 12 year-olds' understandings of symmetry and reflection, see Healy, 2002.

The second aspect is related to the first, and concerns differential performance. Put bluntly, children who may be apparently unable to express *any* relationships about their figures with pencil and paper, are able to express them quite adequately (and sometimes quite elegantly) with the computer.

Reports of differential performance depending on context are commonplace. There are consistent and widely-reported findings concerning the differential performance between adults carrying out tasks in everyday settings, and when given written assessments. For example Scribner's (1985) study of the dairy industry, Lave, Murtaugh and de la Rocha's (1988) investigation of weight-watchers, and the seminal work of Saxe (1991), and Nunes, Schliemann and Carraher (1993) on street vendors, have all shown convincingly that people who are error-prone in tests are mostly error-free in familiar practical contexts, and that there is a major disjuncture between the strategies used in the two settings. More generally, and especially since the work of Jean Lave and Etienne Wenger (1991) and others in broader anthropological contexts, we may more or less take for granted the situated view of knowledge genesis. A key insight is that people construct solutions in the course of action, and that these solutions are structured by activity. In the supermarket, for example, Lave illustrates how people avoid doing what might be classified as school mathematics not because it is too hard, but because the practice of supermarket shopping carries with it its own discourse, and its own mechanisms for meaning-making. One point that is often missed, however, is that we cannot conclude that there is nothing that passes for mathematical in shoppers' activities. The point is that when shoppers *do* use mathematics, it is supermarket mathematics, a mathematics made possible through the resources of the setting.

Since these studies, the situated cognition perspective has become ubiquitous. In its extreme version, it states that 'every cognitive act must be viewed as a specific response to a specific set of circumstances' (Resnick, 1991, p. 2). However, such arguments – compelling as they are – face researchers of mathematical learning with a number of seemingly intractable difficulties. If mathematics cannot be regarded as a decontextualised resource to be learned and then mapped onto settings, if it can only be defined in relation to specific situations, then we seem to have come close to distilling the mathematical essence out of mathematical thought.

As a way out of this *cul-de-sac,* Hoyles and myself proposed, some ten or so years ago, the idea of *situated abstraction* (a first attempt is in Hoyles & Noss, 1992) as a tool to aid in understanding how learners construct mathematical ideas by drawing on the material and discursive components of a particular setting (other attempts have been made by Cheng and Holyoak, 1985; Cheng, Holyoak, Nesbitt and Oliver, 1986; Nunes *et al.,* 1993). Situated abstraction seeks to describe metaphorically how a conceptualisation of mathematical knowledge can be both situated and abstract. It may be finely tuned to its constructive genesis – how it is learned, how it is discussed and communicated – and to its use in a cultural practice, yet can simultaneously retain mathematical invariants abstracted within that community of practice.

The idea of abstraction as a conceptualisation or a piece of knowledge lying in a separate realm from action, tools, language or indeed from any external referential sign system, is important from a perspective of mathematical discourse, since mathematical discourse is normally conceived as self-contained: it forms part of a system that has its own objects and its own rules for transforming them (see Piaget, 2000). This characteristic of formal mathematical abstraction is central to its utility: situated abstraction does not seek to challenge that utility, but questions whether mathematical abstractions can ever be fully separated from the context of their construction or application (see also Wilensky, 1991). Our broader hope is that the idea of situated abstraction will contribute to a theory of how mathematical knowledge is used or 'transferred' across settings (for other contributions to this emerging theory, see for example, Carraher & Schliemann, 2002; Sfard, 2002; Nemirovsky, in press; Hershkowitz, Schwarz and Dreyfus, 2001).

At the point in which Hoyles and myself began to formulate these ideas, they were essentially hypotheses, based only on data derived from children and adults engaged in computationally expressive media. Fortunately, we were subsequently able to test these ideas in studies of mathematics in work, affording the opportunity to focus on the situativity of mathematical meanings by investigating their use rather than their genesis.

INTO THE WORKPLACE

While it is clear that persons studied in their communities yield rich and useful data which describes what they do, it remains desirable to locate and elucidate the mathematical knowledge that they know. To achieve this aim, our group in London has employed ethnographic and interview data to capture meanings created *in situ* and the dialectical relationship of these meanings with mathematical expression on the one hand and professional expertise on the other. This has involved Celia Hoyles, Stefano Pozzi and myself in a series of studies with investment bank employees, paediatric nurses and commercial pilots; more recently, Phillip Kent and myself have been working with a group of structural engineers. The professional groups differ in substantial ways, but there are similarities: in the explicitness of their mathematical training, and in their intolerance – to a greater or lesser extent – of errors. We have developed a map of mathematical workplace activities comprised of documentary analysis, interviews with senior staff in each profession, general and task-based interviews with practitioner volunteers, and ethnographic observation of these subjects in the workplace.

I will now try to summarise some of the outcomes of this research. I will do so by sketching five vignettes, chosen to illustrate the outline of the theoretical position I wish to advance in the form of a set of provisional 'results'. The text of each vignette is based on the relevant co-authored papers that are referenced within it.

Vignette 1: The epistemological fragmentation of the workplace. The first vignette is drawn from a study conducted with a group of bank employees, part of which attempted better to understand the bankers' ways of thinking about quantitative data (see Noss and Hoyles, 1996a).

In responding to tasks involving the interpretation of functional relationships represented as graphs, the responses of the bank employees were surprisingly uniform. Most identified graphs as a visual display of numbers, as a pictorial representation of underlying data rather than as a functional relationship, and as an indication of a trend in data that allowed prediction. Where we saw graphs as a medium for expressing relationships (e.g. between quantity and time) bankers saw a display of data.

The origins of this epistemological diversity are almost certainly to be found in the tools of the system in which the bankers operate. On each employee's desk were several computers. Some, the traders and the operations staff, had three or four. On all but a very few screens, there were columns of data, graphs, and more columns of data: in every sense, graphs *were* pictures of numbers, rather than graphical representations of a functional relationship.

This epistemological standpoint with respect to graphical representations can, it seems, be thought of as the graphical face of a fragmented knowledge structure that characterises the practice of investment banking. We encountered departments specialising in the finest detail on one financial instrument, sharing a common wall but no common language with another – essentially similar – department. Of course, similarity is in the eyes of the beholder: while we might view, say, Nominal

Certificates of Deposit and Treasury Bills as flavours of similar financial instruments sharing the same (or nearly the same) mathematical structure, the bankers saw finely-tuned pragmatic knowledge and strategies, and a discourse that served to reinforce the differences between them.

> *Result 1: There is an epistemological fragmentation of the knowledge structure of the workplace that shapes, and is shaped by, the discourse of the working practice. Strategies are finely tuned to the pragmatic demands of work activities, with little tendency to strive for a theoretical orientation involving generality or appreciation of unifying models.*

Vignette 2: The role of artefacts and tools. The idea that people think and act within sociocultural contexts which are mediated by cultural tools is now commonplace. The work of Vygotsky, Luria and Leont'ev, indeed the entire corpus of work on activity theory, offers compelling evidence that individual and social acts of problem solving are contingent upon structuring resources, involving a range of artefacts such as notational systems, physical and computational tools, and work protocols (Gagliardi 1990).

Workplace settings are, naturally enough, littered with artefacts. These artefacts are, for the most part, a simple expression of work protocols, so that in routine use – and the overwhelming majority of time in working practices is spent on routine – the structure of the artefact is hidden from view. For example, in one study on a hospital ward (Pozzi, Noss & Hoyles, 1998), we found that a seemingly straightforward artefact like a fluid balance chart, contained within it the crystallised activity (Leont'ev, 1978; see also, Wertsch 1985) of the hospital community, shaping in complex – but unnoticed – ways the actions and discourse of those using it. A central part of this crystallised activity was a mathematical model of essential variables and relationships embedded in the activity: evidence for both the complexity and the invisibility of this mathematical model was gained by observing the ambiguity and uncertainty felt by a newcomer to the paediatric ward, as well as the extreme difficulty faced by the old-timers in communicating to her the structure that they had come to take for granted.

The arrival of the newcomer on the ward served to trigger a 'breakdown' or decision point within routine practice, a situation in which the models underpinning artefacts and the representational infrastructures on which their use depends, rise to the surface, and become open for inspection and negotiation by participants (and observation by researchers). That this model is normally hidden should cause no surprise: we have already noted that the purpose of an artefact is to facilitate the pragmatic activities of the workplace, not to learn mathematics or to gain insight into underlying models. Nevertheless, when breakdowns do occur, invisible relationships buried in artefacts do not suffice and there is a need for the community to understand at least some of the workings of the models, to examine their strengths and limitations, and to scrutinise the results of the mathematical labour congealed within them (see Hall, 1998, for a similar finding). At least in breakdown situations, we are abruptly made aware of circumstances that require more than mere procedural routine and the learning of work protocols, but systemic interpretation – the individual is required to make sense of what she does within the broader socio-technical system.

Result 2: Tools and artefacts shape activities and thought in ways that only become visible at times of breakdowns to routine. In disruptions to routine, individuals need to develop a broader interpretative view of the model that underpins their routine practice.

Vignette 3: The anchoring of mathematical meanings in practice. It will help to focus on a specific knowledge domain: I will turn to one of the most widely-researched topics in the field, ratio and proportion. Researchers on proportional reasoning in school and the workplace have distinguished two ubiquitous classes of strategies for making proportional calculations, *functional* (across measure) and *scalar* (within measure): see Vergnaud (1983) for a thorough analysis. Nunes, Schliemann and Carraher (1993), have suggested that scalar strategies offer a mechanism for holding on to situational meaning by keeping only one measure in view. By way of contrast, functional strategies tend to be seen as semantically sparse manipulations of numerical quantities *per se*. It appears that this difference is the reason why people tend to prefer scalar strategies, even when it results in a more computationally awkward calculation, and is the crux of the counterposition in the literature of scalar and functional approaches, in that the privileging of the former has arisen from the apparent necessity in the latter to relinquish meaning in the form of a situational referent. Nunes, Schleimman and Carraher concluded that scalar approaches are drawn from experiences in everyday situations, are more flexible and generalisable than easily forgotten algorithmic approaches, and, most relevant here, allow people to preserve the meaning of the situation by keeping variables separate and not calculating across measures.

Let us see how robust this finding is in the case of a group of paediatric nurses who are similarly expert in their field, but who have had years of school mathematical education as well as professional training. During ethnographic observations and interviews, we noticed that while all the nurses' drug calculations were carried out correctly (unsurprisingly, in written tests, nurses' responses were highly error-prone), the strategies adopted were varied and exhibited a far richer complexity than would be suggested either from our interviews or from the existing nursing literature. Of 30 episodes related to drug administration (out of a total of 250) we collected during 80 hours of observation, 26 combinations of ratios were observed with a variety of drugs, packaging and prescriptions. Of these, only four involved the nursing rule (in which all were endlessly drilled during training), while equal numbers chose scalar and functional strategies. Moreover, the nurses often opted for strategies that would, in the literature, be described as lacking in meaning.

Our interpretation of these findings is that the nurses' knowledge of concentration, that is their appreciation of the invariance of the relationship between mass and volume as evidenced in their drug calculations, was anchored in an intimate knowledge of the drug itself, as well as in the properties of familiar packaging constraints of prescribed doses. The knowledge was mutually constituted and expressed as both mathematical relation and culturally-shared situational noise – the same kind of knowledge that we encountered earlier, in the context of computer worlds and which we called situated abstraction.

Result 3: Knowledge is mutually constituted by a coordination produced in activity of mathematical knowledge and situational noise to form situated abstractions.

Vignette 4: The qualitative restructuring of mathematical knowledge in activity. In a recent study, Phillip Kent and myself have been investigating the ways in which mathematical knowledge is conceived and deployed with employees of a large London-based engineering firm (see Kent & Noss, 2001; Kent & Noss, 2002). We have encountered, even with this mathematically educated group, a ubiquitous view that the majority of structural engineers do not "use mathematics" of any sophistication in their professional careers. So, while all believed that it was important for graduate engineers to have an appreciation for advanced mathematics, it is something they would rarely be expected to use:

> Once you've left university you don't use the maths you learnt there, 'squared' or 'cubed' is the most complex thing you do. For the vast majority of the engineers in this firm, an awful lot of the mathematics they were taught, I won't say learnt, doesn't surface again.

I think that this particular engineer's description of mathematics as not "surfacing" is a fortuitous one. We have seen in the case of the nurses, that mathematical knowledge becomes fused with professional knowledge, as situated abstraction, not as abstraction in its pure form. But it is, particularly for mathematically sophisticated groups such as engineers, this pure form that is readily recognisable as mathematics. Our engineer is right that mathematics does not surface; or rather, that it seldom surfaces in the form it was learned and taught. It has been transformed into something else, something at once more usable, more embedded, more noisy. Only the vestigial traces of the college mathematics taught to engineers remains in the mathematics that they actually use in activity.

The transformation in the character of mathematics appears to be not simply a quantitative one, nor merely a replacement of mathematical activity by professional expertise and experience. It represents a qualitative, epistemological and cognitive restructuring of the mathematics as it becomes 'embedded' in engineering expertise[1]. I claim that engineers' conceptualisations of this restructured mathematical knowledge are legitimately considered as situated abstractions.

I will illustrate with an example. The type of qualitative thinking that characterises the use of 'feel' in the engineering design process is exemplified by the concept of *load path*, the notion that the loads acting on a structure have to "flow down into the ground" like a kind of fluid. It is a powerful, very physical concept, and extremely useful because it provides a way of thinking about a structure before any analysis is done, allowing judgements to be made about the validity of quantitative analysis of the structure.

Formal mathematical analysis, on the other hand, is based on the assumption of static equilibrium, which assumes that nothing is moving in a stable structure, an assumption that appears to conflict with the load path concept[2]. Nevertheless, load path has become a situated abstraction of stability criteria: it allows predictions of behaviour that emerge from fusing together the actual properties of the material (e.g. steel beams) with the associated (mathematically-abstracted) forces.

The relevant point is this: engineering discourse employs, in at least one important way, a kind of knowledge which is at once about mathematical relations and about substance. The idea of flow makes no sense without something to flow through – the beams and struts of everyday engineering practice. Mathematical knowledge has been transformed to the extent that even those engaged in it do not necessarily recognise its existence. This poses sharply two questions: how does the formally-learned knowledge (e.g. the engineers' knowledge of Newton's laws, or the nurses' knowledge of the nurses' rule) become transformed both cognitively and culturally, into something new and more functional within professional practice and what connection, if any, is maintained between them?

I have no data on these questions. For the moment, the key issue concerns the transformation of knowledge, the creation of new epistemologies as a characteristic part of professional expertise. Here, at least, is the explanation of the apparent invisibility of mathematical activity. Here too is a broader, more culturally oriented perspective on the hitherto individualistic notion of situated abstraction that recognises the individual's embedding in an ambient social and cultural space.

> *Result 4: As mathematical knowledge is embedded in new settings and activities, it undergoes an epistemological and cognitive transformation. What is consciously thought of as mathematics by practitioners appears to be only the visible component of a larger, transformed body of mathematics in use that takes the form of situated abstractions.*

Vignette 5: The situativity of abstraction. The final vignette will deal with the most problematic (and so far, under-researched) issue. The challenge is to test the situativity of knowledge, to assess the extent to which knowledge in the form of situated abstraction 'transfers' to new situations (or better still, to find a convincing alternative metaphor for the notion of transfer itself).

In the study of nurses, we undertook a series of task-based interviews, in which the nurses whom we had followed on the ward were faced with situations that were progressively removed from the practices we had observed, yet which retained elements of familiar situations for them (see Noss, Hoyles & Pozzi, in press). We found that when the nurses were faced by a close simulation of their practice, they displayed similar strategies to those identified in the ethnographic studies, together with a strong sense of the invariant relationship of mass and volume – an abstraction. In these cases, the nurses' reasoning was supported by a synergy of their existing (school) mathematical knowledge and their practical experience. By contrast, an analysis of the nurses' responses to a less familiar scenario, derived from a breakdown observed on the ward, showed that when the texture of nursing practice became unavailable for any reason, the mutually constitutive elements of professional and mathematical knowledge became disconnected and the situatedness of their conceptualisation was apparent.

> *Result 5 (conjecture): The noise of a situation forms a core part of a situated abstraction. When it can be called upon in a new situation (and only then?) the mathematical knowledge can be 'transferred'.*

DESIGNING FOR CHANGE

I promised at the outset to draw the implications of the work studies, and to draw some general principles about the design of mathematical practices for learning. The hypothesis is that the ways in which people reconstruct knowledge for use in work is spontaneous, in the sense of deriving from participation in the practices of the community, and, for the most part, not formally taught within the practice. That being so, we might further hypothesise that – given the effectiveness of this kind of knowledge – we might attempt to design and construct activity systems for learning that harness the features of the workplace, at least those that we perceive as constitutive of learning. For reasons of space, I can only schematically summarise the findings and outline some challenges for the design of learning (and learnable) environments that flow from them: see Table 1.

	If ...	we should design to...
1	knowledge is fragmented and strategies pragmatic	demonstrate the power of invariants
2a 2b	knowledge is pervasively structured by artefacts and their underlying models people need to understand the models	supply lots of Really Useful Things make things that people can see inside
3	situated abstractions are mutually constituted by mathematical knowledge & situational noise	maximise situational noise in culturally-relevant ways
4	mathematical knowledge is transformed when it structures new activities	respect the mathematical epistemologies of new representational forms
5	situated abstractions depend on noise for 'transfer'	afford construction of new situations from old ones

Table 1: some schematic challenges of the mathematics in work findings for the design of learnable environments

In our book *Windows on Mathematical Meanings: Learning, Cultures and Computers* (Noss & Hoyles, 1996b), Celia Hoyles and myself argue generally that constructing runnable models in the form of computer programs[3], affords a compelling example of a learnable mathematics, opening unique opportunities for students to interact with a formal system. In modifying or constructing a model of a system, a student must articulate rigorously its salient relationships, describing mathematical structures in a language that can be communicated, extended, and become the subject of reflection.

There are many advocates of a similar perspective (see Hoyles & Noss, in press, for a review). In a recent study, for example, Sherin (2001) proposes that programming-based representations might be easier for students to understand physics than equation-based representations, and that programming-based representations might privilege a somewhat different "intuitive vocabulary", i.e. might tap into *different* things that people 'just know'. I would add a third point: that programming affords a rich set of situated abstractions of physical relationships that I think correspond to what he calls *a physics of processes and causation* (as opposed to algebra-physics which he characterises as *a physics of balance and equilibrium*).

It is not important whether we accept Sherin's conjecture or not: in *Windows* we refer to LogoMathematics or Programming Mathematics to emphasise that it *is* a different kind of mathematics that is at issue (this is an instance of the fifth design challenge). What is important is that we recognise that the switch from one representational form to another, carries with it the possibility of a switch simultaneously in epistemology and learnability.

I would like to add one more crucial element to the consideration of design principles for learning environments: the importance of mathematical models, a proximally social issue that I briefly touched on in Vignette 2. I believe the knowledge economy has massively broadened the number of people who need to understand the system they are using: elsewhere (Noss, 1998; 2002) I elaborate a case that competence in constructing, interpreting and critiquing models has become a core part of social and professional life in the twenty-first century. Sharing, critiquing and representing models is massively under-represented in mathematics curricula, still wedded to the epistemological and pedagogical requisites of the nineteenth century rather than transforming both in the face of the demands, and computational possibilities, of the twenty-first (see Kaput, Hoyles & Noss, 2002).

I contend that manipulating, modifying, constructing and sharing computationally instantiated models of mathematical systems affords the best chance we have for designing a more learnable mathematics, and of realising the five challenges outlined in the previous section. We have recently completed a study aimed at instantiating this approach in the *Playground Project*[4], which has involved a group of researchers based in four European countries who have developed a system with which young children, aged less than 8 years old, can play, share, construct and rebuild computer games. Our goal has been to put children in the role of game designers and game programmers, rather than merely consumers of games programmed by adults.

I can, for limitations of space, only sketch an example (see Noss, 2002 for a full description). Mitchell is an eight year-old boy in an inner-city school who has been helping to design and debug the *Playground* system for about a year. He is playing a game where he controls a character called 'dusty', who shoots out flowers every time the force joystick trigger is pressed. Mitchell collects points by hitting an animated target moving vertically up the left-hand edge of the screen. He finds it very easy to play and achieve high scores since he can move his character as close to the target as he pleases.

Mitchell decides that the game would be more fun if it was competitive, so at his request, Miki, the researcher, adds another player character, this one controlled by the mouse. It is at this point that Mitchell plays a trick: demanding that she avert her eyes, he removes a piece of Miki's program, effectively disabling her mouse by restricting it only to vertical movements, while he has two-dimensional control and can get as close as he likes to the target! Mitchell made use of a surprising and difficult fact: that two-dimensional motion can be instantiated as the vector sum of horizontal and vertical components.

Think for a moment of the knowledge congealed in the innocent phrase "vector sum". Concealed in this phrase, is a taken-for-granted representational infrastructure that includes the definition of a vector, the algebraic system for combining two or more vectors, and a range of properties (e.g. scalar and vector product) that give meaning to the very idea of what a vector is and why it is a conceptually powerful generalisation of a real number. This structure is relatively complex, and is postponed with good reason until the latter stages of compulsory education, if it is taught at all. Yet the complexity is in the infrastructure, not the idea: the latter is, as Mitchell showed, rather intuitive. The point is that what is intuitive is hugely contingent on the representational infrastructure with which the intuition is expressed. In Mitchell's world, the addition of vectors is instantiated not as an algebraic relation but as a natural property of the representational system. The (object-oriented) structures of the system translated, more or less directly, into what kinds of things Mitchell could take for granted as "just so", what meanings he could derive from them, and most importantly, the ways in which he could make the ideas work *for him* in achieving his goal. In short, the change in representational infrastructure transformed not only the learnability of the mathematical knowledge, but the mathematical epistemology *at work* in the activity system.

CONCLUDING REMARKS

This last point brings me to the intention I flagged at the outset, to conclude with the notion of epistemology *at* rather than *in* work. What is the connection between the two? A key link is that the analysis of mathematics in work concerns the transformation of knowledge as it is recontextualized across settings. We have seen how a person's mathematical knowledge is not invariant across time and space; it is transformed into different guises, different epistemologies, more or less visible as mathematics. This transformation seems much more powerful than the traditional notion of "application" or "use" that is often employed as a metaphor to describe this process. If formally-taught mathematical knowledge is transformed in this way, it is at least possible that reciprocal transformations may, in the future, take place for Mitchell, and that he may come to recontextualise his piece of knowledge about vector addition in (for him) novel ways.

More generally, the sketch I have provided offers a further point of connection between cognitive and cultural perspectives. In imagining how mathematical structures can be externalised and manipulated within an appropriate symbolic or linguistic framework, it suggests how abstractions constructed within concrete situations may compensate for their lack of universality by their gain in expressiveness. When general relationships can be expressed, they can be explored and become familiar. In the process, the links with knowledge of lived-in-cultures can be maintained, rather than severed in the quest for ultimate pinnacles of abstraction.

Mitchell was immersed in a world that was, I think, every bit as concrete and real to him as the load path on the components of a bridge are to an engineer. And, like his professional counterparts, Mitchell was engaged in an activity that researchers in the field of mathematical learning may recognise as having a mathematical component, but which were to him part of the ecological system – the totality of relationships between himself and the environment, and the ways in which these were expressed and communicated.

That the Playground and mathematical epistemologies run side-by-side should not be a matter of surprise: there is, after all, no single way in which humans can conceptualise (mathematically or otherwise) their environment, even though some are socially and historically privileged within a given culture. Official, symbolic mathematics is privileged in just this way; and there are good reasons for this. But the compactness and elegance of mathematical expression does not necessarily make it equally functional for learning, and if learning is our prior goal, we would do well to think about new epistemological frameworks in which to embed the mathematics we wish our students to understand. New epistemologies mean new intuitions, new things to be built with them, and new means for combining and reconstructing them. They involve new sets of situated abstractions that are both functional and powerful. I think this is the major challenge for the design of didactical environments, to create new systems which might, I think, be justifiably described as new mathematical epistemologies at work.

ACKNOWLEDGEMENTS

The corpus of data on which this paper is built was the result of several research projects, and some ten or more years of collaboration with colleagues. I acknowledge the support of the Economic and Social Research Council UK (grants R 0222 50004 & R 0002 23420) and the EU (grant 29329). Almost all of the data presented here has been published in collaboratively-written articles, which are appropriately referenced. My thanks to Ross Adamson, Miki Grahame, Lulu Healy, Celia Hoyles, Phillip Kent, Dave Pratt and Stefano Pozzi for their collaboration in the various projects. Lulu offered helpful comments on an earlier draft; Celia helped me redraft (and stay sane) when Version 1 exceeded the page constraints by a factor of 2. *Windows on Mathematical Meanings* has provided a source for the main theoretical ideas, and in this respect – as in so many others – I would like to acknowledge the profound help and support of my co-author and collaborator, Celia Hoyles.

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NOTES

[1] Michèle Artigue (in press) has made a related point deriving from the work of Yves Chevallard.

[2] diSessa (1980) makes a compelling case that a view of force as momentum flow may more easily engage and refine (rather than deny) students' existing intuitions, and therefore present a more learnable physics than that represented by the familiar $F = ma$.

[3] Constructing a computer program no longer necessarily involves writing lines of text.

[4] The Playground project was a consortium across four countries, directed by myself and Celia Hoyles. The London team also comprised (at various times) Ross Adamson, Miki Grahame, Sarah Lowe and Dave Pratt. Ken Kahn, the author of *ToonTalk,* was a consultant to the project.

TAKING THE MODELING PERSPECTIVE SERIOUSLY AT THE ELEMENTARY SCHOOL LEVEL: PROMISES AND PITFALLS[1]

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WORD PROBLEMS AS MODELING EXERCISES IN HISTORY AND TODAY

Mathematics provides a set of tools for describing, analyzing and predicting the behaviour of systems in different domains of the real world (Burkhardt, 1994). This practical usefulness of mathematics for understanding the world around us, for coping with everyday problems, and for future professions, has always provided, and still provides, a major justification for the important role of mathematics in the (elementary) school curriculum (Blum & Niss, 1991). In particular, the inclusion of application and modeling problems was mainly intended to develop in students the skills of knowing when and how to apply their mathematics effectively in various kinds of problem situations encountered in everyday life and at work.

The application of mathematics to solve problem situations in the real world, otherwise termed mathematical modeling, can be usefully thought of as a complex process involving a number of phases. There are many different descriptions of this model (e.g., Burkhardt, 1994; Mason, 2001; Blum & Niss, 1991; Verschaffel, Greer & De Corte, 2000), but, in essence, they all involve basically the following components: understanding and defining the problem situation leading to a situational model; constructing a mathematical model of (or: mathematizing) the relevant elements, relations and conditions embedded in the situation; working through the mathematical model using disciplinary methods to derive some mathematical results(s); interpreting the outcome of the computational work in relation to the original problem situation; validating or evaluating the model(ing process) by checking if the interpreted mathematical outcome is appropriate and reasonable for his or her purpose; and, stating and communicating the obtained solution of the original real-world problem.

Evidently, this modeling process can not be described as a linear activity; it has to be considered cyclic (Blum & Niss, 1991; Burkhardt, 1994; Mason, 1997). Moreover, modeling is not a straightforward activity. In modeling, not all aspects of the reality can, nor should, be modeled. The modeler tries to capture the essentials of the situation in the model, but what is considered as essential, and, thus, taken into account in the mathematical model, is not fixed, but relative to the modeler and to the context wherein (s)he is confronted with the modeling task at hand. Ikeda and

Stephens (2001) point to the pivotal task for the modeler of seeking a proper balance between over-complexity and over-simplification, taking into account the goals of the modeling task, certain personal and contextual constraints, etc. Finally, while modeling is often performed in order to answer one or more well-defined questions, this is not necessarily always the case. Modeling also occurs in situations where no well-defined question(s) has been, or will be, posed, but where the goal is to grasp, understand, make sense of, represent, explain, predict, and so forth, a situation or a phenomenon (Niss, personal communication).

Historically, one major way of teaching modeling process is through word problems, i.e. verbal descriptions of problem situations, typically presented in a school context, wherein a question is raised the answer to which can be found by performing (a) mathematical operation(s) on the numbers in the problem (Verschaffel et al., 2000).

An analysis of the very long and worldwide history of word problems reveals that word problems have been included, and are still being used, with the ostensible aim of accomplishing several goals (Kilpatrick, 1985; Niss & Blum, 1991; Verschaffel et al., 2000). I will focus on their oldest, and probably most important goal, namely to offer practice for the situations of everyday life in which mathematics learners will need what they have learned in school. The (implicit) idea behind this goal is to bring reality into the mathematics classroom, to create occasions for learning and practicing the different aspects of applied problem solving, without the practical (organisational, financial...) inconveniencies of direct contact with the real-world situation evoked by the problem statement. By means of such "best alternatives" for the real-world situations outside the classroom, students become prepared for the mathematical requirements they would face in their (future) everyday lives.

For a very long time, word problems have played this application function without much reflection and critical concern. Of course, there have always been individuals showing (some) awareness of the bridging problem between reality and mathematics, and the risks involved (Lewis Carroll is a marvelous example), but many teachers, textbook writers and researchers, have been using, and still use nowadays, word problems as if there was no bridging problem at all.

During the last 10-15 years, it has been argued by many scholars from different disciplines, such as mathematics education, psychology, linguistics, and anthropology, that the current practice of word problems in school mathematics does not at all foster in students a genuine disposition towards mathematical modeling. First, in-depth linguistic analyses of word problems as a text genre have led to serious questionings of the "unproblematic acceptance of concepts of separable mathematical and real worlds and of word problems as a transparent bridge between the two" (Gerovsky, 1997, p. 22). A second relevant line of research comprises studies on "everyday cognition" revealing remarkable discrepancies and difficulties of transfer between applied mathematics in contexts in and out of school (Carraher, Carraher & Schliemann, 1985). Third, empirical studies, mostly grounded in socio-cultural and socio-constructivist theories, have shown that after several years of schooling many students have constructed an approach to mathematical application problems whereby this activity is reduced to the execution of one or more arithmetic operations with the numbers in the problem, without any serious consideration of possible constraints of the realities of the problem context that may jeopardize the appropriateness of their standard models and solutions (Boaler, 1994; Lave, 1992; Nesher, 1980; Reusser & Stebler, 1997; Schoenfeld, 1991; Verschaffel et al., 2000). Altogether, these three related lines of research have led to a scepticism about word problems as a vehicle for promoting the development of students' disposition towards authentic mathematical modeling.

I will continue with a brief review and discussion of this third line of research carried out to investigate this phenomenon of "suspension of sense-making" (Schoenfeld, 1991) when doing school arithmetic word problems, beginning with ascertaining studies documenting and marking out the phenomenon, and then wending my way to an explanation of the observed effects, which will be found in the culture of the mathematics classroom. This explanation will be followed by a sketchy review of some design experiments wherein researchers have tried to develop a new instructional approach aimed at the development in pupils of more appropriate conceptions about, and strategies for, doing word problems, based on the modeling perspective. Finally, I will discuss a number of educational implications of the research done so far as well as some challenges for the future of teaching mathematical applications and modeling.

EVIDENCE OF LACK OF SENSE-MAKING AMONG STUDENTS

The most spectacular, and probably also the most quoted, case of "suspension of sense-making" is that of the French and German researchers (Institut de Recherche sur l'Enseignement des Mathématiques de Grenoble, 1980; Radatz, 1983) who posed non-sensical problems such as "There are 26 sheep and 10 goats on a ship. How old is the captain?" and found that many elementary school children supplied answers produced by arithmetical operations on the numbers in the text without expressing any concern about the appropriateness or meaningfulness of their computation-based answers on these absurd problems.

Another famous example is the buses item "An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?", that was used for the first time in the Third National Assessment of Educational Progress in the US (Carpenter, Lindquist, Matthews, & Silver, 1983) with 13-years-old students, and that elicited a remarkably large number of non-whole number answers ("31.3 buses") and answers wherein the outcome of the division was rounded to its nearest whole-number predecessor (rather than its contextually much more appropriate successor).

Inspired by these striking examples of suspension of sense-making in school mathematics (as well as some other examples), Greer (1993) and Verschaffel, De Corte and Lasure (1994) carried out pencil-and-paper studies with upper primary and lower secondary school students, using a set of problems including those listed below:

- Steve has bought 4 planks each 2.5 meters long. How many planks 1 meter long can he saw from these planks?
- A man wants to have a rope long enough to stretch between two poles 12 meters apart, but he only has pieces of rope 1.5 meters long. How many of these would he need to tie together to stretch between the poles?
- John's best time to run 100 meters is 17 seconds. How long will it take him to run 1 kilometer?

They termed each of these items "problematic" (P) in the sense that they require (from our point of view) the application of judgment based on real-world knowledge and assumptions rather than the routine application of one or more simple arithmetical operations. Each such P-item was paired with an S-item (for "standard") in which the "obvious" calculation is (we would argue) appropriate. For each item, the students, as well as recording an answer, were invited to comment on the problem and their response. A response to a P-item was classified as a "realistic reaction" if *either* the answer given indicated that realistic considerations had been taken into account *or* if a comment indicated that the student was aware that the problem was not straightforward. For example, a classification "realistic reaction" for the planks P-item would be given to a student who gave the answer "8" or who made a comment such as "Steve would have a hard time putting together the remaining pieces of 0.5 meters". In both studies, students demonstrated a very strong overall tendency to exclude real-world knowledge and realistic considerations when confronted with the problematic versions of the problem pairs. For instance, in Verschaffel et al.'s (1994) study, only 17% of all reactions to the P-items could be considered as realistic.

These initial studies were replicated in several other countries (e.g., Chili, China, Germany, Japan, Switzerland, The Netherlands, Venezuela), using similar methodologies and, to a considerable extent, the same items. The findings were strikingly consistent across many countries (see Verschaffel et al., 2000). Using the same criteria as in the two initial studies, none of the P-items was answered in a realistic fashion by more than a small percentage of students (except for two problems; see below), sometimes to the great surprise and disappointment of these other researcher(s) who had anticipated that the "disastrous" picture of the Irish and Flemish pupils would not apply to their students.

While the results were remarkably consistent across nationalities, there is increasing evidence, both from these replication studies and from several other studies with a somewhat different scope and methodology, that the tendency to respond to school arithmetic word problems in a stereotyped and non-realistic way is related to various kinds of task, subject and context characteristics.

With respect to *task variables*, P-items about the interpretation of a division with a remainder (i.e., the buses item and a similar item about sharing balloons) elicited considerably more realistic answers than the other kinds of P-items in the problem set in every study mentioned above.

With respect to *subject variables*, research evidence suggests that students' tendency to ignore plausibly relevant and familiar aspects of reality in answering word problems, is associated with age, gender, and social class. Children with less years of experience with (traditional) schooling (Radatz, 1983; Yeping & Silver, 2000), girls (Boaler, 1994) and working-class children (Cooper & Dunne, 1998) seem more likely to remain within an "everyday" frame of reference when doing application problems in a school context, leading to less appropriate answers if scored from a traditional point of view.

Finally, several follow-up studies have tested the effectiveness of variations in the *experimental setting*. A first group of studies assessed the effectiveness of making students more alert to consider aspects of reality and to legitimize alternative forms of answer produce. For instance, Yoshida, Verschaffel and De Corte (1997) gave half of the pupils an explicit warning at the top of their test sheet (a translation of Verschaffel et al.'s (1994) test) that some of the problems in the test were problematic, and invited them explicitly to write down and explain these unclarities or complexities. This manipulation did not result in a significant increase in the number of realistic reactions to the P-items in the test. In other studies too, this kind of manipulation provided, at best, only very weak effects (see Verschaffel et al., 2000). In a second set of studies, one or more categories of P-items were presented in a more authentic, performance-based setting. For instance, DeFranco and Curcio (1997) confronted sixth-graders with a version of the buses item in the context of a typical paper-and-pencil math test and, afterwards, asked them to make a telephone call to order minivans to take sixth-graders to a class party. The more authentic setting elicited a much greater percentage of appropriate responses than the restrictive school setting. In contrast to the studies about the effect of alerting pupils mentioned earlier, increasing the authenticity of the experimental setting, as done in the study of DeFranco and Curcio (1997) but also in other related studies (see Verschaffel et al., 2000), yielded much greater improvements in students' inclination to include the real-world knowledge they were so reluctant to activate and use under the previous, more restricted, testing conditions.

WHAT STUDENTS SAY AND THINK ABOUT WORD PROBLEMS

Besides deriving how students think about word problems from their responses to word problems in a paper-and-pencil test, it is, of course, possible to ask them directly.

Interviews carried out by Caldwell (1995) and Hidalgo (1997) suggest that, while unfamiliarity with the contexts involved in the problems and lack of appropriate heuristic and metacognitive skills may provide contributory explanations, (mis)beliefs about school arithmetic word problems constitute the major reason why so many students solve the P-items in a non-realistic way. For instance, a 10-year-old interviewed by Caldwell (1995, p. 39) commented as follows in response to the interviewer's question as to why she had answered a P-item in a non-realistic way: "I know all these things, but I would never think to include them in a math problem. Math isn't about things like that. It's about getting sums right and you don't need to know outside things to get sums right".

In a recent pilot study by Inoue (2001), upper elementary school children failed to give realistic answers to problems similar to the P-items from Verschaffel et al.'s (1994) test, but in the later clinical interview these answers were found not to be so unrealistic if they were judged based on the children's idiosyncratic interpretations of the problem situation. For instance, a student who had ignored the fatigue factor in solving a problem about finding how long it takes to finish a data-entry job based on the given rate (a problem very similar to the running item mentioned above), said that she ignored that fatigue factor since the calculational answer gives us the baseline information to judge how long it takes "theoretically". Her point was that such information is often useful for managing people calculating salary etc. in real life practices. Almost half of non-realistic answers given during the paper-and-pencil test were found to be of such type. This kind of "sensible unrealistic answers" (Inoue, 2001, p. 31) reminds also of Selter's (1994) findings with the infamous how-old-is-the-captain problem, wherein some students came up with highly ingenious explanations for their answers. For example, having responded to "There are 20 sheep and 5 goats on a boat. How old is the captain?" with the answer 25, one child suggested that may-be the captain's parents gave him an animal on each birthday so he would always know how old he is. Unfortunately, it is extremely difficult to distinguish solutions based on idiosyncratic interpretations that took place *during* problem solving from *post-hoc* rationalizations in defense of that response.

LOOKING FOR AN EXPLANATION: GOING BEYOND THE COGNITIVE

The results from the above-mentioned studies suggest that it is not a cognitive deficit as such that causes students' general and strong abstention from sense-making when doing arithmetic word problems in a typical school setting, but rather that they are acting in accordance with the "rules of the game" of the interactive ritual in which they are involved, or, as others would call it, in accordance with the "didactical contract" (Brousseau, 1997), or the "socio-mathematical norms" (Yackel & Cobb, 1996).

Several authors (e.g. De Corte & Verschaffel, 1985; Gerofsky, 1996; Kilpatrick, 1985; Lave, 1992; Reusser & Stebler, 1997) have carried out analyses of the hidden rules that seem to be used (implicitly, tacitly) by elementary school pupils (and by their teachers!) to make the "game of word problems" function efficiently, and have come up with rules such as the following:

- Any problem presented by the teacher or in a textbook is solvable and makes sense.
- There is a single, correct, and precise numerical answer.
- This single answer must be obtained by performing one or more mathematical operations with the numbers embedded in the text.
- The task can be achieved by applying familiar mathematical procedures.
- The text contains all the information needed and no extraneous information may be sought.
- Violations of your knowledge about the everyday world should be ignored.

De Corte and Verschaffel (1985) introduced the term "word problem schema" to refer to the system of beliefs about word problems shared by students and teachers, involving the perceived intent of word problems, interpretation of stereotyped semantic structure, and a complex network of implicit rules and expectations that govern playing of the "word problem game".

The above interpretation of students' non-realistic responses to word problems like the how-old-is-the-captain problem or the P-items from the test of Greer (1993) and Verschaffel et al. (1994), is in line with Schoenfeld's (1991, p. 340) suggestion that the children who produced such bizarre responses were not irrational but were engaged in sense-making of a different kind: "Taking the stance of the Western Rationalist in mathematics, I characterized student behavior... as a violation of sense-making. As I have admonished, however, such behavior is sense making of the deepest kind. In the context of schooling, such behavior represents the construction of a set of behaviors that result in praise for good performance, minimal conflict, fitting in socially, etc. What could be more sensible than that? The problem, then, is that the same behavior that is sensible in one context (schooling as an institution) may violate the protocols of sense-making in another (the culture of mathematics and mathematicians)". At a more general level, these children's seemingly meaningless behavior can also be interpreted in terms of Vinner's (1997, p. 97) particularly rich and relevant notion of "pseudo-analytical thought processes" in mathematics learning, wherein the person is looking for a satisfactory reaction to a certain stimulus and the thought process is guided by uncontrolled associations and superficial similarities, without any serious cognitive involvement.

THE CLASSROOM CULTURE AS EXPLANATORY FACTOR

This brings us to the question: How do these superficial strategies for and beliefs about the solution of school arithmetic word problem develop? As with most other "pseudo-analytical thought processes" (Vinner, 1997), the development of students' tactics for and conceptions about word problem solving is assumed to occur implicitly, gradually, and tacitly through being immersed in the culture of the mathematics classroom in which they engage. Putting it another way, students' strategies and beliefs develop from their perceptions and interpretations of the didactical contract (Brousseau, 1997) or the socio-mathematical norms (Yackel & Cobb, 1996) that determine(s) (explicitly to some extent, but mainly implicitly) how to behave in a mathematics class, how to think, how to communicate with the teacher, and so on. More specifically, this enculturation seems to be mainly caused by two aspects of current instructional practice, namely (1) the nature of the problems given and (2) the way in which these problems are conceived and treated by teachers.

Let's first have a look at the first of these two explanatory factors. In an attempt to summarize the characteristics of traditional word problems that appear in classrooms and textbooks and which lie at the basis of students' beliefs about and strategies for solving word problems as discussed above, Reusser and Stebler (1997, p. 323) wrote: "Only a few problems that are employed in classrooms and textbooks invite or challenge students to activate and use their everyday knowledge and experience. Most word problems used in mathematics instruction are phrased as semantically impoverished, verbal vignettes. Students not only know from their school mathematical experience that all problems are undoubtedly solvable, but also that everything numerical included in a problem is relevant to its solution, and everything that is relevant is included in the problem text. Following this authoring script, many problem statements degenerate to badly disguised equations." If the vast majority of the textbook and test problems have these characteristics, it should not be a surprise that many students develop gradually but inevitably strategies for and beliefs about word problem solving that are characterized by a lack of sense-making.

A second plausible explanatory factor for the development of the observed student beliefs about and tactics for word problem solving is the way in which these problems are conceived and actually treated by teachers in the mathematics lessons. A study that sheds some light on this second factor is an investigation by Verschaffel, De Corte and Borghart (1997), wherein (future) elementary school teachers were asked, first, to solve a set of P-items themselves, and, second, to evaluate four alternative answers from (imaginary) pupils to the same set of P-items as "absolutely correct answer, "partly correct and partly incorrect answer", or "completely incorrect answer". For each P-item, the four response alternatives always included the typical non-realistic answer and the most reasonable realistic answer. Only half of the student-teachers' own answers to the P-items in test 1 were scored as realistic, and, with respect to test 2, their evaluation of the non-realistic pupil answers to the P-items was considerably more positive than for the realistic answers based on realistic considerations!

In sum, the available evidence suggests that students' beliefs about and tactics for word problem solving do not develop as a result of direct teaching, but rather emerge from the nature of the textbook and test problems with which they are confronted and from the permanent interaction between teacher and students around these problems.

TAKING THE MODELING PERSPECTIVE SERIOUSLY

Based on the results of the above-mentioned theoretical and empirical work, several authors have put forward suggestions for improving the quality of the applied part of the mathematics education curriculum.

A minimal and rather easily achievable goal is to improve the quality of word problems as applications in numerous ways that have been suggested over many years such as:

- Break up the expectation that any word problem can be solved by adding, subtracting, multiplying or dividing, or by a simple combination thereof.
- Eliminate the flaws in textbooks that allow superficial solution strategies to be undeservedly successful.
- Vary problems so that it cannot be assumed that all the data included in the problem, and only those data, are required for solution.
- Weed out word problems in which the numbers do not correspond to real life.
- Accept forms of answer other than exact numerical answers.

However, on top of such a list of recommendations, which together constitute a minimal response to the identified flaws in traditional teaching of word problems, we propose in our book (Verschaffel et al., 2000) a more radical solution, namely to reconceptualize word problems *as genuine exercises in mathematical modeling*. In contrast to the truncated caricature of the multiphased and multidimensional model of mathematical modeling that underlies many traditional lessons in applied problem solving, at the basis of our approach is a much more elaborated version of the model of solving application problems, wherein all phases of the genuine modeling process are equally important and wherein

- knowledge about the phenomenon is not suppressed but considered as a valuable component in the initial stage of the solution process,

- the nature and the outcome of the mathematization act is influenced by the goals implicit in the situation, imposed by the teacher, or negotiated,
- the solver can make use of a rich variety of resources (including software modeling tools) in the stages of mathematical modeling and analysis,
- the interpretation and evaluation phase involve comparison and discussion of alternative models, and
- the task requirements may involve a communication phase that goes far beyond the bald reporting of the result of the calculation.

Starting from the above criticisms on the traditional practice surrounding word problems in schools and from the modeling perspective described above, researchers have set up design studies wherein they developed, implemented and evaluated experimental programs aimed at the enhancement of students' mathematical modeling and problems solving along the lines mentioned above. To mention just a few: several developmental research projects from the Freudenthal Institute in The Netherlands (see e.g., Gravemeijer, 1997), the Jasper studies of the Cognition and Technology Group at Vanderbilt (1997), wherein mathematical problem solving is anchored in realistic contexts using new information technologies, Lehrer and Schauble's (in press) experimental curriculum for mathematics and science teaching in young children built upon the modeling approach, and our own study about the design and evaluation of a learning environment for mathematical modeling and problem solving in upper elementary school children (Verschaffel, De Corte, Lasure, Van Vaerenbergh, Bogaerts & Ratinckx, 1999).

Characteristics common to these experimental programs include:

- The use of more realistic and challenging tasks than traditional textbook problems, which do involve some, if not most, of the complexities of real modeling tasks (such as the necessity to formulate the problem, to seek and apply aspects of the real context to proceed, to select tools to be used, to discuss alternative hypotheses and rival models, to decide upon the level of precision, to interpret and evaluate the outcome, etc.).
- A variety of teaching methods and learner activities, including expert modeling of the strategic aspects of the modeling process, small-group work, and whole-class discussions; typically, the focus is not on presenting and rehearsing established mathematical models, but rather on demonstrating, experiencing, articulating, and discussing what modeling is all about (see also Mason, 2001).
- The creation of a classroom climate that is conducive to the development of the elaborated view of mathematical modeling and of the accompanying beliefs.

In most of these design experiments positive outcomes have been obtained in terms of performance, underlying processes, and motivational and affective aspects of learning. After reviewing the available research evidence, Niss (2001, p. 8) concludes that "application and modeling capability can be learnt, and according to the above-mentioned findings has to be learnt, but at a cost, in terms of effort, complexity of task, time consumption, and reduction of syllabus in the traditional sense".

To some extent, these characteristics of the modeling approach are beginning to be implemented in mathematical frameworks, and tests in many countries. However, according to Niss (2001), it is still the case, in general international terms, that genuine and extensive applications and modeling perspectives and activities continue to be scarce in the everyday practice of mathematical education.

PROMISES AND PITFALLS OF THE MODELING PERSPECTIVE

When putting the modeling perspective forward for serious consideration, it must be recognized that it is not without major difficulties and challenges.

A first critical issue is: how far can and should we go in our efforts to make the modeling tasks realistic? How much reference to the complexity of reality is possible and appropriate in the classroom context? I agree with Gravemeijer (1997) that there is, and always will be, an insurmountable difference between solving problems in the out-of-school reality and solving word problems in a mathematics lesson or test. But if we accept that there will always remain some gap between mathematical modeling in a school and an out-of-school context, what is the appropriate level of "reference to the real" that should be established in the mathematics classroom? And is encouraging students to use their real-world knowledge not opening a Pandora's box? I don't think that there is one appropriate level of realism nor that the non-existence of such a clearcut level forms an impassable didactical problem. It only suggests that the question about the model's degree of abstraction and precision may be regarded, not as a difficulty, but as a part of what we want students to learn to make deliberate judgments about, as one crucial aspect of a disposition towards realistic mathematical modeling. Such difficulties with respect to the level of realism and precision are most serious, I believe, when word problems are presented in a context that precludes discussion, such as a student working alone on a textbook problem, or sitting a written test. Within the context of discussion and collaboration, the degree of precision, the reasonableness of plausible assumptions, and so on, may be negotiated (Verschaffel et al., 2000).

Second: does the modeling perspective exclude traditional word problems? I would contend that it does not rule out mathematical teaching and learning activities around classical word problems wherein students learn to apply powerful schemes for identifying, understanding, and solving certain categories of problems. For example, the schemes of addition, multiplication, direct proportionality, etc. are very powerful, and applicable to a wide variety of situations. So, it is important that students (also) learn to master these schemes and to apply them in various contexts during their mathematics lessons. However, the problem arises when such a schema is automatically triggered by superfluous cues, i.e., when students are given no training in discriminating between those cases where it is appropriate and those in which it is inappropriate or, in the extreme sense, nonsensical (Hatano, 1997). So, there should be room for different kinds of word problems with distinct instructional goals. At one time they may be used mainly to create strong links between mathematical operations and prototypically "clean" model situations (with little room for timeless discussions about the situational complexities that might jeopardize this link), whereas at other times they may be used primarily as exercises in relating real-world situations to mathematical models and in reflecting upon that complex relationship between reality and mathematics[2]. Both roles are important in a balanced math curriculum, but they have quite different characteristics and priorities, which must be clear to the student as well as the teacher (Burkhardt, 1994). In this respect, Galbraith and Stillman (2001, p. 301) propose an interesting problem classification, consisting of four categories, with essential differences in terms of the kind of thinking processes they elicit in students, as well as in terms of their underlying assumptions about word problems and their relation to solving problems in the real world:

- injudicious problems, wherein realistic constraints are seriously violated;
- context-separable problems, wherein the context plays no real role in the solution and can be stripped away to expose a purely mathematical question;
- standard application problems, where the necessary mathematics is context-related and the situation is realistic, but where the procedure is (still) rather standard;
- genuine modeling problems, in which no mathematics as such appears in the problem statement, and where the demarcation and formulation of the problem, in mathematical terms, must be (at least partly) supplied by the modeler.

Whereas, I would agree with these authors, injudicious problems are to be avoided, because they strongly reinforce the belief that mathematics has nothing to do with the real world (expect if they are purposely used in the context of making students, or in-service or pre-service teachers, aware of this problematic relationship!), context-separable and standard application problems do have a meaningful role in mathematics education. However, when using them, one should be cautious for the risk that students will tend to overgeneralize the validity and relevance of familiar modeling notions (e.g. addition, multiplication, proportionality) and transfer these to settings to which they are neither relevant nor valid (Niss, 1991), as argued by Hatano (1997) and as amply evidenced, for instance, in a study of De Bock, Verschaffel and Janssens, (1998) for proportionality. Moreover, we should not labor with the latter two types of problems under the illusion that such problems will foster the ability to apply mathematics to solve realistic problems. Only in the fourth type of Galbraith and Stillman's (2001) categorization do we find the need to invoke assumptions integrating mathematical development with the real context as a key to progress.

A final important issue I want to address is whether teaching and learning mathematical applications according to the modeling perspective is important, and feasible for, all students. Over the past few years several authors, such as Keitel (1989) and Mukhopadhyay and Greer (2001), have made strong pleas for engaging *all* students in the modeling perspective, both for the empowerment of the individual and for the betterment of society. For Mukhopadhyay and Greer (2001), this "political aspect" can be considered as a third perspective from which mathematics education in general, and teaching and learning mathematical modeling in particular, should be analyzed critically, besides the (purely) cognitive and the social/cultural perspectives. In relation to this political aspect, the most important reason for introducing the modeling perspective to all students is to help as many people as possible "to become critical thinkers who can use mathematics as a tool for analyzing social and political issues, and can reflect on that tool use, including its limitations" (Mukhopadhyay & Greer, 2001, p. 310). Evidently, once mathematics educators start applying this modeling perspective on a larger scale and allow students to bring in their personal experience when trying to make sense of all kinds of technical, social and cultural issues and phenomena, they will be confronted quickly and inevitably with the diversity of these experiences in terms of gender, social class, and ethnic diversity (see also, Boaler, 1994; Cooper & Dunne, 1998; Tate, 1994). I endorse Mukhopadhyay and Greer's (2001) claim that engaging students in such modeling activities, with careful attention to the relevance of the problem contexts and all the diversity in views and approaches that they elicit, is the best way to prevent students from becoming alienated by mathematics and its authority, and to help them using mathematics as a powerful personal tool for the analysis of issues important in their personal lives and in society. Given the multiphased, multidimensional and non-straightforward nature of the modeling process, it is often viewed as an activity that is only within the reach of older and/or more capable students. The evidence from several recent design experiments reviewed in this paper suggests that it is not only important, but also feasible, to start applying the modeling perspective successfully in mathematics education of all students already from a (very) young age on and with a diversity of learners.

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[1] For this paper I relied partly on a book that I recently wrote together with B. Greer and E. De Corte on *Making sense of word problems*. Furthermore, I thank D. De Bock, B. Greer, M. Niss and W. Van Dooren for their valuable comments on an earlier draft of this paper.

[2] Besides these two major functions, word problems can play other roles too, like in "mathematical puzzles" that are used to train or test people's intelligence or mathematical ability (Verschaffel et al, 2000).

PLENARY PANEL

Theme
Learning from Learners

Coordinator
João Filipe Matos

Panelists
Joop van Dormolen
Susie Groves
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LEARNING FROM LEARNERS

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INTRODUCTION – JOÃO FILIPE MATOS

When looking closely at everyday activity it is difficult to avoid the conclusion that learning is ubiquitous in ongoing activity. Nevertheless, societies came to forms of pedagogical practice that conform with the idea that social arrangements can be made up to provoke specific learning effects in some of the participants in a certain practice. Schooling is perhaps one of the most apparent examples of how societies create and maintain stable social practices. As soon as one enrols in school, at any level, one is positioned and labelled as a *learner*. This is not surprising but it strikes me the fact that as soon as one is positioned as a teacher, at any level, one tends to suspend or forget our condition of learners.

This paper is the reflection of the preparation of the PME26 Panel aiming to discuss the issue of 'learning from learners'. When addressing the topic of Learning from Learners, one of the key issues is what it implies to be a learner. Sinha (1999) frames this issue asking how the developing human is being constructed and positioned in particular and specific kinds of practices, in such a way that they become a learner of the kind required by the culture within which teaching-learning activities occur. This question is asked implicitly in most studies addressing mathematics learning. But within this paper it is formulated in order to reflect on the way we as teachers and/or researchers learn from learners— pupils, teachers, researchers.

The panellists invited to address this topic—Joop van Dormolen, Susie Groves and Rosetta Zan—accepted the challenge of producing a short narrative and a brief analysis of an episode that each one found relevant for their own learning as a teacher and/or as a researcher. They were also asked to comment on each other episode and combine their own analysis with the others'. The result was a joint paper putting together the reflection of the three.

From my point of view, the very idea of conceptualizing the notion of 'learning from learners' contains in itself a rethinking of learning in terms of framing it by context, communication and social practice abandoning traditional views which focus upon *isolated and individual subject* in confrontation with a cognitive or learning task. This implies, for example, the abandonment of the universalistic idea that learners, in all essentials, are the same in all times and places. These notions are reflected in different ways in the several parts of the paper.

EPISODE I: HOW KARIN DISTURBED HER TEACHER'S LINE OF REASONING - JOOP VAN DORMOLEN

The episode

The episode that I want to describe happened during a mathematics lesson in grade 7 that I observed. I was sitting together with a student teacher, let us call him John, in the back of the class, while the lesson was given by John's mentor, let us call him Wilbur. The episode took place in the end of September, four or five weeks after the start of the school year. Wilbur, the teacher, had a 15 to 20 year experience with teaching grades 7 to 12. Students liked him and trusted him as a teacher.

This was the first day that John had started his teaching practice period. It was decided that he should not right away start giving lessons, but first observe lessons given with his mentor. I was there as the John's university teacher. Wilbur and I had agreed that I would be in his class the first time that John was there and that after the lesson the three of us would sit together to analyze the lesson. This analysis in itself was meant to be a learning experience for John.

At the time of this story I had 20 years experience in the same kind of school as Wilbur and for about 6 years had been a teacher educator for secondary school mathematics at the university.

The students

The school was a typical Dutch school in which students would follow either a six years course that prepared for university studies or a five years course that prepared for higher non-university studies. Grade 7 is the first class of such schools. Decisions of which of the two courses the students would follow were not to be made before the end of the second year.

The subject matter

The lesson was about basic algebra rules: the commutative rules $a + b = b + a$ and $a \times b = b \times a$, and the distributive rule $a \times (b + c) = a \times b + a \times c$. During the weeks before the lesson the students had learned the concept of a variable and had had exercises with substitutions of numbers for variables and solving simple first degree equations with one variable.

The event

Wilbur had explained the fact that it does not matter whether you add 3 to 2 or 2 to 3, you end up with the same result, and this not only for the numbers 2 and 3, but for any two numbers. We can express this with $a + b = b + a$ in which a and b stand for any number one can think of. This is not only for additions. Also in multiplication we have the same kind of rule: 3×2 means $2 + 2 + 2$ and 2×3 means $3 + 3$. These are two completely different multiplications, but their product is the same. So for any pair of numbers, hence we can say $a \times b = b \times a$.

Most of the time Wilbur was speaking and explaining with occasional questions to students. The class was attentive and seemed to be willing to learn.

Then Wilbur started with examples of the distributive rule. He had been working on this for some time, say 8 to 10 minutes, when one of the students, let us call her Karin, put up her hand. Wilbur, thinking that she wanted to comment on what he was discussing, asked her what she wanted to say.

She said: "Two time three is six because two times three is one plus one plus one ...*(here a slight pause)* plus one plus one plus one. And three times two is one plus one ...(*slight pause*) plus one plus one ... (*slight pause*) plus one plus one. In both cases you have the same number of one's.". While she spoke she moved her finger as if writing down in the air what she was saying.

In a friendly way Wilbur dismissed Karin's remark with something like: "Yes, you are right, but we saw that already, we are now doing something else." and went on with talking about the distributive rule.

What did I learn from Karin's remark?

Sitting in the back of the classroom I had not Wilbur's preoccupation of having to go on with the lesson, I could reflect on what Karin had said and I decided to make that one of the subject of the discussion with John and Wilbur after class. In that discussion Wilbur said he felt a little irritated, because Karin had said something that had nothing to do with what he was talking about at the time. He also told John and me that Karin was in his opinion one of the weaker students and therefore assumed that she just had been a little slow in following his reasoning. After some discussion he said he was sorry not to have realized that in fact Karin showed a deeper understanding of the commutative rule than just a formal acceptance.

Karin showed that she was really amazed. For many years as a teacher and in my discussions with my student teachers I felt that it is almost impossible to teach young children the commutative and the distributive rule as important mathematical properties. For most of them they are natural phenomena and I feel that making a big fuss about it is pretentious and only shows that mathematics is about making simple things difficult. Only when we are more sophisticated in understanding what mathematics is about, we learn that formally mathematics has to be built up from scratch and to do so we have to formulate axioms. These axioms are not just concoctions that somebody dreams up. They are based on practical experiences. In other words: the commutative and the distributive rules are not, as Wilbur and I and many other teachers had been trying to teach for a long time, *explanations* of phenomena, but indeed *formalizations* of such phenomena. I think that this is just what Karin felt: Karin's amazement was not that you can explain the phenomenon 2 _ 3 = 3 _ 2 with help of the rule, but that you can explain the rule with help of the phenomenon.

I am pretty sure that Karin did not realize this, but for me, as a teacher I learned from it that it pays to present situations to students in which they can get amazed as a starting point for learning to generalize. If Wilbur had realized this during the lesson he could have used Karin's remark in this way.

The same language, different meanings

There are several approached to reflect on this episode. I have chosen one that I found in an article (Klaassen & Lijnse, 1996) of two ex-colleagues of mine. They describe an episode in which a teacher did not succeed to convince a student. They give five different viewpoints from which one can analyze the episode. First, the view that an error was made and thus the student has to learn what was wrong and to repair it. Second, the view that there is a misconception and the teacher should work on that to help the student to get the correct conception. Third, the view that the idea of misconception is not an acceptable way of analyzing. Better is to recognize student's previous learning and talk about pre-conception. It is the teacher's job to find ways to confront that pre-conception and the 'correct' conception so that the student can change. Fourth, the view that the student and the teacher are living in a different world. "In each world, different concepts are used, being part of different kinds of knowledge, with different characteristics and problem-solving procedures" (ibid. p. 121).

The fifth view comes from the authors themselves. They argue that teacher and student do not live in different worlds. In fact they agree with each other, at least as one looks at the language they use from the outside. However they use the same language in a different way, giving the words a different meaning.

Interestingly in our case there was no mistake, so the first three viewpoints are not relevant here. We even cannot think here in terms of different worlds. The last view, however is enlightening.

We have to be clear that in our case the issue was not something about the commutative rule. In fact this rule was not relevant at all; it was just a means to get to the more general concept that certain kinds of rules are formalizations of certain phenomena. Following the line of Klaassen and Lijnse we can say that the language spoken by Karin and Wilbur was the same: Both talked about the fact that 2 time 3 is equal to 3 times 2, that this was so for every pair of numbers which could be expressed by $a \times b = b \times a$. They agreed about that, but each of them gave a different meaning to it. Wilbur did not recognize that. He thought that Karin was just a bit slow in taking up his ideas and therefore in a friendly way dismissed her remark as not relevant any more.

Reflecting on how I learned and its implications

From Karin I learned to distinguish between formalization of phenomena and explanation or a rule. In both cases one can use the same examples but in different contexts: in the context of formalization the examples are paradigms, in the context of explanation they are instances (Freudenthal 1978, pp. 201; Van Dormolen 1986).

Reflecting on my learning experience I found the same-language-different-meaning approach highly illuminating and useful in analyzing other episodes in which there is some sort of teacher-student conflict.

What I learned is interesting and illuminating. I used that later many times in my own classes of pre- and in-service teachers. More general is, however, the question of how I learned. Why was it that I could learn from Karin, while Wilbur and John did recognize her intentions only after I brought the matter up in our after-lesson discussion?

One essential element is (as ever so often) the role of the context. For me the situation was open, I had no restraints for attaining goals, like Wilbur. I was not on edge like John who had to cope with many new experiences. I had no responsibility like Wilbur to keep as many students as possible attentive and motivated. I could start to think about Karen's remarks without paying attention what was happening after that. Wilbur's context generated for him a situation that made him, experienced teacher as he was, react automatically. His experience with Karen's not so bright performances in the past made him - without realizing it - assume that her remarks would again be on the poor side. In the after-lesson discussion he realized and deplored this automatic-pilot-attitude and he had every intention to be more attentive in the future to unexpected interventions of his students.

There is more to say about context. It is noteworthy that in the descriptions of all three episodes, Groves, Zan and I apparently wanted to tell much more than – on first sight – seems necessary to describe the case. After writing down the first draft of my episode, I wondered if I had written too much. Could I not delete, for example, all what I wrote about John? On first sight the presence of John seemed to be irrelevant. Yet I could not bring myself to take him out of the story. Realizing this I found that, intent as I was on finding subjects to discuss in the after-lesson talk, I might not have noticed the importance of Karin's intervention without John's presence. I see similar situations in the episodes of Groves and Zan. Apparently the total context was crucial for the three of us as an opportunity to learn.

Last remark: All this came from reflection (Van Dormolen 1998, 2000), both in the classroom, in the after-lesson discussion and in the composition of these notes for the panel discussion. Crucial as elements such as context, attentiveness, open mindedness, teaching experience etc. are for learning, without conscious reflection I would not have learned at all.

COMMENTS ON JOOP'S EPISODE - ROSETTA ZAN

The episode described by Joop can be read and generalised both by centring on the various actors present, and also by viewing it from a different perspective: one possibility, suggested by Joop, is to focus on the epistemological positions (either conscious or unconscious) of the teacher and the researcher. My reading of the episode comes from a different perspective, which is also the one that comes most naturally to me given my area of research: I will focus my attention on the complexity linked to the presence of two different subjects, pupil and teacher, and to their communication.

I tried to 'put myself in Karin's shoes', and I'll now describe her possible thinking process using a *virtual monologue* (Leron & Hazzan, 1997): this method, which originated from a researcher's attempt to better see the world with the student's eyes, provides us with an instrument for communicating *our own* way of seeing the pupil's point of view to others. When Wilbur starts to introduce the distributive rule, I am still following the previous topic about the commutative rule:

> It's not like before, when maths was simply about doing calculations. No,... now we're not just working with numbers to get a result ... Wilbur is teaching us something important, something 'grown up': we can describe things that seem simple in a more important way, it's not enough any more just to do a calculation. Something that seemed so natural... that 2x3 and 3x2 give the same result ...Wilbur explained to us that this is a property that has a name! It's important to be precise like this in mathematics: as we go on, we've got to learn to be even more precise! ...But... if we really want to be exact (and that's what we're learning to do, isn't it?,) ... who's telling me that 2+2+2 and 3+3 give the same number? Without calculating as we did before! Let's see... If we write 2+2+2 as 1+1..+..1+1...+...1+1, ...yes, that's it! and we write 3+3 as 1+1+1...+...1+1+1... there we are! Now, things are more precise!! I'm going to tell Wilbur... (and she puts up her hand)

Karin's effort reminds me of many mathematics students that are learning to prove some results by starting from the Peano axioms and the definitions of the + and the x operations. It is the effort to distinguish between what can be assumed as true and what has to be proved when we are moving in an environment where everything already seems familiar. Even if this effort is not always successful, and the student's solving processes highlight his confusion, it still testifies that the student is starting to approach mathematics in a new way: a more sophisticated way, linked to an evolving vision of the discipline, moving towards more mature epistemological forms.

Karin's intervention was a missed opportunity for Wilbur, but Wilbur had *his own* project in mind: 'to explain' the commutative and distributive rules. This project is the result of a decision made before he started his interaction with the pupils. There are many of similar decisions to be taken: how to present the topic, what examples to make, how much time to dedicate to the various steps.... As time goes on, such decisions get easier and easier: after all, Wilbur has been teaching for many years, and this is a well defined task for him! But once he starts interacting with the class, Wilbur finds himself having to manage situations that he hadn't completely planned for when preparing the project. He has to deal with the pupils' questions, and he has just a few seconds to decide *how* and *whether* to answer. Since the teacher's decisions are not just influenced by his knowledge, but also by his beliefs and emotions, this influence is particularly strong when there is little time available for deciding what to do. While Wilbur is explaining the distributive rule, Karin intervenes. Wilbur, who is so focused on his attempt to make the students understand distribution, is ready to answer questions or deal with objections on *that* property. But Karin's intervention moves away from the direction he wants to follow and puts him in difficulty: 'Yes, you are right, *but we saw that already...*'

It becomes natural to ask ourselves: what idea was Karin constructing? And how could her teacher have discovered it? What consequences will Wilbur's reaction have had on these new ideas? What consequences will this have had on Karin's desire to understand what was new in this activity? Will Karin try again to 'make sense' of mathematics? Or will she simply conform to the demands and pace of Wilbur, to the demands and pace of the lesson, to the demands and pace of others?

But this is obviously a general problem: how can we understand what sense of mathematics the pupils are making for themselves? How many Karins do we distance from mathematics because we do not know how to listen to them? What consequences does a rigid and limited schedule have on the pupils? What idea of mathematics will be constructed by pupils who haven't had an answer to their questions or haven't been able to ask their questions? And what will be the resulting image that they construct of themselves?

What about Wilbur? Again, this is a general problem: how can the teacher learn to make quick decisions? How can he recognise the consequences that these decisions have had? How can he correct his mistakes if required? And how can he learn to respect the time needed by the students, and to respect their thinking processes?

COMMENTS ON JOOP'S EPISODE – SUSIE GROVES

In one sense it is difficult to go beyond Joop's own analysis of this episode — in particular the discussion of the difference between *explanations* and *formalisations* of phenomena. However, for me, the most striking comment in Joop's description is where he says that he learned that "it pays to present situations to students in which they can get amazed as a starting point to generalise".

There are clear analogies with this view and the notion of creating cognitive conflict as a teaching strategy in science education. Vygotsky (1962) characterised children's *scientific conceptions* as developing downwards, while their *spontaneous conceptions* develop upwards, stressing the importance of the interaction between these, with spontaneous concepts enriching scientific concepts with meaning and scientific concepts offering generality to the spontaneous concepts. This is sometimes described as the vine metaphor. Pines and West (1985) adopted the vine metaphor as a framework, distinguishing different prototypes of learning situations by the extent to which the "upward and downward growing vines" clash or are congruent. From a constructivist viewpoint, they considered meaningful learning to take place when the two vines become intertwined, with the new scientific knowledge serving to make sense of the learner's world of experience. In many ways this appears to describe what was happening when Karen "intertwined" the formal statement of the commutative property with her observation that two times three and three times two "have the same number of one's".

While Wilbur, the teacher in Joop's episode, failed to capitalise on this opportunity — or even recognise it at the time — this is not surprising as he had a whole class to contend with and had already moved on to look at the distributive property. Moreover, if we are to take on board the notion of using students' 'amazement' as starting points for generalising, we not only need to find appropriate situations with the potential to amaze students, we need to acknowledge the idiosyncratic nature of such happenings. For example, who could have predicted Karen's reaction to a standard treatment of the commutative property, which presumably she had met many times before?

There are also implications for the organisation of teaching situations if teachers like Wilbur are to be able to capitalise on spontaneous situations such as the one described in this episode. In our *Calculators in Primary Mathematics* project — briefly described elsewhere in this paper — using the calculator, often in free play situations, prompted young children to be amazed. For example, one four year-old boy is captured on video spontaneously using his calculator to count by ones from a million. When asked whether he could count to one million one hundred he initially says he thinks there is no such number. However as he nears it he begins to have doubts and finally when he gets to 1 000 002 he is truly amazed to see he has "gone right past it". The fact that the rest of the children in the class were absorbed in their own calculator explorations allowed the teacher to spend four uninterrupted minutes talking to this boy and his partner, during which time she was able to "learn" something about the boy's understanding of very large numbers, spontaneously prompt him to try a potentially amazing activity, and engage him in discussion related to the outcome. This would not be easy to parallel in a secondary classroom.

Perhaps the real lesson from Joop's episode however is that we are unlikely to learn from learners if we do not listen to them. A lot of time is spent in teacher education in teaching teachers to "teach" . Perhaps a lot more should be spent on teaching them to listen and learn from their students. In mathematics classes, this is particularly critical in terms of the type of communications we engage in with our students and the social and "socio-mathematical" norms (Yackel & Cobb, 1996) which operate.

EPISODE II: MARCO AND ANNA - EACH FOLLOWING THEIR OWN PATH – ROSETTA ZAN

The episode

The event that I'm about to describe happened during a meeting with teachers. Another University had invited me to give a lecture on the difficulties in mathematics: the audience was mostly teachers from higher education. In order to get them involved, I decided to give some examples that would be within their experience and ask them to perform an interpretative analysis. My first case was the incorrect use of brackets, a very common area of difficulty. I described the behaviour of a student, Marco, who had to multiply x+1 by x+2. He wrote: $x + 1 \bullet (x+2)$, but he went on:

$x + 1 \bullet (x+2) = x^2 + 2x + x + 2 = x^2 + 3x + 2$

My intentions were: 1) To ask for their analysis of the case, start a discussion, etc. 2) If no other opinions had emerged, I would suggest a possible interpretation of this systematic error: maybe Marco interprets the presence of brackets as a sign that indicates the order of precedence for performing operations, but as a personal shorthand for his own use. He doesn't therefore see them as signs to be used to communicate with others, with fixed and shared rules, but as notes to be ignored once the exercise has been completed; 3) to underline that, following this interpretation, Marco's behaviour is completely consistent. Marco had possibly put brackets where he sensed they were needed to highlight a certain order, but didn't use them when he felt there was no need because the order to be followed was easily apparent. In this interpretation, the fact that he performed the operations in the "correct" order shows that *his* use of brackets was correct.

The example caused an immediate reaction with all the teachers present. There was a chorus of: 'It's true ! they do that!' 'Mine do too…' When everyone had calmed down, one teacher raised her hand and asked to make a point: 'It's true: they often do this. But I always take off two points: one because they made a mistake in not using the brackets, there … [and indicates $x + 1 \bullet (x+2)$]; and the other point because they make a mistake in multiplying, here… [and she indicates the equal sign in the expression $x + 1 \bullet (x+2) = x^2 + 2x + x + 2$]'. The other teachers reacted noisily, many burst out laughing. The teacher who had spoken (let's call her Anna) realised she was being derided by her colleagues and she repeated heatedly: 'two mistakes: two points off. *I explain to them* that they have made two mistakes.' Her colleagues looked at me before reacting, trying to understand what I was thinking. In order not to put Anna on the spot, I asked her without irony: 'and in *this way* … do the students understand? Don't they make any more mistakes using brackets?' She replied with resignation, but still convinced: 'No, they keep getting it wrong… but there are *two* mistakes.'

The incident finished there: I continued with my program.

What did I learn from Anna's observations?

My first reaction to Anna's intervention was disconcert and disappointment, even if I hid this behind my question. I had been talking about the constructivist model of learning and everyone (including Anna) had been following me with attention; I had talked about the importance of understanding what is behind a mistake if you are to overcome it, and everyone (including Anna) had agreed ... and then *that* intervention! The other teachers were as shocked as I was: at least they seemed to have understood the message. Thinking back as I was on the train home that evening, even if my intervention had been useless to Anna, it had been of help to the others, so the balance of this exhausting day could be said to be positive. But even so, I couldn't get over Anna's intervention: remembering her expression, her conviction in defending her position, but also her dissatisfaction in recognising the uselessness of it! There was something important under all of this, something that I couldn't grasp. But what? When I thought back to my initial program in which I presented the case of Marco, what I wanted to do was to push the teachers towards 'putting themselves in Marco's shoes', because in this way, they could begin to see the interpretative hypothesis I was making: to decide how to react, it is first necessary to ask yourself '*Why* has the student made this mistake?'. But after Anna's observations, when I tried to put myself in the shoes of *her* Marco, I felt a total sense of estrangement: I had been corrected and penalised, though I hadn't even made a mistake! My attention had moved from *why* Marco had made the mistake to the fact that it wasn't possible for Marco to realise that he had made the mistake. But how could Marco modify his behaviour in a significant and definitive way, if he didn't recognise he was wrong, if the consequences of his error didn't lead to a failure? In that period the Pisa group was doing research about the evolution of attitude towards mathematics: we set an essay on mathematics ('Me and mathematics: my relationship with maths up to now") for students at all levels of school, and then we analysed the essays elaborated. It was a sentence of another Marco that came into my mind at this point: '*...I constantly dedicated a lot of time, and I really wanted to succeed. But seeing that everything I did was wrong, I became resigned and came to the conclusion that maths and me weren't meant for each other ...*'. Marco slowly distanced himself from a discipline that he couldn't control: maybe because the things he did 'well' were corrected and penalised?

Anna and Marco: I couldn't help but thinking of them 'together', as if there was just one Marco, the one with the brackets problem and who *then* went on to write the essay, and Anna his teacher. Together, but each one following *their own* path, and, in the end, both of them in difficulty, or rather -and Anna helped me understand this - *their diversity was the reason* why both of them were in difficulty: each had their own goals, their own criteria for evaluating whether they had been achieved, their own epistemology. But also Anna and I, in the end, had followed different paths: paths characterised by different goals, and different epistemologies. And in the end we didn't manage to find each other either.

Analysis and comments

The sense of estrangement that we feel when we put ourselves in the position of a Marco whose teacher has just taken off two points highlights the possible rift that exists between the perception that the teacher and the student may have of the same situation. Such rift is particularly apparent in the case of Marco and Anna, not just because Marco's result is correct, but also because the fact that the result is correct makes his personal use of brackets 'correct' (in the way we hypothesised before). But this episode was important to me because it forced me to focus my attention on the consequences that are usually generated when two different subjects, the teacher and the student, have to recognise an error or failure, a preliminary recognition that must occur before the recovery operation can start. This focus allowed me to reinterpret significant episodes that at first appeared separate, in a new single coherent perspective.

In the case of Anna and Marco, the rift is so large that Marco doesn't recognise any of the errors that Anna identifies: in the end, Anna recognises incorrect solving processes, while Marco is probably convinced that he has given a correct answer with correct solving processes. But a similar rift can be observed in situations that at first seem very different to this case. For example, if the student sets himself the goal of finding the correct answer, and the error does not affect the outcome, the teacher's perception of failure cannot be shared by the student. But if the student doesn't realise his failure, or that *that* specific behaviour is causing the failure, how can he invest resources to change his behaviour? It is therefore not only important to ask ourselves *why* the student made a specific mistake before intervening, but also and in the first place, *whether* the student realises that his behaviour is at fault. But this is not the only point. The sense of estrangement that a student feels when he sees processes or products being corrected that he doesn't recognise as being wrong can have extremely negative results in the long term. The student begins to feel he has lost control over his success / failure, and mathematics appears ever more as a discipline in which the only person who has any control is the teacher. The belief that mathematics is an uncontrollable discipline has very important and negative consequences. On the one hand, the student is constantly searching for what the teacher 'wants'. On the other hand, negative emotional aspects appear such as anxiety, frustration, and even fear and panic, and in this way a fatalistic attitude takes over, with a consequential passive and resigned behaviour.

These considerations led me to change the perspective under which I approached the problem of learning difficulties in mathematics. The interpretation suggested by this episode allowed me to observe that some elements were common to other significant episodes that seemed to be completely separate problems up to then. It made me also reflect theoretically on the construction of a single coherent framework, which would provide instruments to observe and modify significant phenomena. A first partial and temporary result was the characterisation of difficulty, which also follows on from the comparison with other theoretical studies, and generates new problems and directions for research.

Traditionally, the problem of recognising difficulties in mathematics relates to the recognition of errors. This doesn't mean that the difficulties are identified with the errors: many researchers (see Borasi, 1996) underline the positive role of errors in the learning process. But even so, errors remain a strong indicator of difficulty. The reflections expressed above convinced me that a new *explicit* distinction must be made between errors and difficulty. The diagnosis of difficulty based on the recognition of error is apparently linked to objective elements (in fact the error is weighed and evaluated), thus risking to hide the intrinsic subjectivity of the observation / intervention process, caused by the presence of two different actors, the teacher and the student. Even if the teacher recognises the student's error and intervenes, it is up to the student to modify his behaviour: but if the student is to significantly change his behaviour he first has to be convinced that a change has to be made, that the existing behaviour leads to failure. Talking about a failed process is different to talking about mistakes: it means making reference to a goal that has not been achieved. Therefore it is only by setting things up in terms of goals, rather than as 'objective' errors, that we can highlight the presence of two different subjects, student and teacher, and the complexity that derives from this. In the same way, we can consider managing this complexity only if we recognise explicitly that it exists.

All these considerations led me to conclude that it is better not to identify errors as indicators of difficulty: we should instead concentrate on failure, intended as not having reached the desired goal. The reference to goals takes us back to points underlined in some studies (Cobb, 1986): the problems that students face in the context of learning mathematics are 'more social than mathematical'. We have to therefore consider the possibility that the goal set by the student is not necessarily 'mathematical': it can happen that when setting an exercise, the teacher sets a goal (e. g. find the area of a certain figure) which provokes a different goal in the student (e. g. give the right answer to the teacher). We should also consider that the points made here regarding the didactical value of errors suggest that only *repeated* failures and not a single one should be considered as a difficulty indicator. These observations bring me to the following temporary working definition of difficulty: the repeated failure in problems that the pupil encounters in the context of mathematics education. The effectiveness of this definition should be measured in practice, particularly in terms of its capacity to provide new instruments to detect and modify significant phenomena, opening up new directions for research.

LANGUAGE AND CONTEXT: COMMENTS ON ROSETTA'S EPISODE – JOOP VAN DORMOLEN

Anna and Marco

I would like to comment on the episode presented by Rosetta from the background of same-language-different-meaning approach that I wrote about in my comment on my own episode. Both used the same language, in this case the sequence:

Multiply $x + 1$ with $x + 2$

$x + 1 \cdot (x + 2) = x^2 + 2x + x + 2 = x^2 + 3x + 2$

Both were using the same language but gave it different meaning: Marco considered it right, Anna said it was wrong. Rosetta did not write about this in these terms, but her remark that Anna and Marco followed different paths seems to agree in essence with the same-language-different-meaning approach. I would like to put forward that the language approach may give us a more promising possibility to get Anna and Marco to come to terms with each other. The different-paths metaphor makes us a little hopeless. How can they ever come together? That is the danger with metaphors: They make you understand, but they do not show how to go on. The language approach is not metaphoric at all. It describes real life situations. As such it helps us not only to understand the situation; it also might give us an indication how to repair the discord between the two speakers. How can we get them to speak the same language with the same meaning?

For that we have to analyze the situation a little more. That was what Rosetta had been doing. She wanted the teachers to try and understand "what is behind a mistake...". Yet she did not convince Anna. Apparently Anna agreed as in general terms, but when she imagined that something like the example would happen in her class, she became uneasy and had the courage to give her opinion. What made her unhappy with the Marco example?

I would like to propose two reasons. One, as Rosetta points out, is the inability to put herself in Marco's shoes. The other is the divergence between Anna and Rosetta.

Let me try and analyze the first reason first. I learned from Karen (see my own episode) that thinking in terms of analyzing mistakes is an asymmetric way of working. It is the I-know-better attitude. The same-language-different-meaning approach makes one realize that there are two sides in a conflict. Both sides are right when viewed from each owns standpoint. So let us try and find out what the different standpoints are.

Again I see a similar situation as in the Karen-Wilbur conflict that I described in my own episode. Marco's path is *formalisation* of an experience, while Anna wanted to *explain* the procedure from the rules. Seen from this standpoint both are right in their conclusion: Marco thought he did the right thing, namely reply to the instruction to multiply x + 1 with x + 2 and for that he invented an ad-hoc formalisation, that helped him to find the solution, but is irrelevant for others. From his viewpoint his teacher had no business with the way he worked. He gave the correct answer to the task. Anna thought to explain what he did in terms of allowable rules and as such –from her viewpoint- he went wrong.

Anna and Rosetta

Much more important for the case than the Anna-Marco conflict (they even did not know each other) is the Rosetta-Anna conflict. Analyzing it with the same-language-different-meaning approach one could say that Rosetta and Anna were using the same language when talking about what Marco did, but Rosetta (and the other teachers) were thinking about an explanation of his behaviour, while Anna was thinking in a for her more direct practical direction: What to do about it? At that moment she could not find another solution but the one she was used to. Putting myself in Anna's shoes I would say something like: "All this talk is good and interesting, but it does not tell me how I can make Marco aware of what he has done and why I cannot accept that".

Rosetta, in her reflections, takes a wider approach, which brought me to think about the different contexts in which Rosetta had put herself (and most of the teachers with her) and Anna's context.

Rosetta's context was the constructivist model of learning. As long as this went in general terms it was also Anna's context, but the example of Marco changed that. Just as the other teachers she recognized the situation from her own experiences. Maybe because of that, the situation became more realistic for her than for the others and she wondered what she had to do when she had had Marco in her class. She could not understandingly just accept Marco's work. She knew she had to do something. She asked herself how she had to react. That changed her context completely. She had to make Marco aware that this is not done and therefore she had to punish him.

Anna's context made her say things that – in her fellow teacher's eyes – transformed her in the stereotype of the old fashioned teacher that wants her student just do what they are told. In a way this is similar to Wilbur's generalization of Karin's abilities in my episode. (From reading Rosetta's analysis I am sure that this was not how she saw Anna!). Anna apparently was not aware of her other context and therefore expressed herself rather crude and cruel, but maybe what she said was a clumsy way of asking for help: "Tell me what one can do in such a situation". Was this a (rather extreme) instance of same-language-different-meaning?

What did I learn from Rosetta?

I learned nothing new, but am grateful for her description and her reflection on it, because I saw the same elements as in my and Susie's episode:

- The reason of a discord can often be analyzed with the same-language-different-meaning approach. It makes the situation symmetric. There is no I-know-best situation any more.

- In mathematics teaching-learning such different meanings can sometimes (often?) be explained as difference between formalizing experiences and explaining rules.

- The same-language-different-meaning approach may also prevent us from generalizing other people's abilities.

COMMENTS ON ROSETTA'S EPISODE - SUSIE GROVES

For me the most striking observation about the episode presented by Rosetta is the critical importance of people's beliefs about the nature of mathematics for their behaviour as learners and as teachers of mathematics.

The first of the two Marco's referred to in the episode appears to (not uncommonly) view mathematics as a set of procedures to be mastered in order to obtain correct answers to problems which of themselves appear to have no meaning and over which they have no ownership. Having obtained the correct answer of $x^2 + 3x + 2$ for the product of $x + 1$ and $x + 2$, Marco presumably was not concerned by the fact that his use of notation in the process of obtaining the answer was unorthodox. In my experience, students when asked what they mean when presenting a written or verbal solution often answer "You know". When pressed further they still repeat "You know" as often as needed. This is not surprising because usually the teacher DOES know and is only asking for confirmation from the students that they too "know" whatever it is the teacher knows. Students have very little experience of being asked to solve problems where it is possible that the teacher does NOT know the solution strategy or thinking used by the students — that is situations where they genuinely need to explain their thinking to someone else. They have little idea that mathematics needs to be communicated to other people, both within the community of mathematicians and when applying it to problems in the real world — and of course that this is why mathematical conventions and their correct use are important.

By way of contrast, Anna in this episode has a very strong sense that mathematical conventions are important. But there is no sense that she sees these as important because of the need to communicate mathematics to others. In fact, she appears to share Marco's view of mathematics as a possibly meaningless, but nevertheless rigid, set of rules and procedures which it is her task to transmit to her students, however unsuccessfully. A constructivist model of learning *mathematics* is by and large irreconcilable with such a view of mathematics, so it is not surprising that both she and Rosetta felt confronted by the mismatch between Anna's response to the Marco scenario and her apparent acceptance of the constructivist model. While I realise that constructivism refers to how knowledge is constructed by humans, regardless of the context or the discipline, I have italicised the word *mathematics* in the previous sentence because I wonder what we believe might be the implications of adopting a constructivist model of learning for teaching someone, for example, how to fly an aeroplane — which I am assuming here is strictly a set of skills. Would holding Anna's view of the nature of mathematics simultaneously with a constructivist model for the learning of mathematics lead to an impossible notion somewhat akin to "discovery learning"?

The second Marco in Rosetta's episode presents a sad story of really trying to succeed, but seeing "everything I did was wrong" and coming to the conclusion that he and maths "weren't meant for each other". This story appears to illustrate that this Marco also saw mathematics as a set of rules and procedures which need to be learned verbatim and mimicked in response to an appropriate cue. Again there is no sense of mathematics as a living, evolving discipline, with intrinsic underlying meanings which can be explored and personally constructed into a web of knowledge.

In her description of the episode, Rosetta, while seeing Anna and the two Marco's together, sees them all following their own paths, with their own goals and criteria. In my reading of the episode, I was much more struck by the similarities of their goals and criteria and their common, mechanistic view of the nature of mathematics.

EPISODE III: CALCULATORS – THE FIRST DAY - SUSIE GROVES

The episode

The event I have chosen to describe occurred ten years ago when Jill Cheeseman and I were producing the video *Young Children Using Calculators* (Groves & Cheeseman, 1993) to disseminate some of the findings from the *Calculators in Primary Mathematics* project.

The context

The *Calculators in Primary Mathematics* project was a long-term investigation into the effects of calculator use on the learning and teaching of primary mathematics. Kindergarten and grade 1 children in six schools in 1990 were given their own calculator to use whenever they wished in class. The project followed these children through to grade 3 and 4 in 1993, with new children joining the project each year as they started school. The project was based on the premise that calculators, as well as acting as computational tools, are highly versatile teaching aids which can provide a mathematically rich environment for children to explore. Teachers were not provided with classroom activities or a program to follow, instead they were regarded as part of the research team investigating the ways in which calculators could be used in their mathematics classes.

At the time when this event occurred, video footage had been obtained in a number of classrooms and we were ready to edit it to produce the final videotape. However, we were alerted to the fact that we had missed an important aspect when a teacher at one of our regular teacher meetings showed her "home video" of children in her kindergarten class being given their calculators on the first day of school. For technical reasons we could not include her video footage and it was too late to capture the first day of school in any of the project classrooms. However, for reasons which no one could remember, one of the six schools had decided that the project would commence at grade 1 rather than kindergarten. We approached the school and asked whether they would be prepared to include their kindergarten class in the project and allow us to videotape the lesson in which the children would be given their calculators. The school and the kindergarten teacher, who we will call Clare, agreed.

The event

A week before the videotaping was to occur, we visited the school with the technical producer to discuss what would happen in the lesson, space constraints, etc. For reasons of space it was decided that the class would be split into two and each have a 20 minute session. Unlike the other lessons videotaped (and all other lessons taught in the project) where we left it entirely up to the teachers to plan their lessons, on this occasion we particularly wanted to capture footage similar to what we had seen on the "home video" and in many other project classrooms over three years — i.e. children being given their calculators and asked to explore them and then report back to the class what they had found. Clare, who at that stage had had no previous connection with the project, readily agreed to the proposed plan.

Just before we left, the producer asked me to recap with Clare exactly what would be happening the following week. Clare thought for a while and replied that as it was a Thursday, she would be dealing with number and that as it would be the ninth week of the school year, she would be dealing with the number nine. The children would be doing some colouring and other activities based on the number nine and five or ten minutes before the end of the lesson she would hand out the calculators and ask the children to enter some nines on it. This was not what we had expected and we asked her whether she would consider devoting the entire 20 minutes to the plan we believed had previously been agreed. This time she agreed somewhat reluctantly, saying that she did not believe that children would be productively engaged for such a long period.

During the two 20 minute sessions, extracts of which appear on the final videotape, children excitedly explored what they could do with their calculators, while Clare engaged them in purposeful discussion about what they were doing. One boy opened his calculator and exclaimed: "Ooh, now I can tell the time!" One child entered the numbers 12345678 and was disappointed when he had to clear the display in order to continue with 9101112. Another child showed the teacher a display which included a negative sign and then demonstrated, at Clare's prompting, how to "take the sign away". A girl entered 98765432 and answered Clare's question regarding the significance of the numbers by saying that "that's when a rocket ship blasts off". Most remarkable was the child who entered 92430 and explained that this was the date — the teacher had recorded the date on the board for the first time that morning. But of course it was recorded as 30/4/92.

What we learned

The most striking thing that I learned was the fact that I could not take for granted that each of the parties in a discussion took away the same understanding of what had been discussed — on my part, I "took as shared" the understanding that the children would freely explore their calculators and then report back to the class, while Clare framed the discussion in terms of her previous understanding of what a lesson at this stage of the year would look like.

More importantly, I also learned what an excellent teacher Clare was. Although unwilling at first to follow the proposed plan for the lesson, she did so with enthusiasm on the day and her questioning of the children both during the exploration time and the sharing time resulted in high quality discussion. She was very good at listening to the children and asking probing questions in order to understand the purpose of their often surprising uses of the calculator.

Clare also learned what a wide range of number knowledge was present in her class and how by lock-stepping the whole class into the same, relatively low-level activity, she was only catering for a very small minority of the children. Not only did Clare acknowledge this, but the role of the calculator in revealing this common feature was reported frequently by many teachers throughout the project's duration.

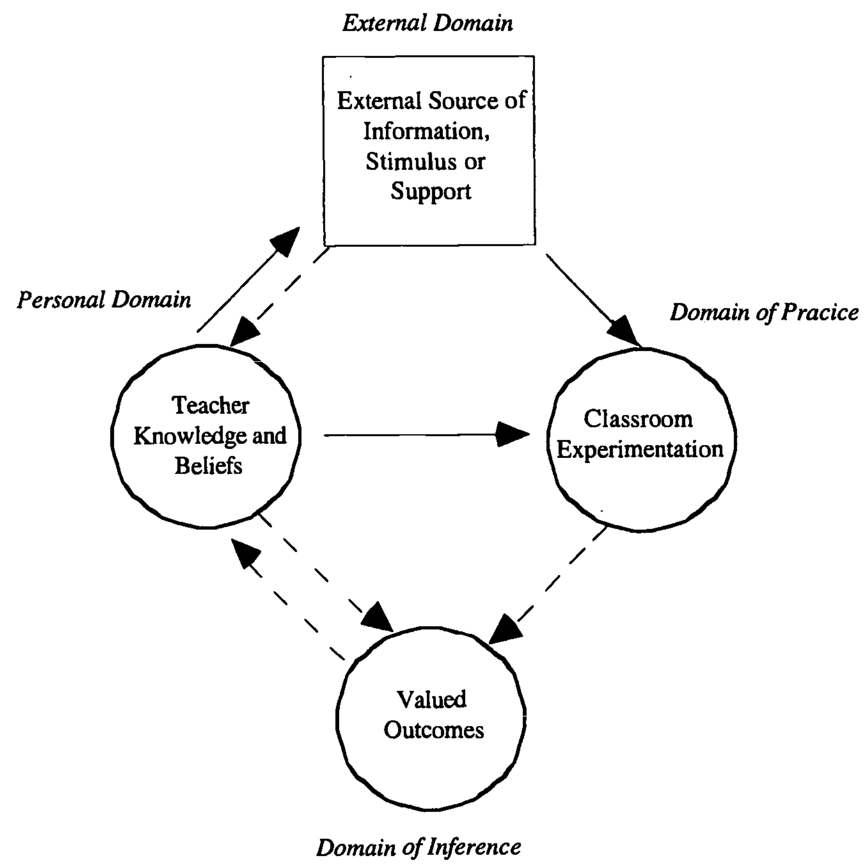
Both Clare and I also learned from the children new ways in which the calculator could be used not just as a computational tool but also as a tool to facilitate mathematical exploration and reflection. In fact, in much the same way as illustrated here, the project had already found that very young children frequently used the calculator as a "scratch pad" to quickly and easily record numbers and number patterns which they found particularly significant. None of us had anticipated this use of the calculator. Once teachers noticed this spontaneous use, many devised innovative learning activities based on using the calculator as a recording device.

Students as learners and teacher professional growth

Since Clare joined the *Calculators in Primary Mathematics* project at the stage when kindergarten classes only continued to be part of the project for a little over six months and she was not one of the seven teachers who were part of our case study investigating the effects of calculator use on classroom practice, I am unable to comment further on what happened to Clare after the event reported here. However, from the wealth of data collected in the project, I believe that Clare's experience was shared in many ways by many of the teachers in the project. In particular, I chose this event because in my opinion it highlights the way in which teachers' learning from their students' learning experiences can play a powerful role in their professional growth. I will try to elaborate on this below.

The two major aims of the project were to investigate the effects of children being given 'their own' calculators to use freely from kindergarten onwards on firstly the classroom practice of teachers and secondly the long-term learning of children. While funding for these two 'investigations' came from separate sources, they were nevertheless seen as inextricably linked from the outset. In particular, an underlying assumption of the project was the belief that teachers would observe children using their calculators – presumably in ways which challenged their existing beliefs about the nature of young children's learning of mathematics – and that reflection on the 'classroom experiment' of calculator use and the consequent observations of children would lead to a change in their teaching practice.

This assumption was originally based on Guskey's (1986) model of teacher professional growth, which proposed that the major motivation for teachers to change is the desire for improvement in student learning outcomes and that changes in teachers' classroom practice need to precede changes in their beliefs and attitudes. Thus teachers learning from children's learning was a central thrust of the project from the beginning.



Key: Solid line = *enactive* mediating process; broken line = *reflective* mediating process

Figure 1. **The Clarke-Peter model of professional growth**

Later in the project, we adopted Clarke and Peter's (1993) dynamic model of teacher professional growth, which traces its origins to Guskey's model. The Clarke-Peter model (see Figure 1) identifies four domains: the *personal domain* (teacher knowledge and beliefs); the *domain of practice* (classroom experimentation); the *domain of inference* (valued outcomes); and the *external domain* (sources of information, stimulus or support). It further identifies *reflection* and *enaction* as two mediating processes which are used to explain how growth in one domain is translated to another. In recognition of its dynamic nature, the model allows for entry at any point in the cycle.

In terms of the Clarke-Peter model, the stimulus of the presence of the calculator, together with the support provided by the project, were changes in the external domain. This was translated into action in the domain of practice through classroom experimentation, which, in this case, took the form of using calculators on a regular basis. As well as their own observations of children learning in the calculator environment, teachers received feedback from the members of the project team who reported their observations of individual children's learning during regular classroom visits. These visits therefore acted as a further stimulus by giving teachers access to a much wider range of observations of children than would otherwise have been possible. This in turn provided teachers with enhanced opportunities to engage in the reflective process in order to bring about change in the domain of inference — that is the valued outcomes. The reflective process was also supported by the discussion which took place at project meetings. This reflection on changes in both the external domain and the valued outcomes mediated change in the personal domain.

Most of the seven teachers who were part of the case study investigating effects of calculator use on classroom practice claimed to have made substantial changes to their teaching of mathematics, with all seven commenting that their mathematics teaching had become more open-ended, and four teachers describing their mathematics teaching as having become more like their teaching of language (Groves, 1993). While it is not possible to determine the long-term effect on Clare of the event reported here, it appears highly likely that Clare's teaching also became more open-ended as a result of her experiences that day and also when subsequently working with calculators with children in her class. Certainly the findings from the project in general support the notion that placing teachers in experimental situations which challenge their existing beliefs by focusing attention on their students' learning has the potential to support teachers' own learning and professional growth.

COMMENTS ON SUSIE'S EPISODE – ROSETTA ZAN

Reading this episode gives us the possibility of reflecting on some significant aspects of the teaching process. Even though the information that can be extracted from a single episode are limited, Clare's behaviour reminds me of behaviour and decisional processes observed in other teachers, and of possible interpretations of these processes.

In a first instance Clare refuses the work proposal of Susie and Jill: she thinks that the children cannot be left for 20 minutes to perform that type of activity, and that a more structured activity under the teacher supervision would be better. I seem to recognise in Clare's behaviour, but above all in her decisions, the influence of her beliefs and emotions. Her beliefs regard both her class and their capacity (the children cannot be left working for a long time on a single activity), and the teaching of mathematics (if we want the children to learn, we have to provide them with structured activities). But I also recognise a strong emotional component in all of this: the fear of losing control of the situation, and the anxiety of not completing a pre-established program.

The importance of the theme of teacher's beliefs in the research on teaching is linked to the shift of this research from the only observable phenomena such as teacher's behaviour, to studies about the teacher's decision processes. In the more recent studies teachers are seen as thoughtful professionals, who make judgements and carry out decisions in a complex environment. The teacher's decisions are influenced (just like the decisions made by a subject involved in a problem posing or solving activity) both by their knowledge and by their beliefs. In particular, the beliefs regarding the capacity of their pupils can push the teachers towards making low demands, and can end up in low results, as suggested in pioneering studies by Rosenthal and Jacobson (1968) on the so called Pygmalion effect. However, research on problem solving highlights that decisions made by a subject who is solving a problem are also influenced by his/her emotions (Mc Leod & Adams, 1989): therefore the teachers' emotions, which are currently studied less than their beliefs, have just as important a role in explaining their decisional processes (Malara & Zan, 2002).

These factors (beliefs and emotions) also have a great influence on the relationship between teachers and researchers, and can be an obstacle to the implementation of the researchers' project. Even if Clare declares, and is convinced that she wants to participate in the project, her co-operation is held back by her resistance to allowing the children to work freely for a longer period of time. This phenomenon has also been analysed in mathematics education with respect to the reproducibility of a teaching experiment (Arsac et al., 1992): even when the teachers' behaviour has been programmed down to the finest detail, some of their non programmed decisions at a micro level are enough to make the activity not reproducible.

Reflection on this type of episode can lead both the researcher and the teacher to modify their practices. As a teacher these reflections convinced me of the importance of being aware of my decision-making processes, and of the deeper motives that bring them about. As a researcher, reflection on the role of the teacher convinced me that the reproducibility of a teaching experiment depends on the training given to the teachers involved in the experiment. Such training cannot simply be limited to communicating the spirit of the project to the teachers, but has to try to work not only on their knowledge but also on their beliefs and their emotions.

Analysis of episodes like this one also gives the opportunity to reflect on the importance of allowing pupils to explore, and not to limit their mathematical experience to very structured tasks. If problem solving is a crucial activity in the development of pupils' abilities and knowledge, the task of problem posing is no less so (Silver, 1993). In particular, the fact that during problem posing the subject solving the problem is the same as the one proposing it, has important implications. It allows certain cognitive obstacles to be overcome, such as those linked to the comprehension of a text, which are often exasperated by the stereotypical nature of school problems. As well as this, since exploration stimulates the production of conjectures, problem posing provides an ideal context for constructing the first steps in proving processes that are crucial to mathematical activity. But above all, problem posing guarantees active involvement of the pupil, reducing the risk that mathematical activity loses its sense and that the pupil loses control, thus generating a negative and fatalistic attitude.

Concluding remarks

What I presented here is certainly only one of many possible interpretations of Clare, Karin and Wilbur's behaviours, but it is also the result of other experiences and analyses performed on other pupils and teachers. What I want to underline is the plausibility of this interpretation and NOT its correctness! Such plausibility is sufficient to highlight that the lack of communication between teacher and pupil (that I see as common to the three episodes) can be or become cause of difficulties. But I think that highlighting the complexity linked to these aspects takes us towards having instruments to make it manageable. The pupil/teacher relationship is asymmetrical: in my opinion managing to listen to and observe the pupils, talk to them, monitor the vision of both mathematics and themselves that they are building up, are all components of the teacher's role. This raises new issues which researchers will have to face: the need of developing a theoretical framework providing teachers and researchers with tools for observing and interpreting pupils' behaviour, and the need of adequate teacher education, in order to develop teachers' knowledge, but also their sensitivity and awareness (Mason, 1998). Theoretical tools, sensitivity and awareness can help the teacher recognise that the pupil is following a different path from the one he or she was expected to follow, and consequently make adequate decisions to reduce the distance between reality and expectations. But knowledge and metacognition are not enough without a crucial emotional aspect: the *desire* of communicating. Hence it is also on such desire that researchers need to work, in order to make teacher education act upon and affect such emotional factor.

Similarly, in my opinion, theoretical tools and metacognitive abilities are also the instruments that allow teachers and researchers to learn from learners: but, again, no tools are sufficient without the *desire* of learning.

WHAT I LEARNED FROM KARIN, ROSETTA AND SUSIE – COMMENTS ON SUSIE'S EPISODE – JOOP VAN DORMOLEN

I started with my own episode, and then I worked on the Rosetta Zan's. Susie Groves's episode is the last of the three on which I have to comment. I cannot but see several common elements in all three descriptions and therefore, next to commenting on her episode I use this, for me last, scribbling for a sort of summing up.

Same language, different meaning

The three episodes of Susie Groves, Rosetta Zan and me have several common elements. One of them is the phenomenon that people can talk with each other, using the same language and yet give different meaning to it. In Susie's story there is the agreement that "the class would be split into two and each have a 20 minute session." These 20 minutes meant for Susie that the whole period would be devoted to exploration of calculator, while for Clare the splitting up was meant just for technical reasons as there was not space enough for the camera crew in a full classroom. For the rest she though she could go on with her planned lesson, be it that five or ten minutes before the end would be devoted to the calculator. The intervention of the producer made it clear that there was indeed a difference in meaning. As ever so often this shows again the usefulness, if not importance, of conducting an evaluation after a meeting in which, between other issues, participants look back on their decisions: Are they clear for everybody? Does everybody attach the same meaning to it? (Johnson & Johnson 1997, Van Dormolen 2000).

The same-language-different-meaning is also clear in the actions and explorations of Clare's children. The language of the numbers in the display was indeed for them very different from the language in which the calculator is a computational tool.

Susie rightly points out Clare's willingness and ability to learn her children's language and to talk with them in that language in order to give them more and deeper experiences.

The context

Another element that is strongly apparent in all three episodes is the influence of the context on learning. All three of us found it necessary to go into detail about the circumstances that leaded to our learning. Each of us could have given the incident about which we wanted to write in less than half a page, but we decided not to do that. The context helped us to learn, so we were compelled to describe the context. For the children it was the context of being able to freely explore the new instrument. For Clare it was the context of being in the classroom with her children and noticing their excitement. For Susie it was the context of having the class as part of her project which gave her the opportunity to observe and learn.

Like ever so often the three episodes showed us again the importance of creating a rich context in order to provoke meaningful learning (Freudenthal 1978, pp. 178-185). The ideas of realistic mathematics education are based on this assumption (or should I say: law?).

Formalization and explanation

Explanation is about rules, formalization is about experiences. Both are elements in language and may help to determine if there is a case of same-language-different-meaning and if yes, to describe the different meanings.

From Susie's description I get the impression, that neither she, nor Clare wanted to explain rules about how to use the calculator. It is not clear what they expected what could happen if there was no explanation, but both of them were open minded to give children the freedom to find their own formalizations. One could ask if the word 'formalization' is a bit heavy in this case. I think not. Formalization is for me the result of some generalization. The children in Susie's episode, at least the children that she mentions, generalized and by telling Clare what they found, they formalized their findings.

Never generalize on another person's abilities

In my episode Wilbur had certain experiences with Karin's abilities to learn mathematics. Unconsciously he generalized these experiences and that prevented him to recognize her intervention as constructive. In Rosetta's episode something similar happened to the teachers who seemingly stereotyped Anne as a rigid teacher. Wilbur's context was not a favourable one in which he could recognize his own bias. The same can be said about Anna's fellow teachers. Rosetta, Susie and Clare were in a different position. Their context was different, which allowed them to be aware that the situation should not be understood automatically. Clare was convinced that her children could not very well cope with the calculator and therefore did not want to spend more than 10 minutes to it. She was however open minded enough to try something else and, in contrast to Wilbur, she had all the time to make a decision to change her plan. Susie might have been shocked by Clare's intentions and did what she could to change Clare's mind. Yet she seemed to allow Clare to act as she looked fit. Susie's context was a very favourable one: an important goal of her project was just to be open and see what happens. Rosetta was shocked by Anna's remarks, did not allow herself to act accordingly. In this case she could not decide on the most favourable reaction. Her context restrained her in the time to find that reaction. Only later in the train she could reflect on the incident.

EPILOGUE – JOÃO FILIPE MATOS

When people find out that they have competing versions of ambiguous events, they often try to negotiate a kind of commonly agreed upon definition of the situation. This we can find in all three episodes just presented and analysed.

One of the features that seem to be equally relevant in the three episodes and subsequent analysis is that authors report learning within a certain situation reconstructing the conditions where it occurred and re-contextualising the learning within their actual frame. It is not surprising that all three authors do believe that learning occurred in the episodes shown and that they are able to identify the sources of learning in interaction with others.

There are also some aspects that emerge from the several analysis presented and that can help us to formulate relevant research questions regarding the issue of learning from learners.

Firstly, learning is seen as a phenomenon inherent to human nature and therefore an ongoing and integral part of everyday life. In fact, it doesn't seem possible to see learning as a special kind of activity separable from the rest of our life. This brings us to questions such as the definition of the unity of analysis in research on learning — focusing on people acting in the world *versus* focusing on cognitive processes.

Secondly, learning is conceptualized as an ability to negotiate new meanings, involving the whole person in dynamic participation in practices. Learning in general (and learning mathematics in particular) is viewed as a constructive activity that is developed more in a process of socialisation rather than one of instruction. It is during this process—which is socially mediated by language and other communication tools—that patterns of thought and action develop and are considered to be (mathematically) legitimate. This brings along for example the question of the contrast that is established between the concerns arising when we intend to teach children to *do* mathematics (emphasis on knowledge as a way of doing) and the concerns arising when we intend to educate them mathematically (highlighting a concern with a form of knowledge).

Thirdly, learning transforms one identity, by transforming one's ability to participate in the world and it does this by changing who we are, our practices and the communities we belong to. This very fact makes emergent issues such as the way the participant in the practice develops a sense of *belonging* and therefore construction of identity becomes a key element.

Finally, learning involves an interplay between the local and the global. The three episodes help us to realize how learning takes place in practice as it defines a global context for its own locality. Learning occurs whatever the educational form that helps to create the context for learning (including the absence of an intentional educational form) pointing to a fundamental distinction between learning and intentional instruction. Obviously this idea does not deny that learning may occur where there is teaching, but it does not make intentional teaching to be the source or the cause of learning.

All these features of learning need to have the attention of researchers in mathematics education. This paper aims to be a contribution for a first step towards an *ethnography of learning from learners.*

ACKNOWLEDGEMENTS

The *Calculators in Primary Mathematics* project was funded by the Australian Research Council, Deakin University and the University of Melbourne. The project team consisted of Susie Groves, Jill Cheeseman, Terry Beeby, Graham Ferres (Deakin University); Ron Welsh, Kaye Stacey (Melbourne University); and Paul Carlin (Catholic Education Office).

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RESEARCH FORUM 1

Theme

Abstractions: Theories about the emergence of knowledge structures

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Contributors

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ABSTRACTION: THEORIES ABOUT THE EMERGENCE OF KNOWLEDGE STRUCTURES

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INTRODUCTION

Tommy Dreyfus and Eddie Gray

Since there is no universally accepted research paradigm in mathematics education, theories and terminology tend to multiply. It is therefore one of the tasks of the research community to critically compare theories that deal with closely related issues and have similar aims. The setting of a research forum at PME conferences is one of the few opportunities where attempts at such comparison can be undertaken in public by a large group of researchers.

Several theories for processes describing the emergence of mathematical knowledge structures (abstraction) have been put forward recently. While these theories differ in many respects, they have a common goal: they aim to provide a means for the description of processes during which new mathematical knowledge structures emerge. Thus, they have the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice.

Three theories of abstraction have been selected for discussion in the Research Forum on the basis of their similar aims but different approaches. The selection was made so as to achieve, within the limited space and time available, a wide variety with respect to the theoretical underpinnings of the approaches, the formal or informal nature of the emergent knowledge, the role of context in the process of abstraction, the importance the theories attribute to contextual factors, and the degree to which they are anchored in instructional design. While a main aim of each of the three theories is to describe processes (rather than outcomes) of the emergence of knowledge during learning activities and a secondary aim is to contribute to the design of learning, the assumptions of the three theories differ considerably. For example, the theory by Tall & Gray is predominantly cognitive (with links to neuro-physiology), while context is accorded a limited role; on the other hand, the theory by Gravemeijer is predominantly contextual. Similarly, instructional design interacts with the theories in very different ways: For Tall & Gray, instructional design is a result of the research undertaking, for Schwarz, Hershkowitz & Dreyfus, it is the location of the research undertaking (in the sense that the research takes place in an environment designed for instruction), and for Gravemeijer instructional design is at the origin of the research undertaking.

A theory cannot possibly be presented appropriately in the limited space allotted to each group of contributors within this written presentation of the Research Forum. Therefore, the following contributions by the three presenting groups only give a summary overview over each of the theories, and provide the reader with ample references for further reading. Many of these references are accessible from the conference website. They are also available from the authors. The papers included hereafter have been written following a set of fairly detailed guidelines. The aim of these guidelines was to define a number of dimensions for comparison of the theories. Specifically, each paper

(i) Gives a description of what their theory is about,
(ii) Identifies the assumptions being made by the theory,
(iii) States the theses of the theory (what does the theory say?), including a detailed discussion of the meaning of the terms that are used,
(iv) Discusses the aim(s) and applications of the theory,
(v) Gives evidence concerning the validation of the theory (for example empirical research).

Authors have been asked to be specific about the meaning of their terminology to enable the identification of cases where either the same term is used with different meanings or where different terms are used to describe closely related phenomena. First and foremost, the term 'abstraction' is likely to mean different things to different people; similarly, the term 'context' may be given a rather narrow or a very wide interpretation. In addition, each theory uses its own idiosyncratic terms.

The two papers by the reactors stress commonalties and differences between the three theories, for example with respect to the underlying definitions of abstraction, the domains of applicability of the theories, the empirical evidence validating the theory, and the role which context plays in the process of abstraction.

Whilst our theme examines from a contemporary perspective theoretical issues that have been of interest to PME members over the past quarter of a century, two broader issues are also relevant to our discussions.

First, there is an issue associated with “having abstracted”. Though we are examining theoretical perspectives of the role of abstraction, its contribution and influence on different modes of thinking displayed by mathematics students leads us to ask to three general questions, which can and should be expected to arise for discussion out of the reaction papers are:

- When do we know students have abstracted and what student behavior attests to this?
- What happens if students do not abstract?
- How do we encourage abstraction?

Secondly, there is an issue associated with coherence and unification. In the opinion of the presenters and the reactors, the domain of theoretical physics has no exclusive right to yearn for a unified theory. Even though mathematics education as a scientific discipline is a few centuries younger than physics, we believe that this is the time to start the work of combining, merging and fusing our theories, and thus to make them more widely known, applicable and applied. One milestone on this road will be the use of different theories to analyze the same data set and thus to directly confront the theories. We hope that the discussions of the research forum will give rise to such undertakings, as well as to some speculation on a possible unification of the theories into a larger framework. It is this hope for progress in the direction of fewer, more widely known, more widely agreed, and more widely applied theories that has motivated this research forum.

ABSTRACTION AS A NATURAL PROCESS OF MENTAL COMPRESSION

Eddie Gray & David Tall

Introduction

The term 'abstract' has its origins in the Latin *ab* (from) *trahere* (to drag) as:

- a verb: to ab*stract*, (a process),
- an adjective: to be *ab*stract, (a property),
- and a noun: an *ab*stract, for instance, an image in painting (a concept).

The corresponding word 'ab*stract*ion' is dually a process of 'drawing from' a situation and also the concept (the abstraction) output by that process. It has a multi-modal meaning as process, property or concept. Piaget distinguished between construction of meaning through *empirical abstraction* (focusing on *objects* and their properties) and *pseudo-empirical abstraction* (focusing on *actions* on objects and the properties of the actions). Later *reflective abstraction* occurs through mental actions on mental concepts in which the mental operations themselves become new objects of thought (Piaget, 1972, p. 70). In Tall et al, 2000, we reviewed ideas in the literature and concluded that elementary mathematical thinking uses reflective abstraction both by focusing on *objects* (for instance, in geometry) and on *operations* on objects represented as *symbols* (in arithmetic, algebra, etc). In the latter case we see symbols used dually as process and concept and have formulated this in terms of the notion of *procept* (Gray & Tall, 1994, see also below). At a later stage, in advanced mathematical thinking, the focus changes to *properties* (of objects and operations) formulated as fundamental axioms for mathematical theories.

Our hypothesis is that different forms of abstraction lead to different type of cognitive development and in turn, to differing cognitive problems. Empirical and reflective abstraction in shape and space lead to a van Hiele type development that we see as the growing dominance of verbal description over visual perception, as language refines our imagery and leads to increasingly sophisticated forms of mathematical structure and proof. Pseudo-empirical and reflective abstraction in arithmetic, algebra and calculus naturally focus on our notion of *procept*. Increasing focus on properties and deduction lead to a property-based axiomatic theory where the process of proof leads to the concept of theorem which may then be used as steps in building up a systematic formal theory.

We have a great empathy for the notion of different *modes* of operation as proposed by Bruner (1966) and, more particularly, in the SOLO taxonomy of Biggs and Collis. (1982). For instance, it is possible to build a holistic embodied mode that relates to the enactive/iconic modes of Bruner or the sensori-motor/ikonic modes of Biggs and Collis, before gaining an insight in proceptual (concrete-symbolic) terms; or, at a later stage in advanced mathematical thinking, in formal-deductive terms. Tall (1999) considers the distinct forms of proof available in these various modes as the child develops cognitively into a mathematical expert. Tall (2002) reviews calculus in terms of an enactive-iconic approach manipulating graphs, symbolic-proceptual representations (manipulating formulae) and formal proof (in analysis).

In this short paper we do not have space to attend to our full theoretical perspective. We focus only on the abstractive processes occurring in constructing *procepts* in arithmetic, algebra and symbolic calculus and how differing types of symbol (whole numbers, fractions, algebraic expressions, (infinite) decimals, limits) give rise to distinct problems of concept construction and re-construction.

Five Aspects

The research forum is designed to focus on five aspects, given in (a)-(e) below.

a. What is the theory about?

Our theory grows as a result of our quest to understand not only *what* students do in constructing symbolic mathematics, but *how* they do it. We believe that abstraction is a *natural* consequence of human brain function. At any given time human thinking occurs dynamically as a process, whereby items evoked in the focus of attention are manipulated mentally as concepts. It is the duality of symbols in arithmetic, algebra, etc as both process and concept that is the basis of our theory.

b. What assumptions are being made?

We assume that abstraction is a natural product of human mental activity, in which a complex parallel-processing organ solves the problem of complexity by focusing on essential structures that enable decisions to be made. Sometimes this process of abstraction is a conscious reflective act, but much of it does, and must, occur unconsciously to enable the brain to focus only on essential elements. There is physical evidence that over time routinising tasks uses less brain capacity:

> As a task to be learned is practiced, its performance becomes more and more automatic; as this occurs, it fades from consciousness, the number of brain regions involved in the task becomes smaller. (Edelman & Tononi, 2000, p.51)

There is also a compression in the nature of the symbolism being used:

> I should also mention one other property of a symbolic system – its compactibility – a property that permits condensations of the order $F = MA$ or $S = ½gt^2$, ...in each case the grammar being quite ordinary, though the semantic squeeze is quite enormous.
>
> (Bruner, 1966, p. 12.)

We do not have the data to link mathematical activity in a one-one mapping to neurophysical phenomena, steps in this direction (eg Dehaene, 1997) are still in their early stages. However, the underlying biological basis of mathematical thinking in a brain ill-built for numerical computation and formal logic, is a vital underpinning for our own reflections on how mathematical thinking develops.

c. What does the theory claim? What terms are used and what do they mean?

The notion of procept (as given in Gray & Tall, 1994) is seminal in what follows.

> An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.
> ... A *procept* consists of a collection of elementary procepts which have the same object. (Gray &

We follow Davis (1983, p. 257) in defining a *procedure* as an explicit step-by-step algorithm for implementing a process and see a spectrum of increasing power through the usage of procedure, process and procept. We do not agree with Sfard or Dubinsky that the development invariably proceeds in a sequence we describe as procedure-process-procept. In particular, as students become more sophisticated, they may sense an intuitive holistic grasp of the overall ideas in, say, an embodied mode before concerning themselves with the specific procedures that may be seen to occupy a particular role within a symbolic or formal mode of operation.

We do not have a theory that tells us how *all* individuals can be helped to move through all of these modes. (Indeed, *no-one* has such a theory at this moment in time.) Instead, in the growth of symbols, we find a bifurcation between those who concentrate more on the procedures associated with symbols, who have a greater cognitive strain to overcome, and those who develop a proceptual system switching flexibly between process and concept to construct a more powerful generative mental structure. This does *not* mean that students necessarily remain in a fixed part of the spectrum. However, we do have considerable evidence that there is a bifurcation in performance between those who remain entrenched in procedures and those who develop more flexible proceptual thinking, so that progress to greater sophistication is more difficult for some and easier for others.

d. What are the aims of the theory and what are its applications?

The initial aim of our theory of the proceptual growth of symbols is to try to explain why some students are so highly successful with symbols, whilst others are procedural at best and could, at worst, be overwhelmed by the complexity of mathematics. To move towards this overall goal we focus on the different ways that procepts arise in cognitive development. These include

(1) ***arithmetic procepts***, 5+4, 3×4, $\frac{1}{2}+\frac{2}{3}$, 1·54÷2·3, all have built-in algorithms to obtain an answer. They are *computational*, both as processes and even as concepts. *Fractional procept*s behave differently because the focus moves from sharing procedures (eg divide into 4 equal parts and take 2) to equivalent fractions, which from our viewpoint are seen as processes that have the same effect (divide into 4 equal parts and take 2, has the same effect as divide into 6 equal parts and take 3).

(2) ***algebraic procepts***, e.g. $2+3x$, can only be evaluated if the value of x is known and so involves only a *potential process* (of numerical substitution) and yet the algebraic expressions themselves represent manipulable concepts (manipulated using algebraic rules of equivalence).

(3) ***implicit procepts***, such as the powers $x^{\frac{1}{2}}$, x^0 or x^{-1}, for which the original meaning of x^n no longer applies but the properties need to be deduced using the power law $x^m \times x^n = x^{m+n}$ (which also no longer has its original meaning!)

(4) ***limit procepts***, $\lim\limits_{x\to a}\dfrac{x^3-a^3}{x-a}$ or $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ etc, have ***potentially infinite processes*** 'getting close' to a limit value that may not be computable in a finite number of steps.

(5) ***calculus procepts***, such as $\dfrac{d(x^2e^x)}{dx}$ or $\int_0^\pi \sin mx \cos nx\, dx$ focus again on finite operational algorithms of computation (the rules for differentiation and integration).

This reveals that each of new form of procept has its own peculiar difficulties that makes abstraction qualitatively different in each case. We believe that knowledge of these specific difficulties is essential to help a wider spectrum of students to succeed in the longer-term process of successive abstractions.

e. How has the theory been validated?

Our data (summarized in Tall, Gray, *et al*, 2001) reveals both general themes and specific information on cases (1)-(5) above. The general themes illustrate diverging approaches from procedural to proceptual in a spectrum of students from elementary arithmetic (Gray & Tall, 1994), through algebra (DeMarois, 1998; McGowen, 1998; Crowley, 1999), symbolic calculus (Ali, 1996), and on to formal mathematical theory (Pinto, 1998). In addition, qualitative differences in imagery emerge from different forms of abstraction (Pitta, 1998; Gray & Pitta 1999), leading to differing levels of success in the longer term, depending on whether children continue to focus on real-world situations and imagery, or move on to a more flexible proceptual hierarchy (Gray *et al*, 1999). The data from the above-mentioned studies reveal how differing contexts pose significantly different kinds of cognitive problems in both the nature of the procepts concerned and the procedure-process-procept spectrum of student activity. We believe that these difficulties are best handled by the learner supported by a mentor who is aware not only of the mathematics but of the underlying cognitive structures.

This aspect of learning is complementary to the desire of Schwarz *et al* (this forum) to theorize about a general strategy for encouraging abstraction in context. We suggest that it is a laudable aim to have a general theory of construction, but we observe that specifics often overwhelm the broad sweep of such a theory. From the learner's point of view, different obstacles occur in different contexts. The acquisition of mathematical knowledge from early years to undergraduate level involves a variety of reconstructions. Each new reconstruction refines that which was established earlier so that effective reconstructions contribute to the organic nature of growth in the embodied and proceptual modes of operation and on to a close harmony between wider aspects of concept image and concept definition in advanced mathematical thinking. Our central concern is not just how we can encourage students to make abstractions, but also to find why some students succeed so effortlessly and others can fail so badly at making the necessary reconstructions. Our empirical evidence provides an insight into a possible answer—inappropriate abstraction from mathematical activity.

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ABSTRACTION IN CONTEXT: CONSTRUCTION AND CONSOLIDATION OF KNOWLEDGE STRUCTURES

Baruch Schwarz, Rina Hershkowitz and Tommy Dreyfus

The construction of abstract knowledge structures is central in human learning, including mathematics education. As practitioners who are informed about recent theoretical research, we have been deeply involved in curriculum design, development, and implementation. Our approach to abstraction is thus a product of our interest both in theory concerning abstraction and in experimental observations of activities in schools in which we judged that a process of abstraction has been evidenced.

Many researchers have taken a predominantly theoretical stance and have described abstraction as some type of decontextualization. For example, Piaget has proposed that abstraction consist in focusing on some distinguished properties and relationships of a set of objects rather than on the objects themselves. Abstraction is thus a process of decontextualisation. According to Davydov (1972/1990), on the other hand, abstraction starts from an initial, undeveloped form of knowledge and ends with a consistent and elaborate knowledge structure.

A Definition for Abstraction, the Nested RBC Model, and Consolidation

Leaning on ideas of Davydov and other researchers, and in view of our experience in classrooms and our need for an operational definition, we translated our theoretical principles into the following more applicable definition:

> *Abstraction is an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure.*

The term *activity* in our definition is directly borrowed from Activity Theory (Leont'ev, 1981) and emphasizes that abstraction is an activity, a chain of actions undertaken by an individual or a group, and driven by a motive, which occurs in a specific context. Context is a personal and social construct, which includes the student's social and personal history, conceptions, artifacts, and social interaction. The term *previously constructed mathematics* refers to two points: One, that outcomes of previous processes of abstraction may be used during the present abstraction activity; and two that a process of abstraction leads from initial, unrefined abstract entities to a novel structure, as posited by Davydov. These two points show the recursive nature of abstraction. The phrase *reorganizing into a new structure implies* the establishment of connections, such as inventing a mathematical generalization, proof, or a new strategy of solving a problem. The novel structure comes about through reorganization and the establishment of new internal and external links within and between the initial entities. We very intentionally used the word *new* to express that, as a result of abstraction, participants in the activity perceive something that was previously inaccessible to them. Finally, we borrowed the term *vertical* from the Dutch culture of Realistic Mathematics Education, in which researchers relate to vertical mathematization as to an activity in which mathematical elements are put together, structured, organized, developed etc. into other elements, often in more abstract or formal form than the originals. It is mainly this integration that comes about by the establishment of new connections during processes of abstraction, which we wanted to express by means of the term vertical.

According to this definition, abstraction is not an objective, universal process but depends strongly on context, on the history of the participants in the activity of abstraction and on artefacts available to the participants. In this sense structure is internal, "personalized".

The study of abstraction raises a methodological challenge. Whichever its definition is, abstraction implies mental activity, which is not observable. Since we want to empirically investigate processes of abstraction, we need to devise a way to make them observable. Put otherwise, we need to use (theoretical) spectacles, which let us see processes of abstraction, as they occur during students' activities. And it is precisely this view of abstraction as activity, which provides us with the desired spectacles: Activities are composed of actions – and actions are frequently observable. The question, which actions are relevant for abstraction, we answer with reference to Pontecorvo & Girardet (1993): *Epistemic actions* are mental actions by means of which knowledge is used or constructed. Epistemic actions are often revealed in suitable settings. Therefore, settings with rich social interactions are good frameworks for observing epistemic actions. Coming back to our experimental research we were able to identify three particular epistemic actions, which are constituent of abstraction, and provide a strong indication that a process of abstraction is happening: *Recognizing, Building-With and Constructing*, or RBC. In summary, we consider these epistemic actions because they characterize abstraction and because they are observable. In other words, they provide us with an operational description of processes of abstraction.

Constructing is the central action of abstraction. It consists of assembling knowledge artefacts to produce a new knowledge structure to which the participants become acquainted. *Recognizing* a familiar mathematical structure occurs when a student realizes that the structure is inherent in a given mathematical situation. *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. The same task may thus lead to building-with by one student but to constructing by another, depending on the student's personal history, and more specifically on whether or not the required artefacts are at the student's disposal.

The three epistemic actions are the elements of a model, called the dynamically nested RBC model of abstraction. According to this model, constructing incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognizing actions are nested in building-with actions and in constructing actions. Moreover, constructing actions may themselves be nested in further constructing actions.

On the basis of observations reported below and elsewhere, we postulated that the genesis of an abstraction passes through (a) a need for a new structure; (b) the construction of a new abstract entity; (c) the consolidation of the abstract entity through repeated recognition of the new structure, building-with it in further activities with increasing propensity, and using it in further constructions.

Stage (c), the *consolidation* of the newly knowledge structure, seems *a priori* to be linked to the following behaviors: (i) the reconstruction of the new structure or its actualization by recognizing it in different contexts, (ii) its use with increasing facility for building-with in different contexts, (iii) its use in the construction of further structures for which it is a necessary prerequisite, (iv) its verbal articulation, possibly during or after an activity of reflection such as reporting or summary discussion in class. Thus the term *consolidation* denotes a progressive familiarization and further use observable through recognizing and building-with actions in these four types of situations.

The validation of the theory through empirical research

We characterized abstraction as a process taking place in a complex context that incorporates tasks, tools and other artifacts, historical background of the participants, as well as the social and physical setting. Abstraction processes are then context dependent. However, we claim that the ways in which these processes are taking place and become operational have a universal structure. This structure was elaborated and partially confirmed in Hershkowitz, Schwarz and Dreyfus (HSD, 2001). Further studies were partially designed for confirmation of the model and partially designed for extending it.

In HSD we showed that the dynamically nested RBC model fits the genesis of abstract scientific concepts acquired in activities designed for the purpose of learning. A first validation of stage (a) and (b) of the genesis of abstraction according to the model was obtained in a case study with a single ninth grade student who was interviewed while solving a problem, a suitable computer program being at her disposal.

We showed that the model describes the mechanism of processes of abstraction. As such it contains the main invariant features of abstracting as a thinking process. Moreover, the model is apt to take context into account.

HSD also revealed a methodological problem: the occurrence of processes of abstraction cannot be ensured; rather, students can only be presented with opportunities for abstraction. The creation of such opportunities presents a challenging design problem since it depends on the contextual factors mentioned above. We tried to elicit strong motives such as the need to justify a just discovered claim, the need for solving a problem as well as conflict situations in order to augment the opportunities for abstraction. This corresponds to stage (a) of the genesis of abstraction according to the model.

Dreyfus, Hershkowitz and Schwarz (2001a; 2001b), tested stages (a) and (b) of the model in a richer context, in which two peers interacted to construct new knowledge. The study focused on the social dimension of the process of abstraction. Two parallel analyses were carried out on the same protocols: the analysis of epistemic actions according to the model of abstraction as well as an analysis of the interaction. The study showed far-reaching parallels between the two analyses. In other words, we enhanced the RBC model of abstraction so as to describe processes of abstraction by interacting pairs of students and patterns of distribution of abstraction between collaborating peers. The parallel analyses led to the identification of types of social interaction that support processes of abstraction.

While research concerning stages (a) and (b) can be done within one activity, investigation of consolidation processes requires at least a medium term research, where one can analyze processes occur among successive activities. Such an analysis demands the elaboration of powerful methodologies with the help of which individual history of individuals evolving in changing learning environments should be traced. A small number of studies in this direction have already been undertaken (Dreyfus & Tsamir, 2001; Tabach, Hershkowitz & Schwarz, 2001; Tabach & Hershkowitz, 2002). A first attempt at an empirically based theory for consolidation emerged from a sequence of interviews about the comparison of infinite sets with a single talented student. It showed that consolidation may occur both as a result of problem solving activities and as a result of reflective activities, and that it can be identified by means of the psychological and cognitive characteristics of immediacy, self-evidence, confidence, flexibility and awareness.

The significance of our theory of abstraction concerns theoretical, psychological and educational issues: Since our research is empirical, it has the potential to yield insights on processes of abstraction and consolidation as they develop, and to confront these empirical insights with the theoretical constructs of the model. Abstraction and consolidation as the central components of construction of knowledge are investigated in relation to the history of the participants, and to series of activities in a social context. On the basis of the theory, we also expect to articulate educational design principles for sequences of activities that are intended to lead to abstractions and their consolidation.

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BUILDING NEW MATHEMATICAL REALITY, OR HOW EMERGENT MODELING MAY FOSTER ABSTRACTION

Koeno Gravemeijer

Abstract

In general, mathematics is thought of as abstract, formal knowledge. Within this view, the key problem for mathematics educators is to shape mathematics instruction that helps students in bridging the gap between informal, situated knowledge at one hand, and abstract, formal mathematical knowledge at the other hand. A rather common view is that students have to abstract from their informal knowledge; they have to decontextualize, or to cut the bonds with reality. In this paper, an alternative view is presented that does not take its point of departure in the metaphor of a gap between abstract, formal knowledge and informal knowledge, but in an emergent approach, within which formal mathematics grows out of the mathematical activity of the students. The latter view is part of work in the area of instructional design.

Abstraction as the Creation of New Mathematical Reality

The deliberations on the issue of abstraction that will be presented here grew out of an effort to further explicate and elaborate the domain-specific instruction theory for realistic mathematics education (RME) (Treffers, 1987). As part of this effort, this domain-specific instruction theory has been recast in terms of instructional design heuristics (Gravemeijer, 1994). The elaboration of one of those design heuristics—concerning emergent models, or emergent modeling—created the need to further investigate the underlying, implicit, notions of abstraction (Gravemeijer, 1999). The emergent-modeling heuristic assigns a role to models that differs from the classical role of models in mathematics education: instead of trying to concretize abstract mathematical knowledge, we try to help students model their own informal mathematical activity. In doing so, we attempt to foster a process, within which a *model of* their own informal mathematical activity gradually develops into a *model for* more formal mathematical reasoning for them. In contrast with the gap metaphor, formal mathematics is not seen as something separate, existing independent of a knowing agent. Instead, formal mathematics is seen as emerging alongside with the model-of /model-for transition.

When speaking of formal mathematics, we hasten to say that in RME, formal mathematics is not seen as something "out there". Instead, formal mathematics is seen as something that grows out of the students' activity. For us, the notion of "abstraction" is tied to a progression from informal to more formal mathematical reasoning, which in turn is tied to the creation of new mathematical reality. So instead of "cutting bonds with (everyday-life) reality", we want to stress "construction". Informal, situated knowledge is the basis upon which more formal, abstract mathematical knowledge is build.

Our claim is that the emergent-modeling design heuristic helps instructional designers in developing topic-specific instruction theories and corresponding instructional activities that support learning processes in which students construe new mathematical reality. In order to clarify the emergent modeling heuristic, we will briefly describe an exemplary instructional sequence.

This exemplary sequence, which concerns linear measurement and flexible arithmetic, was developed in connection with a teaching experiment carried out at Vanderbilt University (Cobb, Stephan, McClain, and Gravemeijer, in press; Stephan, 1998). The underlying idea is that measuring by iterating measurement units can give rise to the construal of a ruler and that the ruler can subsequently support arithmetical reasoning about problems concerning incrementing, decrementing and comparing measures.

After a series of preparatory activities, the students start measuring with stacks of ten unifix cubes. They first iterate units of ten, then adjust by adding or subtracting ones. In this manner, measuring with tens and ones helps the students in structuring the number sequence up to 100. Next, the students create their own paper strip that is ten unifix cubes long. With that, a basis is being laid for the construction of a measurement strip that comprises ten units of ten; each subdivided into ten units of one cube. The idea is that, thanks to the history, measuring with the measurement strip is grounded in the imagery of measuring with units of ten and one. Thus, for the students, measuring with the strip signifies iterating a unit of ten cubes and a unit of one cube. Next, a shift is made from actually measuring items to reasoning about lengths when solving tasks around incrementing, decrementing and comparing lengths of objects that are not physically present (i.e. comparing the measures of the heights of sunflowers in the context of a sunflower contest). These tasks offer opportunities for developing solution methods based on curtailed counting—using the decimal structure as a framework of reference. Numbers close to a decuple, for instance, can be identified by using that decuple as a referent, e.g. 64 = 60 + 4; 40 = 35+5. These relations can be exploited when analyzing patterns that correspond with jumps of 10. An empty number line is introduced as a means for symbolizing measurement strip-specific, arithmetical solution methods that are grounded in reasoning with "tens & ones" (see fig. 1). A jump on the number line describes a move on the measurement strip that in turn can be seen as corresponding with iterating unifix cubes or smurf bars.

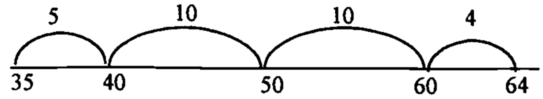

Figure 1. 35-29 on the empty number line.

Finally a generalization is made from magnitudes to numerical quantities in general; the students are asked to solve various addition and subtraction context problems, while using the empty number line as a means to record and support their thinking.

Emergent Models

We will use this example to explicate the emergent modeling heuristic. We may start by noting that the label "model" is used in a metaphorical sense. There is an overarching model that takes on various manifestations. We may characterize the series of symbolizations within which the model manifests itself as a chain-of-signification (Stephan, 1998). In the exemplary sequence the ruler is conceived as the overarching model. The idea is that the ruler emerges as a *model of* iterating a measurement unit (or measurement units). In this sense, the ruler is grounded in the activity of measuring. Gradually, however, the ruler changes character, as the attention shifts from measuring to reasoning about the results of measuring. Finally a schematized ruler becomes a *model for* reasoning about arithmetical relations between numbers up to one hundred.

Key for us is that the shift towards more formal mathematical reasoning is connected with the creation of a new mathematical reality. In the example sequence, we may conceive this new reality as constituted by numbers up to 100 as entities in a framework of number relations. What is expected is, that in the course of the sequence, a shift is taking place in which the student's view of numbers transitions from referents of distances to numbers as mathematical entities. This shift involves a transition from viewing numbers as tied to identifiable objects or units (i. e. numbers as constituents of magnitudes; "37 feet") to viewing numbers as entities on their own ("37"). For the student, a number viewed as a mathematical entity still has quantitative meaning, but this meaning is no longer dependent upon its connection with identifiable distances, or with specified countable objects. In the student's experienced world, numbers viewed as mathematical entities derive their meaning from their place in a network of number relations (see also Van Hiele, 1973). Such a network may include relations such as 37=30+7, 37=3x10+7, 37=20+17, 37=40-3. The critical aspect of this network is that the students' understanding of these relations transcends individual cases. That is, when students form notions of mathematical entities, or mathematical objects, they come to view relations like the above as holding for *any* quantity of 37 objects (including a magnitude of 37 units). We would denote this conception of numbers as mathematical objects that derive their meaning from a framework of number relations as new mathematical reality.

As an aside, we want to remark that we prefer to limit the use of the model-of/model-for terminology to those more encompassing shifts where one can speak of the creation of new mathematical reality. We may further note that this creation of new reality is reflexively related to the model *of* to model *for* transition. On the one hand, the students' actions with "the model" foster the constitution of new mathematical reality (in our example, a framework of number relations). On the other hand, through the students' development of this new mathematical reality, "the model" *can* take its role as a model for mathematical reasoning.

Aims and Applications

The emergent modeling heuristic may guide instructional designers by asking them to think through the endpoints of a given instructional sequence in terms of new mathematical reality; to describe what mathematical objects the students are expected to construe, and how these relate to some framework of mathematical relations. They are further advised to think through the model-of/model-for transition, which for instance means, to indicate what informal situated activity is being modeled, and what a potential chain-of-signification might look like. In connection with the above, the heuristic suggests points of attention for the enactment of the instructional sequence. It highlights that formalizing is not equal to, and cannot be forced by, the use of formal notations. Instead formalizing grows out of a shift of attention towards mathematical relations. The aforementioned considerations will indicate what those relations are, what the mathematical issues are that are to become topics of discussion, and what role the various tools/symbolizations may play.

The emergent modeling heuristic implicitly or explicitly plays a role in various RME designs (e.g. Streefland, 1990). The role of emergent models is older that the explicit characterization presented here. However, more recently this heuristic has explicitly guided design and analysis in a number of developmental research (or design research) projects. In this respect, we can claim that this heuristic is validated in a number of teaching experiments. Next to experiments at the primary-school level, like the aforementioned numberline experiment, we want to mention research in *data analysis* (Cobb, in press), and research on *differential equations* (Rasmussen, 1999).

Acknowledgement

The analysis reported in this article was supported by National Science Foundation (NSF) Grant REC 935387 and by Office of Educational Research and Improvement (OERI) Grant R305A60007. The opinions expressed in this article do not necessarily reflect the view of either the NSF or the OERI.

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REACTION

Anna Sierpinska

This "reaction" will be about my rather unsuccessful attempts at understanding the three proposed theories. Words were familiar, all right. But they were put together in strange juxtapositions and concatenations. I could *recognize* the words or parts of them - activity, perception, abstraction, procedure, process, concept, proCEPT, PROcept, object, structure...- but I couldn't *build-with* them, never mind *construct* anything remotely resembling a *structure* with them.

I was taught in school that "meaning belongs first of all to the world of objective-historical phenomena" (Leont'ev, 1959, p. 223), and that the content of an individual mind is a result of "an assimilation of the experience of the previous generations of people" (ibid.), but here I was, confronted with the meanders of individual consciousnesses of several "Robinson[s], making [their] own independent discoveries on a desert island" (Leont'ev, 1959).

Take, for example, ABSTRACTION. I am used to thinking of abstraction as a *dual* mental activity whereby some aspects of the object of thought are ignored while other are highlighted. For example, if the object of my thought are integers and I decide to ignore multiples of 2, then all that is highlighted are the remainders, 0 or 1, and I end up with the even/odd distinction (or the *concept* of even and odd numbers, if you will). If I now highlight the mental process which led me to the construction of the even/odd numbers construction, and disregard the fact that I was ignoring multiples of 2, and decide to now remove from the field of my attention multiples of 3, 4, or any number n, for that matter, then I end up with the concepts of Euclidean division, and congruence modulo n. I can further ignore the specific nature of particular integers, look at the whole arithmetic of integers from afar, highlight only its ring structure and ask myself if I could not do something similar with other rings as well. I may fancy taking R[x] and decide to ignore multiples of $x^2 + 1$. Then what is highlighted forms a structure strikingly similar to the field of complex numbers. This chain of ignoring and highlighting is usually called generalization: a process of abstractions which starts from some object of thought O_1 and arrives at an object of thought O_2 such that O_1 is a special case of O_2. If abstraction is understood this way, as an act of ignoring/highlighting, then it appears as an "elementary particle" in the process of mathematical thinking. One would hardly want to call the whole process of theory building in mathematics "abstraction". Even the processes of single concept construction involve more than a few acts of abstraction. This concept of abstraction is too elementary to capture what happens in processes of mathematical thinking. It is also not specific to mathematics nor any scientific knowledge for that matter. Abstraction is an elementary operation in any kind of thinking. For example, we engage in abstraction when we move from saying that our neighbors seem to be a happy couple to thinking about happiness in general. Therefore, in speaking about mathematical thinking, we need more specific concepts such as generalization and concretization, formalization and de-formalization, algebraization and geometrization, axiomatization and modeling, etc.

The above socio-cultural notion of abstraction appeared not to satisfy the authors of the RBC theory presented in this forum. SHD (here and in the sequel, G, GT, and SHD will indicate the three groups of authors, according to the initials of the family names of their authors) were inspired by Davydov's definition of abstraction, which they interpreted as, "abstraction starts from an initial, undeveloped form of knowledge and ends with a consistent and elaborate form. It proceeds from the idealization of the basic aspect of practical activity involving objects to cognitive experimentation characterized by the fact that one (a) mentally transforms objects during the activity and (b) forms a system of connections between these objects". I didn't quite see how this description would exclude mental activities such as fantasizing. Imagine a poor man sitting there in his boat with a fishing rod and idealizing the basic aspect of his activity as dreaming about a big catch. Cognitively experimenting with his vision of a big catch, and dreaming about how this would impress his wife upon his return home, his vision was suddenly transformed into a half-fish/half woman. This way, the notion

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of mermaid has been proved to be a socio-cultural consequence of the division of labor in the poor man's household, and fantasizing – a special case of abstraction.

SHD ' own definition of abstraction was as follows: "Abstraction is an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure". Now, this was very confusing because the statement was at the same time very restrictive (restricted to mathematical abstraction) and very general. It was general because it seemed to identify all construction of mathematical knowledge with mathematical abstraction. It was confusing also because it was circular. The circularity was, in particular, in the definition of the genesis of abstraction as "passing through (a) a need for new structure; (b) the construction of a new *abstract* entity; (c) the consolidation of the *abstract* entity...". Thus the product of abstraction should be an "abstract entity", but I was not informed what the authors understood by "abstract entity".

While SHD stressed that, for them, abstraction was a mental *activity*, GT assumed that it is was a *product* of mental activity, "in which a complex parallel processing organ solves the problem of complexity by focusing on essential structures that enable decisions to be made". GT's notion of abstraction had the properties of the notion I was used to, namely those of ignoring some aspects while focusing on some other aspects, and I felt quite comfortable with it. I was less at ease with the rest of the theory and especially with the interpretations of the students' mathematical behavior that this theory afforded. The focus was entirely on the biology of cognition, in abstraction from the social and institutional situation, in which the learning of mathematics normally takes place. For GT, "essential structures" of the problems they presented to the students were always certain mathematical structures, those structures they were themselves most familiar with. In the experiment with children being shown five red cubes and asked what would be worth remembering about them, children who chose to remember that there were 5 cubes happened to belong to the group of high achievers; those who chose to remember the color, the pattern or configuration were from the low achievers group. But who decided that remembering the number of the cubes was the right thing to do? Isn't this a matter of didactic contract? In a mathematics class, numbers are important. In communication or arts class, color and arrangement could have tremendous importance. Aren't some children failing because they have not figured out what is the didactic contract in each particular class? Because they have not figured out how school works and what one is rewarded for? It is extremely dangerous to explain success and failure in mathematics at school by cognitive factors alone. We must take the didactic system as a whole and the student as a "perfinking" person (perceiving, feeling and thinking, David Krech cited in Bruner, 1987) in it as a whole. "Success in mathematics" is an institutional measure, not a measure of cognitive progress or capacity.

The examples of the contrasting behavior between high and low achievers in the area of algebra again suggest that low achievers are those who are bad at noticing what are the rules of the game; in this case – what are the formal conventions of writing algebraic expressions, and when two functions are to be considered the same. They are strangers in the school mathematics culture. They would rather use a different syntax to express things, and how you get a result is important for them. How you get a result is important in programming computers; a(u + v) and au + av, where u and v are vectors and a is a scalar are different functions in a CAS. The first involves n additions and n multiplications; the second - 2n multiplications and n additions. How do you know that in this particular algebra class this does not matter? How do you know what is important and what is not in a particular culture? This is not mathematics thinking; this is socio-cultural thinking and some people are better than others in assimilating into a foreign culture. Some keep their *terrrribel forrreyn akzent* for ever.

G, representing RME (realistic mathematics education) defines abstraction as an activity, which is comprised in the processes of mathematizing and seen as a two-way process: from less formal to more formal and the other way round. As in GT, abstraction is not an important concept in RME. More important is the assumption that, for the purposes of mathematics education, mathematics should be seen as a human activity and not as a library of accomplished and polished theories. RME is a project of curriculum development, not a theory, although the developers have started formulating their epistemological assumptions and principles in view of building techniques (heuristics), technologies and theories of mathematical instruction. SHD claim that their theory is also an outcome of curriculum development activities but their focus in the papers is on a theory of learning. RME researchers focus on the design of tasks embedded in long term curriculum activities. They describe the activities and try to justify the design. Little is said about the classroom experimentations and the notion of "success" of an experiment is not defined.

What are the criteria of success? What have been the proofs of success? Results on TIMSS? G claims that in RME the developmental research is "*evolutionary* in the sense that theory development is gradual, iterative and cumulative. There is no theory with which to start. The initial, global theory is elaborated, refined and explicated during the process of designing and testing" (G, 1998, p. 282). This suggests the hope that the law of the "survival of the fittest" will guarantee progress in the long run. However, the law of the survival of the fittest does not imply that the best curricula and best conditions for learning mathematics are eventually going to be achieved. What is "the fittest" is often governed by the law of least resistance or the tendency of the educational system to short-circuit all scholarly activities that are costly in managerial effort and time and are not directly related to the preparation of students for passing the final examinations.

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ABSTRACTION: WHAT THEORY DO WE NEED IN MATHEMATICS EDUCATION?

Paolo Boero

I will divide my reaction into two parts. In the first part I will follow the grid *("what is the theory about"*, etc.). In the second part I will discuss the need for a theory of abstraction in mathematics education, and the requirements that, in my opinion, such a theory should meet, then I will reconsider the three theories from this personal point of view. In the sequel, G, GT, SHD will indicate the three theories (according to the initials of the family names of their authors).

According to the Grid...

Most of suggested reading criteria (*What is the theory about? What assumptions are being made? What does the theory claim? What terms are used and what do they mean? How has the theory been validated?*) are related to minimal, necessary requirements that each theory (as a "scientific theory") must meet in human sciences (psychology, anthropology, sociology, etc.). The second part of the last reading criterion (*What are the aims of the theory and what are its applications?*) refers to a specific challenge for theorists in mathematics education.

More or less explicitly each presented theory satisfies the first and the third requirement. Here it seems to me that theories G and SHD deal with subjects that are rather close to each other (a common title might be "abstraction in context", yet with a different meaning of the word "context" – see later), while in the case of GT the theory deals with mathematical content and related individual learning processes.

Different ways of satisfying the second requirement are followed within the above presentations of the three theories. In the case of GT, "assumptions" are intended as general assumptions derived from other theories in order to create an environment where the theory can develop and be better understood; in the case of G and SHD, general assumptions are internal to the theory and constitutive of the core of the theory itself. We can observe how G aims at self-sufficiency in the presentation of the theory, while SHD refers to existing general theories (especially "activity theory"). The second criterion poses some problems: in the case of G, what meanings may be attached to crucial terms in a self-sufficiency perspective? (See later). In the case of GT and SHD, what relationships to establish with related theories?

Concerning the latter problem, we can identify different attitudes in mathematics education research as well as in the GT and SHD presentations. In the GT case, an autonomous elaboration (the procept theory) is linked to existing general theories in the field of psychology (Piaget's *"Pseudo-empirical and reflective abstraction in arithmetic, algebra and calculus naturally focus on our notion of procept."*) and used to reinterpret some other theories in the field of mathematics education: *"We do not agree with Sfard or Dubinsky that the development invariably proceeds in a sequence we describe as procedure-process-procept"*. In my opinion the legitimacy of these links and re-interpretations should be carefully discussed. As concerns the specific section "*What assumptions are being made?*" in the GT presentation, in my opinion the need for this kind of discussion becomes ever stronger: for instance, beyond heuristic hints, what are the precise relationships between the "*semantic squeeze*" in Bruner's quotation, the *"reduction in brain area involved"*, considered in neurophysiology studies, and the construction and functioning of procepts? In the SHD case, activity theory is taken as a fundamental reference. In my opinion, the adoption of an activity theory reference paradigm needs to consider the teachers' role as constitutive of the "learning" process (in our case, of the "abstraction" process). Indeed in Vygotsky's seminal work it is well known that a crucial, recurring term is "*obucenie*", that means *"teaching and learning"*. Yet I see (in the article as well as in the SHD forum presentation) that this "joint activity" aspect is not sufficiently developed.

Let us come now to the most critical criterion: *What terms are used and what do they mean?* It is clear that the danger for a person like me, who was educated as a mathematician, did research in mathematics for some years and still teaches mathematics at the University level, is to apply such criterion in the strict way he generally uses when dealing with his students' mathematical performance. It is true that in the human sciences domain it is very difficult to give "definitions" in the same, strict sense. In most cases, definitions are reduced to some evocative words that suggest a meaning, and then the context provides the full meaning. But I think that within the same theoretical construction (a theory), or the presentation of a theory, a crucial term must have a rather precise meaning (in order to establish whether or not an object or a situation falls within its semantic domain) and keep it. From this point of view, I find that in G and SHD the meaning of some crucial terms is not sufficiently clear, while the meaning of other terms seems to change during the presentation of the theory. In particular, I refer to the following terms:

"Formal mathematics" (in G): "formal" according to a high level of formalisation? And/or according to a social (or academic) consensus about ways of presenting relevant concepts, validating statements, etc.?

"Mathematical reality" (in G): what is its psychological and epistemological status? A subjective construction (or re-construction)? A historically shared and inheritable production, rooted in mankind's needs and experiences? A set of shared conventions?

"Structure" (in SHD): one part of the axiomatic organisation of mathematical knowledge (e.g. "the structure of group")? And/or the overall organisation of mathematical knowledge? And/or the organisation of mathematical thought?

"Context" (in SHD): in mathematics education, like in psycholinguistics, the word "context" takes different meanings:

- that of "situation context": those factors affecting the mathematical performance that are related to the situatedness of the students' activity (including social relationships in the classroom, environmental factors, etc.);
- that of "task context": the task evokes specific "realities" and constraints; as a consequence, behaviours, schemes, etc. related to those "realities" are activated;
- that of "inner context": in this case the attention is focused on the (internal) representation of the subject's past and present experience.

These different meanings of the word "context" suggest different perspectives under which teaching and learning mathematics in the classroom can be considered. For instance, in the case of abstraction the second perspective suggests to choose peculiar tasks suitable for it, while the first perspective suggests to take into account the social interactions that the teacher must "orchestrate" in the classroom.

Concerning the *aims* and the *applications* of the three theories, they are very different. Here again there are strong analogies between G and SHD (the theories are intended to provide useful tools to plan and/or improve teaching projects, and better interpret what happens in the classrooms where planned teaching is implemented). In the case of GT, the focus is on interpretative aims and in particular on explaining *"why some students are so highly successful with symbols, whilst others are procedural at best"* (etc.). In my opinion, in mathematics education we need both types of theories, bearing in mind that a theory of the second type can develop in (or support) a theory of the first type, and that a theory of the first type can provide interesting research questions for theories of the second type.

Concerning *validation of theories*, it seems to me that (in relationship with their specific aims) each theory meets this requirement. However I must say that it is met not so much in the above papers as in the articles included in the references: this is an unavoidable, necessary consequence of the space limitations of presentations.

Do we Need a General Theory of Abstraction in Mathematics Education? What Kind of Theory?

Let us consider the following examples:

- a right-angled triangle is drawn on the blackboard; students draw right-angled triangles on their copybooks; the teacher illustrates Euclid's theorem;
- the teacher writes on the blackboard: $(uv)'=u'v+uv'$, then $\int x\sin x dx=$, then illustrates and justifies the well known method of integration of the $x\sin x$ function based on the law of derivation of products of functions;

- the teacher draws a square on the blackboard, then one diagonal, then he proves (by the usual "reductio ad absurdum" proof) that the diagonal and the side of a square are incommensurable: .... $d^2=2s^2$, $(d/s)^2=2$, etc.
- the teacher establishes a 1-1 correspondence between the set of even numbers and the set of all natural numbers, then defines "infinite sets" as those sets which are equivalent to a proper subset.

In each of these cases mathematicians recognise some specific aspects of "abstraction". Mathematics educators have tried to deal with these aspects in different ways. For instance, C. Laborde and B. Capponi define a (geometric) figure as the set of couples (O,d_i), where O is the geometric object (e.g. the right-angled triangle) and d_i is one of the drawings that constitute the 'material representation' of the geometric object. Therefore the figure *"is the product of the abstraction process performed by the subject when, starting from a drawing (signifier) he or she thinks about the represented geometric object"*. This definition of figure is useful to deal with some difficulties that students meet when they approach geometrical reasoning (for instance, the reference to the peculiarities of a drawn right-angled triangle, or the stereotyped representation of the height of a triangle). These considerations are specific to the kind of abstraction inherent in the first situation. For the second situation an entirely different theoretical approach to abstraction is needed. Indeed, the 'material representation' is related to the represented mathematical object in a completely different manner: on one side the 'material representation' is much more distant from the mathematical object, on the other it becomes the starting point for a chain of transformations performed on the written expressions according to general syntactic rules. The third example shows some partial similarities both to the first example and to the second. Let us consider the fourth example: the idea of "epistemological obstacle" was elaborated in order to cope with students' difficulties inherent in "accepting" some cultural, "abstract" constructions like the "equivalence" between a set and a proper subset, which contradict our usual experiences about the sets of objects that we can describe extensively (i.e. by listing all their elements).

I add, as explicitly quoted in GT and in SHD, that general theories of abstraction already do exist in psychology.

A recurrent question for me, when reading the contributions for this panel, was: do we need a **general** theory of **mathematical** abstraction in **mathematics education**, i.e. a general theory suitable for describing and interpreting many typical phenomena of "abstraction" that intervene in teaching and learning mathematics and, possibly, controlling them (i.e. planning teaching in order to get the best results) by selecting pertinent variables and coming to predict the effects of actions on them?

In my opinion, a general theory of mathematical abstraction that would be of interest for mathematics education purposes should:
- cover most forms of abstraction currently met in teaching and learning mathematics at different school levels;

– interpret difficulties met by students in tackling abstraction in their approach to mathematical knowledge;
– point out relevant variables accessible to educational intervention;
– take into account relevant research in the field of epistemology of mathematics (as concerns reflection on abstraction) and cognitive sciences (as concerns general theories about abstraction, or specific theories about mathematical abstraction).

According to the first three requirements that I propose, each of the three theories offers some relevant contributions but also shows important weaknesses. G shows (through the example sketched in the PME contribution and other evoked examples) how some processes of abstraction can work, helping both the designers of teaching projects and teachers to plan and manage suitable classroom situations. But it seems to me that G does not cover processes of abstraction that are needed when a "break" is unavoidable in the transition from mathematical experience in real contexts to "formal mathematics". Moreover, the role of mediation played by the teacher, which is particularly crucial in this case, is not explicitly dealt with. GT is suitable to cover many "abstract" mathematical objects, but it seems to me that in my first example it does not provide much help in dealing with students' difficulties in managing the "abstract" notion of right-angled triangle (e.g. in the case of stereotyped images). And (being mainly a theory about mathematical objects in their relationships with generating processes) I see it might have some difficulties in dealing with the "abstraction" inherent in the activities (e.g. students' mathematical argumentation). SHD is a very general theory and it surely covers abstraction in a wide sense, but its current generality implies that some peculiarities are lost in specific cases. Perhaps in the future this theory will become suitable to deal in a productive way with all the examples provided at the beginning of this Section, but then the subject's individual processes should be better investigated in relation with the "situation context", the "task context" and the teacher's mediational strategies.

As concerns the fourth requirement that I propose, it seems to me that the three theories do not take sufficiently into account important, new streams of research in epistemology of mathematics, psychology and neurophysiology that are developing in different research communities. Recent joint studies in the fields of epistemology and neurophysiology (for an example the project "Geometry and cognition" at the ENS, in Paris) show the possibility of a convergence on the idea that even at the highest level of "abstraction" productive reasoning relies upon very "concrete", body related intuitions. This approach puts again into question, but within a new perspective (neurophysiology investigation), the idea of a purely conventional character of axioms and axiomatic theories; the anti-logicist positions have opposed this idea during the whole XX century under different perspectives (mainly philosophical or ideological or based on introspection). This approach also draws an almost entirely new picture of how high level professional mathematical activities are performed. It seems very interesting and promising that this stream of research goes in the same directions of some other streams of research in different disciplines (in particular, "embodied cognition", in psycholinguistics and psychology).

If it is true that "thinking abstract objects as if they were concrete" is possible, but "thinking in an abstract way" is impossible (as far as productive thinking is considered), then a theory of abstraction suitable for educational purposes should take charge of the whole complexity of the relationships between mathematical objects, the thinking processes concerning these objects, their "situatedness" in the classroom environment (the mediational role of the teacher being a crucial issue), and the most suitable "task contexts" for meaningful mathematical abstraction (both as concerns the mathematical content involved and the body-rooted processes). From this wide and very demanding point of view I think that the presented theories offer some important contributions, but we are still far from a comprehensive theoretical answer to the challenge of mathematical abstraction in mathematics education.

RESEARCH FORUM 2

Theme
The nature of mathematics as viewed from mathematics education research

Coordinators
Lyn English and Gerald Goldin

Contributors
Laurie Edwards
Ruhama Even
Brian Greer
Hartwig Meissner
Anna Sfard
Shlomo Vinner

THE NATURE OF MATHEMATICS AS VIEWED FROM MATHEMATICS EDUCATION RESEARCH

Introduction

Lyn English, Queensland University of Technology

Gerald A. Goldin, Rutgers University

How we conceive of mathematics has a major bearing on our educational efforts. The nature of the mathematical ideas we consider essential for success in the new century, and the ways in which these ideas are conveyed during the school years, can facilitate or impede students' lifelong mathematical understanding, learning, and communicating. Though we have made significant advances in mathematics education research, fundamental issues of intellectual importance having political, social, and economic ramifications, continue to be debated. These issues include: (a) What counts as mathematics? (b) What is the nature of mathematical ideas? (c) What is the relative importance of these ideas to society? (d) What is the nature of the various representations these mathematical ideas may take (both internal and external to the student)? (e) What are the processes by which these ideas are understood by students? (f) How might we maximize students' understanding of these mathematical ideas?

Among the viewpoints are those that consider mathematics and the direction of its growth to be shaped by a complex system of cultural, social, and political forces (e.g., D'Ambrosio, 1999; Skovsmose & Valero, 2002). Lerman (2000) refers to the "social turn" in mathematics education, which came into being towards the end of the 1980s. The social turn saw the emergence of theories that view mathematics and mathematical meaning and reasoning as products of social activity. Social constructivists, for example, emphasize the processes through which consensus develops in determining the nature of mathematical knowledge and how it is constructed.

Complementing the sociocultural perspectives are those that draw on advances in cognitive science to explain aspects of the structure of mathematics and its development. Included here are analyses of external mathematical representational systems (e.g., mathematical symbolism, computer microworlds, structured analogues, diagrammatic structures) and internal systems (e.g., verbal/syntactic systems, imagistic systems, conceptual metaphors, mental models). The nature of the interactions within and between these representational systems is considered to play a powerful role in learners' mathematical growth.

Theories that lie within the broad sociocultural framework, along with the more cognitively oriented theories, are contributing to current debates about what mathematics students should learn, how they should learn it, and the extent to which school mathematics curricula should capture the essence of workplace mathematics (e.g., see Stevens, 2000).

A comparatively recent and controversial cognitive perspective on the nature of mathematics is that of "mind-based mathematics" (Lakoff & Nunez, 2000). Here mathematics is not inherent in the universe, nor is it merely a cultural artifact; rather it is shaped essentially by the nature and structure of human brains and minds via "conceptual metaphors."

There are varying perspectives among both mathematicians and mathematics educators on the aforementioned issues. It is important for educators to consider mathematicians' points of view, especially in light of the current curriculum debates highlighted by the media in several nations. Will mathematics educators and mathematicians find intellectually sound ways of connecting differing perspectives, or will existing gaps widen further?

The papers in this Forum provide critical debate on the issues we have addressed. Brian Greer argues that mathematics, mathematics education, and mathematics education research are situated in "sociohistory, culture, and politics." On a somewhat different note, Laurie Edwards presents a "personal journey" on the nature of mathematics, where she illustrates the perspective that mathematical ideas are shaped in fundamental ways by our embodied experience in the world. The importance of communication is emphasised in Anna Sfard's paper, where mathematics is seen as a discursive activity, that is, "a special way of communicating." Within her communicational framework, thinking is regarded as a special case of communicative activity. Another interesting perspective is presented by Shlomo Vinner who addresses boundaries, identities, and mathematical objects in his discussion of mathematics and mathematics education. Hartwig Meissner, on the other hand, explores the distinctions among "Einstellung" (attitude), "Vorstellung" (internal image) and "Darstellung" (external representation) in addressing mathematics and the processes of mathematics learning. Gerald Goldin extends Meissner's ideas in his discussion of representational systems, and provides critical thought on the sources of the "widening chasm" between mathematics and mathematics education.

Complexity of Mathematics in the Real World

Brian Greer, San Diego State University

Perception of the relationships among mathematics, mathematics education, and mathematics education research used to be simple.

Mathematics was seen as a relatively well-defined, hierarchically structured, body of knowledge. Mathematics education meant transmitting this body of knowledge to each student up to an appropriate level in the hierarchy. Psychological research was expected to provide general theories of cognitive development and learning, with the assumption that these theories could be applied to the learning of mathematics as a domain and the improvement of (mathematics) education through generating hypotheses testable via standard experimental designs. Many mathematicians and psychologists taking a more or less informed interest in mathematics education feel comfortable with this simplicity (for example, in the context of the Californian Math Wars, see the analysis of the unholy alliance between psychologists and mathematicians by Jacob and Akers (1999)).

However, the situation has become more complicated.

First, mathematics continues to grow fast and computers have changed both its content and its methods. Consequently, questions of selection and arrangement arise – what parts of mathematics should be chosen and how should they be reorganized for education? Typically, curricula are largely the result of tradition and inertia and, insofar as growth occurs, it is mainly through accretion without radical restructuring. There is very little by way of principled design – consider the limited adaptation to the new representational systems afforded by computers, for example.

Second, the first wave of the cognitive revolution generated disequilibrium when it became clear that there was "de-emphasis on affect, context, culture and history" (Gardner, 1985, p. 41). The outcome was the "second wave" (De Corte, Greer, & Verschaffel, 1996, p. 497) which mathematics education research both contributed to, and was influenced by, in major ways. Methodologies became interpretative rather than scientific, with results that are liberating or anarchical, depending on your point of view. The work of some researchers now exemplifies Engestrom's proposed methodology for activity theory that puts it to "the acid test of practical validity and relevance in interventions that aim at the construction of new models of activity jointly with the local participants" (Engestrom, 1999, p. 35).

Inevitably, mathematics education researchers' views of mathematics have been complicated by their immersion in activity systems, including exposure to the culture of the classroom, the nature of schooling, and the politics of mathematics education. Mathematics, mathematics education, mathematics education research are all situated in sociohistory, culture, and politics.

To illustrate the foregoing comments, I offer sketchy outlines of three key characteristics of mathematics (revealing my own biases, naturally) and how they play out in mathematics education and mathematics education research.

The Two Faces of Mathematics

> On the one hand, mathematics is rooted in the perception and description of the ordering of events in time and the arrangement of objects in space, and so on ("common sense -- only better organized", as Freudenthal (1991, p. 9) put it), and in the solution of practical problems. On the other hand, out of this activity emerge symbolically represented structures that can become objects of reflection and elaboration, independently of their real-world roots. In the process, common sense is soon transcended, yet, time and again, the results of such elaborations have proved (often after a considerable lag in time) useful in theoretical descriptions of real-world phenomena and solution of real-world problems. (De Corte, Greer, & Verschaffel, 1996, p. 500).

The link between the two faces of mathematics is the activity of modeling. Typically, the modeling of a real-world situation leads to a range of solutions that need to be judged in terms of human criteria such as utility, purpose, and complexity. Introducing pupils early to this perspective may be considered part of the process of enculturation into the practices of mathematicians, yet until relatively recently, it has not received much attention (Niss, 2001; Verschaffel, 2002).

The Developmental Nature of Mathematics

"Mathematics grows … by its self-organizing momentum" (Freudenthal, 1991, p. 15). In the course of the sociohistorical construction of mathematics, several developmental mechanisms may be identified:

(a) The disequilibrium that comes from lack of closure. The obvious example is the extension of the concept of number from its origins in natural numbers. (It seems to me that there is a clear parallel with Piagetian theory but I am not aware of anyone who has explored this idea in depth).

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(b) Metaphorical extension, which has been elaborated in the recent book by Lakoff and Nunez(2000) (and see Edwards). Why are all those different things all called "numbers"? (Poincare defined mathematics as the art of giving the same name to different things).

(c) Variations on the theme of reification (e.g. Sfard, 1991, and see Vinner).

(d) Mediation by cognitive tools, as illuminated by the Vygotskian tradition – language (see Sfard), symbols, representational systems (see Goldin, Meissner).

(e) Systematization, including the development of axiom systems. The history of attempts to teach mathematics on this basis is well known.

It has been pointed out that a major reason for the difficulty of mathematics education is that children are expected to master in a few years concepts that took humankind millennia to develop. All of the above developmental processes have ramifications at the ontological level. In particular, analyses of developmental obstacles represent one broad focus for the continuing relevance and usefulness of cognitive analyses (Greer, 1996).

Mathematics as Cultural Construction

"Mathematics as a human activity" has become a principle cutting across developments in mathematics education, new directions in the philosophy of mathematics education (e.g. Hersh), and influences on mathematics education from critical pedagogy, ethnomathematics, feminist critiques, historical perspectives, and so on.

For balance, it should be remembered that the proof of Fermat's last theorem, and the pages of complex formulae that Ramanujan sent to Hardy also represent human activity and require an account of the coherence and continuity of cognitive processes within an individual brain over an extended period of time however mediated by social environments (Greer, 1996).

Mathematics as a Form of Communication

Anna Sfard, The University of Haifa, Israel

Many different answers have been offered to the question *What is mathematics?* throughout history, but the definition given by Henri Poincare is the one which I find particularly useful. According to the French mathematician, *mathematics is the science of calling different things the same name*. This deceptively simple statement, if interpreted in a way not necessarily intended by Poincare himself, can be seen as a forerunner of the *communicational* vision of mathematics. In what follows, I outline this special approach in general terms. The presentation is organized as a series of questions and answers.[i]

Q_1. What is mathematics?

A_1: It is a kind of discourse (a way of communicating)

The first thing to notice in Poincare's definition is that by putting the issue of *naming* in the center of our attention, it implies that mathematics is, in principle, a discursive activity. In other words, mathematics is a special way of communicating. One can oppose saying that it is *thinking* rather than *communicating* that should be given prominence in the definition. My answer to this is that thinking is already included in the term *communication*. Indeed, according to the basic tenet of the communicational framework, *thinking can be regarded as a special case of communicative activity.*

Q_2. What renders mathematical discourses their unique identity?

A_2. Their use words, their visual mediators, and their special routines.

After bringing the discursive activity to the foreground, Poincare gives a hint as to what makes mathematical communication distinct: It is the mathematicians' special propensity for unifying many different things under the same name which is the hallmark of the mathematical discourse. True, using the same word as a signifier for many different signifieds is not unique to mathematics – this activity is the very essence of *conceptualization*, and as such it is a vital ingredient of any communication. Mathematics, however, exceeds all the other types of discourse in the range of things included under each of its terms. This special tendency of mathematicians to speak of sameness even when what reveals itself to their eyes (and ears) appears different, is known as their propensity for *abstracting*.

Please note that within the communicational approach, the adjective 'abstract' refers to the way words are being used in the discourse, and not, as is often the case within other conceptual frameworks, to a special property of objects that are being talked about. More generally, *the use of words* is the first of several properties that one has to consider while trying to decide whether the given discourse can be called mathematical. While becoming a participant of the mathematical discourse, the learner often modifies her uses of known words and then introduces new words which from now on will serve as common names for sets of things that until now were never considered as "the same".

Two additional dimensions along which mathematical discourse can be distinguished from other types of communication are their special *mediating* *tools* (or simply *mediators*), that is, visual means with which people help themselves while communicating; and their distinct *discursive routines* with which the participants implement well defined types of tasks. Let us say a few words about each of these special discursive features.

Mediators. Unlike in the less abstract, more concrete discourses which can be visually supported with objects existing independently of the discourse itself, mathematical communication is mediated also, and sometimes exclusively, by symbolic artifacts specially designed for the sake of communication. Contrary to what is implied by a common understanding of a tool in general and of symbolic tools in particular, within the communicational framework one does not conceive of the communication mediators as mere auxiliary means that come to provide expression to pre-existing, pre-formed thought. Rather, one thinks about them as a part and parcel of the act of communication and thus of cognition.

Discursive routines are patterned discursive sequences that the participants use to produce in response to certain familiar types of utterance expressing a well-defined type of request, question, task or problem[ii]. In the case of mathematical discourses, the routines in question are those that can be observed whenever a person performs such typically mathematical tasks as calculation, estimation, explanation (defining), justification (proving), exemplification, etc. The routines with which interlocutors react to the given type of request (e.g. "estimate" or "justify") may vary considerably from those employed in response to a similar question asked in everyday setting. One of the special characteristics of full-fledged mathematical discourse is that its routines are particularly strict and rigorous.

Finally, let me explain why the question I am answering now speaks of "mathematical discourse*s*", with the plural form implying that there is more than one type of communication that can count as mathematical. Although the same words can be used on many occasions, the rules that regulate this use may vary from one setting to another. Similarly, although seemingly speaking of the same things (quantities, geometric shapes) discourses may differ in their mediators and in their routine interpretation of what appears as the same tasks. Thus, we have a good reason to speak of different types of mathematical discourse, distinguish between everyday mathematical discourses, school mathematical discourse, and the discourse of professional mathematicians (cf. Rittenhouse, 1998).

Q_3. Why do we need mathematical discourse?

A_3. For the sake of economy of communication, for its maximal effectiveness, and to solve problems that could not be solved before.

The brief answer A_3 above points to three reasons because of which mathematical discourses came into being and developed the way they did. The last of these reasons seems quite obvious, so I will elaborate here only on the other two.

The economy of communication is attained by the very property Poincare was talking about: By calling different things the same name, mathematical discourse *subsumes* several former, independently existing discourses, turning them into discourses "about the same thing" and making it possible to express in the new language everything that can be said in any of them with their own special signifiers. For instance, while saying that "three and two equals five" we simultaneously express a truth about fingers, dollars, kilograms, and infinity of other countable objects. The successive discursive "squeezing" exists also within the mathematical discourse itself. For example, the discourse about *functions* subsumes discourses about *graphs* and the discourse about *algebraic expressions*.

The issue of effectiveness must be considered when one asks why the meta-rules of mathematical discourse developed the way they did. It seems that it has always been an undeclared hope of the mathematicians to create a discourse that would leave no room for personal idiosyncrasies and would therefore lead to unquestionable consensus. Such consensus would imply certainty of mathematical knowledge. The exacting rules of the modern mathematical discourse are the result of unprecedented efforts of 19th- and 20th-century mathematicians to attain this unlikely goal.

Q4. What is mathematics learning?

A4. To learn mathematics means to change one's discourse

Learning mathematics may now be defined as an initiation to mathematical discourse. It is important to note that the introduction to a new form of communication never starts from zero. Whether the discourse to be learned is on fractions, triangles, functions or complex numbers, it will be developed out of the discourses in which the children are already fluent. If so, to investigate learning means getting to know the ways in which children modify and extend their discursive ways in the following three respects: in vocabulary they use, in the mediators they employ, and in the discursive patterns (routines) they follow.

Q5. How does the learning occur and what can we say about teaching?

This is the very central question math ed researchers are asking. The issue is extremely complex and it would be imprudent to try to summarize it in a few sentences. I thus leave this last question without an answer. Here, let me just say a few words about the expected impact of the communicational conceptualization on the vision of learning and teaching mathematics.

Perhaps the most dramatic difference between the more traditional, cognitivist vision of mathematical thinking and the one discussed in this paper lies in their conception of the origins of mathematical learning: The traditional approaches assume that learning results from the learner's attempts to adjust her understanding to the externally given, mind independent truth about the world, and thus imply that, at least in theory, the learning could occur without the mediation of other people. In contrast, the idea of mathematics as a form of discourse stresses that individual learning originates in communication with others and is driven by the need to adjust one's discursive ways to those of other people.

What is the added value of this conceptual shift? First, if we agree that the site of mathematical learning is *between* people rather than beyond them, we also realize that social and cultural factors are those that enable the process of learning in the first place. Second, the communicational conceptualization helps us to see an inherent complexity of learning: The idea of thinking as a form of communication and of mathematics as a kind of discourse, if taken seriously, makes us realize that in the process of learning mathematics, the students' awareness of the proper use of words and symbols must precede their ability to account for this use. This vision of learning is bound to entail a revision of some popular interpretations of the idea of learning-with-understanding. Finally, the communicational approach brings second thoughts about many other pedagogical believes as well. As has been argued in many places, some of these beliefs must be modified, while some others would better be abandoned altogether. Much work is yet needed to examine the practical value of this theoretical change.

The Nature of Mathematics: A Personal Journey

Laurie D. Edwards, St. Mary's College of California

What counts as mathematics? What is the nature of mathematical ideas?

The questions that frame this Research Forum are clearly foundational to the practice of mathematics education. I would like to address these questions not by proposing definitive answers, but by reflecting on my own experience as a researcher over the past 18 years. During this time, my own thinking about the nature of mathematics has evolved, in parallel with the emergence of the theoretical frameworks discussed in this Forum. I hope that the examination of a particular "case" of changing theoretical perspectives in a single body of research may be instructive.

My first major research project involved the creation of a computer-based learning environment for a specific mathematical domain, transformation geometry. As with many studies of students' mathematical thinking, the research revealed "errors" in the children's thinking, interpretations that differed from accepted mathematical truth. An example of such an "error" is described in this passage:

> The rotate bug...is an error in conceptualizing a transformation...Instead of imagining the entire plane rotating around the center point...these students thought that the shape would first slide over to the specified point, and then turn around it in place." (Edwards, 1989, p. 107-8).

The characterization of the students' interpretation of rotation as an error, as well as the entire framing of the research, reflected an objectivist view of mathematics (Edwards & Núñez, 1995); indeed, it exemplified what Lakoff and Núñez call "the Romance of Mathematics" (Lakoff & Núñez, 2000). According to this view, "Mathematics is an objective feature of the universe; mathematical objects are real; mathematical truth is universal, absolute, and certain" (ibid., p. 339). In other words, mathematics has a transcendent existence, apart from any human knowledge of it. The implication of this view is that our role as educators and researchers is to design more effective instruction about, and representations of, this mathematical reality.

Lakoff and Núñez acknowledge that there is no way to determine, empirically, whether mathematics indeed has such a transcendent existence. However, it is clear that the teaching and learning of mathematics always takes place within specific social contexts, and that simply characterizing students' understandings as "correct" or "incorrect" does not go very far in helping to improve learning. Thus what Lerman has called "the social turn" in mathematics education has come to the fore (Lerman, 2000). This change in focus from evaluating the adequacy or inadequacy of individual cognition to investigating the irreducibly social nature of learning and teaching emerged in my own research as well, One specific area in which this framework became important was in the investigation of mathematical explanation and informal proof. I first used the transformation geometry microworld with 11-year-olds. In addition to the occasional "bug" in the students' understanding of the transformations, I also found that few students were able, independently, to generate explanations or informal proofs for the patterns they were guided to discover in the microworld. At the time, I attributed this to the students' age and level of intellectual development. I expected that when I used the microworld with older students, they would be able to, fairly spontaneously, notice and explain these informal theorems that seemed so obvious in the microworld. This turned out not to be the case: the older students behaved very much like the younger children with regard to their mathematical explanations – neither group was able to produce such explanations without some degree of scaffolding and interaction with the researcher. This led me to reconsider the nature of mathematical explanation and proof. Rather than expecting that, given a dynamic and accurate representation of a domain, students would be able to discover and explain pre-existing mathematical truths, I came to think of proving as a social process, one which needs to be explicitly modeled and scaffolded (Edwards, 1997).

Thus, in my own personal journey in thinking about the nature of mathematics, I moved from assuming that mathematical ideas were "out there," waiting to be discovered, to thinking of mathematics as a product of social interaction, a kind of language, a human practice with norms that must be learned over time. Yet the fact that mathematics is learned and practiced within social contexts begs an important question: within a given social context, why is it that mathematical ideas take the form that they do? And how is it that humans, as cognizing creatures, are able to co-construct systems of mathematical knowledge that are mutually intelligible? One answer, of course, might be that mathematical ideas take the form they do simply because of their objective, transcendent reality, that human beings are simply "perceiving" the way things are, mathematically. I found this answer unsatisfying, in part, because it seemed to set mathematics apart from all other products of human history and cognition. Instead, I found work on conceptual metaphor (Lakoff, 1987; Lakoff & Johnson, 1980) and embodiment (Varela, Thompson & Rosch, 1991) to be evocative, in pointing to a deeper level of cognitive structure upon which much of human thought and language is construc-ted. The reason that mathematical ideas take the form that they do, and the reason they are mutually intelligible, is because they are, at a foundational level, built upon the common experience of being humans, with the same kinds of minds and bodies, living and growing in the same physical world (Lakoff & Núñez, 2000; Núñez, 2000, Núñez, Edwards & Matos, 1999).

A concise statement of the implications of embodiment for understanding mathematics can be found in the work of Lakoff and Núñez:

- Mathematics, as we know it or can know it, exists by virtue of the embodied mind.

- All mathematical content resides in embodied mathematical ideas.
- A large number of the most basic, as well as the most sophisticated, mathematical ideas are metaphorical in nature.

(Lakoff & Núñez, 2000, p. 364).

I would like to offer an example of the application of this perspective by returning to the "rotate bug", described above. This interpretation arose after the students were introduced to what was, for them, a new mathematical idea – they had never been taught about geometric transformations before. Yet the "idea" of turning was not new to them – indeed, the embodied experience of moving through the world, from a very early age, includes innumerable instances of turning one's own body. However, this experience of turning is different in an important way from the mathematical version of rotation instantiated in the microworld. This general transformation, or mapping of the plane, could take place around any arbitrary center point, whether this point was part of, or distant from, the block letter L used to show the transformations.

The conceptual construction that the students made of the new mathematical idea of rotation of the plane was shaped by their embodied experience of turning in the physical world: the rotate "bug," in which rotations always take place around a point *on* the L-shape, can be seen as a metaphorical mapping from the experience of turning one's own body in place. It is worth pointing out that this metaphorical mapping was unconscious: there was no socially-communicated introduction of the metaphor; instead, the physically-grounded source domain existed prior to the introduction of the mathematical idea, and shaped its assimilation in the children's minds.

In fact, the researcher did introduce an explicit metaphor or image to help the students extend and generalize their understanding of rotation. I asked the students to think about an object at the end of a string, which could be turned, with the other end of the string being fixed in place. This explicit, socially communicated metaphor helped, I believe, to bridge students' initial "local" interpretation of rotation to the more general or global mathematical one.

These remarks are intended to communicate aspects of a personal intellectual journey, yet this journey is not one in which prior theoretical commitments are left completely behind. Putting aside the question of the objective existence of mathematics (which seems to be something of a religious question), I still believe that much of mathematics is socially constructed, and that in understanding teaching and learning, we must attend to particular social and cultural contexts. However, *what* is constructed, within these contexts is not arbitrary: mathematical ideas, as they exist within, and are shared between, actual human minds, are shaped, in fundamental ways which we are still in the process of understanding, by our embodied experience in the world.

Boundaries, Identities and Mathematical Objects – Should we bother?

Shlomo Vinner, University of Israel

The proposal for this research forum raises the question whether mathematics educators and mathematicians will find intellectually sound ways of connecting their differing perspectives and reinforce each other's ideas, or whether the existing gaps will widen further. I assume that the mathematicians mentioned here are the university mathematicians who teach tertiary mathematics. Some of them are not interested in teaching mathematics since their main interest is mathematical research. Others, in case they care about teaching, have their own views on how to do it and do not believe that mathematics educators have useful advice for them. Usually, mathematicians have vague ideas about who we are and what we do (there might be some exceptions). So, who is going to listen to us? One answer is that we can listen to each other. This is quite common for academic circles. A parody about such circles appears in Davis and Hersh (1981) where a handful of devoted mathematicians who work on the decision problem for non-Riemannian hypersquares is described (pp. 34-39). If we do not want to stay like them in the isolated ivory academic towers the alternative is to look for communities who can use our research findings. Such a community, and perhaps the only one, is the community of mathematics teachers. However, if we want to approach them it should be done within their intellectual frameworks and in their language. The nature of mathematics is undoubtedly an issue with which they have to be involved. But to what extent? Thus, this forum, whose title is *The Nature of Mathematics as Viewed from Mathematics Education Research,* is a good opportunity to raise some questions about mathematics education research that deals with the nature of mathematics. To be more specific, my question is the following: What aspects of the nature of mathematics are relevant to the community of mathematics teachers, and what aspects should be kept for our closed circles where we can discuss any subject at any level of sophistication. Asking that, I am, in fact, raising two questions. One is about boundaries and the second one is about identity. The one about boundaries is: What are the boundaries of mathematical education research that are relevant and meaningful to mathematics teachers. The one about identity is: What is the purpose of mathematical education research? In fact, this is an identity question about our group and it has been raised in the past several times by several people. One of them was in Ballachef's letter from 1996. Questions about boundaries and identity have more than one answer. So, what I suggest here is only one possible answer out of many. I suggest trying to define a restricted domain of mathematical education research which I will call the core and which will have an immediate simple application to the practice of teaching. Some other issues, which imply a level of sophistication that teachers do not have, will be considered as peripheral.

The nature of mathematics has many aspects. One of them is the nature of mathematical objects. Some time ago, a student of mine (she is a junior high mathematics teacher) came to me complaining: "You sent us to take a course in the philosophy of mathematics," she said, "and the lecturer spent three weeks discussing the question: what are objects and what are mathematical objects? What is the point of it?" I was quite irritated by the question but as a teacher, I have trained myself to control my reactions and to try to tolerate and to understand my students' views. "Isn't this question relevant to our research forum?" I asked myself and decided to discuss it here. First of all, I would like to explain why the question of mathematical objects is such a crucial question in the philosophy of mathematics. According to mathematical logic and model theory, mathematics is a collection of theories about mathematical systems. A mathematical system is a set of abstract objects with relations and operations that fulfill certain primary conditions. The mathematician's task is to discover some interesting claims about the mathematical systems implied by these primary conditions. Whether you accept this or not, it can explain why the problem of mathematical objects is so crucial for the philosophy of mathematics. But is it also so crucial to mathematics education research? Even a short literature survey will show us that many mathematics educators are involved in investigations about this issue. In a paper by Tall et al (2000), there is an attempt to draw certain boundaries between some approaches to mathematical objects in mathematical education research. When one speaks about boundaries one has to speak also about territories, but Tall et al. (p.233) speak about scopes, not about territories. Hence, what I say here is my interpretation of their formulation. Their paper discusses three approaches to mathematical objects: In the first two, an attempt is made to explain how mathematical objects come into being in the human mind: it is either by encapsulation (Dubinsky, 1991) or by reification (Sfard, 1991). The third one (Gray and Tall, 1994) does not bother with the question of how mathematical objects come into being. It assumes that people think and speak about mathematical objects. However, it draws our attention to the fact that some mathematical terms and mathematical symbols are ambiguous. These terms and symbols denote both processes and objects or, if you wish, both processes and concepts. This led Tall and Gray to invent the notion of procept. If you look at it this way, the discussion in the above paper (Tall et al, 2000) is, in fact, about boundaries. If you agree to accept mathematical objects without trying to ask how they are formed then some controversies are moved from the *core* to the *periphery*. A serious objection to excluding the mathematical objects from the *core* might claim the following:

People fail in mathematics because they have not constructed in their mind the mathematical objects required in order to perform the mathematical tasks imposed on them. We should lead them through well-designed activities in order to construct in their mind the required mathematical objects. Therefore, these activities should be an essential part of the mathematical education research core. More specifically, we should make our students go through many processes that will eventually become (by encapsulation or reification) mathematical objects.

Since I wish to avoid controversies I will not argue with this claim. I would only suggest, as an alternative working assumption, a different view. Mathematical objects are a special case of abstract objects. The problem of abstract objects is widely discussed in the philosophy of language. There is the classical distinction between concrete nouns and abstract nouns. Please, note that we are speaking here about nouns and not about objects. Some nouns or noun phrases denote well-defined concrete objects. For instance: *The dog of my mother in law.* On the other hand, many nouns do not denote any object. For instance, is there any object in our world that is denoted by the noun *milk*? The question becomes even more embarrassing when abstract nouns are discussed. Are there objects in the world denoted by *love, peace* or *compassion*? Surprisingly enough, in our thoughts we relate to these nouns *as if* they denote objects. This is *reification.* The Webster's Ninth New Collegiate Dictionary suggests that the word "reification" is in use at least since 1846 and *it is the process or result of reifying. To reify is to regard something abstract as a material or concrete thing.* Thus the working assumption which I suggest claims the following:

(1) There is a is a tendency in languages to introduce nouns even when no objects are involved (it probably facilitates talking about certain things.)

(2) Reification occurs spontaneously the moment a noun is introduced.

Because of time limitation I will bring only two short examples to support the above claim. 1. When teaching limits in Calculus, many of us use the term "infinity" instead of saying "increasing unboundedly." However, in calculus (contrary to set theory), there is no object the name of which is infinity. In spite of that, many calculus students think of infinity as an object. 2. Even languages of primitive cultures have abstract terms. Levi-Strauss, the famous French anthropologist, in his book The Savage Mind (1966) illustrates this by some examples. Two of them are the following: *In Chinook, a language widely spoken in the north–west of North–America, the proposition "The bad man killed the poor child" is rendered as: "The man's badness killed the child's poverty." And for "The woman used too small a basket" they say: She puts the potentilla-roots into the smallness of a clam basket"* (p.1). (The issue of reification is widely discussed in the major works of Quine (1960), 1981, 1995)). Finally, I would like to relate again to the claim that people fail in mathematics because they fail to construct the mathematical objects involved in their mathematical tasks. There are so many potential reasons for failure that it is impossible to isolate one factor and to claim that it is the cause for failure.

The way I suggest to understand the procept paradigm by Tall and Gray allows us to speak about processes, objects and concepts as primary notions. Namely, we do not have to explain what processes, concepts and abstract objects are. On the other hand, the procept theory points at a major obstacle in the learning of mathematics - ambiguity. Ambiguity is a serious obstacle in communication. On the other hand, it also enriches communication immensely. I suggest, however, that this issue will not be included in the core of mathematical education research.

Einstellung, Vorstellung, and Darstellung

Hartwig Meissner, Westf. Wilhelms-Univ. Muenster, Germany

Einstellung, ***Vorstellung***, and ***Darstellung*** are keywords to describe the process of learning and understanding mathematics. Analyzing this process we do not rely on philosophical theories (Kant's ontology). We base our theory of learning and understanding mathematics on the following assumptions:

(1) Mathematics is "something" which exists independent from human beings or from human brains like trees, birds, genetic codes, time, space, electricity, gravity, infinity, ...

(2) There are external representations of mathematical ideas, ***Darstellungen***, which we can read, or see, or hear, or feel, or manipulate, ... These *Darstellungen* can be objects, manipulatives, activities, pictures, graphs, figures, symbols, tags, words, written or spoken language, gestures, ... In a *Darstellung* the mathematical idea or example or concept or structure is

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hidden or encoded. There is no one-to-one-correspondence between a mathematical idea, concept, etc. and a *Darstellung*.

(3) Human beings are able to "associate" with these objects, activities, pictures, graphs, or symbols a meaning. That means each *Darstellung* evokes a personal internal image, a ***Vorstellung*** (cf. concept image, Tall & Vinner, 1981). Thus *Vorstellung* is a personal internal representation, which can be modified. Or the learner develops a new *Vorstellung*. A *Vorstellung* in this sense is similar to a cognitive net, a frame, a script or a micro world. That means the same *Darstellung* may be associated with many individual different internal representations, images. Each learner has his/her own *Vorstellung*. And again here, there is no one-to-one-correspondence between a *Darstellung* and a *Vorstellung*.

(4) The process of building up a *Vorstellung* depends very much on the basic mentality of the learner, i.e. on his or her ***Einstellung***. The *Einstellung* includes affective components like attitudes, beliefs, emotions, values (Goldin). The *Einstellung* affects attitudes towards learning in general, towards mathematics in general, towards problem solving, or towards the specific learning "environment". The *Einstellung* is a product of social interactions (with parents, teachers, peers, etc.), of genetic factors, of cultural or historical impacts, etc. Positive *Einstellungen* in the class room are necessary for a successful teaching-learning process. A learner with a negative *Einstellung* probably will not be very successful. *Einstellungen* work like a filter or a catalytic converter in the transformation processes *Darstellungen* $\leftrightarrow$ *Vorstellungen*.

In this paper I concentrate on *Vorstellungen* and *Darstellungen*. The process of building up a *Vorstellung* very much depends on the already existing internal representations ("assimilation, accommodation" according to Piaget, "coherence, connectedness" according to Greeno) and on the already existing "subjective domains of experiences" (Subjektive Erfahrungsbereiche, Bauersfeld). Learning mathematics now means that the learner has to build up a *Vorstellung* which "corresponds" (especially in the sense of Greeno) as much as possible to the mathematical idea / concept / structure. But the learner does not experience the mathematical idea / concept / structure directly, the learner only is confronted with (different types of) *Darstellungen*. Figure 1 presents a summary of these ideas.

We are interested in the cognitive processes[1]. What does "learning" mean? And when do we "understand"? Learning obviously is the process of building up an "adequate" *Vorstellung* of a given mathematical situation (by means of "appropriate" *Darstellungen*). But the *Vorstellung* is individual and cannot be inspected or evaluated directly. With other words, there is no direct way to evaluate "adequate" or, there is no direct way to evaluate the degree of "understanding".

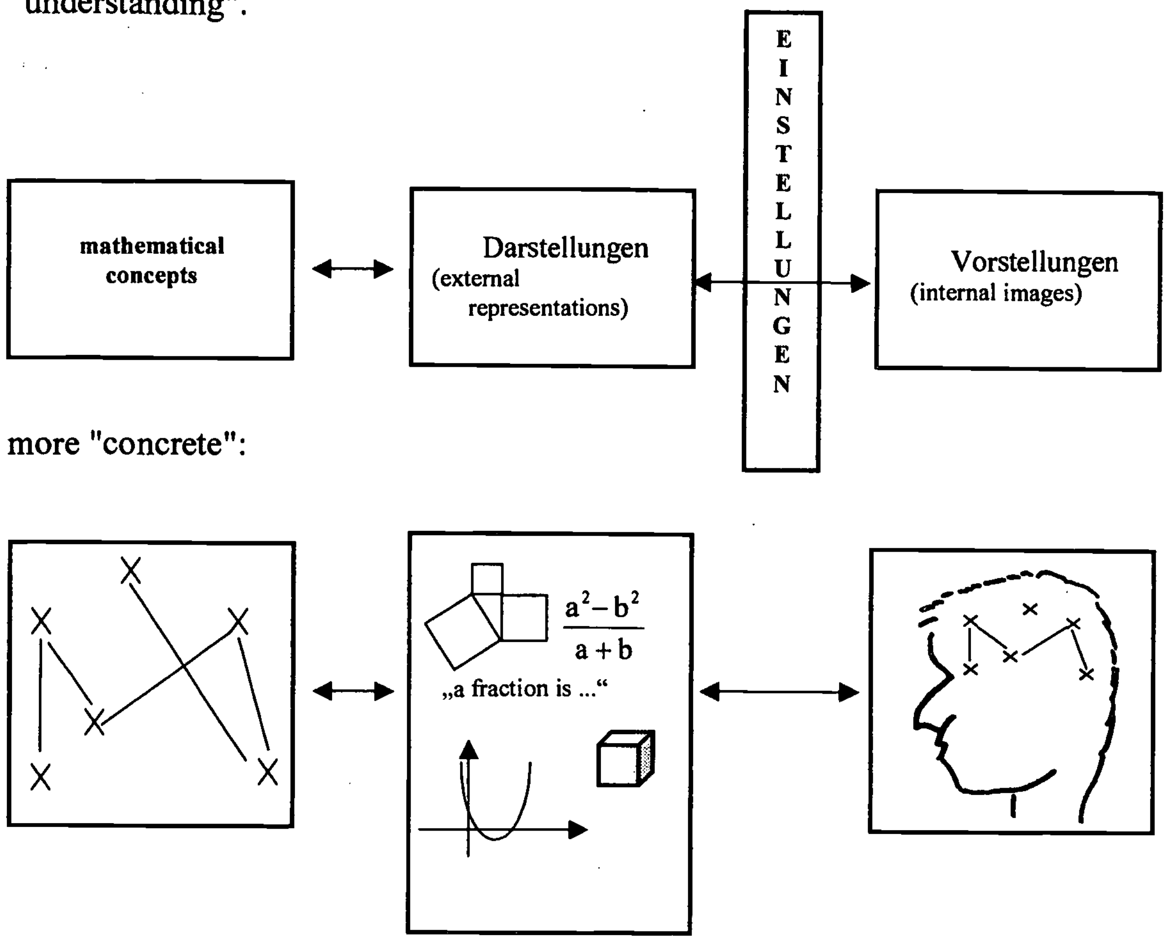


Figure 1: Examples of *Vorstellungen* and *Darstellungen*

We only can judge a *Vorstellung* by a corresponding *Darstellung*, that means "communication" is necessary. To prove understanding the learner must transform the individual *Vorstellung* into a *Darstellung*. And when the learner's *Darstellung* corresponds with one of the expected *Darstellungen* we may assume that the learner did understand. The problem is obvious. We do not judge a *Vorstellung* but we interpret a performance.

[1] Despite a suggestion from the IC to "facilitate greater insight and engagement" we will continue the paper with the German words which are more precise than English translations.

To distinguish the *Vorstellung* from the performance we will speak of a ***conceptual understanding***[2] when the learner has an adequate *Vorstellung*, i. e. his or her internal representation corresponds appropriately to the given situation. A conceptual understanding also may be intuitive or unconscious. And of course, a conceptual understanding only can be demonstrated indirectly up to a certain degree, consciously or unconsciously, see examples.

To detect the student's conceptual understanding we still need *Darstellungen* from the student. But we must allow flexibility in the use of *Darstellungen*. Not the *Darstellung* itself is important but the *Vorstellung* behind that *Darstellung*. When there are misunderstandings concerning a specific *Darstellung* just change the *Darstellung* to clarify if the misunderstanding originates from the *Vorstellung* or from the *Darstellung*. Of course also non-standard *Darstellungen* can be used. Clinical interviews with experienced interviewers - and also experienced class room teachers - can identify the student's conceptual understanding. Written tests - like TIMSS or PISA - usually cannot help to prove conceptual understanding[3].

Skemp distinguished between instrumental understanding and relational understanding. ***Instrumental understanding*** is characterized by selecting and applying appropriate rules to solve the problem without knowing why ("rules without reasons"). In our terminology specific *Darstellungen* are expected and it is necessary to (re-)produce them: "Tell me what to do and I will do so". There is no adequate mathematical *Vorstellung* behind.

An understanding with more *Vorstellung* behind it, was termed a relational understanding by Skemp (1978): "knowing both, what to do and why". Here an adequate *Vorstellung* leads to an expected *Darstellung*. We want to call this a ***communicable understanding***. It combines both, the conceptual understanding with the ability to communicate in a wanted or given format. A person with communicable understanding is able to communicate flexibly in those *Darstellungen* which are expected or - according to the specific problem - in *Darstellungen* which fit best to the problem. We will give some examples to illustrate these various aspects of understanding[4].

[2] In earlier papers we called this a relational understanding (which is different to SKEMP's relational understanding).

[3] HOSPESOVA and TICHA found through interviews examples for both, "good conceptual understanding, but no expected TIMSS-Darstellung" and "correct TIMSS-Darstellung, but no adequate conceptual understanding".

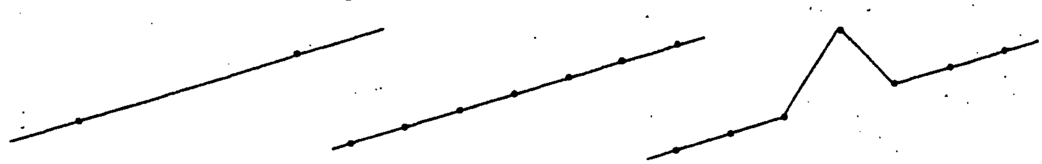
[4] For more details on the role of communication and communicational conflicts see Anna Sfard and Gerald Goldin in this Forum-paper.

Example 1 (Sorting objects)

A teacher and about 25 students (age ~9) are sitting around a set of about 40 geometrical solids. Teacher: "I have a rule in my mind to sort these solids. These two solids follow my rule. Who can find other solids which follow my rule?" Only directed by the teacher's YES or NO without any further explanations the set gets sorted into the set of solids which follow the rule and a second set (of counterexamples). This non verbal process of sorting objects by guess and test may lead to an intuitive concept (of *polyhedra* or *rectangular solid* or *rectangle* or ...), i.e. an (partly unconscious) conceptual understanding develops. But during the process of sorting in the classroom also discussions start to guess the "rule", to verbalize the situation. The concept becomes more conscious and a communicable understanding develops.

Example 2 (Linear functions, discuss possible *Vorstellungen)*

To draw the graph of $1.5x - \sqrt{8}\,y = -\sqrt{3}$ we get the following *Darstellungen:*



Example 3 (Interpreting functions, analyze the conceptual understanding)

Student A draws the correct graph of a given function and determines correctly by computation the ***maximum*** at x=5. Student B draws the same graph and determines the ***minimum*** at x=5 (by applying the correct algorithms with a computational mistake).

Example 4 (Procepts)

A keyword or symbol or tag as a *Darstellung* can serve as a stimuli to evoke proceptual thinking (Gray & Tall, 1994). Here the *Vorstellung* involves both, a

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procedural and a conceptual aspect. E.g. y=f(x) may be seen as an assignment (process) or a function (concept).

Connecting Understandings from Mathematics and Mathematics Education Research

Gerald A. Goldin, Rutgers University

The perspective I bring to this discussion may be a controversial one, but I shall start noncontroversially by building on Hartwig Meissner's accompanying presentation. Meissner highlights differences among the notions of *Einstellung* (attitude), *Vorstellung* (internal representation), and *Darstellung* (external representation) as descriptors of processes in mathematical learning and understanding, and takes mathematics to be something that "exists independently" of these. Before considering aspects of the nature of mathematics, let me continue with two further, important ideas about representation.

First, I would emphasize that individual representational configurations, whether external or internal, cannot be understood in isolation. Rather they occur within *representational systems.* The latter are not mere collections of representations, but have complex structures that in practice may be ambiguously defined or context-dependent (Goldin, 1998). Thus words and sentences occur within natural language systems, having conventional grammatical and syntactic structural features that can be characterized as *external* to any one cognizing individual. *Internal,* verbal representational configurations also occur in each individual, within a personal system of linguistic competencies encoded in the brain that has its own structural features. All these depend on context in various ways. The "communicational approach" in Anna Sfard's accompanying presentation, at least tacitly, involves such structural features of language. Likewise in mathematics, we have conventional, external systems of representation including base ten numeration, rules for arithmetic operations, ways of denoting rational numbers, Cartesian graphs, a system of algebraic notation, etc., with accompanying verbal descriptions, all usefully regarded as external to the individual. And we have the visual imagery, notation-images, kinesthetic encodings, and so forth, occurring within personal systems that may be partially-developed and embody misconceptions, contradictions, and idiosyncratic structural features. We might use the terms *Darstellungsysteme* and *Vorstellungsysteme* to refer respectively to external and internal representational systems.

In my work I have found it useful to distinguish five different types of *Vorstellungsysteme* that come into play in learning and doing mathematics: (1) verbal/syntactic systems, referring to internal natural language competencies, (2) imagistic systems, including visual/spatial representation, kinesthetic representation, and auditory/rhythmic representation, (3) formal notational systems, referring to internal procedural/structural competencies associated with the conventional representations of mathematics, (4) a system of heuristic planning and executive control, including configurations for strategic decision-making that govern problem-solving activity, and (5) an affective system including rapidly-changing emotional states as well as more stable, multiply-encoded constructs such as attitudes, beliefs, and values. A psychologically adequate description of mathematical learning, development, and problem solving requires that we take account of all five types of *Vorstellungsysteme* in interaction with each other and with *Darstellungsysteme.* Here I think Meissner's term *Einstellung* usefully distinguishes certain more stable aspects of affect and related cognition that individuals bring to mathematical situations.

The second idea I want to emphasize is the strong psychological role that the initial assignment of meaning, or semiotic step, plays in the individual's developing *Vorstellungsysteme.* Understanding this is important not only to education, but to grasping how mathematics itself has evolved.

For example, children frequently learn that multiplication of natural numbers ("times") is an abbreviation for repeated addition: i.e. '3 x 5' *means* '5 + 5 + 5' (three fives). The formal notational and imagistic representational subsystems associated with the operation of multiplication then develop structurally, making use of this 'meaning'. The usual multiplication tables are constructed, and patterns found in them. The commutative and associative properties of multiplication, and the distributive property of multiplication across addition, are verified and illustrated. As more structure is built on the initial meaning, its psychological persuasiveness increases. Repeated addition becomes for the learner what multiplication *really is.* But the moment comes when the meaning fails! A child may interpret '3 x ½' as '½ + ½ + ½', but '½ x 3' is problematic—what does it mean to 'add three one half of a time'? The structural commitment to the commutative property suggests a value for ½ x 3, but the absence of the original meaning leaves a gap in understanding—a cognitive obstacle. We have well-documented related misconceptions, such as the idea that "multiplication always makes larger," which may persist until some reconceptualization has occurred.

Similar obstacles occur not only in individuals learning mathematics, but in the history of the mathematical field. They have their origins in the structural extensions of mathematical systems that require *relinquishing* the necessity of the original semiotic connections—so that the mathematical structures are abstracted, and the ways in which the mathematical notations function as representations of imagistic configurations are generalized. Mathematicians of earlier eras struggled mightily with the concepts we today call irrational numbers, imaginary numbers, negative numbers, and non-Euclidean geometries, due in part to the psychological difficulty of abandoning the initial, "real" meanings attributed to numbers, to points and lines, and so on. The necessity for such reconceptualization is well understood now, and has influenced our evolving notions of "mathematical existence" and "mathematical truth."

This brings me to the major point in my presentation—the notion of mathematical truth, its recent unfortunate downplay in mathematics education, and the consequent widening chasm between the fields of mathematics and of mathematics education. I view the gulf that has developed as both damaging and unnecessary. Although the divide has reached a depth that seriously impedes our common educational goals, I do not think the sociological reasons for rivalry are so strong as to generate inevitable conflict. For both communities, and for the next generation of students, the value of achieving meaningful improvement is extremely high, providing a powerful incentive for real collaboration. I want to focus on what I perceive to be *intellectual* reasons for division, intending my comments to be strongly critical.

At the root of the problem in each community is a willingness *to deny or dismiss the very integrity of the knowledge* generated by the other. It is not always apparent to mathematicians when they do this in relation to mathematics education research, nor is it always apparent to non-mathematicians when they do this in relation to mathematics. Sometimes, however, it seems to be done consciously and opportunistically, as a way of inviting attention and gaining a following—with a kind of wilfully-maintained ignorance of the other discipline.

On the one hand, some in the mathematical sciences community insist on imposing—with unwarranted confidence—tacit but naive models of what it means to learn mathematics. It is straightforward for mathematicians to focus on building powerful competencies in *formal notational* systems, as these are culturally agreed to be part of mathematics, and competencies in them can usually be tested straightforwardly. Powerful problem-solving strategies and *heuristic planning* techniques that work in various mathematical domains (Polya, 1954, 1962, 1965), including proofs, are likewise generally agreed to be part of, or closely related to, mathematics, though techniques such as "work backward from the goal" or "solve a simpler, related problem" are difficult to test. But for many mathematicians the traditional view of mathematics as consisting of abstract systems encoded formally accords only casual or unimportant status to all but the most standard of representations. The power of formal, logical reasoning when applied to abstract mathematical entities, together with the fact appreciated in mathematics that visual intuitions can mislead, creates a reluctance to place a high, explicit value on *imagistic* representation, especially non-standard representation with accompanying differences in individual learning styles, or on *affect.* Visualization, metaphor and metonymy, emotions, and the relation between feeling and mathematical imagination, are dismissed or relegated to incidental status, despite growing empirical evidence for their fundamental roles in the learning of mathematics.

One extreme position is to discretize, take as "given", and value very highly in defining the curriculum a collection of standard mathematical material, in disregard of the complexity of the processes through which mathematical understanding develops in students of diverse abilities and motivations. This view has energized the "traditionalist" side of the recent "math wars" in the United States. Skills are seen as prerequisites to conceptual understanding, and are thus to be taught first. Mathematical achievement manifests itself through speed and accuracy in answering test questions. Mathematical ability is seen as an innate, unitary characteristic of individual students, describing the rapidity with which they acquire formal notational competencies when trained in them. Of course I am *not* saying that all or most mathematicians adopt such dismissive positions, though some educators have sought to establish this stereotype. *Some* mathematicians do, and the fact that they do offers a convenient rationale for counterpart dismissive fashions in education.

Fundamental to the integrity of mathematical knowledge is the notion of "truth", which has evolved significantly by virtue of mathematical insights achieved over millenia (Kline, 1980). Let me use this term in a certain way that mathematicians typically use it. The field of mathematics has been characterized by many as the study of pattern (e.g. Sawyer, 1955). This includes pattern detected in the natural world, and pattern in systems invented by human beings. To study patterns, mathematicians seek to characterize them as precisely as possible. One way this is done is to formulate definitions and axioms or postulates that describe a system or class of systems incorporating a pattern. We then have a collection of mathematical statements taken from the outset as true. Further propositions (called theorems), often not at all obvious, can be proven from the axioms by means of well-defined rules of inference and are thus demonstrated to be true. Truths in mathematics occur within systems of assumptions. In developing such systems, our concepts change and evolve. Some lines of reasoning turn out to be valid, while others are demonstrated to be invalid. Often our initial conceptions turn out to be too limited, or even self-contradictory. Sometimes imagistic thinking guides us to mathematical truths, and sometimes it misleads us. In short, there exist essentially *objective* answers to important mathematical questions. Furthermore the system we create is *abstract*, and not necessarily restricted to apply only to the original, motivating conceptual domain—other, unexpected models are likely to exist! And there are fundamental, logical limits—*proven* limits—to the possibility of demonstrating the completeness or consistency of mathematical axiom systems.

The fashionable but dismissive intellectual trends influencing mathematics education research have in the past two decades been *ultrarelativist*. Such views are ideological (in the sense of being nonfalsifiable), since a contrary argument can never be more than an alternative viewpoint. They include radical constructivism, radical social constructivism, and variations of postmodernism, each in its own way *denying the very possibility* of objective truth, knowledge, or validity, and thus dismissing from the outset the central construct of mathematical inquiry. These have energized extremes on the "reform" side of the "math wars." Most recently we have the grand claim that mathematics consists entirely in "conceptual metaphors" (Lakoff, G. & Núñez, R., 2000), predictably attracting favor among some mathematics education researchers. Here there are only conceptions, no misconceptions. Ideas and visualizations (familiar to mathematicians) that underlie and motivate abstract constructions are renamed as metaphors, presented as if newly-discovered, and taken to *be* the mathematics—with those mathematicians who might disagree caricatured as Platonists, naive realists, or empty formalists. In this view mathematics cannot possibly "exist" independently of human metaphors, so the initial point in Meissner's presentation is be rejected entirely.

Of course, the definitive characterization of mathematical truth and the validity of mathematical reasoning are far from a solved philosophical problem. We have a lot to learn about it, and many unanswered questions remain. However, we must distinguish between the assumptions, definitions, conventions, and rules of inference chosen to characterize some visualized or imagined patterns (socially constructed, subject to negotiation in their framing, and possibly "metaphorical") and their logical consequences (now "true" in an important, "objective" sense). Denying or dismissing the very construct—replacing "mathematical truth" by "social consensus" or "stability of human metaphor", replacing "validity" by "viability", and so on—makes *no contribution* to our mathematical understanding. Rather it seems to make deeper mathematical understanding unnecessary. Some version of ultrarelativism may be a tempting response to closed-minded or "absolutist" views among mathematicians. It may seem to justify our being open to students' various ways of thinking mathematically, to our emphasizing in education the *ideas* of mathematics, imagery and metaphor, open-ended problem solving, discovery processes, social and cultural environments, and various systems of belief—all that I strongly favor! But ultrarelativist "isms" undermine what should be central goals of mathematics education—conveying the nature of mathematical truth and the power of valid, objective mathematical reasoning, bringing learners to experience the processes of abstraction and proof, and helping students to identify the same abstract mathematical constructs in a variety of different conceptual domains.

In my studies of mathematics and of students' processes of learning and problem solving in mathematics, I have never found what we learn validly as mathematicians and what we learn validly as researchers in the psychology of mathematics education to contradict each other. Both sets of understandings are needed. Mathematicians who are "absolutists" nevertheless offer important mathematical insights. "Ultrarelativist" educational researchers have designed and reported on ground-breaking studies. Progress is made when mathematicians and educational researchers communicate effectively and learn from each other, so that our understandings of difficult notions are enhanced—not when we erase distinctions or dismiss centrally important constructs.

To conclude, then, it is time that mathematics education researchers exercise far greater discernment than we have in the past. Let us *knowledgeably and thoughtfully abandon* the dismissive fads, fashions, and "paradigms" in mathematics education, in favor of a unifying scientific and eclectic approach.

Concluding Points

Lyn D. English, Queensland University of Technology

The papers of this Forum have presented a range of perspectives on the nature of mathematics as viewed from mathematics education research. The authors have raised controversial and, at times, opposing viewpoints on the issues presented in the introudction. The ideas expressed by each of the authors provide a rich basis for tackling the numerous debates emerging within and beyond our discipline, debates that are being fuelled by the increasing scrutiny of mathematics education by the public, by governments, by mathematicians, and by school systems. Such debates include the nature of the mathematical content we should teach, how and when we should introduce this content, how we can provide all students with access to powerful mathematical ideas, and how we can encourage more students to undertake courses in mathematics. We face many challenges as we attempt to deal with these concerns. The question remains as to how effectively we are meeting these challenges.

Although not denying the importance of diversity in our perspectives and the richness this brings to mathematics education, I believe we have become too divided and too insular in many of our beliefs, theories, and ideologies. We seem to be addressing only those questions that fall readily within our particular ideological stance while ignoring other important issues. While following a particular social-constructivist perspective in exploring children's mathematical learning, for example, we often overlook the inherent structure of the mathematical tasks or at least give it superficial treatment. So we might argue for the richness of children's learning through their classroom interactions, while failing to recognize the mathematical inadequacies of the tasks being explored.

As no one perspective can provide a satisfactory answer to all of our research and teaching issues, we need to be more cognizant of alternative viewpoints and incorporate the best of our own ideas with those of others. This Forum represents an important step in this direction. There still remain, though, many issues in need of attention as we continue to foster the growth of our teaching and research communities. Some of these issues are listed below for further debate.

- To what extent are our theoretical bases addressing the mathematical needs of society in the new millennium?
- How are our existing ideologies impacting on students, teachers, and on the community at large?
- How can we ensure a closer match between what we believe about mathematics and mathematics education and the goods we deliver?
- To what extent are we dismissing one another's perspectives and philosophies, both within our own mathematics education community and with our neighbours, the mathematicians?
- Mathematics continues to grow rapidly. How is this growth changing our views on the nature of mathematics and what we consider important to know and understand?

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[i] Needless to say, this extremely concise exposition cannot possibly count as a proper introduction of the comprehensive conceptual framework. The interested reader may turn for elaboration to Harre & Gillett (1995), Edwards (1997), and Sfard (2000 a, b; 2001a).

[ii] Two discursive sequences will be regarded as being instances of the same discursive routine if they comply with the same set of *meta-discursive rules*. This latter term refers to principles that help the observer to account for the regularities she spots in the behavior of the interlocutors. Rather, than being prescriptions which the speakers follow in a conscious way, these are propositions that help the analyst encapsulate the discursive flow the way the formula of free fall helps physicist to encapsulate the movement of falling bodies (cf. Sfard, 2000b; Sfard & Kieran, 2001a).

RESEARCH FORUM 3

Theme

Measuring mathematics learning and describing goals for systemic reform

Coordinators

Peter Sullivan and Dianne E Siemon

Reactor/Discussant

Stephen Lerman

Contributors

Margaret Brown, Mike Askew, Valerie Rhodes, Hazel Denvir, Esther Ranson & Dylan Wiliam, Barbara Clarke, Andrea McDonough & Peter Sullivan, Robert Wright & Peter Gould, Zemira Mevarech, Marja van den Heuvel-Panhuizen

RESEARCH FORUM

MEASURING MATHEMATICS LEARNING AND DESCRIBING GOALS FOR SYSTEMIC REFORM

Co-ordinators:

Peter Sullivan, La Trobe University, Australia
Dianne Siemon, RMIT University, Australia

FORUM OVERVIEW

In many countries, governments have initiated projects to evaluate and describe mathematical competence. Researchers within such projects make assumptions about what mathematics should be learned at particular ages, how learning goals are described, how learning is measured, and what findings may mean for curriculum and teaching. The forum is not critiquing whether this process is appropriate or necessary, but considers the solutions that researchers have sought and the rationale for those solutions.

Contributors to this forum describe their involvement in such projects, assumptions they made, the tensions they identified, and their resolutions of those tensions. Particular emphasis is on the contribution that such projects make to the mathematics education research.

Some of the decisions connected with this process include: What are the mathematical goals for students at particular ages? How can the learning of this mathematics be measured? What instruments and processes are appropriate for collecting evidence? What evidence would be characteristic of learning? Are there underlying assumptions about the way curriculum is described? Are there underlying assumptions about the way mathematics is taught? What are the tensions (e.g., suitability of instruments for all students, need for attention to have issues arising will be addressed) identified, and how are these reconciled?

MEASURING MATHEMATICS LEARNING AND DESCRIBING GOALS FOR SYSTEMIC REFORM: SOME ISSUES FOR CONSIDERATION

Stephen Lerman, South Bank University, London, UK

Large-scale projects to provide evidence for policy on developing mathematical competence are opportunities for mathematics education researchers to play a part in the improvement of students' learning of mathematics. There have been too many situations internationally where policy has proceeded without evaluative, descriptive or predictive research. Indeed in some countries educational researchers are deliberately ignored. It is most important, therefore, that such projects consider carefully as wide a range of issues concerning the quality of their project and its potential impact before commencement and as an on-going process throughout their work.

What follows is a non-exhaustive list of issues, expressed as questions, that I believe should be considered. These comprise methodological and ethical questions and one concerning engagement with the research community.

I invite the contributors to this Research Forum to respond to some or all of these issues, describing whether and how they feature in their projects. On the other hand you may have reservations about the necessity of responding to one or more of these issues. If so, I invite you to explain your point of view.

- Whose agenda is the project? Whose interests are served: students, parents, teachers, society, Government, the mathematics education community? Who owns the data? How will it be used?
- What are the theoretical frameworks (some of which may be implicit)? Are they drawn from psychological and/or sociological and/or other intellectual resources? What are the methodological justifications for the methods being used?
- What are the assumptions about the relationship between teaching and learning? What are the assumptions about the role of texts and other materials?
- What are the assumptions about the nature of curriculum?
- Where do the norms of student mathematical development come from (since both norm-referenced and criterion-referenced norms are actually norm-referenced)? Will tests be designed to measure actual development or proximal development (or both)?
- Does the sample size do justice to the research questions?
- Who benefits from the project, and how? Will there be any losers? Will the disadvantaged be further disadvantaged?
- How are insiders' perspectives taken into account (students, teachers, parents, etc.)?
- What arrangements are there for gaining, then reviewing and revising the consent of participants?
- How will the findings be critiqued within the community of mathematics education research (or other academic communities)? Will the project be discussed with the research community at the design stage, the interim findings and/or the final outcomes stages? Will the findings be expected to be generalisable internationally?

MEASURING PROGRESS IN NUMERACY LEARNING

Margaret Brown, Mike Askew, Valerie Rhodes, Hazel Denvir, Esther Ranson and Dylan Wiliam

King's College, University of London

LARGE SCALE RESEARCH AND POLICY

We focus on the longitudinal assessment strand of the Leverhulme Numeracy Research Programme in the UK between 1997 and 2002. A broader report is presented in a Plenary paper in this conference (Brown, 2002): we have tried to minimise the overlap in content.

Steve Lerman is right in his challenge to be concerned about large-scale assessment of mathematical competence; there is no doubt that the results of studies such as SIMS and TIMSS (Robitaille & Garden,1989; Reynolds & Farrell, 1996; Beaton et al., 1996; Mullis et al.,1997) have had a major influence on mathematics education in England, providing the justification for an ever-tightening centralised control of curriculum and teaching methods. Two English studies which one of us has previously managed (Concepts in Secondary Mathematics and Science (CSMS) (Hart(Ed.), 1981), and Graded Assessment in Mathematics(GAIM) (Brown (Ed.), 1992) has supplied data informing the structure and content of these reforms. While we regret some of the uses made of the results of these studies, these policies have nevertheless been developed by democratically elected governments and supported by the majority of teachers, parents and voters. We believe that it is part of the collective responsibility of mathematics education researchers in any country, and indeed internationally, to assist in the collection of reliable and valid large-scale data to inform international, national and local policies. We also of course regard critique as part of our collective responsibility and hence value large-scale data which can provide evidence on which rigorous critique of policy can be based (e.g. Brown *et al.*, 1998). Large-scale research can also often provide a basis for and/or evaluation of design and development work, which is a third essential element of the collective responsibility of mathematics educators.

Our support for large-scale work is by no means in opposition to arguing the importance of case-studies; indeed we believe that insight gained from successful small-scale work is not only valuable in itself but also an essential preliminary to large-scale studies. We do not therefore have a problem in being associated with both types of work in a complementary way, as in the Leverhulme Numeracy Research Programme and in many other studies in which we have been involved.

In the next sections, we will try to answer the most relevant of Steve's questions to our project. Sometimes the order is changed to assist the narrative; in some cases the pressure of space has forced rather briefer answers than we would like.

THE LEVERHULME NUMERACY RESEARCH PROGRAMME

1. Origins and Purposes

The Research Programme is funded by the Leverhulme Trust, which is independent of Government and distributes the interest on profits from manufacturing to support charitable work. The research agenda was partly that agreed by the Trust, advised by academics; they announced in 1996 that they would donate £1 million to a study of low achievement in primary schools focusing on literacy and/or numeracy. The Core Project, one of six projects in the Programme, was proposed *'to obtain large-scale longitudinal value-added data on numeracy, in order: to inform knowledge about the progression in pupils' learning of numeracy throughout the primary years; and to assess relative contributions to gains in numeracy of the different factors investigated in the programme.'*

The sample would be all the children from two different cohorts in 40 schools, ten in each of four local education authorities in different and varied areas of England. One cohort would start in Reception (age 4 to 5) and progress to Year 4 (age 8 to 9); the other would start in Year 4 and progress to Year 8 (age 12 to 13), although test data would be collected only in Years 2 to 7.

2. Theoretical Background

In the full research team there are about 16 researchers, although only two work full-time on the programme. In addition to mathematics educationists the team includes a cognitive psychologist and a social anthropologist, since an aim of the research programme is to examine primary school numeracy from multiple perspectives. Most of the group working on the large-scale assessment project are educationists who espouse a broadly social constructivist view of learning, involving progressive generalisation from situated cognition sometimes experienced within overlapping communities of practice (Wenger, 1998), and a dialectic both between the social and the individual, and between the cognitive and the affective. However our separate theoretical positions vary along these continuums.

There are clearly problems about the concept of criterion-referencing which assumes an all-or-nothing capability. The tests are only loosely criterion-referenced in the sense that most items can be related to specific cognitively based skills. Justification for this lies in the fact that the original form of the tests was as part of a diagnostic interview which was developed iteratively as part of a study identifying important steps in learning (Denvir & Brown, 1986). Any inferences from success in items to specific understandings is highly dangerous, but the analysis of associated sets of items will involve some inferences about learning which we recognise as very speculative. But such speculation lays the foundations for science.

As part of the study we have observed teaching in all the classrooms and adapted an analytic framework first used by Saxe (1991) which relates to a broadly activity theory approach focusing on discourse, tools, tasks, relationships and norms.

3. Methods

In Steve's terms the assessment is probably intended to be actual not proximate, although evidence suggests that items sometimes scaffold learning. We would all accept the crudeness of the method of assessing children's progress which we use, i.e. tests with questions read out by the teacher and answered in printed test booklets. It is constrained by the large size of the sample (about 1700 children in over 70 classes in each of the two cohorts) and the consequent need for brevity, simplicity and uniformity since we rely on teachers to administer the tests.

We have evidence through case-studies that a single child's test performance on a particular occasion is not particularly reliable; however our aim is not accurate assessment of individuals. First, we will make inferences about progression in learning of English children at a particular point in time. This will be based on changes in the proportion of the cohort over a particular time span who are able to obtain correct answer to a linked set of items. Thus most idiosyncratic variation in individual children will cancel out but sources of systematic bias will remain. We will however present as much data as possible for others to critique our conclusions.

For example, after a practice item where children discuss with their teacher a quick way of finding 30+21 if they know 30+20=50, they are given on their booklets that 86+57=146 and asked quickly to use this to work out answers to different questions. Different forms of this item were on the tests used from Year 2 to Year 6 (each year was tested in October and June), and the percentages successful were:

Table 1: Percentages successful on linked items (Year 2 & 3 data relating to Cohort 1 in 1999-2001 is in italic font; Year 4, 5 & 6 data relates to Cohort 2 in 1997-2000)

Given	*Year 2*		*Year 3*		Year 4		Year 5		Year 6	
86+57=146	*Oct*	*Jun*	*Oct*	*Jun*	Oct	Jun	Oct	Jun	Oct	Jun
87+57	*35*	*60*								
86+56	*17*	*45*	*60*	*71*	69	79				
57+86			*57*	*67*	66	78	81	80	92	91
85+57					40	56	66	68		
143-86			*15*	*36*	22	36	46	57	71	80
86+86+57+57					21	36	41	54	68	79
860+570	*1*	*5*	*7*	*21*	19	31	39	55	69	76

This information illustrates the second purpose of the testing which is to examine the effect of different factors which may influence numeracy attainment. We have related the average gains each class makes between October and June each year to a variety of other data on pupils, teachers, teaching, and schools. To remove the important prior attainment factor, the tests have been designed so that the average gains over this period are independent of the initial scores. Examples of the results on this analysis are given in the plenary paper (Brown, 2002).

Concerning curricular validity, there is little agreement internationally or indeed even in the UK (Brown et al., 1998) on the definition of numeracy. The tests have in fact been developed and researched over a long time-span. The original diagnostic interview was designed in the light of the research literature on development of knowledge and skills in number which seemed relevant to the 1980s curriculum in most English schools; this evolved and was evaluated in a PhD study (Denvir & Brown, 1986). This instrument has been updated as part of another study (Askew, Bibby & Brown, 2000) and is now published for teachers' use (Bibby & Denvir, 2002). A further study entailed the adaptation of the test for class use and validated it against the individual interview assessment (Denvir & Brown, 1987). Finally the test has been modified and trialled for use in three different age groups (Askew *et al.*, 1997) before being adapted and trialled for 6 age groups in this study (Brown *et al.*, 1996). In each of these adaptations a check was made that the content matched the primary number/numeracy curriculum at that time in England.

There are for practical reasons no long items of a problem-solving or investigative nature, but these are currently little used in teaching in England. An investigative item was abandoned when the analysis of the data revealed large inter-class differences in strategy which suggested that teacher influence was overwhelming. In addition to the much researched and refined nature of the tests we have other reasons for having confidence in the comparative validity and reliability of the tests:

• we have compared performances on specific items on seven successive occasions for our case-study sample of 30 children per cohort: this suggests that the results are generally consistent. The results are also generally consistent with our observations of these children's cognitive behaviour in their normal lessons:

• the overall performance of a whole cohort on successive occasions is consistent;

• the reliability measure on a single test occasion is high (Wiliam *et al.*, 1998);

• we have observed classes and interviewed all teachers each year; this enables us to investigate any results which seem to be inconsistent with classroom performances;

• teachers have been very satisfied with the content of the tests, except that some are concerned that some questions are too difficult and go beyond what has been taught. (This is unfortunate and may affect results for some children, but it is a necessary part of the test design to measure progression for children across the whole range; teachers are asked to explain to children before they start that they are not expected to be able to do all the questions as some are designed for older children.)

4. Consultation, dissemination and application

Headteachers were asked to consult staff before agreeing to take part (and almost all did). Over five years of intense national change only 3 schools out of 40 have withdrawn, due to new headteachers. Permission was sought from all parents.

The study has involved 45-minute interviews with all teachers (about 600 over 5 years, with less than five refusals), annual interviews with headteachers and mathematics co-ordinators, and 45-minute interviews and many informal conversations with the case-study sample of about 60 pupils (Years 3 and 6). In all interviews participants have been asked about their views of the tests and other aspects of the study. There was limited scope for changing the tests for that year group as the model was already set, but tests for other year groups could be and were slightly affected, as they were developed and trialled one year at a time.

Payment for teacher cover during the interviews has been provided so that schools and teachers did not suffer. Annual feedback on test results is provided to local authorities, headteachers and teachers, but names of schools, classes and children are not given in accordance with promises of confidentiality. At the same time we have provided some summary of aspects of the interim project findings. The majority of schools and teachers have voluntarily expressed their regret that the project is now finishing, so presumably they have found participation useful; several have cited specific uses of the feedback e.g. in self-evaluation or inspection.

We have presented papers for discussion and critique at both national and international research meetings, at the stage of design (e.g. Brown *et al.*, 1996) as well as at the results (Wiliam *et al.*,1998; Rhodes *et al.*,1998, Brown *et al.*, 2000; Brown, 2000). However we have generally avoided wide scale national publication of, and publicity for, interim results beyond the research community, as we have preferred to wait until possible effects were confirmed or contradicted in the data for successive years. We are contracted to Kluwer to write a series of four books.

In terms of our aims, the main output is in knowledge and understanding about primary numeracy in order to improve attainment; hence the main beneficiaries should be future generations of pupils and teachers. Specifically from the longitudinal study this relates to progression in learning and factors which affect it. On the basis of reactions to presentations we have given and articles we have published in professional journals (e.g. Brown, 1999) we believe that the large-scale results, illustrated by specific examples, will be of use and interest to individual teachers as well as to researchers, developers and policymakers. Apart from our own uses, both to achieve our aims and to exercise our right of policy critique, we will of course have little control over how the data is eventually used once it is in the public domain. But that is a necessary part of an open society.

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MEASURING AND DESCRIBING LEARNING: THE EARLY NUMERACY RESEARCH PROJECT

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SYSTEMIC MOTIVATION

In the late 1980s, both levels of government in Australia collaborated in specifying the outcomes that could define learning (see Australian Education Council, 1991). While this type of curriculum specification and measurement was comprehensibly criticised (e.g., Ellerton & Clements, 1994) as constraining teaching and restricting achievement of students, it has also been argued that defining of curriculum goals in terms of standards and targets creates pressure on governments to provide resources for initiatives to ensure that the targets are met (e.g., Willis, 1995). It seems that both predictions have come true, although we focus on the latter here.

Governments have taken the standards seriously and initially invested heavily in literacy achievement for early years children with broad community support. Numeracy, which for all purposes at this level is used interchangeably with mathematics, received slightly lesser but still enthusiastic government and community support. Arising from this interest the relevant Victorian government department invited submissions for a large-scale research project, which was won by a team including these three authors. For this forum, there are two relevant motivations to consider: the motivation of the government department in soliciting the tenders and the motivation of the team who won the tender in pursuing their research goals.

For the government's part, they were keen to gather data to allow them to argue in competition against other sections of government for funding support for on-going professional development, school-based grants for co-ordination and resources, curriculum initiatives, and other support for teachers. Recognising that essentially different arms of government both within and outside education compete with each other for initiatives and support, the gathering of data which could justify support in one area as against another is central to the working of the bureaucracy. The government department was also keen to identify the content of professional development and resources which could be produced to support teaching.

From the researchers' perspectives, while the tender required us to serve the above goals, we had other aspirations. We sought insights that could describe student learning over time, perhaps identifying discontinuities that could be the focus of particular resource support, and articulating reasonable expectations for students in schools. We were also concerned about teachers and strategies for developing their confidence in teaching mathematics.

EARLY NUMERACY RESEARCH PROJECT

The Early Numeracy Research Project was initiated in Victoria following the success of the Early Literacy Research Project. The Early Literacy Research Project (Hill & Crévola, 1999) worked with 27 disadvantaged Victorian primary schools to bring about substantial improvements in early literacy outcomes. Part of this research involved the development of models and guidelines for teaching, assessment and additional support for young children learning to read. As a result of the research, Hill and Crévola offered a "general design for improving learning outcomes" (p. 122), which they believed had application in literacy, numeracy, and other curriculum areas.

The Early Numeracy Research Project (ENRP) was established in 1999 by the (then) Victorian Department of Education, with similar aims to those of the Early Literacy Research Project, but with a mathematics focus in the first three years of schooling.

The 35 trial schools were selected from approximately 400 schools who applied to participate. The trial schools were chosen to represent the range of Victorian schools, in terms of geographical location, socio-economic status, language background, school size and indigenous population. The 35 reference schools were matched from remaining Victorian primary schools, seeking the closest match taking into account the above variables.

The stated aims of the Early Numeracy Research Project were

- to assist schools to implement the key design elements as part of the school's numeracy program;
- to challenge teachers to explore their beliefs and understandings about how children develop their understanding of numeracy, and how this can be supported through the teaching program; and
- to evaluate the effect of the key design elements and the professional development program on student numeracy outcomes.

The central part of the data collection was an individual interview addressing nine domains of mathematics. In each domain, previous research was surveyed to identify key issues associated with developmental learning within those domains. These "growth points" were the basis of the quantitative scale used (Horne & Rowley, 2001). While a requirement of the data collection process, these growth points served a function of describing student thinking. It was intention of the research team to focus on student mathematical thinking and strategies. The nature of mathematical understanding was clearly implicit in the nature and form of the growth points (for more detailed discussion see Clarke, 2001). Essentially we believed that teachers can focus on a small number of big ideas in each of the domains and this can guide teaching, learning and assessment in way that can facilitate student learning. Ultimately we hoped that the description of the student achievement at this level would inform curriculum, both in scope and in sequence (see, for example, Sullivan, Clarke, Cheeseman, & Mulligan, 2001). The interview was used on 34 398 occasions with children during the ENRP, with 11 421 children being interviewed at least once. In addition 1152 and 868 interviews were conducted with Grade 3 and 4 children, giving a total of 36 426 interviews in the period 1999-2001.

The key recommendations from the student data overall were:

- The data considered individually and collectively highlighted the independence of the respective mathematics domains, and all domains should be part of the curriculum at each Grade level in the early years of schooling.
- The growth points seem to provide clear indications of standards for learning and targets for teaching at these levels, and teachers could use these to inform planning, teaching and assessment.
- By using the assessment interview, teachers can gain important insights into the learning of their individual children. The growth points then provide important information for teachers to allow them to adjust classroom experiences to address the particular learning needs of each student.
- In a number of the domains, barriers to student learning were identified. These were growth points that it seemed to take students significant time to achieve. Particular experiences should be planned by teachers to address the elements of the barrier explicitly.
- In each domain specific recommendations related to changes in the emphasis and specification of the Victorian were made. In some places serious deficiencies were identified.

The interview, of course, was only one part of a project. Indeed it could be argued that the most informative parts of the project are the intensive case studies conducted with both schools and individual teachers who appeared to have higher achievement of their students over the period of the research. Nevertheless the data from the interviews revealed important information about the sequence and rate at which young children learn those elements of numeracy that were part of the framework, and the results can be used to inform curriculum and resource development, and pre- and in-service teacher education.

We also conducted extensive professional development and gained significant insights into the way that such professional development changes teachers' beliefs and orientation to teaching, as discussed in the next section.

The impact of measuring and describing learning on the participating teachers

A range of data both qualitative and quantitative data was collected from the teachers. There were approximately 250 teachers involved in the professional development program throughout the three years of the project. Support provided to the teachers within the project can be considered as three "opportunities":

1. Opportunity to develop *knowledge of children's learning of mathematics* through a framework of key growth points and a task-based interview that provides insights into the range of strategies children understand and use, as well as an understanding of more sophisticated strategies that the children can be encouraged to work towards.
2. Opportunity to develop *knowledge of students* through one-to-one, mathematically-focused interaction during the interview.
3. Opportunities to explore a range of classroom strategies, in an environment of collegiality, to build upon children's mathematical understanding, through purposeful and informed teaching.

The ENRP growth points were designed to provide teachers with a framework to describe their children's learning in mathematics and the interview measured this learning. The perspective of the research team was of the teachers as co-researchers and teachers as having responsibility for the making of curriculum (Clarke, Clarke, & Sullivan, 1996).

It seems that the growth points provided not only a way to discuss what the children already know but the direction to move. Teachers in the ENRP had a clearer picture of the typical trajectories of student learning (Carpenter & Lehrer, 1999), and can recognise landmarks of understanding in individuals. Such a picture guided the decisions they make, in planning and in classroom interactions, as their knowledge of the understanding of individuals informs their practice. This increased knowledge in turn seemed to build the confidence of the teachers:

> I used to avoid maths when something had to be missed, now I can't find the time to do enough! I'm more confident and I enjoy it more. I'm more flexible and more responsive to the children. I have a better understanding of how children think and reason, due to the assessment. This has impacted on what/how I teach.
>
> The main thing that has changed is my confidence in my maths teaching, because I have more knowledge of how children learn maths, what they should know and some ideas of how to get them there. My lessons are more varied and fun now.
>
> It has given me a greater understanding of why and how I teach maths, therefore increasing my confidence in my maths teaching.

While ultimately the recommendations of the project reflect the statement of the growth points which in turn informed the assessment items, the findings can inform curriculum development, teacher education, and further research. The data were collected from large numbers of students over three years with a carefully selected, stratified sample, to ensure all sections of the community were represented, and can be taken as an accurate representation of the potential of all students to answer the questions asked at these levels. The teachers have been empowered through increased knowledge and a belief in their own teaching and we would argue –the freedom to make their own professional decisions. The project sees the provision of similar professional development for other teachers as a key recommendation for government action.

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LARGE-SCALE STUDIES IN MATHEMATICS EDUCATION: A SEARCH FOR IDENTITY

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International studies, such as TIMSS and PISA, consider how well different countries educate their students. These large-scale studies also highlight associations between student background characteristics and student performance. The important question of these studies is not the rating of the countries. The important questions had to do with attempting to discern what it is about the instructional practices, school policy, the curriculum, and the psycho-sociological environment in certain countries that result in higher levels of student achievement (Willms, 2001).

Along with international studies, many countries conduct also national surveys on students' mathematical achievements. The main purpose of these surveys is to provide insights into the factors that influence the development of mathematical knowledge, skills, and competencies, how these factors interact, and the implications of the findings for policy making. As part of this process, researchers seek to measure and describe the relationships between the "inputs" and the "outputs" as well as between the intended, implemented, and attained curriculum.

Who are the audience of large-scale studies? Whose interests they intend to serve, and how? Whose interests are indeed served, and how? By supporting a shift in policy focus from educational inputs to learning outcomes, large-scale studies aim to provide insights into the factors that contribute to the development of competencies and into how these factors operate. Undoubtedly, such studies seek to combine the P and the E in PME. By that, these studies intend to assist policy makers, as well as mathematics educators and the community in large.

Yet, the ambitious scope of these studies and their tremendous costs raise the issue of the extent to which large scale projects indeed play a part in the improvement of students' learning of mathematics. Repeatedly, PME researchers raise the question of the extent to which such projects can assist educational systems in developing students' mathematical competencies: How can the findings serve as a basis for changing classroom practice? How can they assist in developing mathematics curriculum? Or, how can they set up the infrastructure for effective professional development? The present study addresses some of these questions.

Do large scale projects play a part in the improvement of students' learning of mathematics?

Many think that research has no impact on policy: policy makers do not get assistance from researchers, and if they get it, it is often either too late, or too "theoretical", or too vague. Researchers, on the other hand, have the feelings that policy is not based on empirical findings. The frustration is expressed by many researchers. Husen (1968), for example, claimed that "in retrospective, I am completely aware of the failure of research to influence reforms in schools. One may ask if the policy would be different if the research would not be conducted on the first place." Ten years later, Malmquist and Grundin (in Mevarech & Blass, 1999) argued that "those who are responsible of the educational policy do not expect research to give answers to the ongoing problems that the educational system faces ... policy makers tend to consider research as a marginal activity." Recently, Robitaille et al (2000) summarize the impact of TIMSS by saying: "the results (of TIMSS) seem, at least until now, to have made a greater impact on the general public and in political circles through the mass media than they have on educators."

The impact of national and international studies on the mathematics education research community is also far from being significant. So far, large-scale studies have contributed very little to the development of new theories or to clarifying the "how's" questions regarding teaching, learning, or curriculum development. Gravemeijer (in Arcavi, 2000) explained: "the question with which the researchers are struggling is... how to design instructional activities that (a) link up the informal situated knowledge of the students and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (pp. 278-279). Maybe, large-scale studies are not appropriate for addressing such questions. Probably, studies that are based on qualitative methods or small scales are better able to answer this kind of questions than large-scale studies such as TIMSS, PISA, or national surveys. Why such ambitious studies have (so far) only small impact on policy, even though such studies are frequently cited in the media and often funded by institutions that make the policy? To answer this question we have to consider another series of questions: Whose agenda is the project? Who benefits from the project, and how: students, parents, teachers, society, Government, the mathematics education community? Are there any losers? Unfortunately, in many countries, Israel included, TIMSS findings are not on the agenda of the educational system. Even though the Ministry of Education funded the project, the educational system does not consider itself obliged to the findings. Several reasons may explain it. First, those who made the decisions felt threaten by the findings and thus defend themselves by ignoring it. Second, the immediate implications of the findings are not very clear – the studies did not identify the reasons for the low achievements, nor did they describe the factors that contribute to the good performance in high achieving countries. Third, ironic as it sounds, researchers may also be blamed (explicitly or implicitly) for the outcomes particularly in those countries where researchers are involved in curriculum development. Finally, given the assumption that there is a relationship between the intended, implemented, and attained curriculum, even teachers and principals may be criticized (implicitly or explicitly) for not being effective. Thus, even though students, parent, teachers, and the society in large can benefit from the study, the list of losers is not shorter.

There is, however, another option. The usefulness of large scale studies lies in identifying the factors that contribute to mathematics education. As indicated by Robittaille (2000): "educational systems can learn from one another. They can learn that different approaches to common concerns -- such as the streaming of students by some measure of ability are taken in different countries. They can study the relative success and efficacy of those different approaches and then make decisions about what might work in their setting." (p. 169). The following is an example that shows the usefulness of international studies beyond what can be achieved by any other kind of research.

Mathematics Education, Resource Allocation, and Equity Policy

Educational inequality remains a major challenge to policy makers as well as to researchers. There is a long debate in the educational community regarding the duality between excellency and equality in education. Whereas many believe that the role of society is to reduce inequality even if it may come on the expanse of excellency, others argue that the role of schools is to enable each student to fulfil his/her potential and attain high levels of achievements. To address the issue of the duality between excellency and equality, we investigated the relationship between mean achievement in mathematics and measures of inequality in mathematics performance across countries by using TIMSS-1999 data. Inequality in mathematics performance was defined as the gap in achievement between the 5th and 95th percentiles within each country. The analysis indicated that the correlation between mean achievement and inequality is -.75. In other words, countries in which mean performance is high the level of inequality is low and vice versa. Further analyses showed that countries that improved performance between 1995 and 1999 reduced inequality. It is interesting to note that analyses of economical data from 67 countries reveal similar findings regarding the strong negative relationships between growth and inequality. Additional analyses indicate that the strong negative correlation between achievement and inequality is not a function of investment. These findings led us to examine the contributions of social capital or human capital to excellency and inequality. (More information will be reported at the conference.)

This finding along with additional analyses explaining the factors that contribute to the high correlation reported above has clear indications for policy makers and for mathematics education community: schools can make a difference, curriculum can make a difference, and resource allocation can make a difference. This finding calls researchers in mathematics education to provide policy makers models that enhance mathematical performance along with reducing inequality. Examples of such models are those which utilize metacognitive instruction in mathematics education (e.g., Mevarech & Kramarski, 1997). To my knowledge, several countries have started to work in this direction.

Large-Scale Projects Looking for Identity

Arcavi (2000) distinguishes between two kinds of research in mathematics education: theory-driven and problem-driven. Within problem-driven research, he identifies three kinds of research: interesting/puzzling behaviors, a curriculum or a practice, and didactic opportunities. Large-scale projects do not fit any of these categories. It seems that a new category is suitable for large-scale studies. I suggest "checkup-driven research". (I prefer the term knowledge-driven research, but tend to believe that all research is knowledge driven.) By checkup-driven research I refer to a systematic inquiry into a subject, in our case -- mathematics education, designed to uncover pertinent information about mathematics education and the educational system in large. Checkup-driven research is more than comparing countries' curricula, or analyzing cost-effectiveness between and within countries. It is also more than explaining students' puzzling behaviors, didactic opportunities, or practices. Checkup-driven research has to provide analytical descriptions of the system, usually with respect to certain possible causal connections. [So far, TIMSS, PISA and national surveys provide reliable descriptive information, but their causality part is weak.] Conceiving large-scale studies as a checkup-driven research enables us to develop appropriate expectations of such studies: their main purpose is to provide ongoing evidence on the development of mathematics competencies, and to suggest follow-up research whenever a problem is discovered.

Like medical check-ups, also educational checkups have unique characteristics and methodologies that distinguish them from theory-driven or problem-driven research. Of these characteristics, perhaps the most important ones are those in which evaluation is viewed as a process of identifying and collecting information to assist decision-makers in planning knowledge-based policy, and researchers in understanding the macro picture of mathematics education. Since large-scale research is a relatively new kind of research, its main concern is now setting up an original framework, and developing its own concepts and methods that would satisfy three criteria: relevance to the observable phenomena, exhausitivity with our understanding of mathematical thinking, learning, and teaching, and consistency of the concepts and methodology within the theoretical framework. Large-scale research should include, therefore, the methodological requirements of sharing knowledge, reflections, and curriculum design. These comprise methodological and ethical questions and call for larger involvement of the mathematics education research community, and other academic communities at all stages of the research. Prior arrangements such as these may eventually lead to knowledge-based policy making and may serve as a springboard for changing curriculum, classroom practice, and even national policy in attempts to enhance mathematical education. These issues will be further elaborated in the forum.

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FROM CORE GOALS TO LEARNING-TEACHING TRAJECTORIES AS A GUIDE FOR TEACHING PRIMARY-SCHOOL MATHEMATICS IN THE NETHERLANDS

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This paper is dedicated to the Dutch TAL project. In this project learning-teaching trajectories for primary school mathematics are being developed. Regarding the title of this Research Forum, the focus of this paper is rather on "describing goals" than on "measuring learning". Different from an approach in which large-scale assessment projects are carried out to provide evidence for policy on developing mathematical competence, the TAL project can be characterized as a didactical-phenomenological approach in which the nature of mathematical learning content, its sequencing over the grades and the way it is taught in primary school, are reviewed and overhauled.

INTRODUCTION

The nineties can be seen as the decade of standards. In many countries, at governmental level, decisions are made about what schools should teach their students. Examples of these standards are the NCTM Standards in the United States, the National Curriculum and Numeracy Project in the United Kingdom, and the Learning Framework in Number in Australia. This paper deals with the Dutch chapter of this trend.

Compared to many other countries, the Netherlands do not have centralized decision making for the primary school mathematics curriculum (see Mullis et al., 1997). Nevertheless — or probably one should say thanks to this — the mathematical content taught in primary schools does not differ much between schools. In general, all teachers follow roughly the same curriculum. A very important reason for this is the existence of a qualitatively high-standing base of commercially published mathematics teaching methods. Even the implementation of the Dutch reform of mathematics education has in fact been achieved voluntarily through these methods. This rather informal implementation mechanism is more or less the result of the existence a strong community of researchers and developers of mathematics education in the Netherlands. This provides a strong driving and regulating force for reform.

THE CORE GOALS FOR PRIMARY-SCHOOL MATHEMATICS CURRICULUM

Until recently, there was no real interference from the Dutch government regarding the content of educational programs. There only was a general law containing a list of subjects to be taught. In 1993, however, government policy changed and the

Ministry of Education came up with a list of 23 attainment targets for the end of primary school[1], called "core goals". The goals are split into six domains, including general abilities, written algorithms, ratio and percentage, fractions, measurement, and geometry. Table 1 shows the goals for the first two domains.

Table 1: Part of the core goals for Dutch primary school students in mathematics

By the end of primary school, the students (at age twelve) ...		
General abilities	1	Can count forward and backward with changing units
	2	Can do addition tables and multiplication tables up to ten
	3	Can do easy mental-arithmetic problems in a quick way with insight in the operations
	4	Can estimate by determining the answer globally, also with fractions and decimals
	5	Have insight into the structure of whole numbers and the place-value system of decimals
	6	Can use the calculator with insight
	7	Can convert simple problems which are not presented in a mathematical way into a mathematical problem
Written algorithms	8	Can apply the standard algorithms, or variations of these, to the basic operations, of addition, subtraction, multiplication and division in simple context situations

Compared to goal descriptions and programs from other countries this list is a very simple one. It means that there is a lot of freedom in interpreting the goals. At the same time, however, such a list does not give much support for educational decision making.

In the years since 1993, there have been discussions about these core goals (see De Wit, 1997). Almost everybody agreed that they can never be sufficient to support improvements in classroom practice or to control the outcome of education.

LEARNING-TEACHING TRAJECTORIES — A NEW FACTOR FOR MACRO-DIDACTIC TRACKING

For several years it was unclear which direction would be chosen for improving the core goals: either providing a more detailed list of goals for each grade, expressed in operationalized terms, or a description which supports teaching rather than pure testing. In 1997, the Dutch Ministry of Education tentatively opted for the latter and asked the Freudenthal Institute to work out the description for mathematics. In September 1997, this decision resulted in the start of the TAL Project[2].

Aim of the TAL Project

The aim of the TAL Project is to develop learning-teaching trajectories for all domains of the primary-school mathematics curriculum. In total three learning-teaching trajectories will be developed: a trajectory for whole number calculation, one for measurement and geometry, and one for fractions, decimals and percentages.

The project started with the development of a learning-teaching trajectory for whole-number calculation. This first trajectory description for the lower grades (including K1, K2, and grades 1 and 2) was published in November 1998. The definitive version was released a year later. The following year the whole-number trajectory for the higher grades of primary school (including grades 3 through 6) was published. In 2001, both learning-teaching trajectories were translated in English and published together in one book (Van den Heuvel-Panhuizen, 2001).

In 1999, a start was made on the development of a learning-teaching trajectory for measurement and geometry. This will be finished in the end of 2002.

What is meant by a learning-teaching trajectory?

Giving the teachers a pointed overview of how children's mathematical understanding can develop from K1 through grade 6 and of how education can contribute to this development is the main purpose of this alternative to the traditional focus on strictly operationalized goals as the most powerful engine for enhancing classroom practice. In no way, however, is the trajectory meant as a recipe book. It is, rather, intended to provide teachers with a "*mental educational map*" which can help them, if necessary, to make adjustments to the textbook.

Although a learning-teaching trajectory puts the learning process in line, it should not be seen as a linear and singular step-by-step regime in which each step is necessarily and inexorably followed by the next one. A learning-teaching trajectory should be seen as being broader than a single track and should have a particular bandwidth. It should do justice to differences in learning processes between individual students and to the different levels at which children master particular skills and concepts.

A new educational phenomenon

Compared to the goal descriptions that were traditionally supposed to guide education and support educational decision making, the learning-teaching trajectory as it is worked out in the TAL Project has some new elements that makes it a new educational phenomenon.

First of all, the trajectory is more than an assembled collection of the attainment targets of all the different grades. Instead of a checklist of isolated abilities, the trajectory makes clear how the abilities are built up in connection with each other. It shows what is coming earlier and what is coming later. In other words, the most important characteristic of the learning-teaching trajectory is its *longitudinal perspective* which has a long history in the Dutch didactical (subject-matter connected) approach to mathematics education.

A second characteristic is its *double perspective of attainment targets and teaching methods*. The learning-teaching trajectory does not only describe the landmarks in student learning that can be recognized en route, but it also portrays the key activities in teaching that lead to these landmarks.

The third feature is its *inherent coherence, based on the distinction of levels*. The description makes it clear that what is learned in one stage, is understood and performed on a higher level in a following stage. A recurring pattern of interlocking transitions to a higher level forms the connecting element in the trajectory. It is this level characteristic of learning processes, which is also a constitutive element of the Dutch approach to mathematics education, that brings longitudinal coherence into the learning-teaching trajectory. Another crucial implication of this level characteristic is that students can understand something on different levels. In other words, they can work on the same problems without being on the same level of understanding. The distinction of levels in understanding, which can have different appearances for different sub-domains within the whole number strand, is very fruitful for working on the progress of children's understanding. It offers footholds for stimulating this progress.

The fourth attribute of the TAL learning-teaching trajectory is the new description format that has been chosen for it. The description is not a simple list of skills and insights to be achieved, nor a strict formulation of behavioral parameters that can be tested directly. Instead, *a sketchy and narrative description, completed with many examples*, of the continued development that takes place in the teaching-learning process is given.

Development of the TAL learning-teaching trajectories

In the development of the TAL learning-teaching trajectories "didactical phenomenological analyses" – as Freudenthal (1983) called them – play a crucial role. These analyses reveal what kind of mathematics is worthwhile to learn and which actual phenomena can offer possibilities to develop the intended mathematical knowledge and understanding. Important is that one tries to discover how students can come into contact with these phenomena, and how they appear to the students. This means that problems and problem situations that give students opportunities to develop insight in mathematical concepts and strategies must be identified. Therefore a team, containing all kinds of specialisms in primary school mathematics, has been formed. The group contains experience in research and development of mathematics education, assessment, teacher educating, teacher advice, and teaching mathematics in primary school. The core of the work is formed by the (almost) weekly discussions in the project team, for which input comes from a variety of sources: analyses of textbook series, analyses of research literature, investigations in classrooms, and extensive consultations of experts in mathematics education. An earlier example of such an approach, but aimed at finding the long term learning process for the domain of ratio, can be found in Streefland (1984/1985).

THE TAL TRAJECTORY FOR WHOLE NUMBER CALCULATION

In the TAL trajectory for calculation with whole numbers (see Figure 1), calculation is interpreted in a broad sense, including number knowledge, number sense, mental arithmetic, estimation and algorithms. The trajectory description gives an overview of how all these number elements are related to each other.

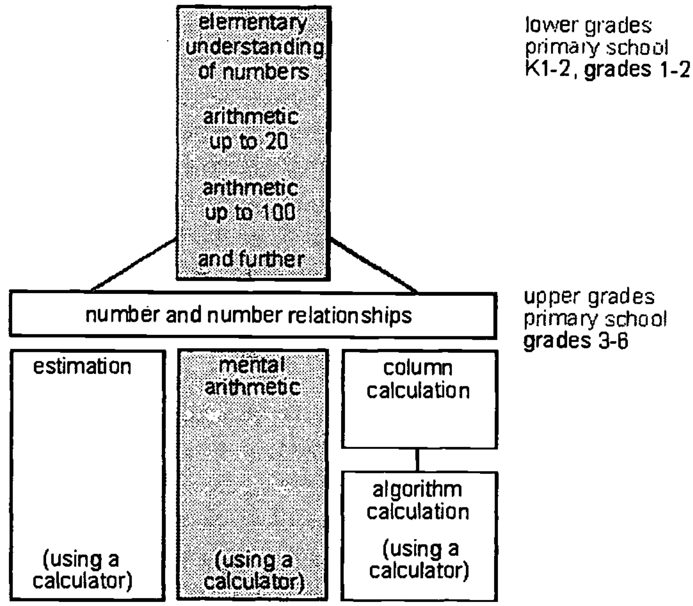

Figure 1: The TAL learning-teaching trajectory for whole number calculation in primary school.

The scheme reflects that the students gradually come from a non-differentiated way of counting-and-calculating to calculations in more specialized formats that fit particular kinds of problems in a particular number domain. Mental arithmetic is considered to play a central role in whole number calculation. It is seen as an elaboration of the arithmetic work that is rooted in the lower grades and forms the backbone in the upper grades.

ESTIMATION AS AN EXAMPLE

New in this trajectory is also the proposed didactics for estimation. Although estimation is now widely acknowledged as an important goal of mathematics education, in most textbooks a framework for how to learn to estimate is lacking. The textbooks at most only contain some problems on estimation, but doing some estimation problems from time to time is not enough to develop real understanding in how an estimation works, and it is not sufficient to comprehend what is 'allowed' and what is not when estimating.

At the Research Forum an overview will be presented of the TAL trajectory that has been developed for the domain of estimation. The proposed sequenced teaching structure with the intermediate attainment targets will be compared with the findings of the PPON, a large-scale Dutch assessment of students' performance in school subjects (Janssen et al., 1999).

Mathematics education researchers should play a part in the improvement of students' learning of mathematics. The question however is how? Providing empirical evidence about student performance is certainly important input, but will students' scores be sufficient to answer the core question about the goals te be achieved?

Notes

1. In the Netherlands, primary school is meant for students of ages 4 to 12 and includes eight grade classes. The first two are kindergarten classes.
2. The TAL Project is carried out by the Freudenthal Institute and the SLO (the Dutch Institute for Curriculum Development), in collaboration with CED (school advisory center for the city of Rotterdam). TAL is a Dutch abbreviation and stands for Intermediate Goals Annex Learning-Teaching Trajectories. Since the beginning of the project the following people have contributed to the development of the learning-teaching trajectory: Joop Bokhove (FI), Jan van den Brink (FI), Arlette Buter (FI/CED), Kees Buys (SLO), Nico Eigenhuis (CED), Erica de Goeij (FI), Marja van den Heuvel-Panhuizen (co-ordinator) (FI), Jan Hochstenbach (FI), Christien Janssen (FI), Julie Menne (FI), Ed de Moor (FI), Jo Nelissen (FI), Anneke Noteboom (FI), Markus Nijmeijer (FI), Adri Treffers (FI), Ans Veltman (FI), Jantina Verwaal (FI). In total the size of the TAL Team was equivalent to three fully employed persons.

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USING A LEARNING FRAMEWORK TO DOCUMENT STUDENTS' PROGRESS IN MATHEMATICS IN A LARGE SCHOOL SYSTEM

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The large school system referred to in this paper is the government school system in the state of New South Wales (NSW), Australia. The school system serves a population of approximately seven million people. In administrative terms, the school system is organised into 40 districts of roughly equal student populations. Approximately half of these districts serve part of the metropolitan area of Sydney and the other half serve regional and rural areas of the state. Almost all aspects of school administration and governance occur at the state rather than the district level.

The school system has approximately 1,700 schools with classes in the K-6 range. The age range of K-6 students is about 4 years 9 months to 12 years plus. In 1989, the government introduced a basic skills testing program (BSTP) in literacy and numeracy. All students in government primary schools in NSW in Years 3 and 5 are required to take these tests. As well, the tests are used by many of the non-government schools in NSW. The stated purpose of these tests is diagnostic. The tests are intended to provide information to teachers and schools about weaknesses in knowledge at individual, class or school levels. These tests are developed using Item Response Theory (IRT), have a predominantly multiple choice format and are computer-scored.

Assessment can have many different purposes and a range of audiences. The audience for the BSTP is parents, who receive individual reports of their child's performance, and teachers, who have access to the diagnostic information from the test — a pencil and paper, single-occasion assessment. The mandatory testing system just described is not the main focus of this paper. The description of the testing system is given to provide contextual background to the main focus of the paper, that is, a novel approach to student assessment that has been adopted by the school system. This approach involves interview-based assessments undertaken by class teachers, for the purpose of documenting students' knowledge and learning.

THE NEED FOR A SYSTEMIC INITIATIVE IN EARLY NUMBER

In 1995, the NSW school system recognised a need to undertake a systemic initiative in mathematics in the early years (Years K, 1 & 2). This initiative was intended to address several issues: (a) There were vast differences in the levels of mathematical knowledge of students at the Year 3 level, as indicated by the state-wide basic skills testing program. (b) Approaches to assessing students' mathematical knowledge in the K-2 range were problematic. (c) Some students (K-2) were not being sufficiently challenged in mathematics. (d) Teaching programs were not catering sufficiently for the range of student knowledge in mathematics. (e) By and large, teaching programs did not reflect particularly strong or useful theories of young children's mathematics learning or teaching. In the early 1990s, school systems in NSW and other states had undertaken significant initiatives in early literacy learning. These initiatives seemed to address issues in literacy, many of which are similar to those listed above for early mathematics. Thus at this time, the NSW school system was ready to focus attention on reviewing and overhauling the teaching of mathematics in the early years.

BACKGROUND AND OVERVIEW OF CMIT

Development of the CMIT initiative involved from the outset, collaboration between system leaders in mathematics on one hand, and university-based researchers in mathematics education on the other hand. An interview-based assessment had already been developed by researchers and had been used for example, to document the mathematical knowledge of school entrants (Wright, 1991) and the development of mathematical knowledge during the first two years of school (Wright, 1994a). As well, during 1992-5, this approach to assessment had been the basis of a major research program, the focus of which was teacher development in assessing and teaching mathematics in the early years of school (Wright, 1994b; 2000; Wright et al., 1996). Thus the initial development of CMIT involved adapting several key aspects of the earlier research program. This included (a) the assessment tasks; (b) the approach to assessment; (c) the theoretical orientation to learning; (d) explanatory models of student learning; and (e) approaches to teachers' learning (Wright et al., 2000; 2002).

The goal of the CMIT initiative is to develop teachers' professional knowledge of teaching and learning mathematics in the early years, and through this to increase their students' learning of mathematics (NSW Department of Education and Training, 2000). Within CMIT assessment provides the central impetus for teacher change. It is for this reason that the main audience for the assessment data is the classroom teacher. The assessment plays a pivotal role in providing information on what each student can do. Assessment in CMIT is viewed as the process of gathering information to provide direction for teaching. That is, assessment is not an endpoint but a starting point to step into the cycle of teaching and monitoring. If the purpose of the assessment is to provide a lens for teachers' practice then the teacher must trust and own the data. The one-to-one assessment interview is used not only to obtain information as to what students can do. Rather, the teacher carries out assessment in order to learn more about their students' thinking. In this way teachers develop a very strong sense of ownership of the assessment data.

The approach to teachers' learning is school-based, problem-based and team-based. In each of the 40 districts a position of K-8 numeracy coordinator was created. One of the key responsibilities of the numeracy coordinator is the support of CMIT in the schools of their district. Typically this involves supporting an implementation of CMIT in 8-10 of the districts schools, each year. The introduction of CMIT was both an outside-in and inside-out process. As Fullan (1999, p.62) notes, 'Two-way inside-outside reciprocity is the elusive key to large-scale reform'. The numeracy coordinator provides 'at the elbow' consultancy support in classrooms working as a learning partner to teachers in the school teams. The classroom becomes the place of learning for both students and teachers. Teachers learn more about how students think and use that information to adjust the planned instruction for students. Often, the desire to participate in the program originates in the school. Over a period of 6-8 years, almost all schools will implement the initiative.

The resource materials for CMIT took the form of a professional development package which provides a basis for the teachers' learning. Key elements of the professional development package are the Learning Framework in Number (LFIN) and the Schedule for Early Number Assessment (SENA). These are described in the next two sections. Development of these resources drew on current research and in particular the constructivist teaching experiment methodology and results of Steffe and colleagues (Steffe & Cobb, 1988) and investigations by Wright (1994a, 2000).

THE LEARING FRAMEWORK IN NUMBER (LFIN)

The LFIN (Wright, 1998; Wright et al., 2000) provides several key ingredients for the CMIT initiative. First, it provides a general orientation to young children's learning of number. This includes highlighting the significant mathematics content to be learned, describing instructional contexts and tasks for assessment and teaching, and detailing the strategies students use to solve tasks which are problematic for them. The LFIN describes a range of significant aspects of students' learning of early number. These include strategies students use in counting, additive and subtractive situations, students' facility with number words and numerals, students' knowledge of simple addition and subtraction combinations (number facts, number bonds), early multiplication and division strategies (Mulligan & Wright, 2000), ability to think about multi-digit numbers in terms of tens and ones, and use of finger patterns and spatial patterns in numerical contexts. As well, the LFIN provides an indication of directionality in student learning. The teacher using the LFIN should be able to determine the extent of the students' learning in various aspects of LFIN, likely progressions in the students' learning, and instructional situations based on the framework which are likely to lead to those progressions. LFIN can be regarded as a rich description of children's early number knowledge. The term 'knowledge' is used here in an all-encompassing sense — everything the child knows about early number, and includes but is not limited to the child's arithmetical strategies.

Another key feature of the LFIN is that it enables profiling of students' knowledge across various aspects of early number learning. This feature involves writing summary descriptions of aspects of early number learning in tabular forms. Aspects of early number learning for which tabular forms are used include early arithmetical strategies, that is strategies for counting, adding and subtracting, early multiplication and division strategies, knowledge of forward and backward number word sequences, numeral identification, and knowledge of tens and ones. These so-called models of aspects of early number learning can be used to document in summary form, students' current knowledge and progressions of knowledge over time of large numbers of students (Wright et al., 2000; 2002).

THE APPROACH TO ASSESSMENT

In CMIT, individualised student assessment is regarded as a key initial part of teachers' learning (Bobis & Gould, 1998). Teachers learn to administer a schedule of assessment tasks, that is the Schedule of Early Number Assessment (SENA). This involves a one-to-one interview during which the teachers' goal is to elicit as much information as possible about the students' arithmetical knowledge and strategies. Reviewing of the interviews involves using a standard procedure to complete an assessment record sheet, which details the child's responses and strategies, and leads to determination of the student's levels. The assessment process results in three levels of assessment information relating to the students' early number knowledge. First, and most detailed is the videotape record of the assessment interview; second is the completed assessment record sheet which contains extensive written information about the child's responses and strategies. Third, is a table showing the levels of the student's knowledge in terms of four or five of the key aspects of LFIN. The latter constitutes a summarising profile of the students' knowledge and is particularly useful when documenting the knowledge of large numbers of students and making comparisons in students' knowledge over time (Bobis & Gould, 1999).

The CMIT assessment procedure recognises that children frequently use strategies that are less sophisticated than those of which they are capable. This may happen for one or more of several reasons, eg (a) it may be easier and although it may take more time, this may not be of concern to the child; or (b) some feature of the child's thinking immediately prior to solving the current task may focus the child's attention on a less sophisticated strategy. Thus an important challenge for the teacher as observer and diagnostician is to attempt to elicit the child's most sophisticated strategy. This is crucial to gaining a powerful understanding of the leading edge of the child's current knowledge. Viewing assessment and teaching through the lens of the learning framework helps to mediate its adoption by classroom teachers. 'Learning about the Learning framework: I can see...understand where individuals fit in, and therefore what I need to teach them in more detail and more specifically than I've learnt from using the curriculum' (Kindergarten teacher, 16-20 years experience).

DISCUSSION

The approach to documenting student learning that is described above can and should be considered in its broader context. CMIT is a program of teacher development and systemic change that seems to be effective. The program is well-regarded by participating teachers and schools, and also by external observers (Bobis & Gould, 1998; 1999). As well, there are good indications that the program has positively influenced the number learning of vast numbers of students (White & Mitchelmore, 2002).

The novel approach to assessment described above is one of the key elements of the initiative. Assessment in CMIT focuses on more than simply 'right or wrong'. The psychological models that are integral to the learning framework determine the nature of the assessment. The methods of solution children use are important to determine the child's location within the models. The nature of the models also provides a sense of direction for instruction. We agree with Simon (1995) in that our conceptual framework with respect to mathematics teaching includes an emphasis on teachers' inquiry into the nature of students' understandings so they can pose tasks in which those understandings might be challenged and extended.

CMIT does not involve prescribing teaching programs and lessons for teachers. Rather, it provides a school-based context for teachers to review their professional practice. When a team of teachers in a given school participate in the processes associated with CMIT, in a very significant way, they invent or construct the program. CMIT recognises that there is not a single mathematics curriculum (designed, delivered, attained, assessed). Rather, the program works on aligning the assessed curriculum, the implemented curriculum, and the attained curriculum. In overall terms, CMIT is a learning community that: (a) includes students, teachers, numeracy coordinators, system leaders and researchers; and (b) incorporates Senge's (1990) five basic learning disciplines — systems thinking, personal mastery, mental models, building a shared vision and team learning.

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RESEARCH FORUM 4

Theme

From number patterns to number theory: issues in research and pedagogy

Coordinators

Rina Zazkis and Stephen Campbell

Contributors

Rina Zazkis and Peter Liljedahl, Nathalie Sinclair, John Mason, Tim Rowland

RESEARCH FORUM 4

Theme

From Number Patterns to Number Theory:
Issues in Research and Pedagogy

Coordinators

Rina Zazkis and Stephen Campbell

Session 1

- Introductions and opening remarks
- Short presentations (all interspersed with questions/answers, problems & discussion)
  "Making a case for number theory" by Rina Zazkis and Stephen Campbell
  "What is number theory? (Part 1)" by Stephen Campbell
  "Repeating patterns as a gateway?" by Rina Zazkis and Peter Liljedahl
  "For the beauty of number theory" by Nathalie Sinclair
- Plenary discussion
- Homework assignment (optional)

Session 2

- Short presentations (all interspersed with questions/answers, problems & discussion)
  "What makes an example exemplary? Pedagogical & research issues in transitions from numbers to number theory" by John Mason
  "Proofs in number theory: History and heresy" by Tim Rowland
  "What is number theory? (Part 2)" by Stephen Campbell
- Checking the homework
- Plenary discussion
- Concluding remarks

MAKING A CASE FOR NUMBER THEORY

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Simon Fraser University

Stephen R. Campbell
University of California, Irvine

This research forum reflects on what number theory is, why it is relevant and important, how it connects with and builds upon number patterns, and with other research pertinent to this area (e.g., Campbell & Zazkis, 2002). Number theory is usually considered as a generalisation of whole number arithmetic or as a whole number specialised adjunct of algebra (Campbell, 2001). As such, it is not considered as a topic of study in K-12 mathematics education, in and of itself. In North America, for instance, the limited profile that number theory enjoyed in the NCTM *Standards* (1989) has been diminished in the NCTM *Principles and Standards* (2000). Nevertheless, introductory problems from elementary number theory typically abide and can be found sprinkled about various K-12 curricula around the world. On the bright side, there has been an increasing emphasis on using problems from number theory with respect to mathematical reasoning and proof in the middle grades, despite usually being presented under the guise of arithmetic or algebra.

The development and continual refinement of the number system is one of the great, and arguably the most important accomplishment in the history of mathematics. Yet, considering the whole numbers as a proper subset of the real numbers is a relatively recent innovation, and one that has evidently become widely incorporated in mathematics education around the world. There can be no denying the importance of rational and real numbers, practically and theoretically, in teaching and understanding arithmetic, algebra, and especially the calculus. Making a sound case for number theory, however, presupposes that one carefully discerns the integers and whole numbers from rational and real numbers, along with their respective properties, relations, and operations. Only in this way can number theory be properly and justifiably distinguished from and related to other topics in the mathematics curriculum.

A case for number theory can be built, and critiqued, from several different perspectives: its formal mathematical nature, particularly with respect to the additive and multiplicative structure of the integers and whole numbers, its profound beauty and mystique, its utility (or perceived lack thereof) and accessibility, not to mention its contribution to the history and philosophy of mathematics. We hope to touch on all these perspectives, and perhaps others, through the course of this forum. After a brief welcome and introduction to the goals and participants, we first consider the question of what number theory actually is. Campbell treats this issue by identifying some of the main concepts involved, illustrated through a classification schema for problems that comprise number theory in the K-16+ curriculum.

A natural access to number theory is provided through the exploration of patterns. Such exploration, which is typically practiced in elementary years, should not stop there. Zazkis and Liljedahl suggest that a more systematic approach to number patterns may serve as a gateway for introducing concepts and relations pertaining to number theory. They discuss trends and difficulties that surfaced in preservice elementary teachers' work with repeating patterns, connecting them with themes identified as troublesome in previous research. In rounding out the first session of this research forum, Sinclair makes a case for beauty, or aesthetic experience in mathematics education. She presents a model for describing the evaluative, generative, and motivational character of aesthetic activity, and illustrates various ways in which it applies to number theory.

It has become customary in mathematics education to argue for teaching the practical utility of mathematics through "real life" applications. We recognise the "practicality" of number theory in more advanced areas of application, such as cryptology, and that elementary number theory does not readily fit in the realm of applicability of "daily life." Knowledge of number theory will not help in calculating taxes or balancing checkbooks. We consider the utility of number theory from a different perspective – its utility for teaching and learning *mathematics*.

Topics from number theory, such as factors and multiples, provide natural avenues for developing mathematical thinking, for developing enriched appreciation and understanding of numerical structure, especially with respect to identifying and formulating conjectures, and establishing their truth. Applying his method of noticing to the question as to how certain task-exercises can lead to theoretical insight, Mason opens the second session by exploring how learners can make transitions from numbers to number theory. Continuing on this theme into more advanced levels, drawing on famous examples such as Gauss's method and Wilson's theorem, Rowland illustrates ways in which generic examples in number theory provide exemplary means for teaching and learning proof. Campbell brings the forum full circle by revisiting the question as to what number theory is by analysing why it is distinct from arithmetic and algebra, and how it can readily be treated as such in the early and middle grades.

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AUDIENCE PARTICIPATION ENCOURAGED AND FACILITATED THROUGHOUT BOTH SESSIONS

WHAT IS NUMBER THEORY?

Stephen R. Campbell

University of California, Irvine

In making a case for number theory, it is helpful to consider various ways in which number theory can be defined and, what is related but not necessarily the same thing, to various ways in which it can be understood. In this paper both are taken to be involved in addressing the question as to what number theory *is*. One way to begin is to consider the kinds of problems involved. A problem classification scheme is introduced to provide further granularity in this regard. Number theory is then considered with respect to the field axioms implicitly underlying the K-12 curriculum in arithmetic and algebra. Whole number division with remainder, as a result of this analysis, is identified as *the* key factor in understanding number theory. This result supports evidence suggesting that understanding number theory is enhanced by quotitive dispositions toward division with remainder, and otherwise diminished by partitive dispositions (Campbell, 2002).

Defining Number Theory: Problem Classification

Number theory in K-12 and teacher education programs is typically restricted to introductory topics from elementary number theory. Such topics include the study of multiples, factors, and divisors, divisibility, divisibility rules, least common multiples (lcm), greatest common divisors (gcd), prime factorisation, prime and composite numbers, prime powers, relatively prime numbers, linear sequences, and so on. More substantive topics such as congruence relations, continued fractions, quadratic residues, Diophantine equations, and so on, are typically found in undergraduate number theory courses offered by departments of mathematics.

Most of these topics, along with the many problems in number theory that they readily give rise to, can roughly be grouped into four main problem classes: multiplicative, additive, linear, and non-linear. Although many problems fall clearly into one class or another, these classes are not disjoint. No claim is made or presumed regarding the comprehensiveness, uniqueness, or finality of this classification.

Multiplicative problems typically derive from the *fundamental theorem of arithmetic*, which asserts that every integer greater than one can be uniquely expressed, independently of order, as a product of prime numbers. Multiplicative problems presuppose divisibility properties of integers, and are often posed in those terms. Problems that require the use, identification, or derivation of various factors, multiples, and divisors are multiplicative. The following are examples of introductory multiplicative problems from elementary number theory:

Consider $M = 3^3 \times 5^2 \times 7$. Is M divisible by 2? 3? 5? 63? (Zazkis & Campbell, 1996a)

Find a number with exactly 13 factors (Mason, this volume).

Determining the number of factors (or divisors) of the integers provides a paradigm case for multiplicative problems in elementary number theory. For an illustration of how this problem can be implemented in the classroom, see Teppo (2002).

Additive problems are typically concerned with determining sums and/or sequences of summands of integers. A classic example here is Goldbach's conjecture that all even numbers greater than four can be expressed as the sum of two primes. Mason (this volume), for instance, invites us to attend to the patterns in the following series of equations involving sequential sums:

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

Other well known additive problems in elementary number theory include determining which numbers can be represented as sums of consecutive numbers, or consecutive odd numbers, or sums of squares, and so forth.

Linear problems are exemplified by the "Division Algorithm" and the "Euclidean Algorithm." These two algorithms are not just methods for dividing integers and determining greatest common divisors respectively, they are two of the most fundamental theorems of number theory. Both of these theorems are characterised by a linear structure combining multiplication and addition. Linear sequences of integers also fall under this category. Zazkis and Liljedahl (this volume) focus on repeating number patterns and their classic relation to linear sequences.

Non-linear problems typically involve non-linear equations in one or more variables such as the classic problem of determining Pythagorean triples, namely, integer solutions to the non-linear equation: $x^2 + y^2 = z^2$. The generalisation of this problem, and the Pell equation, $x^2 - dy^2 = 1$, provide other good examples of classic non-linear problems of elementary number theory, even though in many cases more advanced non-integral methods are required to solve them. Determining integral solutions to non-linear equations are usually referred to as Diophantine problems, in honour of the Hellenic mathematician, Diophantos of Alexandria (fl. ~250 AD).

Although there are many important problems in elementary number theory that can readily be identified with one or another of these four classes, there are also problems that blur the boundaries. Consider Gauss's method for determining the sum of the first 100 non-zero integers (Rowland, this volume). The formulation of this problem is ostensibly additive, but the solution is a non-linear integer function (i.e., $[N(N+1)]/2$).

Aside from linear and non-linear problems, there are many others with combined emphases placed on addition and multiplication. Determining perfect numbers, in which divisors must be determined and also summed, provides a case in point. Other problems involving base representation, divisibility rules, distributivity, and congruence also place combined and deeply interrelated emphases on these two operations.

Defining Number Theory: Fields and Rings

With a sense of the kinds of problems to be found in number theory, let us turn now to some axiomatic considerations that distinguish number theory from the underlying assumptions upon which K-12 arithmetic and algebra are based. The set of integers constitute an algebraic structure called a *ring*, whereas sets such as the rational and real numbers constitute algebraic structures called a *field*. It is helpful in defining and understanding number theory to consider differences between these formal structures.

A field is a (not necessarily numerical) set F with two operations, called *addition* and *multiplication*, which satisfy a collection of axioms, known as *field axioms*. Field axioms can be separated into axioms for addition, axioms for multiplication, and the distributive axiom (i.e., distributive law) relating these two operations (Rudin, 1976).

The field axioms for addition and multiplication define the properties of closure, commutativity, and associativity for these operations. They also include axioms establishing the existence of operational identities and inverses. Uniqueness properties of operational identities (i.e., 0 and 1) and inverses (i.e., $-x$ and $1/x$, such that $x + (-x) = 0$, and $x(1/x) = 1$) can readily be deduced from these field axioms. The existence of unique operational inverses for all elements in the field (with, of course, the notorious exception of zero for multiplication) provides the logical basis for defining the operations of subtraction and division in terms of addition and multiplication respectively (i.e., $y - x \equiv y + (-x)$, and $y \div x \equiv y(1/x)$).

A (commutative) ring is a (not necessarily numerical) set R that conforms to all of the field axioms with the crucial exception of the axiom establishing the existence of operational inverses for multiplication. In a ring such as the integers, without multiplicative inverses, division simply cannot be defined or understood in the same sense as that operation is in a field.

This may not come as a surprise as there is another, more familiar way of reaching basically the same conclusion. It is well known that division as defined by the multiplicative field axioms suffers a lack of operational closure over the integers, when integers are considered independently as a subset of the rational or real numbers (e.g., there is no integral solution for $7 \div 5$). What *may* be surprising is that the lack of closure here is *not* due to the fact that some integers do not have multiplicative inverses. Rather, it is because no multiplicative inverses exist in the integers whatsoever. The apparent "exceptions" are a result of the divisibility property of the integers (e.g., $8 \div 4 = 2$ because $8 = 2(4)$), they are *not* (partitive intuitions aside) due to multiplication by a multiplicative inverse (e.g., $8(1/4) = 2$)).

The above considerations have not been raised in some covert "neo-new math" attempt to resurrect formalisms back into the K-12 curriculum. Rather they have been raised to emphasise, contrary to popular opinion, that division with integers and whole numbers is *fundamentally different* than rational number or real number division. There is evidence to suggest that when the two are conflated, trouble follows (Campbell, 2002; 2001).

Understanding Number Theory

This brief reflection on axioms defining rings and fields clearly and strongly indicates that understanding number theory in relation to and in contrast with arithmetic and algebra primarily involves questions pertaining to division. More specifically, this means, in teaching and learning number theory, that close consideration be given to understanding terms, procedures, and concepts pertaining to division with remainder (Campbell, 2002; Zazkis, 1998), divisibility (Brown, Thomas, & Tolias, 2002; Zazkis & Campbell, 1996a), and prime decomposition (Zazkis & Campbell, 1996b).

There are both reasons and evidence to suggest that when students have trouble understanding number theory, it is often because they are thinking of division with remainder (i.e., integer and whole number division) in terms of rational or real number division. Campbell (2002) noted that whereas 10 out of 10 students with a partitive disposition toward division were unsuccessful in conducting division with remainder using a calculator, 7 out of 8 students with a quotitive disposition toward division *were* successful. Campbell suggested those observations could be accounted for by structural similarities between the division algorithm (i.e., $A = QD + R$, where $0 \leq R < D$), which serves to define whole number and integer division, and the quotitive model of division.

These structural similarities alone indicate the quotitive model would be a much more appropriate model for teaching and learning division with remainder than the partitive model. These considerations, of course, take nothing away from, and can only complement the effectiveness of using the quotitive model for teaching and learning division with fractions. Partitive dispositions towards division, on the other hand, insofar as they allow for the possibility of non-integral quotients, seem at the very least to interfere with, if not completely undermine, students' understanding of number theory

Even in the limited case of divisibility, where one might be tempted to think that it applies, the partitive model seems quite at odds with the multiplicative structures upon which the concept of divisibility is based (i.e., for any non-zero integer D, D divides A if and only if there exists an integer Q such that $A = QD$). Partitive dispositions, at least insofar as they entail a whole number divisor, are more appropriately applied to and should be reserved for teaching and learning rational and real number division, not number theory.

Concluding remarks

Number theory (qua *arithmos*) dates back to the emergence of mathematics as a formal conceptual discipline (Campbell, 1999), and there are grounds to suggest that history can inform the psychology of mathematics education (Campbell, 2001). Logical and empirical grounds provided here further suggest that number theory can, and should be treated as a distinct conceptual field in the teaching and learning of mathematics. There are important ways in which number theory is formally and conceptually distinct from arithmetic and algebra, especially with respect to division.

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Zazkis, R., & Liljedahl, P. (this volume). *Repeating patterns as a gateway.*

REPEATING PATTERNS AS A GATEWAY

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Consider the following problem:

> *Imagine a toy train, in which the first car is red, the second is blue, the third is yellow, the forth is red, the fifth is blue, the sixth is yellow and the same pattern repeats for all the cars. What is the color of the 100th car? If the train has 200 cars, what is the number of the last yellow car?*

Even very young children can engage in activity of continuing with this pattern. They can make a kids-train and declare or pick a label identifying the color. At an early age child's ability to continue the sequence can rely on recursive observations, that is an ability to relate items to adjacent items (such as blue after red, red after yellow, etc.), as well as on a "rhythmic" approach in memorizing the unit of repeat.

Work with patterns is justified in helping acquire mathematical reasoning that is important to learning – as a context for generalization, as a conceptual stepping stone to algebra, as a context for recognition, conjecturing and communication of rules (Threlfall, 1999). However, in order to achieve this relevance it is essential to develop a perception of the unit of repeat in a repeating pattern. Only then can one attend to the question *"What is the color of the 100th car"*. We further suggest that repeating patterns provide a vehicle for directing student's attention to the multiplicative structure of natural numbers, and in such provide a gateway to introducing the concepts of number theory.

So, what is the color of the 100th car? This may be too challenging for a very young child. Let's think, in Polya's tradition, of a similar but simpler problem. What is the color of the 15th car? While young children will play out the sequence explicitly, older ones may start paying attention to the "unit of repeat". Another way to simplify the problem is to consider a two colour train. This may present a wonderful, and for some learners the first, opportunity to consider even and odd numbers. Extending the unit of repeat, that is, the number of colours in a repeating pattern, could introduce or foster a concept of a multiple in elementary school years.

Now let's imagine a 1000 cars toy train in a 7-colours repeating pattern (red, orange, yellow, green, blue, purple, white) and consider the color of the 800th car. A number theoretical analysis of the problem provides a systematic means for predicting the color of any car in the sequence. Strategies for determining such number patterns rely on introductory concepts of number theory, such as factors, multiples, and divisibility. A systematic solution may rely on division with remainder: 800 leaves a remainder of 2 in division by 7, therefore the 800th car is orange. An alternative strategy is to "count up from a multiple": every 7th car is white, therefore the 798th car is white, the 799th is red and subsequently the 800th car is orange. In what follows we describe the themes that emerged in analyzing the solutions to this problem in a group of preservice elementary school teachers and connect them to the findings of prior research on the understanding of concepts and relations underlying elementary number theory.

Multiplication and division, multiplication and addition

The strategy of "counting up from a multiple" was preferred. Even students who recognize and confidently implement both strategies are often unable to describe the connection, that is, to consider "remainder" as a distance from a multiple. Remainder is perceived as one of the numbers you get in performing division with remainder (Liljedahl & Zazkis, 2001). Participants' lack of connection between remainder and "distance from a multiple" is a particular explication of a lack of a more general connection—connection between division with remainder and multiplication. This issue is discussed in detail by Campbell (2002). A very illustrative example is presented in a request to determine quotient and reminder in division by 6 of the number A, where $A=147\times6+1$. Fifteen out of 21 participants in Campbell's study calculated the dividend A and used a long division algorithm in order to answer this question. Generalizing further, a fragile connection between multiplication and division with remainder could be seen as a manifestation of a "weak link" between multiplication and division in general. Consider for example the question of divisibility of M by 7, where $M=3^3\times5^2\times7$. As reported by Zazkis and Campbell (1996), it was not uncommon to calculate the value of M and then divide it by 7 in order to conclude divisibility.

In applying "counting up from a multiple" strategy, it was common not to focus on the closest multiple, but rather on a "convenient" multiple. For example, a student could start "counting up" from 700. However, rather than attending to 770, for example, as the next possible benchmark for a "easy" multiple, a students would engage in a "long" sequence of 707, 714, 721... untill 798 was eventually reached. We see in this strategy a strong preference towards addition, rather than multiplication. This can also be explained as incomplete understanding of the fact that repeated adding of 7 is equivalent to adding a multiple of 7. The theme of additive dispositions has repeatedly appeared in prior research (e.g. Brown, Thomas and Tolias, 2001).

Divisibility and division with remainder

The remainder strategy may appear as advantageous for a "mature" mathematical thinker. However, we observed instances of correct algorithmic applications of the strategy, without an understanding of "why it works". "Counting up from a multiple" strategy, though it may appear as less sophisticated, entails in it an important underlying idea of the number structure: the idea that "every seventh number is divisible by 7", or, in general, that "every *n*th number is divisible by *n*". Prior research shows that this property is not among the properties that students take for granted. For example, participants in a study of Zazkis and Campbell (1996) had difficulty in deciding whether there is a number divisible by 7 in a given interval of ten (they were asked to consider numbers between 12358 and 12368) without explicitly finding such a number in this interval. Furthermore, after correct calculation of remainder in division of 12358 by 7, not everyone could determine, without calculation, what would be the remainder in division by 7 of 12359.

In mathematics the idea of "every *n*th number is divisible by *n*" is naturally extended to the partition of natural numbers into disjoint sets by the relation of congruency modulo n. (That is, elements in each set of such partition are congruent to each other modulo n). This idea is expressed in terms of our pattern as "every seventh car is white". However, depending on where the count begins, it is equally true that every seventh car is blue and every seventh car is orange. While this may appear as "trivial" when considering cars, the mathematical manifestation of this property – that, for example, numbers leaving a reminder of 5 in division by 7 are "7 apart" on the number line – was not applied naturally by some participants. For example, having identified the color of a specific car as red, a participant had to count up in order to decide what would be the number of the next red car.

The ideas of partitioning of natural numbers have been explored further in the study that focused on arithmetic sequences (Zazkis & Liljedahl, 2002). In one of the interview questions students were asked to consider whether a given (large) number was an element of a given arithmetic sequence. Our analysis made a distinction between sequences of multiples (e.g. 3,6,9,12, ...) and sequences of so-called "non-multiples" (e.g. 2,5,8,11, ...). While the divisibility of a number by d (d is the common difference) gave a clear indication of belonging to a sequence of multiples, the lack of divisibility by d left students uncertain about its membership in a sequence of "non-multiples". This is another indication that extending the property of "every *n*th" from division to division with remainder should not be taken for granted.

Notable misgeneralizations

In previous sections we described students' approaches that may be seen as inelegant or mathematically unsophisticated, but at least they weren't wrong. In this section we describe several reoccurring mistakes related to recognition of numerical patterns.

- A notable misapplication of the above mentioned property "every 7th number is divisible by 7" appeared when students assigned to a car, which number was divisible by 7, the color "red", the color of the first car, rather than "white" - the color of the seventh car. These students noted the property of "every seventh", but applied it starting with the first, rather than the seventh, element (Liljedahl & Zazkis, 2001).
- Another improper generalization repeatedly appeared when participants generated the property of the *k*th attribute (blue) to "every *k*th" car, without attention to the unit of repeat. A claim such as "Since the blue car is in the 5th position, all the multiples of 5 will be blue" illustrates this approach.
- For a number of students that clearly and correctly identified that multiples of 7 are the key to solving the pattern, the ways to determine such multiples were improperly generalized. We witnessed claims that "the 137th car is white

because 137 is divisible by 7 as it ends in 7" and also "151 is divisible by 7 because of the sum of the digits" - which are based on overgeneralization of familiar divisibility rules for 5 and 3. Though these claims were infrequent, they confirmed appearance of similar improper generalization reported in prior research (Zazkis & Campbell, 1996).

- Consideration of a multiples of d (d is the common difference) is the key in situations of determining membership and generating elements in arithmetic sequences of multiples (described above). However, this strategy has been improperly extended in consideration of sequences of non-multiples as well. A claim such as "700 is an element in the sequence 8,15, 22, ... because 7 is a factor of 700" illustrates this extension. Similar phenomenon, referred to as "difference product" or "direct proportion", was observed by researchers investigating middle school students generalization of repeating patterns (Orton and Orton, 1999, Stacey, 1989).

Conclusion

There is a recent trend in mathematics education to make mathematics "relevant" to the students by presenting it in "context". "Relevance" and "context" are often interpreted as activities related to "everyday" or "real" life. In contrast, it is argued that numbers themselves offer a "context" for investigation and for rich background of ideas and experiences. We agree with Nemirovsky (1996) that "real contexts are to be found in the experience of the problem solvers" (p. 313), rather than in formulation of the problem.

Focusing on patterns is advocated in the research literature as a stepping-stone in the generalization approach to algebra (Lee, 1996). In particular, Threlfall (1999) argues that linear, or one-dimensional, repeating patterns represent the first step towards number patterns in algebra. Extending these claims, we suggest that attending to number patterns is also a stepping-stone to number theory. We believe that consideration of repeating patterns can either introduce or enhance concepts and relationships underlying elementary number theory, and especially the multiplicative structure of natural numbers.

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FOR THE BEAUTY OF NUMBER THEORY

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Imagine if—instead of following the established formal mathematical structures that dictate the basic guidelines of curriculum—we based our choices on more aesthetic aspects: which mathematics will students find appealing, wonderful, surprising and intrinsically satisfying? Why might we do this? Which topics would we choose? In this short article I will outline some of the potential aesthetic aspects of elementary number theory and thus provide additional support to several researchers' recommendations for its increased curricular emphasis (Campbell and Zazkis, 2002).

Introduction

There are no shortage of quotes, such as Bertrand Russell's assertion that mathematics possesses a "supreme beauty... capable of a stern perfection such as only the greatest art can show" (1917, p. 57), in which mathematicians profess the beauty and elegance of their discipline. Many mathematicians have also claimed that the aesthetic plays an important if not necessary role in mathematical activity, a claim that has had curiously little impact on mathematics education research—partly, perhaps, because we know so little about the roles of the aesthetic in the production and appreciation of mathematical knowledge.

Given its low profile—and even, at times, its outright dismissal—in the mathematics *education* research community (as opposed to the professional mathematics community), why should the aesthetic play any part in determining curricular and educational goals? There are three different though related reasons that I have formulated: a) the aesthetic should play a major role in mathematics education because it plays a fundamental role in research mathematics; b) the aesthetic should play a major role in mathematics education because aesthetic ways of thinking and knowing are central to learning and meaning-making; and c) the aesthetic should play a major role in mathematics education because providing students with access to aesthetic experiences—that is, intrinsically satisfying experiences of mathematics inquiry—is a basic goal of education. [1]

Since I adopt a broad interpretation of the aesthetic that extends beyond restrictions to responses to objects of 'art,' and that encompasses other domains of inquiry through which humans attempt to make their experiences meaningful and 'fitting,' I begin by elaborating a model of the roles of the aesthetic in mathematical activity.

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[1] In the following section, I provide warrants for the first reason. I have provided arguments for the second reason in Sinclair (2002b). The third argument rests on particular goals for education that some may disagree with; it loosely corresponds to the Deweyian (1938) goal of "assisting the personal growth and development of individuals."

Model of the roles of the aesthetic in mathematical activity

Through an empirical/analytical interdependency methodology (Toulmin, 1971) I have shown that the aesthetic dimension of mathematical activity is not merely a fanciful, romanticised, after-the-fact judgement of mathematical beauty; but rather, it permeates mathematical activity and purposively animates mathematical knowledge (Sinclair, 2002b). I have identified three distinguishable aesthetic characteristics that play varying roles in mathematical activity: the evaluative, generative and motivational.

The evaluative characteristic concerns the aesthetic nature of mathematical products and the role of the aesthetic in making value judgements on these products (including theorems, proofs, definitions, diagrams, questions and theories). The evaluative characteristic of aesthetics is not just about objectively deciding whether a proof is elegant. Rather, it is involved in a mathematician's decision-making about truth, understanding and value. Without aesthetic judgement, it would be very difficult to distinguish the select results—of the thousands and thousands produced each year—that are worthy of further attention, of recognition, of passage into the 'canon' of textbooks, and of funding.

The generative characteristic focuses on the role of the aesthetic in inventing or discovering mathematics; it may be the most difficult of the three characteristics to discuss explicitly, operating as it most often does—for mathematicians at least—at a subconscious or tacit level (Hadamard, 1945; Poincaré, 1908). The aesthetic choices guiding mathematicians' processes of inquiry manifest a form of understanding that is qualitative, that is, neither formal nor propositional. These choices are made based on sensitivities to aesthetic qualities such as symmetry, analogy, simplicity and order. Mathematicians as well as philosophers (Dewey, 1938; Polanyi, 1962) have argued that the process of (mathematical) inquiry relies on the extra-logical form of thinking of the generative characteristic of aesthetics as much as on the more frequently cited logical and sequential forms of thinking.

The motivational characteristic relates to the role of the aesthetic in prompting or inspiring mathematical activity. This character is necessary to the process of selection and initiation in the mathematician's work. But more importantly it is directly involved in motivating inquiry as it directs the attention of the mathematician—what will be noticed—and frames the types of questions a mathematician will ask about a certain situation. Thus the motivational characteristic not only determines the field(s) and problems on which a mathematician works, but actually shapes the mathematician's inquiry (Dewey, 1938).

My model of the roles of the aesthetic in mathematical activity also accounts for a more encompassing theme in mathematical inquiry not explicitly mentioned in the literature: the aesthetic qualities of experiences which mathematicians describe having that unify and make memorable some of their encounters with problematic situations. This experiential dimension—pervading feelings and responses that arise out of mathematicians' experiences—contrasts with the more distinctive, cognitive roles of aesthetics identified above (see Sinclair (2002a) for an analysis of the aesthetic as a theme in mathematical experience).

The aesthetic in number theory

The following 1989 NCTM Standards quote nicely draws attention to the aesthetic potential of elementary number theory: "Number theory offers many rich opportunities for explorations that are interesting, enjoyable and useful. These explorations have payoffs in problem solving, in understanding a developing mathematical concepts, in illustrating the beauty of mathematics, and in understanding the human aspects of the historical development of number" (p. 91). What is meant mean by "interesting" and "enjoyable"? What is meant by the phrase "illustrating the beauty of mathematics"? Are these aesthetic opportunities actually valued as fundamental to mathematics learning by the education community or are they seen as epiphenomenal? By analysing five different dimensions of elementary number theory in light of my aesthetic model, I would like to inquire further into these aesthetic qualities of mathematics, as well as better articulate why they should be valued.

<u>Children have number</u>

Non-negative integers—the primary objects of elementary number theory—are pervasive in children's lives, as are the operations of multiplication and division. These are objects and operations that children see, know and use "... students from middle school through college feel comfortable dealing with whole numbers" (Selden and Selden, 2002, p. 214). This level of comfort and acquaintance can help in conjecturing and problem solving, and I claim it can also stimulate both the generative and motivational characteristics of aesthetics. In terms of the generative, students working with number theory situations or activities have the cognitive liberty of concentrating on exploring, guessing, and 'playing with' without being distracted by having to simultaneously extend their conceptions of mathematical operations and objects. Silver and Metzger (1989) have shown how this cognitive liberty is crucial to enabling aesthetic thinking—even in 'expert' mathematical activity.

A higher level of comfort and acquaintance can also stimulate the motivational characteristic of aesthetics, particularly with respect to the potential for surprise and paradox: the more you know about something, the more you will expect certain regularities based on your experiences, and the more likely you will be surprised by contradictions or interruptions. Zazkis (1999) illustrates a simple example of potential surprise with respect to the number for factors of an integer F(n): students expect that the greater A is, the greater F(A) should be (or if $A_1 > A_2$ then $F(A_1) > F(A_2)$), using the reasoning that works well in everyday contexts that 'the more of A, the more of B.' Students are surprised by mathematics when their expectations, even those based on their intuitions, are contradicted. For instance, students can be surprised that a small change can make a big difference, that a random collection of objects can share a common property, or that analogies can prove to be non-analogous. Moreover, as Movshovits-Hadar (1988) argues, surprise can not only provide exciting experiences for students in the mathematics classroom, it is also deeply connected to learning. The feeling of surprise stimulates students' curiosity which can, in small steps, lead towards intelligibility; it makes the students struggle with their expectations—and with the limitations of their knowledge—and thus their intuitive, informal, and formal understandings; and lastly, it provokes students to develop an appreciation of the significance of the concept in question.

Humans have had number for a long, long time

Mathematicians are often aesthetically motivated by a feeling that mathematics is somehow transcendental—providing them with a sense of order and certainty in our otherwise chaotic world. Similarly, there is an appeal to the seemingly transcendental types of problems and concepts that have been with us—or could have been with us—since the time of the Ancient Greeks: mathematicians continue in the tradition of Pythagoras to look for proofs of the irrationality of 2; many questions about prime numbers, factoring, and partitioning are as vibrant today as they were 2000 years ago. The realisation that she is posing a problem, or working on one, that Zeno or Euler could have posed might stimulate in a student an aesthetic motivation—one that is much more profound than the purported motivation factor associated with the 'real-life' problems pervading so many textbooks and curricula.

More than meets the eye!

As Selden and Selden (2002) write, "it is a well-known 'folk theorem' among mathematicians that number theory questions and conjectures can be easy to state but phenomenally hard to prove" (p. 214). For mathematicians this is a frequent source of aesthetic motivation: questions that are easy to state can elicit in them not only with the same sense of wonder that Aristotle claimed grips philosophers, but can provide them with a way of connecting with non-mathematicians and mathematicians alike—everyone can understand Goldbach's conjecture! In a related way, elementary number theory features many of the aesthetic wonders of mathematics—ideas such as infinity and scale, which Egan (1990) has shown have intrinsic appeal to students. Such ideas can thus initiate Whitehead's (1929) romantic stage of mathematics learning during which the "subject-matter has the vividness of novelty; it holds within itself unexplored connexions with possibilities half-disclosed by glimpses and half-concealed by the wealth of material" (p. 29). Whitehead argued that this stage was crucial and necessarily antecedent to the stages of precision and generality which students are often pre-maturely pushed into.

A study of patterns

The types of problems that crop up in elementary number theory are, as Ferrari (2002) notes, particularly well-suited to those actions of pattern-discernment and exploration which are eminently available to students. This can stimulate the motivational, generative, and evaluative characteristics of aesthetics. Humans are good at patterning—it is how they learn about and appreciate their world. When students recognise that they can activate a behaviour they know they are good at, they can derive the same kind of aesthetic appeal that motivates mathematicians to seek out and take on the types of problems that activate the ways of thinking with which they have had success—whether its symbolic juggling or visual representation. Also, many of the problems and objects of number theory can be patterned in concrete, accessible forms (for example: actual numbers as opposed to, say, abstract algebra objects; the patterned numbers in the colour calculator (Sinclair, 2001); figurate numbers; Pascal's triangle[2]) on which the qualitative, aesthetic ways of reasoning can more easily operate. Finally, with such pattern-rich, accessible situations, students can undertake that most crucial of mathematical activities: problem posing (Brown and Walter, 1983). It is with the act of posing problems—deciding whether they are worth pursuing and convincing others that they are—that the evaluative characteristic of aesthetics is first operative for mathematicians, and can be first stimulated in students.

A mathematical showpiece

Of course, the evaluative characteristic is also operative in the judgements that mathematicians make about their results, be they theorems, definitions or proofs. It is this aspect of the aesthetic on which mathematics educators have traditionally focused—concluding that although it is important in mathematics, it cannot be introduced to, or appreciated by students until the 'basics' have been learned (e.g. Dreyfus and Eisenberg, 1986). Their conclusion is based on the fact that students do not have the same aesthetic responses as mathematicians. However, Brown (1973) has shown that students do have aesthetic responses—based on different criteria perhaps than those of mathematicians—which contribute to their sense of the value of mathematics and thus serve a similar animating purpose. While Brown's students responded to the degree with which a solution to a number theory problem revealed their own path through the problem, students can also appreciate, for instance, the efficiency of a solution (as evidenced in their responses to Gauss' purported method for summing the integers from 1 to 100), the cleverness of a proof, and the perspicuity of a diagram; these are all criteria that mathematicians such as Hardy (1940) have proposed for judging the worth of mathematical entities. Geometry has traditionally been the topic area through which students are exposed to that ultimate mathematical entity of proof (and its related arts of conjecturing and verifying), but apart from an exposure to the various proofs of the Pythagorean theorem—maybe—they have little opportunities to respond to the various elegant, brutish, insightful, practical ways that mathematicians have of explaining and convincing. However, through elementary number theory, even students with little mathematical background can encounter many of these methods such as proofs by contradiction, induction, generic examples (see Rowland, 2002), visual proofs and counterexamples.

[2] In fact, Hawkins (2000) argues that by working with a combination of number and form—of the spatial and pictorial with the analytic means of understanding—that students' aesthetic engagement can best be stimulated and supported.

Concluding remarks

This article is but an initial, and very general, attempt at analysing the aesthetic potential of elementary number theory in mathematics education. Although my model of the roles of the aesthetic in mathematical activity is both theoretically and empirically grounded, more inquiry into the particular responses of students would greatly help in designing activities that can best capitalise on this aesthetic potential.

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WHAT MAKES AN EXAMPLE EXEMPLARY?: PEDAGOGICAL & RESEARCH ISSUES IN TRANSITIONS FROM NUMBERS TO NUMBER THEORY

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Worked examples have been used in teaching mathematics since the earliest of historical records, at least. In order for such examples to be useful, learners must see them as exemplifying something. If that 'something' is, for example (sic) 'the mystery of mathematics', 'the impossibility of doing something similar myself', or 'the ridiculousness of all this stuff', then learners are unlikely to make mathematical progress. If on the other hand the 'something' is a class of problems and a collection of techniques and ways of thinking, then the worked examples have served at least some of their purpose. Looking through a particular and seeing generality is a form of, and a building block for, generating a theory. When a learner has constructed their own 'theory', their own seeing of how 'to do' a whole class of problems, then real learning is taking place. In the language of Sinclair (this volume), learners participate in an aesthetic of efficiency and compactness afforded by their awareness of generality, or what the examples are exemplifying. Rowland (this volume) provides further examples of techniques for supporting learners in seeing the general through the generic particular (see also, Mason & Pimm, 1984).

The fact that text authors have, throughout the ages, inserted worked examples and only in some periods tried to start with general rules before illustrating their use in particular, suggests that in most generations teachers have been aware of the difficulty of proceeding from the general to the particular (at least unless very carefully guided: see Davydov 1990).

Method of working

My method of enquiry is to identify phenomena I wish to study, and to seek examples within my own experience. I then construct task-exercises to offer to others to see if they recognise, or can be directed to recognise, what I find myself noticing. Through refinement and adjustment of task-exercises in the light of experience and of reading relevant literature, I both extend my own awarenesses, and offer others experiences which may highlight or even awaken sensitivities and awarenesses for them. These sensitivities and awarenesses may inform their future practice. As task-exercises are developed and shared, actions which exploit what is noticed for the benefit of learners are incorporated. My method does not attempt to capture or cover the experience of readers. Rather it aims to make contact with that experience, perhaps challenging interpretations, perhaps pointing to other features not previously noticed. The data of this method are the experiences generated, the sensitivities to notice which are enhanced. If you recognise at least something of what I am talking about as a result of having worked on the task-exercises, then you may be stimulated to look out for similar experiences in the future, and over time, begin to act upon what you notice. Validity in this method lies in you finding your actions being informed in the future, not in what I say (Mason, 2002).

Examples

Consider the following task:

Find the greatest common divisor (gcd) *of 84 and 90. Find their least common multiple* (lcm) *as well.*

You can imagine a whole page of these sorts of questions, and if you do, then you have a sense of a class of tasks (for more on classifying problems in number theory, see Campbell, this volume). But do learners also have a sense of this class, even when they have worked through the page? Could they make up for themselves versions which 'showed possible difficulties that can arise in some cases' or which 'illustrate what things can happen'? Are they aware of two approaches (factoring and the Euclidean algorithm), and do they have criteria for choosing one over another? If such tasks are augmented a little:

Multiply your two answers together and compare the product with 84×90

something more is suggested, which can be pushed further into the realm of generality with something like

Might this always happen? or Will this always happen? or even, Can you find two numbers for which this does not happen?

The indefinite pronoun 'this' may invite learners to clarify what the 'this' is, or may leave them helpless, depending on their past experiences with these kinds of question. To prompt learners to think of factoring, the task

Find a number with exactly 13 factors

makes use of a largish number (13) which invites simplification (try 3 factors, try 1 factor!) and then re-generalisation beyond the 13, with extensions concerning characterising or describing the class of all numbers with a given odd number of factors, and finding the smallest such number. There are also extensions to even numbers for the bold. This is typical of the approach developed in the 60s and 70s in the U.K. (Banwell *et al.*, 1972) in that learners are invited to undertake a task in which they make choices in order to simplify on the way to re-complexifying for themselves.

Finding a number with exactly 13 factors is not a single task but an entry point into a whole domain of tasks which include finding numbers with a given number of factors, how many smaller numbers less than itself are relatively prime, among others (see Banwell *et al.*, 1972 p. 37, pp. 102-3). Such a task domain includes various variants in presentation as well as in content. It is of pedagogic use only if it becomes a vehicle for learners to use their natural powers to imagine and to express, to generalise and to particularise, to conjecture and to convince, and if their attention is drawn to the fact of these powers and to ways of refining and honing them. For example, although an observer might say that they detected the emergence of a theory about the structure of numbers as products of primes, and about the parity of the number of divisors, participants might be wholly unaware that that is what they were doing. Their attention is likely to be confined to trying to sort out their ideas and to justify their conjectures. Yet what they are doing is what mathematicians do. Becoming aware of the emergence and articulation of a 'theory' can both inspire and support the development of a learner's mathematical self esteem.

Another collection of tasks which promote number factoring are obtained by 'undoing' the first task:

Find a pair of numbers whose gcd *is 6; find another pair; and another ... leading to the learner deciding spontaneously to classify or characterise all such pairs. Similarly, find a pair of numbers whose* lcm *is 1260, leading to, how many different pairs can have a specified* lcm*?*

Participants are not only developing a theory about the structure of pairs of numbers with a given *lcm*. They are also experiencing the use of their power to organise and characterise in a mathematical context, much as mathematician are wont to do. In other words, by being immersed in such tasks learners are likely to develop a 'theory' of what doing mathematics is like, a theory which would be very different from a theory developed as a result of only attending lectures and doing routine exercises (Watson & Mason, 1998).

A further advantage of refining and honing their powers to think mathematically is that learners are less likely to be caught by, or to persist in, spontaneous theories such as that 'all functions are monotonic': if $x > y$ then it is likely that $f(x) > f(y)$ (Zazkis, 1999), or that all functions are linear: e.g., $\sin(A + B) = \sin(A) + \sin(B)$ and $\ln(A + B) = \ln A + \ln B$. These are all too common manifestations, even by learners who when questioned directly know that they are false generalisations. The problem is perhaps that learners are entirely unaware that they have these theories, and they are not in the habit of testing theories and looking for counter-examples. In the midst of a complex problem, they simply do not have sufficient free attention to monitor what they are doing.

The temptation of authors and teachers is to lay out examples, perhaps as tasks, and to expect learners to build on the experience of a succession of tasks in order to become aware of, or to experience that succession. But as Kant has quite rightly pointed out in his *Critique of Pure Reason*: a succession of experiences does not add up to an experience of that succession.

Evidence of this phenomenon can be found in many different contexts. For example, in a professional development session I offered the following sequence of statements, with lots of pausing so that participants would see that they were supposed to check each equation in turn, and to attend to the patterns between equations:

$1 + 2 = 3 \qquad 4 + 5 + 6 = 7 + 8 \qquad 9 + 10 + 11 + 12 = 13 + 14 + 15$

'We' (meaning I wrote what some of the participants said) had written down two more rows. I asked for an expression of generality. One person suggested

$n + (n + 1) = \ldots$ but was then a bit stuck.

It transpired that his attention was entirely on the first equation. The others were not seen as part of 'the experience to be generalised'. Rowland (2000) met the same thing with pre-service primary teachers. Asked to check that:

$3 + 4 + 5 = 3 \times 4, \quad 8 + 9 + 10 = 3 \times 9 \quad \text{and} \quad 29 + 30 + 31 = 3 \times 30$

and then to write down a statement in words generalising these three examples, many wrote nothing or nothing that could be construed, and some wrote a false generalisation such as:

three consecutive numbers added together equals the product of the first two

achieved by attending solely to the first 'example'.

It is quite likely that unfamiliarity with being asked to express a generality produced a tunnel vision effect, so that attention became focused on but one instance. The fact that the two threes play different roles could be overlooked, which would make the conjectured generality at least understandable. In the case of the sequence above, progress was made by asking people to chant the equations out loud but with emphasis first on the first number, then in a second pass, on the last.

Watson (2000) has pointed to the phenomenon of 'reading with the grain' and the necessity of 'reading across the grain' in order to experience structure. Thus in the Tunja sequences (Mason, 1999; 2001), which I developed in order to promote simultaneous work on factoring quadratics and on multiplication of negative numbers, I have found as expected, that non-mathematical audiences are perfectly capable of working with the grain—that is of following a pattern which is closely related to counting numbers and perhaps square numbers. For example:

$3 \times 5 = 4^2 - 1 \qquad 4 \times 6 = 5^2 - 1 \qquad 5 \times 7 = 6^2 - 1$

can be extended 'downwards' to more equations by observing the counting-numbers in sequence. Being directed to read across the grain—that is to relate both sides of each 'equation'—leads to the realisation that the symbolic expressions must be the same, somehow. This is one of the necessary awarenesses that make algebraic manipulations meaningful. Different looking expressions can nevertheless express the same thing, so there ought to be a way to get from one expression to the other simply by manipulating symbols. The sequence can also be pushed backwards, against the grain, to reveal necessary facts about multiplication of and by negatives, given that we want the 'pattern' to continue.

As is well known, specific kinds of questions can lead learners to unexpected meta-generalisations. For example, being asked to express a pattern as a general formula can lead them to a pattern of behaviour which avoids the intended inner-task (Tahta, 1980) and exercises only the outer-task, namely, to find a formula which fits. As mathematicians well know, though intelligence testers seem yet to discover, no sequence, even the Tunja or the 'Consecutive Sums', uniquely defines its next term. There must be some source for a pattern which is agreed. Thus the sequence:

$1, 2, 4, 8, \ldots$

can have many different fifth terms (indeed, any fifth term, but see Sloane (1973) for examples of sequences which count things, and which begin this way). If the sequence is counting some aspect of a sequence of pictures, such as the number of regions of a circle formed by 0, 1, 2, 3, ... chords, or the number of regions of space formed by 0, 1, 2, 3, ...

planes in general position (Polya, 1965), then it is essential to have a statement of that generality before embarking on trying to find a general formula.

But even where learners are frequently engaged in formula-finding, the whole exercise can turn into 'train-spotting' (Hewitt, 1992) rather than productive mathematical thinking which exercises and refines the power to generalise.

Consider then some further options for extension.

The act of finding the *gcd* and *lcm* of two numbers can be seen as functions, but this requires the learner to step out of immediate action-experience, and to contemplate the whole. Having achieved some measure of competence with these calculations, the calculations themselves can be seen as objects. This is the domain of reification, when a process is also experienced as an object. In Mason *et al.* (1985) this was used to characterise one of the major steps in appreciating algebra, when expressions like $3x + 4$ come to be seen both as a specification of a calculation process, and as the object resulting from that process, and this dual nature then becomes the essence of algebraic expressions which replace numbers as the objects to be manipulated. Sfard (1991; 1992; 1994) developed the notion of reification while (Gray & Tall, 1994) used the term *procept* to indicate the evolution of a concept from carrying out a process as a theorem-in-action (Vergnaud, 1981) to seeing the process as an object in itself. Notation for the process helps enormously, for once something is named, it comes into psychological existence.

One way to stimulate learners to experience calculations as processes is some variant of:

Tell an absent friend how to calculate the gcd *and* lcm *of a pair of numbers.*

Program a machine to find the gcd *and* lcm *of a pair of numbers.*

I am going to be given a pair of numbers, but I can't tell you what they are at the moment (or I have a friend who has a pair ...). Please tell me how to calculate their gcd *and* lcm.

Suddenly what seemed almost frighteningly open becomes bounded. A theory might just be possible. Notice that I do not provide my 'answers' nor even my theories in the sense of ways of seeing. For once theories are published, pedagogic value leaks away.

Final comments

Terms such as *investigative teaching* (and its variants such as *discovery learning*) provoke extreme reactions in many audiences, while *lecturing* and *starting from the abstract* provoke similar reactions in different audiences. Neither reactions are helpful as they are based on emotive associations with general labels, rather than precise details of pedagogic strategies. When teaching that is even marginally effective is examined closely, aspects of both pedagogic stances, of starting from the particular and the concrete and starting from the general and the complex will be found to have value. Strict adherence to one format is likely to foster the pedagogic theory that 'this is always how things are done', whereas variation in approach is more likely to broaden learners' views of what mathematics is about, what questions it addresses, and what methods it employs (Watson & Mason, 1998). Above all, the most important theory we want learners to construct is that they do actually possess the requisite powers to do mathematics and to think mathematically. Then they can make an informed choice as to whether to develop and make use of those powers within mathematics in the future.

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PROOFS IN NUMBER THEORY: HISTORY AND HERESY

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My purpose in writing this paper is to advocate the use of particular-but-generic proof strategies in undergraduate classrooms and in textbooks, in order to convince students of the truth of number-theoretic theorems and student-generated conjectures. The domain of number theory lends itself particularly well to generic argument, presented with the intention of conveying the force and the structure of a conventional generalised argument through the medium of a particular case. The potential of the generic example as a didactic tool is virtually unrecognised. Although the use of such examples has good historical provenance, the suggestion that they might be an alternative to formal proof tends to be viewed as a kind of heresy from the perspective of modern proof practice.

Procedures and proofs

The use of examples to point to abstract concepts and to general *procedures* is commonplace pedagogical practice (see e.g. Mason's paper for this research forum). In the field of number theory, a case in point might be explication of the Euclidean algorithm for the greatest common divisor of two natural numbers. Beginning with, say, 194 and 40 the demonstration proceeds:

$$\begin{array}{lcl} 194 & = & 4 \times 40 + 34 \\ 40 & = & 1 \times 34 + 6 \\ 34 & = & 5 \times 6 + 4 \\ 6 & = & 1 \times 4 + 2 \\ 4 & = & 2 \times 2 + 0 \end{array}$$

In order to apply the procedure to another pair of natural numbers, the student needs to become aware of the status of each number in each row of the procedure, and how each row relates to the next. That is, not only to agree that each line is a true statement, but to appreciate how it has been initiated and structured. As teacher, I might assist this by (say) underlining the quotients 4, 1, 5, 1, 2 in red. I might draw diagonal lines joining the divisor and remainder in each line to the dividend and divisor, respectively, in the next e.g. joining the two 40s, the two 34s, and so on. (It is relevant to pause to reflect on how you made sense of the previous sentence: perhaps the example was more illuminating than the somewhat archaic expression of the general procedure that preceded it.) The choice of example (194, 40) was made in recognition of its merits in its own right and relative to some alternatives. I judge it to be preferable to (194, 48), which is a poor paradigm because, for that pair, the algorithm terminates too soon. I would also avoid (144, 89) for a different reason: although it has good 'length', it conveys difference rather than division. Try it, if the intention of that remark is not self-evident. I would resist (97, 20) in recognition of my own liking for coprime pairs despite their particularity.

Much less common is the use of examples to explain why general relationships might hold: in short, to *prove*. One reason why this might be the case is clear enough—because one or more examples cannot prove a statement about an infinite category of cases. Yet there is a sense in which the *presentation* of a single example can speak for some general truth, and for some general argument above and beyond the particularities of the example itself. Such examples, suitably structured to be not just a confirming instance but a chain of reasoning, are known as generic examples. As Balacheff (1988) so clearly and elegantly puts it:

> The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class. (Balacheff, 1988, p. 219)

The generic example serves not only to present a confirming instance of a proposition - which it certainly is - but to provide insight as to *why* the proposition holds true for that single instance. The transparent presentation of the example is such that analogy with other instances is readily achieved, and their truth is thereby made manifest. Ultimately the audience can conceive of no possible instance in which the analogy could not be achieved.

Un peu d'histoire

The story (probably apocryphal, but see Polya, 1962, pp. 60-62 for one version) is told about the child C. F. Gauss, who astounded his village schoolmaster by his rapid calculation of the sum of the integers from 1 to 100. Whilst the other pupils performed laborious column addition, Gauss added 1 to 100, 2 to 99, 3 to 98, and so on, and finally computed fifty 101s with ease. The power of the story is that it offers the listener a means to add, say, the integers from 1 to 200. Gauss's method demonstrates, by generic example, that the sum of the first $2k$ positive integers is $k(2k+1)$. Nobody who could follow Gauss' method in the case $k=50$ could possibly doubt the general case. It is important to emphasise that it is not simply the *fact* that the proposition $1+2+3+ \dots + 2k = k(2k+1)$ has been verified as true in the case $k=50$. It is the *manner* in which it is verified, the form of presentation of the confirmation.

Paul Hoffman recounts the story in his best-seller *The Man Who Loved Only Numbers* (Hoffman, 1998). His comment on it (quoting mathematician Ronald Graham) is a telling testimony to the genericity of Gauss' method.

> What makes Gauss' method so special .... Is that it doesn't just work for this specific problem but can be generalised to find the sum of the first 50 integers or the first 1,000 integers ... or whatever you want. (p. 208)

In introducing the notion 'generic example' to audiences of all kinds - undergraduate and graduate students, mathematics education conference-goers, 'general audiences' - I routinely choose Gauss' method as a paradigm of the genre. We should not be surprised that Gauss, of all people, should have provided it. Ironically, his *Disquisitiones Arithmeticae* established the 'modern' standard for generality in number theoretic proof arguments.

By contrast, Pierre de Fermat (1601-65) was notorious for stating number theoretic results in the absence of formal proof. In particular, it was Euler who gave a general proof of the 'Little' Theorem (if p is prime and a, an integer, p divides a^p-a) some decades after Fermat stated it. In a recent article, Burn (2002) offers some suggestions concerning the kinds of reasoning that Fermat himself might have used to establish the truth of some claims associated with his Little Theorem, made in a letter to Mersenne in 1640. These claims were developed in the course of Fermat's search for perfect numbers. A natural number (such as 6 or 28) is said to be *perfect* if it is equal to the sum of its divisors including 1, but not itself. Around 300BC, Euclid had established that the set of perfect numbers can be identified with integers of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime. In his letter to Mersenne (after whom such primes are named), Fermat claimed that if n is composite then $2^n - 1$ is not prime. The proof amounts to the observation that $2^a - 1$ divides $2^{ab} - 1$. The converse, however, is false: in 1536, Hudalrichus Regius had shown that although 11 is prime, $2^{11} - 1$ is composite, and so $2^{10}(2^{11} - 1)$ is not a perfect number.

Fermat made a claim which was to transform the previously Herculean task of determining whether or not $2^p - 1$ is prime for a given prime p. In effect, Fermat claimed that if an integer of this form has a prime factor, then that factor is of the form $2kp + 1$ (the factor 1 is covered by k=0, and it follows from Fermat's Little Theorem that $2^p - 1$ itself is of the form $2kp + 1$). Thus, to decide whether a proper factor of $2^{11} - 1$ exists, we only need to consider 23, since this is the *only* prime of the form $22k + 1$ with square less than $2^{11} - 1$. In fact, $2^{11} - 1 = 23 \times 89$. (Note that 89 is also of the form $22k + 1$, as expected).

In his letter, Fermat exemplifies this statement about prime factors of $2^p - 1$ with reference to this case i.e. when $p = 11$. Burn (*ibid.*) reconstructs the argument that Fermat might have given with reference to this particular-but-generic case. Burn then continues: "Now we generalise the generic example of factorising $2^{11} - 1$ by expressing the argument algebraically". Of course, this accords with good modern practice, although Burn does not suggest that Fermat, having established the generic, then required the general formulation to be convinced of the general case.

Teaching and learning Wilson's theorem

The obscure Cambridge mathematician John Wilson is remembered to this day on account of a theorem stated in 1770, a century after Fermat's demise:

p is prime if and only if $(p\text{-}1)! = -1 \pmod p$

To be precise, it was Edward Waring, Isaac Newton's successor (and Stephen Hawking's predecessor) as the Cambridge Lucasian Professor of Mathematics, who stated the result of his former student, Wilson. In the best traditions of the time, neither Wilson nor Waring managed to prove the theorem: its status seems to have been a conjecture, the outcome of inductive reasoning from examples. It fell to Lagrange to give the first proof of Wilson's theorem, in 1773. How then, might we approach the genesis of the theorem and the construction of its proof with the hindsight and didactic insights of the twenty-first century? What might a generic proof of that theorem look like?

As a preliminary, we would need to know that ± 1 are the only self-inverse elements under multiplication modulo p. Now consider the prime number 13 (17 or 19 would do equally well) and list the reduced set of residues modulo 13:

1 2 3 4 5 6 7 8 9 10 11 12

Pair each of the numbers from 2 to 11 with its (distinct) multiplicative inverse mod 13: (2, 7), (3, 9), (4, 10), (5, 8), (6, 11). 1 and 12, of course, are self-inverse. [I usually link the elements in the inverse-pairs with lines on a chalk board]. Clearly, the product of these integers from 2 to 11 must be congruent to 1^5, i.e. 1, modulo 13. Therefore $12! \equiv 1 \times 1 \times 12$ (=12) mod 13. The argument is generic, since 13 was in no way an untypical choice: the pairing would work equally well with any prime.

The scene now shifts to a session with class of about 20 first-year undergraduate joint honours mathematics-with-education students. I could have stated Wilson's theorem and proved it formally in five minutes. In fact, it took an hour to make some conjectures and to work on proof. This is what happened.

First, I asked them to evaluate 4! mod 5, 6! mod 7, 10! mod 11, and to write down a conjecture. The most common version of the conjecture was $n! \equiv n \bmod (n+1)$. The 'for all n' seemed to be implicit. I asked them to evaluate 5! mod 6. They did, and they were visibly surprised by the refutation. I asked whether they could modify the conjecture. At first they homed in on the even/odd distinction between moduli, but n=8 led to further refutation and eventual restriction to prime values of $n+1$. n=12 provided a further confirming instance. I proceeded to an interactive presentation of a generic proof, inviting Sonia to pick a prime between 11 and 19. She chose 19. I got them to list 1 to 18 and work on inverse pairs in table-groups, during which Simon spontaneously explained to his colleagues why 18! *had* to be 18 mod 19. I asked him to repeat his reasoning to the class, and wrote his explanation on the whiteboard. He picked out eight inverse pairs, and explained why the product of the integers from 2 to 17 inclusive would have to be 1 mod 19. They dutifully copied Simon's argument. Later, I enquired what would have happened if we had looked at 28! mod 29, and Abby explained why it would have to be 28, again referring to inverse pairings of the integers from 2 to 27, although without feeling the need to identify the pairs this time. "Does everyone agree?", I asked. They agreed. One shouldn't read too much into such consent, however pleasing; nevertheless, Abby, at least, had convinced me that she had appropriated the proof-scheme.

The next day, at a tutorial meeting, I asked five members of the class to write it out the proof (that, for primes p, $p! \equiv p-1 \bmod p$) in conventional generality. Their responses were unaided and individual. It should be borne in mind, as I indicated earlier in this paper, that these students will have had little experience of composing formal proofs. Nevertheless, they all indicated in their writing that the genericity of the case p=19 had been apparent to them. Moreover, their argumentation and use of notation would have satisfied any examiner. Hannah's response, which was typical, was as follows.

$(p-1)(p-2)(p-3)(p-4) \ldots 2 \times 1$

Every element of M_p[1] has an inverse, because M_p is a group.

We know (from work on primitive roots) that only p-1 has order 2. Therefore p-1 is self-inverse. All other members of M_p apart from 1 must have a distinct inverse.

Each inverse pair when multiplied gives 1 mod p.

This gives $(p\text{-}1)(1^{1/2(p-3)})1 \equiv (p\text{-}1)! \bmod p$

Therefore $(p\text{-}1)! \equiv (p\text{-}1) \bmod p$

Only Zoë gave evidence of some insecurity in this intangible world that lies beyond examples. Her proof was much the same as Hannah's, but began with identification of the inverse pairs in the case p=11 (transfer to other examples) and concluded the comment:

> I tried to find a formula for the inverses, for example p-2 has inverse p-6 (but only for p=11). I have been unable to do this.

For Zoë, mere knowledge of the *existence* of distinct inverses in the range 2 to p-2 is not enough. What is not clear is whether it leaves *her* cognitively insecure, or whether she believes that I (in my role as assessor) will expect more.

Abby's proof was elegantly and lucidly expressed, but stated that there are $^1/_2(p\text{-}1)$ inverse-pairs rather than $^1/_2(p\text{-}3)$. A case an error of manipulation, but not one of conception.

Generic arguments and cognitive unity

The domain of elementary number theory lends itself remarkably well to generic argument, presented with the intention of conveying the force and the structure of a conventional generalised argument through the medium of a particular case. One reason for this might be that, in the choice of examples, one seems to be spoiled for choice: there are an awful lot of integers (or primes, or whatever subset is called for) compared, say, with groups, or topological spaces. This is not to say that the choice of a generic example is an arbitrary one: it can be (and in a sense, it *ought* to be) a conscious pedagogical act. Some examples work better than others do for particular purposes - they carry and convey the generalisation rather better because the salient operations on the variable(s) can easily be tracked through the argument. Some tentative principles for the selection of generic arguments and the construction of generic arguments in number theory are given in Rowland (2002). I conclude, however, with some cautionary remarks.

First, the proof of Wilson's theorem given above crucially depends on knowing and being certain that 1 and p-1 are the only self-inverse elements under multiplication mod p. How shall we establish that result? It emerges readily (as a conjecture, of course) from examples, especially when the contrast is made with non-prime moduli. The usual proof runs as follows: if $1 \le a \le p\text{-}1$, and $a^2 \equiv 1 \bmod p$, then $p \mid (a\text{-}1)(a+1)$ and so $p \mid (a\text{-}1)$ or $p \mid (a+1)$. Whence a=1 or a=p-1. The essence of this argument is the solution of a

[1] M_p denotes the group $\{1, 2, 3, \ldots, p\text{-}1\}$ under multiplication mod p.

quadratic equation in a multiplicative modular group, which seems to rule out a generic presentation entirely free of algebraic symbolism. It is true that I could take p to be 13, and argue that a must be 1 or 12, but I can't seem to side-step arguing from (and with) $13 \mid (a-1)(a+1)$. Perhaps someone will delight me by convincing me that I'm wrong on this point.

Secondly, the converse of Wilson's theorem [if n is composite then $(n-1)! \neq n-1 \bmod n$] *appears* to lend itself wonderfully well to generic exposition. Take the case $n=10$. Now $9!=362880$, so $9!\equiv 0 \bmod 10$. Yes, but why? Because 9! includes factors 2 and 5. Since 10 is composite it can be decomposed into the product of two factors, both strictly between 1 and 10, so both occur as terms in 9! It is thus apparent that if n is composite then:
$(n-1)! \equiv 0 \bmod n$

However, whilst this conclusion is certainly true, the argument does not, in fact, transfer to all composite numbers. In those special cases when n is the square of a prime p, the only possible decomposition of n into the product of two factors, both strictly between 1 and p-1, is $n = p \times p$. The factors are not distinct and it is not the case that both occur as terms in $(n-1)!$ It is not difficult to make a separate argument for these cases, but they can easily be overlooked, and caught in the shadow, as it were, of the earlier generic argument.

Notwithstanding these cautionary words, there seems to be a good prospect of developing and offering a systematic didactic technology of formal proof in number theory, building on skilfully-constructed generic examples. There is evidence that such an approach to proof is supportive of the *cognitive unity* of theorems, that is to say "the continuity … between the process of statement production and the process of its proof, as well as providing meaningful examples". (Mariotti, Bartolini Bussi, Boero, Ferri and Garuti, 1997).

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DISCUSSION GROUPS

Imagery and Affect in Mathematical Learning

Coordinators: Lyn English, Queensland University of Technology (Australia)
Gerald A. Goldin, Rutgers University (USA).

Traditional views of mathematics as an abstract, formal discipline have tended to relegate visualization, metaphor and metonymy, emotions, and the relation between feeling and mathematical imagination to incidental status. Continuing the discussions begun at PME-24 in Hiroshima and at PME-25 in Utrecht, our primary focus is on imagery, affect, and how they interact: their interplay with natural language, formal notations, heuristics, beliefs, and especially with each other. Representation in learning mathematics includes not only external structured physical configurations, but also internal systems that encode, interpret, and operate on mathematical image and symbol configurations (Goldin, 2002). We have a strong case for the centrality of imagistic reasoning, analogies, metaphors, and images in such representation (English, 1997; Presmeg, 1998). Lakoff and Nunez (2000) even (controversially) seek to recast the foundations of mathematics in terms of conceptual metaphors. Essential roles of affect in encoding information, influencing learning and performance have also been noted and studied (Evans, 2000; Goldin, 2002; McLeod, 1992).

For PME-26, we will extend our discussions on the nature and role of affective and imagistic representational systems in mathematical learning and problem solving. In doing so, we will explore an embodied perspective on perception, cognition, affect, imagination, and reasoning. This issue generated much discussion at PME-25 and participants expressed a keen interest to pursue the topic.

In the first session, we will review the Discussion Group's progress to date, and then continue our discussion on the embodied perspective and other perspectives that participants raise. In the second session, we plan to focus on specific classroom examples of the issues in question and consider some of the difficult points in their empirical investigation. Intending participants are asked to bring along problem-solving data (including videotaped activities and transcripts) in which the group can identify examples of imagery, affect, and especially their interplay.

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STOCHASTICAL THINKING, LEARNING AND TEACHING

Co-ordinators: Brian Greer, San Diego State University, USA
Chris Reading, University of New England, Australia

In general, the group will continue to discuss the relationship between stochastical and mathematical thinking, learning, and teaching from multiple perspectives. Specific themes to be addressed will be:

- curriculum issues – syllabus, textbooks, software, assessment
- making connections within mathematics and with other subjects
- the wider social significance of stochastics education

We will begin with a commemoration of the contribution of John Truran to stochastics education, in particular his leadership of this Discussion Group.

Participants in the Sixth International Conference on Teaching Statistics and in the Second International Research Forum on Statistical Reasoning, Thinking and Literacy will report on those meetings.

Short contributions to initiate discussion will include:

- Andy Begg (New Zealand) talking about rich learning activities in syllabuses, textbooks, and assessment linking statistics with other parts of mathematics and with other subjects
- Chris Reading (Australia) talking about the "Profile for Statistical Understanding"
- Swapna Mukhopadhyay (USA) talking about the use of statistical data for critical analysis of social issues
- Chris Nisbet (Australia) talking about students' graphical represention of statistical data

DISCUSSION GROUP
SEMIOTICS IN MATHEMATICS EDUCATION RESEARCH

Coordinators

Adalira Sáenz-Ludlow, University of North Carolina, USA — sae@email.uncc.edu
Norma C. Presmeg, Illinois State University, USA — npresmeg@msn.com

For at least ten years, mathematics educators have incorporated different theoretical semiotic frameworks into an already rich set of psychological, anthropological, linguistic, and sociological theories to analyze the teaching practice and the learning of mathematics in-and-out of the classroom. The purpose of the group is to continue the discussion on how different semiotic perspectives contribute to our understanding of cognitive, linguistic, and social issues in the teaching and learning of mathematics. Six scholars, who have been using semiotics as a theoretical framework in their research activities, have been invited to contribute to the discussion.

In the first session, the contributors will discuss different semiotic aspects of classroom communication namely the role of signs for coding, indicating, and constructing mathematical knowledge; the semiotic reciprocity between referring, expressing, and addressing in educational settings that takes into account curricula, textbooks and classroom in a holistic manner; and the power of Buber's "I-Thou-We" triad to coordinate the wide spectrum of students' experiences and arguments in undergraduate and graduate courses on problem-solving and design.

1. The communicative and cognitive role of mathematical signs: An epistemological perspective on mathematical interaction
   Heinz Steinbring — Universität Dormund, Germany

2. From utterances to genres in mathematics education (and vice versa): A dynamic triadic (socio-) semiotics exemplified
   Sigmund Ongstad — Oslo University College, Norway

3. The significance of dialogue in problem-solving using Buber's triad "I-Thou-We"
   C. Jotin Khisty — Illinois Institute of Technology, USA

In the second session, the contributors will pursue different semiotic venues to offer an enriched view that balances social and individual aspects of mathematical knowing and the role of signs in learning and doing mathematics. The papers will discuss the nature of the writing activity of mathematical text on the part of mathematicians in contrast to that of secondary school students; the complementarity of diagrammatic reasoning and conceptual thinking; and the pivotal role of signs in reading, writing and thinking mathematically.

1. What does social semiotics offer to mathematics education research?
   Candia Morgan — University of London, UK

2. How diagrammatic is mathematical reasoning?
   Willi Dörfler — Klagenfurt, Austria

3. A semiotic perspective on mathematical activity
   Paul Ernest — University of Exeter, UK

At the beginning of June 2002, the papers will be posted in the web site <http://www.math.uncc.edu/~sae/> that was established for the PME25 discussion group on semiotics so that participants will have the opportunity to read them beforehand. Contributors will give the participants a birds'-eye-view of the main ideas in their papers and will elicit discussion on different issues. Each presenter is allowed 25 minutes to achieve these two goals. Five minutes are saved for transition between presenters.

RESEARCHING THE SOCIAL AND POLITICAL IN MATHEMATICS EDUCATION FROM A CRITICAL PERSPECTIVE: DILEMMAS FOR RESEARCHERS

Co-ordinators: Tansy Hardy and Hilary Povey
Sheffield Hallam University, England

Researching the social and political in mathematics education from a critical perspective raises a particular set of dilemmas for researchers. These may include the choice of what to research and who decides; the purpose of the research and the purposes to which it might be put; the methodology of research, traditional or alternative; the practice of the research, particularly as it relates to the people involved; and the reporting of the research, the format, the language, the authorship.

It is the intention to continue the work that was undertaken on this subject last year at PME 25. A number of individuals offered the whole group a response to the invitation to reflect on their own research into political and social issues in mathematics education and to highlight some feature of the research which had been problematic for them. Issues considered included

- researching sites which promote values opposed to those of the researcher;
- reporting in the language of the participants? the language of the researcher? the most powerful language in mathematics education?
- resisting the attribution of the 'label' expert northerner researching the south;
- the centrality or otherwise of a narrowly defined focus on mathematics education in the research;
- reporting to and with the participating community;
- the importance of seeking strategies that impact on policy;
- contributing to a shift within the dominant discourse of the mathematics education research community itself.

The co-ordinators will bring to the first session some specific and focused, possibly fictitious, dilemmas on which small sub-groups will work. Depending on their complexity, these may be circulated electronically before the conference. The second session will be adapted to meet the outcomes of the first session. It may include sharing reflections on the first session; writing reports of dilemma resolutions; considering dilemmas which are actual for the group members; or planning continuing work for the group.

FACILITATING THE CONCEPTUAL CHANGE IN MATHEMATICS

Erkki Pehkonen, University of Turku, Finland
Kaarina Merenluoto, University of Turku, Finland
Marianna Tzekaki, Aristotle University of Thessaloniki, Greece

The crucial idea in the conceptual change is the radical reconstruction of prior knowledge which is not very well observed in traditional teaching. The theories of conceptual change (see Vosniadou 1999; Duit 1999; Reiner, Slotta, Chi & Resnick, 2000) define two levels of difficulty in the learning process. The easier level of conceptual change means *enrichment* of one's prior knowledge structure. In this case the prior knowledge is sufficient for accepting the new specific information. The student needs only to add the new information to the existing knowledge. The more difficult conceptual change is needed when the prior knowledge is incompatible with the new information but needs a reconstruction. There seems to be at least two basic kinds of directions for problems in conceptual change. The one is the knowledge and operations which are relevant on a certain domain, but need to be revised on the other. This is the case with the mistaken transfer from natural numbers to the rational numbers. For example the students seem to face considerable difficulties in struggling to sort fractions (Hartnett & Gelman 1998) because of their spontaneous use of the logic of natural numbers. The other kind of problems are the beliefs and conceptions caused by experiences with mathematics. The need to make drastic changes to the prior thinking may not even occur to the students unless the needed change is made very explicit in the teaching.

The purpose of the discussion group is to explore the specificity of the nature of conceptual change in mathematics concept formation (wide conceptual field needed, different representations used etc.), to discuss about the role of prior thinking and to find ways to facilitate this conceptual change.

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Examining Theses Discussion Group

Kath Hart (retd. University of Nottingham)
Peter Sullivan (La Trobe University)

The production of a thesis is the major task in most Ph.D programmes in mathematics education. Very often the work is closely supervised by one individual although in some institutions a committee is provided, for this purpose. Ideally the student could be working within a research project and calling upon the support of a number of individuals. The responsibility for producing the thesis rests with the candidate but the supervisor influences the work at all levels and so also bears a responsibility. The work is eventually judged by others, probably members of the mathematics education community. On what bases are these judgements made? Does a student know how the work will be assessed?

The discussion group plans to consider criteria for judging the worth of a thesis and also to engage students in the conversation to find their preferences and experience.

DISCUSSSION GROUP: SYMBOLIC COGNITION IN ADVANCED MATHEMATICS

Principal Co-ordinator: Stephen J. Hegedus,
University of Massachusetts Dartmouth, USA

Assistant Co-ordinators: David Tall, University of Warwick, UK.
Ted Eisenberg, Ben-Gurion University, Israel

The study of Symbolic Cognition builds upon the work on representations in mathematics learning and communicating in the classroom (Janvier, 1987; Cobb, Yackel & McClain, 1997). We refer to symbolic cognition as the construction of mathematical signs and symbols and the processes involved in manipulating such objects into meaningful concepts, procedures and representations.

Following our very first meeting of the group at PME25 in Utrecht, rich and varied discussion has continued through the implementation of an on-line email forum.

At PME26, we aim to continue discussion focusing on seven important areas of interest which arose out of preliminary inquiries. These include:

1. The mediation of human experience by purposeful use of symbol systems;
2. The co-evolution of language and symbol systems;
3. The role of orality and literacy in the development and use of mathematical symbols;
4. The inter-relationship of signs and symbols (e.g. the work of Pierce);
5. On producing the symbolic – processes, actions and manipulation;
6. New technologies, new representational structures;
7. Meta-processing – the role of reflection and monitoring in using symbols.

Each day discussion will be led by two or three short and focused presentations of 10 minutes each by members of the group aiming to further concentrate our ideas. Our aim is to generate specific lines of inquiry, which lead to the formation of a working group at the next meeting of PME. We see the group working in small sub-groups, concentrating on some of the main themes already outlined and new ideas deemed important and significant. This set-up will support the formation of special interest groups, which reflect the individual members' lines of expertise in symbolic cognition.

On-going work can be found at the group's website at www.symcog.org.

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Learning disabilities in mathematics

Coordinators: Petra Scherer (University of Bielefeld/Germany)
Elisabeth Moser Opitz (University of Freiburg/Switzerland)

The aim of organising a discussion group about *learning disabilities in mathematics* is to share research results and perspectives concerning this topic. Learning disabilities in mathematics seem to become an increasing problem in many countries. Too often these problems are assigned to experts who are not specialized in mathematics education. We intend to establish more contact between mathematics educators who work in this field.

The following aspects should be discussed:

- Is there a common understanding of learning disabilities in mathematics?
- What are the experiences in different countries and school systems?
- Are there new research findings concerning the diagnosis of learning disabilities in mathematics?
- Do children with learning disabilities in mathematics need a specific curriculum?
- What kind of teaching and instruction is appropriate for these students? Which standards of actual research in mathematics education are valid as well for low achievers as for high achievers? Which standards can be adapted?

During the sessions some specific thesis will be discussed. Moreover a reaction of experts is planned (asked: Rene Parmar/USA; Anne Watson/UK).

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WORKING SESSIONS

John Mason, Open University, Milton Keynes, UK

Anne Watson, Dept. of Educational Studies, University of Oxford

AIMS

The principal aim of the workshop is to pursue conjectures concerning the psychology involved in the construction of mathematical objects meeting specified conditions. Participants will engage in mathematical tasks which involve the construction of objects, which, although familiar in general, may lead to extending their sense of classes of objects.

A subsidiary aim of the workshop is to demonstrate in practice how it is possible to learn from reflection upon, and abstraction from, your own experience, informed and supplemented by accounts offered by colleagues,

TOPIC

Our starting point is that one characteristic of an expert is the ability to re-construct what they need, suitably modified and adapted to the current situation. This is in contrast to remembered and recited phrases, incantations and techniques. Our concern is with the construction of mathematical objects. We have discovered that many teachers at all levels report that they sometimes get their learners to construct mathematical objects for themselves. We are collecting examples of how learner-generated-examples alter the learners' relationship with mathematical objects and affect the kind of understanding which they display.

PLANNED ACTIVITIES

Activities will be based around mathematical tasks constructed on the basis of studies currently in progress and so are not specifiable in detail at this stage. They will be appropriate for a wide range of participants.

Participants will be invited to work individually and then in groups on a minimum of two different tasks per session, and to engage in small-group and plenary reflection and abstraction from that experience.

LEARNING FROM EACH OTHER.

Breen, Chris, University of Cape Town, South Africa

Hannula, Markku S., University of Turku, Finland

AIMS

The workshop aims at increasing researcher sensitivity in the interpretation of classroom incidents. Participants will engage in analysis and role-play in order to generate different emotional and critical interpretations of a chosen classroom interaction.

ACTIVITIES

The working session will use both group work and whole class discussion methods. The seed for the working session will come from a classroom incident, which will be accessed through either videotape or transcript. Several groups of participating researchers will be asked to role-play the same situation. Our aim is to facilitate the creation of different interpretations of the thoughts and feelings of the teacher and students during the episode and use this as a platform to discuss the various interpretations. In addition, we will try to get the class to identify some critical teacher decisions in the excerpt and then generate the possibilities contained in alternative possibilities for teacher action.

COMPOSING TRAJECTORIES

Ricardo Nemirovsky
TERC
Cambridge, Massachusetts

Apolinario Barros
Jeremiah Burke High School
Boston, Massachusetts

This working session will focus on the problem of how to compose a path on a plane given two functions over time (e.g. x(t) and y(t) in a cartesian plane), or, the inverse problem of decomposing a given path into two functions over time. The problem of composing and decomposing planar paths have been the subject of recent individual and classroom teaching experiments conducted by the coordinators and their colleagues. Through these activities we tried to enrich high school students' capability to imagine motion in space as defined by parametric functions. Gestures, utterances, and tool-use were focal aspects of our analysis.

The working session will be divided in two parts. During the first half the participants will work with tools and activities that involve the composition and decomposition of paths on a plane. During the second half the participants will analyze selected episodes filmed in individual and classroom teaching experiments. The coordinators will bring videotapes and transcriptions for the selected episodes.

The research questions for the session will be about the roles of physical experiences in mathematics learning and about the development of imagination. This question will be explored through the participants' engagement with the tools and activities as well as through the analysis of classroom interactions.

Gesture, Metaphor and Embodiment in Mathematics

Coordinators:

Laurie Edwards, St. Mary's College of California

Janete Bolite Frant, PUC-Sao Paulo

Jan Draisma, Catholic University of Mozambique, Nampula

One perspective on mathematics as a form of human cognition is that its roots lie in common human experience, both social and biological. In particular, our experiences as embodied, conscious beings provide the "raw material" needed to construct mathematical concepts. From the perspective of embodied mathematics (Lakoff & Nunez, 2000), both mathematical objects and processes can be analyzed in terms of more basic conceptual structures such as image schemata and conceptual mappings (Fauconnier, 1997). A common type of mapping is the conceptual metaphor, in which the logical and inferential structure of a source domain is utilized in making sense of a target domain (an example would be the embodied understanding of how objects can contain each other which underlies, unconsciously, the mathematical notion of set inclusion).

A complementary perspective on cognition and communication utilizes the analysis of gestures to help reveal how people think about mathematics. Coming out of the work of David McNeill and other psychologists and linguists (McNeill, 1992, 2000), this perspective views gesture as an integral part of language, not simply an embellishment. In advance of the session, readings on embodied mathematics as well as the analysis of gesture generally and in mathematics teaching and learning (e.g., Goldin-Meadow et al. 1999) will be made available via a website. During the session, videotaped data will be analyzed interactively in terms of embodiment, as expressed in both metaphorical language and gesture. Through the discussion group, we hope to continue building a common vocabulary, theoretical perspective, and methodology for understanding mathematics as an embodied phenomenon.

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WRITING RESEARCH PAPERS FOR PUBLICATION
[with attention also to the review process]

Janet Ainley and Barbara Jaworski

This Working Session will aim to engage participants in discussion and activities concerned with various aspects of writing for publication. In particular we shall try to work on the following issues.

What are the key elements of a research paper?

What is the nature of such elements? How can they be presented effectively? What issues are involved?

> Should all papers include some theoretical framework? What forms does/can such a framework take? What would be included? What are the associated issues?
>
> What kind of examples of data should be included? How would they be selected? How are examples related to findings and results?

In what ways might a research paper be structured?

What do readers need to be told, how, and in what order?

> e.g., dealing with the complexity of findings: use of bullet points, subsections, tables etc.

What is involved in a review process?

How do reviewers look at papers? What are the key criteria? What guidance is given? What questions are asked?

We hope that PME members who are editors and/or reviewers for the main research journals with which PME is associated will join this group and contribute to activity and discussion. Special issues of a journal arising from work at PME could be a consideration.

The two sessions will be based around detailed examination and discussion of a number of issues of relevant journals. Participants will be encouraged to work in small groups identifying characteristics of articles and raising questions.

Participants will be given an article or articles to read between sessions and asked to analyse this in terms of the questions and characteristics raised. Discussion of this activity will follow in the second session.

SHORT ORAL COMMUNICATIONS

CHANGING ENGINEERING STUDENTS ATTITUDES IN CALCULUS

Roselainy Abdul Rahman (UTM), John H. Mason (OU) & Yudariah Mohd. Yusof (UTM)

This presentation will give the preliminary results of an on-going research on changing students' attitudes towards Calculus. The main objectives of the research were to identify suitable approaches to encourage students to use their mathematical thinking powers and develop materials to further support these activities. The research was conducted on a group of first year engineering undergraduates taking Basic Calculus.

This research was motivated by the main findings from research conducted on mainly engineering undergraduates in Universiti Teknologi Malaysia (UTM) which found that these students had difficulties in coordinating procedures and manipulating concepts (Liew Su Tim & Wan Muhamad Saridan, 1991;Tall & Razali, 1993) and that they could not organise known facts effectively as well as master the mathematical language and symbols (Mohd. Yusof & Tall, 1994; Khyasudeen, et al, 1995). Furthermore, Yudariah (1995) found that the lecturers in UTM had little confidence in their students' abilities to cope with a formal mathematics course and designed their teaching accordingly. Students were encouraged to learn procedurally and given routine tasks to ensure success in examinations. However, they felt that there was a need to develop students' mathematical thinking and problem solving abilities.

The study adopts suitable qualitative research methods that studies the learning situation in context, in particular, an action research perspective of "practitioner as researcher".

Here we will report on some of the findings which includes changes in the teaching methods to invoke students' mathematical thinking powers such as specialising and generalising, the use of symbolic and multiple notations, technical terms and unfamiliar examples and expressing mathematical ideas. Difficulties encountered in the research implementation will also be discussed.

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THE EFFECTIVENESS OF A QUESTION-BASED FIRST COURSE IN ANALYSIS

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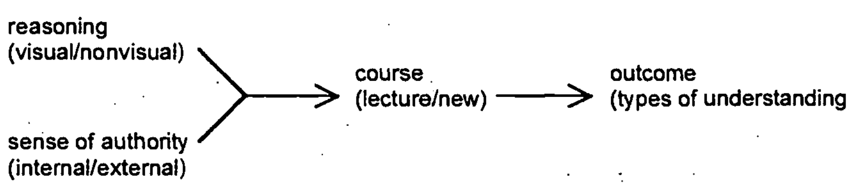
This presentation will review a substantive theory explaining different students' progress in learning Real Analysis, and thereby integrate established theoretical constructs in this area.

This theory was developed through inductive analysis of interview data. Participants were attending either a lecture course, or a new, classroom-based course in which they worked in groups through a sequence of problems in order to prove results themselves (based on Burn, 1992).

Learning outcomes are characterized as demonstrating *instrumental*, *relational*, *logical*, or *formal* understanding, where these are closely related to how the student justifies statements about sets of mathematical objects (Skemp, 1979, Tall, 1995). Factors found to be causal in student development are their *visual* or *nonvisual* reasoning style (Presmeg, 1986), and their *sense of authority* regarding the mathematics, where this is characterized as *internal* or *external* (Copes, 1992, Skemp, *ibid.*). The new course did not precipitate changes in these predispositions, but did promote improved student reasoning in restricted ways. Hence, the relationship between these factors may be represented as below:



The presentation will define the terms in more detail, and provide illustrative examples demonstrating the ways in which these factors interact to lead to the different types of understanding.

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SECONDARY TRAINEE-TEACHERS' KNOWLEDGE OF STUDENTS' ERRORS AND DIFFICULTIES IN ALGEBRA

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This study investigated trainee-teachers' explanations of students' errors in algebra and their suggested ways for addressing such errors. The theoretical framework for the study is derived from Shulman's (1986) idea that teachers' knowledge consists of several types of knowledge. In the model developed for this research, and informed by the work of Askew *et al* (1997) and Ma (1999), teachers' knowledge about students' errors and difficulties and teachers' belief about mathematics and how it can be taught have been incorporated.

Data gathering involved administering an open-ended questionnaire to a national sample of 251 trainee-teachers of secondary mathematics in 12 institutions in the UK, followed by a small number of semi-structured interviews for the purpose of validating and extending the data from the questionnaire survey.

The analysis of the completed questionnaires revealed that the majority of trainee-teachers explain students' errors in terms of the incorrect application of the procedures for working out algebra problems. Hence, they recommend re-teaching these procedures to the students so that the students might overcome their difficulties. Other parts of the analysis showed that most secondary mathematics trainee-teachers are able to suggest a teaching sequence that takes into account the hardest and the easiest algebra problems. Finally, most of the trainees were able to predict the most likely errors in a set of five algebra problems. However, less than fifth of them obtain $R^2>0.6$ when correlated with Küchemann's (1981) suggestion in regards to the facility level (percentage) of some algebra problems. This indicates that only a small proportion of trainee-teachers understand the sort of characteristics that determine the complexity of an algebra problem such as the number of variables in the problem, the nature of the elements involved and, importantly, students' interpretations of the letters.

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PRIMARY MATHEMATICS IN ENGLAND AND QATAR

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Three levels of mathematics curriculum were identified: the intended, the implemented and the attained curriculum (Dirks and Robitaille, 1982). Alexander (1992) suggested a framework for aspects of practice in primary classrooms that serves as a useful prompt to those involved in the judgement of that practice (the implemented curriculum) which are the *content*, the *context*, the *pedagogic process* and the *management* of teaching and learning. This study focuses on the similarities and differences in the context and the pedagogical practices of teaching primary mathematics between the English and the Qatari classrooms.

Eight primary classes in each country were selected to cover the entire primary age range from classes whose teachers were willing to participate in the study in both countries. Each classroom was observed for three lessons using a structured observation schedule.

The box-and-whisker plot and one-way ANOVA were performed in order to assist the analysis and comparisons of the duration of different activities used in the two countries. In contrast to the findings of the TIMSS (Third International Mathematics and Science Study) with regards to England (Wiliam, 1998), the analysis of data showed that teachers in both countries devote a high proportion of lesson time to whole class activities. Individual, group work and off-task activities were experienced in both countries, while paired work was evident only in the English lessons. The most salient differences between the two countries were in the differentiation by setting (despite the current trend in England of reducing the range of pupils' attainment in a class), seating arrangement and using information technology.

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THE TALL- VINNER PROBLEM. AN OPERATIVE REFORMULATION

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In order to identify conceptual comprehension problems, Tall & Vinner [1], [2] introduce the notions of *concept-image, evoked concept image,* and the differences between the *formal definition* and the *personal definition (PD)* of a mathematical concept. We will define these problems as the Tall-Vinner problem (TVP). This problem is manifested as incoherencies among the *meanings associated to the concept* or when the *meanings* associated with the name of the concept do not agree with its *formal definition.* This is observed when the subject responds to cognitive demands in different situations.

In [3][1], we construct an *operative definition of the basic comprehension of function. (BCF),* inspired by Vergnaud's definition of concept [4], in order to identify the measure in which a student possesses a comprehension level of function that would allow him to successfully overcome the TVP.

We will present and discuss quantitative indicators which make the definition of the BCF operative, defining the concepts of a *stable PD, a mathematically well-adapted PD, the coherence of a stable PD in relation to function.* Our purpose is to provide an instrument of TVP analysis that can be generalized to the study of other concepts.

NOTES

1. This work was supported by COLCIENCIAS. Code: 1106- 11- 202- 96.

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ALGORITHMIC MENTAL MODELS AND METACOGNITION

Aviva Barash

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Research in mathematics education has focused in recent years on students' conceptions and reasoning processes in mathematics. One explanation to students' mathematical misconceptions is the existence of intuitive mental models acting uncontrolled in the reasoning process (Fischbein, 1987). Dealing with algebra, we refer to algorithmic mental models, basic algebraic procedures that tend to become mental models. For instance, students who know the correct formula: $(a+b)^2 = a^2 + 2ab + b^2$ consider, inspired by the model of the distributive property, that $(a+b)^2 = a^2 + b^2$ (Fischbein & Barash, 1993).

Relying on recent research dealing with the affect of metacognitive methods on mathematical performance (Tirosh & Tsamir, 1997; Swan, 1998), we tried to rule out students' mistakes made due to the influence of tacit algorithmic mental models, by using metacognitive skills. For instance, using the "conflict method" (analysing the incorrect answers, raising a conflict, solving similar problems), to increase students' awareness of the existence of the models and control their impact. This was the main idea of our study unit: "Multiplication formulae and their use in reducing algebraic fractions"

One hundred and ninety four students in the 9th grade answered a pre-test to explore their formal knowledge of multiplication formulae and their error frequency in reducing algebraic fractions. Then, the study unit was applied. The same students, answered, a year later, a mathematical knowledge questionnaire, in order to examine the influence of the intervention program. The findings assert the hypothesis that learning by metacognitive methods, improves the formal knowledge of multiplication formulae and reduces the use of mental intuitive models which conflicts with formal knowledge.

I believe it is very important, that algebra teachers are aware of the impact of algorithmic mental models and recognise the importance of metacognitive skill applications. In my presentation I shall deal with the theoretical frame, the findings and some examples from the study unit based on general and specific psycho-didactical aspects.

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INSTRUCTIONAL STRATEGIES AND TASKS DEVELOPED IN A TEACHING EXPERIMENT ON PROBABILITY MODELING

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This study developed probability tasks and used them in a teaching experiment to investigate the impact of instruction on the modelling strategies of children in grades 3 and 4. Instructional strategies that were used are reported.

The research addressed the void in research-based knowledge of students' thinking in probability modelling by investigating the thinking of 6 students in grades 3 and 4 on contextual tasks that incorporated probability modelling. The teaching experiment consisted of 7 sessions: pre-assessment; 4 instructional sessions; short-term post-assessment; and long-term post-assessment. During the teaching experiment, the students' own cognitive strategies were used as a basis for planning and implementing instruction. The teacher-researcher posed tasks developed from this process, with some probes from the witness.

Instruction and tasks were influenced by four main teaching/learning patterns found in analysis of the transcripts and artefacts: (a) subjectivity in probability reasoning, when it exists, is difficult to counteract through instruction; (b) having students record their matchings of sample spaces for generators and tasks made the correspondence strategies they used more transparent for these students; (c) students use and think about discrete models differently than continuous models; and (d) two-dimensional modelling requires substantial accommodations for children.

Results indicate that young children use idiosyncratic and correspondence strategies, and that even a student whose idiosyncratic reasoning is deeply lodged can progress to using correspondence strategies with assistance from carefully designed tasks and probes. External representation may play a role in students' flexible use of correspondence strategies and probability generators, both discrete and continuous.

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LESSON PLAN STUDY AND THE GROWTH OF UNDERSTANDING OF SCHOOL MATHEMATICS AMONG PROSPECTIVE HIGH SCHOOL TEACHERS

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North Carolina State University

The focus of this preliminary study was to examine the growth of prospective teachers' understanding of high school mathematics while engaged in "lesson plan study" [LPS]. The tasks of LPS were used as research tools to examine pre-service teachers' growth of understanding of what and how to teach high school mathematics. It is the intent to show the importance of engaging pre-service teachers in the conceptually rich mathematical tasks of LPS as the tasks relate to teaching high school students. Two frameworks were used in this study to examine the growth of mathematical understanding. One provided a frame for the growth of understanding (Pirie & Martin, 2000) and the other provided a frame for the school mathematics (Berenson, Cavey, Clark, & Staley, 2001). In this study, prospective teachers were asked to connect their lessons on rate of change to ideas of ratio and proportion. Four to five weeks were need to complete one cycle of LPS and included the following tasks:

- Preliminary LPS Interview – Individual (20 min.). The pre-service teachers fold back to their primitive knowledge of algebra 1, linear functions, and slope to begin **making their images** of rate of change.
- Lesson Planning – Individual (45 min.). This activity promotes **collecting other ideas** of rate of change, and teaching activities to modify and change images of content and pedagogy.
- Post Planning LPS Interview – Individual (30 min.). Through explanations, pre-service teachers clarify their images to begin **having images** and **noticing properties** related to the content and pedagogy of LPS.
- Preliminary Group Interview – Group of 4 (10 min.). Pre-service teachers review their notes from the individual lesson planning to **formalize** their understanding.
- Lesson Planning and Post Planning Interview – Group of 4 (90 min.). Pre-service teachers share their LPS images and recursively **collect** and **make new images of** their understanding of what and how to teach rate of change.
- Lesson Plan Presentations – Four Groups (30 min each). Listening to the ideas of other groups, the pre-service teachers collect additional understanding at the **formalizing and observing** level of understanding.
- Written Lesson Plan – Individual (Variable time). Each pre-service teacher writes a final plan, **formalizing** the ideas **collected** over five weeks of LPS.

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LOOKING AT A SQUARE THROUGH ANOTHER LENS: A SQUARE AS A LOCUS

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Locus is a salient mathematical concept that reflects one of the basic and general ideas of modern mathematics. Thus, the aim of our study was to identify sites in the geometry curriculum where the notion of locus can be interwoven, in particular with respect to definitions of geometric objects. There are various ways of defining geometric concepts (Shir and Zaslavsky, 2001). The most common way of defining a geometric object in school mathematics, as appears in most textbooks, is by a structural definition, namely, by characterizing properties of the object. Another way to define a geometric object is by a common property of its points, namely, as a locus.

There are several geometric objects that are usually associated with the notion of locus. Some are actually defined as a locus (e.g., a circle) while others are not ordinarily defined as locus, however, are looked upon as locus in terms of a unifying property of their points (e.g., a angle bisector, and a mid-perpendicular of a segment). On the other hand, most geometric objects are not normally associated with locus at all, let alone defined as one.

A square is an example of a concept that is not usually treated as a locus, although it can be defined as one. In our study we examined how mathematics educators (teachers, graduate students and researchers) think about the possibility of defining a square as locus. More specifically, we studied the following three questions: 1. To what extent do they think that it is possible to describe a square as a locus of points? 2. For those who think it is possible, to what extent can they describe a square as a locus? 3. To what extent is a (given) statement describing a square as a locus acceptable as a definition of a square?

Data was collected through written questionnaires and group discussions. There were a total of 74 participants in our study, who worked in two groups in two separate workshops. Our findings point to the reluctance to accept the proposed definition of a square as a locus, mainly for the participants' unfamiliarity with such approach and for their views of the notion of locus as an abstract concept not within reach for students. In our presentation, a detail account of the findings will be reported. In addition, implications to teacher education and curriculum development will be discussed.

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UNDERSTANDING NUMBER MEANINGS AND REPRESENTATIONS OF DIRECTED NUMBERS*

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An initial study (Borba and Nunes, 2000) - using Vergnaud's (1982) theory of acquisition of mathematical concepts - showed that number meanings, conceptual invariants and symbolic representations affect reasoning about directed numbers. Controlling for number meanings and invariants, solving problems orally was significantly easier than having to make the representations of numbers explicit and operating on these explicit representations. Thus, performance in written assessments might lead to underestimating what children understand about directed numbers.

A second study investigated the possibility of teaching young children to represent directed numbers explicitly, enabling them to attain the same level of performance they showed orally. The participants were 80 children (mean age: 7y8m), pre-tested on problem solving using explicit representations of directed numbers. The children were matched for pre-test performance and then randomly assigned to four taught groups and one control group. The instruction the children received differed in number meaning (measure x relation) and form of explicit representation (writing x use of manipulatives). At post-test the taught groups performed significantly better than in the pre-test, but the control group did not. Children instructed on relations improved significantly more than the children instructed on measures. At post-test, the children transferred what they had learned in one form of explicit representation to other forms, and the representation used (writing or use of manipulatives) had no significant effect on performance.

Instruction was effective in teaching children to make explicit their implicit understanding of directed numbers. Instruction was more effective for the number meaning children had more difficulty in understanding. This suggests that instruction does not always have to start at the lowest level but rather that starting at an intermediate level children can build on their previous understanding of a concept.

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* This research was sponsored by CAPES (Fundação Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) as part of the first author's doctoral studies.

MATHEMATICAL NETWORKS AND THEIR TRANSFORMATION IN TEACHING AND LEARNING PROCESSES

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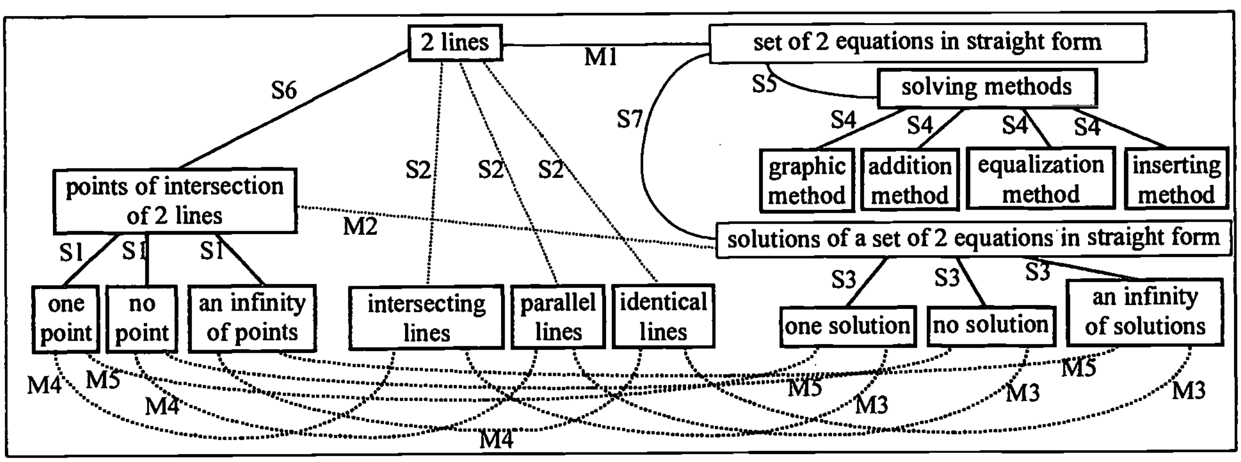
This contribution presents part of a study with the aim to investigate how a mathematical network as it is stated by teachers to have been presented in middle grade classes is transformed when carried over into students' minds. The study particularly focused on the topic "sets of two equations in straight form" and restricts on the investigation of some network relations according to subject systematics and a special relation according to the application of mathematical objects, the model relation. (For the definition of main network categories with relevance for mathematics education in school see Brinkmann, 2001.)

The participants of the study were 3 experienced and proficient teachers of different schools in Germany and altogether 137 of their students. The teachers' statements in respect to the network they implemented in classroom were investigated by interviews. In order to map out the networks learned by students, there were developed tests demanding from students activities closely related to those of concept mapping.

The figure below shows the network teachers stated to have taught. Those linkages that turned out in the tests to be achieved only by less than 50% of the students are drawn with interrupted lines. The different links according to subject systematics are marked with S, model links with M.



The study reveals the incompleteness of the transfer of implemented networks into students' minds. The mainly learned connections by students are part of the links according to subject systematics, model links are hardly known. Conclusions in respect to the teaching of the different sorts of connections have to be drawn.

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IT'S NOT WHAT YOU DO, IT'S WHY YOU DO IT: REFLECTIONS ON THE ROLE OF TEACHERS IN RESEARCH ON ALGEBRA

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Generalisation is accepted as an activity supporting students' use of algebraic notation and language. One class of such problems used in the research literature involves finding a general rule for, say, the number of matches used to make a row of n squares. There seem to be four styles of research with such problems:

1) students' responses (problems given without teacher intervention, but often with support of interviewer or teacher) are analysed (e.g. Friedlander et al, 1989)
2) a large-scale survey testing mathematical reasoning which, after analysis, is to be followed up with case studies of effective outliers (Küchemann & Hoyles, 2001)
3) discussions of personal pedagogical practice (e.g. Hewitt, 2001, Mason, 1988)
4) exploring details of teaching strategies alongside student work (Radford, 2000, Brown & Coles, 1999).

How is it possible for us, as researchers, to question assumptions that we are making and to become aware of different teaching approaches? Reading research papers does not seem to be sufficient experience. At the recent 12th ICMI study conference, in a discussion amongst researchers from many countries, it was possible to surprise each other as we discussed the research and teaching related to this class of problems. The surprises came as we experienced the detail of the teaching practices of others. At the heart of research are the developing awarenesses and actions of teachers and their students. If the aim of research is to improve teaching and learning, then we need to report on what is *possible* and include the complexities of teachers' purposes, knowing that these cannot be copied but with the hope of extending our practice. This paper comes with a plea that we make the role of the teacher and the related students' work central to our research projects and that we report, in some detail, case studies of examples of practice that surprise us.

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COGNITIVE PROBLEMS OF UNIVERSITY STUDENTS WITH THE CONCEPT OF DISTRIBUTION

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The concept of distribution, and specifically the notion of the Dirac's Delta Distribution, was found to be difficult for university students who deal with it in the advanced course of Mathematical Analysis. It is shown that these difficulties are related to mental models and schemes that the students have or develop at the moment when the concept of distribution is first presented to them.

In this study carried out with a group of 30 students from the Engineering Faculty of the University of Buenos Aires, those difficulties are described and discussed within the framework of the cognitive theory of intuitive models (Fischbein, 1987).

A modelling activity was presented to the students with the aim to observe how they established the isomorphism between the mathematical theory and a physical model. A questionnaire and interviews allowed us to study the students' beliefs, their ways of reasoning, and the intuitive models evoked during the problem-solving.

From the data analysis we could conclude that the use of the distribution for modelling has turned out to be unacceptable for the majority of the students. The students preferred to conceive the Dirac's Delta as a function of numerical variable – null everywhere except at zero where it is infinite, but which at the same time has a non-zero integral. However, their misuse of the familiar model of numerical function became an obstacle in understanding such general ideas as function spaces and functions defined on these spaces. The students have ignored these contradictions by giving the Dirac's Delta a status of "special" or "different function".

In designing didactical interventions, it seems highly recommendable that teachers know students' intuitive models. Modelling activities of real situations seem to be highly helpful to get this information and make the students to be aware of their own intuitive models, enhancing their meta-cognitive skills.

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TAIWANESE INSERVICE MATHEMATICS TEACHER'S BELIEFS AND VALUES CAN CHANGE

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Alan Bishop has studied values in mathematics for more than two decades (Bishop, 1988). Recently, he says (2001, p.245) "… there had been so little research attention paid to values teaching, and none related to student teachers' values. … the sooner we have more research focusing on values education in mathematics the better". In 1997, invited by Fou-Lai Lin, Alan's student, I joined their values research group as a director on junior high school level. In the previous case studies (Chang, 2000, 2001), it was found that the Taiwanese inservice junior high school mathematics teachers believe and value Platonism or absolutism in mathematics, score-ism in education, specialism in mathematics education. These beliefs and values the teachers held are incompatible with constructivism, and become the main obstacle for teacher to teach mathematics from a constructivist perspective. Hence, the research problem of the present study was how to change teacher' conventional beliefs and values in mathematics education.

The purpose of this article was to present how one teacher's beliefs and values in mathematics education changed over a period of two years. In this case study, a junior high school mathematics teacher, I-Fen, joined the program of in-service teacher education and the project of values teaching, and conducted a collaborative action research of value teaching with researcher in her mathematics classrooms. Data collection included classroom observations, interviews, questionnaires, and self-reports, teaching journals, teaching material or tasks designed by I-Fen. The findings showed that her teaching improved, and her beliefs and values in mathematics education changed.

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* This paper was supported by NSC grant NSC 90-2521-S-018-001, Values Teaching in Mathematics Classroom Project. The National Science Council or its staff does not necessarily share opinions and conclusions expressed here.

The affective domain and the achievement of the school children under two different mathematics instruction

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In Taiwan, the school mathematical curriculum and the associated textbook used to be unified. The new curriculum stressed student centered discoursing while all the previous curriculums stressed teacher centered lecturing instruction. The reformation was so great that it needed special justification for the sake of politics and education. Therefore, the main researcher, being a member of the curriculum develop group and the principal of the National Taipei Teachers College (NTTC) laboratory school carried out this minimum pair comparison of the two curriculums on 1998 and 1997 class students of NTTC lab school. We made a questionnaire on Mathematics Learning Attitude (MLA) to test students of both graduation years from grade 3 to 6. We also used a battery of Mathematics Achievement Test (MAT) based on the 1997 class grade 5 textbook to test all students at grade 5 and 6.

The test results made us to conclude that:

1.All the students thought that they all learned a lot of mathematics.

2.Experiment students (1998 class) liked more to do mathematics independently and to discuss with classmates.

3.Experiment students were more critical on the solution strategies and more aware of the written mathematical expression of the solutions.

4.All students showed equal achievement in taught scopes. For the experiment students, if the test items were not taught yet, they first showed inferior scores at grade 5, then jumped to the level equivalent to the compared students (1997 class) as long as the test items were then taught.

Based on the above conclusions we could carefully argue that the new teaching mode that the teacher poses a problem, then the students solve it voluntarily, then the students announce and defense the solutions, then the teacher steers the students to some sound patterns and the needed measures of restructuring of the materials to write a completely new textbook was an advance in school mathematics education. This study was a biproduct of six years of action research in NTTC lab school from 1992 to 1998 (Chung, 2000; Chung & Chu, 2001) to help teachers to cope with the challenge of teaching new curriculum. The scope and the depth of this curriculum reform, and the wisdom of the researcher to mobilize NTTC lab school to carry out this minimal pair comparison of the curriculums made the case of the curriculum reform in Taiwan worthwhile.

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Argumentation in the context of a didactic sequence in elementary algebra

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Mônica Maria Lins Lessa, Mônica Oliveira Osório

A growing number of researchers accept the view that the teaching-learning process is about the development of some shared understanding that comes out through discourse (Leitão, 2000; Douek & Scali, 2000; Steinbring, 2000; Yackel, 2001). The aim of the present work is to contribute to the discussion about the interest of discursive-argumentative activities for the conceptual development in mathematics. The analysis of an excerpt from a teacher-led argumentation will illustrate an analytical procedure designed to capture the process of knowledge updating through argumentation. This analytical procedure is based upon the analysis of three types of discursive actions: ***pragmatic actions***, concerning the offer of necessary psycho-social conditions to the emergence of *divergence* among pupils, as well as the encouragement of debate as the way to negotiate divergence and attain a consensus; ***argumentative actions***, concerning the presentation of opinions / points of view, with their respective *justifications*, as well as the consideration of opposite arguments; and ***epistemic actions***, referring to certain forms of reasoning and informational topics specific to an epistemic domain, like mathematics. Starting from these views, we analyze a fragment of a didactic sequence for the early introduction to algebra among 7-8 year-old Brazilian students (Da Rocha Falcão, Brito Lima, Araújo, Lins Lessa & Osório, 2000). This analysis seeks to demonstrate that this fragment of didactic sequence can be characterized as a process mainly oriented to formulation and justification of points of view, the counter-argumentation being present in only one utterance of the teacher. A possible general conclusion allowed by the analysis above is that the argumentative process remains *partly* implemented in the discursive fragment of didactic sequence that is reported here. This is a relevant aspect for research concerning didactic and conceptual development in mathematics.

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BROADENING THE SCOPE OF THE ILLUSION OF LINEARITY: A CONCEPTUAL REANALYSIS OF PROBABILISTIC MISCONCEPTIONS

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Linear (proportional) relationships are undoubtedly one of the most common models for representing and solving both pure and applied problems in mathematics. But according to several authors, the attention given to linear models in current elementary and secondary mathematics education creates in students a tendency to use these linear models also in situations wherein they are not applicable, e.g. the belief that if the side of a square is doubled, the area also will be doubled (NCTM, 1989). This phenomenon is referred to as the illusion of linearity.

This short oral communication reports the results of a review of the literature about the illusion of linearity in the domain of probability. This literature survey has shown that there are several probabilistic misconceptions that have a remarkable common feature: the fallacious reasoning can be conceptually linked to and possibly explained by an improper application of proportions or linear functions when approaching probabilistic situations. The obtained inventory of linearity-related misconceptions contains both misconceptions that have already been studied intensively before as well as phenomena of a more anecdotal nature found in the literature.

Within the inventory of linearity-related misconceptions, we identified a special category of very closely related erroneous reasonings, wherein there is an improper assumption of a linear relationship between the determining variables of a binomial chance situation. This framework appeared to be a powerful tool to analyse the possible linear reasoning that might be elicited in binomial chance settings. It offered a new viewpoint for looking at some historical and well-known misconceptions and for seeing conceptual links between these – apparently very different – struggles with the laws of probability.

In a next step in our research, we set up an empirical study aimed at investigating to what extent the overreliance on the linear model is indeed the mechanism underlying students' faulty thinking when they suffer from the probabilistic misconceptions that we described in this short oral. The preliminary results are reported in Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel (2002).

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RECOGNISING, RECALLING AND DOING IN THE 'SIMPLIFICATION' OF TRIGONOMETRIC EXPRESSIONS

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I consider students' 'simplification' of trigonometric expressions and present an operational model of how they simplify expressions. The model has three components: recognising, recalling and doing. I describe the interaction between these components and link this model to other models, e.g. Saxe (1991), of doing mathematics.

Data on students' manner of simplification was obtained using concurrent verbal protocols as students solved simplification items to gain an insight into their thinking. revealed a uniformity of approach with regard to approaches to simplification.

Students begin by reading the question and then focus on a subexpression or an algebraic form – I call this 'recognising'. They then 'recall' trigonometric or algebraic properties. In the schematic of my model (see Figure 1 below) read, recognise and recall are grouped together under the term 'recognition' because they all rely on sign association. Students then 'rewrite' the given expression with some form substituted for another. They then examine the rewritten form and recognise/ recall another subexpression/property and enter a further manipulate/ rewrite phase or accept their rewritten form as the result, the simplification. Recall, manipulate and rewrite are group under the term 'doing' because they all rely on transforming signs. The model, in its current form, has 'recall' in both the 'recognition' and 'doing' groups which reflects the dialectic between recognition and doing in this context.

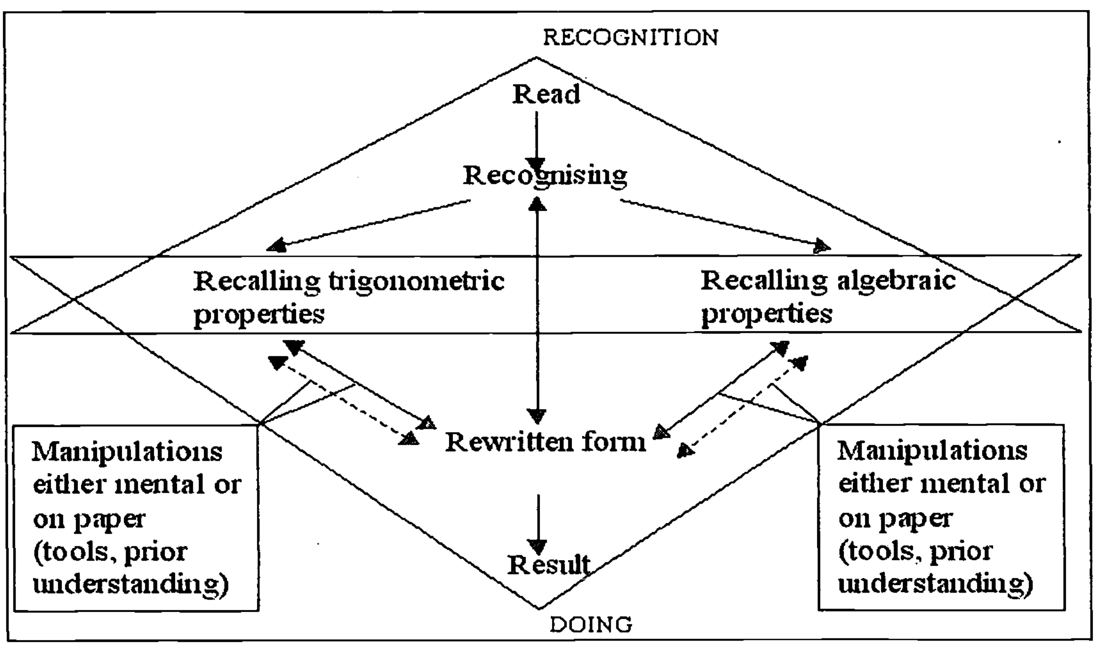

Figure 1 *Schematic model of students' simplification of trigonometric expressions.*

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The context of this study was development of teaching leading to development of students' conceptual understanding in secondary school mathematics classrooms in Pakistan. The aim of this study was to explore teachers' learning in their classrooms as they participated in a co-learning partnership (Wagner, 1993; Jaworski, 2000) with a teacher educator. Collaborative work with the teachers focused on supporting the teachers in developing their teaching while investigating both the teachers and the teacher-educator's learning. The teacher participants taught secondary school mathematics and had resumed teaching in their schools after attending a 'Visiting Teacher' mathematics education programme at a university in Pakistan. Data was collected through maintaining field notes from the classroom observations, audio-recording interviews in pre and post-observation meetings and written comments in reflective journals. The analysis uncovered issues of practicality and limitations of a co-learning partnership in teacher development in the real classroom context.

This study shows that responsibility of a teacher educator to teachers' developing teaching cannot be ignored in a context where teachers have never been encouraged to question or analyse their own or others' actions within their school. In addition, teachers have limited knowledge and understanding of mathematics relating to new practice, and might not be aware of their own mathematical misconceptions. Therefore, nature of collaborative partnership cannot be achieved by the singular influence of any ideology or the theoretical assumptions of collaborative work. It is utterly dependent on the needs of the teachers and the reality of their context. Developing an attitude in which teachers see and experience questioning as learning should be integrated with the provision of adequate interventions.

This study suggests conceptually, that a commitment to learning establishes a teacher educator as a learner along with the teachers. The teachers' commitment to learning activated their understanding of change incorporation in the practical reality of school. The teacher educator revisited her theoretical principles of collaborative partnership within the school's reality in order to support teachers struggling with issues that impede change. The differences in knowledge and understanding were not viewed as teachers' deficits or a teacher educator's superiority but were appreciated as resources of co-learning. However, issues relating to differences in knowledge and understanding between partners, and the constraints of school encouraged the teacher educator to take a leading role in the teachers' learning.

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MATHEMATICAL ROMANCE: ELEMENTARY TEACHERS' AESTHETIC ONLINE EXPERIENCES

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We explore the aesthetic as an integral dimension of mathematics education and we report on our study of elementary school teachers' experiences with the aesthetic dimension of mathematics in an online mathematics teacher education course. The study examines the online transcripts of the thoughts and discussions of 20 teachers participating in the online course. We look primarily at the first of six modules of the course, which offered teachers the opportunity to reflect on the nature of mathematics and mathematics pedagogy. The readings for the first module included three interviews with mathematicians who related their love for mathematics, two articles on the value of children inventing personal algorithms for arithmetic operations, and a text reading of problem-based mathematics activities. In the online discussion for the first module, teachers shared their personal views of mathematics, to describe the methods they used to mentally solve the arithmetic problems 16 x 24 and 156 + 78 + 9, and to reflect on the implications for teaching.

Our research depends on teachers' stories about mathematics, particularly their romantic stories. How they came to love math, or hate it, and in some cases, how an old passion was sparked and re-ignited. We rely on the idea of romantic story with its ideas of overcoming difficulty and finding happiness (or wholeness) at the end of the day. We suggest that research on the dialectic relationship between teachers' beliefs and practice may be expanded to include their stories about childhood and other experiences of mathematics. As well, we look for the mathematics stories embedded or perceived by teachers in curriculum documents, in classroom practice, and in mathematics activities. Learning about, and being in tune with our aesthetic sensibilities, may in part guide and bind us to stories of mathematics and to our underlying beliefs and practice.

The stories of mathematics articulated by teachers in our study were based on personal experiences with mathematics that were aesthetically informed. This was manifested in statements such as "I LOVE math. I always have. I, like many of you, had many problem-solving car trips. I still get excited when I see a licence plate that I can make ten with (using any means)." Note the teacher's verbal expression of delight and the use of capital letters to convey her emphasis. Teachers appear to have been affected by the online course experiences with mental arithmetic in conjunction with the journal article readings. As their personal views and beliefs were enlarged some teachers interpreted curriculum through 'new spectacles'. These new insights into mathematics and mathematics teaching were aesthetically informed and story-based. The interplay between the various stories that were available in the online course is a first step in incorporating new--hopefully romantic--plots.

Local Magnification and Theoretical-Computational Conflicts

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The aim of this research is to understand how conflicts between computational and non-computational representations can contribute to enrich students' concept image of derivative and limit. We consider *theoretical-computational conflict* as any situation where a computational representation for an object is (at least potentially) contradictory with the associated mathematical theory. For example, numerical calculation with machine accuracy cannot be performed in a way which corresponds exactly to mathematical theory of limits.

To perform the investigation, we designed a computer-based Calculus course having the notion of local straightness as a cognitive root for the derivative concept, as suggested by Tall (2000). On this approach, we give emphasis to theoretical computational conflicts, instead of avoiding them, particularly those related with the impossibility of computational representations for limits and infinite processes in general. This course was tested with a first year undergraduate class and a sample of six students was selected for weekly clinical individual interviews, when they are given theoretical-computational conflicts.

Partial results show that students develop different mental strategies to cope with this kind of situation. The experiment also seems to indicate that, in general, students can build up more sophisticated theory and establish richer links between cognitive units to answer questions that emerge from theoretical-computational conflicts.

The full paper can be downloaded from www.dmm.im.ufrj.br/~victor.

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MATHEMATICAL LANGUAGE AND PUPILS' PERFORMANCE

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This paper reports on an investigation that was carried out to establish the effects of specialised mathematical language on the performance of pupils at secondary school level. Since Zimbabwe's independence in 1980, the lowest pass rates in national examinations at secondary school level, have been in English and Mathematics (Riddel and Nyagura 1991). The pedagogical question arising from this scenario is whether or not the performances of pupils in these two subjects are related. Official syllabus documents, stipulate that one of the aims of mathematics education is to make pupils understand, interpret and apply mathematical information. The question of interest here is whether this is being achieved. The majority of students in Zimbabwe fall into the category which Barwell (2000) called English Additional Language. This means that mathematics is taught in English, to pupils whose first language is not English. In such situations there is always a possibility of lack of equivalents in the pupils' language for some mathematical expressions. In addition to this, a mathematical fact that appears obvious to adults may be a highly structured and stylised statement to pupils Southwell (1994).

Two mathematics tests were administered to Form 3 classes (15-16 year olds) in four randomly selected secondary schools in Harare, Zimbabwe. The first test established how pupils interpreted specialised mathematical terminology, while the second one tested the applications of the interpretations in word problems. The question- by-question analysis of the pupils' performance, revealed that many pupils used the same interpretation in answering corresponding test items in the two tests. Thus most of the pupils who failed to interpret mathematical expressions correctly, also failed to solve the corresponding application problems. For such pupils, incorrect interpretations led to incorrect application solutions. On the other hand correct interpretation resulted in correct application solution strategies. Therefore the study concluded that specialised mathematical language has an effect on the performance of pupils.

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USE OF A COMPETENCE MODEL TO DESCRIBE STUDENTS' UNDERSTANDING OF CONCEPTS RELATED TO IMPROPER INTEGRATION[1]

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Improper integrals constitute a concept of great utility for Mathematics degree students. However, it appears that students are unable to assimilate this concept within the wider system of concepts they learn in their first year of Mathematics studies. In this oral communication we describe a competence model used in a preliminary and exploratory study of the kind of understanding students possess about improper integral calculus when two registers of representation come into play. We understand competence to be the coherent articulation of different semiotic registers: to be competent in mathematics is to be able to articulate coherently the different representations of a mathematical concept when having to solve "non routine" problems. According to us, the evaluation of our students' knowledge means that this should be analysed on the basis of activities that aim to clarify possible connections (articulations) made by them while constructing a given concept. In order to design our competence model we adapted different stages of development noted in cognitive representation systems in the case of algebraic language (Socas, 2001) to our own concept. After analysing the results to a questionnaire, six students were selected to be interviewed on the basis of their overall results and the significance of some of their answers. The main difficulties we detected are due to the lack of meaning or of knowledge of previous concepts. We can also see that, although some of our students have managed to make representations in the graphic register to some extent, going from graphic to the algebraic remains more difficult than going from the algebraic to the graphic. We will also discuss the formal competence model used.

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[1] This work has been partially supported by grant AP2000-2106 (*) and the contract of DGI BXX2000-0069 of Spanish MCT

PERCEPTIONS OF A GROUP OF PRIMARY TEACHER TRAINEES AND MATHEMATICS TEACHER TRAINES ON THE USE CALCULATORS AS A COGNITIVE TOOL

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Hand-held personal technology (H-hPT), e.g. calculator, can be used as a cognitive and mathematical tool in enhancing mathematical understanding, and give the opportunity to the user to confront with the real life mathematics. As a result, students at each grade level achieve a higher degree of mathematical thinking and save time for problem solving. In the present study, which is an ongoing project on H-hPT, a group of teachers' and teacher trainees' perceptions of calculators as a cognitive tool in mathematics instruction is searched, and their views were reflected to some extend. In the study, the researchers have looked for the answers to following two questions: (a) What are the differences between the perceptions of teachers and teacher trainees about the use of calculators, (i) calculators as a cognitive tool, (ii) using calculators in maths classroom, (iii) use calculator in problem solving? (b) What are the differences between the development problem-solving skills of teachers and teacher trainees before and after the seminar and workshop in mathematics teaching and learning? In the first year, the planned activities were programmed and scheduled as seminar and workshop for the participating mathematics teachers from various schools, teacher students and teacher trainees from Balikesir University in Turkey. A rather detailed questionnaire was administered in the two-days seminar and workshop to the participants, and an interview protocol was designed and applied after the activities. In the interview, 6 teachers and 12 teacher trainees were participated.

The statistical analysis of the designed questionnaire is in the process, and some results have been obtained. The first result shows that although the teachers and teacher trainees had negative opinions about the use of calculators in mathematics instruction and student-centred activities at the beginning of the seminar and workshop, their attitudes and beliefs were changed towards positive at the end. Interview results are also supporting this notion. The detail of two-day seminar and workshop and their reflections to the interview will be reported, and teachers' and teacher trainees' perception and views will be reflected in presentation of the paper.

IMAGERY IN GEOMETRICAL PROBLEM SOLVING

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The essential role of imagery on students' geometric reasoning has been researched extensively (Bishop 1989). Different types of imagery have been proposed (Owens & Clements 1998, Persmeg 1986). Usually students are not restricted to only one type of imagery, they use or combine different types of imagery simultaneously, according to the situation.

Nevertheless, the school system discourages visualization and imagery as proper tools for the formal solution of mathematical problems, especially after 9th grade, giving emphasis to abstract thought and typical proofs. As a consequence, "geometric- type" students (according to Kruteskii's typology) are assessed as "low-ability" ones.

The present contribution is based on the case study of a 10th grade student with a poor theoretical background, whose mind is "dominated by images". Faced with four geometry problems, demanding different types of construction, she is led to unexpected solutions. This ability is altogether unknown both to her school and her, since she is assessed as one of the weakest students of her class and she believes she has no aptitude in Geometry.

Analysis of the student's responses shows that imagery can function positively both for the understanding and solving of geometrical problems, even in cases of students with a relatively poor theoretical background.

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HOW THE GEOMETRIC CONSTRUCTIONS HELP THE UNDERSTANDING OF COMPLEX NUMBER?

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Introduction. The objective of this research was to analyze how the geometric constructions used in activities of dynamic geometry help students and teachers to produce meanings for the body of complex numbers. The research took place during the 2000 and 2001 years in an institution that prepares undergraduation teacher located in the western part of Rio de Janeiro, Brazil.

Participants of the research. Participated in the work two students and two teachers from the Undergraduat Course in Mathematics of the University. The two students had different preparation levels. One already knew complex numbers in an algebraic and automatical way and the other one never had studied this subject. The two teachers were engaged in mathematical investigation and were much interested to learn new methodologies for the construction of the meanings of mathematical concepts.

The framework. To analyze the development of the activities we used Tall (1981) who treats the formation of concepts of mathematical objects and also Hershkowitz(1994) who formulates an outline about use of the definition of concept to students. We tried to understand how the learner appropriates the mathematical concepts. We observed that human mind do not act exactily in accordance whith symbolic logic. The concept is built by different experiences and approaches arriving to what Tall calls the "Conceptual Image". Tall created also, joining the words "procedure" and "concept", the name "procept" to designate the process to the construction of a concept. Since we worked with dynamic geometry we created activities using, as a tool, Cabri Géomètre II software. The environment provided to students the possibility to creat, to conjecture, to eleborate and to represent contents.

Conclusion. The student who had not previously learned about complex numbers built its concept better and in more appropriate way than the other one. We conjecture that use of visualisation in the geometrical approach is more effective than the algebraic approach. The theachers were glad to learn a new way to introduce complex number in the classroom.

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A TEACHER EMPOWERMENT PROGRAM- A MODEL OF TEACHING

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A new curriculum for teaching elementary school mathematics is being developed in Israel. The new curriculum stresses (in light of the NCTM 2000 standards) inquiry and investigation in learning mathematics, learning for understanding, using connections among mathematical ideas, applying a variety of appropriate strategies for problem solving and developing number sense.

The ministry of education requires that from now on all mathematics teachers be specializing in mathematics (mathematics being the main subject they are teaching) starting already at the first grade of elementary school.

In light of all the above, a broad in-service program for the empowerment of elementary teachers is being processed aimed at disseminating the reform. In a former PME paper, Becker (2001) pointed out that there is little information available about the design features of in-service programs which maximize changes in teacher practices.

A model for enhancing teacher development was designed for the in-service program. In our short oral presentation we will describe this teaching model that we use throughout the in-service program for each teaching unit. This model will be demonstrated on a specific unit dealing with numbers and operations. A task (of an inquiry type) will be presented. We will describe the mathematical objectives and content and its suitability to the various goals of the new curriculum.

The process the teachers underwent will be described, starting with working on the given task, finding various appropriate strategies for solving it, using different types of reasoning, methods of proof and documentation, recognizing and making connections among mathematical ideas and reflecting upon the process of problem solving. Then, adapting the problem to their class level, presenting it in class, evaluating students' work and documentation, promoting students' discourse and monitoring students' discussions and debates.

Teachers' activities as well as students' work will be presented and linked to the knowledge required for teaching (Shulman, 1986).

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RESEARCH ON THE PROCESS OF UNDERSTANDING CONCEPTS OF TRIANGLE AND QUADRILATERAL IN A CLASSROOM

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In order to understand the process of understanding mathematics, Koyama (1992) presented the so-called "two-axes process model" of understanding mathematics as a useful and effective framework for mathematics teachers. The model consists of two axes, i.e. the vertical axis implying levels of understanding such as mathematical entities, relations of them, and general relations, and the horizontal axis implying three learning stages of intuitive, reflective, and analytic at each level. By analyzing an elementary school mathematics class in Japan, Koyama (2000) demonstrated the validity and effectiveness of this model and suggested that a teacher should make a plan of teaching and learning mathematics in the light of "two-axes process model" and embody it with teaching materials of a topic in due consideration both of the objectives and the actual state of students, and that she/he should play a role as a facilitator for the dialectic process of individual and social constructions through a discussion with students.

The purpose of this research is to examine closely the 38 second-graders' process of understanding concepts of triangle and quadrilateral in a classroom at the national elementary school attached to Hiroshima University. These students are characterized to be at transitive stage from the first to the second level of geometrical thinking in terms of van Hiele Model (1986). In order to promote and improve their understanding concepts of triangle and quadrilateral, with a classroom teacher, we planned the teaching unit of "Let's make figures with geo-board" and in total of 11 forty-five minutes' classes were allocated for the unit in the light of "two-axes process model". The data was collected in the way of observation, videotape-record, and pre- and post-tests during these classes, and analyzed quantitatively and qualitatively to see the change of students' thinking level and the dialectic process of individual and social constructions through discussion among them with their teacher in the classroom. First, as a result of first two classes of them, the quantitative analysis of pre- and pre-tests showed the remarkable improvement of students' understanding concepts of triangle and quadrilateral from the first to the second level. Second, as a result of the qualitative analysis of students' discussion on whether a "concave quadrilateral" is triangle or quadrilateral in the fifth class, we found that students could investigate and communicate the reason of their own judgment with geo-board, geo-paper or matchsticks, and that such a new idea/definition was emerged that a "concave quadrilateral" is not triangle but quadrilateral because it is divided into two triangles by the line segment connecting two opposite vertices.

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PROBLEM SOLVING THE CHALLENGE FACING SOUTH AFRICAN MATHEMATICS TEACHERS

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Mathematics Curriculum reform in South African classrooms has been on the minds of mathematics teachers since the introduction of Curriculum 2005, South Africa's version of Outcomes Based Education. The curriculum has introduced new perspectives and challenges on the teaching and learning of mathematics. Amongst others the shift to a problem centred or constructivist approach to the teaching and learning of mathematics is implied in the new curriculum documents.

Despite the rigorous training provided in the implementation of Curriculum 2005, affecting change to teacher's philosophies or beliefs about the teaching and learning of mathematics has been marginal.

Although a few teachers have engaged in problem solving as a context for teaching mathematical concepts, the vast number of teachers still rely religiously on routine textbook problems, for the introduction or consolidation of mathematical concepts. These teacher's classrooms are dominated by traditional teaching practices.

As a response to the need to promoting mathematical problem solving in schools, a project was initiated to providing training and support materials to teachers in selected schools. This report will present the results of this initiative. The research conducted makes an exhaustive analysis of the attitudes and responses of both learners and teachers involved the project.

The results of the project have had significant implications for future training initiatives for Curriculum 2005, and writing of learner support materials to be used in the mathematics classroom.

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DIFFERENT VIEWS OF KOREAN AND GERMAN TEACHERS BELIEFS ABOUT MATHEMTICS

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Since the eighties in mathematics didactics, a branch of research so-called Beliefs ("Mathematische Weltbilder") is developed.(cf. Törner, 1998) Results of the "Beliefs"-research have showed a further possible reason of the difficulty in proving and solving the mathematical tasks.(Pehkonen, 1995) According to Pehkonen, the competence of the problem solving does not depend on the students' mathematical knowledge or mathematical competence, but also on the beliefs about mathematics. In his opinion, beliefs about mathematics should be included as an explaining factor, which influenced on problem solving.

Our research aims are to find the role of beliefs in problem solving, specially related with proving as followed:
(1) Are beliefs the factor for the mathematical competence of proving? (2) Do the teachers Beliefs about mathematics influence student to develop their Beliefs about mathematics? (3) How differences do Korean and German teachers have in Beliefs about mathematics?

We asked 58 Korean and 27 German middle school teachers and 192 Korean and 659 German 7th grade students. We used the questionnaires of Klieme(2001) who refined and categorized the questionnaire of Grigutsch(1996). We have identified them as three broader categories, Application, Formalism(Exactness) and Process(Process as creativity). The view of German students about mathematics is dominated with the fact that mathematics is Formalism(F), Application(A) and Process(P) in the order named, the view of Korean students is similar as them, however, we can find quite different of their value. For German students, A=0.30, F=0.54, P=0.21, for Korean students, A=0.58, F= 0.63, P=0.55. While Korean students and teachers both have similar views about mathematics, German teachers have quite different view from students. We did not identify a significant effect concerning the influence of Beliefs about mathematics on mathematical competence of both Korean and German students.

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MAKING AND PROVING CONJECTURES: CIRCLES

Eva Tsz Wai, LAM

This study explores how the Sketchpad environment may help in developing students' mathematical thinking in relation to proof. My intention is to create situations in which students make a conjecture from what they have discovered and then try to prove it. This approach is supported by Bell, Mason, Burton, Stacey and Cockcroft (as described in Hoyles, 1997). They all argued that students should have opportunities to test and refine their own conjectures, thus gaining personal conviction of their truth alongside the experience of presenting generalizations and evidence of their validity.

Five Secondary 4 students (age 16) participated in this study. Their mathematics abilities were average. In the workshops, students learnt some basic properties of circles. They interacted with the relevant dynamic figure, then made conjectures and finally tried to prove them deductively. A video camera was set up to record the participants' behaviour. Three sets of worksheets were given to all participants to guide them how to do the task and write their proofs. An in-depth interview was conducted after each workshop. In this study, the following questions were addressed:

1. How do students discover and form conjectures using Sketchpad?
2. How do students prove their conjectures?
3. How do students interact with each other and with the computer?
4. What do students feel about this approach?

From my observations, students went through the following process as they made and proved their conjectures: Intuitive Observation (based on the impression of one figure to make a conjecture), Inductive Hypothesis Making (based on some common properties of many figures to make a conjecture) and Deductive Explanation (based on theorems to prove a conjecture). It was found that Sketchpad can motivate students to learn and helps them to think inductively. However, there was no evidence that Sketchpad can help the students to think deductively. They all needed some hints from the teacher. Moreover, students working as a pair on one computer seemed to have more courage to drag the points around, thus it might be the most effective situation to carry out such activity.

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AN AVERAGE WITH UNIMAGINATIVE WEIGHTS WHEN THE WEIGHTS EQUAL THE VALUES

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The **self-weighted mean,** denoted *SW*, is a weighted mean in which the weights are the values themselves:

$$SW = \frac{x_1^2 + x_2^2 + \ldots + x_n^2}{x_1 + x_2 + \ldots + x_n}$$ where $x_1, x_2, \ldots, x_n$ are n positive numbers.

It is easy to see that *SW* is greater than the arithmetic mean (A), because in *SW* the larger values get larger weights. *SW* is linearly (and positively) related to the *variance* of the n values. Lack of distinction between the two means is sometimes referred to as "sampling bias" (Stein & Dattero, 1985).

SW is called for whenever the probability of sampling a given value is proportional to its size, that is, under self-weighted sampling (SWS), usually known as "size/length-biased sampling". SWS is encountered in diverse areas: demography, medicine, management science, and many others (Patil, Rao & Zelen, 1988). Therefore, understanding this concept, and gaining reasoning ability about the mean in SWS situations is important for both information consumers and data producers.

Our research focuses on learning and analysing people's intuitions: How do people intuitively fare in different SWS tasks (e.g. assessing the expected waiting time for a bus that arrives at varying intervals, or the mean class size obtained by questioning students)? Are there typical fallacies, and how could they be overcome? We first hypothesised that some people might fail to understand the need to weight the averaged values, thus calculating *A* instead of *SW*. This hypothesis was confirmed in our experiments. The implicit assignment of equal weights to all the values is compatible with Tversky and Kahneman's well-known description of heuristic principles that people rely on when assessing the value of an uncertain quantity. In particular, tacitly assuming uniformity is in harmony with people's predilection for symmetry and equality (Zabell, 1988). We also found that experiencing the problem's procedure – even only via a thought experiment – provides a useful corrective instrument, as does emphasising the weights by using a roulette with variable angles, and asking about the mean of these angles, for repeated turnings of the roulette.

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THE USE OF STUDENT QUESTIONING FOR ASSESSING MATHEMATICAL UNDERSTANDING

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Abstract of the Study

The issue I am most concerned with is how to enhance the mathematical communication that helps the students acquire mathematical concepts better. Conceptualization of thinking serves as an instance of communication, which in turn provides useful insights for an understanding of human cognitive process (Sfard 2000). Self-questioning defines an active role for the students to construct the solution strategies based on their own personal knowledge during the process of solving mathematical problems. In this study, I intend to make use of student questioning for assessing mathematical understanding.

The Methodology

Two groups of students (aged around 14-16, Fourth Formers) would be asked to solve a set of mathematical problems. The number of students included in each group would be decided on the basis of time constraints, with one being classified as expert-student group (the top students in my class) and the other as novice-student group (the students with below-average performance in mathematics). During the problem solving episodes, the verbalization process of students' thinking would be recorded and the think-aloud protocol is analysed for identifying the key questions being raised. (Sample of data would constitute part of my presentation.)

The Significance of the Study

The mathematical communication is an essential component of curriculum reform in Hong Kong for enhancing the teaching and learning of mathematics. The present study serves the purpose of enriching our understanding of the phenomenon of communication in term of self-questioning during the problem solving episodes. Understanding the self-questioning processes provides us with a key to enhancing the effectiveness of students' learning strategies in doing the self-study.

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ANALYSIS OF CHILDREN'S ARGUMENTATION WHILE MAKING SENSE OF FRACTIONS.

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Argumentation and articulation of ideas are viewed as important processes in the development of children's mathematical thinking. Analysing children's argumentation is therefore important.

Wood (1999) states that "...conceptual change and progression of thought result from mental processes involved in the resolution of conflict" and Yackel (2001) underlines the importance of challenging and justifying explanations. Our analysis of an episode where a group of three Grade 5 children are working on a problem to find a fraction of a fraction, shows an absence of disagreement or challenging of others' ideas.

We used Toulmin's scheme as a methodological tool. According to Toulmin, an argument consists of four parts. A *claim* is made and *data* is offered as grounds for the claim. A *warrant* explains why the data supports the claim, while the *backing* finally links the core of the argument to collectively accepted assumptions (Yackel, 2001).

In the episode, several claims were made and data and warrants offered as support. The claimants tried to find a backing, thereby trying to justify their claims, even though they were not challenged to do so by one another. They were unable to find backings for those claims that were mathematically invalid. The researcher then focused them on a mathematically valid claim that one of them had made earlier. This minimal facilitation enabled the claimant to provide a backing and she made an effort to have the backing accepted by the rest of the group. Eventually a frame switching, from a numerical backing to an iconic backing, made it possible for the whole group to accept the full argument.

We argue that it was the acceptance of the obligation to explain and justify explanations by the whole group that drove the discourse and kept the group from reaching early closure and not disagreement or challenging of others' ideas. It was clear from their claims, data and warrants that their fraction concept was not yet stable, making it difficult for them to challenge others' ideas.

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TEACHERS' THINKING ON INCORPORATING GRAPHING CALCULATORS INTO MATHEMATICS TEACHING

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Research on graphing calculators has focussed predominantly on students with the teacher's role being largely neglected. This study examines graphing calculator use from the perspectives of advanced-level mathematics teachers working in English schools; adding to recent research on teacher's beliefs (Doerr & Zangor, 1999) and teachers' perceptions (Simonsen & Dick, 1997) in relation to graphing calculator use.

Since the intention of the study is to elicit in-depth accounts of teachers' perceptions and experiences of graphing calculator use in their classrooms, an interpretative framework has been adopted. Semi-structured interviews and lesson observations with six teachers from two institutions form the data corpus.

Findings from the study indicate a set of interrelated considerations and motivations for use. Moreover, the process of incorporating graphing calculators into their classroom practice involved several forms of *teaching* (technical, functional and conceptual), *guidance* (indirect and direct), and *support*. Teachers were found to develop knowledge of a range of pitfalls students were likely to encounter when using the graphing calculator, and from this a repertoire of troubleshooting techniques appeared to evolve.

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DEFINING A RECTANGLE UNDER A SOCIAL AND PRACTICAL SETTING BY TWO SEVENTH GRADERS

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1、Purpose of Study

§ What kind of activities could initiate 7th graders into defining?
§ How would the students reason when they involve in the defining activity?

2、Method: designing a Social and Practical Setting

§ Social: interviewers' intervention and peer discussions
§ Practical: the swimming pool task

3、Findings

§ propositions: e.g., if two opposite angles are right angles in a quadrangle, it is a rectangle.
§ concept/theorems-in-actions (Vergnaud, 1998) : e.g., stereotyped concept image of right angles is horizontal and vertical.
§ apprehensions (Duval, 1995) : perceptual, operative, sequential and discursive

4、Conclusion

§ Students are stimulated to develop definitions by the innovative task.
§ Implicit theorems/concepts support students' apprehensions of figures.
§ Defining can be the beginning of learning to prove.

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A TIME OF CHANGE

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Pupils' lack of progress in mathematics as they make the transition from primary to secondary school is well documented (Fullarton, 1996; Murdoch, 1986; SOEID, 1996) and often recurs as an emotive headline (Daily Mail, 2001; Daily Telegraph, 2001; Sunday Telegraph, 2000) following the publishing of reports or statistics.

This study is in the process of investigating:

- groups of pupils who may be affected by the transition and
- the effects on pupil progress from factors within the classroom/school

Factors considered are those over which mathematics staff have some control, and the power to adapt to meet pupil needs. The factors are those related to the classroom: classroom climate, teacher and curriculum.

The research study takes place over three years in two cycles, following two samples of pupils as they move from primary (9 schools) to secondary school (3 schools). In each cycle a sample of pupils is observed and tested during the last year in primary school and the first year in secondary school. All data will be available from May 2002.

In this presentation, results from both cohorts will be discussed. Pupil progress over the period will be considered and related to pupil and teacher perceptions of the classroom interaction. Groups of pupils will be identified linking their progress to influences in the classroom.

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THE CALCULATOR'S ROLE IN PROBLEM SOLVING: A CASE STUDY IN A FOURTH GRADE CLASSROOM

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In Portugal, the fundamental aims of mathematics teaching are based on the development of three skills: (a) reasoning; (b) communication; and (c) problem solving (DEB, 1998). The curriculum supports that the solving of problematic numeric or non-numeric situations should be proposed in all teaching topics. This document also refers the calculation methods that should be used in the classroom. Written procedures, mental calculation and the calculator should be present in daily routine classroom. It is important for children of this grade to be able to compute using one of these methods, as well as to be able to decide which means is more suitable for a specific situation (Abrantes, Serrazina & Oliveira, 1999; NCTM, 1991).

However, in the majority of Portuguese elementary schools, there is a certain degree of resistance to the use of calculators, which raises difficulties in its employment in problem solving situations. As a matter of fact, the Portuguese document "Matemática 2001", which reports diagnostic and recommendations for mathematics teaching and learning, presents some results on the use of calculator in first grades. For instance, almost 74% of the teachers do not use calculators to solve complex problems, 71% do not use it to explore numerical concepts and 63% do not use it to verify results (APM, 1998). It seems that many elementary school teachers ignore how powerful a calculator can be in a problem solving context.

Making an effort to improve this situation, a qualitative case study research was developed in order to understand the role that pupils attribute to calculator as a support tool for computation in a problem solving context. The study was implemented in a ten years old class in Braga, Portugal. In this study the calculator's use in solving tasks of estimation, numerical investigations and application of mathematics to real life situations was analysed, attending to strategies used and to interactions between pairs.

This presentation intends to show how was developed the research and to present and discuss its results.

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PROMOTING STUDENTS' AWARENESS IN APPLYING BIJECTIONS IN ENUMERATION TASKS

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Much mathematical knowledge is retained in the mind in a factual mode that is not in the form that can be readily accessed for problem solving activities. This was noted as far back as 1929 when A. N. Whitehead talked about 'inert knowledge'. We conjecture that this phenomenon may occur for some very basic notions. In this communication, we will specialize in one such notion. A 1:1 and onto function cognitively corresponds to the pairing off of each element of the domain with one of the co-domain such that no elements are left unmatched after the process ends, so bijections preserve set order. (Here the more overtly definitional case of infinite sets is not considered.) However, if students are not aware of this knowledge as a potential problem- solving tool, they could not avail themselves of approaches involving the construction of bijections in enumeration tasks.

The study has two aims. The first is to provide some evidence that indeed the knowledge of the preservation of set order under bijective correspondence is inert. The second is to test whether the above situation may be ameliorated through a particular teaching approach. What motivates the teaching approach is the following. To try to make the students more aware of applications of bijections in enumeration tasks, we propose a general framework about developing techniques to address inert knowledge. A technique lays down several standard stages to achieve, but their accomplishment may require substantial problem solving. An important factor in a technique is a 'cue', i.e. a description of the circumstances where the technique may be regarded as being worth considering. The rationale behind the cue corresponds to the 'releasing' of some inert knowledge. When the understanding of why the technique works is transparent, the awareness of the technique depends mostly on the familiarization with its cue. The idea of the teaching approach is to prompt the students' attention to the cue (in its contextual form), and after this to (largely) leave the students to carry out the application itself. In our case, the cue would either be two sets whose orders we wish to compare, or one set whose order we wish to determine by identifying another (better understood) set for a correspondence.

In the presentation, we shall briefly indicate how the above aims were implemented in fieldwork conducted on college students, and shall outline the results and conclusions reached.

HAVE WE DONE A GOOD JOB? WHAT DO TEACHERS THINK ABOUT THEIR TRAINING TO TEACH MATHEMATICS IN ELEMENTARY SCHOOL

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What should preservice teachers study during their college years in order to become good mathematics teachers in elementary school? Should we emphasize subject matter knowledge since "without adequate content knowledge student teachers spend much of their limited time learning content, rather than planning how to present the content.. (Brown and Borko, 1992).

Should we emphasize pedagogical content knowledge, since as we know, "..pedagogical content knowledge is unique to the profession of teaching [and] we expect it to be relatively undeveloped in novice teachers, and thus be a primary focus of their educational experiences" (Brown and Borko, 1992).

Should we emphasize field experience, since "this is the real practice" and "if novice teachers get too little practice or if they receive inconsistent feedback from their university teacher educators and classroom supervisor, their expertise may not be adequately developed" (Vacc and Bright, 1994).

Can we emphasize all three aspects, and if so to what extent? What, however, do teachers think about the way they were prepared to teach mathematics in the elementary school?

275 teachers (from three different colleges) in their first years of teaching, answered an open-ended questionnaire aiming to reveal their opinions about their preparation in college from the point of view of beginning teachers. The questions asked about: relevant and irrelevant courses that teachers studied, subjects that should have been but were not included in their preparation, difficulties and dilemmas of teachers in their first years of teaching mathematics, sources of help, participation in inservice courses and the involvement that should or should not have taken place to guide them in their first years in school. In addition they were asked to describe a good math teacher in elementary school.

Findings add valuable information from the teachers' points of view, and should be taken into consideration when revising preservice programs.

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AN INTEGRATED VIEW OF CHILDREN'S NUMBER UNDERSTANDING
A multi-component analysis from early counting to place value

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Previous research has tended to focus on the development of separate number components (e.g. counting; addition; multiplication) and so, cannot comment on how development in *one* component affects development in *others*. In this study, 152 children from three different cohorts (aged 4, 5 and 6) answered to several maths tasks - three times along one school year - assessing their understanding of 5 separate number components, such as: counting and number-word sequence; structure of the decade numeration system; addition; multiplication; and written multi-digit numbers (place value). In the analysis, special emphasis was given to the separate role of each component and the developmental inter-relations amongst components, in children's understanding of progressively more complex ideas about number.

The evidence suggests that no progress seems possible without the inter-related development of several components, at key times. For example, no child could understand the structure of the numeration system without a previous *combined* understanding of advanced counting skills (i.e. continuation of counting), addition and multiplication. Also, children who failed to continue counting could not display understanding of addition or multiplication, although they all mastered the counting principles. Based on this longitudinal data, it was possible to outline a preliminary proposal about children's number development, which included the identification of important predictors. Evidence that over 97% of the participants fit the proposed model, suggests that the present approach is relevant and worth further investigation.

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METACOGNITIVE DISCOURSE IN MATHEMATICS CLASSROOMS

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Research in the area of mathematics emphasizes the importance of discourse as an integral part of doing mathematics (The National Council of Teachers of Mathematics, 2000). The discourse in mathematics classrooms includes at least two factors: mathematical discourse and metacognitive discourse. Mathematical discourse includes the abilities to construct mathematical conjectures, develop and evaluate mathematical arguments, and select and use various types of representations. Metacognitive discourse refers to using self-regulating behaviors.

The present study investigates the discourse in mathematics classrooms under two conditions: cooperative-learning with or without metacognitive instruction. The metacognitive instruction was based on the IMPROVE method (Mevarech & Kramarski, 1997). Under this condition, students were guided to activate metacognitive questions in small groups that focus on : (a) the nature of the problem/task (b) the construction of relationships between previous and new knowledge; (c) the use of strategies appropriate for solving the problem/task and understanding why; and (d) reflection on the solution process.

Participants were 122 eighth graders who studied in six heterogeneus classrooms. Data were video-taped and analyzed by using qualitative and quantitative methods. Discourse analayzes indicated different discourse characteristics under these two conditions. Students who were exposed to the metacognitive instruction within cooperative settings were better able than their counterparts in the COOP condition to express their mathematical idea.

Their mathematical discourse was more fluent and involved a richer battery of mathematical concepts. In addition, their discourse involved self-regulating behaviors (e.g., prove, check) than students who studied in cooperative settings with no metacognitive instruction. The practical implications of the study will be discussed on the conference.

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DIAGNOSTIC TEACHING AND RATIO: THE EFFECT OF PICTORIAL REPRESENTATIONS ON STRATEGIES

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We believe that an essential aspect of improving the effective teaching of ratio involves raising teachers' awareness of their students' strategies and misconceptions. A good diagnostic test can be a helpful tool for evoking teachers' knowledge about their children (Williams & Ryan, 2001). This communication is part of a research project which involves the construction, analysis and scaling of two parallel forms of such a test for 10 to 14 year olds (sample, n=236), one in which children are provided with hypothetically helpful diagrams, models and referents and one without.

For instance, six pairs of parallel items differing in numerical structure and context were presented in two versions: one accompanied by a pictorial representation aid in the manner of Lamon (1993) and one without. Here we report the results for these items.

The analysis of the results shows that the addition of pictures in each task affected the kind and the frequency of strategies that students employ. A notable finding is that several students answered the items correctly, based only on the pictorial aid-as shows their work on the scripts. Equally interesting is the fact that, as Santos (1996) commented about illustrations in mathematics textbooks, pupils did not always see the accompanying pictures as aid.

Acknowledgement

We gratefully acknowledge the financial support of the Economic and Social Research Council (ESRC), Award Number R42200034284

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STUDENTS' APPRECIATION OF 3-DIMENSIONAL SOLIDS: A STUDY IN HONG KONG

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ABSTRACT

"Spatial sense" has always been seen as important. This is the case in Hong Kong as well as many other different places in the world. Hong Kong primary students, besides learning simple geometric figures and solids in their mathematics lessons, also have a chance to develop their spatial sense further in their experience of art appreciation in their art lessons. There are opportunities of employing skills such as identifying, describing and classifying 2-dimensional shapes and 3-dimensional solids; intuitive recognition of reflective/symmetrical objects in the visual art curriculum.

A teaching package "Buildings in the future" was developed. The package consisted of three parts. First, different kinds of buildings were presented in screen images and students were asked to recognize the geometric forms and shapes from the screen images. Second, students needed to draw a sketch of a building which they would like to construct. In this part, the students needed to present the concepts in their sketches. Finally, the students constructed models of their own buildings according to their sketches and present their models.

The package lasted for four lessons of 70 minutes each. The package was tried in 4 primary-four classes (age 9) and the lessons were videotaped. The subsequent analysis aimed to find out what roles the students' understanding of the geometric concepts might play in their perception of the buildings in pictures, as well as in the design and construction of models. The framework developed by Gutiérrez (1996), in which visualization is seen as "a kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems and properties", was used as conceptual tools in the analysis.

Some interesting results have been observed. For example, despite the fact that the students had been taught the basic geometric concepts, they had confusion between the shapes and the forms, prisms and pyramids. Some students made their mental image explicit in the presentation of their own sketches. In constructing their own buildings, a few modified their designs as a result of the difficulties in constructing a particular solid.

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CHILDREN'S STRATEGIES TO SOLVE PROPORTION PROBLEMS IN A REAL WORLD CONTEXT

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Traditional education of proportionality has a strong algorithmic orientation by Portuguese teachers. Rate and proportions makes part of the 6^{th} grade students curriculum (11 years old) and most of the students show difficulties in using the equation $a/b = c/x$ to find the missing value.
This study was carried out with 4^{th} graders pupils (9 years old), following a work project methodology. Children negotiated with the teacher the projects which they will be developing, they moved out in real world in order to find the information they needed, and returning to the classroom where they tried to answer the questions they had related to their problem. After that they had to communicate the project to their colleagues, explaining all the procedures. The problem chosen by a group of 4 children which will be presenting in this communication is "How much cost to travel by car?" During the development of the project pupils were not taught by the teacher any formal or informal process to solve the problems that children faced. They just had to mobilised basic arithmetic skills they already had.
During this communication will be analysed students strategies as well as the development of ratio tables they invented to systematise the reasoning. Rogoff (1984) argues that thinking is deeply related *to the context of the problem* to be solved which includes the problem's physical and conceptual structure as well as the purpose of the activity and the social milieu in which it is embedded. This may explain the level of performance shown by these children in solving the proportion problems directly related to their lives and interests. Lamon (1993) and Hart (1983) consider that formal instruction and conventional symbolism should take advantage of students' invented strategies in solving rate and proportions problems and that the best time to teach a child a new skill or idea was when it was needed, because the child was faced with a situation that could not be dealt with using the previously used methods.
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"I went by twos, he went by ones" Multiple Views of Graphs

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In this presentation, I use a situated and sociocultural perspective (Forman, 1996; Gee, 1996 and 1999) to examine a classroom discussion about graphs of motion stories. I address the following questions: What were the multiple perspectives and ways of talking about this mathematical representation? What resources did the participants use to socially construct an understanding of the scales on the graphs?

The data is taken from a series of classroom observations and videotaped lessons collected in a bilingual eighth-grade mathematics classroom. In this classroom students represented stories of bicycle trips using tables and graphs. I examine a mathematical discussion between two students and a teacher as they explained the scales on the graphs they had constructed for a problem about a bicycle trip.

The analysis shows that there were multiple views of the meaning of the marks on the vertical axes of the graphs and multiple ways to describe these scales. I use transcript excerpts to illustrate multiple meanings for the phrase "I went by" and to describe multiple ways of talking about the scales on the graphs. This discussion did not explicitly focus on different interpretations. Instead, each student described what he had done to construct his graph and how he saw the results of his actions. These two students developed an understanding of the scale on the vertical axis by engaging in a discussion detailing their different perspectives of the scales on the two graphs. The clarification and negotiation of these multiple views and descriptions was one way that understanding of this representation was socially constructed. The analysis also shows how the graphs, the teacher, the verbal descriptions, gestures, and the multiple perspectives themselves provided resources for socially constructing the meaning of this representation. When students are making sense of a problem and have some intellectual authority, then verbal descriptions can signal the times and places where views of a representation are or are not shared. In this discussion multiple verbal descriptions marked a place where inter-subjectivity broke down. But this break down was not an obstacle. In fact, the multiple views themselves were a powerful resource for constructing mathematical understanding.

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SIX CHARACTERS IN SEARCH OF AN ANSWER

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Most people find school math disconnected from the reality of their lives. Word problems, that ostensibly are intended to make a connection, are (stereo)typically artificial and unrealistic. Yet it is also recognized that mathematics permeates the everyday life of common people. Jean Lave and others have shown how mathematics situated in daily activities, such as choosing the best buy at a supermarket, represents a very different practice from that of school math.

We carried out an investigation based on a mathematical problem that relates to a situation that most adults in contemporary society have encountered many times, namely: "If you buy something in a store on which you have to pay tax and there is a percentage discount on the item, does it make a difference to what you have to pay if they add the tax first and then take off the discount or take the discount off first and then add the tax?".

This problem has been posed to 230 adult students in lower level college math classes designed to prepare them to be elementary school teachers. Students first give an immediate, intuitive, individual response and then work on the problem in groups. After whole-class discussion of the problem, the students are then set an assignment to interview five people, report on their responses, and reflect on those responses.

Less than 20% of initial responses among the students, and likewise among their interviewees, are correct. Incorrect answers are generally supported either by the argument that adding the tax first means that the discount is greater or, conversely, that taking the discount off first means the tax is less. The linguistic phrases "adding tax" and "taking off a discount" suggest that the problem is additive; in fact, if tax of 8% and discount of 15%, for example, are seen as equivalent to multiplication by 1.08 and 0.85 respectively, it is clear that the order of operations makes no difference.

When students discuss the problem in groups, or when the interviewees are allowed to work with pencil and paper or with a calculator, almost all calculate the results for one or two examples and on this basis conclude that the order does not matter. Almost nobody attempted a general proof using algebra.

The students comment on the impact that carrying out the interviews had on them, in particular their surprise that people with extensive knowledge of monetary transactions neither knew nor were able to logically infer the correct answer and were usually surprised by the result of carrying out calculations. The results have profound implications for the way in which people generally fail to perceive mathematics as a potential tool for analysis of aspects of their lives.

ESTABLISHING AND MAINTAINING AUTHENTIC MATHEMATICAL ACTIVITY IN CSCL COMMUNITIES

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Establishing and maintaining knowledge-building communities of practice with Knowledge Forum and other Computer Supported Collaborative Learning (CSCL) software systems in the domain of mathematics has been found to be a rather intractable problem (Scardamalia & Bereiter, 1996).

In this study, the major aim was to investigate whether having students engage in model-eliciting mathematical problems (Doerr & Lesh, in press) with collective discourse mediated by Knowledge Forum would achieve the kind of authentic, sustained, and progressive knowledge-building activity that has been achieved in more content-rich discipline areas such as science, history, and geography.

The twenty-one participants in this study were students from a Grade 6 class at a private urban Canadian school for girls. The model-eliciting problem posed to the students was to devise an alternative model that could be used for ranking nations' performance at Olympic Games which de-emphasizes the mind-set of "gold or nothing". The production and improvement of the alternative models proceeded in five phases: 1) Warm-up Activity, 2) Initial model-eliciting, 3) Sharing of initial models, 4) Revision of Models, and 5) Dissemination and Revision of Models in Knowledge Forum database.

Four important elements of knowledge-building activity which have been noted in mathematical research communities were observed during the course of the study: (1) redefinition of problem, (2) inventive use of mathematical tools, (3) posing and exploration of conjectures, (4) incremental improvement of the mathematical models. Much of the success we had in establishing and maintaining the mathematics knowledge building community in this study can be attributed to the rich context for mathematizing provided by the problem and to the contexts and scaffolds for both intra- and inter-group discourse provided by Knowledge Forum.

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ARGUMENTATION AND PROOFS IN MATHEMATICS TEACHING[1]

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In the last decades, the teaching of mathematics in Brazil at secondary school has been very superficial: students are not exposed to demonstrations, and are not asked to justify their answers, or the truth of an assertion. This happens in Geometry, when students are given ready definitions and are trained to apply formulas. In the Algebra classes it is even worse: the focus is on procedures – manipulation of algebraic expressions and solution of equations. As a consequence, students are getting to the University without being able to think or to discuss the solution to a problem.

Trying to change this framework, a group composed by two university teachers, five secondary school teachers in Rio de Janeiro and four university students from the Mathematics course carried on an investigation in order to:

- identify, at first, what kind of argumentation and justification students at various school levels, from 11 years on, are able to give;
- suggest trends and strategies to enhance the levels of argumentation of students, as well as of prospective and in-service mathematics teachers;
- develop activities based on these strategies to be tested by the teachers in the group;

This research has been motivated by the project developed by Professor Celia Hoyles, at the Institute of Education, University of London (Hoyles, 1997), and was based on well-known research about the teaching of proofs (Balacheff, Bell, Clements & Battista, De Villiers, Galbraith, Godino, Hanna, Hoyles, van Hiele).

As reported in Nasser and Tinoco (1999), the first trials of our investigation pointed that "the majority of Brazilian mathematics teachers do not require their students to justify their answers..."(p.303), which is the crucial point for the difficulties shown. Several questions asking for justifications have been developed and tested, based on the strategies adopted, in order to improve the ability of argumentation (Nasser and Tinoco, 1999). With these strategies, the teachers are getting great progress in the students' ability to give justifications. Some of the activities used in the research, and samples of students' justifications will be shown at the oral presentation.

The first conclusions show that there is a great relation between argumentation ability and content domain, as well as between that ability and good domain of the mother language; activities to develop the level of argumentation and logical reasoning must be included in the curricular planning since the first years of schooling, and at least two schoolyears of work are necessary to get students writing justifications.

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[1] Reaserch partially supported by CNPq (Brazil)

EXPLANATIONS BY MATHEMATICS TEACHERS: AN ANALYSIS

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One of the fundamental role of the mathematics teacher is to provide learners with understandable explanations. 'Explanations are demonstrations of understanding and provide a window to a person's thinking' (Zuzovsky & Tamir, P.1101, 1999). Studies in explanatory frameworks reveal that they provide an avenue to enhance learners' understanding of scientific phenomenon. Regrettably, less has been written about the role of explanations in mathematics teaching and learning (Dagher and Cossman 1992, Horwood 1988, Martin 1970, in Treagust, 2000).

This paper presents an analysis of twenty-three lesson observations of 10 secondary mathematics teachers in three township schools. This is done with the aim of identifying, classifying and analyzing the spoken and written explanations of a sample of mathematics teachers. The analysis of results was guided by the framework of Curtis & Reigeluth (1984) and Ogborn et al. (1996). The findings revealed among other things that mathematics teachers explanations are influenced by various factors, that they often do not consider learners' prior knowledge, are incomplete and do not always lead to understanding. The findings will be used to guide professional development intervention programs to help teachers improve their mathematics instruction.

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USING COMPUTERS FOR DIDACTIC ACTIVITIES ON AREAS: A CASE STUDY

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The teaching of area of simple geometric shapes (circle, rectangle) in elementary schools is characterized often by the difficulties that pupils face in understanding some basic concepts concerning them. For example, the confusion between perimeter and area of a shape or the "mysterious" number π.

Within the framework of teaching areas we tried to cope with two special didactic problems that are related to the π-estimation and the change of the area of a square or a rectangle when we alter their dimensions with the support of a computer. Specifically for the teaching of these matters in the 6^{th} grade of an elementary school we used the Cabri-geometer, Excel and in some cases freeware like a Java applet. Especially for the π-estimation we used, in Cabri-geometer, a combination of inscribed and circumscribed normal polygons as we were increasing the number of their sides (Archimedes's method), we used Excel for performance and recording relevant computations in a matrix and a Java applet for the Buffon's needle experiment which is connected with π. The same software had been used for the processing of resultant data when dimensions of a square or a rectangle had been changed (e.g. they had been doubled or increased by adding a constant length).

Pupils easily accepted the notion that when we increase the number of sides of the inscribed and circumscribed polygons we could "approximate" π, namely we could find out that the average of the areas of the two polygons is "probably" the area of the circle, which divided by r2, equals the desirable number. However some pupils claimed that when we increase the number of sides of the circumscribed polygon the area increases while others used perimeter instead of area. Of course this method doesn't permit an accurate estimation of π but we think it is important the fact that pupils could "use" (even with teacher's guidance) approximating methods. We also simulated via a Java applet the Buffon's needle experiment for the π-estimation. The simulation offered the opportunity to pupils to toss as many needles as they wanted.

In rectangles and squares pupils easily concluded that if we multiply their dimensions with a number, then the second power of this number indicates how many times its area increases. When they had to estimate such an area, they generally responded positively. Some of them wrote correctly the power (e.g. 4^2) but they failed in calculating it (e.g. $4^2=8$). A difficulty showed up when we increased the dimensions of a square adding a constant length. The teacher had to guide them with Cabri-geometer's help to find out graphically the similarity with the previous case.

WHAT IS GEOMETRY FOR FRENCH PRESERVICE ELEMENTARY SCHOOLTEACHERS ?

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This communication describes the first results of an ongoing research about preservice elementary schoolteachers in French IUFM (Academic Institute for Teachers' Training). Concerning most of these students, trainers agree on their knowledge in geometry being far from sufficient for future teachers. This can be attributed partly to their previous curriculum: even if they all have spent at least three years at university, many did not attend scientific courses ; however, all were secondary students, and as such had geometry courses until the age of 15-16.

Our theoretical frame, based on our previous researches [Parzysz 1991], distinguishes, among others, two geometrical paradigms which seem relevant to account for our students' relationship with geometry : roughly speaking, the first one (G1) deals with physical objects and perceptive proofs, and the second one (G2) deals with theoretical objects and deductive proofs (or, rather, 'informal deductive", according to Godino & Recio [Godino & Recio 1997 p. 317]).

In order to find suitable means to improve our students' training, we started with an investigation into their beliefs, through a questionnaire (N >700). A major aim of this part of our study was to get an insight into their knowledge in elementary geometry (especially about construction techniques) and more precisely about how they connect a construction (G1) to the underlying geometrical properties (G2). We also wanted to know whether –and how- they can use their geometrical knowledge (G2) to overcome a material difficulty (G1), in the present case, material constraints preventing them from using a routine construction technique.

The analysis of the results through various statistical techniques (among which implicative analysis) allowed us to find that, even if our students have a good knowledge of elementary geometrical construction techniques, many of them cannot relate these techniques to the corresponding geometrical properties. Moreover, the study of the techniques used by the best-performing students show that these techniques have more 'degrees of freedom' and, as such, can be adapted more easily, when necessary, to overcome possible constraints.

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TEACHER STUDENTS' MATHEMATICAL UNDERSTANDING FROM THE VIEW POINT OF CONCEPTUAL CHANGE

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Mathematics is often considered to form a hierarchical structure where all the new concepts logically follow from prior ones, which allow students to enrich their knowledge step by step. The very fundamental idea of successor, for example, is necessary for learning the notion of natural numbers. It is, however, seriously conflicting with the understanding of the very character of both rational and real numbers (cf. Sowder, 1992). According to the theories on conceptual change (cf. Carey, 1985; Vosniadou, 1994; Duit, 1995) the relationship between learners' prior knowledge and new information to be learned is one of the most crucial factors in determining the quality of learning.

The objective of this paper is to analyze the results of a level test in mathematics done by elementary teacher students at the University of Turku (Finland), and to explain these from the viewpoint of conceptual change theories. Elementary teacher students' basic skills and attitudes towards mathematics were measured in November 2000. The results refer to serious problems in the understanding of basic concepts of numbers and to low level of students' mathematical thinking. The thinking of whole numbers and everyday experiences seems to be restrictive to a higher level of thinking, which refers to the problems of conceptual change. This suggests also that the extension of the number domains has not been dealt with thoroughly enough from a theoretical viewpoint, but it has been rushed into practice with new numbers.

Within the theories of conceptual change, it has been recently emphasized how a process of a radical change seems to be very slow. One of the presumptions for the conceptual change is the increasing meta-conceptual awareness of one's thinking. Thinking of whole numbers dominated the students' answers that refer to problems of conceptual change. As a result, we suggest that the perspective of conceptual change would be profitable in the mathematics education of teachers' pre-service and in-service training.

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THE EFFECT OF THE SOCIO – CULTURAL FACTORS DURING AN INTERDISCIPLINARY APPROACH OF TEACHING MATHEMATICS & TECHNOLOGY

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This research examines the effect of cultural factors during the implementation of an activity that was carried out by 6th grade students in the framework of an interdisciplinary approach of Mathematics and Technology courses. Two four-membered mixed teams formed the sample of students. The activity concerned the design of a window of the belfry of an Eastern Orthodox Church that laid nearby the particular High school. The students were asked to draw a window twice the size of the one appeared in an under scale photograph of the belfry. The activity was carried out by the students without the presence of the researchers and the discussions between the members of both teams were video and audio recorded. In the analysis of the discussions and the drawings collected, particular emphasis was given to the effect of the dominant cultural elements. The discussions as well as the drawings reveal the degree to which the cultural context of the problem activates previous experiences of the students. We find that during the implementation of the activity their discussion incorporated influences from themes that are related to modern television forms of entertainment. However, it is the traditional cultural factors that mainly influence the drawing proposals. With respect to the mathematical part of the activity, the students proceed with off-hand numerical calculations and with vague and spontaneous use of the notion of scale. The students do not appear to be familiar with this notion although they were taught this subject the previous year. The analysis of the recordings shows also that the boys are more familiar with the application of certain mathematical and technological ideas. The incorporation of "decorative" elements in the final drawing by the girls leads the first team to the adoption of this particular drawing and to the rejection of some other appreciable proposals that present remarkable harmonies. The second team does not manage to adopt a unique proposal since its members had been separated tacitly from the beginning in two subgroups: boys and girls. Thus, this team proposes two different drawings, both characterized by originality and by traditional cultural effects.

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THE ROLE OF REPRESENTATION DEVICES IN THE ALGEBRAIC CONCEPTUALIZATION OF CHANGE: A study with 10 to 11-year-olds

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Initiating the learning of algebra concepts by describing, exploring and analysing graphs, tables, and numbers without using –at first instance- the algebraic code is said to be likely. The aforementioned is possible by the use of representation devices generated by a mediation tool such as a simulator (in this case Math Worlds). In order to do so, an early-introduction-to-algebra model, applied to 10-11 year old students who have not received any former education on the subject, is tested. Motion backgrounds are employed profitably so as to introduce algebra concepts; i.e., the starting point is the mathematics of change by using simulations generated by computer environments such as SimCalc Math Worlds. The algebra concept being examined and reported in this article is the functional relation such as velocity, from reading the position and velocity graphs as well as creating tables; in other words, by exploring and elaborating different representation devices.

The methodology used is basically longitudinal case study developed throughout 3 stages: 1) Design and/or selection of teaching activities aimed at promoting cognitive activities of *formation*, *treatment*, and *conversion* among devices (according to R. Duval's theory of representations), by using simulations of motion phenomena in Math Worlds. 2) An exploratory study aimed at testing a diagnostic instrument and the first sequence of teaching activities. 3) The main study aimed at analysing cognitive processes involved in the construction of algebraic knowledge, which is the result of *conversion* and *treatment* activities among devices.

We will discuss issues arising from two cases (Isabelle and Mario) within the framework of R. Duval's theory, in relation to the use of different representation devices in an early introduction to algebra. In one case, the representation devices and the simulator were used to express concepts that the student (Mario) already had started to build up e. g. the velocity concept as a functional relation between distance and time. While in the other case (Isabelle) the devices were used to get information and to start constructing the velocity concept that will be strengthened in the future. In both cases, we will stress the role that a variety of representations played when constructing or explaining the notion of a functional relationship, in the particular context of motion phenomena (specifically, with a focus on velocity as a functional relationship).

Literature's review was according to early introduction to algebra, the role of representation, and experimental studies with SimCalc Math Worlds.

We want to thank The National Council of Science and Technology in Mexico (Conacyt) for funding the research project "Incorporation of IT to the school culture: the teaching of math and physics in the secondary school" (grant No. G26338S).

CATECHETICS OBSERVED IN MATHEMATICS CLASSROOMS

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During an audit of the teaching of mathematics within a group of schools spanning the age-range 4 to 16 years, attention was paid to observing the 'catechetics' of the lesson, i.e. the use of question-and-answer interactions as a means of teaching, and its inverse, the students' use of question-and-answer with their teacher and their peers in order to enhance their learning. The incidence and significance of catechetic interactions in 12 mathematics lessons is the basis for this paper.

THE LESSONS OBSERVED

Lessons seen ranged from those with students aged 4 to 14 years, across 12 schools, all but one of which was implementing the English 'National Numeracy Strategy'. By then NNS had been in place for 4 terms, and the focus of much professional development for all of the teachers observed. Lessons lasted 45 minutes or so and were observed throughout, incidents being noted on a proforma derived from observable processes and from modes and styles (ref. Gardner, 93).

Each teacher interpreted the NNS' 3-part lesson differently. The 'mental/oral starter' sometimes involved high concentrations of catechetic teaching, sometimes almost none. In only a few lessons was the inverse-catechetic form observed: students leading question-and-answer episodes with their teacher, or with their peers.

DISCUSSION

Much emphasis has been given to such interactions within NNS, so it might be assumed that this aspect of lessons would play a significant part in observed teaching. This study suggests that a rather more complex set of practices is in use. Prior practices seem more enduring than out-of-classroom discussions would indicate. All these teachers seemed prepared to use NNS approaches in prior conversation and lesson-planning documents. There were catechetic-rich episodes in 3 teachers' lessons, but almost no catechetic activity in 3 lessons. Yet the latter teachers self-evaluated their lessons as following the NNS approaches as planned.

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Analysis of young children's achievements in measuring 'time'

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The English national curriculum includes 'time' as a key element within the measures section with the expectation that seven year old children achieving above average in their Key Stage One assessment tests (attaining level 3) will be able to "use standard units of time in a range of contexts" (DfEE/QCA 1999). An initial analysis of the results of these tests shows that many children who attain level 3 are still unable to fulfil the requirements with respect to time, for example to draw the hands on a clock face accurately to indicate the time "half a hour after nine o'clock", or to calculate a finishing time given the starting time and duration, where the calculation requires movement across the next hour. The conclusions drawn (e.g. QCA 2001) are that teachers are not teaching children sufficiently in the early years of schooling to read the clock face and to calculate time. However an alternative interpretation could be that we are asking something that the majority of children at this age are unable to understand.

Time is the measure of an abstract quality, which cannot be observed. It is difficult to assess the child's understanding, as indeed Piaget found - resorting to activities which related more to speed than time (Piaget 1969). Experience of time is subjective, some hours seem to go on for ever, others pass in an instant, depending on activity. Secondly, 'telling the time' requires the interpretation of a complex continuous scale, which bears little relationship to any other measuring instrument. Reading the clock face requires attention to two separate markers (the hour and the minute hands) and the relationship between them and fixed marks (hours) interpreted as two different scales (hours and minutes). Thirdly, calculation of time requires calculation with a set of different bases (60 minutes in an hour, 24 hours in a day, etc.) which do not accord with base ten calculation that the children are learning at the same time (Cockburn, A. 1999).

The research sets out to analyse children's understanding of time and to identify children's procedural and conceptual misconceptions. Data collected and analysed from standard assessment test papers and from observations of classroom lessons will be presented and discussed and the implications for teaching 'time' to young children drawn out.

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HOW DO FUTURE MATHEMATICS TEACHERS DEAL WITH DIFFERENT TYPES OF IMPLICATIONS?

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The issue of how advanced mathematics students evaluate assertions with a false premise was considered in a study presented at PME25 (Rogalski & Rogalski, 2001). Seven items of a logical test proposed to 107 future mathematics teachers were analysed, all asking for the evaluation of the value of an implication. Students were classified depending on the orientation in their answers to three non computable assertions with a false premise: 'Logic' ('*the assertion is true as the hypothesis is false*'), 'Relevance' ('*the assertion is stupid*', '*non sense*'), 'Falseness' ('*hypothesis always false, then assertion false*'), 'ND' (non dominant type of answer). When focusing on 'logic' students also answered logically to a social contract question. Both 'logics' and 'relevance' orientations led to correct answers to a mathematical computable assertion, and to the two Wason's selection tasks —but only 'logic' was related to correct answers to both items.

Are these results stable? Are the answers sensitive to changes in the wordings of implications ? What is the role of the computability of a mathematical implication in its assessment? We shall report here on a new study aiming at highlighting these questions, with the general purpose of a better understanding of how future mathematics teachers use the logical tool.

In this new empirical study 71 students were asked to assess the truth of assertions in 4 identical items, 3 items where the canonical "if ..then" was substituted to the original formulation, and 4 new mathematical computable assertions with false premise. We observed the following results (numerical data in a related poster):

1) stability of the distribution of patterns and stability of the correlation between patterns and correct answers to similar items;

2) effect of wording only on the classical Wason's task (searching data for assessing an implication), due to an increased use of the contrapositive;

3) effect of the type of mathematical computation needed to derive the 'target' consequence from the false hypothesis: there were more logical answers to the item "if 1=2 then 2=3" than to items such as "if $x^2+1\leq 0$ then $(x^2+1)^2\leq 0$".

Globally, even if future teachers are reasoning more logically than generally observed in the psychological litterature, they are far from a mastery of the logical tool, while this use is a determining factor of how they will manage the teaching/learning process of this tool in their future classrooms.

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MATHEMATICS TEACHING AND LEARNING PRACTICES IN TRANSITION

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From the perspective of the socio-cultural approach, learning is predominantly seen as the transformation of social practices and of individuals. This view raises three issues of particular importance: the relationship between knowledge and the value certain social groups attach to it, the positioning of the learners and the importance of contextual factors in learning (Bliss and Saljo, 1999).

Empirical research carried out so far within the socio-cultural tradition has been mainly focused on the understanding of skills in everyday use, paying very little attention to issues of valorisation of knowledge and conflict or resistance. However, these issues are essential, especially in understanding the difficulties encountered by both pupils and teachers when new learning frameworks, which make possible novel kinds of learning interactions, are introduced.

Recent research has shown that both children and teachers find it hard to adopt changes in the classroom. For example, in a study that set out to explore scaffolding strategies of primary teachers in mathematics among other subjects, it was found that scaffolding either did not exist or did not work in classrooms (Bliss, et al., 1996). The teachers interpreted or translated pupils' contributions into their own thinking often in line with the task goals, thus requiring pupils to 'ignore' all they know or believe, thus minimising the role of pupils' contributions.

The study reported here focuses on the transformations in teaching and learning practices, following the introduction of a new approach to teaching mathematics in a number of Greek primary schools. The activities introduced were mainly of an investigative character, and the teachers were asked to adopt a role that would keep them on the periphery of the learning process, providing support to the pupils only when necessary. The results of the study so far indicate that the majority of the teachers were unaware of many aspects of their practice and tended to fall back on traditional strategies, particularly when considering that the mathematical meaning constructed by the pupils was 'at risk'. Pupils, on the other hand, tended to pay little attention to the instructions provided in the activities, often looked for the teacher's approval of their work and were unable to see the overall mathematical scope of the activities.

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UNDERSTANDING AND USING GRAPHS: A DIDACTIC SEQUENCE FOR PRE-SCHOOL CHILDREN*

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Several studies (Nunes, 1997; Meira, 1998; Da Rocha Falcão, Brito Lima, Araújo, Lins Lessa & Osório, 2000) have observed the importance of symbolic representations in mathematics conceptualization. We are here particularly interested in investigating the bars graphs comprehension among kindergarten in the exploration of additive-structure problems among six year-old children. We proposed a didactic sequence to children based on graphical representation of quantities derived from tridimensional histograms built with Lego©-blocks. Twenty-four children from a public school in Recife (state of Pernambuco-Brazil), working in pairs took part in a six-week long didactic sequence covering three modules of activities. In the first module, only tridimensional blocks were used to represent and compare quantities, in order to solve additive problems concerning combination, equalization and comparison. A set of blocks was used to construct bars serving as precursor representations of bar graphs. The second module explored aspects of the Block-histogram, like the notion of scale (the previously established relationship between each elementary piece and a specified quantity), and finally the third module explored the activity of passage from Block-histograms to draw and interpret representations in simplified Cartesian plans. In this oral presentation we will analyze the relationship between the previous activity based on blocks and the interpretation and construction of bar graphs on the paper, in the context of addictive structures (Vergnaud, 1985). Preliminary results seemed to suggest that the activity provided by modules 1 and 2 helped children to understand information represented by Cartesian graphs in terms of quantities, their relationship and eventually their pattern of change over time.

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* This paper was supported by Capes and CNPq research grants.

SUPPORTING MATHEMATICAL LITERACY FOR KINDERGARTEN CHILDREN AT-RISK

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Children entering first-grade are expected to solve written addition and subtraction exercises early in the school year. Therefore, the quality of informal number knowledge which the children bring from kindergarten prepares them for initial success or frustration with school mathematics. Evidence has been found for differences in informal arithmetic understandings by kindergarten children from different socio-economic backgrounds (Ginsburg, Klein, & Starkey, 1998). A research-based program was introduced by our staff to enhance the development of mathematics in a group of children of immigrants, identified as being at-risk for school failure.

The program provided informal instruction to small groups of children, an hour a week per group of four children, including structured game-based activities for all four, or for two, three, or one child. The teachers were advised how to extend these activities over the week.

The project included up to 115 children each year in 16 kindergartens. The children participated in an initial interview with tasks of counting, identifying numerals and using a mental number line, knowledge critical for developing understanding of addition and subtraction (Griffin, Case, & Siegler, 1995). Data from the interviews were updated regularly for an "on-line" assessment of the children's progress. The staff was engaged in a continuous process of examining together the children's thinking about numbers, and exchanging ideas for appropriate activities to most effectively match the children's development. The project was an adaptation of the "Classroom conceptual research" model of Fuson, Sherin, & Smith (1998).

Results at the end of each year (May, 2000 and May, 2001) showed significant improvement in the children's arithmetic knowledge. A follow-up is planned to record the children's experience in first-grade.

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REPRESENTATIONS AND SOLVING PROCEDURES IN WORD DIVISION PROBLEMS: COMPARING FORMAL AND INFORMAL KNOWLEDGE IN CHILDREN

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When solving word division problems children have to consider three basic elements: the size of the whole, the number of parts, and the size of the parts (quota) which must be the same for all the parts (Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 1997). It is important to know how children deal with these elements when solving word division problems presented under situations that involve different systems of signs (Nunes, 1997) and manipulative material, which are often present in classrooms. It is also important to explore the role played by formal and informal knowledge in children when solving word division problems.

This study aimed to investigate how children before learning division in school (Group 1: 20 children) solve word division problems by using informal knowledge. In this research we compared their solution procedures and representations to those used by children who had been formally taught about division in school (Group 2: 20 children). Each child was individually asked to solve two word division problems (partitive and quotitive) by using paper and pencil, and manipulatives (objects). The analysis of children's performance took into account the graphic representation used, the solving procedures adopted and the types of errors presented. Differences were found between groups in relation to the graphic representations adopted ($p<.01$) and to the way problems were solved ($p<.001$) in each situation (graphic and manipulative material). The results suggest that in the classroom close attention should be given to: (a) the role played by different systems of signs children adopt when solving word problems; (b) the differences between formal and informal knowledge in children; and (c) the importance of informal knowledge to the teaching of division.

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CHILDREN'S MISUNDERSTANDING OF AN INVERSE RELATION

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One difficulty that young children encounter in understanding division problems is in understanding the inverse relation between the divisor and the quotient. Recent studies (e.g., Correa, Nunes & Bryant, 1998; Sophian, Garyantes, & Chang, 1997) have shown that children find it difficult to judge that when a given quantity is shared out, the larger the number of recipients (divisor), the smaller each portion (quotient). The current studies investigated whether children's understanding of this relation might partly be dependent on the numbers presented in the problems.

Children were presented with problems about two parties of rabbits (with the same total amount of food, i.e., the same dividend). Children had to judge whether a rabbit in one party would get the same (or more) food to eat as a rabbit in the other party. In some trials the number of rabbits (divisors) in each party was the *same* (e.g., 8) and in the others there was either a *small* (e.g., 6 vs. 4) or a *large* (e.g., 12 vs. 2) difference in size between the divisors. Previous research had not investigated whether a large size contrast between the divisors would aid children's understanding of the inverse relation. It was found that children were successful in the same-divisor trials, but only 25% of five-year-olds performed significantly better than chance in the different divisor trials. A common mistake that children made was to say that in the party with more recipients, each rabbit would receive more to eat. There was no significant difference between the small and the large contrast conditions in either five- or six-year-olds. In a second study, children were again given problems involving small or large divisor contrasts but this time they were shown pictures of the resulting portions in each party and children had to decide which sized portion corresponded to each party. It was found that almost 80% of six-year-olds consistently matched the *larger* portion to the party with a *greater* number of recipients, i.e., they inferred a direct rather than an inverse relation.

These studies provide evidence that some young children actually have a *misunderstanding* of the inverse relation between the divisor and that this is independent of the size of the numbers in the problems. Suggestions for future adaptations of the tasks and possible educational implications of the results will be discussed.

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ABILITY GROUPING, SEX AND SOCIAL CLASS – EFFECTS ON ATTITUDE TOWARDS AND CONFIDENCE IN MATHEMATICS

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The substance for the Short Oral will be based on results from a questionnaire given to Year 9 pupils (640 in total) in three schools to investigate attitudes towards and experiences of learning mathematics. The profiles of the three schools used in the survey are similar in that they are all successful, mixed comprehensives in a small English city and were chosen opportunistically. The statistical results presented therefore act as a snap-shot picture of particular pupils' attitudes and cannot be reliably generalised. The results appear to confirm existing research yet prompt further questions as to the extent of pupil disaffection, its causes and the particular groups of pupils affected.

There is a view that mathematics as a discipline is objective and value-free. Yet research at school-level shows that sub-sections of students have different experiences of mathematics lessons and can be prevented from realising their potential or from gaining ownership of the mathematics they do. In particular, the attitudes of pupils in two such sub-groups, namely girls and those from lower socio-economic groups will be presented. The effects of grouping by perceived mathematical ability will also be considered.

1) Girls

In the UK, while girls' performance compared to boys in mathematics examinations at age 16 has improved this improvement is relatively minor compared to other school subjects. More importantly girls remain under-represented in non-compulsory mathematics courses post-16. Much research around gender issues has shown that confidence is a key factor in determining attitudes to the subject and its choice. Leder suggested that 'bright' girls demonstrate 'fear of success' in mathematical situations more than boys. There is also evidence that girls 'experience' school mathematics differently and do not value the type of mathematics that is widely taught. (Boaler, 1997).

2) Social class and ability groupings

There is little research within mathematics education, into the effects of social class on pupil's experiences and attitudes. Cooper has suggested working class children perform less well on national tests at age 11 which results in these pupils being placed in lower ability groups with lower expectations of their mathematics potential. In England nearly all secondary schools teach mathematics to such 'setted' groups. Boaler, Wiliam & Brown have detailed the effects of this policy and argue that pupils' experiences of learning mathematics often depend on the ability group in which they are placed.

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EQUATIONS AS AN EXAMPLE OF A "PROCEPT"

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Gray and Tall coined the expression "procept" as an amalgam of process and concept (Gray 1994). This amalgamation reflects a flexible way of representing mathematical notions and is characteristic of students' thinking. Sfard writes also about mathematical notions being at the same time a process and an object (Sfard 1991).

Piaget's theory about acquiring a concept and the connections between the concept and the actions, which are linked to this concept, are applicable also to mathematics (Piaget 1972). According to Piaget the abstraction (or conceptualization) is drawn not from the object that is acted upon, but from the action itself.

In my research I chose the concepts of "a root of an equation" and "equivalent equations." Both concepts have a formal definition but are naturally linked to a process, the process of solving equations. The main activity in algebra lessons is dedicated to solving equations. My research question was: After several years of practicing this action, are students capable of conceptualizing these mathematical concepts in a mental way or are they still bound to the process, thinking only through algebraic manipulations?

One hundred and ten 10^{th} graders were given a set of seven pairs of equations and had to decide which of the pairs were equivalent. This was done after a discussion in class about equivalent equations.

The results show that when the answer was the outcome of standard algebraic manipulations, most of them answered correctly. But when there was no clear process to connect the two equations and one has to go back to the formal definition, students had difficulties. In this case there was no real interiorization in Piaget's sense. There is, of course, a certain justification to this behavior. In these cases the correct answer has to do with a semantically abstract meaning of the concept and is rather detached from the "real life" meaning.

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RELATIONSHIP: SIGNS – OBJECT – CONCEPT IN THE PROCESS OF THE CREATION OF THE CONCEPT OF SIMILAR FIGURES (CASE STUDY)

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During my research lasting 3 years I analysed the process of forming of the concept of similar figures. One of the aims was connected with the problem of selecting a set of standard representations for the similarity. I tried to check the following criteria (in the frame of the "epistemological triangle"):

- Does the reference context and object capture the fundamental mathematical ideas underlying the given domain?
- Does the given sign (word, name, expression) is connected in the right way with the mathematical concept?

During the first stage of my research I used different tools for learning the meaning of the statement: "the same shape". Among others there were the transparent sheet with the picture of a car. We have manipulated those sheet on the overhead projector and observed the picture on the screen. We used the expressions: "figure have the same shape", or "figure has a changing shape".

After three years I used the same tools during the investigation. This situation caused bringing up new associations. The tools were recognised, but the reference context was interpreted not in direction of ideas of similar figures. The representation, which give the possibilities of the action on the object does not open the way to the domain of similarity. They allow two different interpretations:

1. in the direction of non-proportional change of the figure (deformations);
2. in the direction of isometries.

Both of those domains are different. Similarity is not connected with any of them. The used object and reference context does not satisfy my expectations. Also the expression: "the same shape", connected with such context does not help is creating the right intuitions – it has completely different meaning. Going after the children though – preservation of shape is connected with such action of the object which does not change the object as the whole.

It seems that the procedural change the one figure into the other one is not a basic root for that mathematical concept.

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FROM NAÏVE DRAWING TO CARTESIAN REPRESENTATION

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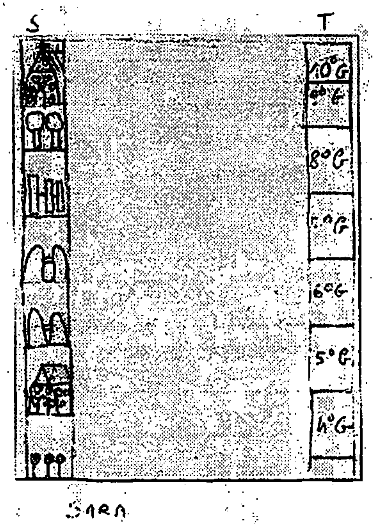
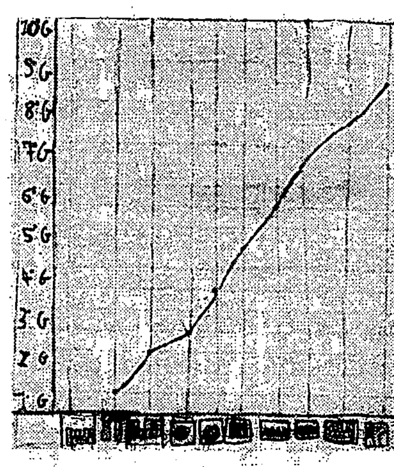
A very important goal in mathematical education is the ability to connect different representation *registers* (Duval); but this attitude, far from being spontaneous, must be promoted in pupils. In particular the ability to represent a link between two variables by means of a Cartesian diagram, exactly as happens for every mathematical object, is the result of a long process rooted in natural thinking and growing through successive structuring stages. But in order to grasp the "cultural" meaning of a representation, children must experience the necessity of proposing their own naïve representations, and possibly of modifying them, if they want to communicate, while sharing achievements and emotions.

Following this idea, we have proposed to 6-7 aged children to dramatize a short novel by D. Buzzati *The seven Messengers*. The aim was a) to promote collective *discussions* (Bartolini) and reflections about the conceptual space-time tangle involved in any experience of motion. Ilaria says: "Space and time run together because time goes on as I'm walking."; b) to favour a natural evolution from spontaneous drawings toward bidimensional representations (the three selected pictures give an idea of this transition).





During the classroom activity a major care of the teacher was to let the pupils: a) evoke and utilize all their previous knowledge; b) freely go back and forth from a register to another (bodily, verbal and graphical), whenever useful to understand. As a result of this activity, focused on different levels of representations, a Cartesian diagram is understood not only as a descriptive tool but also as a powerful mathematical object (from a cognitive point of view).

A detailed account of the whole activity, together with theoretical background and samples of tape-recorded classroom discussions, will be the content of a paper in preparation and will be ready for the time of communication.

INTRODUCING REASONING IN EARLY CHILDHOOD

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Argumentation and reasoning are introduced in mathematics education from early years. This demand the design of experience and activities that should help children to discuss their opinions, to explain their interpretations and justify their ideas in different situations (English, 1997). Based on this orientation, we explored the reasoning abilities of 4-5 years old children and implemented teaching intervention to improve these abilities.

The paper presents the results of a teaching experiment in pre-schoolers. First, the children have been tested in argumentation and reasoning activities including i) recognition of patterns and rules, ii) explanation and justification of their ideas iii) making conjectures (Hanna, G.& Janke, 1996). The intervention process lasted six weeks and consisted of 35 tasks. The designed tasks were problems, games, stories and constructions, extended in situations (such as shapes, space relationships and transformations, qualitative and quantitative characteristics and relations) that encouraged argumentation and reasoning procedures (Alexander, 1997, Edwards, 1998, Maher & Martino, 1996). After the end of the intervention, the children were tested again in the same abilities.

The results of how interventions of this kind can help young children enhance their reasoning abilities provide interesting evidence about mathematics teaching in early years. Pre-schoolers can be encouraged to reason based on their own life experience and their background knowledge, but their improvement in this field depends on the situations the educator will employ. This point will be discussed during the presentation of the research.

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CHALLENGES IN INCORPORATING NEW TECHNOLOGIES TO THE MATHEMATICS CLASSROOM

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Department of Mathematics Education, Cinvestav

We present issues arising from a national project [1], which aimed to incorporate new technologies to the middle-school (12-15 years olds) mathematics curriculum of Mexico. The use of Spreadsheets, Cabri-Géomètre, SimcCalc, Stella and the TI-92 calculator were researched with nearly 90 teachers and 10000 students, over more than 3 years. The project incorporated results from international research in computer-based mathematics education, to the practice in the "real world". In particular, the pedagogy underlying the design of mathematical microworlds (Hoyles & Noss, 1992) was considered in the design. The computational instruments were conceived as mediational tools for students' construction of concepts. The setting and classroom structure were also emphasized (Ursini & Rojano, 2000): from the physical set-up of the equipment, to the collaboration between students, to the role of the teacher, to the pedagogical tools (e.g. worksheets). The project was evaluated considering the role of teachers, headmasters and parents; and students' learning and use of tools. Although it was groundbreaking in changing the role of the teacher and the traditional passive attitude of children, opening the door for richer ways of incorporating technology in schools, the project had challenges and difficulties. Factors not present in laboratory settings come into play, when implementing a project such as this one, "out in the real world". The more outstanding issues were: lack of adequate mathematical preparation on the part of the teachers; lack of experience working with technology by both teachers and students; difficulties in adapting to the proposed pedagogical model; teachers' lack of free time to prepare anything outside the established curriculum (all of these factors contributed to make the activities much more directed than we would have liked); bureaucratic difficulties: teachers and schools had to be provided with permits to fit in the project activities to the curriculum; and lack of communication between the different levels of authorities. We believe that a fundamental facet to emphasise for overcoming many of the above issues, is in the training and continuous support of teachers: to implement a project such as this one requires long-term training in the use of technology, in linking mathematics and technology, in the pedagogical approach expected; as well as creating a support link between teachers and experts.

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[1] The project is known as EMAT (Teaching Mathematics with Technology). Its evaluation is financed by Conacyt: Research Grant No. G26338S.

WHAT MENTAL MODELS DO STUDENTS USE REGARDING THE STRUCTURE OF THE DOMAIN OF RATIONAL NUMBERS?

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In teaching mathematics, it is often assumed that, due to the hierarchical structure of Mathematics, new ideas about numbers will smoothly follow from prior ones. However, learning difficulties in this area suggest that the conceptual shift from prior to new mathematical concepts is not as smooth as often supposed. Theories of conceptual change suggest that the enrichment of a conceptual structure by simple addition of new information is not enough to achieve understanding of certain mathematical notions.

In this study, we monitored the changing understanding of "number density" while students acquire expertise on rational numbers. We claim that we detect the need of a radical kind of conceptual change.

We interviewed twenty students of the ninth grade. They were given tasks regarding the internal structure of the set of rational numbers. (e.g. "How many numbers can you find between … and …?"). Both the prior knowledge about the natural numbers and their properties, and the use of the number line, influenced clearly the way the students responded. We report our first results in identifying and categorizing the mental models that students construct when dealing with questions about density of rational numbers.

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JAPANESE CHILDREN'S UNDERSTANDING OF TEN AS A UNIT

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International studies, both large and small scale, have shown that students from Far Eastern countries such as Japan, Korea and China consistently outperformed their counterparts from the West. Some researchers have noted that the number words in those countries reflected the base-10 numeration system very closely, and such language structure may be a contributing factor for their superior performance. For example, Fuson & Kwon (1992) discussed how Korean language structure may have helped Korean children to develop efficient methods for simple addition and subtraction problems. Miura & Okamoto (1989) suggested that the Japanese children's cognitive representation of numbers positively affected their understanding of place values.

Although a number of researchers speculated that the number word structures in Far Eastern languages may have contributed to their children's performance, there has been no study that investigated Asian children's understanding of ten as a unit. Thus, a small scale, exploratory study was conducted to analyze Japanese young children's concepts of ten.

A convenient sample of 12 Kindergarteners, aged 5 years and 5 months to 6 years and 3 months, and 12 first graders, aged 6 years and 8 months to 7 years and 3 months, participated in this study. These students attended the Kindergarten and the Elementary School affiliated with the College of Education of a national university in Japan.

Each child was individually interviewed. There were four types of interview questions for children in both age groups. These tasks were adopted from Steffe and his colleagues' work (e.g., Steffe & Cobb, 1988) and involved individual counting squares and ten-strips. These interviews were semi-structured in that there were a set of common questions for each age group. However, based on children's response, the interviewer asked additional questions, and or or altered the size of numbers. Each interview took approximately 20 minutes. All interviews were videotaped for later analyses.

In our presentation, we will present our analysis of these interviews, focusing specifically on these children's understanding of ten as a unit.

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ABLE AND GIFTED: A CASE STUDY OF YEAR 6 CHILDREN

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The aim of this exploratory research was to identify, examine and draw recommendations from analysis of the similarities and differences between able and gifted 10-11 year-olds, where 'able' refers to high achievers, and 'gifted' to exceptionally high achievers. The research draws upon Szabos' (1989) characteristics of 'bright learners' and 'gifted learners', and Krutetskii's (1976) structure of mathematical abilities. It addresses the questions: What distinguishes a child who is gifted in mathematics from the larger group of able, above average children? Are these differences in degree or differences in kind?

Based on the above definitions, two schools were asked to identify an able and a gifted 10-11 year-old child. Using a case study methodology, clinical interviews were then conducted with each of the four children. The interviews were video taped, transcribed and analysed using a grounded approach.

Many broad similarities were found between the able and gifted, including their approaches to procedural tasks. The results presented here will focus on the more subtle qualitative differences in kind observed in the children's approaches to conceptual tasks. Key differences appeared to be the tendency of the gifted to plan a strategy, solve problems in an efficient and elegant manner, and justify solutions for themselves. Drawing on the children's responses to conceptual tasks, a model was developed to characterise the strategically based methods of the gifted. It will be presented and illustrated with examples of children's work. This research supports the use of mathematically difficult tasks to enhance the identification of giftedness (Niederer & Irwin, 2001; Span & Overtoom-Corsmit, 1986). It also raises many questions, calling for further data collection and evaluation of the model.

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STUDENT TEACHERS' EXPERIENCES OF BECOMING 'EXPERT' WITHIN THE PRACTICE OF PRIMARY SCHOOL MATHEMATICS TEACHING

Peter Winbourne, South Bank University

In this session I will report on the first year of a three-year research project in which I am working with a group of student primary school teachers through their one-year post-graduate training and then, I hope, the first two years of their teaching careers. The students and I are working to characterise their sense of their identity as learner and as teacher of mathematics and track how this changes and develops over time.

I am using concept mapping (Mwakapenda, 2001) as a methodological tool to provide a framework for representing, discussing and probing students' developing pedagogic content knowledge and, from the perspective of situated cognition and identity theory, their developing identity as 'teacher of mathematics' (Lave and Wenger, 1991, Wenger 1997, Winbourne and Watson, 1998, Winbourne 2002). We have begun the process of accounting for development in identity in terms of students' participation in university mathematical and pedagogical practices and increasingly practices in their placement schools; later, the scope of these accounts will be expanded to include the practices of the schools in which students start work as newly qualified teachers (NQTs).

Through hermeneutic analysis of data (Van Manen 1990) emerging themes are identified, explored and used to construct biographical narratives that are represented to the students/NQTs themselves for further discussion and validation

Amongst the questions I hope to consider are these: if students/NQTs become practitioners who are critical of their practice (meaning the mathematics teaching practices within which they are becoming experts) what in their experience may turn out to have been central in the development and maintenance of this disposition? If not, what features of their experience may be similarly identified?

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Winbourne, P (submitted for publication 2002) *Looking for learning in practice: how can this inform teaching?*

SOME CHALLENGES OF THE USE OF COMPUTER SIMULATIONS IN PROBABILITY TEACHING

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The use of computer simulation in probability teaching raise questions about its use that is worthy to search. In this sense, Coutinho (2001) reports some difficulties the students had when several situations were given for building urns models for simulating some probabilistic games. These difficulties are associated with the use of the software, the resistance to use the simulation for solving a problem that can be solved directly by calculation and the difficulty on accept simulation data that has not obtained by themselves in order to estimate probabilities.

We are developing a searching for knowing the understanding that the students generate as a result of an instruction supported on the use of the computer simulation. The first part of this work, besides of looking for students' intuitive ideas about probability, look for the spontaneous ideas they have about the frequentist approach and, in particular, the difficulties they face solving probability problems using the computer simulation. Furthermore, we pretended find the strategies they used for estimating the values of the requested probabilities. Some of the results obtained are the content of this paper (Yáñez, 2001).

Twelve university engineer students in Mexico City participated in this study. The activity was made in four sessions of three hours each. In the two first sessions we started working with Fathom solving some problems with computer simulation. In the third session the students worked in pairs for solving three problems of conditional probability; in the fourth session the students individually solved three problems of conditional probability.

For the results obtained, it is seen that the students have difficulties in the modeling of the random experiment and in its programming in the simulation computer language. Other difficulty deals with the interpretation of the graphics of the relative frequencies for estimating the probabilities.

Almost all of the students used the *last value strategy* that identifies the last value of the relative frequency with the probability requested. Another strategy deals with modeling for solving only the questions and not the random experiment, trying to transfer the theoretical analysis to the simulation computer language. It might be that the students make distinction between a theoretical probability and simulated one, even some of them believe there are as many probabilities as generated values of the relative frequencies.

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POSTER PRESENTATIONS

LOGICAL ERRORS OF STUDENTS IN THE SOLUTION OF MATHEMATICAL PROBLEMS

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Peter Samovol, Ben-Gurion University of Negev, Beer- Sheva, Israel

One of the major objectives of mathematics teaching at schools is the development of advanced logical thinking in students. Pedagogical experience convinces us that the process of logical thinking development in students was and is the major weak point of mathematics teaching in school. Studies of the nature of students' logical errors during solving mathematical problems show that such errors result from certain gaps in the students' system of fundamental knowledge. Elimination of these drawbacks requires from the students assimilation of basic notions of mathematical logic such as: a necessary and sufficient condition, corollary, equivalence, etc. The proposed technique is based on the idea of students' involvement in the research and expertise activity.
A student is offered: to check the "problem solution" proposed by the teacher and try to find the logical error in it; to highlight the error; to study the error nature; to offer correct solution technique; to formulate the rule preventing further mistakes of the kind.

This instruction technique promotes:

- Practical study of mathematical logic elements;
- Finding and elimination of logical errors;
- Development of critical thinking, which is the integral part of mathematical culture and scientific outlook;
- Maximal rapprochement between standard mathematics teaching process and the students' teacher-instructed self-education;
- Realization of basic ideas of the collaboration pedagogic;
- Advancement of learning motivation (a student is warned against "the potential danger", and he, naturally, strives to test his abilities).

The authors have tested the of the report major statements on high achievers of various grade levels for a number of years. The report indicates typical examples illustrating the technique, substantiates and demonstrates certain methodological types which are efficient for generating the respective teaching process.

INDIVIDUAL DIFFERENCES IN COGNITION AND AFFECT AND SECONDARY SCHOOL STUDENTS' MULTIPLICATIVE KNOWLEDGE IN BASIC ARITHMETICAL PROBLEMS

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The study was set in a Brisbane metropolitan secondary school that draws students from a mix of demographic and socio-economic areas. Individual differences in cognition and affect in secondary school students' multiplicative knowledge structures of basic mathematics problems were examined. Procedural and Conceptual knowledge aspects of multiplicative structures of basic mathematics problems were analysed within a proceptual framework described by Gray and Tall (1994). An expanded model of Proceptual Multiplicative Knowledge was proposed.

OVERVIEW

Four quantitative studies examined models of cognition, affect and knowledge structures of basic mathematics problems and their inter-relationships. Luria's (1973) 'whole brain' theory describes how information is processed successively (sequential and primarily temporal) and simultaneously (continuous and primarily spatial). Marsh's (1990) multi-dimensional model includes academic dimensions as measures of Self-Image. The academic dimensions of Maths and Verbal Self-Esteem can be linked directly to performance in mathematics problems. Students' responses to basic mathematics problems provided student performance data on the arithmetic operations involving integers and non-integers. A qualitative study, incorporating semi-structured interviews, focussed on individual differences in cognition and affect and multiplicative knowledge structures that students use in solving basic mathematics problems.

Ten A4 sheets of paper in varied colours will be displayed in the allocated area. Text, tables and figures will show the findings of the study and the proposed expanded model. A one-page handout of the highlights of the poster will be provided.

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TEACHERS' KNOWLEDGE ABOUT THE EQUATION CONCEPT

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The purpose of this study is to analyse how teachers in mathematics understand the equation concept. Furthermore my goal is to investigate what kind of previous experiences the teachers have had of their own concept learning. Ten teachers in mathematics in the secondary school participated in the study. Five teachers were newly graduated and five were experienced. Data was gathered by interviews and questionnaires. Tapes were transcribed into protocols and interpreted to categories of conceptions by the phenomenographical research method (Marton and Booth 1997). The results indicate that the teachers have different kinds of misconceptions of the equation concept. They are insecure both to the mathematical symbols, the letter expressions and the solving procedures. Their experiences at school suggest that they have spent most of the time to develop procedural skills instead of mathematical understanding. Their acquired experiences of the concept learning have formed their concept images (Vinner 1991). These are not identical with the concept definition. For some of my teachers an equation does not constitute a mathematical statement. The equation concept is for them closely linked to other difficulties like variables, unknown factors, the roles of the equality sign, the meaning of the mathematical symbols, the role of the formal definition and the solving procedures. The process-object duality of the mathematical notation (Sfard 1991) creates fundamental problems for teachers. They have considerable problems in leaving the process level and entering the object level and there is no difference between the newly graduated and the experienced teachers. Concept definitions help us to form a concept image, but they do not guarantee understanding of the concept. Teaching mathematical concepts with understanding requires different kinds of metaphors, various examples, non-examples, language, situations and so on. It presupposes that teachers have mathematical and pedagogical knowledge and skills and a rich image of mathematical concepts.

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EARLY EXPERIENCES IN DYNAMIC VISUAL REASONING IN A COMPUTER ENVIRONMENT

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Transformation geometry is considered of great importance for the elementary school by a number of researchers (Williford, 1972; Del Grande, 1990; Edwards, 1991; Johnson-Gentile & Outros, 1994) but there is some controversy about how children think about transformations (Shah, 1969; Kidder, 1976; Lesh, 1976). The learning of transformation geometry is intimately connected to the presence of dynamic visual reasoning. We are interested in the ways technology can enhance the dynamical visual reasoning of pupils. We designed a computer environment, microworld, to try to show ideas like translation, reflection and rotation, through the motions slide, flip and turn, to elementary school pupils. The microworld allows with some ease a number of actions on some visual representations. Two pre - built shapes can be moved (slide, flip or turn), some of its characteristics can be modified, a set of geometric concepts may be explored, discussed and communicated. The microworld will give opportunities to develop the ability to learn to imagine dynamically and to perform mental experiments of spatial reasoning. With this environment to learn mathematics children may combine the practical ways to mathematise those motions with higher mathematical activities.

The poster will: - elaborate on the types of learning sought; - explain the design of the microworld; - describe how the microworld works; - give details of the use of the microworld with pupils of 4th grade and in-service teacher training (grades k-4).

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HOW DOES A TEACHING PRACTICE PROJECT INFLUENCE THE TEACHER-STUDENTS'ACTIONS

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Inside all of disciplines of the Mathematics' course to students who intend to be a teacher the teaching practice is responsible for the articulation between the pedagogic speeches and the formal speeches.

If we want to stimulate changes to teacher-students' procedures we have to influence the future teachers that they need to develop their sensibility about the functions and purpose of mathematical speech. This implies a total change in the nature of their actions and concepts about mathematics teaching.

In Faculty of Teacher Training – FFP-UERJ, a public University in Brazil, the Mathematics' Department develop a new model of mathematics teacher training breaking the isolation of the individual subjects. For that the teachers' preparation include a large reflection about how does the teaching practice influence the teacher - students.

This poster presents some favorable results about this project that gives priority to the didactical and methodological action of the mathematics students in their teaching practice.

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DO INDIVIDUAL STUDENTS CONSISTENTLY APPLY THE SAME INTUITIVE RULE?

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In recent years, the intuitive rules theory has received growing attention in the mathematics and science education research community because, according to its advocates, it can explain and predict various kinds of responses of students to a wide variety of tasks from scientifically different content domains (see, e.g., Stavy and Tirosh, 2000). Two major intuitive rules are manifested in comparison tasks and are called "More *A* – more *B*" and "Same *A* – same *B*". A recent Belgian replication and elaboration study by De Bock, Verschaffel and Weyers (2001) has challenged the predictive power of that theory. Hundred-and-seventy-two students of grades 10, 11 and 12 solved five problems, stated in a multiple choice format with three alternatives: the correct answer, an incorrect answer in line with "More *A* – more *B*" and an incorrect answer in line with "Same *A* – same *B*". The results showed that Belgian students are less affected by the intuitive rules than their Israeli peers: for most problems, the results in line with the intuitive rules were far below chance level and several other misconceptions proved to be at the origin of students' erroneous answers.

A different kind of analysis on these data was executed to provide an answer to the following research question: *are individual students consistent in their choice for one of the intuitive rules?* We shifted from a data analysis at the item-level to an analysis of the *answering profiles* of the 172 participants. Theoretically, there are 21 ways to answer the five problems (e.g. two answers in line with "More *A* – more *B*", two in line with "Same *A* – same *B*" and one correct answer is one of these ways). By means of three-dimensional frequency diagrams, the poster shows and compares the observed distribution of the students' profiles and the theoretical (multinomial) distribution that would be expected if students answered at a random base. This analysis clearly indicated that the typical "More *A* – more *B*" or "Same *A* – same *B*" student does not exist (or is extremely rare). E.g., none of the 172 students answered more than three times in line with "More *A* – more *B*". Confronted with a problem they cannot solve, students rather seem to guess than to apply a specific intuitive rule, a finding which once again challenges the predictive power of the intuitive rules theory.

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Preservice teachers, geometry and proof

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Some preservice mathematics teachers reveal difficulties and negative attitudes when they study geometrical concepts and relations. Some years from now I' m having a problem: what can I do in the classroom to foster understanding and performance of students when they face proof in geometry? I need more knowledge about proof and teaching of proof. I have access to papers from Gila Hanna, Michael de Villiers, Tommy Dreyfus, P. Goldenberg, ... After that, I start designing environments where students play active roles in trying to develop geometrical *habits of mind*. The goal is to develop in preservice teachers reasoning processes, meaning that the variety of actions that students take in order to explain to themselves, to others, what they see, what they do, what they conjecture and why they do it. I decided to use *The Geometer's Sketchpad* (GSP) with the students to contribute to experimentation, to conjecture, to convince of the truth of the conjecture and to help them come to see proof as a form of explanation and understanding why, rather than convincing. One aim of the study is to analyse the performance of preservice teachers in proving, in order to answer to the question: what level of performance reveal preservice teachers in doing proofs? The participants in this qualitative case study are a whole class of preservice teachers in the second year of the maths and science course in a School of Higher Education, during a whole year. Data has been collected through geometric problem solving tasks and observation. To solve the tasks the participants need to use GSP, to make conjectures, to examine and to reformulate those conjectures and to answer why they are true. We are in the beginning of data analysis, that is holistic, descriptive and interpretative. This presentation will try to answer the question and to draw some conclusions.

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JOINED UP THINKING

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A group of students must *"know how to determine the equation of a straight line in the form y = ax+b from its graph"* (Intermediate 2 Mathematics, SQA, 1999). So starting with the particular and moving to the general, we begin with an example...

> *"It's hard to measure the length of a winding road on a map. Can we estimate its length by counting how many times it crosses a gridline?"* We each try out the idea by laying different lengths of string randomly on 1 cm squared paper and collecting data. *"A graph may be helpful."* We draw graphs by hand and, using a transparent ruler, we fit a line. *"The line doesn't quite go through all the points but it's the best fit we can find."* Find the equation of the line. Laurence gets *y=1.1x*. *"The equation doesn't quite fit all our data ...but it's the best fit we can find." "We all got more or less the same equation: y=1.25x, y=1.1x, y=1.2x." "Why weren't they the same?" "Why is it a straight line again? Oh yes, all the pairs of numbers fit the equation / lie on the line".* Natasha says she was told that a line of best fit couldn't go through the origin. *"But a 0cm piece of string must cross 0 gridlines".* We agree that length = crossings × 1.2 (roughly). The graphic calculator gives the equation of the line as *y=1.26344086x+0*. *"That looks different – oh no, if we round it's about the same."* We try to predict the length of a string with 22 crossings. *"So how far is it from Edinburgh to Perth on the M90?" "We can often fit data to a straight line in this way. It allows us to predict..."* (The poster includes illustrations of this activity)

From the particular to the general...

Some teachers like teaching (school) Mathematics because it is tidy and complete and logical. The experience of this group suggests that learning Mathematics is neither a linear nor a tidy process. Its very untidiness pushes it forward. *(Why is the equation on the GC different?)* Learners work to assimilate new ideas to the concepts they are building *(eg a linear relationship can be used for prediction)* and accommodate the concepts to make sense of the inconsistencies they find *(eg. a best fit line* ***can*** *go through the origin).* Using a holistic approach to bring together different concepts motivates the learner to tidy up, reconcile, rationalise, understand better. It is not possible to 'tick off' the linear relationship as completely 'understood' by a learner. Intelligent learning (Skemp,R.(1987) Psychology of Learning Mathematics) involves building a network of links between concepts. This is an ongoing organic process.

By the way...

We also covered co-ordinates, graphing, gradient, ratio, decimals, rounding, algebra, substitution, why Maths is a powerful language, why Maths might be worth learning... And we joined up our thinking.

REFLECTING AT THE M.I.R.R.O.R: TEACHING STUDENTS WITH LEARNING DISABILITIES IN MATHEMATICS

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The awareness of students with special needs is getting more and more central in educational decision making. Students with learning disabilities are a significant group within this population because they share some qualities which may contribute to their society. The definition of learning disabilities refers to students that show a gap of almost 2 years between their actual ability and their potential. Most of them have an average and even more intelligence quotient and one of the most problematic subjects is mathematics. Mathematics is a very unique subject matter because of its special language, its inquiry systems and its rigid and step by step concepts and principles construction. Students who are diagnosed having dyslexia or ADHD or writing difficulties suffer much from mathematics demands. Some of these learning disabilities are due to psychological aspects and also to teaching disabilities.

I developed, tried and evaluate a humanistic model for teaching mathematics to students with learning disabilities at grade 7 -12. The model is based on mutual reflection between the coach and the student. The components of the model are in initials: M.I.R.R.O.R:
M- for Mathematics; I- for Interaction; R- for Relevance; R for Realistic; O-for Openness; R- for Respecting; The rational for this model us base on psychological-educational theory of Dewey(1933), Piaget(1969) and Rogers (1963).

The poster will show graphically the M.I.R.R.O.R model. In the middle of the poster there will be a mirror, student and teacher with connections to the 6 phases in the margins of the poster: Mathematics; Interaction; Relevance; Realistic; Openness; Respecting - with examples for each phase and explanations

Dewey, J., 1933, How we think, Boston: D.C. Health & Co.
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FORMING COGNITIVE SCHEMES IS ONE OF THE CONDITIONS OF DEVELOPING STUDENTS' INTELLECTUAL ABILITIES

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M. Kholodnaya, Institute of Psychology of RAS, Moscow

Educational process is the system of constructing links between teaching and learning. Its contents, methods, skills and techniques should be, first of all, directed to exposure and usage of every student's experience. M.A.Kholodnaya (Kholodnaya, 1997) singles out such forms of intellectual experience as cognitive, metacognitive and intentional. Cognitive experience includes such mental structures as the ways of information coding, cognitive schemes, semantic structures and, finally, conceptual structures. Cognitive schemes are responsible for receiving, collecting and transforming information. Here arises a question: "What are the conditions which help students to form cognitive schemes in process of teaching?"

In the course of long experiment we have pointed out the types of school texts, which help forming a repertoire of students' cognitive schemes. First of all, we pointed out some types of cognitive schemes. These are, for example, "focus-examples" (Bruner, 1971). Focus-examples are schematized images, which are used as the starting point by a person who solves this or that problem. To create generic focus-examples is necessary to have a set of specific focus-examples, which are used to form a concept.

Another variety of cognitive schemes is a frame (Minsky, 1979). On the one hand, to create frames is possible with the help of texts, which form students' ability to perform the procedure of identifying objects, phenomena, properties and links between them. On the other hand, there should be school texts, which could develop students' flexibility of thinking and ability to reconstruct a frame.

Special attention in our research was paid to forming controlling cognitive schemes-plans of actions in a problem situation (Pascal Leon, 1970, 1987). Here we pointed out the types of texts which help students to construct plans of their activity, their fixation, pointing out separate elements of these plans, correlation of different plans and so on. It is very important for a teacher to be well aware of the activity, which is connected with forming cognitive schemes: if the familiar cognitive scheme becomes stronger, larger and reaches or a new cognitive scheme is created.

Success of students' school activity depends on the repertoire of student's cognitive schemes and how a student can analyze them in a certain situation.

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THREE MEASURES FOR CONFIDENCE IN MATHEMATICS

Hannula, Markku S.; Maijala, Hanna; Pehkonen, Erkki; Soro, Riitta

University of Turku, Department of Teacher Education

In this poster we shall report some preliminary results of the research project 'Understanding and self-confidence in mathematics'. The project is directed by professor Pehkonen and funded by the Academy of Finland. It includes a two-year longitudinal study for grades 5-6 and 7-8 with two quantitative surveys for approximately 200 mathematics classes and a qualitative study of 40 students. In this poster we will present the instrument and some results of the first survey.

We designed a survey to measure the level of self-confidence and understanding of number concept in a large population. The survey was administered by teachers during normal 45-minute lessons during the fall term 2001. It consisted of five parts: student background, 19 mathematics tasks, success expectation for each task, solution confidence for each task, and a mathematical beliefs questionnaire.

The students' confidence was measured with three different measures. Before the students did the actual tasks, they estimated on a 5-point Likert scale whether they can solve the task or not ('success expectation'). After solving each task they answered on another 5-point Likert scale on their confidence on their solution ('solution confidence'). Furthermore, the 25 items in the mathematical beliefs questionnaire included ten items that were adopted from the confidence scale of the Fennema-Sherman Attitude Scales (Fennema & Sherman, 1976) ('self-confidence'). Success expectation and solution confidence responses tended to be skewed (high confidence), especially for seventh graders. For further analyses we chose eight (most difficult) tasks that produced less skewed responses

As expected, the three confidence measures were correlated. The success expectation and solution confidence had a correlation of 0.67. Self-confidence correlated with the other measures on the level 0.44 – 0.50, weaker correlations being for the fifth grade. Somewhat surprisingly correlations between confidence measures and success in test were weaker, more so among fifth graders and there especially for girls (0.06-0.09).

Another interesting result was found in gender differences. On fifth grade the differences in both success and confidence favoured boys. However, on seventh grade no gender differences were found.

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Semantic types of an anchored addition and subtraction[1]

Milan Hejný, Charles University, Prague

The poster will be split into six main sections: illustrations of fundamental terms; definitions; tasks; student's solutions; examples; didactic applications of theoretical ideas.

As a continuation of G. H. Littler's poster about semantic types of numbers (Ad = address, St = state, Co = comparison, Ch = change) the most frequent semantic types of the addition and the subtraction are discussed:

Type	Example
St + Ch	I had 10 Kč. I found 5 Kč. How many Kč do I have now?
St – Ch	I had 10 Kč. I lost 5 Kč. How many Kč do I have now?
St + Co	Ed is 124 cm high, Eva is 3 cm higher. How tall is Eva?
St – Co	Ed is 124 cm high, Eva is 3 cm lower. How tall is Eva?
St + St	Here you see 5 girls and 4 boys. How many children are here?
St – St	Here are 9 children. 5 of them are girls. How many boys are here?

We asked pupils to indicate the semantic inverse of these cases. There were no problems with the issues St ± Ch and St ± Co, since words *to lose* and *to find* as well as *higher* and *lower* have been identified as opposite by all pupils. However the issue St ± St was a difficulty. The most frequent opposite to *put together* was identified as *to part the whole* or *do not touch it.* And the most frequent opposite to *take the complement* was identified either *take the whole* or *take the starting part* or *do not take the complement.*

<u>Consequence</u>. To be prepared to grasp the idea of subtraction a child has to be familiar with cases St + Ch and St + Co. If the majority of his/her semantic experiences with the addition is of the type St + St a child is not well prepared to understand subtraction.

Types Ad ± Co (I live on the 5rd floor and Fred lives 2 floors above/below. On what floor does Fred live?) or Ad ± Ch (Our family moved from the 5rd floor 2 floors up/ down. On what floor do we live now?) are of the same difficulty as St ± Ch and St ± Co. A child must to meet these cases as well.

More demanding are types Co ± Co and Ch ± Ch (Ann is 2 years older than Bert and he is 1 year older than Cindy. How many years is Cindy older than Ann?) Here some children ask for the starting state (the age of Ann). Even more demanding are the cases Co ± Ch and Ch ± Co (During the last year Mark grew 4 cm taller. Now he is 1 cm shorter than the table. How many cm was Mark shorter than the table before?)

[1] The research was supported by VZ J13/98/114100004 (CZ)

OLD TUNES, NEW TECHNOLOGY: HOW COMPUTERS MEET HISTORY IN THE MATH CLASSROOM

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Kibbutzim College of Education
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Technology and especially computers are essential when teaching and learning mathematics; they influence the mathematics that is taught and enhances students' learning (NCTM 1989,2000). In addition, mathematics educators have long emphasized the importance of integrating the history of mathematics into the curriculum (Arcavi, 1987; Freudental, 1981). The integration of computers and historical aspects contribute to the understanding, interest and enjoyment in learning mathematics.

The poster will demonstrate a teaching topic that combines historical and computational aspects: ***The number π and it's computation to a desired accuracy*** (Hoffmann, 1996), intended for elementary and secondary school mathematics prospective teachers. The calculation of π is characteristic of the history of mathematics in general and, according to Beckmann (1971), "The history of π is a quaint little mirror of the history of man". Its importance goes beyond its historical origin in geometry, and expands into the various fields of mathematics. Calculation of this number has engaged the greatest mathematicians from the dawn of history (Kline, 1972). Today, using computer power, more than 200 billion digits of the number have been extracted.

There exist several numerical methods to calculate π which cover a range of mathematical and computer-science concepts. Each method generates beneficial by-products: Analysis of the effect of round-off errors, rates of convergence (Breuer & Zwas, 1993), probabilistic reasoning, computer simulation (Hoffmann, 2001), the uses of series and so forth. We will show how current technology enables integrating these methods into the school curriculum and review our experience in teaching the topic in teachers' colleges and to university students.

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Implementation of Content and Language Integrated Learning (CLIL) in Mathematics Classroom

Jarmila Novotná, Marie Hofmannová, Charles University, Prague

The work was inspired by the European project TIE-CLIL developed over the past four years. CLIL refers to any teaching of a non-language subject through the medium of a second or foreign language (Pavesi et al., 2001). It aims to create an improvement in both the foreign language and the non-language area competence.

The described experiment is one of the outcomes of an optional CLIL course aimed at the Faculty of Education students wishing to enhance their teaching qualifications from the original Diploma in Mathematics (to be taught in Czech) and English (as a foreign language) to a new, integrated competence of teaching mathematics through English to Czech learners.

Four final year teacher trainees were placed at two upper secondary schools – one in Prague and the other in Olomouc. The Prague school provided a typical learning environment. Students, experimental group (n = 30), learn all the subjects in Czech. The Olomouc school, control group (n = 29), belongs to the network of twelve schools where selected subjects are taught in a foreign language. For the period of two weeks both groups were exposed to the same subject matter (graphs, ellipse) in English. The aim of the experiment was to monitor the teacher's role and specify the amount of his/her verbal and non-verbal support through the stages of the students' cognitive development. Observation sheets, videotapes and questionnaires became the primary source of data. The nature of support could be seen partly in the interaction of three languages of information processing: Czech (L1), English (L2) and the language of mathematics (L3). Teacher talk as an insufficient instrument was combined with a variety of pedagogical practices including the use of teaching materials and aids.

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Acknowledgement: The research was supported by the Research Project GAČR 406/02/0809 Language Forms and Their Impact on the Cognitive Processes Development.

FOCUS ON MEDIATION OF ACTIVITY TO DEVELOPING CLASSROOM DISCOURSE

Verónica Hoyos

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Abstract. In this work is endeavoured to contribute in the search of significant tools to learn mathematics (significant in the sense that they necessarily alter the character of knowledge, in according to Confrey, 1993). We are presenting the results of some empirical observations about performed activity of a group of 18 student's 14-aged, who were working on certain scenarios of learning. The pupils were mainly handling certain cultural artefacts for the elementary geometric transformations learning. Here, our main concern was to analyze, using a communicational learning approach (Sfard, 2001), the way that learners approach to mathematical terms using scripts for the activity and describing their accomplished tasks.

INTRODUCTION

Some recent research results show that in sessions of practical work, secondary pupils, using recent processing development, are acting to access to notions, properties or mathematical procedures that were previously subjects of access for no one but professional Mathematicians (Hoyos, 1999; Hoyos, Capponi and Geneves, 1998; Moschkovich, Schoenfeld, and Arcavi, 1993).

On the other hand, theoretical expectations and alternative approaches in didactics of mathematics (Bartolini, 1998; Bartolini and Boero, 1998; Boero et. al., 1997; Mariotti et. al., 1997) argue for introducing "voices" or historical contexts of recreation of scientific experience in the classroom, as a way to generate ideas or complex mathematical notions.

Here, we attempt to contribute further information in a perspective of research related to the use of computerized scenarios and the introduction of historical contexts in the search for significant tools for mathematical learning. Particularly, it was interesting to analyze the different descriptions that pupils are effected at the end of performed activity. The analysis of the advancement of students was based in a communicational approach to learning (Sfard, 2001). From our point of view, those pupil descriptions reflect how they coordinate obtained experiences.

Acknowledgments: The work presented here was based in a research partly supported by the National Council of Science and Technology (CONACyT) of Mexico, No. 30430S.

COMPUTER-BASED MATHEMATICAL GAMES FOR PRESCHOOL CHILDREN

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University of Ioannina-Greece

Computer games play nowadays a significant role in the culture of children and adolescents. Given the strength of their influence, it is likely that these games may affect the knowledge children could get either from school or at home (Bruner 1985, Wark, 1994).

The spread of computer software as mediator in the development of mathematical concepts (Anderson 2001) and the Internet as a teaching tool (Pange 1998), provide an extra chance to children to play and learn mathematics using computer-based games either at home or at school, as most of these computer-based games are easily accessible to children and there are at no extra cost.

A survey was conducted amongst Greek pre-school teachers and children aged 5-6 years, to examine whether they use computer-based games in the Internet to teach mathematics.

Results revealed that very few Greek children are using the Internet and also very few Greek pre-school teachers know how to use computer-based games, in order to teach mathematics. From a group of 500 pre-school teachers only 10 knew how to use the Internet at school. These teachers found that computer games allow children maximum freedom in approaching mathematical tasks in their own manner. But, mathematics have to be the main stage of the software, the software has to encourage children to play again the computer game, all children with any learning style, can enter into the computer game, and children can explore new mathematical ideas from the game. Without these characteristics of the software, children become bored and non-enthusiastic in playing the mathematical computer game.

This study concludes that the use of Internet computer-based games from pre-school children has to be guided by an experienced teacher. All teachers have to be well acquainted with computers and also to be quite experienced with Internet in order to be able to teach mathematics to children and to evaluate the computer-based games.

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ELEMENTARY MATHEMATICS TEACHERS' BELIEFS ABOUT THE USE OF TECHNOLOGICAL TOOLS IN MATHS TEACHING IN RURAL AND URBAN AREAS

Denizhan Karaca and Hulya Gur
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In this paper, elementary mathematics teachers' beliefs about the use of technological tools in mathematics teaching in rural and urban areas were analysed. The researchers develop a questionnaire. The questionnaire was conducted in Balikesir, Turkey. An interview protocol is also developed. An interview is conducted with twelve mathematics teachers in both rural and urban areas. Initial findings show that mathematics teachers in rural areas said: "*technological tools are insufficient*". Mathematics teachers in rural areas have also lack of knowledge about how to use technological tools in mathematics teaching effectively. Although, teachers in urban areas do not keen on to use technological tools, even though technological tools are available.

SUPPORTING CHANGE THROUGH AN IN-SERVICE COURSE:

Activities to Promote Thinking in the Mathematics Classroom

Berinderjeet Kaur

National Institute of Education, Singapore.

The primary aim of the mathematics curriculum in Singapore schools is to enable pupils to develop their ability in mathematical problem solving. The conceptualisation of the mathematics syllabuses is based on a framework that emphasises the interplay of five components – *Concepts, Skills, Processes, Attitudes and Metacognition* to achieve this aim (MOE, 2000). Prior to the year 2000, *Processes* in the framework referred to *Deductive reasoning, Inductive reasoning* and *Heuristics* (MOE, 1990).

In 1997, responding to the Prime Minister's launch of the THINKING SCHOOLS, LEARNING NATION vision (Goh, 1997), the Ministry of Education initiated changes to curricular subjects to implement the Thinking Programme in schools (MOE, 1998). A revision of the Mathematics syllabuses in 1998 saw a significant change to the component *Processes* in the framework. It now referred to Thinking Skills and Heuristics. At the secondary level, the list of *Thinking Skills* suggested include: Classifying, Comparing, Identifying Attributes & Components, Sequencing, Induction, Deduction, Generalising, Verifying and Spatial Visualisation. However, there was no change to the list of *Heuristics*.

To support this change a 30 hour in-service course: *Activities to Promote Thinking in the Mathematics Classroom* is conducted by the author twice a year for secondary school mathematics teachers. Participants come together once a week for 3 hours to design mathematical tasks that infuse thinking skills and also to share their experiences about tasks they have attempted to use in their classrooms.

The poster presents the outline of the course, tasks designed by the teachers' and findings of a questionnaire administered to the teachers at the beginning and end of the course.

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OPENING THE MATHEMATICAL HORIZON USING OPEN-ENDED TASKS

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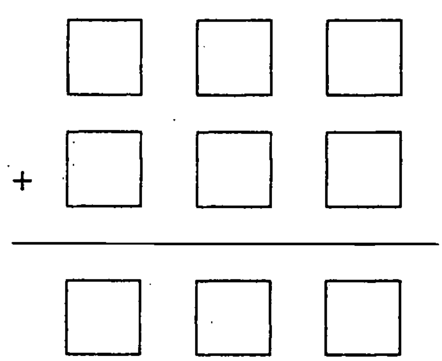
Boaler (1998) claims that open-ended tasks and projects help students develop conceptual understanding of mathematics. Zevenbergen (2001) supports Boaler's claim with a caution that if the tasks are too open-ended it might lead to confusion and ambiguity. Because of a huge potential to enhance student learning, I have been using open-ended tasks in my teaching of elementary and secondary school preservice teachers. This poster focuses on how open-ended tasks can convince preservice teachers that mathematical problems can have multiple correct solutions. This is very important for preservice teachers because many of them believe that a mathematical problem has one right answer and there is usually one correct way to solve it.

In this poster I will provide how preservice teachers struggled and eventually learned a great deal from the following problem. It is noteworthy that one of the preservice teachers claimed that there are 32 solutions to this problem, all of which will be provided in this poster in a pictorial format.

Problem. Use the 9 digits 1, 2, 3, 4, 5, 6, 7, 8, 9 to fill the nine squares below. Each digit may be used only once to make a true addition statement.



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A TEACHING EXPERIMENT ON HOW CHILDREN UNDERSTAND THE MEANING OF ½ FRACTION.

Dimitrios Kontozisis, Maria Kaldrimidou
University of Ioannina

Our purpose of this paper is to describe a teaching experiment conducted in an "early childhood mathematics classroom" about teaching fractional-related concepts. The paper describes the responses of 16 children of four and five years old to a problem requiring the sharing of a square pattern. Data analysis is focused on the material and its properties and on the situations created by experimenting with it in order to enhance children's knowledge about fractions.

Fraction learning is a major obstacle for children. It seems to be a very complicated issue in the teaching and learning mathematics. Despite the wide range of research reports many issues are still to be answered.

While a certain number of researches dealt with the epistemological nature of fractions, most researches focused on the material elaboration and the methods used. Most of these approaches tried to teach fractions either with partitioning of continuous and discrete quantities or with subdivision of sets, length, area and volume (Miller, 1984, Davis et al, 1993). Regarding to early childhood education, there is limited research on children's knowledge about fractions (Hunting and Davis, 1991). The material used in these approaches refers to sharing of discrete and continuous quantities.

In our teaching experiment the task was designed to establish a conceptual field for the ½ fraction. The task, which seemed to be a powerful and challenging activity, is the one we call "Fill in the square". Four different material sets were used to "Fill in the square". The uneven material (6 triangles 1 square) using during the episode resulted to be the most effective in generating the notion of the half fraction and the patterns' equivalence among different shapes of the material.

Results support the conclusion that where some fractional knowledge exists, the use of well-organized and challenging activities can promote and develop children's previous knowledge.

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$IMST^2$ – A NATION-WIDE INITIATIVE FOR UPPER SECONDARY SCHOOLS IN AUSTRIA

Konrad Krainer, University of Klagenfurt

Following the poor results of Austrian high school students in the TIMSS achievement test, a research project was set up in which the results were analysed and additional investigations into the situation of mathematics and science teaching were started. As a consequence, the initiative $IMST^2$ – Innovations in Mathematics, Science and Technology Teaching was launched to support teachers' efforts in raising the quality of learning and teaching in mathematics and science. In the school year 2000-01, 126 Austrian schools participated in total with about one quarter collaborating more intensively with the $IMST^2$-team and documenting one or more innovations.

Four priority programmes – S1: Basic education, S2: School development, S3: Teaching and learning processes and S4: Practice-oriented research – have been established. The four teams support initiatives at schools as well as develop corresponding theoretical concepts. *Innovations* are the key feature of $IMST^2$. They are not regarded as singular events that replace an ineffective practice but as *continuous processes* that lead to a natural further development of practice. Participation in the initiative is *voluntary*, teachers and schools have the *ownership* of their innovations. *Writing down* the experiences in a systematic way means a second cycle of reflection and opens the opportunity for more people to learn from those experiences. Another important feature of $IMST^2$ is the emphasis on supporting *teams of teachers* from one school.

Evaluation is an integral part of $IMST^2$ whereby three different functions have been defined:

- The *process-oriented evaluation* should generate in a continuous feedback process steering knowledge for the project management and the project teams.
- The *outcome-oriented evaluation* should work out the impact of the project at different levels of the educational system.
- The *knowledge-oriented evaluation* should generate new theoretical and practical knowledge which will form a basis for improving support to innovations at schools.

The poster shows the first outcome of the process-oriented and the outcome-oriented evaluation.

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DIFFICULTIES ARISE WITH IDEA OF COMPLEMENT

Jana Kratochvílová, Charles University, Prague, Czech Republic

On the basis of experiences from long-term experimental teaching in a primary school and other experiments M. Hejný suggested that further research on the phenomenon of a complement with pupils needed to be carried out. The aim of my poster presentation is:

1. To show several concrete examples where the phenomenon occurs.
2. To outline prepared research.

Illustration of the Idea of Complement

Many areas of mathematics need the idea of the complement to reduce the work involve in the normal arithmetic or measuring circumstance. I list below three such possibilities.

A. Pupils are given Fig. 1 and they have to find the number of triangles. The pupils count it in the following way: 2 x 5 = 10. Then when given Fig. 2 they have to find the number of triangles as well. They count each triangle separately and the black dot is disregarded. Most pupils at 8 years do not discover the idea of complement, that is to find the number of all the triangles and then to subtract the one black dot.

Fig. 1	Fig. 2
∇ ∇ ∇ ∇ ∇	∇ ∇ ∇ ∇ ∇
∇ ∇ ∇ ∇ ∇	∇ ∇ • ∇ ∇

B. Most pupils do not use the idea of complement in the processes of addition and subtraction even though they are aware it. For example the task 14 + 29 using the idea of complement could be solved as 14 + (30 – 1) but pupils usually solve these tasks using decomposition of the second number as in this example 14 + (20 + 9) or even 14 + (20 + 6 + 3).

C. The idea of complement is also present in the finding the area of triangles on squared paper. Pupils getting Figure 3 find the area by a cutting method – two 'half square' triangles give one square and one plus one square gives two squares. However with Figure 4 pupils see that the cutting method does not work and they usually struggle to find a solution. It is difficult for them to discover the idea of complement – the triangle lies in a 2x2 square and when we cut the marked triangle from this square pupils can more easily find the area of remaining parts. They then subtract these from area of the 2x2 square to get the required area of the triangle.

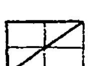
Fig. 3

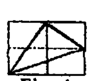
Fig. 4

I intend to develop the research which I have already undertaken involving the Abracadabra problem (Polya, 1966). This research did not involve the phenomena of the complement. To be able to research ideas of the complement it will be necessary to reconstruct the research tool so that it is suitable for pupils in the age range 9 to 14 years.

Reference:
Polya, G., 1966: 'Mathematical Discovery'. John Wiley & Sons Inc. USA

Acknowledgement: The contribution was partially supported by the project GAČR No. 406/01/P090.

Levels of understanding of letters in the word problem solving processes

Jarmila Novotná, Marie Kubínová, Charles University, Prague

In (Arcavi, 1994) the deep analysis of manipulations with symbols is being discussed: "Even those students who manage to handle algebraic techniques successfully, often fail to see algebra as a tool for understanding, expressing and communicating generalisations, for revealing structure, and ... ."

In this contribution, we are comparing the levels of dealing with letters during the period of grasping the meaning of the assignment (encoding phase), and the period of mathematisation of the solving process (transformation and calculation phase). The use of letters in students' written records are analysed and classified in two concrete situations.

Situation 1 – The assignment does not contain algebraic elements (Novotná, 1997): The following four stages of the transition from an arithmetical to an algebraic way of using letters in the written record of assigned information were identified:

1a) Solvers use one letter for labelling several values, the letter is a symbol of a general unknown for them.

1b) Solvers use one or more letters in the encoding stage without working with them in the transformation stage, the unknown is only used as a label for something that is to be found.

1c) Solvers consciously use letters for labelling required values and for describing assigned relationships, arithmetical models are more important and thus the arithmetical solution is used.

1d) Solvers use letters for labelling the values and algebraic operations are carried out and solved. The conditions for the successful use of algebraic methods have already been created.

Situation 2 – The assignment contains algebraic elements (Novotná – Kubínová, 2001): The following four stages of dealing with the assignment were identified:

2a) Solvers ignore data which are not assigned as concrete numbers, their ability to work with algebraic representations is not developed.

2b) Solvers use letters only as labels for something that is to be found by calculations, the value assigned by a letter is handled as an unknown.

2c) Solvers are aware of the nature of data assigned as letters; by substituting a concrete number a letter, they change the problem into a pure arithmetical one. The symbolic algebraic description of the situation is not yet fixed in their knowledge structure.

2d) Solvers are able to work successfully with data assigned in both arithmetical and algebraic languages.

The correspondence between the two situations will be documented by concrete examples of students' solutions.

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Acknowledgement: The research was supported by the projects GAČR No. 406/02/0829 Student-focused mathematics education.

A WEB-BASED MODULE ON FRACTIONS FOR PRE-SERVICE TEACHERS

Amanda le Roux, Hanlie Murray and Alwyn Olivier

University of Stellenbosch, South Africa

This presentation outlines and analyses a web-based module for pre-service elementary teachers on the learning and teaching of fractions. The objectives of the course include developing their mathematical content knowledge, pedagogical content knowledge, and their beliefs about teaching and learning.

The course is structured to elicit optimal reflection. Students work through the course material in their own time (and to a certain extent at their own pace), but are required to attend contact sessions where whole class discussions are held and uncertainties clarified. Students have to submit assignments (which include several group assignments) by e-mail and can participate in on-line discussions throughout the module. Individual feedback to assignments is given electronically.

The content of the course includes reflection on the nature of understanding (Carpenter and Lehrer, 1999), studying research articles describing children's informal knowledge of fractions (Murray, et.al., 1996) and their misconceptions about fractions and the analysis of learning materials and video clips of children working on these materials (MALATI, 2000).

Almost the whole class indicated that they had not previously understood the underlying rationale for the rules for performing the operations with fractions and it only became clear to them during the course. There was a marked improvement in the students' formulation of their ideas which may be an indication of improved understanding. The electronic medium made more communication between lecturer and individual student possible, which lead to an increased awareness of students' misunderstandings.

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WISE DESIGN OF MATHEMATICS SOFTWARE

MARCIA C. LINN
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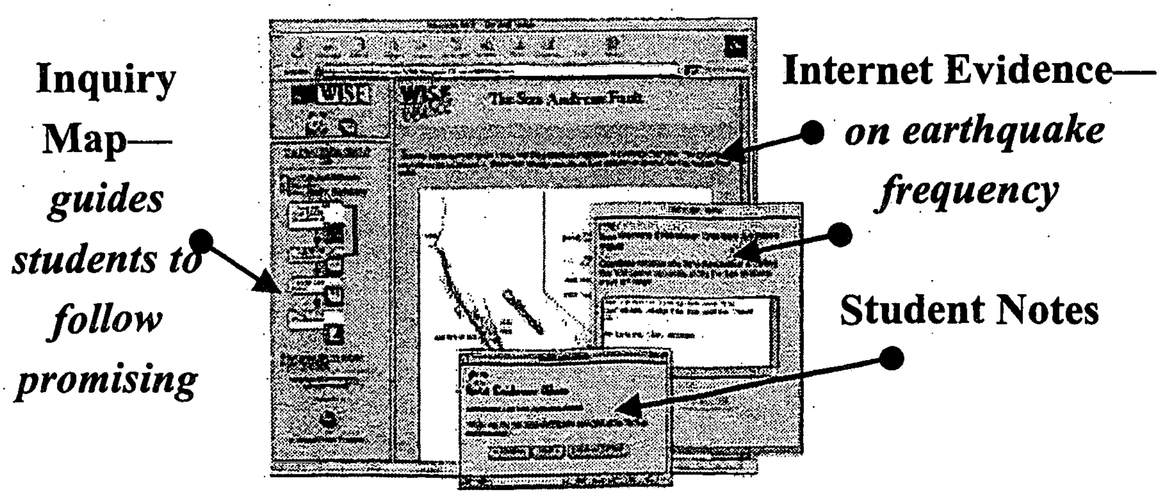
The Web-based Inquiry Science Environment (WISE) offers software designers a flexibly adaptive learning environment that implements the scaffolded knowledge integration instructional framework (Linn, M. C. and S. Hsi (2000). Computers, Teachers, Peers, Lawrence Erlbaum Associates.), incorporates proven technology features, and supports promising instructional patterns. This presentation describes WISE mathematics projects (see http://wise.berkeley.edu), illustrates WISE data analysis tools, reports results from classroom trials, and discusses how WISE projects might improve mathematics instruction.

By supporting flexibly adaptive design, WISE can empower instructors and curriculum designers to test alternative ideas, analyze weaknesses in instruction, and redesign instructional materials to meet student needs. Teachers can also easily customize WISE projects, adding relevant data – such as water quality information from a local stream – and adjusting mathematics demands – by using qualitative or quantitative tools. WISE offers designers graphing tools, real time data collection options, connections to data collected by PDA and more.

The Scaffolded Knowledge Integration Framework has four main tenets that jointly support knowledge integration (Linn & Hsi, 2000). WISE has developed features such as discussion tools and patterns that combine the features to implement the framework. Research with WISE shows that students learn more when instruction combines making thinking visible using a graphing tool with making predictions, reconciling results with expectations, and testing ideas against established criteria. Figure shows data collection options for an earthquake project:



SEMANTIC TYPES OF NUMBER

G.H.Littler, University of Derby, UK.

The poster will be compartmentalised to contain illustrations of fundamental terms; the table given below; student's tasks and solutions and didactic applications of the theoretical ideas presented.

The world of numbers emerges from children's everyday experiences. During the first years at school this world reaches its autonomy, a natural consequence of mental development, very often loosing its linkage to the real world - a cause of low quality future mathematical knowledge in students. This paper classifies different types of **anchored numbers**, - numbers linked to the real world, using a slightly improved version of the classification defined in Hejný, Stehlíková (1999), as follows:

<table>
<tr><th colspan="2" rowspan="2">Class</th><th rowspan="2">Sub-class</th><th rowspan="5"></th><th colspan="2">Examples</th></tr>
<tr><th>Count</th><th>magnitude (is measured)</th></tr>
<tr><td colspan="2" rowspan="2">Identifier</td><td>name</td><td colspan="2">Bus number 15. Telephone number 37. (is observed)</td></tr>
<tr><td>address</td><td colspan="2">Newton was born 1643. Beethoven's fifth symphony.</td></tr>
<tr><td rowspan="5">Q
u
a
n
t
i
t
y</td><td>State</td><td></td><td>Six apples on the plate.</td><td>Mary is 152 cm tall.</td></tr>
<tr><td rowspan="4">Operator</td><td rowspan="2">Compa-rison</td><td>A</td><td>Tom has £5 less than Ed</td><td>Doug is 7 cm taller than Debby</td></tr>
<tr><td>M</td><td>Michael did 3 times more steps than Karin</td><td>The Rhine is twice as long as the Seine.</td></tr>
<tr><td rowspan="2">Change</td><td>A</td><td>I put two more apples into the basket.</td><td>She shortened her hair by 10 cm.</td></tr>
<tr><td>M</td><td>The number of participants was four times those expected</td><td>During the flood, water in this brook increased 90 times.</td></tr>
</table>

A – additive; M- Multiplicative.

An anchored number is either an **identifier** or a **quantity**.

A number as an identifier is used to denote some object, person, time, location, event, … If it is part of a structured set like the scale on a thermometer, a seat in a theatre we will call it an **address.** If there is not a structure to the set, numbers on ice-hockey player's shirts, bus destination numbers, it is a **name.**

A number as quantity can be classified from the points of view of its **function** and of its **quality**. As a function a number describes either a **state** – it quantifies some object, situation, event.. or an **operator** – which is either a **comparison,** comparing two different states or a **change,** turning a given state to another. The quality of a number expresses either a **magnitude** – such as length, weight, pressure, etc. which is measured by means of some unit, or the **count** of the elements of a set. Boundaries between the classes, sub-classes are not well defined.

Reference:

Hejny, M., Stehlikova, N. (1999) *Ciselne Predstavy Deti* Charles University, Prague

THE DESIGN OF A COMPUTATIONAL ENVIRONMENT TO EXPLORE CHILDREN'S DEVELOPMENT OF PLACE VALUE

Ana Tiburtius Lucena & *Richard Noss*
Mathematical Sciences
Institute of Education, University of London

This paper describes the design process of a computational environment created to explore the development of 5 and 6 year-old children's understanding of place value, particularly their ability to explain the relations between the units of different sizes (i.e. hundreds, tens and ones) used in written multi-digit numbers.

The goal of the game was to integrate the epistemological principles involved in children's understanding of place value in order to (1) discriminate their conceptual development, and (2) promote their reflection about the relations between the different units in the multi-digits, in a meaningful way. Children's conceptual development was explored through the refinement of several number-comparisons categorising the span of relations between different units ("2̲00-3̲00" and "23̲5-24̲5" are examples of different categories), while the use of multimedia promoted their reflection about these categories dynamically.

After a series of game-playing sessions, all participants were able to explain the relations between the different units in multi-digits, and used place value correctly, according to post-task assessments. The analysis of case studies suggests that this computational environment can help to describe individual trajectories of development by opening a window into children's *thinking-in-change*, from idiosyncratic ideas to more sophisticated conceptions about place value.

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THE CALCULATOR'S ROLE IN PROBLEM SOLVING: A CASE STUDY IN A FOURTH GRADE CLASSROOM

Ema Mamede – University of Minho - Institute of Child Studies

In Portugal, the fundamental aims of mathematics teaching are based on the development of three skills: (a) reasoning; (b) communication; and (c) problem solving (DEB, 1998). The curriculum supports that the solving of problematic numeric or non-numeric situations should be proposed in all teaching topics. This document also refers the calculation methods that should be used in the classroom. Written procedures, mental calculation and the calculator should be present in daily routine classroom. It is important for children of this grade to be able to compute using one of these methods, as well as to be able to decide which means is more suitable for a specific situation (Abrantes, Serrazina & Oliveira, 1999; NCTM, 1991).

However, in the majority of Portuguese elementary schools, there is a certain degree of resistance to the use of calculators, which raises difficulties in its employment in problem solving situations. As a matter of fact, the Portuguese document "Matemática 2001", which reports diagnostic and recommendations for mathematics teaching and learning, presents some results on the use of calculator in first grades. For instance, almost 74% of the teachers do not use calculators to solve complex problems, 71% do not use it to explore numerical concepts and 63% do not use it to verify results (APM, 1998). It seems that many elementary school teachers ignore how powerful a calculator can be in a problem solving context.

Making an effort to improve this situation, a qualitative case study research was developed in order to understand the rôle that pupils attribute to calculator as a support tool for computation in a problem solving context. The study was implemented in a ten years old class in Braga, Portugal. In this study the calculator's use in solving tasks of estimation, numerical investigations and application of mathematics to real life situations was analysed, attending to strategies used and to interactions between pairs.

This presentation intends to show how was developed the research and to present and discuss its results.

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RELATIONS FOR FIFTH TO EIGHTH GRADERS

Cristina Maranhão; Sonia Igliori; Elizabeth Soares

Pontifícia Universidade Católica de São Paulo, Brasil

Igliori et al. (2000) checked, in a late study, that 5th graders restricted the meanings of the relations *come before than* and *not come after than* in solving problems. For these students, *come before than* meant *come immediately before than* and, in the ordering relation *not come after than* (equivalent to come before than or at the same time as), they did not admit *come at the same time as* as an ordering. The authors also checked if it was possible to see an improvement in the knowledge of students subject to a didactic intervention. Bearing in mind that establishing relations is adamant for the learning of mathematics in the various teaching levels, the authors elaborated the present study, which widened the former, based on the following question: Did 5th to 8th graders have the same problems diagnosed in the late research? And if they had, was the evolution of a restricted conception to a broader one different for each grade for the same didactical intervention? To this end, questions about enunciations of the following kind were proposed: *A teacher could not figure out the exact order of her students' arrival. Give the possible orders of arrival according to the statements students gave her: João said that he came to school before Eni. Eni said that she came before Bia. Rita could not remember the arrival of her mates, but she was sure she came before Eni.* The research was conducted among 4 groups of 5th to 8th graders (10 to 14 years old), totalling 64 students, from a school in the state of São Paulo, Brazil. There were students of the same grade in each group taking a pre-test, a class and a post-test dated a week apart, based on Brousseau (1997). These 3 phases were conducted by the same researcher. The pre-test results highlighted no difference in the answers' standard among the grades for all students. They presented the restricted meaning of the investigated relations. This proves that the existent difficulties disregard the grade. Furthermore, a single class could improve the students' knowledge. For all instances, the values for t_{crit} were regarded highly significant. This means that there actually was a statistical shift in the students standard answer in favor of a better comprehension of the broader meaning of the relations under study. This poster will present the pre and post test data organized in tables for each group, including the statistical analysis for discussion and reporting some debates developed in the aforementioned class.

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What is between Lafontein's quarter and the rational number quarter? or What is the differnce between math justice and jungle justice?

Shoshy Millet, Achva College of Education, Israel
Dorit Patkin, Hakkibutzim College of Education, Israel
Hanna Ezer, Levinsky College of Education, Israel

This study is focused at the positive effect of integration of fables as literary genre for improvement the math language, reduction in the fear of math teaching, increase in the satisfaction and strengthening of the relationship between use of daily language and math language.

Professional literature emphasizes that teaching math along with literacy activities enables teachers to reach a variety of students (Fulwiler & Young 1990; Ackerman 1993).
Also, a study among different populations of math teachers reveals that integration of literacy in teaching may contribute to the understanding of this subject and reducing fear of it (Ezer, Patkin& Millet 1999).
This concept is integrated in the call for strengthening the interdisciplinary connections in different systems including the educational one.
This approach emphasizes that understanding of the subject is achieved by multi-disciplinary systems.
The research is consisted of 60 student teachers for elementary school.
At the beginning of this study 75% of the participants testified that to teach the subject of quarter is considered difficult.
Following learning of the term fraction using the fable of "The Lion and the Hunting" by Lafontein, 90% changed their position and said that they feel more confident in teaching of the term fraction.

The findings indicate that the use of intriguing fables that stimulate the math thinking and lead to an educational lesson promotes the positive approach to teach math and the use of math language.

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LOST IN SPACE: PRIMARY TRAINEE TEACHERS' SPATIAL SUBJECT KNOWLEDGE AND THEIR CLASSROOM PERFORMANCE

Claire Mooney and Keith Jones

Centre for Research in Mathematics Education, University of Southampton, UK

The knowledge of mathematics that elementary teachers bring to their teaching is recognised as a significant influence on how successfully they teach mathematics (Fennema and Franke, 1992; NRC, 2001) yet this is more complex than simply requiring a grasp of mathematics content (Ball, 1990; Ma, 1999). A number of studies have examined trainee elementary teachers' knowledge of number, and how this knowledge is related to their teaching competence (for example, Rowland *et al*, 2000). This study extends this work to examine graduate primary school trainee teachers' knowledge and understanding of spatial concepts.

The theoretical framework being developed for this study builds on suggestions that Shulman's (1986) model of teacher knowledge may be too simplistic (see, for instance, Cochran, DeRuiter and King, 1993) and incorporates Ma's (*ibid*) notion of "profound understanding of fundamental mathematics" (PUFM). One aim of the study is to determine what form of geometrical knowledge is needed for the effective teaching of spatial concepts.

Data comes from audits of trainee teacher knowledge and confidence together with assessments of their teaching competency. Initial analysis of this data indicates that the trainees' knowledge of geometry is quite poor, certainly poorer than their knowledge of number or algebra. They appear not to recall some topics, may never have met other (for example, the nets of solids), and are unable to solve relatively simple problems such as calculating the surface area of a triangular prism.

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CONCEPTS OF OPERATIONAL SYMMETRY IN THE THINKING OF FOUR ELEMENTARY SCHOOL STUDENTS

Norma Presmeg and Jeff Barrett

Illinois State University

The research described in this poster is part of a 4-year longitudinal study of the thinking of four students who are presently in grades 4 and 5. We describe some of their concepts of symmetry, defined broadly as iteration of patterns, and involving rotation and translation as well as 'mirror reflection'. Interview tasks with these students included working in 2 and 3 dimensions. Preliminary results suggest that degrees of symmetry (in the sense of "this is more symmetrical than that") is a natural idea for these students; that moving from the metaphor of folding to point-wise operations in reflective symmetry is a complex but necessary process; that notions of rotational symmetry in two and three dimensions are less well developed but may be fruitfully taught at this age; and that all of these developments facilitate and enhance performance on tessellation tasks.

Our objective is the establishment of a cognitive basis, through literature review and clinical interviewing of students, for investigating how symmetry and pattern recognition may be used as generative processes in the teaching and learning of mathematics at various levels. Here we focus on the notions of symmetry that evolved in the four students as evidenced by data collected in the fall semester of 2001, based on interviews using the following tasks. Preliminary results are reported.

Interview tasks

1. Compare six clay solids and rank them in order of apparent symmetry: a spherical ball; a cylindrical "snake"; a coiled basket made by spiralling a long, thin, cylindrical piece; a rectangular prism; a triangular pyramid; and an irregular lump of clay.

2. Compare pair wise, four large photographs of landscapes. In each of these six comparisons, decide which picture is more symmetrical than the other, giving reasons for your choice.

3. Two-dimensional symmetry:

(a) Given a drawing of a geometrical figure, can you "make the figure symmetrical" (rotation or reflection) and draw the corresponding position of the figure?

(b) Given a drawing of a parallelogram, with a diagonal drawn, or a line through midpoints of opposite sides, explain with reasons why this is a line of symmetry, or why it is not.

4. Three-dimensional visualization task: Can you take a two-dimensional region and rotate if about an axis to generate a solid of revolution by prediction?

WHAT DO FUTURE MATHEMATICS TEACHERS EVALUATING AN IMPLICATION WHEN FACING THE CASE OF PREMISE FALSENESS?

Marc Rogalski, Janine Rogalski

AGAT Lille1-CNRS & IMJ Paris6-CNRS

Laboratoire Cognition & Activités Finalisées Université Paris8-CNRS

We propose a poster linked to a Short Oral communication by J. Rogalski and M. Rogalski, organised around the main results of a study about how future mathematical teachers deal with different types of implications, when confronted to the falseness of the premise. Such situations not only appear in advanced mathematical thinking but also when teachers are confronted to students' reasoning: their control is part of the competence of a mathematics teacher in secondary and high school. It will present:

• 1. examples of the variety of items: factual non computable implications with a false premise; assessment of a formal rule (two versions of the "classical" selection task of Wason); implication with a false premise in a social contract ("*if somebody solves the problem, I will give sweets to everybody*") computable mathematical implications: hypothesis always clearly false (such as: "*if* $1=2$ *then* $2=3$"); hypothesis unknown: the implication may be proved without looking at the value of the premise (which will be proved to be always false); the case of the falseness of the hypothesis has to be considered in a twofold quantified implication (such as: "*for every x and for every m, if* $x^2-2mx+2m+3\leq 0$ *then* $|x|\leq(m-1)^2-4$");

• 2. examples of typical students answers oriented toward "logic" (expected correct answer), "relevance" of the implication, or "falseness" due to the false premise;

• 3. results of a test proposed to 71 future mathematics teachers, aiming at: a) testing the generality of results of a previous study with 107 other teacher-students (Rogalski & Rogalski, 2001); b) evaluating the effect of changes in wordings (introducing the canonical form "*if ... then...*" in some critical items); c) identifying factors involved in the management of the falseness of the premise in mathematical implications (proposition *vs* predicate, premise always false *vs* falseness for some values).

• 4. discussion of the global results: the rationality of future mathematics teachers in their use of implication appears not to be resilient to atypical situations, interactions with a somehow difficult mathematical content, or unusual students' arguments. Consequences for their use of logic as an indispensable tool in teaching mathematics (Hanna & Jahnke, 1993) will be discussed.

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THE STRUCTURE OF MATHEMATICAL ABILITIES

Valery A. Gusev, Ildar S. Safuanov,

Moscow Pedagogical State University, Pedagogical Institute of Naberezhnye Chelny

We present a systematization of all components of mathematical abilities based on previous research in the area. Consider one of possible classifications of components of mathematical abilities of pupils.

Block 1. Components of mathematical abilities, influencing the development of general abilities of pupil.

1.1. Components of mathematical abilities describing inherent qualities of the person and singularities of mental activity.

1.1.1. Qualities of the person: strong-willed activity and capacity of working hard; persistence in reaching the purpose; good memory; arbitrary control of attention; introvertness; intellectual inquisitiveness.

1.1.2. Qualities of mental activity; skill of abstract thinking; economy of thought; exactness, conciseness, clearness of verbal expression of a thought; quickness; ability of analyzing.

1.2. Components of mathematical abilities helping to raise the effectiveness of any educational activity of pupils.

1.2.1. Possession of basic means of educational activity: habit of working regularly; skill of schematizing; ability of independent extracting knowledge; skill of making conclusions.

1.2.2. Possession of means of research and creative educational activity; the art of consistent and correctly partitioned logical reasoning; skill of raising new problems; skill of comparing conclusions.

Block 2. Components of mathematical abilities, ensuring effective mathematical activity.

2.1. Components describing mathematical activity of the pupils.

2.2. Components describing "mathematical style" of thinking: flexibility of mental process; a reversibility of mental process during mathematical reasoning; economy of thought, strictness of a thought and its expression; clearness, simplicity and beauty of solutions.

2.3 Components describing qualities of the person of pupils as mathematicians: Inclination to discovering the logical and mathematical sense in all phenomena of the reality; a habit to rigorous logical argumentation; speed of mastering of an educational material; geometric imagination or "geometric intuition"; possession of sufficient patience in mathematical problem solving; mathematical memory.

THE EURO AT SCHOOL

Laia Saló Nevado
University of Helsinki

From the beginning of 2002, shops have displayed prices in euros. Eleven EU countries have adopted the Euro. It has become a reality within our society, at the market, at the bank and even at school. A fact we cannot avoid.
We take for granted the measures adults take to deal with the new currency such as new pocket calculators, little mathematical tricks, government information, etc. But how do first and second year children acquire the necessary skills to cope with the Euro Philosophy? How can a child who can barely count up to 1000, add and subtract deal with such abstract concepts like the correlation between money, value and price?

Our project has been carried out by first and second graders (7 and 8 year olds). They had to discover for themselves and get to grips with the new currency as the rest of society. Our goal has been to introduce the Euro in our school bringing to the students the reality from outside. The students built a "Euro Shop". They brought in real product packages and worked out prices. At first, we did not have any prices.

The students were free to invent, create and imagine. The starting price for a single milk carton was over 1000 euros. Within a couple of weeks the prices had come down drastically. The teachers were observing without interfering. Everyday interest and the role play brought new values for the money and the products. The milk carton went from 1000 euros to 500 euros, to 10 euros and finally to 50 cents. No direct teaching was involved.

Our project development showed us how the very young ones learn by themselves how to deal with unknown and new quantities (for example decimal numbers). The quantities where related to blocks of units, tens and hundreds. The decimals where treated as another part of the value. Our big standard was: 1 EURO = 100 CENTS. They did not need anything else. No explanations.

Our presentation shows some results from our students and shares the excitement of a very new Europe-wide-project. At the same time, we wanted to prove once more that mathematical thinking can be turned into a simple game, an everyday situation or a challenging test.

NEW WAYS OF DEVELOPING MATHEMATICAL ABILITIES

Mihaela Singer, Institute for Educational Sciences, Romania

From informational islands to structured knowledge

According to the traditional philosophy of learning, which is still very strong everywhere in the world, information represent the essential in mathematics teaching, the mathematical thinking being just a mechanical consequence of their assimilation. The results of this philosophy are expressed in the present-day fact: the average student's failure to learn mathematics. This average student possesses, at best, just "informational islands". Researches carried out on secondary-school students reveal major flaws exactly on the level of mathematical thinking; but these flaws have their roots precisely at the beginning of the informational stairway, that is as early as the first grade. The teaching that is focused on developing mental capacities implies an extremely structured *knowledge organization*.

From problem solving to learning the generators

It is equally important to teach the child how to solve a problem and how to communicate the problem's solution, i.e. how to present a series of items of reasoning or operations accepted by *consensus* as explaining the way to get to the result. This consensus represents the accumulation of an historical experience, so it is a cultural acquisition. Therefore, how numerous must the problems be in order to ensure learning? How should these problems be dosed in a textbook so that understanding might occur without an exaggerated effort? To make learning more effective, it is necessary to have an inventory of *generative* situations and to offer children as many opportunities as possible in order to have them interfere with various learning specific environments. *Generator,* in this context, means that on its basis, *by combination, substitution, by enlarging-narrowing the domain, by varying the actions, changing the topic*, etc., a great variety of problems can be created. Generators learning as opposed to problem solving has the advantage that it structures the understanding of the mathematical phenomena hidden behind particular statements. If, together with the ability to solve problems, the student gets the ability to understand and use the generators of a class of problems, then the child's cognitive acquisition is definitely superior and it refers to the arising of an over-learning phenomenon.

From "drill and practice" to "practice and structure"

Understanding word problems is closely connected with *analyzing and transforming the problems*. The child is stimulated to create word problems using various starting points. The technique is practiced loudly, silently, in written form, maintaining the interest in exercising as many capacities as possible and stressing the passage from one type of activity to another. There is also a systematic training of *becoming aware of errors*. The teacher "hides" some errors in exercises or problems, in series of numbers, in comparisons, etc. These errors are analyzed with the purpose to eliminate the pupils' typical errors, as well as to improve the analysis ability. *Estimations and approximations* are also systematically taken into consideration. The objective is focused on understanding the significance of numbers size and on checking the computing validity. Finally, two tasks are left to be solved: raising the internalized structure to the level of formal representations and transforming the internalized structure into a dynamic one.

METHODOLOGICAL CONSIDERATIONS ON GENDER STEREOTYPES

Leila Pehkonen
University of Helsinki

Riitta Soro
University of Turku

The aim of this poster is to present a new methodological tool for analyzing the Who and Mathematics (WM) -instrument developed by Forgasz (2001) and Leder (2001) and to get an integrated view about mathematics as a gendered domain. Our methodological contribution is based on diagrammatic model inspired by Donald Wolfe (1959), who investigated family authority structure. The data used for this methodological development was gathered from two Finnish schools. The total number of 7th grade students was 508.

The gendered domain is divided into two dimensions. The x-axis represents the gender-free values (GF). The GF -score is determined by the number of items which are answered „no difference between boys and girls". The possible maximum score on this dimension is 30. It would indicate that mathematics is completely gender neutral domain to the student. The y-axis represents the male - female -dimension (MF), with female dominant in the top and male dominant at the bottom. The MF-score is based on the sum of the numerical codes across the 30 items in the instrument and it ranges from 30 to 150.

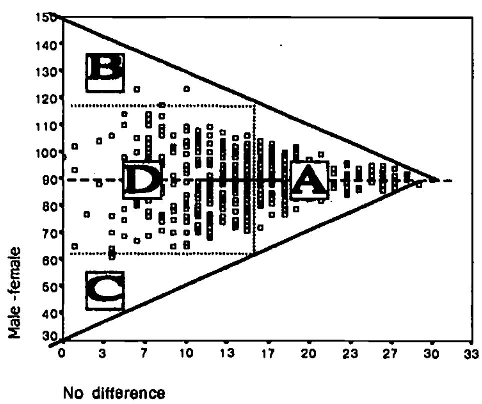

Figure shows the diagrammatic model of the theorical distribution on genderness on these dimensions. The dotted line divides the genderness into four types. Domain A can be seen as neutral domain. Domain B represents female and domain C male domain. Domain D can be seen as a indifferent domain – some aspects of mathematics are seen more female or more male. With help of this model we can consider, how genderness is divided in our sample on the whole. We can make diagrams on various groups. We believe, this model gives a promising tool to compare international samples, too.

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Forgasz, H. (2001). *Mathematics: Still a Male Domain? Australian Findings.* Paper presented at the AERA Annual Meeting, Seattle, USA, April 10-14, 2001

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REMEDYING SECONDARY SCHOOL STUDENTS' ILLUSION OF LINEARITY: A DEVELOPMENTAL RESEARCH

Wim Van Dooren[1,2], Dirk De Bock[2,3], An Hessels[2], Dirk Janssens[2] and Lieven Verschaffel[2]

[1]Research Assistant of the Fund for Scientific Research – Flanders (F.W.O.)
[2]University of Leuven and [3]EHSAL, Brussels; Belgium

A series of ascertaining studies by De Bock, Van Dooren, Verschaffel and Janssens (2001) has shown a strong and irresistible tendency among secondary school pupils to overgeneralise the linear (or proportional) model when working on applied geometrical problems about the lengths and the area/volume of similar geometrical shapes. These studies have shown that even with considerable support (such as self-made and ready-made drawings or metacognitive stimuli calling students' attention to the problematic character of the word problems), only very few students appeared to make the shift to the correct non-proportional reasoning. Moreover, a lot of information was obtained on the actual process of problem solving: although some students seem to really 'believe' that quantities are always linked proportionally, their improper use of linearity often results from superficial and intuitive reasoning, influenced by specific mathematical conceptions, habits and beliefs leading to a deficient modelling process. Altogether, these studies yielded valuable building blocks for the final stage of our research program, namely the design, implementation and evaluation of a powerful learning environment aimed at overcoming the illusion of linearity by developing in students a deep conceptual understanding of proportional reasoning including the disposition to distinguish between situations that can and cannot be modelled linearly.

This poster illustrates the major design principles of the learning environment by means of a sample of the instructional materials that were constructed in this developmental research. The learning environment is interspersed with various realistic problem situations, both linear and non-linear ones, which typically challenge students' mathematical (mis)conceptions, socio-mathematical norms and habits leading to stereotyped modelling. The relevant mathematical concepts and insights are explored and clarified from a variety of perspectives (using verbal, numerical, graphical, algebraic, and pictorial representations) serving as different lenses through which students can interpret these problems and their solutions. Finally, the learning environment relies on a combination of instructional techniques that have proven to be successful in enhancing students' deep understanding and higher-order thinking skills, such as coaching, scaffolding, articulation and reflection.

De Bock, D., Van Dooren, W., Verschaffel, L., & Janssens, D. (2001). Secondary school pupils' improper proportional reasoning: An in-depth study of the nature and persistence of pupils' errors. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of PME 25* (Vol. 2, pp. 313–320). Utrecht, The Netherlands.

HOW TO READ A MATHEMATICAL TEXTBOOK

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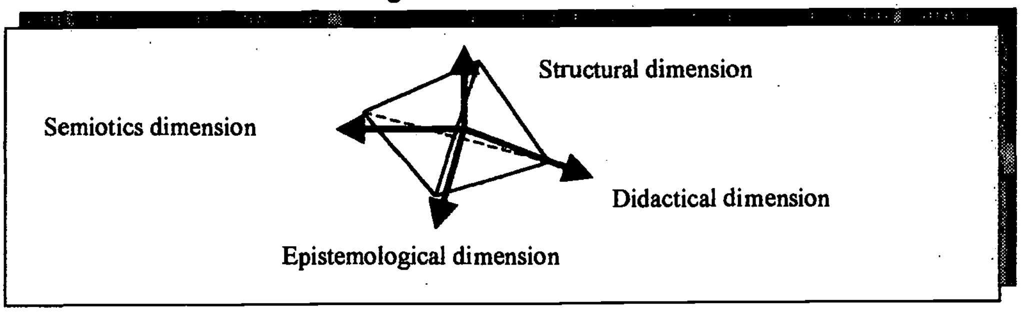
Universidad Nacional de General San Martín. Fundación Cultural Glaux
Buenos Aires- Argentina.

During the development of a Mathematics´lesson, specially in those which geometry is the main topic, pupils bring interest in some subjects. When this situation happens, teachers take decisions about how to explain some contents.These decisions grow up during the class and need solutions, which have been generated at the same place. When this appears, the textbook shows as an instructional resources, which suggests to the teacher different ways of class engineering. What pupils learn about geometry is conditioned by the contents of the lesson which appear in the textbook because itself brings us the most important geometrical acknowledgement to be studied during the school class. Although, teachers and pupils find a lot of different activities in the textbooks. Due to this, we propose an "Analysis Form of Textbooks". This form is based on four dimensions or categories showed in this tetrahedral:



These categories bring information about:

The way by which the author of the textbook presents his production to the pupils (structural dimension); What is mathematics and which is its epistemological frame (epistemological dimension); The author' thinking about mathematics and its didactics (didactical dimension); The mathematics' concepts and its relations which a semiotic frame (semiotic dimension).

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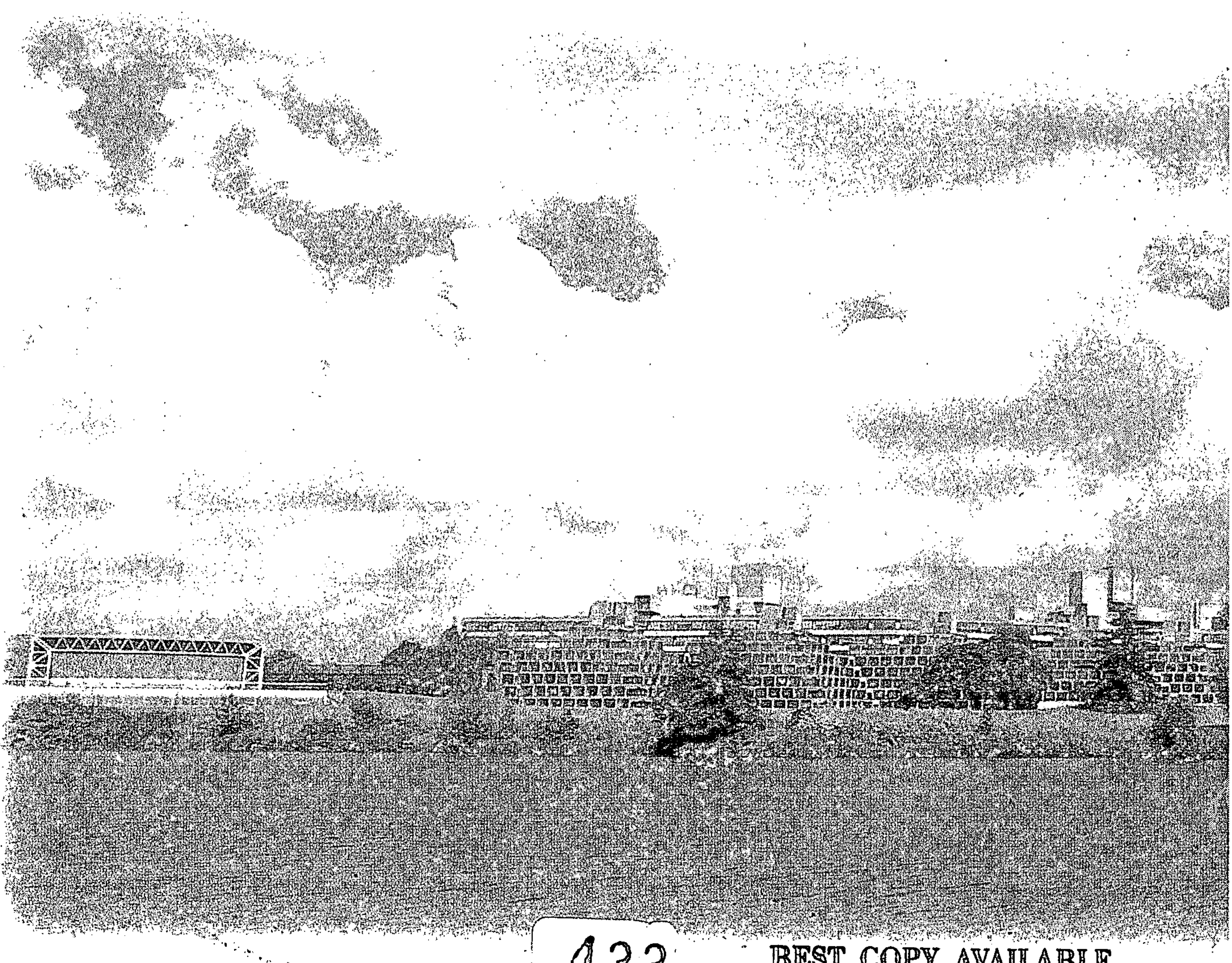

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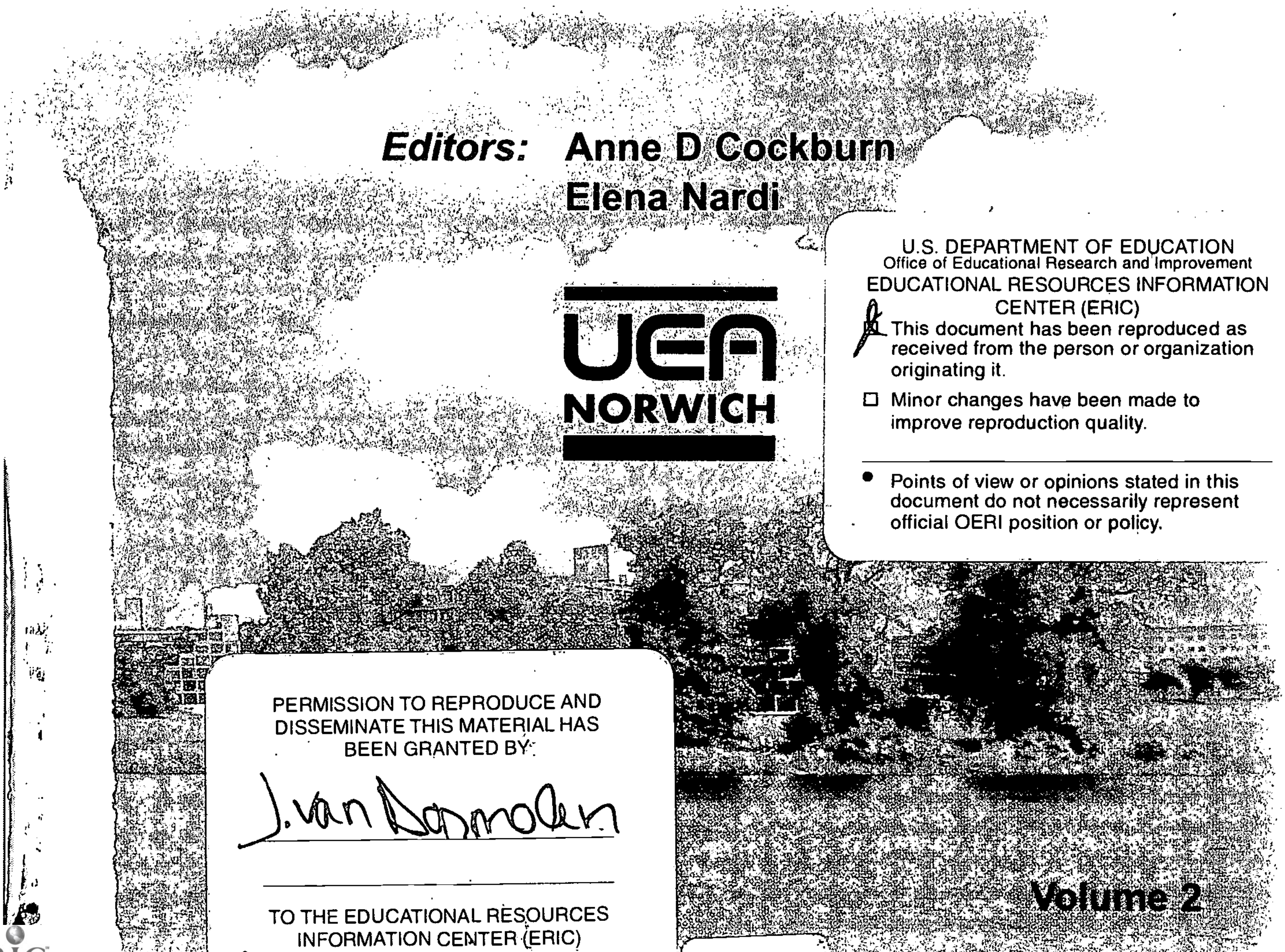

Editors: **Anne D Cockburn**
Elena Nardi

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Volume 2

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INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

PME26

UNIVERSITY OF EAST ANGLIA • UK

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PROCEEDINGS

Volume 2

Editors

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School of Education and Professional Development

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Proceedings of the 26th Conference of the International group for the Psychology of Mathematics Education

Volume 2

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Fax: + 44 1603 593446
Website: www.uea.ac.uk/edu
The Proceedings are also available on cd-rom

ISBN number 0-9539983-6-3

Design of the PME26 logo: Steve Giles

Printed by the University of East Anglia, Norwich, UK

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RESEARCH REPORTS

HIGH SCHOOL STUDENTS' IDENTIFICATION OF EQUAL SLOPE AND Y-INTERCEPT IN DIFFERENT STRAIGHT LINES

CLAUDIA ACUÑA
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CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS IPN-MÉXICO

Abstract

In this study I use a semiotic perspective to analyze with three student samples from different schools identification tasks of equal slope and y-intercept in different straight lines. Some of the students strategies take into account a visual conception that lies upon Duval's vision concept (Duval, 1999). When using these strategies, the students set aside the previously known slope and y-intercept definitions and the graphical aspect of the line, and limit themselves to the use of the coordinate axes as a visual reference, serving as an anchorage for the visual orientation for the identification of the line.

Resumen

En el presente estudio utilizo una perspectiva semiótica para analizar el trabajo de identificación de la ordenada al origen y la pendiente de tres muestras de estudiantes provenientes de diferentes escuelas. Algunas de las estrategias usadas por los estudiantes tomaron en cuenta una concepción visual que se ajusta al concepto de visión de Duval (1999). Al utilizar tales estrategias, los estudiantes hacen a un lado sus conocimientos previos sobre las definiciones de ordenada al origen y pendiente de una recta y su aspecto gráfico, y se limitan al uso de los ejes de coordenadas como referencia que sirve como anclaje de la orientación visual para la identificación de la recta.

Introduction

I want to mention two remarkable misconceptions about the understanding of the straight line. The first one was mentioned by Janvier (1978), quoted by Leinhart, et al, (1990). According to Leinhart et al, Janvier says that between the findings about the construction and interpretation of a straight line are those about slope/height confusion: "students have been found to confound these two graphical features on both interpretation and construction tasks" (op. cit. 1990, p. 37). "No consensus exists regarding the cause for such errors … A common interpretation is that students are confusing two graphical features: highest value versus slope" (op. cit., p. 39).

The second one is mentioned by Schoenfeld, Smith and Arcavi (1993). In their fine-grain analysis, they found in an interviewed subject that she had separate ideas about slope and y-intercept notions. This was evident when she had to develop construction and interpretation tasks. In their study they say that she knows that $m = \frac{y_2 - y_1}{x_2 - x_1}$, but this knowledge is nominal and although it is used to compute the line slope, this computation has no graphical entailments, and although she knows that b is the y-intercept, her understanding is nominal too and it is not tied to the underlying structure. The authors think that this is caused by the absence of what they call the Cartesian Connection, that is, "a point is on the graph of the line L if and only if its coordinate satisfy the equation of L" (Moschkovich, Schoenfeld and Arcavi , 1993, p. 73).

In regard to the above "misconceptions", I rather prefer the term "alternative students' conceptions". These conceptions appear when students interpret or read the graph. For Mavarech and Kramarsky (1997, p. 229) interpret or read the graph means the "students' ability to read a graph (or a portion thereof) and make sense or gain meaning (of) it ".

In the first "misconception" it is necessary to compare the slope and height graph; and in the second one, in order to make the Cartesian Connection, the student needs to translate between two different representations, one algebraic and other graphic.

In my research I used a questionnaire with items that require the only visual strategies. Unlike the results mentioned above, my work lies upon the visual aspects of the straight line, and the elements that take place in the treatment are the semiotic signs, their meanings, and their relations.

But, what is a graph? And what do we have to see in it so as to read it? From Bertin's (1968) perspective the visual perception (in graphs) consists of the perception of three factors: the variation of shadows and the two dimensions on the plane, regardless of the time variable.

In a similar way, Duval (1995, p. 142) claims that "a figure is an organization of sharp contrast of the brightness. It emerges from a background through the presence of 'traces' or 'spots', governed by Gestalt laws and perceptual clues".

The visual information in the graph, must be an important part of the graph comprehension. Some researchers, like Friel, Curcio and Bright (2001, p. 132), claim that "by graph comprehension we mean graph readers' abilities to derive meaning from graphs created by others or by themselves", but they do not explicitly consider, in their graph structure comprehension, the importance of the meaning of sign. They missed the consideration of the semiotic meaning of the graph.

From Duval point of view, the function of a graphical representation lies upon two figures: the ground-figure (axes, grid and orientation) and the form-figure (the graph). This relation is completely defined by this representation forming gestalt rule (Duval 1996, 2001).

From his interpretation I take the idea of representational units (Duval, 1999), as well as the gestalt relation, and I consider two of the three units: the slope and y-intercept. I am not considering the third unit, the angle with the x-axis, because it is not included in the high school math curriculum.

In my early observations of this kind of tasks that require visual interpretation, I have found that the students use some specific visual strategies. In this research, they used two different strategies to identify whether the slope and the y- interception of straight lines are the same or not, without taking into account the geometric or algebraic definition. These strategies are:

S1. The student can make a visual movement, in general translation or rotation, from one line to another. They can make a parallel translation in the case of the straight line position, they move the approximate measure of the angle related shape slope or of the position of the y-intercept respect to the horizontal axis.

S2. The student can take into account some real or imaginary tokens, or apparent relation, to help the orientation of the graphic elements, for example the quadrants that the path of the straight line observed and that make sense to the line position.

I observed that both strategies are neutral, this means that you can use them either with right or wrong results, in this way it is useful to mention Duval's idea (op.cit, p. 12) about vision and visualization concepts: vision refers to a visual perception, it gives direct access to the physical object; it consists of a simultaneous apprehension of several objects or a whole field. Visualization is based on the production of a semiotic representation. A semiotic representation shows relations between two representational units.

In my work I found that students used the mentioned visual strategies, regardless the algebraic or proportional definitions, to do some identification tasks. They used a vision oriented conception. This conception has a strong reference on the interpretation of the axes like an anchorage (Mesquita and Padilla, 1990).

Methodology

In this work I used a questionnaire answered by high school students in the 3thd. semester (17 years old). I worked with three samples of students (A,41; B,23 y C,35) in three different schools located in two industrial cities. The intention was that the samples were different only in their location because I wanted to see if this factor affected the answers.

Previous to my study, the students had some knowledge about straight line, namely, its point by point graph construction from a known equation, a slope definition coming from algebraic and proportional issues. They had some skill in calculating straight line slops and y-intercepts. They knew that between two points only on straight line can pass and they calculated slope and y-interception from visual information on a graph.

In my questionnaire I asked for the construction of different straight lines with different slopes and y-intercepts. I requested some explanation about possible changes in the construction in order to make two different lines equal. The students also needed to identify equal straight lines among different lines. In general, the students did well. Most of them showed that they can easily recognize the visual shape of slope and y-intercept.

But, in some items, I found two kinds of answers that caught my attention. In this paper I will report one of them. This result seem to show that under specific situations the student quit the definition and limits herself to consider exclusively the shape. This kind of answer were detected in two groups of exercises in which the students have to identify slope and y-intercept among several options; before they answered the questionnaire, I gave to the students a definition of slope and y-interception because I did not want this to become an obstacle in their performance. The referred items are supported on visual information only, and the indications were posed in natural language.

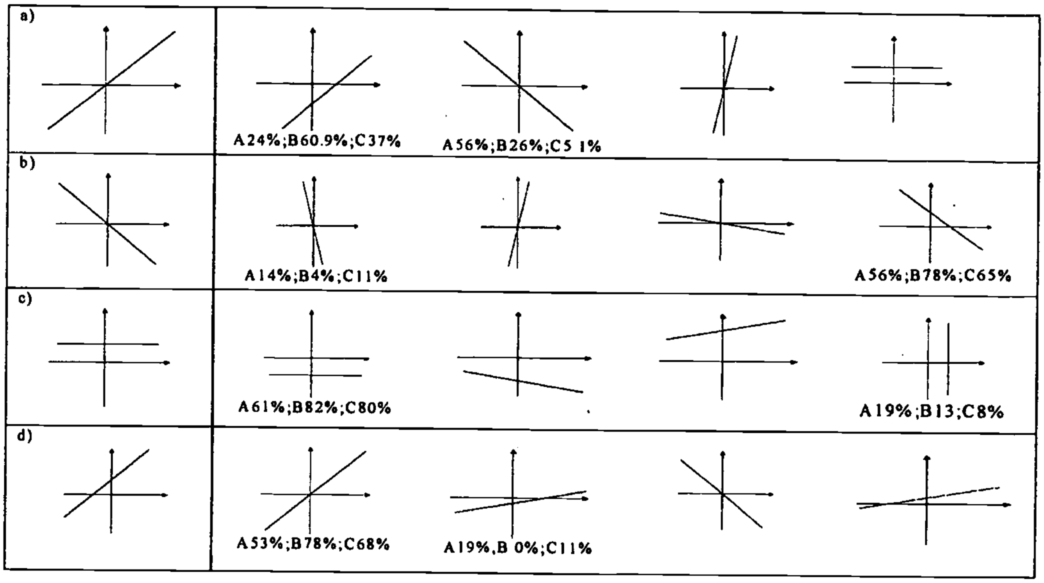
In this qualitative research I expect to give an approximate answer to the following questions: Can the student make a consistent visual identification of same line with the slope and y-interception among different straight lines?, to what kind of signs do the students pay attention to find the difference?

Question Analysis and Results

My observation is based in the following two groups of questions. The student have to choose among different graphs using visual criterions posed in natural language. In the table we have written the frequency of incidence of right answers and the some of most frequent wrong answers, in order to compare the results .

The first group of exercises (belonging to Item 9 of the questionnaire) follows:

9. For each graph on the left, mark whit with an X the graph on the right that has the same slope. The slope is the slant that the straight line has with the horizontal axis.



Frequencies of right answers of the samples A, B and C for item 9.

In Item 9(a) it seems that the term "same slope" is associated to "same angle" between the straight line and the x-axis, but in doing the identification they consider an non-oriented angle. In this case they followed strategy S2 and the tokens used are, in one hand, the x-axis and the "opening" that it forms with the line, both tokens are part of the global aspect of the graph. And in the other, there is the point (0,0) that appears as y-intercept on both lines.

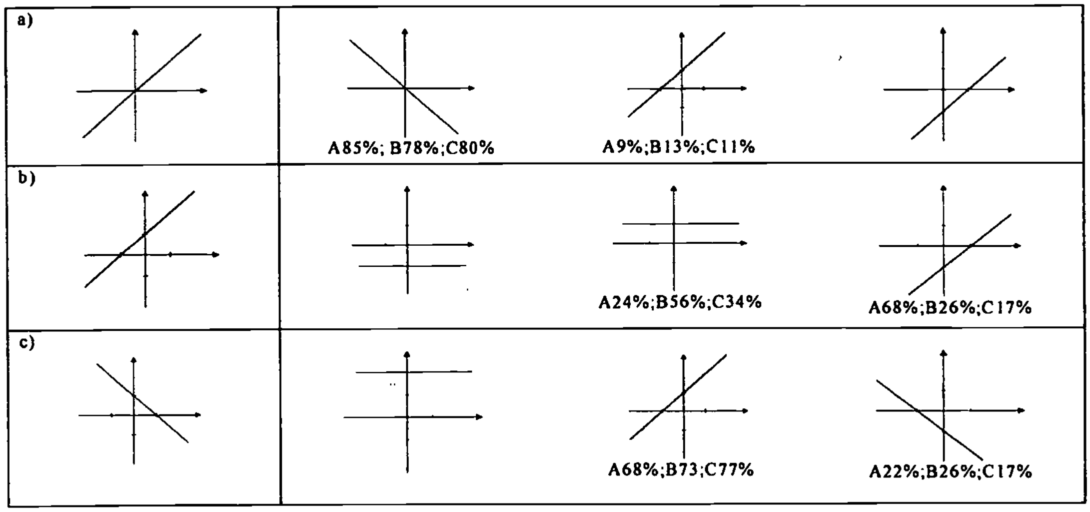
In this case, their use of S2 is oriented by the vision that refers to a visual perception, and it gives access to the physical object and give us the simultaneous apprehension of several objects or a whole field (Duval, 1999). Students focus their attention on the x-axis and the point (0,0), and these tokens are considered like an anchorage (Mesquita and Padilla 1990). Their solution does not have a semiotic treatment, that is, a treatment in which the semiotic representations show their relations between representational units, the gestalt relation is missing.

The other group of exercises (belonging to Item 10) are :

10. For each graph on the left, mark with an X the graph on the right that have equal y-intercept. The y-intercept is the point where the line crosses the y-axis (see diagram on the next page).

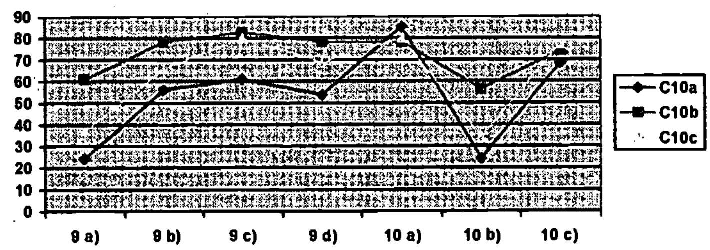
In this exercise the students' identification is more or less successfully in 10(a) and 10(c) (A,85% and 68%; B, 78% and 73%; C 80% and 77%

respectively) nevertheless, the situation in 10(b) is quite different, this is shown in the frequencies graph that follows (see next page). The wrong answers are supported by a consideration of the global shape of the graph, oriented by the vision, instead of a visualization. It seems that a oblique line has nothing to do with a horizontal one.



Frequencies of right answers of the samples A, B and C for item 10

The following graph shows the frequencies of correct answers of the two discussed items, the frequencies are similar in the three different samples.



Frequencies of both groups of exercises

Discussion

The present research lies only on the visual treatment of a straight line graph, for this reason the students' conceptions, evident from wrong answers, are different from those that have a functional treatment like in Janvier (1978) and Schoenfeld et al (1993); the visual application of the conceptions are different too.

The students' difficulties can appear even in the visual situations when we ask the students to make graphs without explaining nor working the gestalt relations, that is, the relation between form-figure and ground-figure, or how we can plot graphs considering the spatial relation ruled by the axes, the grid and the orientation.

In my opinion this students' conceptions where the strategies S1 and S2 take place, were hidden because the visual treatment was not sufficiently considered in the school curricula.

The identification tasks about y-intercept and slope are quite far to be simple applications of a definition, the results suggest that the students can make almost always a consistent visual identification but the wrong answers in items 9(a) and 10(b) suggest that in tasks with a strong situational component, this visual identification fails, so there is evidence that students are not entirely consistent in making visual identifications in the case of y-intercept and slope.

The answers permit us to know some students' conceptions about y-intercept and slope. In fact, our students pay attention to the marks on the axes, and the axes themselves are used as important visual tokens, but in some critical situations this mechanism do not work adequately because the election is arbitrary. In these cases, the axes function like a strong anchorage in the general shape of the graph.

Finally, the students used their own "gestalt" relation that took place on the visual identification despite the previous training or definitions about slope or y-intercept, and they make their own interpretations in cases in which there was not an intentional intervention of the teacher, this interpretations are supported by their own ideas.

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A developmental scale for assessing probabilistic thinking and the tendency to use a representativeness heuristic

Thekla Afantiti Lamprianou and Julian Williams
University of Manchester

We report a study of children's probability conceptions and misconceptions due to the representativeness heuristic. Rasch measurement methodology was used to develop a 13-item open response instrument with a sample (N=116) of 12-15 year olds. A hierarchy of responses at two levels is confirmed for this sample, and a third level is hypothesised. Each level is characterised by the ability to overcome typical 'representativeness' effects, namely 'recency', 'random-similarity' (at level 1), 'base-rate frequency' and 'sample size' (at level 2-3). Our interpretations were validated and anomalies identified through clinical interviews with children making the errors (n= 8), suggesting another measure, which we named the 'representativeness tendency' from 11 multiple-choice errors.

Introduction

This study builds on previous work on children's understandings, intuitions, use of heuristics and misconceptions in their probabilistic thinking (Fischbein, 1975, 1997; Kapadia & Borovcnik, 1991; Shaughnessy, 1992) and especially the significance of the representativeness heuristic (Green, 1982; Kahneman, Slovic & Tversky, 1982; Amir & Williams, 1999; Amir et al, 1999). The misconceptions based on the representativeness heuristic are some of the most common errors in probability: children tend to estimate the likelihood of an event by taking into account how well it represents its parent population and how it appears to have been generated.

In this study we aim to contribute to teaching by developing an assessment tool which can help teachers diagnose inappropriate use of the representativeness heuristic in responses to questions relevant to the probability curriculum. Williams and Ryan (2000) argue that research knowledge about students' misconceptions and learning generally needs to be located within the curriculum and associated with relevant teaching strategies if it is to be made useful for teachers. This involves a significant transformation and development of research knowledge into pedagogical content knowledge (Shulman, 1987), which requires its own study. The development of the assessment instrument involved tuning of, or development of diagnostic items from the research literature: thus the instrument provides a 'boundary object' between the research practice and innovative practice of assessment for teaching and learning.

Thirteen items were used to construct the instrument (the instrument can be seen in full on the web at http://www.education.man.ac.uk/lta/tal). The items identify four effects of the representativeness heuristic; the *recency* effect, the *random-similarity* effect, the *base-rate frequency* effect and the *sample size* effect. Most of the items have been adopted with slight modifications of those used in previous research by Green (1982), Kahneman, Slovic & Tversky (1982), Konold et al (1993), Batanero, Serrano & Garfield (1996), Fischbein & Schnarch (1997) and Amir, Linchevski & Shefet (1999). Other items were developed based on findings of previous research.

Items called *recency1, 2 and 3* tested for the negative recency effect and the gambler's fallacy. According to this effect, a long sequence of one outcome must be followed by the other outcome in order to equilibrate the proportions. Items called *random-similarity 4, 5, 6 and 8* tested for the effect which expects a sample to appear similar in proportion to the parent population and apparently randomly-generated. These items were developed from Kanheman, Slovic & Tversky (1982), Fischbein & Schnarch (1997), Green (1982) and Shaughnessy (1992), respectively.

Items called *base-rate 10, 11 and 12* were written to examine the effect of prior probability or the base-rate frequency of the outcomes in contexts appropriate to this age group. According to this effect, prior probabilities are effectively ignored when misleading irrelevant but stereotypical information is introduced. As Kahneman, Slovic and Tversky (1982, p.5) mentioned, "when no specific evidence is given, prior probabilities are properly utilised; when worthless evidence is given, prior probabilities are ignored". Finally, items called *sample size 7, 9, 13* tested for the tendency to neglect sample size in estimating probability: the first from Fischbein & Schnarch (1997), the second from Shaughnessy (1992), the last ours. These items examined the belief that the probability of a certain proportion in a sample is independent of the sample size, contradicting the central limit theorem, i.e. the probability of getting a certain empirical result tends to approach the theoretical prediction as the sample gets larger.

Method

In order to be able to administer more items to the same sample of pupils two separate test-forms with common linking items were constructed. Test A, designed to be easier, consisted of eight items - items 1 to 6, 9, 10 - and Test B, intended to be more difficult, consisted of ten items - items 3, 4, 6 to 13. Five of the items were included in both tests. Test A was administered to pupils in Year 7 and Test B was administered to pupils in Year 8 and 9.

The tests were administered to 116 pupils from two schools in the North West of the United Kingdom. Before administering the tests to the pupils, the teachers of the six classes were asked to read and comment on the suitability of the tests for their classes. They found the wording of the items acceptable for the pupils' age, but they commented on the degree of difficulty of question 13 (*sample size 13*).

For the analysis of the results of the tests, a Rasch common calibration was used. Since all items had both a multiple-choice and an open-ended question, a common item Partial Credit analysis (Wright and Matsers, 1982) was run. One mark was given for the correct multiple-choice answer and another one for the correct explanation of the open-ended question for each of the 13 items. The result is a single scale consisting of a 'difficulty' estimate for each scored point and an 'ability' estimate for each child consistent with the Rasch measurement assumptions. Item 13 fell outside a model infit statistic value of 1.3 (see Wright & Stone, 1979) reflecting the difficulty of this item for the sample. The term 'ability' is defined by the performance of the pupil in this particular test, which we consider to imply an ability to avoid inappropriate representativeness effects, and so a particular ability with probabilities: it is NOT a measure of general ability, or even of mathematical ability. We will use the term *ability* italicised in this way throughout this paper.

In addition to the test analyses, we drew on structured clinical interviews with 8 children about the test items to gain insight into the cause of the effects described above, to confirm the literature, validate the items and identify anomalies.

Results

According to the table below, the test and sample can be interpreted as falling into a hierarchy of three levels. At level 1, (-3.0 to -0.5 logits) children can succeed on questions that tested for the *recency* effect and easy questions that examined the *base-rate frequency* and the *random-similarity* effect. At level 2 (-0.5 to 2 logits) children attain higher performance and they can explain their answers to the easier questions that tested for the *random-similarity* effect, they can manage harder *base-rate* and *random-similarity* questions and they are beginning to answer some *sample size* questions correctly. Very few children manage to attain level 3 by answering the hardest questions on *sample size* or explaining their answers to the harder questions that tested for the *random-similarity* effect. In order to establish level 3, it is suggested that a more able sample would be required.

By averaging the ability estimates of those children who made an error, we are able to plot errors on the same logit scale in the table. Pupils who gave responses indicating the *recency* misconceptions had a rather low *ability*. Answers indicating misconceptions based on the *random-similarity* effect were given by a broader range of *ability* pupils (averages ranged from -1.95 to -1.25 logits). On the other hand, the mean *ability* of the pupils who gave responses based on the *sample size* and *base-rate* effect was near the average *ability* of the sample, reflecting the fact that these errors were made by so many children (36%, 49%, 73%, 78% and 86% !!).

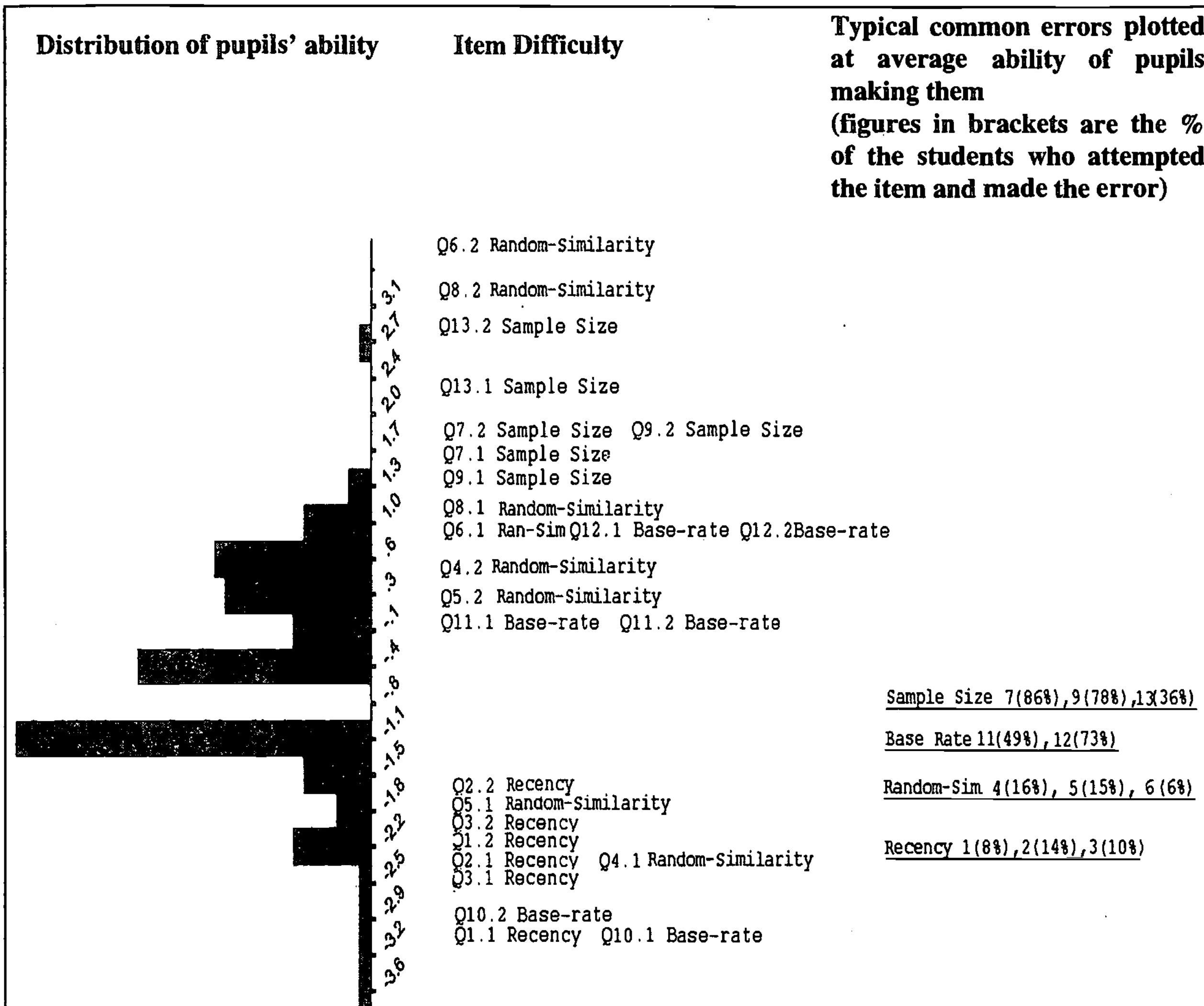


Table: Scale of performance and errors

The errors listed above are most likely to be made by children at the ability adjacent or below

This encouraged us to consider building a 'representativeness tendency' measure from those errors which we can authentically attribute to this heuristic, as a diagnostic measure of tendency to inappropriately apply this heuristic. The main purpose of the pupil interviews was validation of the test, in particular our interpretation that the errors in the test are symptomatic of the representativeness effects discussed in the literature. In this section we illustrate with the interviews of children the *random-similarity* effect, which was examined by *random-similarity 4, 5, 6 and 8*. For example, *random-similarity 4* is illustrated below:

Random-Similarity 4: A fair coin is tossed five times. Which of the following sequence of outcomes is the most likely result of five flips of the fair coin? (H: Heads, T: Tails) (a) HHHTT (b) THHTH (c) THTTT (d) HTHTH (e) All four sequences are equally likely. (Explain why).

Child10: … I'll probably pick (b), because it's a mixture of answers.

Teacher: (b), because it's a mixture of answers and -
Child10: - Because it is more realistic, because it's been a good ... em mixed. ...
Teacher: OK. But we have the same here in answer (d). There are three Heads and two Tails. Why did you choose that answer?
Child10: Because it's a bit more...em... because that's a Head, Tails, Head, Tails, Head, Tails and I think... I don't think it will be ... to get that most times ...

When this child was asked to solve a similar problem (*random-similarity 5*), his choice of frequency was representative to its parent population (it consisted of equal number of boys and girls) and it appeared to be a random mixture of boys and girls:

Random-Similarity 5: In a family of six children which sequence of births is the most likely? (B: Boy, G: Girl) (a) BBBGGG (b) BGBGBG (c) GBBGBG (d) GBGGGB (e) All four sequences are equally likely. (Explain why)

Child10: I'll probably think (c).
Teacher: Why?
Child10: Because you would more likely to em... pick that one because it's more mixture, but sometimes with different people they have all boys or just all girls. But some people have a mixture.

However, some items proved problematic. *Random-similarity 6 and 8* (the multiple-choice answers were classes of sequences, i.e. 6 Heads and 6 Tails) were developed from an item from Green, in which, as Amir, Linchevski & Shefet found, the majority of the pupils chose a different incorrect answer to the expected *random-similarity*. This was based on the conceptual error which reflected children's failure to discriminate between sequences and classes of sequences (combinations), between ordered sets and unordered sets in probability. An example from the interviews:

Teacher: ... So. Why did you answer (e) "All have the same chance"?
Child61: Because they are all likely. You can't guess what ... which **one** will be it. But they all have the same chance, because it's half. ...

Teacher: OK. ... (Child62 writes WWWGGG) ... Is there any other sequence for three white and three grey?
Child62: No.

We therefore came to the conclusion this misconception was distracting from the *random-similarity* effect, which was only given by 6% of children in *random-similarity 6*. Indeed this fell to 0% for *random-similarity 8*, which suffered similarly. Worse still, children chose the correct option but for the wrong reason, arguing from a representativeness perspective, that '6 Heads and 6 Tails' would be most likely, because:

C33: Because, em, there's two sides on the coin and you get, em... even chances of getting six heads and six tails, because if you divide by two... added to two sides of the coin there's six on each, like heads and tails.

This also explains why so few children could give a correct explanation for their answers to these items. We suggest these items may need therefore to be deleted, redesigned or developed and re-scored.

Base-rate 10 might also be discarded when using the test diagnostically for this age group. Almost all pupils answered this question correctly. In addition, all eight interviewees gave the correct answer and justified their responses correctly. From the answers that children gave, it seemed that the irrelevant information did not distract any of the children and they did not ignore the prior probability. This item seemed as if it had no diagnostic value. Analysis of *base-rate 11 and 12*, however, suggested that the more the distracting description is related to a stereotype, the stronger the *base-rate* effect is.

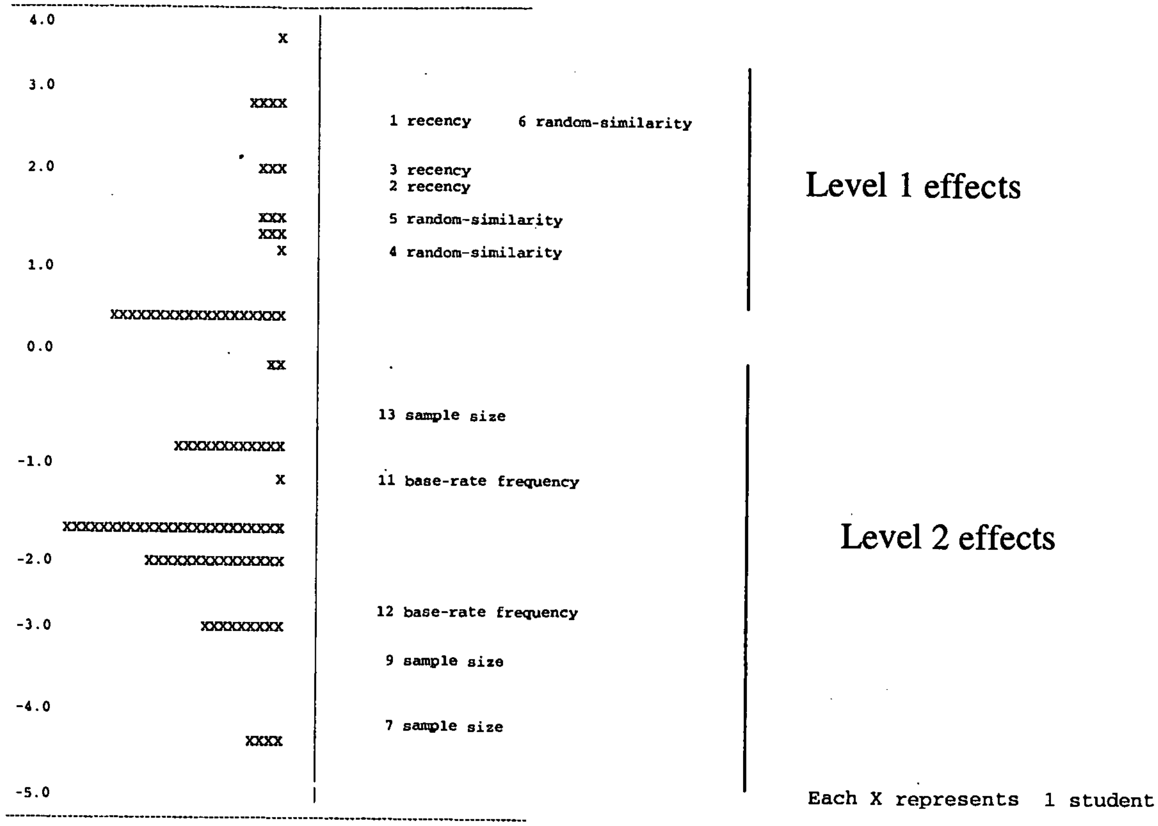
Construction of a representativeness measure

Since the purpose of the diagnostic instrument was to assess whether the representativeness heuristic influenced children's thinking when solving probability problems, a second Rasch model analysis was run. One mark was given only for the multiple-choice answer that indicated the effect of the representativeness heuristic but no marks were given for any other responses. The result was a single scale of items (none of the mark points fell outside a model infit value of 1.3), indicating that the *sample size* effect and the *base-rate* effect were very frequent among the pupils. The *random-similarity* and the *recency* effect influenced a small number of pupils of this sample (see graph below). *Random-similarity 8* and *base-rate 10* were removed from the Rasch analysis because all pupils gave different responses to the expected representativeness effects.

The result is a measure of 'representativeness tendency' for each person, and this naturally correlates negatively with their *ability* as measured previously (rho=- 0.64). However, the outliers are interesting: these represent three children who either found ways of scoring relatively well despite their tendency to use the representativeness heuristic, and vice versa. These might be the focus of further case study.

Children higher up the scale are more likely to make representative-effect related errors, and items higher up the scale are less commonly occurring, i.e. only made by those with a strong 'representativeness tendency'. Note that these fall into the two levels of questions identified previously in the table, with *recency* and *random-similarity* effects generally occurring at level 1 and *base-rate frequency* and *sample size* effects at level 2.

Graph: Item and Person Estimates of the 'representativeness tendency', using 11 multiple-choice responses only.



Conclusions and discussion

We have managed to develop two scales measuring children's responses to the instrument which is revealing about their probabilistic knowledge, especially as regards their inappropriate use of a representativeness heuristic in responding to test questions which are relevant to their curriculum. We have further identified some previously unknown interpretations of children's responses.

While most of the particular items in the scale are not new, the development, validation and calibration of the measures around this heuristic for 12-15 year old children is. We expect these to be useful research tools, but also to impact on teaching practice and teacher education, as discussed in Williams & Ryan (2001).

Having collected responses of some teachers to the instrument, we are doubtful that teachers are aware of these common misconceptions or of the significance of the representativeness heuristic, and we suggest that many teachers might benefit from using such an instrument in their assessment and teaching. The knowledge that teachers would collect from these scales, might enrich teachers' mental models of their learners and help them improve their classroom practice. We will be studying this aspect in the next stage of the work.

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PURPOSE AND UTILITY IN PEDAGOGIC TASK DESIGN

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In England and Wales, the introduction of a National Numeracy Framework for the teaching of mathematics from ages 4 to 13 has placed a very strong emphasis on teachers' planning of objectives. By looking retrospectively at the design of computer-based tasks that have underpinned our research for many years, we recognise a theme of purposeful activity, leading to a planned appreciation of utilities for certain mathematical concepts. We discuss how the identification of objectives needs to go hand-in-hand with careful consideration of the planning of tasks, and propose two constructs to guide that planning.

BACKGROUND

The National Numeracy Framework (NNF) (DfEE, 1999a) was introduced to England and Wales in 1999. By defining how teachers should plan, assess and teach mathematics, this initiative extends the previously established National Curriculum for Mathematics (DfEE, 1999b), which merely set out to define a syllabus of content. The NNF is organised around sets of yearly objectives based on the content of the National Curriculum. The NNF also emphasises short-term planning based around tasks and activities:

> short-term plans: weekly or fortnightly notes on tasks, activities, exercises, key questions and teaching points for 5 to 10 lessons, including how pupils will be grouped, which of them you will work with, and how you will use any support. (p 41)

For clarity, within this theoretical paper, we use the term "task" for what is set by the teacher, and reserve the term "activity" for what subsequently takes place in the classroom setting. The planning emphasis in the NNF makes it opportune to reflect on the complexity of connecting objectives to the design of tasks. We believe that, though this reflection has stemmed from a phase in the development of curricula in England and Wales, in fact the arguments and principles elaborated below will have relevance to mathematics educators and teachers across international boundaries. We begin by problematising the procedure of setting objectives in mathematics lessons, and later we look retrospectively at the design of tasks that have played a significant part in our own research in order to propose two constructs that can guide the connecting of objectives to the design of tasks.

THE PLANNING PARADOX

We begin with a statement of what we call *the planning paradox*.

If teachers plan from objectives, the tasks they set are likely to be unrewarding for the children and mathematically impoverished. But if teachers plan from tasks, the children's activity is likely to be unfocussed and learning difficult to assess.

To elaborate a little, we offer two contrasting examples. The teacher begins the lesson by saying, "Today, we are going to work on adding two-digit numbers together". The lesson proceeds with some explanation and then practice. The task has been determined by the objective in a narrow and constrained way. Such a teacher has fallen foul of the first part of the planning paradox. Now consider a teacher who, as the focus of the mathematics lesson, asks the children to design their ideal bedroom. The children may become highly engaged in a meaningful activity, but the teacher may find it difficult to monitor any mathematical thinking.

TASKS INSIDE AND OUTSIDE THE CLASSROOM

In recent years, mathematics educators have taken a great interest in situated cognition research (for example, Lave, 1988, and Nunes et al, 1993). Such studies argue that everyday tasks (*street* mathematics) lend an authenticity to activity that provides not only purpose but also meaning. A possible implication is that we should attempt to offer such authenticity to children in classrooms. Whilst we would agree with Schliemann (1995) that 'for meaningful mathematical learning to take place in the classroom, reflection upon mathematical relations must be embedded in meaningful socially relevant situations', we see the provision of 'authentic' tasks as inherently problematic.

Teachers often provide children with tasks that may superficially offer authenticity. For example, a teacher of young children may set up a play shop in the corner of the classroom to encourage some mathematical learning. However, the structuring resources provided by this situation will be very different from those offered when the child is really shopping with a parent. For example, in the play-shop task, emphasis may be placed on number, and so the prices of items on sale are simplified to an extent that even young children will recognise as unrealistic. Even with an element of role-play, the social interactions of the play-shop will not provide the structure and constraints experienced in a real shopping trip.

Situated cognition research has typically studied the activities of master and apprentice. In such situations, the master and the apprentice have a common goal, which in the short-term is typically to make some product (so the tailor and his apprentice may aim to make a waistcoat), and in the longer-term is to make a profit. We question whether teachers and children can have common goals. The teacher's agenda must be to focus on their pupils' learning (and the children know this), whereas the children's agenda will be to complete the task, hopefully to the satisfaction of the teacher. This leads us to question what kinds of products teachers and children might make together. We now consider approaches that do place product creation at the forefront of children's activity, though in ways which differ significantly from the tailoring workshop.

Our focus over many years has been on the use of technology in the learning of mathematics, and so it is natural for us to look towards the literature in that field for inspiration.

The constructionist movement (Harel & Papert, 1991a) has proposed that tasks in which children make products, generally through programming computers, are particularly conducive to learning. Thus, in one sense the constructionists replace the "waistcoat" with a product that is programmed by the child into the computer. In our experience (for example, as reported in Pratt & Ainley, 1997, and Ainley et al, 2000), such programming tasks can generate activity that has some of the characteristics of everyday activity studied in the research of situated cognitionists. For example, teachers often become engaged in working with the child to make the virtual product, reminiscent of the tailor and apprentice collaborating to make the waistcoat. The task, rather than the externally set objectives, takes on the role of being the arbiter of what counts as progress. Thirdly, any mathematical learning that takes place is contextualised within the activity of making the product, which provides meaning for the mathematics, but perhaps limits its apparent range of applicability.

Harel and Papert (1991b) also recognise the connection between constructionism and situated cognition, and at the same time signal some differences in emphasis.

> We see several trends in contemporary educational discussion such as 'situated learning' and 'apprenticeship learning'... as being convergent with our approach, but different in other respects... our emphasis (is) on developing new kinds of activities in which children can exercise their doing/ learning/ thinking...(and) on project activity which is self-directed by the student... (p42)

We believe that the constructionist approach recognises that the classroom is not the market place, and does not attempt to place emphasis on authenticity, but, by placing emphasis on the creation of products, it positions consideration of meaningfulness and motivation high on the agenda for the design of tasks that are likely to promote mathematical learning. Our aim is to draw out from these ideas constructs that inform the teachers' problem of connecting objectives to task design (the planning paradox).

CONNECTING OBJECTIVES AND TASK DESIGN

Schliemann (1995) concludes a discussion of the problems of bringing everyday mathematics into the classroom with the statement that 'we need school situations that are as challenging and relevant for school children as getting the correct amount of change is for the street seller and his customers'. In considering both constructionist approaches, and 'authentic' settings for learning mathematics, we identify a common feature, which may provide this challenge and relevance, as the *purposeful* nature of the learners' activity. We see this feature of purpose for the learner, *within the classroom environment*, as one key construct informing pedagogic task design. It is important to note that *purpose*, as we use the term here, is not necessarily linked to 'real world' uses of mathematics. Indeed, there is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide purpose (see for example Ainley, 2001, Cooper and Dunne, 2000). We define a purposeful task as one which has a meaningful outcome for the learner, in terms of an actual or virtual product, or the solution of an engaging problem.

Thus the purpose of a task, as perceived by the learner, may be quite distinct from any objectives identified by the teacher. In a classroom situation, this maybe true in a trivial sense: learners may construct the purpose of any task in ways other than those intended by the teacher. However, we are saying something more than this: within our framework for task design, *purpose* is a distinct element that needs to be considered separately from, but in parallel with, objectives.

However, a focus on purpose in isolation may produce tasks which are rich and motivating, but fall foul of the second part of our planning paradox, by lacking mathematical focus. We therefore introduce into our framework a second construct of *utility*. Within pedagogic tasks that are designed to have purpose for learners, we have found that it is possible to plan for opportunities for learners to appreciate the *utility* of mathematical concepts and techniques. Whilst engaged in a purposeful task, learners may learn to use a particular mathematical idea in ways that allow them to understand not simply how to carry out a technique, but how and why that idea is useful, by applying it in that purposeful context. This parallels closely the way in which mathematical ideas are learnt in out-of-school settings.

A TASK DESIGNED WITH PURPOSE AND UTILITY

An example may help to clarify the related constructs of purpose and utility. A task which we have used (and written about) on a number of occasions, is that of designing a paper spinner, or 'helicopter' (see Ainley et al, 2000). In this task the purpose for the learners is clear: to make a spinner that will stay in the air for as long as possible. In investigating aspects of the design, for example by changing the length of the wings, children record results of test flights on a spreadsheet, (for example, the wing length and time of flight). Their activity offers opportunities to use a number of mathematical ideas, including measurement of length and time, decimal notation, graphing. From these we now describe two examples of *utility*.

Initially it is difficult for children to see patterns in the numerical data on the spreadsheet, partly because the data is not usually collected in a systematic way, and partly because of experimental inaccuracy. Using a scatter graph to display the results at intervals during the experiment makes it easier to see emerging patterns in the time of flight as the wing length varies. Information from the scatter graph is used to make conjectures about the effects of changing the wing length, and which spinners will prove most efficient, and also to identify further areas for experimental investigation. Using a scatter graph in this purposeful way offers opportunities to learn about the conventions of this particular graph, but also to understand that graphing is an analytical tool, which can inform the process of doing an experiment: that is, the children are given clear opportunities to construct a *utility* of graphing.

Discussion of the experimental inaccuracies in the activity leads us to introduce the idea of taking the mean value of several experiments with each wing length to produce a 'better' graph. This can be done quickly and easily using the AVERAGE function of the spreadsheet. Children are able to use their everyday knowledge of the meaning of 'average' to understand enough about this process to appreciate a *utility* of average (which does indeed produce a clearer graph), even though they do not know the detail of how the mean was calculated.

These two instances of introducing the utility of mathematical ideas both involve situations in which the use of technology means that mathematical ideas (graphing, average) can be *used* without children having learned the skills and techniques that underpin them (constructing a graph, calculating the mean). This is not co-incidental, and we return to further discussion of this point later in the paper. What we wish to emphasise here is that the opportunity to understand the utility of these ideas arises because of the purposeful nature of the task set, and of the learners' activity in response to these tasks. Without the underlying purpose of producing an efficient spinner, graphing experimental results and using average values could only have been introduced as isolated techniques. Their usefulness might be described through imagined applications, but could not be experienced in ways that allowed learners to construct rich meanings for the mathematical ideas.

RESOLVING THE PLANNING PARADOX

The two constructs of purpose and utility offer a framework for task design that may resolve the planning paradox. Designing tasks that are purposeful for learners ensures that the activity will be rich and motivating. Such purposeful tasks provide opportunities to learn about the utility of particular ideas, which will give the focus that may otherwise be absent.

It is widely recognised that constructing meaning for a mathematical idea involves many related elements. The distinction is often made between those elements relating to procedures or techniques, and those concerned with conceptual or relational understanding. We propose here a third cluster of elements: those relating to the utility of an idea. A rich understanding of a mathematical idea involves procedural and conceptual elements, but also understanding why that idea is useful, how it can be used and what it can be used for. We conjecture that understanding mathematical ideas without an understanding of their utility leads to significantly impoverished learning. Unlike 'street' mathematics, ideas in school mathematics are frequently learnt in contexts where they are divorced from aspects of utility. Within the classroom, opportunities to understand utility can only be provided through purposeful tasks.

However, the design of tasks that offer both purpose and utility is challenging. It requires the teacher to imagine the trajectory of a learner's activity, taking both a mathematical and a learner-centred perspective. In order to tease out aspects of the design of such tasks, we will discuss our own struggle to create a task with purpose and utility in a particular environment (for an extended discussion of this task see Pratt and Ainley, 1997).

As part of a long-term project, a group of children in our research school had been given access to dynamic geometry software (the original version of Cabri), and had explored its use as a drawing package with little intervention from their teacher. The children had explored many of the features of the software, and produced impressive drawings, but had actually made no use of construction. Their (self-selected) tasks were purposeful, with clear end products (drawings of a football pitch, a clown, and motorcycle and rider), but we felt that the children were not learning any mathematics.

We set out to design tasks that would be equally purposeful for the children, but also introduce the idea of geometric construction in a way that allowed children to understand its utility. This proved problematic. It was easy to design tasks that involved construction (such as creating a square which couldn't be 'messed up') but such tasks had no purpose for the children – except to satisfy the teacher. We knew that the children found creating drawings purposeful, but for such tasks placing points by eye, rather than constructing them, was perfectly satisfactory.

Eventually, we came upon the idea of harnessing the children's interest in drawing by setting them the task of producing a 'drawing kit' for younger children to use. This entailed them writing macros to produce a range of basic shapes (triangles, squares, diamonds etc.) from which young children could create their own pictures. For this task, there was a real purpose behind constructing a square, which could be reproduced many times and manipulated without being messed up by the younger children. As the children worked on making their drawing kits, the utility of construction for producing 'perfect' robust shapes became clear.

The design of the drawing kit task embodies some elements which we offer here as heuristics for creating purposeful tasks.

i) It has **an explicit end product** that the children cared about (a feature in common with the 'spinner' task described earlier).

ii) It involves **making something for other children** to use. In this case, the fact that the children for whom the product is intended were younger added an implicit further dimension of teaching other children.

iii) It was well focussed, but still contained **opportunities for children to make meaningful decisions**: although most of the drawing kits produced contain a similar set of basic shapes, many groups added their own designs, such as wheels or roofs.

To these, we add two other heuristics, which we know to be effective.

iv) Purposeful tasks are often based on **an intriguing question.** The spinners task may be seen as an example: how does wing length affect the spinner? Questions like this, which are solved through optimisation, seem to be particularly rich.

v) Tasks which involve **children arguing from a particular point of view** can engage children purposefully in contexts which may be unfamiliar (see for example McClain and Cobb, 2001, Ben-Zvi and Arcavi, 2001).

A SHIFT IN EMPHASIS: INVERTING A PEDAGOGIC TRADITION

The two examples given above of tasks designed using the constructs of purpose and utility exemplify the role of technology in supporting a shift in pedagogic emphasis, which we see as lying at the heart of opportunity for powerful mathematical learning offered by this approach. Mathematical ideas (such as average, graphing or geometrical construction) are rich and complex, composed of different elements, which here we categorise very roughly as procedures (techniques and algorithms, specific rules of formulae), relationships (links within mathematics, internal structure and consistency), and utilities (why, how and when the idea may be useful). As a learner constructs meaning for a new mathematical idea, mental connections will be made with existing knowledge, but the pedagogic emphasis placed on the different elements will affect the ways in which those links are made.

It is generally acknowledged that pedagogic approaches that focus mainly, or exclusively, on procedures will result in impoverished learning. However, even approaches that emphasise relationships tend to give little attention to utilities. The pedagogic tradition, embodied in textbooks around the world, is to begin with procedures and relationships, and to address utilities as the final stage in the pedagogic sequence (if at all). We suggest that this results in mathematical knowledge becoming isolated as weak connections are made to the learner's existing knowledge of the contexts in which it may be usefully applied.

In learning mathematics in out-of-school contexts, we believe that immediate connections between the learner's existing knowledge and the utilities of the new idea are established in ways which enrich mathematical learning. In school mathematics, the initial links are generally made to procedures and relationships. Pedagogic design based on the framework of purpose and utility, with the support of technology, inverts the pedagogic tradition of school mathematics by placing the emphasis primarily on the utilities of a new mathematical idea. Thus the learner is able to construct meanings that are shaped by strong connections to the application of that idea.

This inversion is made possible largely (though not exclusively) by the power of technology to offer opportunities for *using* a mathematical idea before you learn about its procedures and relationships. Technology affords the possibility of pursuing purposeful tasks by working with mathematical tools, instantiated on the screen, whilst simultaneously coming to appreciate the utility of those tools, in ways which lead to powerful mathematical learning. Ongoing research is developing and refining this framework for the design of pedagogic tasks in various areas of the mathematics curriculum.

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THE SIMPLICITY, COMPLEXITY AND COMPLICATION OF THE FUNCTION CONCEPT

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In this paper we focus on students' understanding of the core concept of function, which is mathematically simple yet carries within it a rich complexity of mathematical ideas. We investigate the linguistic complexity that reveals itself through the mathematically simple notion of a constant function and the representational complexity involved in different representations. Results reveal a spectrum of performance in which a few students may be able to grasp the function concept as a rich highly connected cognitive unit, but others are overwhelmed by a complexity that is highly complicated with many separate aspects weakly connected together.

INTRODUCTION

To a mathematician, the notion of function is a model of *simplicity*. What could be simpler than the idea that 'we have two sets and each element in the first is linked to precisely one element in the second'? The definition is not only mathematically simple, for the mathematician it provides access to a huge *complexity* of mathematical ideas. Some students are able to build this subtle combination of simplicity and complexity. For others, however, the situation is quite different. As they respond to being introduced to the notion of function, they bring their implicit understandings of language and all their previous experiences to bear on the task. The result for them is a highly *complicated* array of personal meanings that both help and hinder their interpretation of the mathematical concept.

LITERATURE REVIEW

The complexity of the function concept has been a focus of attention for almost a generation. Vinner (1983) drew attention to the distinction between the *concept definition* that mathematicians use to define a mathematical concept and the *concept image* which people generate in their mind. He also showed that most students use their own *personal concept definition* for the notion of function, giving highly idiosyncratic meanings to the term. Subsequently, two distinct lines of enquiry have occurred in the literature, on the one hand the nature of function as process and mental object (Dubinsky, 1991; Breidenbach et al, 1992; Sfard, 1992), and the multiple representations of a function (Confrey, 1994; Kaput, 1992; Keller & Hirsch, 1998; Leinhardt et al., 1990). Others have since combined these two aspects into a broader perception of 'horizontal growth' (between representations) and 'vertical growth' (in compression from process to concept) (Beineke et al., 1992; Schoenfeld et al, 1993).

THEORETICAL FRAMEWORK

Our study focuses on the nature of the concept definition and its interpretation in a variety of representations as presented in the Turkish mathematical curriculum. We are particularly interested in the subtleties of meaning attributed to the (personal) concept definition, and the meanings inherent in the various representations.

Thompson (1994), questioned the meanings given to representations seemingly shared by the mathematics education community:

> ...the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of a representation... the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.
>
> (Thompson, 1994, p. 39)

In practice, the simple core concept (given essentially by the definition) proves cognitively elusive and students are introduced to examples in various forms:

- a verbal representation of a function in formal or colloquial (everyday) language,
- a set diagram (representing a function by two sets and arrows between them),
- a function box (representing of an input-output relationship),
- a set of ordered pairs (considered set-theoretically),
- a table of values (often computed using a formula or computer procedure),
- a graph (drawn by computer or by hand),
- a formula.

Each of these has its own peculiarities that contribute to the complication of the student's concept image. To describe the manner in which the human mind copes with these images, the literature has developed a language of *prototypes* (typical instances) and *exemplars* (more specific cases), the latter often being seen as *clusters* of examples. (See Makin & Ross (1999) for a detailed discussion). Our analysis suggests that different representations are presented and interpreted in subtly different ways. For instance, set diagrams are often introduced as prototypes to represent general ideas, whilst graphs and formulae are met successively in clusters (linear functions, quadratics, trigonometric, exponential, and so on). This presents a dilemma for the curriculum developer:

> The learner cannot construct the abstract concept of function without experiencing examples of the function concept in action, and they cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept. (Bakar & Tall, 1992, p. 13)

BACKGROUND OF OUR STUDY

The purpose of this paper is to explore the nature of this subtle mathematical blend of simplicity and complexity and to focus on problems that it presents to many students, with particular reference to a specific curriculum for the function concept. The Turkish curriculum, in which our study is situated, begins with the following:

Formal Definition

Let A and B two sets other than empty set, and f is a relation defined from A to B. If

i) $\forall x \in A$, $\exists y \in B$ s.t. $(x,y) \in f$,
ii) $(x,y_1) \in f$ and $(x,y_2) \in f \Rightarrow y_1 = y_2$

Then the relation f from A to B is called a *function* and denoted by $f: A \rightarrow B$ or $A \xrightarrow{f} B$. A is called the domain and B is called the range.

This mathematically simple notion is followed by a four part *colloquial definition* to help the student focus on the essential properties of a function. These explain properties (i) and (ii) and add (iii) different elements in A can be related to the same element in B, (iv) some elements in B may not be related to an element of A.

The students are then given experience of functions in different representations as

set diagrams, *ordered pairs*, *graphs* and *formulae*,

(but not as function boxes or tables). Our research is to investigate and analyse the use of formal definition and the concept images that arise (in the sense of Tall & Vinner, 1981), with particular interest in the role of *language* and the *prototypes* and *exemplars* that arise in the students written and spoken responses.

METHODOLOGY

One hundred students from four different upper-secondary schools in Turkey were given a questionnaire which asked whether various graphs, equations, correspondences between two set diagrams, sets of ordered pairs are functions or not. Based on an analysis of the responses, eight students were chosen to represent a spectrum of performance for individual interviews. The interviews were semi-structured with certain questions asked of all students with a flexible continuation to take account of individual responses. In this paper, we focus on two specific areas in the interviews: responses to the mathematically simple case of a constant function and responses to a range of questions relating to graphs (with marked domains), set diagrams, ordered pairs, and formulae.

DATA ANALYSIS

The constant function

Mathematically, the simplest function is a constant function that assigns to every element of the domain a single element in the range. Cognitively, however, this example is far from simple. It is a singular case violating strong links in the concept image (such as the idea that f(x) 'depends on' or 'varies with' x).

The following questions were asked to eight students from grade 9-10-11 (aged 15-16-17) in four different upper-secondary schools in Turkey:

I will show you some expressions. Can you tell me whether they are functions or not?

They were then, in succession, shown cards with the following written statements:

$y = 4$,
$y = 4$ (for all values of x),
$y = 4$ (for $x \geq 2$).

The subtle differences in these three expressions greatly affect the student responses so that there is a wide range of responses. Six of these are as follows:

	$y = 4$	$y = 4$ (for all values of x)	$y = 4$ (for $x \geq 2$)
Eren	*Yes*: This is f(x) = y ... it can be a constant or a variable, this goes to a constant, thus it is a function.	*Yes*: It is a function which is not dependent on x.	*No*: not a function, there isn't anything for values less than 2 ... [When the domain was mentioned, he changed his response to *yes*.]
Can	*Yes:* It always goes to 4 in the range.	*No:* There are elements left in the range …[see below]	*Yes:* It goes [is assigned] for less than two. [see below]
Aysun	*Yes*: It is a constant function ... without x … dependent on y … $y = ax+b$ … without x it is constant.	*No*: There isn't any value for x…I can't see x here…there isn't any x on the other side of y.	*Yes*: I have to see two sets, this is function. In the domain, elements greater than 2 are assigned to 4.
Murat	*No*: not a function … I can't say the reason…I don't know.	*Yes*: function…because it says "for all values of x".	*Yes*: function, since 4 is greater than 2.
Irem	*No*: there should be an equation, but there isn't… there should be an unknown.	*Yes*: this may be a function … because it says for all values of x, thus there is y being a correspondent of x.	*Yes*: it can be function, then there was an equation, thus a function.
Deniz	*No*: there should be *f* in the front.	*Yes/Not Sure*: we used to put y instead of x … but we find x then.	*Yes/Not Sure*: function … x greater than 2, then I am not sure.

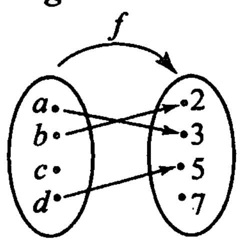
Table 1: Student responses to the various formulations of a constant function

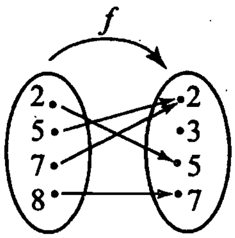
None of the students responded *yes* to all three examples, with the possible exception of Eren, who gave a simple definitional response to the first two examples, but was troubled by the third because some elements were not assigned. The interviewer drew his attention to the notion of domain and he immediately changed his response to declaring it to be a function using the colloquial definition. Student Can responded to the first part using the colloquial definition but drew a graph with only positive values of x in the second part, becoming confused. In the third he gestures to the values of x greater than two and says 'values less than 2', a possible slip of the tongue. Aysun correctly identifies the first example, but sees it as a special case of a linear function, is thrown off course by the absence of an 'x' in the formula in the second case, but correctly identifies the third using the colloquial definition. The other students (represented here by Deniz, Murat, Irem) respond in a variety of ways that reveal a range of complications. They respond to each separate example by noting certain aspects in isolation without at any stage referring to any form of function definition.

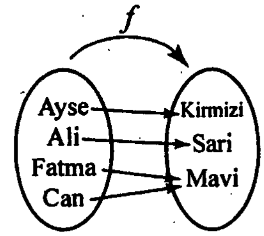
Different representations of functions

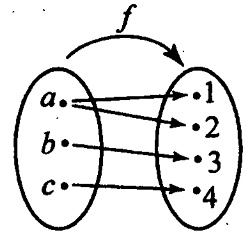
Students were given different representations of functions in isolation. They were given the following items and asked whether they were functions or not.

Set diagrams:

a) 

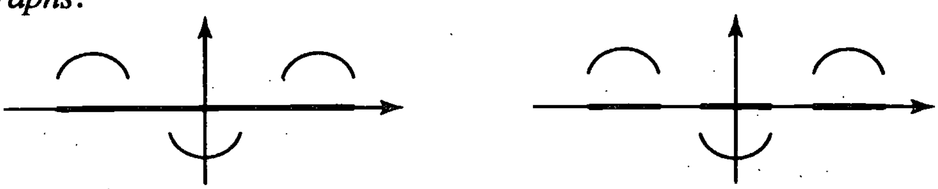
b) 

c) 

d) 

Set of ordered pairs: f: $\{1,2,3,7,9\} \rightarrow \mathrm{R}$ $f = \{(1,3), (2,5), (3,2), (7,-1), (9,1)\}$

Graphs:



[*On the original, the thicker parts on the axis were marked red and denoted the domain.*]

Formulas:

(a) $f\colon \boldsymbol{R} \rightarrow \boldsymbol{R},\quad f(x) = \sqrt{x}$ (b) $f\colon \boldsymbol{R} \rightarrow \boldsymbol{R},\quad f(x) = \frac{1}{x}$.

In the interviews, students were asked to explain why they consider these as functions or not. This gave the responses of eight students on four representations. These were then analysed first by considering each representation in turn, then by considering and comparing the performance of individual students.

Set diagrams were answered by seven students in terms of the colloquial definition. Four of these were successful and seemed to use the colloquial definition as their personal concept definition. For instance, Can said:

> (a) is not a function because there is an element left in the domain ... (b) is function because for every element in the domain there is an element in the range ... (d) not a function because there are two values for a (in the domain). (Can)

Three others used the colloquial definition, but were unsuccessful. For instance, Sena was confused about the properties of a function, saying 'one can go to two'.

Ordered pairs again evoked the colloquial definition, this time from six students. Three of them successfully checked the definitional properties using the colloquial definition as their personal concept definition:

> Here there shouldn't be anything left, 1,2,3,7,9 all of them are used and used once, thus I considered this as a function and all of the elements here are reals. (Eren)

> Function, because every element in the domain goes to an element in the range. (Can)

> All these in the domain are joined with different elements in the range. (Murat)

Graphs were more likely to provoke recall of examples rather than the definition:

> Function…there used to be graphs like that in the tests. (Fatma)
>
> If we join these (three pieces) then it looks like a function. (Irem)

The vertical line test was used by three students (Eren, Can, Aysun) who were able to focus on the role of the coloured domain for each graph. The other five students' responses were highly complicated and based on recalled exemplars, and selected details such as absence of numbers on the axes, as in the case of Sena.

Formulae also evoked examples rather than the definition. Only Aysun used the definition (in a colloquial form) to respond successfully to the question. Eren and Can were able to consider the domain of the formulas, but the other five student responses were complicated and did not use the core properties of function at all.

This first part of the analysis suggests that the responses to set diagrams and sets of ordered pairs act like prototypes that are less complicated than the responses to graphs and equations which are seen more as clusters of exemplars.

The second part of the analysis considered the performance of each individual student. Responses indicated that there is a spectrum of performance for students which we illustrate in terms of four categories which may overlap.

In the first category are two students (Eren, Can) who use their personal concept definitions to give a greater unity as they pass from one context to another, not only for set diagrams and ordered pairs, but also for graphs and equations.

In the second category is Aysun who used the colloquial definition of function for set diagrams and formulas and the vertical line test for graphs. She was unsuccessful with the function properties for ordered pairs.

In the third category are the two students (Fatma and Murat) who use their personal concept definitions for one of the two prototypes (set diagrams or ordered pairs). Their responses to the other representations were complicated combinations of exemplars and selected details without any reference to any definitional properties.

In the fourth category are the other three students who could not focus on the core concept for any of the representations. They produced complicated responses affected by the subtle differences in the particular representations without any reference to the definitional properties at all.

CONCLUSION AND REFLECTIONS

In this paper, we have considered how students deal with aspects of the linguistic and representational complexity of the function concept. Figure 1 illustrates our observations, with the positions of two starred items (function box and tables) inferred from other research (Crowley and Tall, 1999; McGowen & DeMarois & Tall, 2000). The data in this paper is drawn from a curriculum in Turkey, which begins by mentioning the formal definition, interprets it in a colloquial form, then

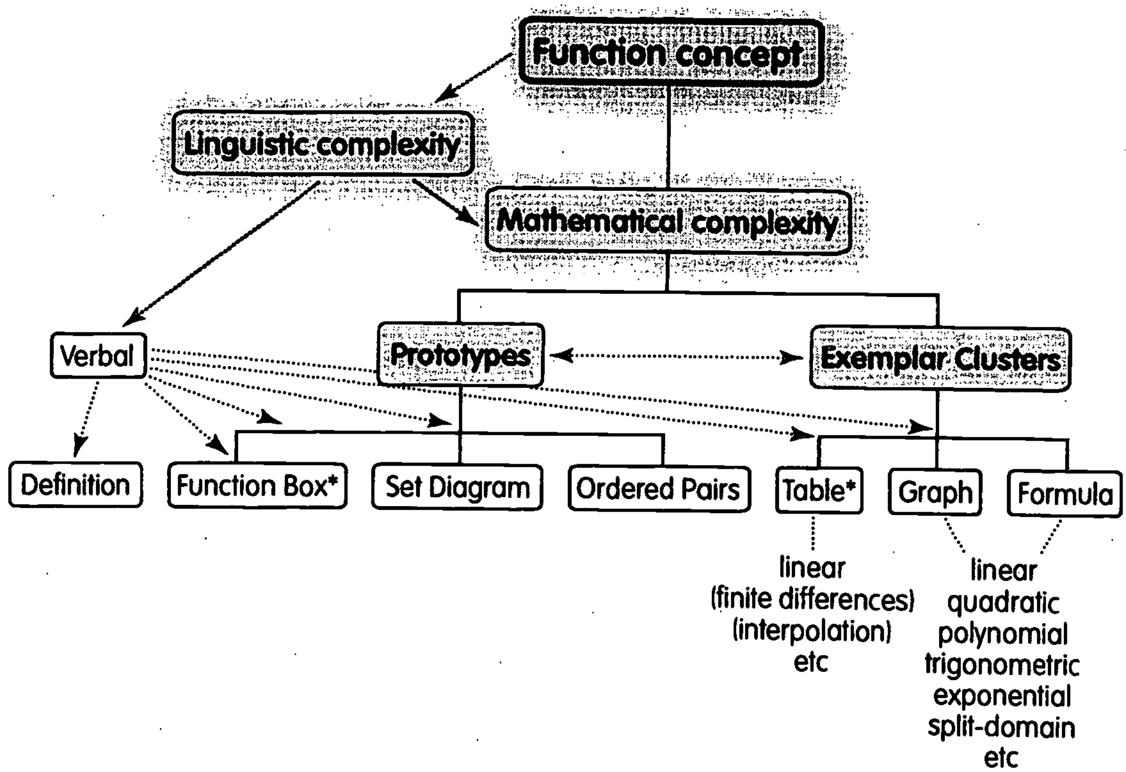


Figure 1: The complexity and complication of the function concept

focuses on set diagrams, ordered pairs, graphs and formulas. The case of a constant function in several variant expressions reveals that the simplicity of the core function concepts eludes most students. The empirical data suggests that a small number (represented by Eren and Can) generally use the core function concept definition in a colloquial form to link ideas across a range of different representational forms into a rich cognitive unit. Errors that they make from time to time are usually amenable to reconsideration through discussion. Essentially, for them, the core concept has a simplicity that can be applied in a variety of contexts.

The other students do not yet have the sophistication to see this simplicity. Some of them focus on the core concept in the form of a colloquial definition for the prototypical forms of set diagrams and ordered pairs. However, graphs and formulas are—at best—seen more in terms of exemplars, relating to types of functions they have met or, in the case of the less successful students, in terms of almost arbitrary aspects of the examples that they happen to focus upon.

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A FRAMEWORK FOR THE STUDY OF INTUITIVE ANSWERS TO RATIO-COMPARISON (PROBABILITY) TASKS.

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ABSTRACT. This paper describes a framework for the study of strategies used in ratio-comparison problems, which was constructed for the analysis of adults' responses to double urn probability tasks. This framework involves two systems, one for the interpretation and classification of answers (strategies), and one for the planning of the numbers involved in ratio-comparison questions (situations). It was applied in an experiment with university students. Some results are reported, which refer to the relative occurrence of strategies, the difficulty levels of different situations and a classification of the subjects according to their performance.

The purpose of this article is to describe a framework for the classification of answers given in ratio-comparison tasks and for the categorization of the numbers involved in questions for such tasks. It was constructed in the frame of a research aiming at understanding how Mexican university students solve the "quantification" problems designed by Piaget and Inhelder (1951) in the context of a double urn probability task. In this research it became clear: a) that many of these adults cannot correctly solve what Piaget and Inhelder claimed to be solvable by the age of 15, and b) that the Piagetian categories were ineffective for the explanation of their behaviours.

Although the research was based upon the classical definition of probability, its main purpose was not to see whether the participant subjects use it or not, but to understand how they deal intuitively with the tasks. Piaget and Inhelder's (1951) work concerning children's conceptions of chance is at the foundations of the study. Fischbein's (1975) definition of intuition is used: it is a form of immediate knowledge which can appear to a person as obvious; it responds to a biological need for action and for certainty, from where it gets its characteristic reliability, stability and coercing nature. Tversky's et al (1982) concept of heuristics is also considered, as is their notion that people use a limited amount of heuristic principles to simplify the complex tasks of estimating probabilities and predicting values, but which lead frequently to serious and systematic errors. In the construction of the framework several references were considered, among which Noelting's (1980) classical orange juice experiment, and Falk et al's (1980) and Maury's (1986) researches on probability learning.

The adult subjects who participated in this study were shown two sets of cards, A and B, that were black or white on the obverse and red patterned on the reverse. When the subject had clearly seen all the cards, they were turned upside down and each set was shuffled separately. He or she was then asked: "Suppose there was a big prize if you took out a black card, but you only have one chance of choosing one card from one of the two sets. From which set would you choose a card: A, B or is it the same?" (No actual prizes were to be given, however). The subject had to make a decision and to justify it. When this was done, the number of black and white cards in each set was changed and the same question was posed. After some such experiments with actual cards, the subject was just shown drawings representing the sets as they were before the cards where turned upside down and shuffled, such as the one in figure 1.

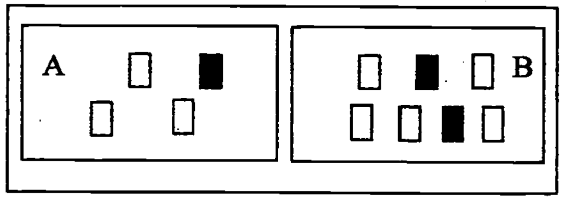

Figure 1: Array (1,3)(2,5)

A framework was constructed, which consists of two ways of envisaging these problems. One of them focuses on the answers given by the subjects; this generates a system of strategies. The other one focuses on the particular quantities involved in the questions; this generates a system of situations. Although different in nature, both systems have a correspondence with each other. The two sections of this article are dedicated to the presentation of the framework constructed and to the description of some results obtained from its experimental application.

THE FRAMEWORK

Each question is defined as an *array*, which is a pair of sample spaces S1 and S2; each of them is in turn characterized by a pair of *favourable* (f: black cards) and *unfavourable* (u: white cards) cases. An array is thus an expression of the form $(f_1,u_1)(f_2,u_2)$. Also defined for each array are the *total cases* ($n = f + u$), the *differences* ($d = f - u$) [1] and the *probabilities* ($p = {}^{f}/_{n}$) [2]. For instance, the question in figure 1 has f_1=1, u_1=3, n_1=4, d_1=–2, $p_1={}^{1}/_{4}$; f_2=2, u_2=5, n_2=7, d_2=–3, and $p_2={}^{2}/_{7}$. These definitions are used in the constructed framework, which will be described here: the systems for strategies and situations, as well as the correspondence between them.

Strategies

Strategies are understood here as solution mechanisms used to solve problems, that have a certain structure or logic and that can be reproduced for other problems of the same type. They are structured heuristics in the sense of Tversky et al (1974) [3]. Strategies are classified as simple or composed; simple strategies are in turn grouped in centrations and relations [4].

Centrations. The subject concentrates on only one class of elements of the array: the total, the favourable or the unfavourable cases. In each instance, the subject can decide to choose the side where there are more elements of the class (*positive centration*), or where there are less elements (*negative centration*), or he or she can answer "it is the same" if in both sides there is the same amount of the considered class of elements (*equality centration*). There are nine different centrations, grouped in three families:

{CN}. Centrations on total cases: Choosing the side where the total amount of cards is smaller ({CN–}) or larger ({CN+}), or saying "it is the same" because in both sides this amount is the same ({CN=}).

{CF}. Centrations on favourable cases: Choosing the side where the amount of black cards is smaller ({CF–}) or larger ({CF+}), or saying "it is the same" because in both sides this amount is the same ({CF=}).

{CU}. Centrations on unfavourable cases: Choosing the side where the amount of white cards is smaller ({CU–}) or larger ({CU+}), or saying "it is the same" because in both sides this amount is the same ({CN=}).

Relations. The subject considers simultaneously two classes of elements, and after establishing two relationships between them, he or she compares the results. These relationships can be based on order comparisons, or they can have an additive or a proportional nature [5].There are ten different relations, grouped in three families:

{RO}. Order relations: Choosing the side where the black cards prevail over the white ones, whereas in the other side either the white cards prevail ({ROwl}: win-lose) or there are as many black cards as white ones ({ROwd}: win-draw). Or choosing the side where there are as many black cards as white ones, whereas in the other one either the white cards prevail ({ROdl}: draw-lose) or the black ones prevail ({ROd}: draw). Or saying "it is the same" because in both sides the black [white] cards prevail ({RO=}: win-win [lose-lose]).

{RD}. Difference relations: Choosing the side where the difference of black minus white cards is the largest ({RD+}) or the smallest ({RD–}), or saying "it is the same" because the difference is the same in both sides ({RD=}).

{RP}. Proportionality relations: Choosing the side where the quotient $^f/_n$ or the quotient $^f/_u$ is the largest ({RP+}), or saying "it is the same" because the quotient is the same in both sides ({RP=}).

Composed strategies. Two or more simple strategies can be linked in a logical juxtaposition; each of them may be dominant or dominated. If *X* and *Y* are two strategies, there are four possible compositions between them:

{*X* & *Y*}. Conjunction: Both *X* and *Y* lead to the same decision (S1, S2 or "same") and they support each other. Both *X* and *Y* are dominant.

{*X* ¬ *Y*}. Exclusion: *X* leads to the election of one side or to the decision "it is the same", and *Y* leads to the election of the other side, but *X* prevails. *X* is dominant and *Y* is dominated.

{*X* * *Y*}. Compensation: *X* leads to the election of one side, and *Y* leads to the decision "it is the same", but *X* prevails. *X* is dominant and *Y* is dominated.

{*X* ⊥ *Y*}. Counterweight: *X* and *Y* lead to the election of different sides and they cancel each other, causing the decision to be "it is the same". *X* and *Y* are dominated.

There are also multiple compositions: compositions of strategies, one (or both) of which is in turn a composed strategy.

Examples. Table 1 shows the most common strategies that would be applicable in figure 1, the decisions they lead to, justifications that could have been prototypically given by subjects, and the mathematical description of simple strategies.

Centrations	{CN–}	A	A has fewer cards.	4<7
	{CF+}	B	B has more black cards.	2>1
	{CU–}	A	A has fewer white cards.	3<5
Relations	{RO=}	Same	In both sides there are fewer black cards than white ones.	1<3 & 2<5
	{RD+}	A	If I take out couples of black and white cards, less white cards remain in A than in B.	1–3 > 2–5
	{RP+}	B	A has three white cards for each black card, and B lacks one white card for that, so it has "fewer" white cards	$^2/_7 > ^1/_4$
Compositions	{CN– & CU–}	A	A has fewer cards altogether *and* it has fewer white cards.	
	{CF+ ¬ CU–}	B	B has more black cards, *although* A has fewer white cards.	
	{CF+ * RO=}	B	B has more black cards, *and* (or *although*) it has, as A does too, fewer black cards than white ones.	
	{CN– ⊥ CF+}	Same	*On the one hand* A has fewer cards altogether *but on the other hand* B has more black cards.	

Table 1: Simple strategies and some compositions applicable in the array of figure 1

Situations

A *situation* is a set of arrays such that the different simple strategies lead to the same decisions in all the arrays. Each situation is characterized by two subsystems, which define a crossed pattern; the two subsystems are combinations and locations.

Combinations. A combination is the succession of results obtained when an order relationship is established between the following pairs of an array: the total cases n, the favourable cases f, the unfavourable cases u, the differences d, and the quotients p. For instance, the array $(f_1,u_1)(f_2,u_2) = (1,3)(2,5)$ of figure 1 has $n_1<n_2$, $f_1<f_2$, $u_1<u_2$, $d_1>d_2$ and $p_1<p_2$, which gives the combination (<<<><). Seventeen such combinations exist [6], which are the following: K0 (=====): identity array; K1 (<<><<), K2 (<><>>): arrays without discrimination (most strategies lead to the same side: S2 in K1 and S1 in K2); K3 (=<><<), K4 (<<=<<), K5 (<=<>>): equalities of total, unfavourable or favourable cases; K6 (<<<<<), K7 (<<<>>): {RD} and {RP} lead to the same side; K8 (<<<><), K9 (<<<<>): {RD} and {RP} lead to different sides; K10 (<<<=<), K11 (<<<=>): equality of differences; K12 (<=<>=), K13 (<<=<=), K14 (<<<>=), K15 (<<<<=) and K16 (<<<==): equality of probabilities, with probabilities respectively equal to p=0, p=1, p<½, p>½ and p=½. (In K6 through K16, centrations lead to the election of different sides). Thus, the array of figure 1 is K8 [7].

Locations. A location is a non-ordered pair of the following alternatives for both probabilities of the array: *c:* certainty (p=1); *w:* win (1>p>½); *d:* draw (p=½); *l:* lose (½>p>0); and *i:* impossibility (p=0). There are also seventeen possible locations [6]: *c=c* $(1=p_h=p_k)$, *cw* $(1=p_h>p_k>½)$, *cd* $(1=p_h>p_k=½)$, *cl* $(1=p_h>½>p_k>0)$, *ci* ($1=p_h>p_k=0$); *w=w* ($1>p_h=p_k>½$), *ww* ($1>p_h>p_k>½$), *wd* ($1>p_h>p_k=½$), *wl* ($1>p_h>½>p_k>0$), *wi* ($1>p_h>½>p_k=0$); *d=d* ($p_h=½=p_k$), *dl* ($½=p_h>p_k>0$), *di* ($½=p_h>p_k=0$); *l=l* ($½>p_h=p_k>0$), *ll* ($½>p_h>p_k>0$), *li* ($½>p_h>p_k=0$); and *i=i* ($p_h=p_k=0$). Location *c=c* is called "double certainty", *dl* is "draw-lose", *ww* is "win-win", etc. For instance, the array (1,3)(2,5) of figure 1 is *ll* ("lose-lose") [8].

Not all locations exist in all combinations; there are 86 intersections of both subsystems of categories, and thus 86 different possible situations [6, 9].

Correspondence between the systems of strategies and situations

One of the most important features of these systems lies in the correspondence between them, which makes it possible to know, for each problem posed, which strategies may be occurring and which lead to each of the three different decisions (S1, S2 or same). The combinations account for the possible decisions induced by all simple strategies except the {RO} family, and the locations account for the ones induced by the {RO} family. For instance, the applicable strategies, the decisions and the justifications depicted in table 1 could be the same for all K8-*ll* arrays.

The correspondence also permits a classification of all strategies according to their correctness in each situation. A *correct strategy* is one that coincides algebraically with the formal probability, and an *incorrect strategy* is a behavioural pattern associated with an inadequate intuition, one that may eventually lead to a decision different from that prescribed by the formal probability. Three groups can be defined:

- Correct strategies: always correct when applicable. Relations {RP+} (applicable in K1–K11), {RP=} (K0, and K12–K16); {ROwl} (locations *wi, ci, cl, wl*), {ROdl} (*di, dl*), and {ROwd} (*wd, cd*). Ten composed strategies such as {CF+ & CU–} (K1–K3) and {CF+ * CN=} (K3). Compositions with correct dominant and incorrect dominated strategies, such as {RP= ¬ N–}, are also considered correct.
- Eventually correct strategies: correct only in certain situations. Centrations {CF+}, {CF=} (only correct in *i* locations), {CU–} and {CU=} (only correct in *c* locations). Four composed strategies such as {CU– * CF=} (only correct in K5, but also incorrectly applicable in K12, which only exists in *i=i*).
- Incorrect strategies: always incorrect when applicable. Families {CN} and {RD}. Strategies {CF–}, {CU+}, {ROd} and {RO=}. Also, most composed strategies.

All the correct strategies other than {RP+} or {RP=} may be thought of as Vergnaud's (1981) *théorèmes en acte*: the subject using them may be perceiving and using properties of the relationships between elements of the array without necessarily being able to make them explicit or to justify them. A particular case are the correct composed strategies, such as {CF+ * CN=} in K3, because when a subject uses {CF+} in K3 the question could arise that he or she might be considering {CN=} but is not stating it, because it is too obvious. In such cases {CF+} is considered a *potentially incomplete expression of a correct justification* (abbreviated PIECJ).

AN APPLICATION OF THE FRAMEWORK; SOME RESULTS

In order to prove its effectiveness, the framework was applied in six experiments in Mexico City involving 64 university students who had not previously taken probability courses [10]. A test was designed in each experiment [11]; each test had between 20 and 40 items of the exposed form, all with n_1 and $n_2 \leq 10$, and covering a variety of situations. Some of the tests were applied in paper and pencil form, and some in taped interviews. A total of 1630 answers was obtained, of which 1144 (70%) were interpreted, 405 (25%) had a decision (S1, S2 or same) but were non-interpretable (mainly because of poor or none justification), and 81 (5%) had to be cancelled [12].

The evaluation of the experimental results is carried out from three viewpoints: strategies, situations, and subjects. Firstly, the frequency of each strategy or group of strategies is calculated. Secondly, the amount of correct responses in each situation permits to define levels of difficulty for situations. Thirdly, each subject can be classified according to his or her performance in each of those levels.

Occurrence of strategies. In the analysis of the 1144 interpreted answers, each strategy's occurrence was calculated as a quotient of the times it was observed over the times it could possibly be observed (for instance, {ROwd} could have been observed only in items with locations *wd* or *cd*, which were 200). Table 2 shows the relative occurrence of the most common strategies in simple or composed forms.

Correct strat.	*N*	S	C	Eventually correct strat.	*N*	S	C	Incorrect strat.	*N*	S	C
{RP+}	*813*	16	*	{CF+}	*925*	25	10	{CN–}	*911*	6	7
{RP=}	*331*	28	5	{CF=}	*219*	26	15	{CN=}	*233*	1	8
{ROwl}	*44*	50	11	{CU–}	*859*	11	5	{ROd}	*122*	5	3
{ROwd}	*200*	15	1	{CU=}	*285*	12	12	{RO=}	*396*	17	11
{ROdl}	*227*	21	2					{RD+}	*945*	2	1
								{RD=}	*199*	6	2

Table 2: Possibilities *(N)* and relative occurrence (%) as simple (S) or composed (C) forms (including dominant and dominated) of the most common strategies [13]

Among these results, the following may be highlighted. The high occurrence of {CF} among these young adults contradicts many authors' assertion that the centration in favourable cases disappears by the age of 11 (e.g. Falk et al, 1980). Also noticeable is the high occurrence of the incorrect {RO=}, greater than the additive {RD}. The proportionality reasoning was easier when the ratios were equal: {RP=} was more frequent than {RP+} (to these, a 3% of arithmetically incorrect attempts at {RP} may be added).

Overall correctness distribution. Of the 1144 interpreted answers, 37% had incorrect justifications (12% with and 25% without a correct decision), 53% had correct justifications and 10% were PIECJ. Among the correct justifications, 38% came from the simple proportionality strategies, 16% from simple correct order relations, 44% from *théorème en acte* strategies (simple or composed) and 2% from other correct compositions. It is noticeable that *théorème en acte* strategies, to which the PIECJ could be added, amount for approximately one half of the answers.

Evaluation of situations. The distribution of correct justifications varies among situations. All 1549 non-cancelled answers were used to analyse the correctness distribution of each situation, and six difficulty levels were defined according to the percentage of correct justifications [14]: I: Situations with impossibilities (including K12 and K13) (93%); II: Situations with certainties and identity situations (77%); III: Situations around p=½: locations *wl, dl* and *wd* (59%); IV: Proportionality situations: combinations K14 to K16 (35%); V: Combinations K1 to K5 in *ll* or *ww* locations (35%); VI: Combinations K6 to K11 in *ll* or *ww* locations (13%) [15].

Evaluation of subjects. These levels also permit the classification of subjects, with the following criterion: A subject is assigned level L if he or she can correctly solve and justify at least 50% of the items of level L and all levels prior to it, but cannot reach 50% in the subsequent levels. Of the 64 subjects, 23% were in level I, 28% in level II, 14% in level III, 16% in level IV, 11% in level V, and 8% in level VI.

Thus, half of the adult subjects who participated in these experiments could only correctly solve and justify the items corresponding situations with impossibilities and certainties; the "one variable" problems (level V), which according to Piaget and Inhelder (1951, Ch VI-4) can be solved by the age of 8, could only be solved by 19%, and the "two variables" problems (level VI), supposedly solvable by the age of 15 (*ibid,* ChVI-6), could only be solved by 8%. A question that remains to be answered is whether university students of this kind encounter also difficulties in solving other ratio-comparison tasks, or if the difficulties were enhanced by the probabilistic nature of the task, as happens with younger subjects (Cañizares et al, 1997).

NOTES

1. The term *difference* is used in this paper as f–u and not as |f–u|, thus possibly having positive or negative values.
2. For the sake of non-repetition, $n_1 \leq n_2$ and if $n_1 = n_2$ then $f_1 \leq f_2$. Thus, in figure 1, side A is S1 and side B is S2.
3. Other unstructured heuristics were also found in the experimentations with adults, but they were rare and were not considered in the construction of the system of strategies. Some examples: attraction ("It is more fun to choose A") and graphic presentation ("The way in which they are put begins with a black card…"). Usually, subjects who used them did so only at the beginning of the experiment, and then settled into the strategies considered in the system.
4. Centrations and relations have also been respectively called "single variable" and "two variables" strategies (Cañizares et al, 1997). Positive centrations correspond to the "more A-more B" intuitive rule identified by Stavy et al (2000) and equality centrations to the "same A-same B" intuitive rule.
5. All relations are described and exemplified in this paper in their "within" forms. However, all of them can also happen in a "between" form (See Noelting, 1980).
6. The proof of these assertions requires simple (albeit long) algebra; they can be consulted in Alatorre (1994).
7. Some other examples: KO: (3,5)(3,5); K1: (1,5)(4,3); K2: (2,3)(1,5); K3: (4,3)(5,2); K4: ((2,4)(3,4); K5: (3,1)(3,2); K6: (1,2)(4,3); K7: (2,1)(3,4); K8: (1,2)(4,6); K10: (1,2)(2,3); K12: (0,1)(0,2); K14: (1,2)(2,4); K16: (2,2)(3,3).
8. Some other examples: *c=c*: (2,0)(3,0); *cw*: (2,1)(3,0); *cl*: (1,0)(2,3); *ci*: (1,0)(0,2); *w=w*: (2,1)(6,3); *ww*: (2,1)(4,3);

wd: (2,2)(3,2); *wl*: (2,1)(2,3); *d=d*: (1,1)(2,2); *di*: (2,2)(0,5); *l=l*: (0,1)(0,3); *ll*: (1,3)(2,5); *li*: (0,3)(2,4).

9. Some examples: combinations K1, K2 and K3 have arrays in all locations except *c=c*, *w=w*, *d=d*, *l=l* and *i=i*. K12 – K16 only exist in one of these = locations (which only exist in them and in K0). K8 and K10 only have *ll* and *li*.
10. See Alatorre (1994, 1999). For an application in other non-probabilistic contexts, see Alatorre (2000).
11. In each experiment, some items were repeated identically in different parts of the test, and some were used as controls for A and B sides of the same array (see note 2). There was a general inconsistency, such as observed by Koch (1987), although not more in the latter than in the former.
12. Of the cancellations, 51 were due to a misunderstanding of the task, 18 to the use of heuristics such as described in note 3, and 12 to other reasons.
13. Not included in table 2 are {CN+}, {CF–}, {CU+} and {RD+}, because they had very low occurrences (S+C<2%).
14. These percentages were calculated in each group of situations as the mean of the percentages of only correct justifications and correct plus PIECJ plus non-justified-correct-decision answers.
15. These results generally coincide with those of authors like Falk et al (1980), Maury (1986) and Noelting (1980), although the comparison is difficult because of the differences in subjects' age, settings and framework (for instance, all of Noelting's K3, K5 and K7 items were in *wl*, *dl* or *wd* locations). Also, it has been shown that there is a strong influence of context in the results obtained in ratio-comparison tasks (Karplus et al, 1983).

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AUTHORITY IN THE MATHEMATICS CLASSROOM AND ITS INFLUENCE ON STUDENTS' ABILITY TO REFLECT

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This paper presents some findings on students and authority in the mathematics classroom. It is shown that students create a web of authorities that extends not only to their teachers but also to their fellow students. While it is not asserted that relationships based on authority should be abolished, it is shown that such relationships can interfere with the formation of intellectual partnerships necessary, for example, in collaborative learning. It is also suggested that the dominance of authority relationships may hinder the development of reflective thinking about mathematical ideas. A brief description is also given of the Learners Perspective Study, in which these observations were made.

Much research in recent years has been dedicated to the social context of mathematics education (e.g. Clarke, 2001; Cobb & Bauersfeld, 1995; Edwards & Mercer, 1987). Indeed, it has become almost a truism that understanding the social context of learning is essential to understanding mathematical learning itself. Thus, there is good reason to study students' relationship to authority: who becomes authority figures for students, what their authority means to students, and how does the relationship to authority affect learning. These are the big questions towards which this paper hopes to make some small steps.

THE RESEARCH SETTING AND METHODOLOGY: THE LEARNERS' PERSPECTIVE STUDY

The considerations on students and authority presented in this paper emerged from a more extensive, and still ongoing, study of the students' point of view, called the Learners' Perspective Study (Clarke, 1998, 2000). The goal of that study is to explore a number of questions concerning the ways students conceive mathematics classroom practice and mathematics learning. Since the answers to such questions are likely to have a cultural dimension, the project adopts an international approach, with research teams working simultaneously in Australia, Germany, U.S.A, Hong Kong, Japan, Israel, Sweden, and South Africa. The authors of the present paper are the principal investigators for the Israeli team.

The Learners' Perspective Study arose out of the Third International Mathematics and Science Study (TIMSS). TIMSS not only established national profiles of student achievement, but also sought to identify national norms for teaching practice that might account for "poor" or "high" achievement scores by videotaping and analyzing a statistically representative sample of eighth-grade mathematics classes in Japan, Germany and the USA (Fernandez, C. et al., 1997). Although this component of the TIMSS study was impressive and unprecedented in international comparative studies, the validity of the "effective scripts" discerned in the TIMSS videos was widely debated and not universally accepted (e.g. Keitel & Kilpatrick, 1999; Stigler & Hiebert, 1997, 1998, 1999). One of the major objections was that the TIMSS Video Study focused exclusively on the teacher and ignored the important role students have in the learning process. The present project, accordingly, overcomes this objection and expands on the work done in the TIMSS study by focusing on *student actions* within the context of whole-class mathematics practice and by adopting a methodology whereby student reconstructions and reflections are considered in a substantial number of videotaped mathematics lessons.

Two specific cases formed the basis for this paper. The first was a sequence of 15 lessons on systems of linear equations taught by a dedicated and experienced teacher, whom we shall call Danit. Danit teaches in a comprehensive high school. Her 8th grade class is heterogeneous and comprises 38 students, mostly native born Israelis, but also new immigrants from the former Soviet Union and one new immigrant from Ethiopia. The second case was a sequence of 4 lessons on geometry taught by a teacher, whom we shall call Sasha. Sasha is a new immigrant from the former Soviet Union with several years experience teaching in Israeli schools and much experience teaching in Russian schools. His 8th class is a high-level class and comprises 30 students.

As specified in Clark (2000), all the classroom sessions were videotaped using an integrated system of three video cameras, one viewing the class as a whole, one on the teacher, and one on a "focus group" of two or three students. Following each lesson, the students in the focus group were interviewed, and their notebooks, containing the notes for that particular lesson, were photocopied. Moreover, once a week the teachers themselves were interviewed. Although we had a basic set of questions for both the student interviews and the teacher interview, we allowed the interview protocol to remain flexible so that we could freely pursue particular classroom events; in this respect, our interview methodology was along the lines of Ginsburg (1997). An important aspect of the interviews was that during the interviews the students could view and react to the videotape of the lesson for which they were the focus group. Needless to say, the interviews themselves were also videotaped.

Among the research questions for the study originally set out in Clarke (2000) was the question of whether teacher and learner practices are conflicting or mutually sustaining. This led us to ask the students about the circumstances in which they request help from their teacher, and, from there, whether they request help from other people as well. These questions, among others, gave rise to a new set of questions for us concerning students and authority, namely, Who is an authority for students? What is the extent of the authority of various people? How pervasive is the influence of authority in students' mathematical lives? What effect does students' relationship with authority have on their mathematical practice? Our preliminary findings regarding these questions are the subject of this paper.

RESULTS

Students' Sources of Assistance

When students were asked to whom they turn for help when they run into difficulties, they provided always the following sources of help: their teacher, their friends, their parents, or their siblings. Of these, the students' teacher and friends were the dominant sources of help spoken about in the interviews. When asked to whom they turn *first*, some students said the teacher and some their friends. The reason given for turning to friends first was almost always that the "teacher is too busy and can't get to everyone." Often, however, it seemed to us from our observations of the class that students turned to their friends first simply because their friends were near, for usually they were students sitting at the same desk.

Teachers, friends, parents and siblings form a web of sources of assistance; when one source is unavailable or unable to help, one turns to another. For example, if Yarden in Sasha's class cannot get help from Sasha, for one reason or another, she turns to one of her friends:

Interviewer: Whom do you turn to for help with homework?

Paulina [the second girl in the focus group]: There's a "dialogue hour" with the teacher.

Interviewer: What if that's not possible?

Yarden: I call a friend.

Interviewer: And if your friend doesn't know?

Yarden: If my friend doesn't know, I ask someone else—or my father.

Can this web be characterized as a hierarchy? One sense in which it can has already been alluded to, namely, as a hierarchy according to *physical proximity*. Thus, in this hierarchy friends, since they sit at the same table, come first, then the teacher, then parents and siblings. However, more significantly, the web forms a hierarchy according to the *degree of authority* possessed by the sources. By "the degree of authority" we mean the degree to which a person's statements are to be taken unchallenged (this sense, in a way, is already built into the word "source"; indeed, the Greek word for "source," *arche*, also means "sovereignty," and in the plural, *hai archai*, "the authorities"). Conversely, turning to an authority means turning to a person for an answer or for instructions, not, by contrast, for a discussion. Moreover, we use the word "authority" rather than, say, "expertise" because the reason a person's statements are not to be challenged is, as we shall soon see, not always dependent on the degree of the person's knowledge, though it may be perceived that way. Now, in this hierarchy, there is no question, the teacher comes first.

The Teacher's Authority

The teacher's tremendous authority, in every sense of the word, was evident in all of the student interviews in both Danit's and Sasha's class. For example, at one point in our interview with two students in Danit's class, Moshe and Sharon, we asked whether a graphical method or algebraic method of finding the solution to a system of equations was more reliable. Here is the exchange:

Moshe: If I get a answer for one and a different answer for the other, then you've got to check. If I get the same answer, then I'll believe it's correct. But if there's, maybe, still some doubt in my mind, I ask Danit.

Interviewer: What does Danit have that other people don't?

Moshe: She's a teacher, she can help; if you make a mistake, she corrects it!

Interviewer: And if she errs?

Moshe: She doesn't err.

Sharon: She studies everything at home before she comes to class.

Moshe: Otherwise she couldn't correct—she's a teacher!

Interviewer: But she did make a mistake at the board [during the lesson, Danit had made a careless error at the board].

Sharon: She got mixed up because she substituted wrong.

Moshe: Those are nonsense things she gets mixed up about, but real things [gestures to show the weightiness of the things he has in mind]—if two exercises [systems] are supposed to get the same answer or not, it doesn't seem to me she'd get mixed up about that.

In this exchange, one is impressed by the extent to which Moshe and Sharon are willing to see Danit as nearly infallible, and the extent to which they are willing to defend her authority, even when she is seen to make a mistake. The students view her, apparently, not only as one who knows more than they do, but also as a strong figure with powers they lack. When Moshe says, "She's a teacher, she can help; if you make a mistake, she corrects it!" he sounds as if he is speaking of a healer, a miracle worker, rather than of his 8th grade math teacher. Similarly, when we asked Sivan and Shimrit, also Danit's students, what exactly do they expect from the teacher when they ask her for help, Sivan said simply "That she will explain to us better," to which Shimrit added immediately, "When she comes over to me, when she explains to me, *suddenly I understand better*...[emphasis added]." Consistent with this image of Danit, was the importance the students seemed to place on the mere fact of Danit's coming over to help them when they worked on exercises. When we asked what the climax of the lesson was, Elah, in the same interview in which Sivan and Shimrit participated, answered, "When I was having trouble with the book and I called [Danit]." In a different interview, another girl in Danit's class, Gal, answered the same question in precisely the same way. Conversely, on two different occasions we came across a student in Danit's class who also appeared to be having trouble with the exercises, but who did not ask Danit for help. When we asked them why not, we received the same response both times: "The teacher doesn't want to help me." Such a statement presents a picture in which the attention the students receive from the teacher is dependent on the teacher's whim. The teacher becomes, in this interpretation, a dictator, though, surely, for most students, a beneficent one who willingly helps them when they need help. Nevertheless, conceiving the teacher as a creature of whim is to conceive the teacher as a creature with terrific power.

The Web of Authorities

That the teacher should be given this degree of authority by the students is perhaps not very surprising. However, we were surprised to see how easily students are willing to see other people as authorities to a degree similar to that to which they see their teacher as an authority. For example, we were interested in seeing how students understood the significance of "showing their work," whether this was only a requirement of students or of mathematics itself. So, we asked whether a salesman who explained to customers how much they should pay given such and such a discount had to show his/her work. To this, Boaz, again from Danit's class, replied: "No, I can rely on him…I can rely on him—for sure lots of people come to him—there must be those who know percentages and things, and they rely on him, so I can rely on him too." It is worth noting here that the Hebrew word Boaz uses for "rely" is *somech,* which is closely related to the word *somchute* meaning, literally, "authority."

Students tend to see authorities at every turn. Their web of sources of assistance becomes, in this light, a true web of authorities. What is particularly striking, though, is that this extends also to the students' friends. As mentioned above, friends in the class are a dominant source of help. But when the students turn to their friends they tend to turn to them only for answers. And, as we saw with Sasha's student Yarden, when one friend does not know, she turns to another. In one interview in Danit's class, we asked a student why he did not ask his friend for help at a certain point during the lesson. He replied, "I knew Uri wouldn't know the answer…" Thus, when students are perceived by their fellow students as knowing the answer to some question *they are treated for that instant as an authority*, that is, the answer is accepted and not discussed. When students are not perceived as knowing the answer, they are usually not asked. In fact, in the classroom videos it can be seen quite often (though less so in the geometry classes) that students sit together, occasionally speak together, but do not really work together, even though they are not necessarily encouraged to work individually.

To understand the significance of this tendency of students to treat one another as authorities *ad hoc*, one needs first of all to see what alternative stands opposed to it. This need not be considered hypothetically, for in Sasha's class we found an exception to the tendency. During his lesson, Sasha gave a geometry problem to the class; we watched as two girls, Yulia and Roni, solved the problem in a truly collaborative spirit. Roni showed her diagram to Yulia; Yulia commented and pointed to her own diagram; they discussed the problem together, and, finally, came to a solution. Yulia and Roni happened to be our focus group for that lesson, and throughout the interview we saw how different their behavior was from other students we observed: they consulted with one another, raised possibilities on their own, revised opinions, and seemed to arrive at common conclusions. In other words, rather than treating one another as possible authorities, that is, only as possible sources of answers, Yulia and Roni treated one another as intelligent interlocutors who could work together to make progress on the question at hand. We should stress that this was, indeed, behavior different not only from that of students in Danit's class, but also from that of other students in Sasha's class. For instance, at one point in our interview with Yarden, we asked if she could draw a triangle having two acute exterior angles; she said she could, and she proceeded to draw a diagram, which, obviously, could not be correct. When we asked Paulina, the second girl in the focus group, whether Yarden's diagram was ok, she assented immediately and with no further remark.

CONCLUSIONS—AUTHORITY AND STUDENTS' ABILITY TO REFLECT

Speaking about argumentation, Simon (2000) claims that, because students treat teachers as authorities who can give a stamp of approval on the students' mathematical arguments, it does not follow that students lack the ability and willingness to engage in the validation process themselves. This is a reasonable claim, and, certainly, our interview with Moshe, quoted above, gives some confirmation of it. Simon compares the students' relationship to the teacher, in her/his capacity as an authority, to the situation of a mathematician who is working in a field not her/his own and who seeks the evaluation of an expert in that field (Simon, 2000, p.166). But the evidence presented above shows that the way the teacher is an authority for the student is much stronger than this. In this connection, we tend to agree with Lewis-Shaw when she writes "[The students'] perspective consists in relating to the world with less awareness of the nature and extent of their personal authority or control. Consequently, they strive for a sense of belonging, count on the teacher for some level of security and authority and regard him as the holder of knowledge and expertise" (Lewis-Shaw, 2001, p.182).

This strong authority relationship is bound to effect the way students engage in mathematical thinking. An indication of this is given in Helme and Clarke (2001). They report that when a teacher approached a small group of students working on a problem there was a different pattern of interaction in which the teacher's questions became the central focus. Moreover, "With the teacher asking virtually all the questions, there was little opportunity for students to initiate ideas or spontaneously express and resolve uncertainty" (Helme & Clarke, 2001, p.146). Such a situation cannot be conducive to reflective thought about mathematical ideas.

In Danit's class, we observed an interesting instance that demonstrates, we believe, that the teacher's authority can create conditions in which it becomes very easy for students to use concepts unreflectively. At a certain stage while discussing the system,

$$\begin{cases} 2x + 3y = 18 \\ y = 3 \end{cases}$$

Danit asked what allows us to go from 2x+3y=18 to 2·3+3y=18, that is, what allows us to substitute 3 in place of x? She prompted the class, saying, "It begins with 'A'…" Finally, Uriel, whom we interviewed afterwards, said "axiom." But which "axiom"? Danit gave the answer the "axiom of replacement." We asked Uriel about what went on in that instance. Uriel told us how he remembered the word "axiom" when Danit said that the word began with "A". We discussed the meaning of axiom, which he seemed to grasp in a very rudimentary way. However, when we asked Uriel if he had any idea why such an axiom might be needed, he had difficulty grasping what we were asking him. It became clear that Uriel recognized an axiom here mainly because Danit said there was one. In other words, the need for an axiom was accepted on Danit's authority only. This was striking because more than once during the lesson Danit emphasized that things in mathematics were not true because she said so, or the because the book said so, or because anyone else simply said so.

That the pattern of authority existing between teacher and student can also come to characterize the relationship between students, even in a small measure, clearly has important implications for collaborative learning. For as Johnson and Johnson say, "Simply placing students in groups and telling them to work together does not promote greater understanding of mathematical principles or the ability to communicate mathematical reasoning to others. Group efforts can go wrong in many ways" (Johnson & Johnson, 1989, p.237). Where students are accustomed to treat one another as authorities one student will simply listen and assent to the other, or, if not, will turn to a different authority, just as Yarden turns to another friend when the first "doesn't know." The point is, where students are accustomed to treat one another as authorities no true dialogue takes place between them, and where there is no true dialogue there can be no true collaborative learning. Thus, the establishment of an authority relationship between students becomes one of the most potent ways in which "Group efforts can go wrong"! The point would be merely an academic one if it were not that collaborative learning seems to be one of the more effective means of encouraging reflective thinking (e.g. Johnson & Johnson, 1989, p.236). In this way, by helping to develop a relationship between students not based on authority, but on intellectual partnership, we go far towards encouraging also the kind of thoughtful approach to learning mathematics that most of us strive for.

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SCAFFOLDING PRACTICES THAT ENHANCE MATHEMATICS LEARNING

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It is just over 25 years since Wood et al.(1976) introduced the idea of 'scaffolding' to represent the way children's learning can be supported. Despite problems, this metaphor has enduring attraction in the way it emphasises the intent to support a sound foundation with increasing independence for the learner as understanding becomes more secure. It has resonance with the widely accepted notion of construction and the constructivist paradigm for learning. The discussion that follows will characterise some classroom practices that can be identified as scaffolding, revisiting some of the original classifications, and identifying further scaffolding strategies with particular reference to mathematics learning.

BACKGROUND

The metaphor of scaffolding was introduced by Wood et al.(1976) to explore the nature of adult interactions in children's learning, in particular, the support that an adult provides in helping a child to learn how to perform a task that cannot be mastered alone. Such interactions are also informed by Vygotsky's (1978) concept of the Zone of Proximal Development, and writing based on the relationship between these two ideas have been extensively developed (Rogoff and Wertsch, 1984).

The notion of scaffolding is used to reflect the way support is adjusted as the child learns and is ultimately removed when the learner can 'stand alone'. Wood et al. identifying six key elements: recruitment - enlisting the learner's interest and adherence to the task; reduction in degrees of freedom - simplifying the task so that feedback is regulated and can be used for correction; direction maintenance - keeping the learner in pursuit of a particular objective; marking critical features - accentuating some and interpreting discrepancies; frustration control - responses to the learners emotional state and demonstration; or modelling a solution to a task. In discussing the last of these, a hint is given to the complexities that may not be apparent in this classification. In demonstrating or 'modelling' a solution to a task, for example, "the tutor is 'imitating' in idealised form an attempted solution tried (or assumed to be tried) by the tutee in the expectation that the learner will the 'imitate' it back in a more appropriate form" (Wood et al.1976:98).

Tharpe and Gallimore (1988) use the term '***assisted learning***' in relating educational practices with Vygotsky's notion of a Zone of Proximal Development. They identify six interdependent means of assisting performance: modelling - offering behaviour for imitation; contingency management - rewards and punishment arranged to follow on behaviour; feeding back - information resulting from experiences; instructing - calling for specific action; questioning - calling for linguistic response; cognitive structuring - providing explanations and belief structures that organise and justify. Of these they claim cognitive structuring, which provides a 'structure for thinking and acting', is the most comprehensive and most 'intuitively obvious'. They note, however, 'study after study has documented the absence in classrooms of this fundamental tool: assistance provided by more capable others that is responsive to goal-directed activities' (Tharpe et al. 1988: 42).

The dependence on adult interaction is qualified by Rogoff and colleagues who use the notion of '***guided participation***' and regard children's development as 'occurring through their active participation in culturally structured activity with the guidance, support and challenge of companions who vary in skills and status' (Rogoff et al. 1993:5). They noted two patterns of interactive behaviour: one in which the adult structures children's learning by organising children's attention, motivation and involvement and by providing lessons taken from the context of an ongoing activity; the other in which children take primary responsibility for learning by managing their own attention, motivation and participation with adults providing assistance that is more responsive than directive.

SCAFFOLDING AND CLASSROOM PRACTICES

Research that attempts to characterise scaffolding practices in the classroom has suggested that teaching situations are more complex than small group settings and 'contingent responding requires a detailed understanding of the learner's history, the immediate task and the teaching strategies needed to move on' (Hobsbaum et al. 1996). Despite this difficulty they support the notion of scaffolding and characterise key elements identified in a Reading Recovery scheme:

- a measured amount of support without reducing the child's initiative;
- careful selection of the task at just the right level of difficulty with right balance of general ease but some challenge;
- child must be able to make sense of task using every available source of information;
- strategies made explicit - drawing explicit attention to strategies and processes.

(Hobsbaum et al. 1996: 22).

In their analysis of 'talk cycles' they found 'the predominant teacher strategy, by a long margin, for leading the move onto the next word cycle was telling' (p26). Teachers did, however, 'structure the internal setting so that the child develops increasingly more complex actions independently'.

In studying classroom teaching sequences in mathematics, science, and design and technology, Bliss et al.(1996) looked for instances of scaffolding but report 'a relative absence of scaffolding in most lessons'. Some '*actual scaffolds*' are identified as: approval, encouragement, structuring work, and organising people. '*Props scaffolds*' are also identified where the teacher provides a suggestion that will help pupils throughout the task and '*localised scaffolds*' providing specific help 'where a teacher finds it difficult to help the pupil with an overall idea or concept simply because it is too large and complex'. Two further scaffolds which Bliss et al. suggest were 'really more like cueing' were *step-by-step* or *foothold* scaffolds (often in a series of questions) and *hints and slots* scaffolds (narrowing questions until only one answer fits). They report few actual scaffolds and suggest reasons for absence of scaffolds in 4 categories: pseudo-interactions or bypassing (which accounted for the majority of classroom instances recorded); scaffolding precluded by directive teaching; scaffolding excluded by initiative being given to the pupils; conditions for scaffolding present but not noticed by the teacher (Bliss et al.1996:46).

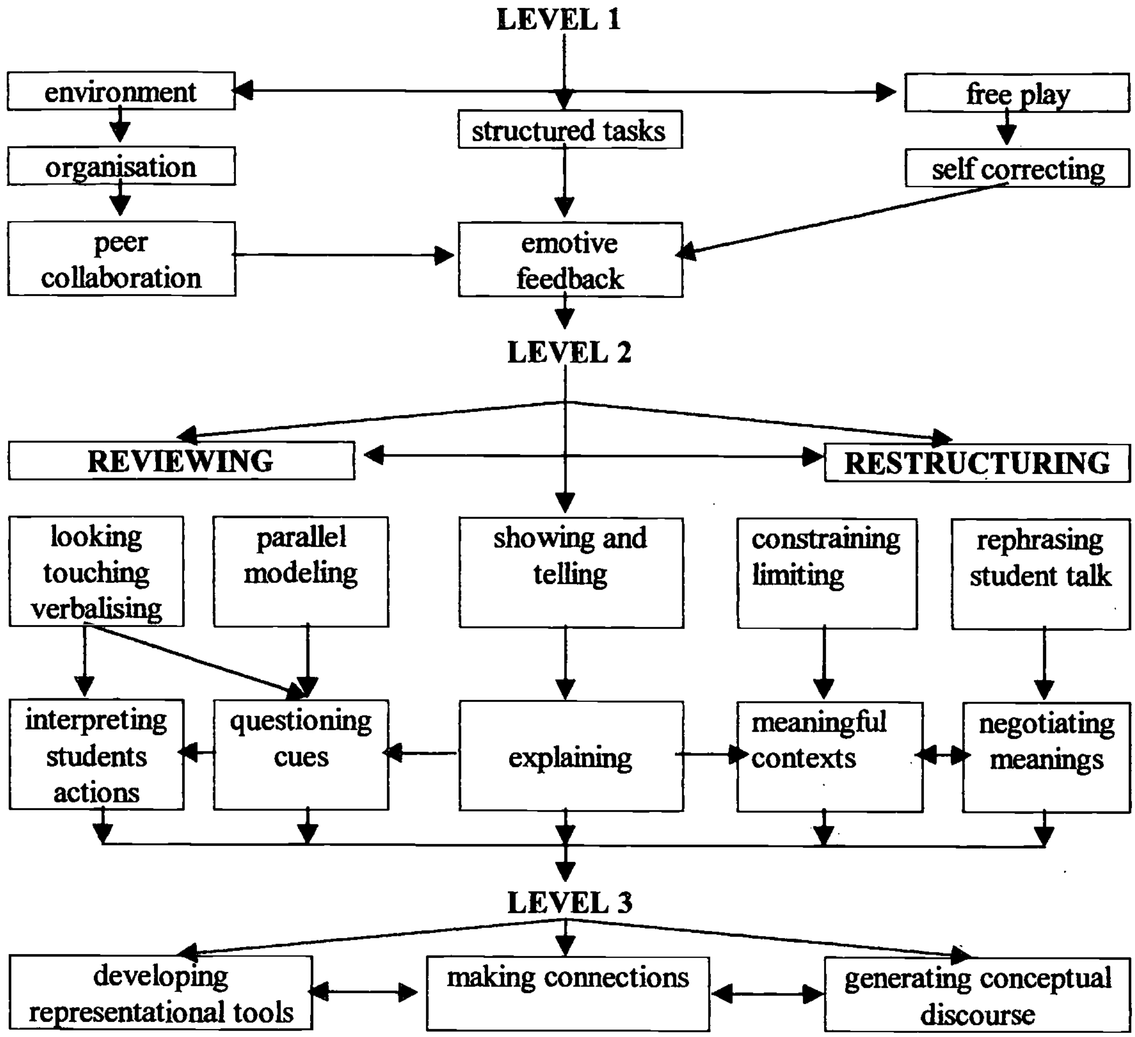
Despite the negative connotations of the latter study, teachers are integral to the learning process and the following discussion will identify strategies that can be classed as scaffolding, with illustrations taken from research studies in mathematics. Some different aspects of the scaffolding process will be illustrated in relation to geometry in the early years of schooling and arithmetic in later elementary years.

IDENTIFIABLE SCAFFOLDING PROCESSES

Some scaffolding practices are found in every classroom while others may be lacking completely. The classroom environment, for example, is prepared by a teacher as a resource for learning with materials available for the students to see, touch and use in their work. Other forms of scaffolding that may be less evident include peer interactions which is minimised where students work from texts in class and at home. Explaining and questioning will be common in most teaching approaches but analysis of these practices shows complex variations in the balance of interactions intended to support pupils' learning. In the following discussion, scaffolding strategies will be identified with further classification as *reviewing* and *restructuring* in a hierarchical inter-relationship.

Teaching strategies that focus on the provisions for the learning environment but do not directly relate to interactions between students and teacher are classed as *Level 1* scaffolding. Also classified at this level will be emotive interactions that are general in their nature. *Level 2* involves direct interactions between teachers and students specifically focused on the task in hand. Such strategies vary from direct instruction - showing and telling - to more collaborative meaning making. At *Level 3* the fundamental aim is to establish connections between what students have within their experiences and new mathematics to be learned. Mathematical thinking is supported through conceptual discourse and the establishment of representations. At any stage, mathematical learning is enhanced by scaffolding at each of these levels, and the following hierarchy reflects not only the progressive (and often circular) supporting strategies that can be used, but also the way effective interactions may be bypassed in more direct teaching approaches.

Figure 1: Teacher Strategies for Scaffolding Learning



LEVEL 1 SCAFFOLDING

Before interacting with the children, teachers will create an environment for learning with a choice of wall displays, puzzles, tasks, appropriate tools and classroom organisation (Tharpe et al.1988). This form of scaffolding has not always been explicitly acknowledged in research (e.g. Bliss et al.1996) but is a crucial element of any school practice. Opportunities for using building blocks in *free play*, for example, results in improved performance on geometry tasks with this performance further enhanced by interactions with a 'more experienced other' (Wood et al.1976; Coltman et al.2002). *Self-correcting* elements provide feedback that supports pupils, not only in finding a solution, but also in reflecting on the processes involved in such a solution and so becoming self regulating in their actions (Tharpe et al.1988).

The tasks that teachers select require *organisational* considerations, for example, appropriate *sequencing* and *pacing*, and groupings for *peer collaboration*. Light and Littleton (1999) report 'compelling evidence for the benefits in terms of learning of peer collaboration' and establish that peer collaboration need not involve 'a more experienced other'. Doise et al.(1984) showed that children of slightly different pre-test level, working together in pairs or threes, tended to perform at a higher level when working as a group than children working alone and this benefit carried over to post-test performances. Large differences in terms of the children's pre-test levels were associated with less progress than small differences in initial ability.

Also at Level 1, but less specific to mathematics learning is *emotive feedback* like gaining individual's attention or offering praise. Bliss et al. included 'approval (and) encouragement' in the majority of what they called 'actual scaffolds' along with 'structuring work' and 'organising people' (Bliss et al.1996: 47).

LEVEL 2 SCAFFOLDING

Mathematics can be presented as a series of skills and processes that are known to the teacher and transmitted to students through showing and telling. On the other hand, students can be more involved in the development of mathematical knowledge through interactions that progressively develop their own understandings.

Showing, Telling and Explaining

Showing and telling need little introduction as they have been traditional in classroom teaching for generations and continue to dominate classroom practice (Hobsbaum et al.1996; Pimm, 1987). With this strategy teachers retained control and structure conversations to take account of the 'next step' they have planned. Students can make their own sense of instruction sometimes in very different ways from the teacher (Bliss et al.1996: 41). Closely related to the classroom practice of 'showing and telling' is the notion of 'explaining' where the teacher amplifies a process or concept, or elaborates why solutions are inaccurate or inappropriate. In contrast to this teaching approach which is built upon teachers' explanations, social norms can be established in the classroom where the students actively participate by explaining and justifying their own thinking (Cobb et al., 1991). Askew et al.(1997) found that for effective teaching it is crucial to encourage pupils to 'develop strategies and networks of ideas by being challenged to think, through explaining, listening and problem solving'.

Scaffolding that goes beyond showing, telling and explaining

In contrast to telling students what to do and how to think about a problem, there are practices which provide support for developing students' own understanding of mathematics. <u>REVIEWING</u> relates to interactions that: encourage students to look, touch and verbalise; use questions, cues and parallel modeling; interpret student actions. <u>RESTRUCTURING</u> involves teachers making adaptations: by constraining or limiting the tasks; by giving meaningful contexts; by sensitively rephrasing students' talk and solutions; and by negotiating meanings.

Looking, touching, verbalising, and parallel modelling

In arithmetic teaching the encouragement to "tell me what you did" can help a student to be self correcting or more efficient. 'It seems that the act of attempting to express their thoughts aloud in words has helped pupils to clarify and organise the thoughts themselves' (Pimm 1987:23). Handling materials and verbalising observations can provide a different orientation and lead to better understanding of mathematical tasks (Coltman et al. 2002). When such reflective interactions are not sufficient there can be a temptation to 'tell' a solution. However, an alternative strategy, evident in studies of geometric block tasks, can be 'parallel modeling' where an equivalent task is demonstrated that helps develop transferable skills in the student (Coltman et al.2002).

Questioning

Pimm (1987) discusses a framework of questioning which 'locks the teacher into 'center stage', acting as controller as well as *heavily* influencing the *types* and range of spoken pupil contributions in class. Tharpe et al.(1988) note that 'it is easy to feel "in sync" with the students when the yes/no answers flow smoothly.....(when) she (the teacher) has inadvertently 'fed' them lines rather than assisting comprehension'. At the other extreme, questions can be posed that encourage the children to construct their own mathematical understanding and to determine independently whether they have reached mathematically valid solutions. Chappell et al.(1999) propose that 'when we modify the questions that we ask ... our assessment of students' thinking refines our instructional practice and indicates to students that we value their ability to communicate about mathematics'.

Interpreting students actions/ Making strategies explicit

Wood et al. (1976) note that 'the learner must be able to *recognise* a solution to a particular class of problems before he is himself able to produce the steps leading to it without assistance' (p90). This recognition of the relevance of actions can involve a teacher interpreting students actions and making strategies explicit. Hobsbaum et al. include 'drawing explicit attention to strategies and processes (which) provides a model of behavioral regulation for the learner, which may become internalised, a 'voice in the head' for future situations'(Hobsbaum1996: 22).

Constraining, limiting and simplifying

In Wood et al.'s original paper one element scaffolding was identified as: '*reduction in degrees of freedom* - simplifying the task so that feedback is regulated to a level which could be used for correction'. An example is given in reducing from three blocks in a repeating sequence to two blocks (Coltman et al. 2002) or simplifying the numbers in a calculation can make a task accessible before building towards the more complex task.

Identifying meaningful contexts

Meaningful contexts can help students find solutions to tasks where they cannot solve related abstract problems. Young children can more successfully solve tasks given in context and then transfer their learning back to abstract tasks (Coltman et al. 2002). In arithmetic, the shift from an abstract calculation, for example, '$6 \div 12 =$ ', to a contextual setting: "Six pizzas are to be shared among 12 people. How much does each person get?", can take a problem from inaccessibility to the construction of a meaningful solution (Anghileri 2000).

Re-phrasing students talk and negotiating meanings

As a teacher pays close attention to the utterances of pupils, many 'spoken formulations and revisions will often be required before an acceptable and stable expression can be agreed upon by all participants' (Pimm 87:23). The teacher's role is to highlight processes involved in solutions, sometimes re-describing students efforts and making clear the mathematical aspects that are most valued. Considerable sensitivity may be needed to 'unpick' the essence of students' talk, rephrasing where necessary to make ideas clearer without losing the intended meaning, and negotiating new meanings to establish mathematically valid understandings (Anghileri 1995).

LEVEL 3 SCAFFOLDING

Making connections

Through both REVIEWING and RESTRUCTURING students can be supported in making connections with their previous experiences. Askew et al. (1997) use the term '*connectionist*' to characterise highly effective teachers who believe that pupils develop strategies and networks of ideas with teacher interventions to connect existing understandings with mathematics to be learned. Lack of connections can hamper progress. In arithmetic, for example, discontinuity between informal approaches and taught procedures can result in little progress while teaching approaches that develop progressive connections lead to better improvements (Anghileri 2000).

Developing representational tools

Much of mathematical learning relates to the interpretation and use of systems of images, words and symbols. Such representations, in addition to providing a means of communication, can also be developed as tools for structuring knowledge and to index the systems in which they arise. Cuoco et al. present a wide range of perspectives about the nature and purposes of representation with distinctions made between external representations (marks on paper, geometric sketches, equations etc.) and internal representations (images we create in our minds for mathematical objects and processes) (Cuoco et al. 2001:x). With teacher guidance, a symbolic record can facilitate discussions, and representations including pictures, diagrams chart and tables can be used not only for recording but as tools for thinking.

Generating conceptual discourse

Teachers play a vital role in shaping classroom discourse through signals they send about the knowledge and ways of thinking that are valued. Cobb et al. (1991) identify classroom discourse as critical in supporting the students' development and focus on 2 characteristics that relate specifically to math learning: the norms and standards for what counts as acceptable math explanation (conceptual not computational) and the content of the whole class discussion. With a conceptual orientation students are likely to engage in longer, more meaningful discussions and meanings come to be shared as each individual engages in the communal act of making mathematical meanings.

SUMMARY

This brief review of scaffolding processes is an attempt to identify a hierarchy of classroom interactions that can enhance mathematics learning by diverging from the narrow approach that has been typical in the past.

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PROVING OR REFUTING ARITHMETIC CLAIMS: THE CASE OF ELEMENTARY SCHOOL TEACHERS

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Abstract: We examined elementary school teachers' justifications to number-theoretical "for all" propositions and existence propositions, some of which are true while others are false. We also assessed whether teachers regarded their justifications as mathematical proofs. About half of the teachers produced formal algebraic proofs. A smaller number of teachers produced non-formal proofs appropriate for presentation in elementary school classes. However, a substantial number of teachers applied inadequate methods to validate or refute the propositions. Finally, many teachers were uncertain about the status of the justifications they gave.

The processes of examining the validity of conjectures, proving correct ones as well as refuting wrong ones are at the core of any student's mathematical development. It is therefore essential that teachers are intimately familiar with and confident in producing and reacting to arguments that purport to prove or refute mathematical statements that are discussed in their classrooms. Prior research (Jones, 1997; Martin and Harel, 1989; Movshovitz-Hadar and Hadas, 1988; Simon and Blume, 1996) has shown that not all is well in this respect.

This paper reports on a study aimed at examining elementary school teachers' justifications to number-theoretical propositions, some of which are true while others are false. We also examined teachers' views of the status of their justifications, that is, whether they regarded them as mathematical proofs. In this paper we shall focus on describing the various justifications that the teachers provided to "for all" propositions and to existence propositions.

METHOD

Participants

A class of 27 in-service elementary school teachers (25 women, 2 men) participated in the study. All were in their first year of a three-year professional development program at an Israeli university, a course that mainly focused on introducing the mathematics topics that are part of the elementary school mathematics curriculum from an advanced mathematical viewpoint. The participating teachers were engaged in teaching mathematics, as well as other subjects, in the upper grades of elementary school (grades 3-6). Their teaching experience varied considerably: Eight were in their first five years of teaching, eight taught between six and fifteen years and twelve taught more than 15 years.

Tools and Procedure

A questionnaire, including the following six, number-theoretical propositions, was administered to the teachers during a 90 minute session:

"For all" propositions

1. The sum of any five consecutive integers is divisible by five.
2. The sum of any four consecutive integers is divisible by four.
3 The sum of any three consecutive integers is divisible by six.

Existence propositions

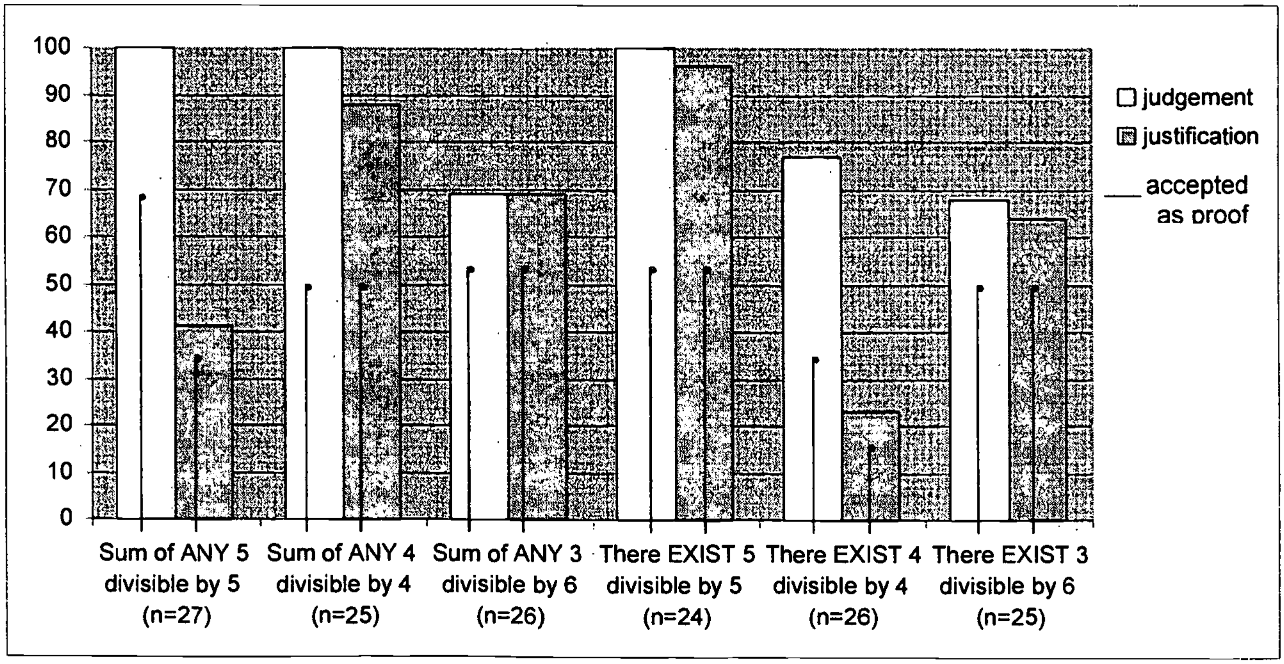
4. There exist five consecutive integers whose sum is divisible by five.
5. There exist four consecutive integers whose sum is divisible by four.
6. There exist three consecutive integers whose sum is divisible by six.

Propositions 1 and 4 are both true, because the sum of any sequence of five consecutive integers is divisible by five. Propositions 2 and 5 are both false, because no sum of four consecutive numbers is divisible by four. Proposition 3 ("for all") is false and Proposition 6 is true, because the sum of some sequences of three consecutive numbers is divisible by six while the sum of others is not. Half of the teachers responded first to the "for all" tasks while the other half responded first to the existence tasks.

The teachers were asked to consider each proposition and (1) decide whether it is true or false and justify their claim; (2) determine whether, in their opinion, the lecturers would consider their justifications as mathematical proofs. It was assumed that the lecturers were perceived by the participants as representatives of the mathematical community. Therefore, this was a measure of the extent to which the teachers viewed their own justifications as mathematical proofs.

RESULTS

The results of both the "for all" and the "existence" tasks are summarized in the following Figure. The figure presents, for each proposition, the number of teachers who responded to it, the percentage of correct judgments (left, white bar) and the percentage of correct justifications (right, shaded bar).



Furthermore, the figure presents the percentage of teachers who expected their judgments to be acceptable by the lectures (line inside left bar) and the percentage of teachers who had given a correct justification and expected it to be acceptable by the lecturers (line inside right bar). It should be noted that justifications were classified as incorrect only if they contained mathematical errors or if they were inadequate in terms of the methodology that was used to prove or refute a proposition (e.g., using supportive examples to prove a "for all" proposition or using a counterexample for refuting an existence proposition).

"For all" Propositions

Proposition 1: The sum of any five consecutive integers is divisible by five.

All participants correctly stated that this proposition is true. However, only 41% accompanied their correct claim by correct justifications (see Figure). Almost all these justifications were algebraic proofs (33%) and they were commonly regarded as proofs by those who provided them (30%). A typical proof was:

> Shelly: x+x+1+x+2+x+3+x+4=5x+10. Any number substituted for x is multiplied by 5, 5x is divisible by 5, 10 is a multiple of 5, so 5x+10 is always divisible by 5.

Two teachers, Odette and Anna, provided non-algebraic proofs. Odette attempted (and succeeded) to "cover all possibilities":

> Odette: I first tried all the possible examples of 5 consecutive numbers within the first ten numbers: 1+2+3+4+5=15, 2+3+4+5+6=20, 3+4+5+6+7=25, 4+5+6+7+8=30, 5+6+7+8+9=35, 6+7+8+9+10=40, 7+8+9+10+11=45, 8+9+10+11+12=50, 9+10+11+12+13=55, 10+11+12+13+14=60.
> All other sums of five consecutive numbers are created by adding one of these sequences to certain multiples of 10 (for instance, 44+45+46+47+48 is composed of 5 times 40 and the sequence: 4+5+6+7+8=30 that we had before. The number 40 is composed of 10 times a number and since 10 is divisible by 5, 40 is also divisible by 5). The same holds for three digit numbers (since 100 is divisible by 5), for four digit numbers, etc. So each such sum is divisible by 5.

Odette regarded her justification as a mere example, insufficient for proving the proposition. She wrote: "I still need to find a rule, beyond my examples".

Anna examined the general structure of any sequence of five consecutive numbers and gave the following, generic proof (Balacheff, 1987):

> Anna: When we add five consecutive numbers, for instance, 1,2,3,4,5, we can look at these numbers in the following way: We have the number in the middle (3), then, the sum of the two numbers that stand next to it from both sides (2 on one side and 4 on the other) is 3-1 plus 3+1. The -1 cancels the +1 and we get 2*3. Similarly, 5 is 3+2 and 1 is 3-2 so we have, again, twice the number in the middle. All in all, the sum is 5 times the number in the middle, and 5 times any number is divisible by 5.

Unlike Odette, Anna was sure that her justification is a mathematical proof.

Most incorrect justifications consisted of providing one or more supporting examples (52%). These justifications were perceived as proofs by 33% of the teachers. Two teachers provided an improper algebraic justification. They reached the expression 5x+10, but then "solved" it in the following manner:

> Sofie: x+x+1+x+2+x+3+x+4=5x+10. 5x=-10, x=-2. The answer is (-2, -1, 0, 1, 2). I got an answer; therefore the sum is divisible by 5.

While one of these two teachers viewed her justification as a mathematical proof, the other expressed reservations, stating that she was unsure whether one example (-2, -1, 0, 1, 2) is sufficient to prove a proposition.

Proposition 2: <u>The sum of any four consecutive integers is divisible by four.</u>

The figure shows that in this case, all teachers correctly stated that the proposition is false and all those who justified this assertion provided correct justifications to their claim (three teachers stated that the proposition is false but provided no justification). Most justifications consisted of one or more counterexamples (72%). Some correctly commented that a single counterexample is sufficient for refuting a claim while others doubted the status of such examples. Indeed, half of those who provided counterexamples regarded them as proofs (36%).

Sixteen percent of the teachers provided algebraic proofs and regarded them as proofs. A typical proof was:

> Ramit: x+x+1+x+2+x+3+x+4=4x+6, 4x is divisible by 4 but 6 is not divisible by 4. Therefore the sum 4x+6 is not divisible by 4.

Proposition 3: <u>The sum of any three consecutive integers is divisible by six.</u>

Here, unlike the previous cases, not all the teachers correctly judged the proposition as false (see Figure). While most teachers (69%) correctly stated that the proposition is false, 23% incorrectly claimed that it is true and 8% were unsure about its status.

All those who correctly argued that the proposition is false accompanied this judgment with correct justifications. Most correct justifications (42%) consisted of counterexamples. Maria, for instance, explained:

> Maria: 2+3+4=9, 4+5=6=15, 7+8+9=24, 8+9+10=27, 10+11+12=33, 14+15+16=45, 16+17+18=51, 27+28+29=51. I found many examples for which the sum is not divisible by 6. Therefore, the statement is incorrect.

Unlike Maria, Lily provided only one counterexample and noted that one such example is sufficient for refuting a proposition of this kind:

> Lily: This is a general proposition. We need one counterexample to prove that it is false: 0+1+2= 3 and 3 is not divisible by 6.

Of the 50% who stated that the proposition is false and provided one or more counterexamples, most (38%) expected the lecturers to accept them as mathematical proofs. Those who did not (11%), were either unsure that examples are sufficient to refute a given proposition or stated that examples are not mathematical proofs and further explained that they do not know the proof and therefore all they could do was provide some examples.

The remaining teachers who correctly stated that the proposition is false provided algebraic justifications (19% of the teachers) and all but one of them regarded these justifications as proofs. The most prevalent algebraic justification was:

> Lima: x is the first number in the sequence x+x+1+x+2=3x+3. The sum of this sequence is divisible by 3. We have to show that the sum is also divisible by 2. If x is even then 3x is even but if we add 3 we get an odd number, so 3x+3 is not divisible by 6. If x is odd then 3x is odd and 3x+3 is even and then it is divisible by 6. The proposition does not hold for all sequences. Therefore, it is false.

Miki gave another argument that used algebraic notation:

> Miki: 1+2+3=6, 6 is divisible by 6; 0+1+2=3, 3 is NOT divisible by 6; 4+5+6=15, 15 is NOT divisible by 6; 7+8+9= 24, 24 is divisible by 6.
> The sum of all sequences of three consecutive numbers x-1, x, x+1 is divisible by 3, because, x-1 + x + x+1 is 3x (-1 and +1 cancel each other and we get 3 times x). But 3x is divisible by 6 only if the middle number, x, is even.

All teachers who incorrectly claimed that the proposition is true provided supportive examples. Only one of them expected his justification to be accepted as a mathematical proof. Those who were unsure about the status of the proposition provided both supportive and counterexamples. They did not expect that their justifications would be accepted as mathematical proofs.

Existence Propositions

Proposition 4: <u>There exist five consecutive integers whose sum is divisible by five.</u>

All participants correctly stated that this proposition is true and all but one participant accompanied their correct claim by correct justifications (see Figure). The most prevalent justification consisted of one or more adequate examples (50%), yet, only half of those who gave such examples regarded them as mathematical proofs.

Two teachers explicitly related to the type of this proposition, arguing that one supportive example is sufficient for proving it. Ora, for instance, wrote:

> Ora: If the claim starts with "there exist" it is sufficient to provide one example to prove existence. In this case, 1+2+3+4+5=15, and 15 is divisible by 5.

Algebraic proofs were given by about a third of the teachers (38%). Odette and Anna, who provided non-algebraic proofs to the matching, general proposition (Proposition 1), repeated the same justifications for the existence proposition. This time, however, both viewed their justifications as mathematical proofs. We note that none of the teachers referred to the fact that Proposition 4 trivially follows from Proposition 1, even though 50% of them had worked on proposition 1 shortly before working on Proposition 4. Vinner (1983) has observed similar behavior among senior high school students.

Proposition 5: There exist four consecutive integers whose sum is divisible by four.

Most teachers (77%) correctly stated that this proposition is false. Yet, 15% incorrectly claimed that it is true and 8% were unsure about its status. This was the only proposition to which less than a quarter of the teachers provided both correct answers and correct justifications. Most correct justifications (19%) were algebraic proofs that were presented by the same teachers to both this proposition and to the matching "for all" proposition (Proposition 2). While 12% of the teachers viewed their algebraic justifications as mathematical proofs, 7% were unsure that such justification refutes the proposition.

Sami provided an original, correct justification:

> Sami: There are no four consecutive numbers whose sum is divisible by 4. I'll prove it in the following way:
> The sum of the first sequence of numbers: 0+1+2+3=6, is not divisible by 4.
> The second sequence is formed by enlarging each number in the sequence by 1. Instead of 0, 1, 2, 3 we get 0+1=1, 1+1=2, 2+1=3, 3+1=4 so we get 1, 2, 3, 4. The sum this time is 10 because we added 1 to each of the four numbers in the previous sequence. Since 6 is not divisible by four, and we added four, the sum (6+4) is also not divisible by 4. The next sequence 2, 3, 4, 5, is created in a similar manner, and the sum is 14. Consequently, the sum of each sequence is larger by 4 than the previous sum. Since we start with a sum that is NOT divisible by 4, the jumps by four always bring us to new numbers that are-not divisible by 4.In this way we can show that the sum of no such sequence is divisible by 4..

Most teachers who incorrectly claimed that the proposition is true explained that they tried several examples and none of them fulfilled the condition. Yet, they felt that if they continued their search they would eventually find a supportive example. The teachers who were unsure about the status of the proposition went through a similar process (namely, trying, with no success, to find supportive examples) but they remained uncertain whether such an example exists.

Proposition 6: There exist three consecutive numbers whose sum is divisible by six.

Most teachers (68%) correctly stated that this proposition is true, and all but one of them accompanied their correct claim by correct justifications (One teacher provided no justification). The most prevalent justification consisted of one or more supportive examples (40%) most of which (28%) were regarded as mathematical proofs. The remaining justifications were algebraic and, much like in the case of propositions 1 and 4, they were identical to the algebraic justifications that were presented by the same teachers to the matching, "for all" proposition (Proposition 3).

The teachers who incorrectly stated that the proposition is false (24%) provided as justification one or more examples of three consecutive numbers whose sum is not divisible by 6. Most of them (20%) regarded these examples as counterexamples and expected the lecturers to accept their justifications as proofs. Amit, for instance, wrote "a counterexample is sufficient to refute any proposition, including this one". It seems that Amit, and others, overgeneralized a scheme that holds for refuting "for all" propositions and used it for refuting existence propositions as well. Two teachers (8%) could not decide whether the proposition is true or false, because they found both supportive and counterexamples.

FINAL COMMENTS

1. There is a wide consensus in the mathematics education community that teachers should encourage students to make mathematical conjectures and investigate them (e.g., NCTM, 2000). A growing number of studies identify conjectures, generalizations, refutations, and even proving among elementary school students (e.g., Ball and Bass, 2000; Lampert, 1990; Maher and Marino, 1996; Zack, 1997). Students present conjectures of different types (e.g., "for all", existence, true, false) and apply different ways to verify them. Teachers, the representatives of the mathematical community in class, have a crucial role in establishing the various sociomathematical norms in general and those related to justifications, argumentation and proofs in particular (Yackel and Cobb, 1996). To fulfill this role, it is crucial that teachers be knowledgeable about different types of propositions and ways of proving or refuting them. Our paper shows that a substantial number of elementary school teachers applied inadequate methods to validate and to refute various propositions. Two salient cases relate to the use of examples: Supportive examples were used by about half of the participants to prove "for all" theorems and about 20% of the participants refuted an existence proposition by counterexamples. These findings call for more attention to this aspect of elementary school teachers' knowledge in professional development programs and in research.

2. Teaching mathematical reasoning and proof in elementary school is a very demanding job. When an elementary school teacher is confronted with a student's conjecture, such as the ones given here, he/she should be able to both examine its validity and to find suitable, non-formal ways to communicate about this conjecture with his students. The propositions that were presented in this study could relatively easily be proved or refuted with algebraic tools. Thus, algebra is a powerful tool that could be used by teachers in elementary schools to determine the validity of number-theoretical propositions and other propositions that could be raised by students in class. However, our data show that only about half of the teachers used algebraic tools in their attempts either to prove or to refute at least one of the propositions. These teachers usually did so in an appropriate manner and most of them were confident that their justifications are mathematical proofs. Moreover, about 20% of the teachers were aware of a need for a more general proof for some propositions, but noted that they did not have the required knowledge. Thus, it seems essential that professional development programs attempt to enhance elementary

school teachers' algebraic reasoning so that they would be able to use this knowledge to determine the validity of their students' conjectures.

3. As stated previously, it is essential that elementary school teachers be able to present non-formal proofs in their classes. A number of such proofs were presented by some teachers in our study (e.g., Odette, Anna and Sami). It is important to find out how students and teachers conceived these justifications: Do they regard them as proofs? How do they explain their decisions? Do teachers who have better algebraic reasoning accept these justifications as proofs? These and other related issues are currently under investigation.

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NUMBER SENSE, PLACE VALUE AND "ODOMETER" PRINCIPLE IN DECIMAL NUMERATION: ADDING 1 TENTH AND 1 HUNDREDTH

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Having a facility with numbers (whole numbers and decimals) is essential for life skills. It requires an understanding and coordination of several powerful mathematical concepts (e.g., place value, additive and multiplicative fields, the role of zero, seriation) and principles (e.g., "odometer", associative, commutative distribute) (Baturo, 2000). This paper reports on 329 Year 6 students' proficiency with seriation of tenths and hundredths when instruction in these two places was complete. They were asked to add 1 tenth and 1 hundredth to a variety of given numbers with the overall results of 53.6% and 54.6% respectively. These results were evaluated in terms of the students' responses to basic place value tasks. This paper analyses the mathematical structure of the tasks, discusses some common error patterns discerned, and draws inferences for teaching.

The decimal number system is superficially simple but is structurally very complex. It is derived from the notion of a unit (Steffe, 1986), the importance (and complexity) of which is often not considered because of its deceptively simple association with one object. The decimal number system consists of simple yet powerful patterns within the domain of whole numbers and decimal numbers, patterns which should encourage the development of succinct yet global mental models within and between each of these domains. Within the domain of whole numbers, students are exposed to processes such as counting, comparing, grouping, regrouping, and approximating, most of which can be transferred without adaptation to decimal numbers (see Baturo, 2000, for a mathematical analysis of these processes). However, comparing is a process that must be adapted for decimal and common fractions (see, e.g., Resnick et al., 1989; Stacey et al., 2001). Furthermore, a study of whole numbers should develop an awareness of the commutative, associative, and distributive laws which can then be transferred without adaptation to decimal numbers (and to common fractions, percents etc.).

Another global principle that occurs in the decimal number system is referred to as the "odometer" principle in this paper. It underlies the counting process and requires an understanding that a place is "full" when it has 9 units (which could be ones, tens, tenths, etc) and that recording the next number requires a new position to the left of the place under consideration. Embedded in this principle is the notion of place value which requires an understanding of the role of the base, the order of places, and that numbers increase in value as they "move" to the left (and, conversely, decrease in value as they "move" to the right).

What makes the decimal number system powerful is the notion of place value and the use of 10 symbols only (base) to represent any number no matter how large or small

(Baturo, 1998). This capability is possible because of the multiplicative relationships (×, ÷) that relate adjacent (and nonadjacent places) places. However, Baturo (2000) found that top-performing students only have an understanding of the bi-directional multiplicative relations.

Research (Baturo, 1997, 2000; Bednar & Janvier, 1988; Fuson, 1990; Hiebert & Wearne, 1992; Jones et al., 1996; Ross, 1990) has produced a plethora of evidence that students (particularly from Years 1 to 4) have great difficulties in acquiring an understanding of place value. The general consensus seems to be that "children find place value difficult to learn and teachers find it difficult to teach" (Ross, 1990, p. 13). Baturo's numeration model (2000) gives some indication of the complexity of place value in that she indicates that there are three hierarchical levels which take account of Halford's (1993) complexity model in which unary relations (e.g., memory objects such as position, base, order) are less cognitively difficult than binary relations (e.g., equivalence such as 1 ten = 10 ones) which, in turn, are less cognitively difficult than ternary relations (e.g., transitivity where 10 ones = 1 ten, 10 tens = 1 hundred so 100 ones = 1 hundred).

In Baturo's model, Level 1 place value knowledge is baseline knowledge associated with position, base, and order, cognitions which are required for all numeration processes. Therefore, without these cognitions, students have no chance of processing decimal numbers with understanding. On the other hand, having Level 1 knowledge alone is not sufficient for a full understanding of all decimal-number numeration processes. Level 2 comprises unitising and equivalence, one or both of which are required for Level 3 knowledge which is associated with reunitising, additive structure and multiplicative structure. These cognitions appear to provide the superstructure for integrating the other levels and, for this reason, were defined as *structural knowledge*

Several researchers have pointed to the difficulty students have with grouping/unitising (Level 2) which involves quantifying sets of objects by grouping by 10 (in a base-10 system) and treating the groups as units (Fuson, 1988) and using the structure of the notation to capture the information about the groupings (Hiebert & Wearne, 1992; Ross, 1990). For example, Bednarz and Janvier (1988) viewed grouping as the basis for recognising and constructing multidigit numbers. Inherent in the process of grouping to form larger units or partitioning to form smaller units (called "superunitising" and "subunitising" respectively by Baturo, 2000) is the notion of base which gives rise to the positional aspect of place value.), From their study, Bednarz and Janvier noted that few children see the relevance of grouping tasks and the validity of doing and undoing groupings (partitioning) to solve multidigit number problems. Moreover, they found that some children form groupings, but only to count collections of objects (by tens in the decimal number system). That is, children do not appear to abstract the role of the base in grouping to form larger places.

In their study with Year 1 students, Jones et al. (1996) developed a framework for nurturing and assessing multidigit number sense. The framework incorporated four components (*counting*, *partitioning*, *grouping and estimating*, and *number relationships and ordering*) which they saw as representing a sequence of understandings related to place value. They predicted and confirmed that counting was the pivotal component of their framework with grouping the key concept. Their study found that solution at the higher end of their framework was dependent on facility with (or understanding of) the lower components. Their framework is attuned to Steffe's (1986) framework with respect to units, namely, that there are four different ways of thinking about a unit, namely, *counting (*or *singleton) units*, *composite units* (grouping), *unit-of-units* (regrouping*)* and *measure unit,* with each type apparently representing an increasing level of abstraction.

Counting is the repeated addition of an iterated unit (counting forwards) or the repeated subtraction of an iterated unit (counting backwards). These notions hold true for counting whole numbers, decimal numbers, common fractions, etc. Counting from a place value perspective requires understanding of the odometer principle (i.e., grouping to form a new place after 9 ones, tenths, hundredths, etc is reached). Therefore, to be able to count/seriate successfully, students require an understanding of place value, order across and within places, the ability to group, and an awareness of when to group and of the effects of grouping.

THE STUDY

The study reported in this paper is part of a doctoral dissertation in which Year 6 students' understanding of the numeration processes (identifying numbers from a variety of representations, counting, comparing, ordering, approximating) relating to tenths and hundredths were assessed (Baturo, 1998). To this end, a diagnostic test was constructed to determine the robustness of Year 6 students' understanding of the numeration processes. The test was trialed with a convenience sample of 8 students, modified and retrialed with 156 students from two large suburban schools (referred to as Trial 1 in this paper), modified and administered to 173 students a year later (Trial 3) at the Trial 2 schools. All item directions were read to the students (but not the actual subitems) and special instructions issued. For example, for Item 1 (see Figure 1), a sample number (2.3) was written on the board and the students were told that answers such as "two point three", "two decimal three", or "two dot three" would not be accepted. They were told that they had to write the name of the decimal part. At the time of the test (early Year 6 in both trials), the students had received two years of instruction in tenths and one year of instruction in hundredths.

Item analysis

This paper reports on the students' performances on five of the diagnostic test items (see Figure 1). The first three were focused on assessing the baseline knowledge of position and order (i.e., Level 1 in Baturo's model) whilst the last two items focused on seriating, with most subitems at Level 1 but some at Level 2 (requiring grouping and thus more difficult). All subitems ranged from prototypic to nonprototypic cases (see, e.g., Baturo & Cooper, 1997) in order to assess the robustness of the students' knowledge. With respect to the items in this study, prototypic tasks are classified as those that are given so frequently in class that students can acquire syntactic rules for their solution (e.g., adding 1 to the rightmost digit when finding the number that comes next after a given number, a rule that can develop from an overuse of counting on with whole numbers). Thus, in this study, nonprototypic tasks were seen as those that would not normally be part of the students' experiences and could not be solved by syntactic rules. Figure 1 describes the tasks and the difficulty level of each subitem in terms of Baturo's (2000) numeration levels and expected prototypicality.

Item 1: Write these numbers in words: (a) 4.7 [1P]; (b) 6.39 [1P]; (c) 0.8 (1NP]; (d) 5.02 [1NP].
Item 2: Write these numbers: (a) nine, and 5 tenths [1P]; (b) 6 tenths [1P]; (c) four, and thirteen hundredths [1P]; (d) sixty hundredths [1NP]; (f) seven and one hundredth [1NP]; (g) three hundredths [1NP].
Item 3: Write the number that has: 2 tenths, 5 hundredths, 4 ones [1NP].
Item 4: Write the number that is 1 tenth more than: (a) 9.4 [1P]; (b) 2.9 [2P]; (c) 4.06 [1NP]; (d) 1.94 [2NP]; (e) 5 [1NP]
Item 5: Write the number that is 1 hundredth more than: (a) 2.76 [1P]; (b) 3.9 [1NP]; (c) 0.09 [2P]; (d) 6 [1P]; (e) 4.91 [1NP]

Note. 1 and 2 indicate Baturo's (2000) numeration levels; P = prototypic; NP = nonprototypic.

***Figure 1.* Items [and difficulty level] designed to assess Year 6 students' understanding of place value and seriation.**

Items 1, 2 and *3* assessed the students' baseline knowledge of *place value* (Level 1). Items 1 and 2 focused on position and naming whilst Item 3 focused on the order of the places. In Item 1, the first two numbers were considered to be more straightforward than the last two which incorporated a zero. Of these latter two, the last number was expected to produce lowest performance because of the internal zero. It was also expected that naming hundredths would produce a lower performance than naming tenths. *Item 3* was included to determine whether students used the semantic understanding of the places or whether they relied on the syntactic "left to right" order in which the digits of a number are usually written.

With respect to the seriation tasks, *Item 4* required the students to record the number that was 1 tenth more than the given number and, as such, could be considered to be a place value item because the students had to identify the place to change first. The subitems included prototypic and nonprototypic cases. Prototypic cases included 9.4 where the digit to change first is in the rightmost place and does not involve grouping, and 2.9 where the digit to change first is in the rightmost place and does involve grouping. Nonprototypic cases included numbers where the digit to change first is not in the rightmost place either because it is an internal place (4.07, which requires no grouping; 1.94, which requires grouping) or because the given number was a whole number (5).

Item 5 was similar to Item 4 except that students were required to add 1 hundredth to the given numbers. With respect to the prototypic examples (2.76, 0.09, 4.91), although they are structurally similar in that hundredths are provided and therefore 1 can be added to the rightmost place, they were deliberately selected to identify syntactic learners. For example, does a student see a "9" and automatically elicit the regrouping process (probably because of having been exposed to prototypic examples)? The nonprototypic examples (3.9, 6) were expected to produce 4/4.0/4.00 and 7 or 6.1 respectively.

The seriation items reported on in this paper were based on Jones et al.'s (1996) lowest component (counting) in their place value framework whilst the subitems required an application of Steffe's (1986) simplest notions of unit, namely, counting/singleton units and composite units (grouping).

Results

Table 1 shows that the means per item were quite poor across both trials. With respect to Items 1,2, and 3, the results indicate that the students did not have the prerequisite baseline place value knowledge (i.e., the names and order of the decimal-fraction places) that will enable them to process decimal numbers with understanding. Thus, not having baseline knowledge of the place names, it is then not surprising that the students were unable to add 1 tenth and 1 hundredth to the given numbers (Items 4 and 5 respectively). Table 1 also shows that, for the place value items, students generally performed much better on Item 2 than on Item 1, both of which were assessing knowledge of place names/position. (See Discussion.)

Table 1: ***Means (%) for All Items for Individual Trials and Overall Study***

	Items 1-5 Means				
Study	Item 1	Item 2	Item 3	Item 4	Item 5
Trial 2 (n = 173)	66.8	79.4	61.3	56.4	56.0
Trial 1 (n = 156)	55.6	71.9	61.5	50.4	53.1
All (N = 329)	**61.5**	**75.8**	**61.4**	**53.6**	**54.6**

Table 2 shows that, within Item 1, the subitems relating to tenths generated markedly better performances in Trial 3 than those related to hundredths (80.3, 79.8 as opposed to 56.1, 50.9 respectively).

Table 2: ***Means (%) for the Subitems in Items 1 and 2 for the Trials and Overall Study***

	Items 1 and 2 responses											
	Item 1 (write in words)					Item 2 (write in digits)						
Study	4.7	6.39	0.8	5.02	**M**	9.5	0.6	4.13	0.60	7.01	0.03	**M**
Trial 2 (n = 173)	80.3	56.1	79.8	50.9	**66.8**	85.0	85.0	83.2	83.2	71.7	68.2	**79.4**
Trial 1 (n = 156)	68.6	50.0	50.6	53.2	**55.6**	85.3	77.6	80.8	71.8	57.1	59.0	**71.9**
All (N = 329)	**74.8**	**53.2**	**66.0**	**52.0**	**61.5**	**85.1**	**81.5**	**82.1**	**77.8**	**64.7**	**63.8**	**75.8**

In Trials 1 and 2, the tenth item with zero, 0.8, generated much lower performances than the tenth item with no zero (4.7). With respect to Item 2, Table 2 shows that, generally, performance decreased markedly over the last three subitems, each of

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which was classified as nonprototypic because they included a zero, particularly the last two which required an internal zero.

With respect to Items 4 and 5, Table 3 shows that the students performed best on the prototypic examples, namely, adding 1 tenth to 9.4 and 1 hundredth to 2.76 (where 1 could be added to the rightmost digit). Although Item 4b (2.9) and Item 5d were also prototypic in this regard, they had the extra distraction of "9" which either didn't evoke the odometer principle in Item 4b or erroneously evoked the principle in Item 5d.

Table 3: ***Means (%) for Items 4 and 5 for the Trials and Overall Study***

	Items 4 and 5 responses											
	Item 4 (add 1 tenth)						Item 5 (add 1 hundredth)					
Study	9.4	2.9	4.07	1.94	5	**M**	2.76	3.9	0.09	4.91	6	**M**
Trial 2 (n = 173)	81.5	64.2	50.9	38.2	47.4	**56.4**	61.8	47.4	60.7	59.0	50.9	**56.0**
Trial 1 (n = 156)	74.4	58.3	42.9	43.6	32.7	**50.4**	64.1	60.9	68.6	31.4	40.4	**53.1**
All (N = 329)	**78.1**	**61.4**	**47.1**	**40.7**	**40.4**	**53.6**	**62.9**	**53.8**	**64.4**	**45.9**	**45.9**	**54.6**

The students' responses were analysed for error patterns and very few consistent errors were made by the same students across the range of subitems within an item, indicating that these students' responses were contingent on the "look" of the number. Generally, individual error patterns were discerned in Item 1. For example, if a student wrote "four point (or decimal/dot) seven" for Item1a, s/he invariably wrote the remaining three numbers using the same syntax. However, across the 337 students, some errors occurred frequently. For example in Item 1, the prototypic examples (4.7, 6.39) produced an error rate of 22.8% and 46.8% respectively. The errors could be classified as "syntactic reading" (e.g., four point seven) or misidentification of places (e.g., 4 tenths or 4 tenths and 7 hundredths). Syntactic reading accounted for 14.9% of the errors whilst misidentification accounted for 11.2% of the errors. For Items 4 and 5, students' errors on all subitems could be categorised as syntactic responses to overpractised tasks. That is, students either added 1 to the rightmost place or added 1 to the ones place irrespective of the number given. For example, when adding 1 tenth to 4.06 (4c), the most common responses were 4.07 (30.4%) and 5.06 (6.1%).

Furthermore, each subitem across the five tasks generated at least 20 different types of errors. The disparity between the number of error types within each item and the percentage of most common errors revealed the idiosyncratic nature of the responses as the students strived to make sense of the tasks. For example, in Item 2c, students gave 23 different erroneous responses but the most common error (4.013) occurred in only 2.7% of the 17.1% of these responses.

CONCLUSIONS

The results on all of these elementary place value and seriation tasks are of concern considering the length of school time devoted to developing students' understanding of decimal numbers. Almost 40% of these students could not read or write decimal numbers to hundredths, and approximately 50% could not add 1 tenth or 1 hundredth to a variety of decimal numbers. Without adequate foundational knowledge, students cannot develop the facility for numbers (i.e., number sense) that is the focus of current mathematics syllabi (e.g., NCTM, 2000).

These students' lack of ability to read and write decimal numbers can be explained by the "short-cut" method of reading fractions (e.g., "four point seven" rather than "four and seven tenths") that predominates in Queensland schools. Hence, students are not practised in reading decimal numbers so that the name of the fractional part is heard. This practice is also prevalent in reading common fractions, for example, "two over three" rather than "two thirds". Baturo and Cooper (1999) suggested that this form of syntactic reading in the early years of learning is detrimental to students' ability to process fractions with understanding. Moreover, the poor performances on such a low-level task as Item 3 (61.4%) suggest that the students have come to rely too heavily on the "left to right" syntactic rule when recording numbers (as indicated by their responses to Item 1, namely, 61.5%).

The seriation items evoked low performances, even on the level 1 prototypic tasks (4a–78.1%; 5a–62.9%) suggesting that adding 1 to a decimal place was, in fact, a nonprototypic task for these students. The most common error responses, namely, to add 1 to the ones place indicate that, for these students, seriation tasks had been limited to whole numbers. Table 3 revealed that when the difficulty level was increased, the results decreased. For example, the generally poor results for the Level 2 grouping items (4b, 4d, 5c) indicate that grouping is not a simple process for young students. The results of both of these seriation tasks indicate the fragile nature of students' understanding of place value and seriation. Students cannot seriate without good place value knowledge and are limited if they cannot employ the odometer principle because of lack of understanding of grouping.

With respect to teaching numeration processes, Ross (1990) claimed that children learn to represent numbers with concrete manipulatives (as practised in Queensland schools) through following the teacher's directions rather than from thinking about what they have constructed. The results of this study suggest that teachers need to be more creative in the types and levels of examples they provide to ensure that students have the robustness and flexibility of knowledge that is required for number sense.

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THE INFLUENCE OF SUBJECT MATTER ON TEACHERS' PRACTICES: TWO CASE STUDIES.

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Abstract

In this article, the general principles and practices adopted as guidelines to a graduate course for secondary Mathematics teachers are discussed. We argue that once the curriculum is designed both to complement the teachers' subject content knowledge and to provide activities promoting links between their conceptual knowledge and their actual practices, it can have a strong positive influence in teachers' subsequent practices. Two teachers who completed the course were subsequently followed during a two year period. They were interviewed and their practices were observed. We analyse their reflections on the main differences in their practices before and after attending the course.

Introduction

The need to create opportunities for regular in-service training courses for secondary Mathematics teachers in our geographical area led the team in charge of the undergraduate teacher-training course in our University (the oldest established in Brazil) to create a two year graduate course. The basic principles underlining the present versions of the graduate and the undergraduate courses is that the sum of solid mathematical formation with basic pedagogical and didactical knowledge is essential in training teachers, but it cannot be considered enough. Both courses should also include activities especially planned to allow the trainees to reflect upon the importance for their classroom practices of establishing connections between their own learning experiences in Mathematics at an advanced level and their practice as secondary Mathematics teachers.

Specific disciplines aiming at establishing these connections have been included in our undergraduate teacher-training curriculum few years ago. The experience on writing materials and developing one of these disciplines (Algebra and Arithmetic) is reported in Belfort, Guimarães and Barbastefano (2001a). This kind of experience has also reflected on the development of the curriculum for the graduate course.

We discuss in this article the general principles and practices adopted when designing the curriculum for the graduate course, and their consequences for the teachers' subsequent practices. The main aims of the course are to complement the subject content knowledge preparation of the teachers and to associate this knowledge with a series of activities specially prepared to "close the gap"* between the Mathematics they are learning and the Mathematics they teach as secondary teachers.

In order to obtain data on how these experiences changed the practices of the teachers involved in the program, two teachers who completed the course were followed during a two year period. During this time interval, we had plenty of opportunities to observe their practices and to interview them. We discuss here their reflections on the differences in their practices before and after attending the course.

Background

There were many reasons leading us to conclude that it was necessary to include experiences specially designed to highlight the connections between the mathematical disciplines developed for the trainees in the program and the subject matter to be taught at secondary schools. There are indicators from research that a solid subject content knowledge may be essential for a successful teacher (for examples, see Ball and Feiman-Nemser, 1988; Grossman et al., 1989; Ball, 1991; Leinhardt et al., 1991). Nevertheless, Ball and McDiarmid had already taken one step further: they began to investigate the special qualities prospective teachers' actual subject content knowledge might lack in order to foster good classroom practices. Ball (1988) stated that "...[secondary teacher candidates' additional studies] do not seem to afford them substantial advantage in explaining and connecting underlying concepts, principles, and meanings" (pg. 24), while McDiarmid (1992) argued that "elementary and secondary teachers frequently lack connected, conceptual understandings of the subject matters they are expected to teach" (pg. 1).

Ma (1999) discusses the importance of the "*profound understanding of fundamental mathematics* (PUFM)" demonstrated by some elementary teachers. She defines this concept as "an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough" (pg. 120). As a consequence of PUFM, some elementary teachers "do not invent connections between and among mathematical ideas, but reveal and represent them in terms of mathematics teaching and learning" (pg. 122). According to her, the work of these teachers displays the following characteristics: connectedness, multiple perspectives, awareness of the basic ideas, and longitudinal coherence.

Our practice in teaching subject matter in undergraduate and graduate courses directed at secondary Mathematics teachers suggests that the difficulties our trainees face with subject content are of a similar nature (Belfort, Guimarães & Barbastefano, 2001a, 2001 b). If we add our experiences to other researchers' descriptions, we can

* The notion of 'gaps' between different learning stages can be found in Hart (1987), where she discusses the use of concrete materials in elementary education.

at least conjecture that this may be a widespread problem.

If we attempt to transpose Ma's ideas to the situation of the study of Mathematics at secondary level, we are faced with the twofold connection of this concepts: to the previous knowledge of elementary Mathematics the students would have and, in the other hand, to the mathematical structures studied in their undergraduate course. Issues such as Cognitive Units (Barnard & Tall, 2001) and Flexibility (Gray & Tall, 1991) also have to be taken into consideration, as explanations for and justifications of some of the mathematical procedures studied at secondary school may be provided only if one resorts to knowledge of structures and properties studied in advanced Mathematics (a simple example: several of the divisibility criteria often taught in the initial years of secondary school depend for their justification on the properties of the integer numbers as an integrity domain, and of the general properties of congruence in this set).

So it seems that, if the teaching of secondary Mathematics is to display characteristics of connectedness, multiple perspectives, awareness of the basic ideas and longitudinal coherence, the teacher would have not only to know Mathematics at an advanced level, but also to be able to reflect on its connections with the contents of secondary school. Such were the premises guiding the design of the curriculum for the graduate course.

Curriculum Guidelines

Several teachers enrolled in the graduate course were previously involved in a summer in-service course by our Institute. So, when designing the curriculum, we considered not only the research-based reflections mentioned above but also the difficulties these teachers seemed to face when dealing with secondary mathematics. Our experiences with these teachers had confirmed the lack of connected and conceptual understandings of the subject matters commented by McDiarmid (1992). These teachers also demonstrated different levels of mathematical reasoning (for example: inductive reasoning, incomplete and/or misleading arguments were found among their justifications to mathematical secondary results).

The curriculum included disciplines to allow the teachers to reflect upon results coming from Mathematics Education research and pedagogical matters. A discipline aimed at fostering the use of technology for teaching Mathematics was also included. But the majority of the disciplines offered these teachers opportunities to review, from an advanced standpoint, the contents they have to teach (algebra, geometry, number structures, calculus and statistics). Based on the principles discussed above, the development of all disciplines included activities designed to allow teachers to make connections between their learning experiences and their practices. We can summarise the general guidelines for developing the courses as:

- we offer the teachers several opportunities to work with (and reflect upon) the basic properties of the mathematical objects;
- we punctuate the courses with commentaries on mathematical reasoning and justification: what it means to show that a result is false (use of counter-examples) and different methods of proof;
- formal proofs are often accompanied by a previous discussion of the need for it and of the strategy in searching for it;
- we encourage the teachers in establishing connections between the topic under study and examples of actual practice in the secondary school classroom;
- every mathematical concept is correctly defined, and possible difficulties in its construction by secondary students are often discussed;
- we furnish ample opportunity for the critical analysis of texts and questions found in Brazilian textbooks;
- we present several examples of wrong and/or incomplete problem solutions to be analysed; and
- we often ask the teachers to present more than one solution to a problem, as a way to encourage flexible and consistent mathematical reasoning.

We expected the course to have a strong positive influence in teachers' future practices. In order to verify this hypothesis, it was necessary to follow some of them in the subsequent years. In fact, it was not a difficult task to select a sample for the case studies. A partnership was established with many of the teachers engaged in the graduate course. They keep coming back to discuss their practices, to ask for comments on their work, to use the library or simply to have a chat.

Teachers' Reflections on their Practices

We selected two teachers, with very different backgrounds, to be followed during a two year period after finishing the course. Their practices were observed and they were interviewed several times. The main aims of the interviews were to ask these teachers the reasons behind their teaching decisions and to verify how these differed from the ones they used to make before attending the course. Observation was used as a mean to verify the conformity between discourse and practice.

Ada is a very experienced and enthusiastic secondary school teacher, working in schools in Rio de Janeiro for the last 20 years. She had a secondary level training as a primary teacher and an undergraduate teacher training course for secondary Mathematics in a private college in Rio de Janeiro. Even though she had been enrolled in several in-service courses before joining our course, she had to work hard to overcome several difficulties with subject content matters during the graduate course.

Ricardo is also an experienced teacher, working in secondary schools in smallish town near Rio de Janeiro for 12 years. Ricardo did his teacher training course in a private college at his home town. He also had been enrolled in several in-service courses, and joined the one we offered in the summer of 1997. The year after that, he enrolled for the graduate course. He commuted to and from his home town every week for the duration of the course.

There were several changes in their careers in the last two years, and both declared them to be related to the course. Ada still works at a state school in Rio, as she has been doing for almost 20 years. She works in adult education during the evening period in this school. She got a new job in a highly respected experimental school in Rio, and became Head of the Mathematics Department there. She is also in charge of introducing the use of computers in Mathematics lessons. Apart from that, she is working as a trainer in in-service courses for primary teachers, and she was recently invited to lecture didactics of mathematics for prospective primary teachers at a private college in Rio. Ricardo also has changed jobs in his home town. He is now working in one the most prestigious private schools in the city and became the Pedagogical Head of this institution. He is in charge of preparing secondary students for the universities' entry exams and he is also training the school's primary teachers at Mathematics.

Both teachers report several changes in their practices. For instance, Ricardo has changed the school adopted Mathematics textbook. He says:

"The old textbook had many exercises to train the kids, but did not teach them mathematical concepts... and most teachers usually only teach what is in the book. I know that because I used to do it myself (laughs). I used the textbook as my only source of study. I usually prepared all the exercises beforehand and, whenever I could not do one of the exercises, I took it out of the list I proposed to the kids. I never discussed solutions of the exercises with other teachers. I guess I felt too insecure to do that ... anyway ... we had a meeting to discuss the textbook. It was not easy to convince the other teachers to change the book ... probably because they were used to the old one, and they did not feel secure adopting the new one. It took me a while to convince them that we would be working as a team, and that we would be discussing concepts and exercises. I guess they finally believed me because I was already bothering them during the breaks to help me with difficult exercises (laughs)."

Ricardo's school accepts students for the last three years in secondary school. Ricardo says that the group of students joining the school this year came from different backgrounds, and most of them demonstrated gaps in their previous mathematical preparation. He has implanted a "Saturday morning of work". He is very enthusiastic about this experiment:

"All pupils in the group are volunteers ... well, sort of! (laughs). In fact, I asked them to solve selected exercises during an interview. It was easy to convince them that it would be a good idea to take time to review some of the concepts they were expected to know at this level. I have been planning these activities carefully, using different sources of materials, because I want the kids to enjoy each lesson and to come back for more the following week. Lots of games and competitions - they work in small groups most of the time. Whenever they finish an activity, they tell the others what they did. It is very noisy, but it is amazing to see this kids discussing their different solutions. This is also the moment of the lesson in which I reinforce the main results and generalise the mathematical concepts. The experience is still at its very beginning, but I believe that these kids would do well in their regular Maths lessons ... and they keep telling the others that the "Saturday lessons" are great fun."

Ada not only changed the textbook she adopted but also she radically changed the way she uses the book. She usually gives the students a series of activities she prepared herself before she asks them to read and to do the exercises in the book.

"This is a much better textbook than the one I used to adopt. Nevertheless, the pupils do not enjoy using it all the time ... also, I believe that different experiences are necessary to promote mathematical learning. So, I usually get them started with a group activity. I use different sources to prepare these tasks, so they would have a great variety of experiences. This strategy also keeps them interested and it gives me the chance to complement the viewpoint presented in the book ... it is very helpful for the students."

Ada declares that the course was a turning point in her career. She says:

"I had been enrolled in several in-service courses before, and I was almost giving up! It was a terrible experience to listen to the trainers statements: you should do this and you should construct that!... *But I did not know how to do it and they most certainly did not teach me. I realise now that my mathematical knowledge was mostly instrumental, and I had little conceptual understanding. So how could I possible teach to promote it? ... I even had difficulties understanding the textbook I adopt now, and several activities proposed there were meaningless to me ... so, I avoided it - and every other one of the kind. After these in-service courses I kept doing instrumental teaching, relying on a bad textbook - and feeling very uncomfortable."*

Ada's comments about her activities as a primary teacher trainer also show the importance she is now attributing to conceptual understanding:

"As a trainer in in-service courses I realised that most primary teachers could not justify their work... they know how *to do it, but they don't know* why *they*

do it. I guess this is probably the way they were taught - and it is also the way they teach the kids! Mathematics has become a series of meaningless rules for them. On the other hand, it is interesting to observe their reactions to a series of activities aimed at constructing the concepts instead of instrumental Mathematics or Pedagogical discussions detached from content matters. Once they overcome their initial surprise, it seems that they start learning ... at least, they work hard and enjoy the experience! It is gratifying to see how quickly they adapt most of the activities of the course into activities for their pupils. Lots of good ideas discussed in the class! It feels like starting a machine: you turn on a button, and it goes on and on ... but then again, I believe it was the same process with me! (laughs)"

Comments and Reflections

Cooney (1988) described teaching as "an interactive process", in which conscious decision making is needed not only during the planning of the lessons but also "on the stage". Models to explain the role of the teacher usually consider the interaction between teacher's knowledge and beliefs as the basis for their decisions (see, for instance, Fennema et al., 1989). The data from the case studies described above point to the strong influence of subject content knowledge in this process.

The case studies suggest that there were substantive changes in the ways these teachers work. Both are now in charge of special programs of study in their schools. Both have changed not only the adopted textbooks but also the ways they use this resource. They are designing complementary materials, which suggests they are much more confident about making decisions about their pupils' learning experiences. As a consequence, the textbook is used as a helpful source of reading and activities, but it is no more determinant of the curriculum developed in their classrooms.

Their reflections on their work before attending the course are also notable. Because of the relationship developed during the course, they are very open to discuss their previous difficulties. Ricardo made clear that his previous difficulties were not related to class management or pedagogical issues, but were due to a lack of content knowledge. Similarly, Ada was almost giving up on in-service courses. Even though she was not able to make these reflections at the time, it seems that she felt uneasy about learning methodological issues she could not apply, mainly because they should be associated to a conceptual content knowledge she did not possess.

We can say that our main goal was to convince these teachers that subject content matter knowledge is at the very root of their jobs - and that it is also the most powerful of all tools they can apply as mathematics teachers. To use it effectively, it is necessary to develop a deep understanding of the mathematics they teach, making it more flexible and establishing connections among different mathematical topics. We hope to have made them aware that this is a never-ending job.

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Mathematical and Pedagogical Knowledge of Pre-and In-Service Elementary Teachers Before and After Experience In Proportional Reasoning Activities

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Abstract - *In this research, we investigated the cognitive domain of proportional reasoning mathematical knowledge of pre-and in-service elementary mathematics teachers – including its three components: the intuitive' the algorithmic and the formal. This investigation was carried out by assessment of change in participants' procedures while they were presenting possible solutions to authentic problems and revealing their reasoning in dealing with ratio and proportion problems. The affective/behavioral domain was also investigated by measuring change in the attitudes toward teaching proportional reasoning following exposure to and experience of investigative proportional reasoning activities.*

Theoretical Background

Recent research findings all over the world have indicated many gaps in the content knowledge of pre-and in-service teachers in mathematical subjects taught in the elementary school, including the topics of ratio and proportion. Frequently, existing knowledge is technical, schematic, unconnected and incoherent and, as a result, difficulties arise which are evidence of the pre-service teachers' lack of understanding of mathematical concepts and feeling that they are incapable both, coping with the material and teaching it. This is usually based upon a negative attitude toward the specific material (Keret, 1999; Fischbein, Jehiam and Cohen, 1994; Tirosh and Graeber, 1990).

A review of the literature about proportional reasoning by Tourniaire and Pulos (1985) indicates a number of factors relating to the context and number structure of proportional reasoning problems that are responsible for much of the variability and difficulties in the performance of both students and teachers. The sources of the difficulties are mainly cognitive and they are consequences of the operational scheme's second order status – meaning that to understand proportion one needs to make a comparison between two rates, each of which needs to be able to coordinate between two variables (Inhelder & Piaget, 1958).

Findings of many research studies indicate that appropriate practice, which includes realistic contextual problems that are interesting and challenging and which encourage students to construct their own conceptual and procedural knowledge, can help them build their proportional scheme (Harel and Confrey, 1994). The American Connected Mathematics Project (CMP) has developed learning material which incorporate these ideas for grades 6-8. The findings indicate that such students are capable of developing their own repertoire of sense-making tools to help them produce solutions and explanations (Ben-Chaim et al., 1996a). Findings of research studies with pre-service mathematics teachers indicate similar results (Keret, 1999).

The conclusion reached from these research studies is that, in order to improve the situation, we need to implement the research findings in teaching ratio and proportion topics in the elementary school and, in parallel, to deal with these topics differently in in-service teachers education. The in-service training should include three central components: the cognitive component which includes the mathematical content knowledge and the pedagogical-didactical knowledge; the affective component which includes attitudes, beliefs and feelings of being capable (that may prevent or encourage the readiness to teach the subject); and the component of teacher behavior which one can see in the teacher's readiness to cope with constructing teaching units and planning the teaching of the subject. In this research, after exposing the pre-and in-service elementary mathematics teachers to investigative and authentic activities in ratio and proportion, we followed the way these teachers changed in regard to the above three components.

Methodology

Sample: 49 pre-service teachers from Israeli teachers colleges as part of their training to teach mathematics in elementary schools and 14 in-service mathematics teachers in elementary schools. The study was conducted during the academic year 2000/2001.

Research Instruments: 1. A proportional reasoning questionnaire that was administered twice: first at the outset of the course and second after gaining experience with investigative activities (19 activities in 10-12 teaching sessions). There were 3 versions of the questionnaires each consisted four parts:

Five rate and density problems. Problems 1&2 deal with unit price- where problem 1 needs numerical comparison and problem 2 has missing value. Problems 3&4 deal with proportional relations between distance, time and velocity – both using numerical comparison and the main difference between them is in their numerical structure. One problem includes only integers and the other includes fractions and decimals to present the factor of time. The fifth problem deals with information of population density, also of numerical comparison type, but with relatively larger numbers. In all numerical comparison problems, the ratios are not equal and in this case the problems are generally considered to be more difficult compared to those with equal ratios (Karplus et al., 1983a, b).

Five ratio problems. The first two problems deal with finding the ratios between the sizes of two groups of children. There is no need to solve the problems, but to specify as many different ways as possible to find the ratio between the sizes of the groups. The responses for these problems will be analyzed in a separate section. The third problem deals with comparison between two ratios. The fourth problem deals with the comparison made between different types of ratio representation. In the fifth problem there is a need to find the whole value by using a given ratio.

Five scaling problems. All five problems relate to finding a ratio and its application after enlarging or reducing the dimensions of pictures. Each problem refers to a different aspect; the first one deals with finding the scaling factor; the second one deals with comparing two ratios; the third one is of a missing value type; in the fourth one there is a need to relate to a quadratic enlargement (areas); and the fifth problem relates to a situation with 2 consecutive enlargements.

Six exercises that include work with fractions. This part consists of six exercises with plain numbers/fractions/decimals similar to those included in the rate, ratio and scaling problems.

The questionnaires are based on versions administered to large populations in U.S.A. and that were translated into Hebrew (Ben-Chaim et al., 1998) with some correspondence of units, and local situations.

2. Attitude questionnaire. The attitude questionnaire includes 22 items and 3 open questions (more details are given in the results section).

3. Authentic investigative proportional reasoning activities. Nineteen activities were developed in order to assess the influence of exposing pre-service teachers to authentic investigative proportional reasoning activities. The development of the activities was based on research findings from around the world (Ben-Chaim et al., 1998; Keret, 1999). The activities were conducted in one of the pre-service mathematics education courses in each of the two participating teachers colleges.

4. Observations. These were aimed at the formative evaluation of the activities created and at a follow up of the learners' procedures.

5. Personal Interviews. These aimed at in-depth and comprehensive examinations of the questionnaire's findings. For this purpose, a representative sample of 14 participants was interviewed. The interviews served as a validation of the findings achieved from the observations and the questionnaires.

Results and Discussion

The layout of the research is one group pre-test – post-test. The questionnaires were administered before and after conducting the ratio and proportional reasoning activities. A special rating form was created to analyze and score the students' responses. Three major response categories were identified and each of them was accordingly subdivided into subcategories (see Table 1).

As indicated before, six exercises in fractions were included to examine if the students had difficulties in comparing fractions and/or acting with naked numbers/fractions similar to those that appear within the verbal problems. In this part, there were no differences found between pre and post testing results. The students had no special difficulties in solving the fractions exercises.

The following report relates to the performance of the pre-service teachers on 15 problems, before and after exposing them to a process of solving authentic problems in ratio and proportion.

It should be noted that the assessment tasks are different from those that appear in standard tests. They stem from familiar situations such as: buying soft drinks, riding bicycles, population density, bussing to school and a visit to the photo shop.

Moreover, the problems and the situations are different from those presented within the investigative activities during the teaching sessions.

Table 1: Overall Pre-Post Results on 13 Problems (5 Rate, 3 Ratio, 5 Scaling) N = 32 Pre-service Elementary Mathematics Teachers

		Correct answer			Incorrect answer			Blank
		only the correct answer	correct support work	incorrect support work	only incorrect answer	correct thinking but wrong conclusion	correct support work	
Rate	Pre	1	56	3	2	11	8	19
	post	-	86	3	-	9	1	1
Ratio	Pre	-	38	10	-	6	7	39
	post	1	81	6	-	10	-	2
Scaling	Pre	1	51	3	-	8	18	19
	post	1	70	3	-	15	8	3

* **All of the numbers in the table are percentages.**

Table 1 presents the overall results of the pre-service teachers' performance, before and after exposure to the authentic activities. It can be seen that, before exposure to the activities, 56% (Rate), 38% (Ratio) and 51% (Scaling) of the pre-service teachers correctly responded with correct support work. After the exposure to the activities, the performance of the students improved dramatically, to 86% (Rate), 81% (Ratio) and 70% (Scaling) of the pre-service teachers answering correctly with full support work. Furthermore before the experience with the proportional reasoning activities, 27% (Rate), 46% (Ratio) and 37% (Scaling) of the pre-service teachers did not attempt the problems or responded incorrectly using incorrect thinking. In contrast, the percentages after the exposure to the activities were only 2% (Rate), 2% (Ratio) and 11% (Scaling). The majority of the participants attempted to answer the problems with support work, both before and after the exposure to the activities. If we ignore the subcategories "only correct answer" and "only incorrect answer" and the category "Blank" then before the exposure: 78%, 61% and 80% of the participants and after the exposure: 99%, 97% and 96% of the participants provided support work for their answers. Hence, when examinees are asked to "show your work", or "how do you know", or "explain" they do add written support for their answers. Yet, the quality of writing is important. Analysis of the written support work clearly showed that after the exposure – during which the participants were guided to explain and discuss their reasoning and ideas, the arguments were more detailed, clearer and of a better quality.

The subcategory of "correct answer with incorrect support work" showed up before exposure to the activities 3% (Rate), 10% (Ratio) and 3% (Scaling). Thus, on a typical mathematics exam that does not ask for justification of answers, such examinees would be marked correct and their misconceptions would be unrecognized and thus uncorrected. On the other hand, within the subcategory of "incorrect answer with partial understanding/thinking" there were 11% (Rate), 6% (Ratio) and 5% (Scaling) before the exposure, and 9% (Rate), 10% (Ratio) and 10% (Scaling) after the exposure. This subcategory of response often occurred when the problem called for calculations followed by reasoning, as is the case for most of the problems presented. The examinees can think correctly, but draw the calculations incorrectly, or they might get the correct number and information, but draw an incorrect conclusion about measurement units. It can be assumed for students in this subcategory that they are beginning to understand the content, but their understanding is still shaky. Certainly in this case, complementary teaching/learning is needed. All of the above is true and characteristic for each of the problems.

As indicated for the first two ratio problems, the participants were asked to specify different ways to present ratios and compare between them. The results show, that, before the exposure 32% (prob.1) and 50% (prob.2) of the pre-service teachers did not succeed in finding even one way to present ratios for problems 1 and 2 compared to only 3% (prob. 1) and 3% (prob. 2) after the exposure. Before the exposure, 68% (prob. 1) and 50% (prob. 2) of the examinees successfully solved problems 1 and 2 in at least one way; after the exposure there was a visible gain in success, while 97% of the participants correctly solved the problems using at least one way with the majority of them using more than two different ways.

Regarding scaling problems, the findings indicate that the pre-service mathematics teachers had difficulties especially in solving problem # 4 in which they were required to deal with quadratic enlargement – area enlargement. Before and after the exposure to the activities, the rate of success was 40%. Obviously, the quadratic enlargement here was not assimilated successfully. This is in agreement with findings of previous studies regarding pre-service teachers (Keret, 1998), as well as elementary school students (Ben-Chaim et al., 1996b). In contrast, the performance on the fifth scaling problem was much better one – 65.5% before the exposure and 80% after. In this problem, the examinees were asked to relate to double enlargement. This problem was presented in two versions: one version included a figure of segments (no numbers) and the problem was to assess the overall enlargement; the second version included a description of a machine that enlarges double and, then triples the size and the problem was to find the resulting enlargement. In both versions the rate of success was almost the same.

As indicated before, the attitudes toward the topic of ratio and proportion were also evaluated in order to detect changes that took place between pre and post exposure to the activities. The assessment was conducted by a Likert attitude questionnaire with a scale of 1-5. The questionnaire included items dealing with attitudes toward: mathematics teaching in general, confidence in the ability to deal with ratio and proportion, difficulties in teaching ratio and proportion, and the importance of teaching ratio and proportion.

Table 2 indicates that pre-service teachers' attitudes toward teaching mathematics in general, before the exposure to activities, are quite positive - they like and enjoy teaching mathematics. Following the exposure, they were even more positive. The standard deviation measure of post testing indicated that all the participants enjoyed teaching mathematics. Another finding shows that the pre-service teachers are aware of the need for mathematics teacher to have much more content knowledge than the material taught in school. This finding is supported by the interviews, as one of the interviewees noted: " I would like to go deeper, there are additional subjects to learn, such as inverse ratio; in my opinion, teachers should know much more than pupils in order to teach". A positive change occurred regarding the feeling of being capable of dealing with ratio and proportion. After the exposure the participants felt that they were more confident in their capability to teach the subject and felt capable of planning and creating relevant mathematics activities.

Table 2: Summary of Attitudes Toward Ratio and Proportion: Scale (1-5)

# Of Items		Mean Before Exposure N = 49	Mean After Exposure N = 12
4	**Items relating to attitudes toward teaching mathematics in general**	4.127	4.777
7	**Items relating to confidence in ability to deal with ratio and proportion**	2.948	3.746
3	**Items relating to attitudes toward difficulties in teaching ratio and proportion**	2.993	3.666
6	**Items relating to attitudes toward the importance of teaching ratio and proportion**	3.691	4.241

To those who were exposed to the activities, teaching ratio and proportion seemed to be more complicated than before the exposure. The interviews strengthened this finding, for example: " I thought that the topic of ratio and proportion was easy and that I knew how to teach it, but today after I have learned, I realize that it is very complicated and that I still lack of knowledge"; or another quotation: "It exposed me to a topic about which I did not know how much I don't know and how unready I am to teach it".

Another change that occurred following the exposure to ratio and proportion activities is connected to the need to include ratio and proportion as part of pre- and in-service teacher training. Before the exposure, not all the examinees thought it was very important to include a course related to ratio and proportion; however, after the exposure, almost all the participants thought that it was very important to include it. As one of the interviewees said: "It is very important to teach this topic of ratio and proportion in college since it is the A B C of mathematics. If you are not familiar with this topic, it is impossible to teach the other topics". These findings indicate a significant improvement in the participants' attitudes toward the overall components and aspects of ratio and proportion topics. The observations during teaching the activities and the interviews conducted after the teaching, clearly indicated that the teaching style caused the participants to construct their understanding of the concepts of ratio and proportion through an enjoyable and efficient process; as one said: "Before, I learned the topic by formula and I could never remember which formula to use. This time the inquiry enabled me to develop it in my own way regarding the topics of ratio and proportion ...". Another one related to the authenticity of the activities: "The topic of the presented problems was new to me and topic that I did not like, but it brought it closer to me. I especially liked the recipes enlargement and reduction of a drawing, and the cutest of all was solving the detective's puzzle – such activities raise curiosity and after you solve them, you suddenly know that you have advanced in learning ratios. It is wonderful". Another one related to teaching in the elementary school: "The activities are close to our world and it is possible to also fit them into the world of children and I am sure that they will also enjoy solving them".

Conclusions

The findings of this study indicate several crucial points:

1. Authentic activities are essential for teaching the topic of ratio and proportion in pre-and in-service elementary teacher education.
2. The type of activities developed and used in this study were found to be suitable, effective and contributory to pre-and in-service teachers for understanding the topics of ratio and proportion.
3. After exposure to the investigative ratio and proportion activities, the pre-and in-service teachers were more successful in solving the problems presented to them in the knowledge questionnaire. It is important to note that the problems and situations included in the questionnaires were different from those presented during the teaching cycle.
4. The majority of the students provided support work and explanations for their solutions before and after the teaching. Thus, when examinees are asked to add arguments and explanations most of them generally meet the requirement.
5. After exposure to the activities, the participants succeeded in finding more methods to solve the problems and their explanations for their methods were of a better quality.
6. The request to add support work is justified, since a correct answer with explanations, frequently identifies an examinee who has a lack of understanding and who would otherwise be considered to be successful. Furthermore, an incorrect answer accompanied by support work can reveal an examinee that already has partial or unstable understanding and just needs complementary tutoring.
7. The pre-and in-service teachers do not have difficulties in solving exercises with plain numbers/fractions and decimals.

8. Following learning ratio and proportion through the authentic activities, the pre- and in-service teachers improved their attitudes toward mathematics in general and all the components and aspects of ratio and proportion.

In summary, the concepts of ratio and proportion are basic concepts in mathematics. In addition they are important for other disciplines in which different phenomena are described as ratio and proportions. Conceptualization of the ratio and proportion concepts and the ability to implement them in problem solving arising from different disciplines can assist learners to construct their mathematical knowledge. More important however is that it will lead to the development of the ability to apply the proportion scheme – meaning proportional reasoning, something that is vital for developing analytical and mathematical thinking.

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HOW TO PROMOTE CHILDREN'S UNDERSTANDING OF FRACTIONS? AN EXPLORATORY STUDY

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This study involves a teaching experiment aimed to improve children's understanding of fractions. Fifty seven children (8-10 years old) were divided into three groups: Experimental and Control Groups: 3rd graders with no previous instruction in fractions; and Reference Group: 4th graders who were taught in a traditional way based on a mechanistic approach to fractions and focused on application of rules. Children in the Experimental Group were given a teaching experience based on problem solving situations and on activities that enabled them to think and discuss their thought processes when solving problems involving fractions. All participants were given a pre and a post-test. The Experimental Group showed a more expressive improvement than the other groups. The results are discussed in terms of the nature of instruction in fractions understanding.

Introduction

This study is based on conceptual field theory in which the fraction is considered one out many other concepts that takes part of multiplicative conceptual field, or simply multiplicative structures, such as multiplication, division, proportion, linear function, vectorial space. According Vergnaud (1983, 1988, 1997) multiplicative conceptual field is a set of situations that requires a diversity of concepts, actions and symbolic representations consistently linked to each other.

When dealing with fractions, for instance, children have to deal with many other concepts and with a large range of situations involving part-whole relations; equal or equivalent units; exhaustive division/sharing (that is, nothing remains); fair sharing; and equality or equivalence of shares. Also, they have to deal with the fractional language and its mathematical representation (*a/b*). Thus, fraction is not an easy concept to be understood and to be meaningfully mastered by children.

Children's difficulties with fractions are well known among researchers and educators (e.g.; Kerslake, 1986; Koyama, 1997; Nunes and Bryant, 1996; Watanabe, Reynolds & Lo, 1995). They have difficulties to understand fractions as mathematical ideas (a number or a quantity) and construct meanings of fractions. According to Streefland (1991) some of the sources behind the failure of fraction instruction reside in the complexity of this concept, and, also, in the traditional approach to fractions which is formal and mechanistic from the very start. Also, the teaching is often detached from reality, consists of rigid rule-oriented instruction, and the informal knowledge children have acquired is ignored entirely.

Thus, one may ask whether children would overcome their difficulties and develop an understanding of fractions if they were given an instructional experience that emphasises realistic and meaningful problem solving situations and that allowed children to show and discuss their informal knowledge (strategies used, informal representations of fractions).

THE STUDY

This study aimed to explore the role played by an instructional experience to improve children's understanding of fractions through a variety of problem solving situations. The idea was that third grade children who were taught about fractions for the first time would benefit more from this alternative teaching programme than fourth grade children who were taught in a traditional approach to fractions. Thus, a study was carried out in which a pre and a post-test were applied to children who were divided into three groups: Experimental Group (alternative instructional approach to fractions); Control Group (pre and post test only); and Reference Group (traditional approach to fractions).

Participants and groups

The participants were 57 low-class children aged 8 to 10 years old attending a public school in São Paulo, Brazil. These were divided into three groups:

Experimental Group (EG): 19 third graders (mean age: 8 years old) with no previous instruction in fractions were given a teaching experience based on a sequence of instructional activities that enabled them to think, discuss and communicate verbally their thought processes when solving problems involving fractions

Control Group (CG): 20 third graders (mean age: 8 years old) with no previous instruction in fractions were given a pre and a post-test only.

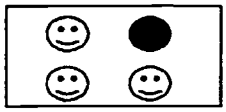
Reference Group (RG): 18 fourth graders with previous formal instruction in fractions were taught about fractions in a traditional way based on a mechanistic approach to fractions, detached from realistic situations and focusing on rigid application of rules.

Pre and the post-test

The pre and post-test consisted of a series of 15 items involving fractions related to continuous and discontinuous quantities (Figure 1).

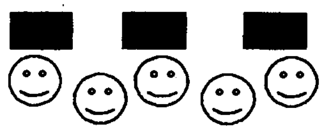
Figure 1: Examples of some of the items in the pre and in the post-test.

João has painted one of the four faces in a set. How can you represent the number of faces he had painted in relation to the total number of faces in the set?



Response:

Could you divide equally 4 bars of chocolate between 5 children? How much chocolate does each child get?

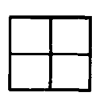


Response:

Make a circle around one third of the collection of hearts bellow. How can you represent the number of hearts you circled in relation to the total number of hearts in the collection?

Response:

Could you paint half of the figure bellow? How can you represent the number of parts painted in the figure bellow?



Response:

How can you represent the amount of parts painted in the figure bellow?

Response:

Some of these items involved part-whole relations and others involved sharing. In these items children had to solve the problem and then numerically represent its result in a form of a fraction (a/b). Both tests had the same level of difficulty and were mathematically equivalent. They were applied to all participants and solved by means of pencil and paper.

Overview to the teaching experiment

The teaching experiment offered to third grade children in the Experimental Group comprised ten 2-hours sessions, two sessions per week over five weeks. The sessions involved whole class activities focusing on a problem task and follow-up questions and discussion posed by the researcher who acted as a teacher according to a previously planned didactical sequence. Children also worked in pairs and in small groups. The role of the researcher was to provide assistance to the students, understand and clarify their thinking (insights and difficulties) and, at the end of each session, offered them a conclusion about the whole activity and its solution. In this instructional experience the approach to fractions was based on (a) realistic and meaningful problem solving situations; (b) student's own fragmentary and informal knowledge; (c) interactive situations: discussion and collaborative work; and (d) student's thinking about their own thought processes when solving problems and communicating verbally their strategies and ideas. Diagrams, manipulative and concrete materials were widely used in a large range of situations involving continuous and discontinuous quantities in which fractions were taught through partitioning and part-whole relations.

RESULTS

The results in the pre and in the post-test, in the three groups of children, were analysed according to the number of correct responses and according to the types of errors children made.

Correct responses

Table 1 presents the general scores in this study. It can be noted that no significant differences were found among the three groups in the pre-test. This result was confirmed by means of Mann-Whitney test (EG vs. CG: $p = 0.6080$; EG vs. RG: $p = 0.7330$; and CG vs. RG: $p = 0.7502$). However, there were marked difference between groups in the post-test (Man-Whitney - EG vs. CG: $p = 0.0000$; EG vs. RG: $p = 0.0000$; and CG vs. RG: $p = 0.0000$). This was due to children in the Experimental Group and in the Reference Group being more successful than children in the Control Group; and children in the Experimental Group performing better than those in the Reference Group.

Table 1: Number and percentage of correct responses.

	Experimental Group (n = 285)	Control Group (n = 300)	Reference Group (n = 270)
Pre-test	29 (10.2%)	36 (12%)	29 (10.7%)
Post-test	199 (69.8%)	37 (12.3%)	86 (31.8%)

The performance of each group in the pre and post-test was compared by Wilcoxon test. This revealed that there was no significant difference between pre and post-test for the children in CG ($p = 0.5014$). In contrast, EG and RG children performed significantly better in the post-test than in the pre-test ($p = 0.001$ and $p = 0.003$, respectively). This suggests that both groups of children improved their performance after instruction. However, it is important to know whether the progression children had was similar in both groups.

Looking at each child individually it was possible to identify that there were children who showed a progression from pre to post-test, while others showed a regression in their scores or did not showed any change in their performance (stability over tests). Table 2 and Table 3 presents the incidence of children in each group according to their progression, stability over tests and regression.

Table 2: Number and percentage of children in each group according to their improvement, stability and regression from pre to post-test.

	Experimental Group (n=19)	Control Group (n=20)	Reference Group (n=18)
Improvement	19 (100%)	8 (45%)	17 (94%)
Stability	0	4 (20%)	1 (6%)
Regression	0	7 (35%)	0

Children in the Experimental Group and in the Reference Group were able to benefit from the instruction given, independently of whether the instruction was a traditional way of teaching (RG) or an alternative one (EG). Regression in performance was only to be found among children in the Control Group. However, when we look at the improvement observed in more details, we can verify that the level of improvement was higher among children in the Experimental Group than among those in the Reference Group, as shown in Table 3.

It can be seen that children who took part in the teaching experiment had a greater improvement than those who were taught about fractions in a traditional way. This was particularly so in relation to the number of children who had Level 3 improvement (EG: 52.6% and RG: 0%).

Table 3: Number and percentage of children who progressed from pre to post-test.

Level of Improvement	Experimental Group (n=19)	Control Group (n=8)	Reference Group (n=17)
Level 1 (1 – 30%)	0	8 (100%)	12 (70.6%)
Level 2 (31 – 60%)	9 (47.4%)	0	5 (29.4%)
Level 3 (61 – 100%)	10 (52.6%)	0	0

Types of errors

Three main types of errors were identified in the pre and in the post test[1]:

Type 1 – children interpret fractions as natural numbers when symbolic representing their answers. This error was more often when discontinuous quantities were involved.

Type 2 – children interpret fractions as natural numbers when solving the item, during the solution procedure. This error was more often when discontinuous quantities were involved.

Type 3 – children do not recognise the need of having equal shares when solving the items that required the division of the whole into parts. This error occurred when continuous quantities were involved.

Type 1 errors were related to the representation children used when writing down the results of the items; while Type 2 and Type 3 were related to the solution procedure adopted. On the other hand, Type 1 and Type 2 errors expressed their difficulty to discriminate fractions from natural numbers; while Type 3 expressed their difficulty to divide the parts into equal shares. These errors were reported in previous studies in the literature. Table 4 shows the frequency of these errors in the Experimental Group and in the Reference Group in the pre and in the post-test.

Table 4: Incidence of each type of error.

Types of error	Experimental Group		Reference Group	
	Pre-test	**Pos-test**	**Pre-test**	**Post-test**
Type 1	159	29	128	49
Type 2	32	29	23	51
Type 3	18	5	24	26

[1] Children in the Control Group had the same high incidence of errors in the pre and post-test.

As a whole, the main difficulties children had in the pre-test decreased in the post-test. However, children in the Experimental Group showed a greater decrease of errors than those in the Reference Group. This was particularly so in relation to Type 1 and Type 3 errors. Thus, even though both groups seem to have overcame some of their initial difficulties, children who took part in the teaching experiment had a greater improvement than children who were taught in a traditional way. It is noteworthy to stress that children in the Experimental Group were younger and were attending a more elementary grade than children in the Reference Group.

CONCLUSIONS AND DISCUSSION

The results in this study revealed that the teaching experiment improved children's understanding about fractions. This intervention led third graders to have a better performance than (a) children of the same age and grade who did not receive any instruction on fraction; and (b) children who were older and attending a higher grade of schooling. It is interesting to mention that performance among groups was much the same in the pre-test. This means that children who were not instructed about fractions (third graders in the Experimental and in the Control Groups) showed a similar understanding of fractions as those who already had received instruction on fractions in a traditional way (fourth graders in the Reference Group). After instruction, children who took part in the teaching experiment increased the number of correct response and overcame the initial difficulties they had in the pre-test. Also the number of children who progressed from pre to post-test was greater among them.

The conclusion that can be drawn from the results was that even though both groups benefited from instruction, the alternative approach to fractions given to children in the Experimental Group was likely to be more successful than the traditional approach given to children in the Reference Group.

Another aspect that should be stressed was that a short period of teaching intervention (10 sessions) proved to be beneficial. The teaching appears to have achieved some success. However, one must be cautious about the fact that the teaching proposed here did not explore all the features and situations that are relevant for the understanding of fractions in a broader sense.

To conclude, it is crucial to find ways to reduce some of the difficulties children encounter when dealing with fractions. Also, it is important to encourage classroom discussion: talk to children about their errors, ideas and the strategies they use (independently of whether they are correct or not); and listen to children by encouraging them to communicate verbally their ways of thinking and their interpretation of fractions. Actually, teachers and researchers should, in collaboration, raise and test hypotheses about how to promote children's understanding of fractions.

Further intervention studies should be conducted with elementary school children. It would be of interest, for instance, to develop a teaching experience in which students would be given the opportunity to explore fractions in a wider context, in different situations, using different forms of representation and in connection with other mathematical concepts. Fractions should be introduced and explored through a large variety of situations (including 'part of a whole model' since it showed to be useful to establish some of the basic ideas about fractions) that allowed children to understand a fraction as a number, to relate it to division and to recognise the equivalence between fractions.

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LINGUISTIC POINTERS IN YOUNG CHILDREN'S DESCRIPTIONS OF MENTAL CALCULATION

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Children's descriptions of their mental calculations can reveal more to the teacher than merely the strategy that has been used. An analysis of nearly 2000 responses given in interview by 26 children aged between 6 and 9 years old has revealed that the language they use may point to individual differences in their modes of thought. Different use of pronouns, tense and causal connectives was associated with different levels of achievement. Descriptions of calculations which accompanied correct answers were more likely to be in the present tense, to be expressed in terms of what "you" do and were more likely to include explanatory words of deduction : "if", "then", "so" and "because".

INTRODUCTION

This paper focuses on the different styles of language that young children use when describing a mental calculation they have just performed. I have previously described this language as 'image-like' (Bills and Gray, 1999) because the words used were related to previous classroom activities. I have suggested that the language could indicate the influence of these activities on the children's thinking. Analysis of the descriptions of 45 mental calculation questions performed in six interviews has also revealed that higher achieving children tended to use a language of generalisation. Lower achievers simply described the particular calculation (Bills and Gray, 2001).

The data presented in this paper suggest that there are other linguistic characteristics of descriptions of mental calculations that are associated with levels of achievement. The speech style observed in the classrooms in this study used the present tense and the pronoun "you" to describe calculation procedures. Language of deduction (causal connectives: "if", "then", "so" and "because") was used in explanations by teachers and pupils. In my interviews with children they most frequently adopted this style when describing a calculation they had performed successfully.

All the children in this study had the same teaching yet differences in their learning was apparent in the differences in their language. I suggest that this is evidence that commonalities in language may simply reflect the speech style of the community yet differences in the use of language may be indicative of differences in mental constructions. The children in this study had been exposed to language of procedure and explanation and they all demonstrated that they could use it in non-calculation situations. The more frequent use of this language style in association with correct answers suggests that its use was not simply a linguistic trait but that it was used when children had a successful procedure. The conclusion is drawn that teachers may learn something about learners' procedural knowledge from the language that they use. Language can act as a 'pointer' to different qualities of thinking.

LINGUISTIC POINTERS

Deictic terms: "this", "it", "you", (from the Greek 'deixis', meaning to point), serve to identify objects, people, times and places without reference to particular things. Rowland (1995) has thus suggested that the use of these words could be indicative of a generalisation. Rowland proposed as a 'deictic principle' that language is a code to express and 'point to' concepts, meanings and attitudes. So that not only deictic terms themselves but use of other words could be indicators of cognitive structures.

Rowland (2000) noted that the use of the pronoun "you" to refer to generalities, i.e. what usually happens, is common in non-mathematical situations where "you" is frequently used in place of the more formal "one". This is particularly true of children in their description of rules of games. Rowland has also suggested, however, that the use of "you" is an effective 'linguistic pointer' to a quality of thinking. For the pupil in Rowland's study the shift from "I" to "you" in a problem solving situation signified her move from working with particular numbers to expressing a generalisation.

The quality of the language used in explanations may also be an indication of the quality of understanding. Donaldson (1986) questioned whether young children's inability to give an explanation is due to poor understanding of the concept to be explained or to a lack of the linguistic ability to explain. She argued that adequate linguistic competence for explanation is demonstrated by appropriate use of the causal connectives "because" and "so". Her studies show that children do use "because" and "so" appropriately and by the age of eight years can use them in the deductive mode. She concluded that the linguistic ability of understanding causal connectives and the cognitive ability of understanding causality are interdependent.

In a study conducted by Vygotsky (1962) 80% of children in the sample, at both 7 and 9 years, were able to correctly complete sentence fragments ending in "because" when related to 'scientific concepts'. Only 60% of the 7 year-olds could do this with sentences related to 'spontaneous' everyday concepts. He attributed this to the fact that scientific concepts had been learned in collaboration with teachers. He assumed that the ability to use "because" appropriately in everyday concepts is improved by being able to do so in scientific concepts. Luria (1982) noted that children may understand the ordinary features of their familiar social experience but will need activities to provide experience of the unfamiliar scientific concepts encountered in more formal school settings. He suggested that at school children may initially use words or perform actions without knowing why. Thus children can use words and perform actions initially without having the same understandings as the teacher.

Piaget (1928) referred to Claparede's 'law of conscious realisation': the more automatically an idea is handled, the harder is the process of its conscious realisation. Piaget suggested that when asked how they had performed a calculation, after giving an answer by some automatic process, children may invent something which would give rise to the same answer. In my study (Bills and Gray, 2001) it seemed that when the high achievers performed the calculation automatically (the answer was given very quickly) they simply stated a rule that would give the answer. The lower achievers tended to describe, in detail, the particular calculation they had performed. This adds supports to Piaget's suggestions that initially children can only reason about the particular (Piaget, 1928) and also that the child only imitates that which he can understand (Piaget & Inhelder, 1971). Higher achievers may have expressed themselves in terms of a generalisation because they understood the rule in this way whilst lower achievers only gave the steps in the particular procedure performed.

Rowland, Donaldson and Piaget suggest that language may indicate individual mental constructions whilst Vygotsky and Luria maintain that language is a social construction. The data presented in this paper suggest both perspectives are valid.

METHOD

Lesson observations and pupil interviews were first conducted with two classes from Year 3 (pupils aged 7 and 8 years) in a school for children aged 5 to 11 years in a large middle-income village near Birmingham U.K., from September 1998 to July 1999. The same pupils were also observed and interviewed in the following year. The 80 children in the whole year had been placed in one of three groups for mathematics lessons based on their previous attainments. Lessons with the high attainment and the middle attainment groups were observed and a sample of 14 pupils from the first and a sample of 12 pupils from the second was interviewed in December, March and July in each year. The samples were chosen to represent the spread of attainment levels in each group.

Over the six interviews 78 questions were used. They were classified into 10 calculation types and 6 non-calculation type. Each was presented verbally and followed by the question "What was in your head when you were thinking of that?"

Type	Description	Examples of questions
1	1-digit addend	17 + 8, 17 + 9 (repeated in each interview)
2	Missing addend	13 + * = 18, 30 + * = 80, 27+*=65
3	2-digit addition	48 + 23 (repeated in each interview)
4	Addition of multiple of 10	97 + 10, 597 + 10, 1097 + 10, 1197 + 10
5	Counting	What comes before 380, 2380, 12100; after 12386
6	Rounding	Round 2462 to the nearest ten, 239 to nearest hundred
7	Recent topic	What is difference between 27 and 65, 0.6+0.7
8	Recent topic	65 subtract 29, Read time (11:40), 0.1 times by 10
9	Division and fractions	quarter of 40, third of 48, 140 divided by 3

10	Multiplication	48 multiplied by 3, 47 multiplied by 5
11	Numerical procedure	Tell me how to: add 23, find a third, times by ten,
12	Non-numerical procedure	Tell me how to: cross road, tell the time, do subtraction
13	Maths concept, first	First thing in head when I say: centimetre, three, million
14	Maths concept, more	What else about: centimetre, three, million
15	Non-Maths concept, first	First thing in head when I say: shadow, ball, adjective
16	Non-Maths concept more	What else can you tell me about: shadow, ball, adjective

Each response to "What was in your head?" was analysed for its use of pronouns, tense and causal connectives.

Responses involving "I" and "you" were categorised in to three types which conform with the 'particular', 'generic' and 'general' classifications described in Bills and Gray (2001). The examples given here are responses to the question "What was in your head?" following the mental calculation "What is 48 add 23?":

Category 1: responses related to the particular, expressions such as "I/you did this"

Bobby **I** added the 40 and the 20 to make 60 then **I** added the 8 and the 3.

John 48 add 20 comes 68 and then **you** add 3 on.

Category 2: responses related to examples, expressions such as "I/you do with these"

Simon say **I** was adding 30, **I** go 40, 50, 60

Irene **you** put the one down in the units column in your head and **you** carry one

Category 3: responses related to the general, expressions such as "I/you always do"

Ellain **I** count it all in my head, and **I** sometimes use my fingers

Elspeth **you** just take the ten of there and **you** put it back and **you** put it together

RESULTS

Linguistic characteristics of correct calculations

There were 1158 mental calculations performed in the interviews and 63% of these were correct. A comparison was made between the responses which accompanied correct mental calculations and the responses which accompanied incorrect calculations. Tables for the distribution of correct and incorrect responses involving different uses of pronoun, tense and causal connectives were compiled and chi-squared tests of significance conducted. Only tables for statistically significant results will be given here.

The following table suggests that pupils whose calculation was described exclusively using "you" were more likely to have been correct than those who used "I" exclusively or a mixture of the two. Notice that 97 responses used "you" only and

79% of these were correct. In comparison 520 responses used "I' exclusively and only 65% were correct. The differences in proportions of correct and incorrect responses was statistically significant ($p<0.05$):

	"I" only	"I" and "you"	"you" only
Percentage correct	65%	66%	79%
Number of responses	520	166	97

Analysis of the categories of pronoun use is also informative. When "I" was used to describe specifically what the individual had done in that particular question the answer was less likely to have been correct than if "I" indicated what the individual did in general ($p<0.005$):

	"I" category 1	"I" category 2	"I" category 3
Percentage correct	63%	68%	79%
Number of responses	489	47	133

By contrast the difference in the distributions for different categories of "you" was not statistically significant. These two results for categories of pronoun use suggest that the adoption of the common classroom speech style involving use of "you" may be less of an indicator of understanding than the use of the personal pronoun "I". The use of "I" may point to ownership of the rule whilst "you" may simply follow the classroom convention.

The difference between the distribution of tenses in which children chose to phrase their explanations after they had given a correct answer and after an incorrect answer was also statistically significant. It appears that exclusive use of past tense was accompanied by a lower proportion of correct answers than present tense and mixed tense expressions ($p<0.05$):

	past tense	present tense	mixed tenses
Percentage correct	62%	71%	71%
Number of responses	391	244	259

There is again an indication that descriptions simply of what had been done were associated with less successful calculations than descriptions phrased in the rule giving, present tense, mode.

When any of the causal connectives were used it was more likely that the correct answer had been given than those responses where no causal connective had been used. This could indicate that use of language of explanations is more frequently associated with correct mental calculations and thus with greater procedural competence.The differences in the proportions correct were all statistically significant ($p<0.005$):

	% correct when used	% correct when not used
"because"	79% (of 189)	59% (of 969)
"so"	72% (of 284)	59% (of 874)
"if"	82% (of 135)	60% (of 1023)
"then"	68% (of 448)	59% (of 710)

Linguistic characteristics of high and low achievers

The sample of 26 children achieved between 12 and 38 correct answers out of 45 mental calculation questions. The 13 pupils above the median were classified as 'high achievers' and the remainder as 'low achievers'. Comparisons were made between the distributions of responses involving the linguistic indicators (pronouns, tense and causal connectives) for the two groups. The differences were all statistically significant ($p<0.05$)

	Percentage of responses involving the indicator	
	Higher achievers (n=578)	Lower achievers (n=580)
"I" only	42%	49%
"you" only	11%	6%
past tense	41%	34%
"because"	20%	13%
"so"	30%	19%
"then"	45%	32%
any connective	66%	49%

The proportions of 'category 3' pronoun use were also compared. A higher proportion of the use of "I" was category 3 for the higher achievers than for the lower achievers. Similarly with use of "you":

	Percentage of responses involving the indicator	
	Higher achievers	Lower achievers
"I" category 3	32% (of 244 uses of "I")	19% (of 284 uses of "I")
"you" category 3	25% (of 157 uses of "you")	17% (of 89 uses of "you")

The differences were statistically significant ($p<0.05$). The responses to non-calculation items in the interviews were also analysed for use of the linguistic indicators. There were no statistically significant differences in the distributions for the two groups of children for any of these indicators in non-calculation contexts.

DISCUSSION

The fact that the two groups of children had the same linguistic characteristics in non-calculation questions suggests that the differences apparent in response to calculation questions were not simply a matter of linguistic ability. All the children were capable of using causal connectives in non-calculation context and the two groups did so in similar proportions. Their use of pronouns and tense was also similar in these non-calculation contexts. The differences between their response after calculations had been performed may thus be an indication that they had different ways of thinking about those calculations. The higher proportion of use of causal connectives to accompany correct calculations may indicate that the procedure was well understood.

The use of the pronoun "you" and present tense was common in the classrooms observed. All the children were exposed to phrases such as "you add the tens and you add the units". Some used phrases like this when describing what was in their head after performing a mental calculation. Others talked only of the particular calculation and used phrases such as "I added 40 and 20 and I added 8 and 3". Those who adopted the teachers' mode of language were the most successful. This suggests that the children were not simply copying the language. They tended to use this mode of language when they were using a procedure which was successful. Notice that my question "What **was** in **your** head?" would seem to require a response from the individual in the past tense and in the first person. The fact that they chose to use the classroom speech style is an indication that they were following a familiar procedure.

Rowland has suggested that children's use of "you" may be a pointer to their progress toward generalisation. In this study the use of "you" is an indication that a familiar procedure had been followed but rules that the individual used were also expressed in general terms using "I". Expressions involving "I" to describe what the individual **does** rather than simply what they **did** were more likely to accompany correct answers. This mode of expression could thus be a powerful indication that the child has formed their own generalisation since it is not couched in the ubiquitous classroom style using "you". The individual pupil may demonstrate their individual mental construction based on their experience of the classroom activities through their use of "I" in expressing general rules.

It is not surprising that children adopt the common classroom speech style but the results presented above seem to suggest that pupils are more likely to do so to accompany a successful calculation. The social constructivist perspective of Luria and Vygotsky could suggest that all children might simply adopt the classroom speech style. This study indicates that the pupils were most likely to do so when they were successful. There is an indication in this that the children's different language use was a pointer to their different mental constructions. The pupils were capable of simply copying their teacher's style of language but were less likely to do so when they had given an incorrect answer even though they were not told if their answer was right or wrong.

IMPLICATIONS FOR THE CLASSROOM

Use of "I" in the sense of 'what the individual does in general' was not common after mental calculations had been performed (8% of responses) but on nearly 80% of these occasions the answer was correct. Similarly exclusive use of "you" is unusual (11% of responses) and has the same high association with correct answers. The causal connectives are also strong indicators, particularly "because" and "if", with low proportions of occurrence but high associations with accuracy.

These words thus act as linguistic 'pointers' to a quality of thinking which the teacher will wish to encourage. In the classroom children's use of these linguistic pointers can alert the teacher to differences between individuals. They may also give indications of development in thinking. A single response can not, in itself, give an assessment of a pupils procedural competence but responses over a period of time can point to a changing quality of thinking. Identifying differences between pupils does nothing to improve the learning of pupils who are less successful in mathematics unless the information is used by teachers to help these pupils gain greater understanding. Identifying the pupils' differences through the use of the linguistic pointers can be a first step.

Listening to the words that children use can be more informative than listening simply to the strategy that has been employed.

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DESIGN PRINCIPLES FOR TASKS THAT SUPPORT ALGEBRAIC THINKING IN ELEMENTARY SCHOOL CLASSROOMS[1]

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The purpose of this work is to describe design principles for mathematical tasks in elementary school that have been transformed from arithmetic tasks to those that exploit algebraic thinking. In particular, we discuss design elements of these 'algebrafied' tasks that can elicit students' activity of generalizing, where our particular mathematical focus is on generalizing from numerical patterns to describe functional relationships. We also consider ways in which arithmetic, particularly number and operation, are embedded in the creation and analysis of patterns. Classroom data in which third-grade students solve the Handshake Problem is used as a context for elaborating these ideas.

BACKGROUND FOR THE STUDY

Algebraic Reasoning as a Way of Thinking Mathematically

Traditionally, the focus of elementary mathematics has been deeply oriented to arithmetic and computation, with little attention given to the relationships and structure underlying simple arithmetic tasks. However, there is a growing recognition that algebraic reasoning can emerge from and simultaneously enhance elementary school mathematics (NCTM, 2000) and can consequently build habits of mind that will prepare students for the more complex mathematics of the new century (Romberg & Kaput, 1998). By algebraic reasoning, we refer to students' activity of generalizing about data and mathematical relationships, establishing those generalizations through public conjecture and argumentation, and expressing them in increasingly formal ways (Kaput 1999). Indeed, algebraic reasoning extends far beyond a traditional view of algebra as an act of syntactically-guided symbolic manipulations and can occur in various interrelated forms, including (a) the use of arithmetic as well as mathematical and non-mathematical situations as domains for expressing and formalizing generalizations; (b) generalizing numerical patterns to describe functional relationships (including covariation); and (c) generalizing about mathematical systems abstracted from computations (Kaput, 1998). Our interest in fostering the development of algebraic reasoning in elementary school mathematics prompted us to consider the kinds of tasks that might be used to exploit mathematical ideas that the discipline recognizes as "algebraic" and to examine characteristics of these tasks that signal this. We discuss our partial results here and report on them more widely in Blanton and Kaput (2001).

The GEAAR Professional Development Project

The work reported here is drawn from our multi-year, district-wide professional development program, Generalizing to Extend Arithmetic to Algebraic Reasoning (GEAAR), designed to develop elementary teachers' ability to identify and strategically build on classroom opportunities for algebraic thinking. In particular, we engage in year-long professional development with teachers in a strategy directed toward classroom-grounded change along three dimensions: (1) building opportunities for algebraic thinking from teachers' available instructional resources; (2) building teachers' capacity to identify spontaneous opportunities for algebraic thinking in the classroom through a focus on student thinking; and (3) building teachers' capacity to create a classroom culture that can support active student generalization and formalization within the context of purposeful conjecture and argumentation. As part of the project, teachers are asked to contribute "algebrafied" tasks that they have developed or selected for use in their own classrooms. During one of our sessions with teachers (Kaput and Blanton, 1999), we and the teachers came to value the feature that generalizing a functional relationship could be achieved through an analysis of the *form* of non-executed sums describing a total for some prescribed quantity (e.g., the number of 'right-side-up' triangles in a large triangle subdivided into smaller triangles). As a result, we began to focus in our sessions with teachers on a genre of tasks with this characteristic and to explore how these tasks could further support algebraic thinking and simultaneously leverage students' experiences with arithmetic, particularly number and operation.

We base the findings reported here on a one-year case study of a third-grade classroom, the teacher of whom has been one of our central participants in the GEAAR Project. The data from which we draw our results were taken from a 45-minute classroom episode that is part of a larger data corpus including student work, the teacher's (Jan – pseudonym) reflections, and classroom field notes, all collected during one academic-year period. In the classroom episode, students solved an 'algebrafied' version of the Handshake Problem, that is, they solved a task traditionally formulated as a single-numerical answer problem that had been transformed to the following:

> *If 5 people in a group shake hands with each other once, how many handshakes will there be? What if there are 6 people in the group? Seven people? Eight people? Twenty People? Write a number sentence that shows your result. Show how you got your solution.*

We selected this problem because it reflects a genre of problems used in our seminars with teachers, and subsequently in their own classrooms, that illustrate the potential for arithmetic tasks to support algebraic thinking, particularly as it relates to generalizing numerical patterns in order to describe functional relationships. In what follows, we draw on students' experiences with the Handshake Problem to describe several of the principles of these tasks that we see as supporting algebraic thinking in conjunction with students' experiences with arithmetic, particularly number and operation.

PRINCIPLES OF PATTERN-ELICITING "ALGEBRAFIED" TASKS

Principle 1: The tasks promoted the use of number and number sentences as objects for reasoning algebraically.

The tradition in elementary school mathematics of finding "the answer" (usually a single numerical value) can disguise the algebraic potential of tasks such as the Handshake Problem by diverting students to a process of finding patterns in a sequence of numbers rather than looking for a functional relationship between varying quantities. In the Handshake Problem, when students rushed to compute the total number of handshakes for groups of varying size they were left with a sequence of numbers, not number sentences, which limited their capacity to develop a predictive model. While finding patterns in a sequence of numbers is a useful task, we argue that it does not engage the full algebraic character of the task because it limits the scope of predictability to consecutive states. The power of functions rests ultimately in their capacity as mathematized objects to allow predictions beyond the scope of known data. To this end, we were interested in suspending computation so that students could reason with the *forms* of the number sentences and thereby deduce a functional relationship. With the Handshake Problem, we found that by using sequences of number sentences and deliberately not computing them, students were able to attend to the sums for their shapes as inscriptions, rather than as instructions to perform procedures. By analyzing these forms, they were able to determine a relationship between the size of any group and the number of handshakes. In essence, treating these non-executed sums as algebraic objects allowed for an analysis of co-variation in the data so that students were able to generalize that the number of handshakes in a group of any size would be "the sum of the numbers from one (or zero) up to one less than the number of people in the group".

Another important feature of these tasks is that they allowed for the *algebraic use of number*. A mature understanding of the concept of function ultimately requires students to attend to the notion of arbitrariness. Numbers can be used algebraically when the type or size of number chosen requires the student to think about structure and relationship between quantities, not simply arithmetic operations on quantities. With the Handshake Problem, by asking students to determine the number of handshakes for a group of 20 people without knowing the previous cases, we maintain that '20' was treated algebraically because it called on students to analyze the structure embedded in the *forms* of a sequence of arithmetic models, or number sentences.

Principle 2: The tasks involved sequences of computations that could be exploited to engage students arithmetically.

The dominant role of arithmetic in elementary schools requires that we think about how to make its role in the more complex processes of algebraic thinking explicit. Tasks such as the Handshake Problem can deeply support arithmetic while simultaneously engaging students in algebraic processes. For instance, mathematizing the act of shaking hands required students to understand the correspondence between a collection of counted handshakes and the number representing it, as well as how to operate additively to find a total amount. These particular aspects of the task were more significant for those teachers in earlier grades where counting and numeracy were in earlier stages of development. Students' facility with number and operation were also strengthened by looking for efficient counting strategies, as the following excerpt illustrates. In it, students were calculating the number of handshakes in a group of size 12, that is, they were computing the sum '11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1'.

1 Teacher:	Did anybody change the order of those numbers in any way when you added them?
2 Student:	Yes. You can change them to tens.
3 Teacher:	You made tens out of these?
4 Student:	We put 11, 10, 9, 8...(voice trails off). You don't need to [add 1 through 7] over again.
5 Student:	All you have to do is put 11, 10, 9, 8 on top of all the numbers we had.
6 Student:	You don't say 12.
7 Teacher:	You don't say 12? Why don't you say 12?
8 Student:	Cause you can't shake your own hand.
9 Teacher:	Now, wait a minute. You just said...
10 Student:	...that we didn't have to count 7, 6, 5, 4, 3, 2, 1, 0...because we knew they already equaled 28.
11 Teacher:	Okay, that's pretty good!...Zolan did something that I really liked.... What did you do?
12 Zolan:	I added the 11 and 10 together. (From the result of 21, he then subtracted 1.) Then I added the extra 1 of the 11 to the 9 and that made a 10.

Zolan had commuted the numbers in such a way that he could add groups of 10. Later in the episode, another conversation occurred about a strategy for computing the sum of the number of handshakes for a group of size 20. At students' suggestion, Jan had written '0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19' on the board. She asked, "Is there a way that we can change the order of these numbers and make them easier to add up?"

13 Student:	You can take the zero and put it, you can change it around by putting the 19 where the zero was and that would be easier. (We infer that the student meant to add '19 + 0'.)
14 Teacher:	Can I do this? If I do this, 19 + 0….
15 Student:	That equals 19.
16 Teacher:	What else could I do?
17 Student:	You could do 18 + 1.
18 Student:	What about 17 + 2?
19 Teacher:	What about 17 + 2? Wait a minute. Let me just try this. So I've used those two (19, 18) and these two numbers (0, 1). (Jan drew lines on the board connecting the numbers in each pair.) Let me see 17 + 2. Okay. These are pairs aren't they? How many pairs of numbers do you think we're going to be able to make out of these 20 numbers? …Anthony, how many do you think we are going to make?
20 Anthony:	Ten.
21 Teacher:	Why?
22 Anthony:	Because you need to get 1 from each side.
23 Teacher:	Let's see if we do make 10 pairs of numbers.
24 Student:	Each one (pair) is equal to 19.

After they had established that there would be 10 pairs of numbers, the sum of each being 19, Jan said, "Now I've got to add up all these 19's. What is this?

25 Student:	Repeated addition. You could do times.
26 Teacher	I could do times?
27 Student:	Nineteen times 10….190.
28 Teacher:	How did you figure that out so quickly?
29 Student:	I just changed that to 9 and added a zero.
30 Teacher:	Why?
31 Student:	Because the one is a 100 and 9 is 90.

We found that when asked to find efficient strategies for computation, students looked for relationships in the arithmetic models in order to organize numbers in ways that exploited productive pairings. For example, recognizing that they could generate a sequence of sums of 19 by adding 0 to 19, 1 to 18, 2 to 17, and so forth, students used this to more effectively 'add' by multiplication. Moreover, students' proficiency with these maneuvers required an understanding of operation, particularly the generalized commutativity of addition and a sophisticated concept of 'counting on'. Indeed, while data from observations earlier in the year showed that 'counting on' was an emerging part of these students' cognitive schemes, the

Handshake Problem seemed to more fully form this concept in their thinking. In particular, students were able to count on from a previously determined sum and interact with that sum as an independent object. All of these processes, such as counting on, commuting numbers, and mathematizing handshakes, occurred in meaningful contexts in which students explored issues of numeracy and increased their facility with number and operation. Moreover, they were simultaneously engaged in the more complex processes of algebraic thinking about relationships and properties of whole numbers and, especially, attention to forms of the number sentences from which they could deduce general relationships. As a result, the arithmetic was not neglected but became an integral part of students' algebraic reasoning.

Principal 3: The tasks allowed for the enactment of actions and situations familiar to students.

By this we mean the tasks allowed students to construct a sequence of number sentences, or arithmetic models, from a context that accessed their everyday experiences (e.g., shaking hands) and in such a way that sequential cases could be explicitly examined to see how perturbations in one case affected other cases. We conjecture that to have these explicit cases, or number sentences, available as permanent artifacts of the task enabled students to move more easily between the abstraction of the models and the source for interpreting the models, namely the physical context of the problem. In contrast, reducing a model to a computed sum would have concealed the physical actions embodied by the representation. In essence, we maintain that students' analysis of the relationship between the number of handshakes and the amount of people in the group was facilitated by symbolizing a familiar action and by keeping that action explicit. To the extent that the sequential nature of these tasks and the explicit number sentences they generated made functional relationships more recognizable, we maintain that this principle supported students' opportunity for algebraic thinking.

CONCLUSION

Tasks such as the Handshake Problem represent one (of many) genre of 'algebrafiable' tasks that can enrich elementary school mathematics. We found that tasks which leveraged students' arithmetic knowledge and included some act of mathematizing a phenomenon so that a mathematical abstraction had its representation in a familiar context and where that abstraction was left in an explicit form (e.g. '1+2+3+4+5+6') could help access students' capacity for algebraic thinking. One significant feature of such tasks is that they embed the arithmetic in a context that requires more complex mathematical thinking, such as (a) knowing to and knowing how to represent data; (b) using arithmetic (number and operations) to model a phenomenon that involves variation; (c) examining how perturbations in a phenomenon affect a model; (d) reasoning algebraically about the forms of sequences of number sentences; (e) deepening arithmetic reasoning to support the appropriate use and choice of operations and to understanding relationships between numbers in order to facilitate computation; and (f) the algebraic use of numbers and number sentences. Thus, it is our claim that these types of tasks offer students a mathematical experience where arithmetic and algebraic processes interact symbiotically.

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NOTES

[1] The research reported here was supported in part by a grant from the U.S. Department of Education, Office of Educational Research and Improvement to the National Center for Improving Student Learning and Achievement in Mathematics and Science (R305A600007-98). The opinions

expressed herein do not necessarily reflect the position, policy, or endorsement of supporting agencies.

THE DEVELOPMENT OF DISCIPLINARY RELATIONSHIPS: KNOWLEDGE, PRACTICE, AND IDENTITY IN MATHEMATICS CLASSROOMS.

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In this paper I will trace a path through three different studies of mathematics learning in England and the United States. As different analytical lenses are cast upon students' experiences I will propose that notions of mathematical capability expand beyond knowledge, to include mathematical practices and the identities these encourage. Further that different classroom practices encourage students to develop different relationships with the discipline of mathematics that impact their capability in profound ways.

Researchers of mathematics education have, for many years, focused upon the knowledge students develop in classrooms, and the ways such knowledge is influenced by a number of different variables. Recently, situated theories of learning have led to the recognition that the practices of classrooms – the repeated actions in which students and teachers engage as they learn – are important, not only because they are vehicles for students' knowledge development, but because they come to *constitute* the knowledge that is produced (Cobb, 1998). Thus the field has moved to greater recognition of the intricate relationship between knowledge and practice and the need to study the practices of classrooms in order to understand students' mathematical capability in different situations. In this paper I will attempt to expand notions of mathematical capability yet further, to go beyond both knowledge and practice, to the identities students form in relation to different mathematical practices, and the *disciplinary relationships* that are afforded by different classroom experiences. Thus I will suggest that when students approach a new mathematics problem, the extent to which they are able to use mathematics depends partly on the knowledge they have developed, partly upon the practices in which they have engaged as they have learned, and partly upon the relationships they have developed with the discipline of mathematics. This focus on students' *disciplinary relationships* combines important mathematics education research on knowledge and practice (Cobb, 1998; Hershkowitz, 1999) with that on belief, disposition and identity (Christou & Philippou, 1998; DeBellis & Goldin, 1999).

The inter-relationship between the practices in which students engage in classrooms and the subsequent knowledge they develop, was demonstrated by a three-year study of students learning mathematics that I conducted in England

(Boaler, 1997). In that study I monitored 300 students who learned mathematics in very different ways. Students at one school learned through 'traditional' methods of watching and repeating standard algorithms and procedures, students at the other schools learned through open-ended projects. One of the findings of the research was that students' knowledge development in the two schools was *constituted* by the pedagogical practices in which they engaged (Boaler, 1997, 1999). Thus it was shown that practices such as working through textbook exercises, in one school, or discussing and using mathematical ideas, in the other, were not merely vehicles for the development of more or less knowledge, they shaped the forms of knowledge produced. One outcome, was that the students who had learned mathematics working through textbook exercises, performed well in similar textbook situations, but found it difficult using mathematics in open, applied or discussion based situations. The students who had learned mathematics through open, group-based projects were able to use mathematics in a range of different situations, partly because the classroom practices in which they had engaged were represented elsewhere.

In that study the students who had learned through open-ended projects outperformed the other students in a range of assessments, including the national examination. One conclusion that may be drawn from such a result, that would fit with cognitive interpretations of learning, would be that the students in the traditional school did not learn as much as the students who learned mathematics through open-ended projects, and they did not understand in as much depth, thus they did not perform as well in different situations. That interpretation is partly correct, but it lacks important subtleties in its representation of learning. A different analytical frame that I found useful, was to recognize that the students learned a great deal in their traditional mathematics classrooms. They learned to watch and faithfully reproduce procedures and they learned to follow different textbook cues that allowed them to be successful as they worked through their books. Problems occurred because such practices were not useful in other situations (Winbourne & Watson, 1998). When the students did not use mathematics effectively in different assessments, it was not because they had not met and 'learned' the mathematical knowledge, but because they tried to repeat the mathematical practices they had learned in the classroom and which shaped their knowledge (such as searching for cues and repeating standard methods) and these did not help them in non-standard assessments. Thus I came to understand the students' mathematical capability as an intricate relationship between mathematical knowledge and mathematical practices.

That study revealed an important dimension of mathematical capability that extended beyond knowledge, to the practices in which students engaged, but two subsequent studies that I will briefly summarise have raised further aspects of mathematical capability that extend beyond knowledge and practice. In the first study I and fellow researchers interviewed eight students from each of 6 Northern Californian high schools (Boaler & Greeno, 2000). The 48 students we interviewed were all attending advanced placement (AP) calculus classes. In that study four of the schools taught using traditional pedagogies – the teachers demonstrated methods and procedures to students, who were expected to reproduce them in exercises. In the other two schools, students used the same calculus textbooks, but the teachers did not rely on demonstration and practice, they asked the students to discuss the different ideas they met, in groups. In that study we found that students in the more traditional classes were offered a particular form of participation in class that we related to Belencky, Clinchy, Goldberger, & Tarule's notion of 'received knowing' (1986, p4). Mathematics knowledge was *presented* to students and they were required to learn by attending carefully to both teachers' and textbook demonstrations. The mathematical authority in the classrooms was external to the students, resting with the teacher and the textbooks, and the students' knowledge was dependent upon these authoritative sources. In these classrooms the students were required to receive and absorb knowledge from the teacher and textbook and they responded to this experience by positioning themselves as *received knowers* (Belencky et al, 1986).

The students who were learning in these traditional classrooms were generally successful, but we found that many students experienced an important conflict between the practices in which they engaged, and their developing identities as people (Wenger, 1998). Thus many of the students talked about their dislike of mathematics, and their plans to leave the subject as soon as they were able, not because of the cognitive demand, but because they did not want to be positioned as received knowers, engaging in practices that left no room for their own interpretation or agency. The students all talked about the kinds of person they wanted to be – people who used their own ideas, engaged in social interaction, and exercised their own freedom and thought, but they experienced a conflict between the identities that were taking form in the ebb and flow of their lives and the requirements of their calculus classrooms:

K: I'm just not interested in, just, you give me a formula, I'm supposed to memorize the answer, apply it and that's it.

Int: Does math have to be like that?

B: I've just kind of learned it that way. I don't know if there's any other way.

K: At the point I am right now, that's all I know. (Kristina & Betsy, Apple school)

Most of the students who told us about their rejection of mathematics in the 4 didactic classrooms – 9 girls and 5 boys, all successful mathematics students – had decided to leave the discipline because they wanted to pursue subjects that offered opportunities for expression, interpretation and human agency. In contrast, those students who remained motivated and interested in the traditional classes were those who seemed happy to 'receive' knowledge and to be relinquished of the requirement to think deeply:

J: I always like subjects where there is a definite right or wrong answer. That's why I'm not a very inclined or good English student. Because I don't really think about how or why something is the way it is. I just like math because it is or it isn't. (Jerry, Lemon school)

The students in didactic classes who liked mathematics did so because there were only right and wrong answers, and because they did not have to consider different ideas and methods. They did not need to think about 'how or why' mathematics worked and they seemed to appreciate the passive positions that they adopted in relation to the discipline. For the rest of the students in the traditional classes, such passive participation was not appealing and this interfered with their affiliation and their learning.

In the other two calculus classes in which teachers engaged students in mathematical discussions, a completely different picture emerged. In the discussion oriented classes the students had formed very different relationships with mathematics that did not conflict with the identities they were forming in the rest of their lives. The students in these classes regarded their role to be learning and understanding mathematical relationships, they did not perceive mathematics classes to be a ritual of procedure reproduction. This lack of conflict was important – it meant that the students who wanted to do more than receive knowledge, were able to form plans for themselves as continued mathematics learners. The type of participation that is required of students who study in discussion-oriented mathematics classrooms is very different from that required of students who learn through the reception and reproduction of standard methods. Students are asked to contribute to the judgment of validity, and to generate questions and ideas. The students we interviewed who worked in discussion-based environments were not only required to contribute different aspects of their selves, they were required to contribute *more* of their selves. In this small study we found the notion of identity to be important. Students in the different schools were achieving at similar levels on tests but they were developing very different relationships with the knowledge they encountered (Daskal & Simpson, 2000). Those students who were only required to receive knowledge described their relationships with mathematics in passive terms and for many this made the discipline unattractive. Those who were required to contribute ideas and methods in class described their participation in active terms that were not inconsistent with the identities they were developing in the rest of their lives.

That was a small interview study but it served to illuminate the importance of students' relationship with the discipline of mathematics that emerged through the pedagogical practices in which they engaged. In the final study that I will describe, a team of researchers is monitoring the learning of approximately 600 students as they go through three different high schools in the US. Two of the schools offer a choice of mathematics curriculum, which they describe as 'traditional' and 'reform' oriented. In the 'reform' classrooms we observe very different patterns of interaction than those in 'traditional' classrooms and as we work to understand the capabilities that are being encouraged by these examples of classroom interaction we are finding notions of identity and agency to be important.

The students in the reform classrooms we are studying, as in the project based school in England, are given the opportunity to use and apply mathematics, a process which confers upon them considerable amounts of human agency. Students are required to propose 'theories', critique each other's ideas, suggest the direction of mathematical problem solving, ask questions, and 'author' some of the mathematical methods and directions in the classroom. We are finding that the *nature* of the agency in which students engage in these classrooms is related to the discipline of mathematics and the practices of mathematicians in important ways. Such insights have emanated from an analytic frame proposed by Andrew Pickering (1995). Pickering studied the work of professional mathematicians and concluded that their work requires them to engage in a 'dance of agency' (1995, p116). Pickering considers some of the world's important mathematical advances and identifies the times at which mathematicians use their own agency – in creating initial thoughts and ideas, or by taking established ideas and extending them. He also describes the times when they need to surrender to the 'agency of the discipline', when they need to follow standard procedures of mathematical proof, for example, subjecting their ideas to widely agreed methods of verification. Pickering draws attention to an important interplay that takes place between human and disciplinary agency in mathematical work and refers to this as 'the dance of agency' (1995, p116).

Pickering's framework seems important for our analyses of the different practices of teaching and learning we observe. 'Traditional' classrooms are commonly associated with disciplinary agency, as students follow standard procedures of the discipline. 'Reform' classrooms, by contrast, are associated with student agency, with the idea that students use their own ideas and methods. The idea that students use their own ideas instead of learning standard methods is part of the reason that many are concerned about 'reform' approaches in the US, but we do not see students failing to learn standard methods in our observations of 'reform' classrooms. Rather than a group of students wandering unproductively, inventing methods as they go, we see a collective engaged in the 'dance of agency'. The students spend part of their time using standard methods and procedures and part of the time 'bridging' (Pickering, 1995, p11) between different methods, and modifying standard ideas to fit new situations. In many of the traditional classrooms I have studied, in this and previous years, students have received few opportunities to engage in the 'dance of agency', and when they need to engage in that 'dance', in new and 'real world' situations, they are ill prepared to do so. When I interviewed a class of students in the fourth year of the reform program at one of the schools, the students all described an interesting relationship with mathematics that contrasted with the students who had learned calculus in traditional classes. As part of the interviews we asked students what they do when they encounter new mathematical problems that they cannot immediately solve. In the extracts below the students give their responses:

K: I'd generally just stare at the problem. If I get stuck I just think about it really hard and then just start writing. Usually for everything I just start writing some sort of formula. And if that doesn't work I just adjust it, and keep on adjusting it until it works. And then I figure it out. (Keith)

B: A lot of times we have to use what we've learned, like previous, and apply it to what we're doing right now, just to figure out what's going on It's never just, like, *given*. Like "use this formula to find this answer" You always have to like, change it around somehow a lot of the time. (Benny)

These students seem to be describing a 'dance of agency' as they move between the standard methods and procedures they know and the new situations to which they would apply them. They do not only talk about their own ideas, they talk about adapting and extending methods and the interchange between their own ideas and standard mathematical methods. The student below talks in similar terms:

E: Like, if nothing else, it's breaking out of the pattern of just taking something that's given to you and accepting it and just going with it. It's just looking at it and you try and point yourself in a different angle and look at it and reinterpret it. It's like if you have this set of data that you need to look at and find an answer to, you know, if people just go at it one way straightforward you might hit a wall. But there might be a crack somewhere else that you can fit through and get into the meaty part. (Ernie)

Many of the students in the traditional classrooms I have studied frequently 'hit a wall' when they were given mathematics problems to solve. They tried to remember standard procedures, often using the cues they had learned. If they could remember a method they would try it, but if it did not work, or if they could not see an obvious method to use, they would give up. The students we interviewed in the reform classes described an important practice of their mathematics classroom – that of working at the interplay of their own and disciplinary agencies – that they used in different mathematical situations. Additionally the students seemed to have developed identities as mathematics learners who were willing to engage in the interplay of the two types of agency. The students had developed what we are regarding as a particular *relationship with the discipline of mathematics* that meant that when they met new mathematics problems they *expected* to adapt and apply methods to solve the problems. It seemed that their capability in different situations depended partly upon the knowledge they had learned, partly upon the practices in which they engaged and partly upon the relationships they had developed with the discipline of mathematics that emerged through the practices of their classrooms.

A number of researchers have written about the importance of productive beliefs and dispositions (Schoenfeld, 1992; McLeod, 1992) but the idea of a 'disciplinary relationship' serves to connect knowledge and belief in important ways. Herrenkohl and Wertsch (1999) have suggested a notion that addresses this connection, that they call the 'appropriation' of knowledge. They distinguish between mastery and appropriation, saying that too many analyses have focused only upon students' mastery of knowledge, overlooking the question of whether students 'appropriate' knowledge. They claim that students do not only need to develop the skills they need for critical thinking, they also need to develop a disposition to use these skills. In claiming that students need to 'appropriate' knowledge, they suggest a connection between the content students are learning and the ways they relate to that knowledge. The fact that the students who learned through open-ended projects in England were able to use mathematics in different situations may reflect the similarity in the practices they met in different places, but it also reflects the fact that they had developed a positive, active relationship with mathematics. They expected to be able to make use of their knowledge because of the opportunities they had received to engage in a disciplinary dance. Thus they were able to 'transfer' mathematics, partly because of their knowledge, partly because of the practices in which they engaged, and partly because they had developed an active and productive relationship with mathematics. This idea seems to pertain to theories of learning transfer and expertise in important ways, expanding notions of capability beyond knowledge and practice to the dispositions they produce and the relations between them.

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AUTHENTIC LEARNING CONTEXTS AS AN INTERFACE FOR THEORY AND PRACTICE

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Our research over the past three years has explored the belief that authentic learning contexts in teacher education can provide an interface for theory and practice. We discuss key features in our teacher education program: constructivism, situated learning, and multiple authentic learning contexts. Data drawn from our research will be used to support our contention that the more authentic the contexts the more effective the learning.

Teacher education is in crisis in many countries of the world, with pressure to move to more school-based approaches an indication of the dissatisfaction with traditional approaches (Korthagen & Kessels, 1999). A major point of dissatisfaction is the apparent inability of beginning teachers to transfer the theory-based knowledge into classroom practice. This link—or lack of it—between theory and practice in teacher education is well documented (Richardson, 1996; Wubbels, Korthagen, & Brekelmans, 1997).

Research by Brouwer (in Korthagen et al. 1999) found that the degree to which teacher education programs integrated and alternated theory and practice was an important factor in determining the extent to which beginning teachers could translate their knowledge into practice. How educational programs and systems have tried to accommodate this factor into their courses has varied greatly. Few details pertaining to actual program structures is publicly available. However, our own experience within various institutions in Australia and contacts with other teacher educators indicates that programs most commonly try to achieve theory/practice integration via field-based assignments such as action research, by adopting realistic-based approaches such as constructivism, and the development of reflective skills. These approaches are not mutually exclusive, but implemented on an individual basis have not normally proved to be as successful as expected (Foss & Kleinsasser, 1996; Fuson, Carroll, & Drueck, 2000). For example, in response to the growing support for teaching mathematics from a constructivist perspective (Australian Education Council, 1990; Department of Employment, Education and Training, 1989), a number of teacher education programs adopted a constructivist approach in the way their courses were delivered. According to constructivist principles, this translated into establishing a learning environment in which students construct their own knowledge by linking prior experiences (including knowledge, beliefs and personal theories) to new knowledge (Jones & Vesilind, 1996), creating 'learning communities' in which students engage in rich discourse about important ideas (Putman & Borko, 2000) and using reflection as a vehicle for reconceptualising knowledge and beliefs (Beattie, 1997). It was anticipated that such modelling of a constructivist approach within teacher education programs would translate into classroom practice. Unfortunately, accumulating evidence suggests that such initiatives have failed to strongly influence the practices of beginning teachers (Klein, 1999) and are often broken-down during practical experiences (Zeichner & Tabachnick, 1981).

The investigation reported here was undertaken after anecdotal evidence relating to the mathematics education courses taught by us directly contradicted the disappointing findings reported in the literature (e.g. Klein, 1999). Our research over the past three years has explored the belief that authentic learning contexts in teacher education can provide a theory/practice interface which will support graduates translate their knowledge into classroom practice. The following discussion describes the context for our study. It provides an introduction to the Master of Teaching program and outlines key features in the mathematics education component: constructivism, situated learning and multiple authentic learning contexts. Data drawn from our study will be used to support our contention that the more authentic the contexts the greater the integration of theory and practice. We also believe that multiple learning contexts will further enhance this interface.

CONTEXT OF THE STUDY

The Master of Teaching (MTeach) program within the Faculty of Education at the University of Sydney is a postgraduate initial teacher education course that adopts an inquiry and case-based approach. Such approaches typically provide students with opportunities to research their own teaching and learning. Generally, the students are mature-aged, have established careers in areas other than teaching and bring with them a variety of life experiences. Central to the MTeach program's philosophy is the importance of developing reflective practitioners.

Key features of the mathematics components

We have designed our primary mathematics education courses to integrate and encourage reflective practice within a constructivist approach to support the philosophy of the MTeach program. Inquiry into our own practices has led us to believe that adhering to constructivist principles in our teaching is a necessary but certainly not sufficient factor for engendering change in the way our graduates view and teach mathematics to primary school children. We now recognise the crucial role played by our use of multiple learning contexts, situated in a variety of environments.

A constructivist approach to teaching and learning recognises the importance of students' prior experiences and uses these experiences to build knowledge of the field. It aims to create a learning environment in which peer tutoring and collaborative learning is highly valued. Associated with this is the idea that students need to have opportunities to engage in rich discourse as they share and build their knowledge. Another vital aspect of such an approach, and central to the MTeach philosophy, is the emphasis placed on reflection. For us, reflection is seen as a vehicle by which students make connections between theory and practice.

A situated perspective on learning (Greeno, 1997) acknowledges that: all knowledge is situated, that some types of knowledge are best constructed in one context rather than another and that the more authentic the context, the more effective the interplay between theory and practice (Aldridge & Bobis, 2001). This fits within the constructivist paradigm and supports our use of multiple learning contexts.

Our prospective primary school teachers experience four contexts whilst undertaking our mathematics education courses. The first context places students in workshops modelled on constructivist principles. A second context relates to in-school experiences associated with practice teaching. Students work in a school over an extended period (12-15 days) under the supervision of a practising teacher. This not only reflects the context in which they will ultimately work, but it is also a fairly typical experience provided by teacher education programs across various institutions.

Prior to undertaking their first traditional-style practical teaching experience we also provide our students with another vital context for learning—one that is not typically used by other institutions responsible for teacher education. This is best described as school-based small group teaching and is situated in schools (an authentic context) during normal tutorial timeslots. Students work in small groups (encouraging collaboration and reflection) with two or three children who are assigned to them for four weekly sessions. They individually assess the children, analyze the results and collaboratively prepare a program of work to build on each child's level of thinking. Our role is to facilitate the process by providing advice at each stage of the process. After each teaching episode, there is a debriefing session. This provides opportunities for rich discourse stimulated by attempts to connect their practice to theory. It provides them with an authentic context in which to situate their new knowledge prior to undertaking their first extended practice teaching experience.

The fourth context is situated at the University in a clinical setting. Students work twice a week over a six week period with children from local schools in a one-on-one situation. This experience provides another context in which students' learning is authentically situated. As facilitators in this context, we may model teaching if requested, or simply offer guidance and support students in their decision making. Students also work together to share ideas, discuss teaching strategies, team-teach and reflect upon the theory/practice integration.

DATA COLLECTION

Data was collected using two tools—concept mapping and semi-structured interviews. Concept mapping helped the students organise their knowledge and beliefs about mathematics and was used as a stimulus in the interviews to elicit explanations of their personal theories about mathematics and their teaching. Figures 1 and 2 were both constructed by the same preservice teacher, Cindy. While each participant's maps were unique, we have selected Cindy's as representative of an initial concept map and a map constructed at the third key point in the program. The first concept map (see Figure 1) was undertaken prior to any courses in mathematics within the MTeach program and the second map was constructed after completing her first practice teaching experience. Each map provides a visual representation of how an individual's thinking about mathematics is organised at different points in the MTeach program. Key ideas are clearly identified for later discussion in semi-structured interviews.

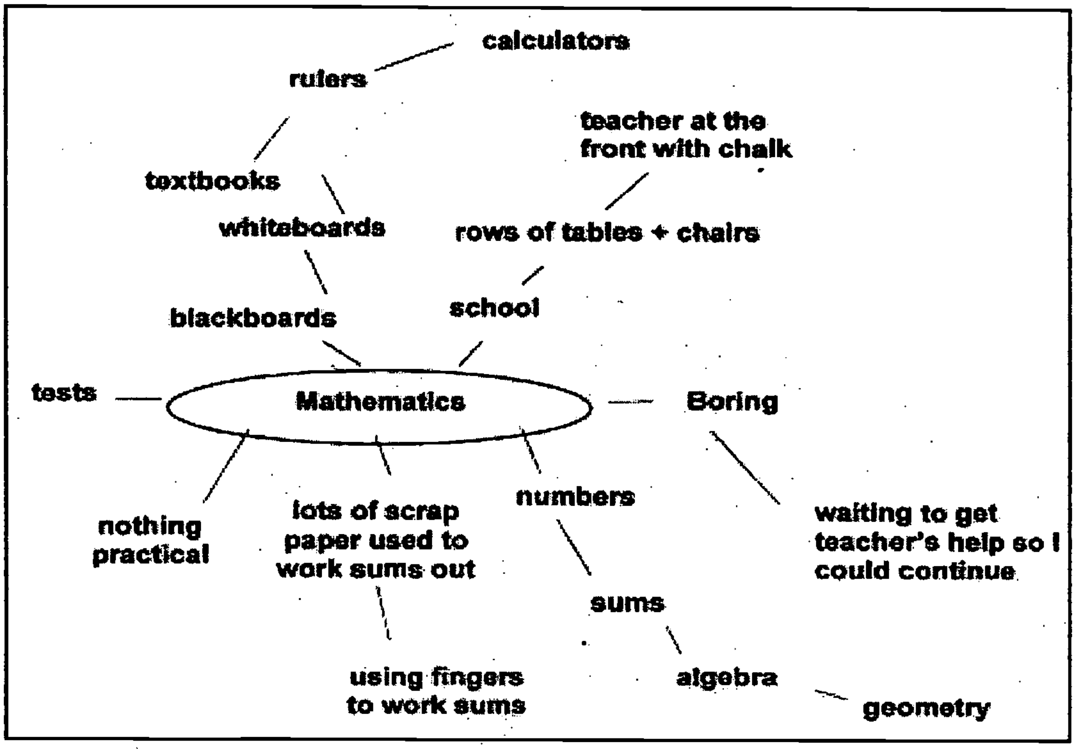

Figure 1. Cindy's first concept map, constructed prior to undertaking any mathematics education units of study.

Comparing an individual's concept maps enabled us to see how each prospective teacher reconceptualised their knowledge during the course of the MTeach program. Figure 2 reflects a more complex understanding of mathematics and its teaching. This is evidenced by the increase in the number of concept 'nodes' and the links between separate branches of the map.

At four key points within the MTeach program, twelve preservice teachers undertook a concept mapping exercise followed by a semi-structured interview. These key points corresponded to the four different learning contexts within the mathematics education program. The results from the concept maps and the semi-structured interviews over the time of the project were combined and compared for analysis. Patterns in responses and changes in emphasis were also identified and coded for analysis.

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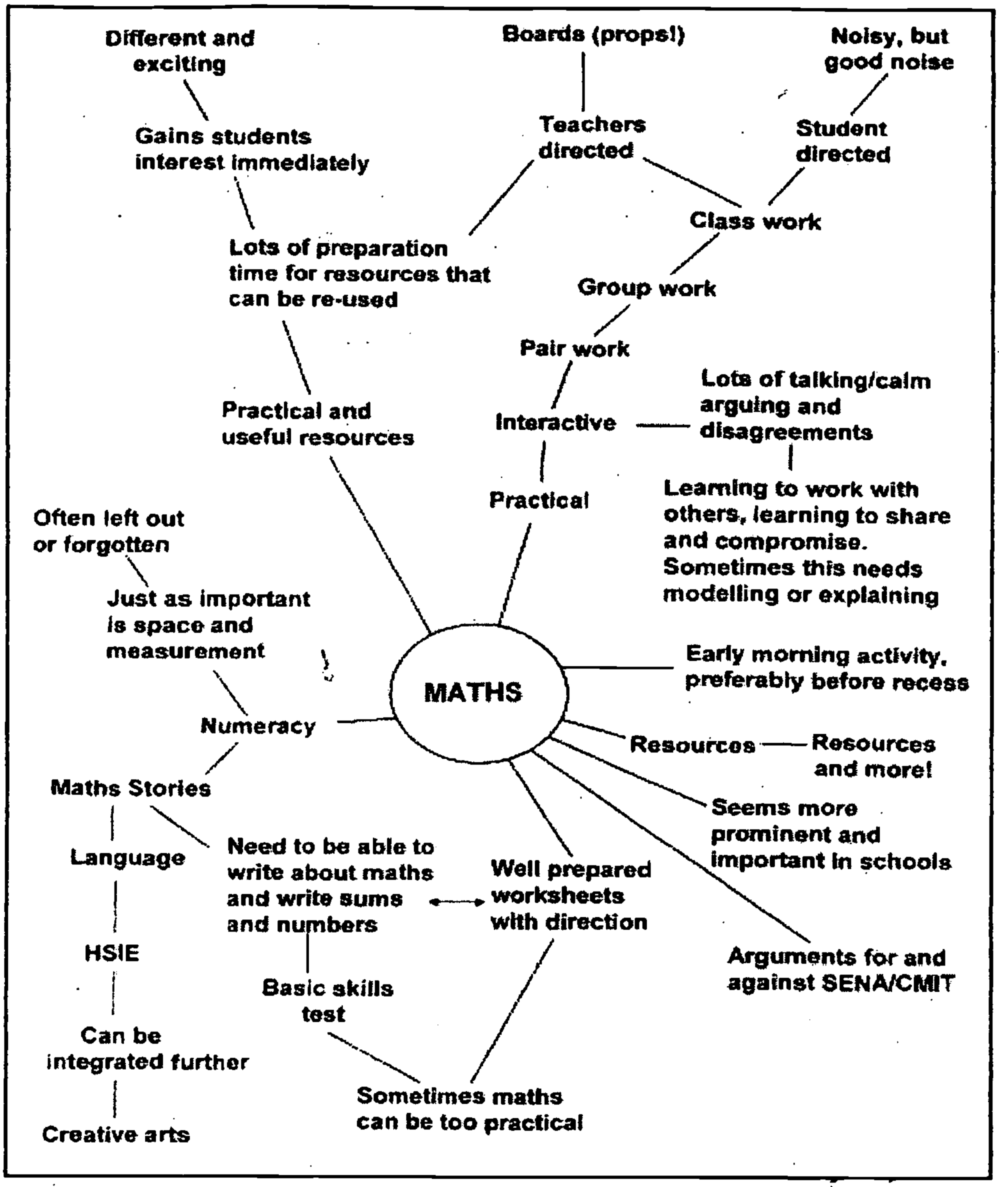

Figure 2. Cindy's third concept map, constructed after her first practice teaching experience.

KEY ASPECTS FOR A THEORY/PRACTICE INTERFACE

Analysis of the concept maps and semi-sturctured interviews revealed the emergence of a number of themes. For the purpose of this paper we will focus on two key aspects. Both aspects have been identified in the literature as playing significant roles in the integration of theory and practice (Korthagen, 1999, Wubbels, Korthagen & Brekelmans, 1997) and are supported by our findings.

The first aspect that was identified in the majority of interviews was the value of the school-based context. Examples of students' comments reveal an awareness that this context provided a valuable link between the theory of the University-based work with their first practice teaching. For instance, Cindy commented in her third

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interview that "we learnt it straight away, blew our minds, applied it and could see the benefits...that was a stepping stone and it wasn't so daunting when we went into the classroom". Similarly, Carmel remarked in her fourth interview that "the school experience was good because you get to use what you are learning". She considered it influential because "when we go out there and have to do it...that is reality." In her third interview, Carmel commented on the benefits of the course structure. She considered the change in contexts beneficial because they "required her to teach on a more complex scale" towards the end of program.

Comments from student teachers reflect what Brouwer (in Korthagen et al. 1999) describes as a teacher education curriculum that has an integrative design. This refers to the extent to which there is an alternation and integration between theory and practice within a program. Brouwer claims that programs designed in this way promote transfer from theory to practice.

A second theme to emerge from the interviews and supported in the concept maps was a reconceptualisation of mathematics and the teaching of mathematics. For example, Cindy's first concept map (see Figure 1) reflects an emphasis on mathematics as content based (e.g. numbers, sums, algebra, geometry), a transmissive model of teaching and an attitude to mathematics that can be described as negative (e.g. boring). Cindy's third concept map (see Figure 2) shows a much broader view of mathematics and reflects a growing awareness of her own personal theory about how to teach and how children learn mathematics.

One aspect of this reconceptualisation relevant to the current discussion involves the change in participants' attitudes and beliefs about mathematics (see Aldridge & Bobis, 2001). Changing attitudes and beliefs is recognised as a vital aspect in a person's ability to transfer theory to practice (Corporal cited in Korthagan et al. 1999). This is particularly important because student teachers begin their courses with a history that corresponds to their initial personal theories. Often this theory contains notions about mathematics and its teaching that are very different from the theories espoused in the University-based context. The need to acknowledge the student teacher's history and construct a teacher education theory that builds on and challenges this is seen as important if transfer between theory and practice is to take place.

A NEW PARADIGM FOR TEACHER EDUCATION

Evidence provided in this paper supports our contention that the utilisation of a multiplicity of authentic learning contexts, combined with a constructivist perspective can provide a vehicle for connecting theory and practice in teacher education programs. Such a theoretical base is akin to the 'realistic approach' to teacher education characteristic of the program at Utrecht University (Koetsier, Wubbels, & Korthagen, 1997) and represents a paradigm shift in teacher education. However, such an approach not only requires that the types of contexts be thoughtfully considered, it also has organisational implications for the program administrators. For example, to alternate between university-based and school-based contexts requires close co-operation between institutions. Flexibility in time-tabling, the proximity and suitability of the school-based setting, as well as the mutual benefits from such arrangements must all be considered.

Another element we consider important is our own credibility. To feel 'comfortable' in both school-based and university-based contexts our own knowledge and experience must be credible in our student teachers' eyes.

The paradigm shift in teacher education has benefits to us as teacher educators. For example, researching and reflecting on our own practice has enabled us to anticipate barriers to the successful integration of theory and practice and has allowed us to deal with such obstacles prior to them surfacing in practical situations. For example, the reliance on mathematics textbooks in Australian primary classrooms is perceived to be an obstacle for the translation of theory into practice by our students.

In addition, utilising authentic learning situations allows us to "stay in-touch" with the realities of the school context and with the needs and concerns of our prospective teachers. Implications of such an approach extend beyond initial teacher education and have ramifications for the professional development of all teachers.

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GEOMETRIC SIGNS AND STUDENTS' VERBAL REPORTS: THE CASE OF THE GEOMETRIC MODEL OF SUN SHADOWS

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I analysed a large amount of written verbal reports, produced by Grade IV and Grade VI students as a response to the task of communicating their knowledge about sun shadows. The texts were produced immediately before and soon after the introduction of the elementary geometric model of sun shadows by the teacher. Some relevant changes were detected; they concerned a much more frequent production of hypothetical and causal sentences expressing geometrical links in the 'after' texts. A discussion about related cognitive, cultural and educational issues is sketched.

INTRODUCTION

In a Vygotskian perspective (see Vygotsky, 1978, Chapters I and VI), important changes can intervene in students' thinking strategies when the teacher introduces some peculiar signs as tools to solve problems. In particular, the signs introduced may allow students (by themselves, or with the help of more competent peers, or under the guidance of the teacher) to solve previously inaccessible problems. Still in a Vygotskian perspective (see Vygotskij, 1990, Chapter VI), spontaneous students' conceptions can develop towards scientific conceptions when scientific conceptions are made accessible to them by the teacher through appropriate (external) representations. In the study reported in this paper I am interested in the effects of the introduction by the teacher of a specific sign (the elementary local geometric model of sun shadows – shortly, ELGMS) on the *ways of thinking* about the sun shadow phenomenon.

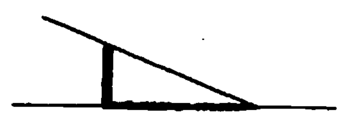

Fig. 1: The ELGMS

As reported in Boero et al (1995), and as we will see in details in the next Subsection, the ELGMS is not spontaneously produced by students neither in Grade IV, nor in Grade VI. Moreover, it is very far from the graphic representations of the sun shadow phenomenon, which are spontaneously produced by most of them (see later). And the ELGMS appeared relatively late in the historical development of human cultures as a relevant invention, whose cultural implications were very rich (see Serres, 1993). The research hypothesis underlying this study is that *the appropriation of the ELGMS as a tool to solve elementary geometrical modelling problems concerning the sun shadow phenomenon can deeply change the way of thinking about this phenomenon.* Some experimental evidence will be provided to support this hypothesis. In particular, by analysing students' verbal reports concerning the sun shadow phenomenon we will see that an important change can be traced in students' reports after the introduction of the ELGMS: explicit hypothetical and causal links between the height of the sun and the lengths of the cast shadows become much more frequent. The final Discussion Section will elaborate on this result. Some cultural and educational implications will be discussed (in particular, as concerns the meaning of the ELGMS as a prototypical thinking tool belonging to the 'rationality' of the Western civilisation). In general, my hypothesis (together with the related cultural issues) agrees with recent developments of research in the Vygotskian perspective. Stetsenko (1995) wrote:

"The originality of the Vygotskian approach to children's drawings is primarily that it addresses and clarifies the functional role of drawings in the overall development of the child – that is, in the entirely of cognitive, emotional, communicative and other aspects of this development." (page 147). "What the cultural-historical theory strives at is a precise specification of the unique ways making and looking at pictures help a child both to understand the world and come to terms with it"(page 148).

METHOD

The Students' Educational Background

Since the second half of the 70s, both Genoa Group Projects for mathematics and science education in primary school (6-11) and lower secondary school (11-14) have devoted a wide interest to the sun shadow phenomenon: from early non-geometric conceptions to local geometric modelisation, till global considerations of the phenomenon on the sun system scale. The level of sophistication of the mathematical tools introduced and the difficulty of the mathematical problem situations tackled at the end of the activities is obviously different in the two Projects. Most of 'our' primary school students do not join classes that adopt the Lower Secondary School Project and most of 'our' lower secondary school students do not come from classes involved the Primary School Project, so the initial steps in the approach to the sun shadow phenomenon are the same in both Projects (even if they are introduced at a different pace). The activities concerning sun shadows take place over a very long period of time in primary school (from the second half of Grade III to the beginning of Grade V). They take place over the whole school year in Grade VI. In both cases, the activities start by provoking students to write and draw what they think about sun shadows; a number of games played in the courtyard follows; then the 'shade space' (between the object and the cast shadow) is discovered; finally, more and more systematic observations are organised (at different times of the same day) and verbally reported by students. In particular, these are common initial steps:

- standardised questions about the sun shadow phenomenon, and discussion about the answers. In particular, the following question is posed both in grade III and at the beginning of grade VI :

«*Have you ever noticed that when you are walking in a sunny place your body casts a shadow on the ground?* (YES / NOT option). *Is your shadow longer at 9 a.m. or at noon ?* (9 a.m. / NOON option). *Why ?*» .

It is interesting to remark that the majority of students in Grade III, and still more than 40% in Grade VI, chooses the 'NOON' option *"because the sun is stronger"*, *"because I see it better"*, etc. (for details, see Boero, 1999)

- production of drawings representing the sun shadow phenomenon. We can notice that most of these drawings (both in Grade III and in Grade VI) are very far from the ELGMS and correspond to non-geometric conceptions. Here we present only some examples; for further details, see Boero, 1999.

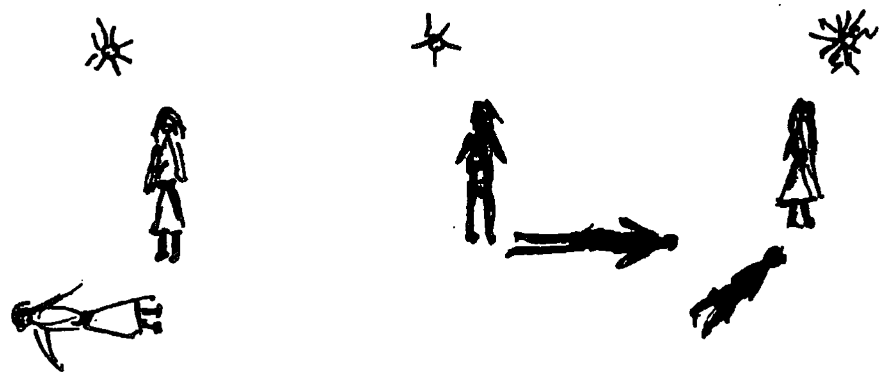

Fig.2: Three examples of students' initial drawings

As remarked in Boero et al (1995) (for further details, see Boero, 1999), the ELGMS is spontaneously produced by very few students, both in grade III or IV and in grade VI, even after all the previously mentioned activities; and it does not spread spontaneously across the classroom. It can be observed that this fact is in accordance with the importance attributed to the invention of the ELGMS by historians of Science: the first traces of the model go back to the late developments of Babylonian and Egyptian civilisations and the early developments of Greek geometry, between the VII and the V century b.c. (Serres, 1993). In the Primary School Project classes, the teacher introduces the ELGMS at the beginning of Grade IV (after 3-4 months of activities in Grade III); in the case of the Lower Secondary School Project classes the introduction of the ELGMS takes place one month after the activities described above. The teaching strategies adopted by teachers to introduce the ELGMS may vary according to personal preferences, theoretical motivations and occasional circumstances: in some cases the teacher exploits the proto-geometrical drawings produced by some students, asking the other students to use them to solve other problems (but it can happen that no student produces an 'exploitable' drawing!); in some cases the teacher introduces the ELGMS as a device to solve problems; in other cases the teacher tries to guide the production of the sign by the students through suitable observations (e.g. the visualisation of the upper border of the shade space, and the task of drawing such situation), then some applications follow (for a discussion about these educational strategies see Scali, 1998).

Available Data and Selected Data

The fact that the sun shadow phenomenon was a crucial subject for both Projects for more than twenty years, together with some methodological choices (in particular, the systematic practice of written verbal reporting for all activities since the end of grade I, and the practice of frequent classroom discussions guided by the teacher), offered a big amount of interesting materials from the classroom: rich individual texts, videotapes and recordings of classroom discussions, etc., concerning the activities described in the previous Subsection. In particular, 24 teachers gathered individual students' texts from 81 classes with detailed information about the activities performed! Given that I am interested in the consequences of the introduction of the ELGMS on students' ways of thinking the sun shadow phenomenon, I looked at the texts produced immediately before, and/or soon after, the introduction of the ELGMS. I considered only the texts produced as a response to a standardised task:

> *«Write a letter to a friend of yours in order to explain him what you know about sun shadows at this moment».*

I took into account the classroom activities between the introduction of the ELGMS and the production of the 'after' text, as reported by the teachers. My preliminary choice was to consider only texts coming from classes who had worked *individually* in that period on two or three applications of the ELGMS over a period of no more than two weeks. According to the collected information, in all these classes, after the introduction of the ELGMS, the students could look at it on the walls and also in their personal copybooks. Then only texts including explicit reference both to the sun and to the shadows were considered (about 60% in grade IV and 68% in grade VI).

Some preliminary analysis performed on relatively small samples of students (two or three classes in each case) had shown that:

- the way chosen by the teacher to introduce the ELGMS does not seem to affect the quality of the 'after' texts;

- concerning the use of texts produced by the same students or not (before and after the introduction of the ELGMS), the only difference was that the second production of the same student was in some cases less rich and less suitable to use than the first one (the repetition of the identical task at the distance of 10-15 days did not help motivation!). So I decided to select a rather big number of classes who had written *only one* report, immediately before *or* soon after the introduction of the ELMGS, for a large scale comparison between 'before' and 'after' texts. Selection was made in order to get similar social environments between the different groups of students. I have considered 202 'before' texts and 206 'after' texts for grade IV; and 182 'before' texts and 170 'after' texts for grade VI. I performed a more detailed analysis on other two smaller groups of students (80 for grade IV and 66 for grade VI) who had produced *both* the first *and* the second text, in order to trace the personal evolution of their performances (see table 3, concerning grade IV).

Criteria for Classifying Students' Reports

- **Conditional** texts: at least once in the report, the student expresses conditional links between the height of the sun and the length of the sun shadows, e.g.: *«If the sun is high the shadows are short».*

- **Causal** texts: at least once in the report, the student expresses causal links between the height of the sun and the length of the sun shadows: *«At noon the shadows are shorter than at nine because the sun is higher»; «At noon the sun is higher than at nine, and so the shadows are shorter; "At noon the shadows are short because the sun is high"*, etc.

- **Descriptive** texts: all the other reports. The student reports what he/she saw at different times of the day, with no explicit 'conditional' or causal link between the height of the sun and the length of the sun shadows: *«At nine a.m. the sun is low and the shadows are long; at noon the sun is high and the shadows are short».*

Some comments about the proposed classification follow.

- Some reports can belong both to the first and the second category. We can remark that causality is very close to 'conditionality' in many situations of communication. In particular, in everyday life situations people frequently use the clauses *«If B, then A», «B, and so A»* and *«A because B»* as if they were equivalent. I preferred to make a distinction between Conditional reports and Causal reports because I had observed that in our specific situation Conditional reports are usually produced by students in order to express a general 'conditional' link (*«if the sun is high the shadows are short»*), while most of the causal reports refer to a specific situation (*«at noon shadows are short because the sun is high»*).

- In Italian, like in other languages, the *«A and B»* clause can be used to suggest the idea of a causal link between A and B. In the sentence *«The driver was running very fast and the car got out of the road»* we can see the intention of implicitly stating a cause-effect relationship. As a consequence, part of the Descriptive reports (before as well after the introduction of the ELMGS) may have been produced by students who were thinking about a causal relationship between the height of the sun and the length of the shadows.

- There would be some reasons for the inclusion of reports containing a 'when' clause (*«when the sun is high, the shadow is short»*) in the category of Conditional reports: the '*A when B*' and the '*if B, then A*' clauses often have similar uses in everyday language; and the 'conditional' clause seems to represent a de-timing of the 'when' clause (see Arzarello, 2000). But in most of students' reports presenting only the 'when' clause the whole linguistic context gives the impression that the 'when' clause merely expresses a temporal coincidence with no 'conditional' link between the information about the position of the sun and the information about the length of the cast shadow. Some interviews (performed immediately after the production of a written report containing one 'when' clause) confirmed this impression.

SOME RESULTS

The following tables display some outcomes of the analysis performed according to the criteria listed in the previous Subsection.

	Total number	Conditional	Causal	Descriptive
'Before' texts	202	38 (19%)	28 (14%)	146 (72%)
'After' texts	206	72 (35%)	75 (36%)	81 (39%)

Table 1: 'Before' texts and 'after' texts of two different groups of Grade IV students

	Total number	Conditional	Causal	Descriptive
'Before' texts	182	40 (22%)	27 (15%)	127 (70%)
'After' texts	170	56 (33%)	70 (41%)	64 (37%)

Table 2: 'Before' texts and 'after' texts of two different groups of Grade VI students

	'Before'texts	Conditional	Causal	Descriptive
'After' texts	80	26	28	34
Conditional	15	8	5	2
Causal	9	1	7	1
Descriptive	59	17	16	31

Table 3: 'Before' and 'after' texts of the same 80 IV-grade students

Table 3 provides information about the evolution of students' productions within the same group of students. For instance, let us consider the last row. 59 students had produced Descriptive texts 'before' the introduction of the ELGMS. 'After' the introduction of the ELGMS, 17 out of them produced a Conditional text; 16 produced a Causal text; while 31 still produced a Descriptive text. By considering these numbers on the last row we get 64 texts: it means that 5 texts became " Conditional"*and* "Causal".

Data concerning the comparison of 'before' and 'after' texts of the group of 66 VI-grade students were similar to those displayed in Table 3.

All tables show a common trend: a statistically relevant increase in the percentage of both Conditional and Causal texts between the 'before' texts and the 'after' texts. The increase is more relevant for Causal text (more than 100% increase in all tables).

DISCUSSION

By analysing students' verbal reports concerning the sun shadow phenomenon we have seen that, after the introduction of the ELGMS, an important change happened in students' reports: explicit hypothetical and causal links between the height of the sun and the length of the cast shadows became much more frequent. The fact that this change happened over a very short period of time (no more than two weeks), both in grade IV and in grade VI, suggests that spontaneous maturation cannot explain the change. The fact that no *collective* classroom activity intervened between the introduction of the ELMGS by the teacher and the second verbal report seems to exclude the possibility of the adoption of those *«modes of saying»* by imitation. And the very nature of the change suggests that the introduction of the ELGMS affects not only the *way of describing* the sun shadow phenomenon, but also the *way of thinking* about it (although in general it is difficult to infer the underlying thinking processes from verbal traces: cf Ericsson and Simon, 1981). It is interesting to examine this *way of thinking* more closely. Its importance does not seem to rely much on the fact that the ELGMS fits better with the sun shadow phenomenon in everyday life experiences, in comparison with other ways of conceiving it. Let us consider an example taken from some observations performed by Claudia Costa, a teacher working in the Italian school in Asmara (Eritrea) (see Boero, 1999). From individual interviews and classroom discussions it emerged than half of the students of two VII-grade classes thought that the shadows were longer in the early morning and in the late afternoon because then the Sun was less strong (or less bright), and so the shadow (a manifestation of the darkness, the opposite 'entity') succeeded in being longer. This non-geometrical conception fits rather well with many everyday life experiences! The cultural importance of the ELGMS relies on the fact that the 'rationality' inherent in it is different from the 'rationality' inherent in the non-geometrical conceptions that agree with empirical evidence. In order to understand this difference let us consider the following statements, produced by one of 'our'students and by an Asmara student:

"At noon the shadow is shorter than in the early morning because the sun is higher";

"At noon the shadow is shorter than at 9 because the sun is brighter and beats the darkness"

The validity of the first statement relies on geometrical necessity, while the validity of the second statement relies on the consideration of the increasing strength of the light, which changes the 'strength equilibrium' with the opposite 'entity'. The introduction of the ELGMS (a product of the cultural evolution) brings in a new kind of 'necessity' in the way of thinking about nature: a geometrical necessity. Western 'rationality' has strongly developed towards this direction over the last twenty five centuries, not only as concerns the geometrical models in astronomy, but also as regards for example the differential models, the stochastic models, etc.: it is like if 'something' happens, and will always happen in the future, because the inherent variables are constrained according to a given mathematical model. This provides us with a very efficient tool to solve quantitative problems, forecast the evolution of many phenomena, etc.

At the beginning of the activities concerning sun shadows, many students of 'our' classes produce 'causal' texts when they describe the sun shadows phenomenon, but causality does not concern the links between the height of the sun and the length of the cast shadows (e.g.: "*The shadow is longer at noon, because the sun is stronger*"; or "*The shadow is longer at noon, because I see it better*". The contradiction with empirical evidence and systematic observations gradually brings most of them to write that "*At noon the shadow is short and the sun is high*". This could be explained well in terms of the Piagetian "adaptation" of mental representations (see Piaget, 1926). Afterwards, students use the ELGMS (introduced by the teacher) in few problem situations. This is sufficient to change the quality of the verbal reports of many of them (see our data). The following activities reinforce this trend. An entirely different situation happens with the Asmara students: even the Asmara students wrote (in their individual initial texts) a lot of rather complex causal and hypothetical sentences. The difference is that they produced those sentences coherently with a way of thinking about the sun shadow phenomenon, which seems to be strongly related to the environmental culture (and not in immediate contradiction with empirical evidence). This means there is a potential richness that schools should not waste. Why, and especially how, to introduce another way of thinking (the one based on the ELGMS) in such a situation, without destroying the existing one, remains a difficult challenge for nowadays intercultural educational perspectives! (cf Barton, 1996)

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LEARNING THROUGH IDENTITY: A NEW UNIT OF ANALYSIS FOR STUDYING TEACHER DEVELOPMENT

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Here we develop a unit of analysis for our study of new teacher development across varied communities of practice (Lave and Wenger, 1991). We do so by combining Shulman's (1987) heuristic of important teacher knowledge with Wenger's conception of learning as identity development while practicing in community. To align ourselves with the view that learning takes place across a spectrum of locations from in-the-mind to socially-embedded (St. Julien, 1997), and with Lerman's call for researchers to consider a unit of analysis that is "person-in-practice-in-person" (2000, p. 38), we develop the overlapping aspects of self-in-mind and self-in-community. These span the spectrum of the cognitive-social continuum, and comprise what we consider to be a person's mathematics teacher identity.

THE STUDY OF LEARNING

How people learn is clearly an open question, and for good reason. Learning how to do things like cry to get what one wants as an infant, add in one's head, participate productively in a work environment, and split atoms certainly are complex and involve many forms of thinking. These can be viewed as happening in different contexts with different intentions, and done within different relationships between physical and social surroundings and mental effort. The learning of developing teachers – which entails the areas of content, pedagogy, and professional participation, and which happens in many contexts including university classrooms, school classrooms, and professional community interactions – involves a large cross-section of the various modes of thinking. As such, it presents a particularly difficult challenge to researchers of learning. This paper presents a framework that we are using to study this complicated landscape of things to be learned by mathematics teachers and ways to learn them.

Researchers have studied learning in mathematics along a broad spectrum of modes of thinking, from the abstracted types that happen largely in one's brain, to those that happen in relation to a very specific context of use. Such work has been productively undertaken using various theoretical frameworks and different units of analysis. Carpenter and Fennema (Carpenter, Fennema et al., 1988) used a fully cognitive model to understand young people's learning of basic addition and subtraction of integers. Greeno (1994) expanded somewhat on the focus of strictly cognitive studies by adding aspects of the physical/structural environment as factors affecting individuals' mental processes. Cobb and his colleagues (Cobb and Bauersfeld, 1995) moved further away from wholly psychological accounts of learning to include interactions between groups of individuals learning mathematics in classrooms, and the norms of interaction that develop. Taking a further step away from the individual as a unit of analysis are situative theorists, led by Lave (1988), who have studied everyday mathematics and the ways that the learning and usage of such mathematics is completely tied to the embedding contexts.

Such research provides a continuum of entry points for studying learning. However, there has been much debate among leading scholars about what can be learned from each perspective (Anderson, Reder et al., 1996; Greeno, 1997). We agree that there are valid concerns. However, we follow others (Greeno and MMAP, 1998) in our belief that the different perspectives on this continuum each offer potentially valuable information about learning. The problem seems to be that in order to be truly useful for educators, cognitive studies need to consider more of the texture of specific contexts, and sociological studies need to provide a more specific analysis of individual cognition (Kirshner and Whitson, 1997). This paper presents our attempt to conceive a unit of analysis, mathematics teacher identity, that will provide us a means of considering a broader swath of the cognitive-social continuum than is normally taken into account in studies of mathematics teacher development. In doing this we draw on the work of situativity theorists, as well as on previously established ideas about what types of knowledge are important for teachers to develop.

OUR RESEARCH AGENDA

Our work focuses on the learning of soon-to-be and new mathematics teachers attempting to become teachers of reform mathematics in the United States. Teaching in schools is a form of social participation that takes place within distinct contexts of classrooms. Classrooms, and the teachers leading them, are themselves embedded in school and societal community contexts, and in relation to professional education communities. Each of these contexts provides different, sometimes competing, constraints and affordances (Greeno and MMAP, 1998) for teachers to make sense of and work within. To learn to teach, then, involves the coalescing of an individual's history of interacting in other communities playing other roles – student in the classrooms of one's own schooling, student in academic teacher education communities, and intern in other teachers' classrooms – into a new role as professional teacher in its various attendant contexts (Bohl and Van Zoest, 2001). The interconnected nature of participation in pre-teaching communities, combined with multifaceted relationships between the various communities within which practicing teachers participate, makes for a highly complex web that previous studies and sets of learning theories have been unable to fully illuminate.

In our own attempts to understand how learning takes place within and across these contexts and communities, we have been faced with the need for a theoretical framework capable of providing (a) a sufficiently detailed framework for our qualitative case study analyses, and (b) a sufficiently broad view of the social embeddedness of teachers' practice to provide a full and textured accounting of their development. Lerman (2000) has explicated the need for such a framework for the field of education in general, as well as the motivation behind an overall move to a social direction in mathematics education research. Our work is consistent with that direction, although it is couched in our study of the learning of teachers rather than the learning of mathematics.

LEARNING, IDENTITY, AND DIMENSIONS OF KNOWLEDGE

Wenger (1998) provides a social theory of learning within communities of practice that lends focus to our work. He posits that to learn is to develop an identity through modes of participating with others in communities of practice. Such communities are both defined and cohered by shared goals, mutuality in working to achieve them, and a shared set of social and physical resources to achieve them with. As an individual grows and learns, she develops an identity. Wenger's (1998) take on identity is broader and more socially-oriented than the colloquial term, which tends to focus on the idea of self-identification and on the existence of a personal cache of characteristics. For Wenger and others (Holland, Lachicotte et al., 1998; Boaler and Greeno, 2000), identity is, in the biggest sense, the who-we-are that develops in our own minds and in the minds of others as we interact. It includes our knowledge and experiences, but also our perceptions of ourselves (i.e., our values, beliefs, desires, motivations, and self-identifications), others' perceptions of us, our perceptions of others, and our perceptions of others' perceptions of us that develop as we participate in communities with one another. As such, our identities exist not only within ourselves, but are also strung across a continuum between ourselves and others. They are defined as we interact with others and react or reformulate our participation in response to others' reactions to us. This development of identity in practice is done through "negotiated experiences of self" (Wenger, 1998, p. 150). These are experiences wherein people develop beliefs, commitments, and intentions with regard to the form and content of a particular community and how they ought to interact within it. They give individuals a sense of who they are in relation to the community and its goals, how they might best participate, and where they belong and what they are becoming in the community.

We take Wenger's proposition that development of identity is the same as learning within communities (Lerman, personal communication, December 2001) as the basis of our analytic framework. However, like much work in situative theory, this characterization leaves something to be desired in terms of concrete reference to individual cognition (Kirshner and Whitson, 1997) as well as in terms of reference to particular areas in which people develop. Taking a measure of mathematics teacher development in terms of effective competence requires a set of explicit dimensions upon which to focus. We use Shulman's (1987) broadly accepted conceptions of important teachers' knowledge as a starting point.

For Shulman, the areas in which competent teachers learn fall into seven categories: 1. content knowledge, 2. curricular knowledge, 3. general pedagogical knowledge, 4. pedagogical content knowledge, 5. knowledge of learners, 6. knowledge of educational contexts, and 7. knowledge of educational ends. For our purposes, we have fit these forms of teacher knowledge into three broader dimensions. First, content knowledge and curricular knowledge both deal what is to be taught. We collapse these two into one new domain we call the content area and curriculum dimension. This dimension relates to the types of knowledge that cognitivists might be most likely to focus on because it is relatively clearly defined. Next, pedagogical knowledge, pedagogical content knowledge, and knowledge of learners all relate to who is to be taught, and how they should be taught. We combine these three into a domain called the pedagogical dimension. This dimension relates to the competencies required for participating as the leader of a classroom community and orchestrating activities to ensure the broadest possible development of student understanding. This is the dimension of teacher knowledge that most directly impacts students, and thus is most often the focus of studies in mathematics teacher education. Finally, we encompass knowledge of educational contexts and knowledge of educational ends into a broader domain we call the professional participatory dimension. This dimension includes the arenas of knowledge required to participate productively within the various communities outside of the classroom that are related to the act of teaching. These communities include one's own mathematics department and school communities, as well as the broader professional and university communities that support one's efforts to teach. This dimension is key to our research since we are studying the impact on teacher learning of individuals' participation across a range of communities.

The above understanding of Wenger's conception of identity and list of particular dimensions across which to track development form the foundation for our construction of a unit of analysis. In the remainder of this paper, we attempt to wed the conception of identity development as learning with ideas about the things that mathematics teachers in particular must learn.

IDENTITY AS ASPECTS OF SELF-IN-MIND AND SELF-IN-COMMUNITY

Lerman (1998) suggests that a unit of analysis for educational research must allow one to zoom in and out, changing one's focus to take into account the full spectrum of locations of cognitive development, from in-the-head to socially dependent. Bernstein's conception of identities as composed of "relations within" as well as "relations between" suggests endpoints on such a continuum of locations for identity development (2000, p. 205). Bearing this in mind, Shulman's delineation of knowledge types is problematic as it does not address the relational nature between knowledge and social context. That is, it leaves open the possibility that all the knowledge referred to is entirely "in-the-head" (St. Julien, 1997, p. 267). We believe that many of the bases for teachers' everyday participatory decisions are fully socially connected and value based, and fall outside of what one might term "objective knowledge" (of the type "if I curse at a student, I will be disciplined"). Thus it is appropriate to distinguish a continuum of types of cognition (not unrelated to that of identity) from largely in-the-head to largely socially-negotiated (St. Julien, 1997). For that reason, we will use the term knowledge to refer to ideas that are universally socially accepted (or nearly so) and not open for public debate (although socially constructed). We refer to ideas that are not universally held and thus are subject to public debate as beliefs. Beliefs include values and conceptions founded on values, and provide justifications for action in particular ways in response to particular types of knowledge in given situations. The place for these in Shulman's heuristic is unclear. Other important types of cognition that are clearly not included are commitments and intentions. These encompass one's desires to either act or not in response to particular situations, as well as impetus and justifications for doing so.

Thus it is not only the development of teachers' knowledge across three dimensions that is important, but also the development of their beliefs, commitments, and intentions with regard to them. This is a fundamental aspect of our formulation of identity: each dimension relating to teacher development consists of aspects of knowledge as well as parallel aspects of beliefs, commitments, and intentions. Taken together, we refer to these as aspects of self-in-mind. Figure 1 provides a depiction of this part of identity as the portion within the single bolded border.

To illuminate this we offer an example of the contents of the table in Figure 1 that represents aspects of self-in-mind. Consider the row Pedagogy Dimension. Related knowledge that teachers ought to have is a variety of means of orchestrating classroom activity. This might include an understanding in the broadest sense of what "small group work" and "lecture" are. These fall under the Knowledge column. Related to this knowledge, teachers hold beliefs (e.g., what types of discourse patterns are most effective during small group work), commitments (e.g., promoting those discourse patterns), and intentions (e.g., to improve student understanding). These related concerns fall under the Beliefs, Commitments, and Intentions column. Knowledge (or lack thereof) of different means of orchestrating the classroom can impact what one believes, commits to, or intends to achieve. Likewise, one's Beliefs, Commitments and Intentions affect the knowledge that one seeks. This is represented by the dual-direction arrows between the columns.

The aspects of self-in-mind related to the contents of the three dimensions make up the cognitive portion of a person's identity. In keeping with his desire for a unit of analysis that allows an adjusting of zoom from individual to social, Lerman suggests the "person-in-practice-in-person" (2000, p. 38). As a step in this direction, we introduce another aspect of teacher identity that we call aspects of self-in-community. In Figure 1, these are encompassed within the double bolded border. Part of competent practice is reaction and adjustment based on our perceptions of others, and of their perceptions, in practice. As one becomes more familiar with the modes of participating that are productive and/or acceptable in the eyes of others, one adjusts to those perceptions by either modifying one's actions, or changing one's beliefs, intentions, or commitments. Perceptions, and the adjustment to feedback from them, are depicted as the large back and forth arrows positioned across the community of practice (the shaded oval), and represent Wenger's negotiated experiences of self. It is important to note the overlap in the aspects of self-in-mind

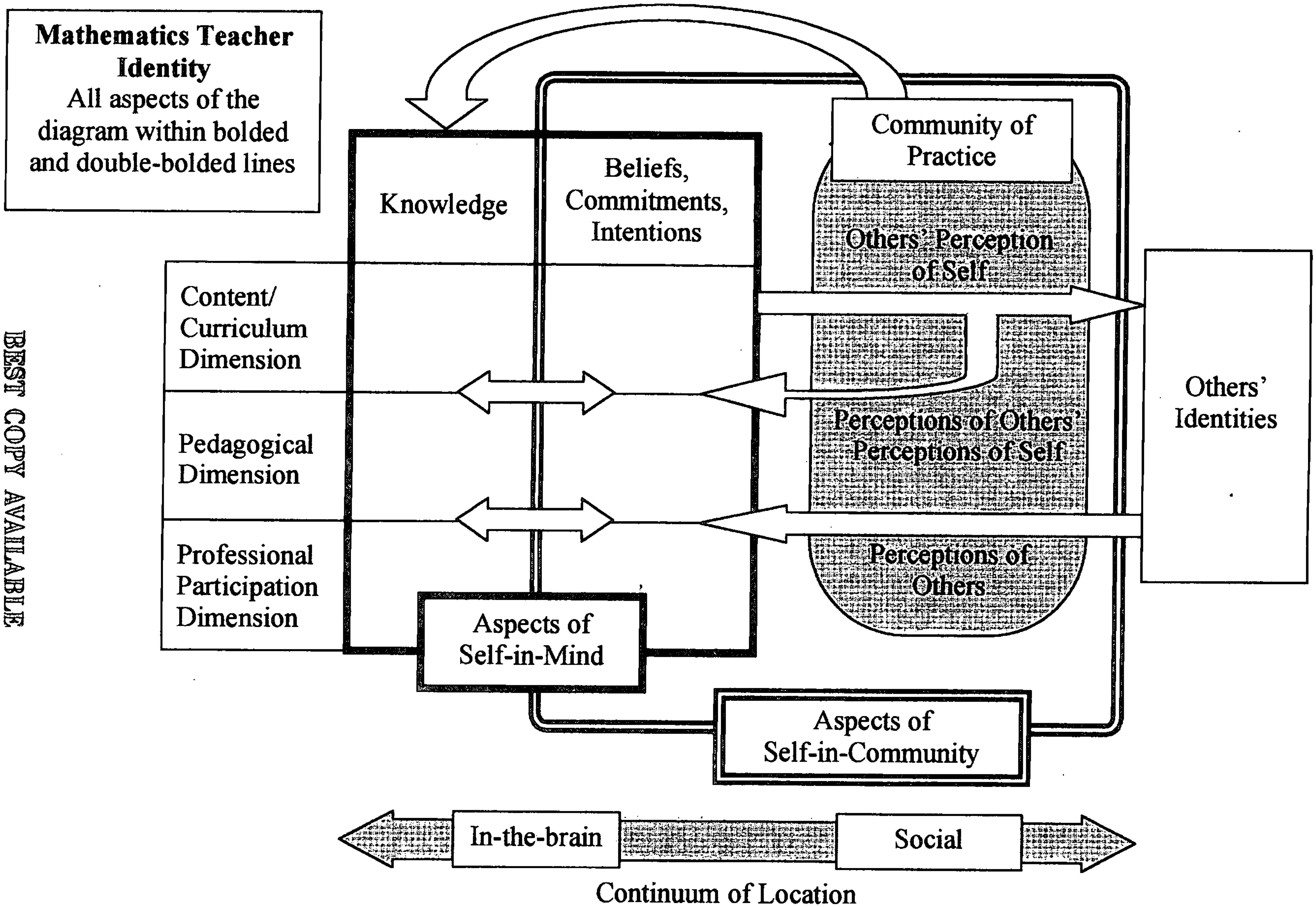

Figure 1: Identity as Combination of Aspects of Self-in-Mind and Aspects of Self-in-Community

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and self-in-community in the area of Beliefs, Commitments, and Intentions. This is the arena in which we conceive of the cognitive aspects of thinking as being stretched over and into the social sphere. Also, as people learn to participate productively (or not), their knowledge is affected. This is depicted by the arching arrow at the top of the diagram from the community of practice to knowledge.

As an example, consider a teacher whose beliefs, commitments, and intentions related to effective discourse patterns in class do not align with those of the school in which she is newly employed. Depending on the strength of her commitments, and on the types of feedback she receives through her perceptions of others' (e.g., students, peers, administrators) perceptions of her own practice, she may adjust her beliefs to align with those of the community, or may work to convince the community that her preferred discourse methods work. In either case she learns in the process, and her identity changes as a result.

A further critical concern for studies of learning in practice is its embeddedness in greater social spaces. Lerman (in preparation) points to the need to zoom out further than depicted in our diagram so as to understand the impact of broader social factors. Such zooming can help answer questions such as "How are the accepted practices within a community impacted by other communities and broader social structures that overlap or subsume the community in question?" Such communities and structures would be depicted in the diagram as more shaded ovals overlapping and/or subsuming the depicted community of practice. Thus, they can be understood as intertwined with, and in some ways defining, practices within the local community. The effects on individual identities of such other communities and social structures could then be considered by studying the constraints and affordances (Wenger, 1988) they help make available to participants in the local community.

Our conceptualisation of mathematics teacher identity as aspects of self-in-mind and self-in-community within a practice spans the continuum from in-the-brain to socially embedded learning. We believe that this will allow us to zoom our analytic gaze further out as we attempt to describe the various factors impacting the development of our subject teachers across multiple communities.

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WHAT DO THE PUPILS THINK ?

PUPILS' PERCEPTIONS OF THEIR MATHEMATICS LESSONS

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Abstract: *The work described here was carried out in two primary classrooms. This research forms part of a wider study investigating the nature of teacher-pupil interaction in the mathematics classroom and focuses on pupil's perceptions about mathematics lessons as seen in the pictures drawn by the children. The data suggests that the pupils perceive their lessons in very different ways to each other, often choosing different things to include in their pictures. The findings in this paper, which are part of an ongoing doctoral study, will contribute to the wider discussion of teacher-pupil interaction within the mathematics lesson.*

BACKGROUND

Theories about how children think and learn have been put forward and debated by philosophers, educators and psychologists for centuries. E. B. Castle, in his book "The Teacher" (1970) explores the historical influences that have helped to shape modern views about children. One of the observations Bruner (1966) made in his book "Toward a Theory of Instruction" was that schools and the social roles they have created (such as 'teacher' and 'pupil') are relatively modern inventions. However, now that we do have these inventions, accounts from children, teachers and researchers all contribute to the image of classroom events. Each view is necessary for a full picture, and each perspective should be treated equally.

My previous research towards an MA dissertation considered the importance of identifying particular opportunities within the classroom from a teacher's perspective, and examined the impact of these moments for both teacher and pupil (Borthwick 1999). Jaworski (1994) recognises the importance of the relationship between teacher and learner, while Bauserfeld (1985) also addresses this teacher-pupil partnership when he writes of "the delicate process of negotiation about acceptance and rejection." This paper begins to look at the pupils and their perceptions of the mathematics classroom in the hope to contribute to the wider discussion of teacher-pupil interaction.

Desforges and Cockburn (1987) reported that to understand how children respond to mathematical activities in the classroom, the teachers need to grasp the children's responses: "Were the children happy and successful with their work? What work practices did they employ? What skills did they evince? What feedback was potentially available to the teachers as their children set about the tasks assigned?"

These are important questions, and while some of them are addressed in this paper, Doyle (1986) is also one to acknowledge that children's responses to their work might play a part in informing the role of the teacher.

METHOD

The findings reported here form the first part of a study into pupil's perceptions concerning their mathematics lessons. The project will consist of three phases. Phase one involved the children drawing a picture of their mathematics lesson. Phase two will be discussions with the children about their pictures, and phase three will involve discussions with the teachers and will consider their response to the children's drawings. While the first two phases focus on the children, phase three is intended to return to the teacher to consider their views of the study. Phase one is being conducted at the time of writing this paper, while phase two is due to be completed in the next 3 months.

The choice of methodology for phase one of this project was inspired from reading Patricia Palmer's book "The Lively Audience" (1986). This is a study of children around the TV set, where she examines the relationship children have with the television and what this means in today's media driven world. Various methods of research were used, but paramount to this project was the perceptions the children had. She writes that "the perspective of children should be sought and used in delineating research questions" (1986). The book includes children's drawings, which form part of the analysis of their views and perceptions.

My own data was collected in two Primary schools in Norfolk, UK. One hundred children from four different classes (Years 5 and 6; ages ten and eleven) were asked, by their class teacher to draw their perceptions of a mathematics lesson. The written brief to the teachers specified that no guidance or details must be given; the only criterion necessary was that the children must include themselves within the picture. Each teacher was given a set of A4 paper, which had several different computer drawn borders on them. The borders were intended to act as the frame for the picture. The choice of border was left up to the children. They were given approximately an hour to draw their pictures. The lesson took place in the children's usual learning environment.

While these single types of observations yield interesting and informative data, I recognise it provides only a limited view of the situation. As Smith (1975) says 'research methods act as filters', however, phase two of this project is an attempt to validate the data further through talking to the artists about their drawings. The drawings are, however, a rich source of enquiry, and allow the research to focus in and narrow down on the significant issues (Delamont and Hamilton 1984).

One of the features of this research is the way the data will be derived from three different sources, but integrated in a way which makes sense and allows the teacher-pupil interaction to be viewed coherently. Jick (1979) is one of the many researchers to write about this process. His view that the researcher is like a "builder or creator, piecing together many pieces of a complex puzzle into a coherent whole" suggests that this enables the researcher to "capture a more complete, holistic and contextual portrayal of the unit(s) under study." The final report will be written as a single account of pupil's perceptions within the study of teacher-pupil interactions.

In this paper I focus on two characteristic drawings from the set of one hundred collected from the children for the purposes of Phase 1.

FINDINGS

Five questions were written prior to the collection of the data:

1. Is the teacher included in the picture?
2. Is there any mathematical equipment included?
3. Is there any mathematical notation drawn?
4. Is it a happy picture?
5. Are the children working together?

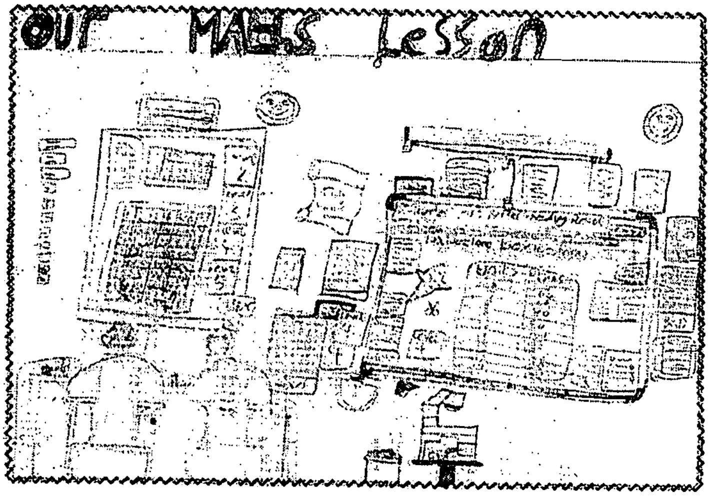
(These questions –their meanings and origins - are described in more detail further on in this paper). Each drawing was then analysed against these questions and the overall responses were collated.

	Yes	No	Can't Tell
Is the teacher included in the picture?	24	76	
Is there any maths equipment included?	8	92	
Is there any maths notation drawn?	75	25	
Is it a happy picture?	35	11	54
Are the children working together?	20	80	

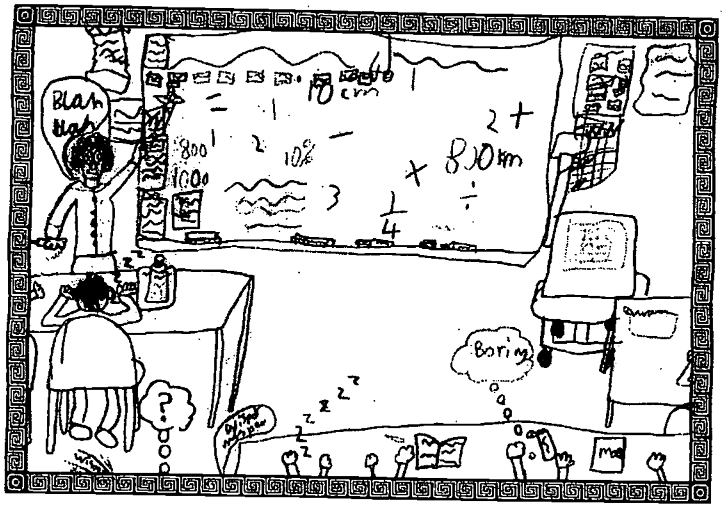
These questions are written in a closed format ensuring a 'yes/no' response as an initial source of quantitative data. If a 'yes' answer is produced further questions are then asked to provide more specific data. For example, if the teacher is in the picture, what does s/he appear to be doing? All the questions bear a certain relevance to today's mathematics curriculum and the influence of the National Numeracy Strategy (DfEE 1999). The Strategy talks about the teacher as key to the teaching process, and encourages 'interactive teaching' throughout the lesson; an increase in the use of mathematical equipment used is encouraged, such as hundred squares, number lines and digit cards; the use of mental jottings as aide memoirs to solve calculations is central to the approach on calculation strategies; the importance on children enjoying their mathematics is supported through both the Strategy and also outside projects such as 'Count On' (the government endorsed project which aims to change the public image of mathematics); children working in different groupings, such as pair or group work is also a feature of the Strategy.

For this paper, I examine two drawings in more detail. These two drawings are quite contrasting in their content and are characteristic of the many views the children expressed in the whole sample.

Picture A



Picture B



Is the Teacher Included in the Picture?

For this question, I had to be able to identify the teacher in the drawing by their name, the clothes they are wearing (if the other persons were wearing school uniform for example) or the written speech included.

Picture A does not include a drawing of the teacher, although the work on the board suggests that s/he is present somewhere in the lesson. The two children also have their hands up and are holding some type of card with possibly an answer on which would suggest a question has just been asked. Furthermore, one child has actually turned around in his chair, perhaps to look at the teacher, with his answer.

In picture B I assume the person at the board is the teacher, determined from the clothes, the body language, the speech bubble and the children's reactions. However, in contrast to picture A, the teacher does not appear welcome in the room, or is seen to be valued as an important part of the learning process – notice the children's comments, "boring, whisper, whisper" and also their reactions – 'zzzz' indicating that two pupil's appear to be asleep!

Is there any mathematical equipment included?

By mathematical equipment, I am referring to items such as number fans, hundred squares, number lines and calculators, not pencils and rulers.

Picture A includes three pieces of mathematical equipment: a hundred square, a board protractor and the number cards the children are holding. The way the protractor has been drawn – on its own hook – indicates its permanent residence. The portrayal of the hundred square is interesting as it represents the most common type of hundred square currently seen in primary classrooms. I.e. one which starts with one in the top left hand corner, and then consecutive numbers follow up to ten and a new line begins with eleven. However, this square, while following these conventions, appears to 'run out of space' at seven! The third piece of equipment is the cards the two children are holding. Many teachers will give out cards for children to select, in response to questions, while others will ask children to write down answers or parts to questions they want. Either way, it is interesting to see these included in the picture, as it suggests their regular use and association with the mathematics lesson.

Picture B, in contrast, has no equipment included, even though it is quite a detailed picture.

Is there any Mathematical Notation Drawn?

By mathematical notation I am referring to universal signs and symbols, either drawn individually or within a calculation. If any notation was drawn, I categorised it into three areas of primary mathematics: number and calculation, shape and space and handling data.

Picture A contains lots of mathematical notation within it. On the right hand board an example of the commutative law for multiplication is drawn, while underneath, place value for tens and units appear. The signs and symbols drawn all appear to be presented in a neat and organised way and suggest it is part of the current lesson. To the left of the picture, mathematical signs are displayed and include addition, subtraction and the equals symbol. These appear to be drawn on card and therefore suggest that they are permanently displayed on the wall.

Picture B also contains mathematical notation. The signs and symbols are all drawn onto the board, with no evidence of a wall display. The notation is drawn in a rather haphazard and disorganised way, but perhaps they are questions or answers to a test, which may explain the range of notation included. Percentages, metric distances, fractions, decimals and whole numbers are drawn with the subtraction, multiplication, division and equals signs also represented.

Is it a Happy Picture?

For this question the children needed to look happy (e.g. with smiles on their faces) or have included positive comments. If they appeared unhappy (e.g. down turned mouths) or negative comments were written, this was counted as a 'no'. If neither conclusion could be drawn, it was attributed a 'can't tell'.

Picture A appears to be a very happy picture. The child turning around has a smile on his face, both children have their hands in the air, indicating their desire to participate, and there are even two 'faces' drawn on the wall with huge smiles on them. Work is neatly displayed on the boards. The display is a central part of this picture, suggesting that pupil's work is valued and worthy of display. Even the waste paper bin is 'steaming' with the amount of work (perhaps jottings now not needed) produced this lesson, which indicates a good working environment.

Picture B, in contrast, portrays the children as either asleep or bored, suggesting scenes of unhappiness, while the teacher seems to be the only happy person. Other indications are the scribbles on the books and the writing on the board, which is written without care. While there are several pieces of work on display, the artist has managed to convey scenes of a bare undervalued classroom by the lack of work on display. Another contributory factor to this interpretation of unhappiness is the speech bubble coming from the teacher. The "blah blah" phrase could suggest either boredom on the part of the children listening to it, the monotony of the teacher's voice, or even the length of time she has been speaking for. Perhaps she has been speaking for so long, the words pale into insignificance as their concentration wanes.

Are the Children Working Together?

For this question, working together meant children sitting together in either pairs or groups. Often whole classes will work together, although the children will still be working alone, and therefore this was not included in this phrase.

In picture A the two children appear to be sitting next to each other at one table, although from their individual cards I assume they are working alone in this activity. However, the fact that a pair of children are sitting together indicates this group work occurs.

In picture B children also are drawn sitting next to each other, but as in picture A, there is little indication that they are working together currently.

DISCUSSION

The evidence examined in this phase of the study and exemplified in this paper raises several questions about how pupils perceive their mathematics lessons. At this stage of the study, the analysis provided is preliminary and further commentary from the artists themselves will support further interpretation. While Palmer (1986) provided the inspiration to use drawings as a source of evidence, the ongoing research will look to form a theoretical framework from which to analyse further the visual data.

Perceptions remain a fascinating source of data. What do you think you are thinking? While you may think you are drawing an accurate representation of your thoughts, perhaps you are not. Perhaps you are clouded by other influences, such as the pencil you are using, the border you have chosen or even how you are feeling at this particular moment? Often thoughts are only challenged when you are questioned about them and are asked to justify and exemplify them. This is what makes the data at this stage so interesting to analyse.

However, while the data remains somewhat tentative, it still offers much to celebrate. The images offered are rich and quite versatile. There are many different interpretations to draw upon in this initial stage. For example, have the children chosen one aspect of mathematics to draw, such as a problem solving lesson or even taking a test, or did they choose to represent a holistic image of their lesson? This would certainly influence the outcome of their picture. The decision to include the teacher or otherwise may have been a conscious decision depending on the type of lesson they are thinking about, or simply a forgotten element. Only talking to the children can verify or falsify these points. However, this absence of the teacher or equipment for example, is arguably attributable to the pupil's own experience and perceptions of mathematics.

These findings suggest different children perceive their mathematics lessons differently. These two drawings suggest very different views. Picture A appears to emit many of the values the mathematics curriculum aspires to today – children enjoying their lesson while actively contributing to it and using mathematical equipment, unlike picture B, which does not. It is possible through my analysis that I have focussed attention on some things to the neglect of others, as observed by McIntyre and Maclead (1978) and so the information must be treated with caution. However, the two pictures do raise issues, which based on how pupils perceive their mathematics lessons raises concerns for the teachers in them.

Research shows that "pupils learn more when their teachers know their attainment and can act on this information" (Askew and Wiliam 1995). Picture B suggests the teacher was not aware of the children's desire not to learn. There is an increasing interest into consulting children. Kings College in the UK, as part of its Leverhulme Project (Brown 1997 – 2002), interviews children into what makes an effective teacher, while Essex County Council undertook a year long action research project into the use of children's perceptions as a tool for school self-evaluation and development (McCarthy 2001). They concluded that asking children does make a difference, both for celebration and development issues.

All teachers have considerable scope to influence the quality of experience in their classrooms. One way is the recognition of potential teaching and learning experiences (Borthwick 1999) and another is the value that the perceptions of pupils can offer. The ongoing research will continue to explore possible answers to the issues raised above.

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SUPPORTING STUDENTS' REASONING WITH DECIMAL NUMBERS: A STUDY OF A CLASSROOM'S MATHEMATICAL DEVELOPMENT

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The focus of this paper is on students' developing conceptions about decimal numbers in a fifth grade classroom. To collect our data we organized a teaching experiment that lasted four weeks. Our students' progress was significant. Their development was supported by activities in the context of metric system measurement and with the model of the number line. In the analysis of our data we delineate the mathematical practices institutionalized by the classroom community. The shifts in this classroom's practices confirm our anticipated learning path.

Introduction

Research shows that students have many misconceptions about decimal numbers. In a recent comprehensive review of these studies (Stacey, 1998), we can notice that these misconceptions are mostly accounted for in psychological terms and they are not analyzed in relation to students' instructional experiences from their classrooms. In this way, teaching decimal numbers can not be adequately informed by available research data. How could teachers by just knowing the prospect of these misconceptions appearing avoid them? The only alternative we can imagine is for them to give their students detailed instructions at the risk of making mathematics more algorithmic than it is. On the other hand, if teachers try to follow the suggestions which are usually given for remedying these misconceptions, they will end up with a top-down structuralist teaching approach. Notwithstanding that, this approach has repeatedly been criticized as insufficient to assure quality in students' learning experiences (Gravemeijer, 1994).

A relational understanding of decimal numbers might be built if teaching had a different orientation. Such new teaching approaches are developed by researchers to promote a meaningful understanding in several mathematical areas (Carpenter, 1997; Cobb, 1999; Lampert, 1989; etc.). In these attempts, teaching is based on students' informal understandings and strategies, it supports the gradual mathematization of their contributions, and encourages communication in the classroom. We tried to develop a similar orientation in teaching fractions and decimals in a fifth grade classroom. At the end of our classroom teaching experiment our students did not show any misconceptions about decimals and their understanding was meaningful. Accounting for students' learning in this classroom could be useful for instructional design research. Also, the results of such an analysis might help teachers to reflect on their current instructional practices. In fact, it was on the basis of preliminary analyses of our students' learning that we were trying to support their developing understandings of decimals. Therefore, this paper presents a first attempt to account for our students' learning in the social context of their classroom by documenting their increasing understandings of decimals.

Theoretical framework

Our focus in presenting our results will be on the ways that taken-as-shared meanings of decimal numbers emerged in our classroom. This is particularly important if we consider these meanings, which originate in students' contributions, as constraining or enabling their individual activity. This consideration stems from the emergent perspective developed by Cobb and his colleagues (Cobb & Yackel, 1996). In this perspective mathematical learning is treated as both a social and an individual activity. More specifically, these two aspects of learning are taken as reflexively related. Individual students reorganize their activity as they participate and contribute to their classroom's mathematical activities. On the other hand, the students' collective activity does not exist separately from individual students' diverse ways of reasoning.

Methodology

The data we will analyze come from a fifth grade classroom in a school of Athens. These data include: 1. the videorecordings of all 17 lessons on decimal numbers, 2. the videorecorded interviews taken from most of the students upon the completion of our teaching experiment, and 3. the 21 students' notebooks. The presenting author taught most of the lessons in this unit, immediately after a fraction instructional sequence.

The activities used in our teaching experiment were a crucial factor for our students' developing understandings of decimals. But our students' activity as well as their further mathematical development was also influencing the selection and development of appropriate activities. Thus the set of instructional activities used in this teaching experiment (see the results section below) was not fixed in advance. However, our overall purpose guided the selection of an initial set of activities. This purpose was to support students' flexibility in reasoning with decimal numbers. For example, we wanted them to compare decimals like 0,06 and 0,6 by reasoning that 0,6 is bigger because it is ten times larger than 0,06, or by reasoning that 0,06 is very close to 0,5, while 0,06 is far from it as it is closer to 0,1. Gravemeijer's (1998) idea that students can reinvent decimals through the activity of repeated decimating along with his heuristic of emergent models were instrumental in helping us to foresee a path through which we could guide them to develop their understanding of decimals.

The lack of a decimal monetary system in our country at that time was a serious difficulty. We thought that our students could be offered the opportunity to see decimals as describing the results of repeated decimating in the context of activities related to the standard metric system measures, which are commonly used in our country. Before involving students in activities in the context of measuring lengths, we decided to start from a fictitious scenario, similar to one found in the "Mathematics in Context" curriculum unit "Measure for Measure" (Britannica, 1997). Our students heard a story about a visit to an amusement park where a competition was organized on a special wheel of power. The winner would be the person who managed to turn this wheel more than the other participants would. In the context of this narrative the problem of finding the winner brought to the fore the issue of precisely measuring the turn of the wheel. Students made a number of suggestions including: halving the wheel and then again halving the half of the wheel, or making divisions like those on a clock face. Next, the teacher announced that this wheel was repeatedly divided by ten and students accepted this as an efficient way of measuring precisely the players' strength. The relationships among decimal fractions were then explicitly discussed. This was meant to prepare the exploration of the subunits' relationships on the meter stick, which was familiar from previous grades. So, when the meter stick was introduced students compared it to the decimally subdivided wheel and easily identified their similarities. These discussions concerning the wheel of power and the meter stick together with the students' learning history from fractions constituted the background for their engagement with the decimal instructional sequence. Also, we took into account that students had had a few lessons on decimals in the fourth grade. We hoped that students' reasoning with the meter stick, would come to function as a model of their activity on the wheel, i.e. of measuring in "ones", "tenths", "hundredths", etc. Eventually, we anticipated that reasoning with the meter stick would serve as a model for reasoning with decimal numbers. Through this shift, by embodying the results of repeated decimating, the symbol of decimal number would evolve into an experientially real mathematical object independent of their activity.

The above mentioned hypothesized path will be documented by utilizing the construct of practice (cf. Cobb&Yackel, 1996). Thus, the criterion we will use in describing the mathematical practices that emerged in our classroom community will be the legitimacy / acceptability of the students' explanations in the classroom as well as their content.

Results

After the introductory discussions on the wheel of power and the meter stick, the teacher asked students how they would find 75 cm on a double number line that she had drawn on the board as a simplified form of their meter sticks. Students offered several partitionings (see Figure 1). The variety of students' partitionings was not totally unexpected. Students had already connected the idea of refined measurement with the use of common fractions. From the outset, it seems that students' prior experiences from fractions were going to influence the course of their decimal understanding, just like in history where the origin of decimals can not be separated from fractions.

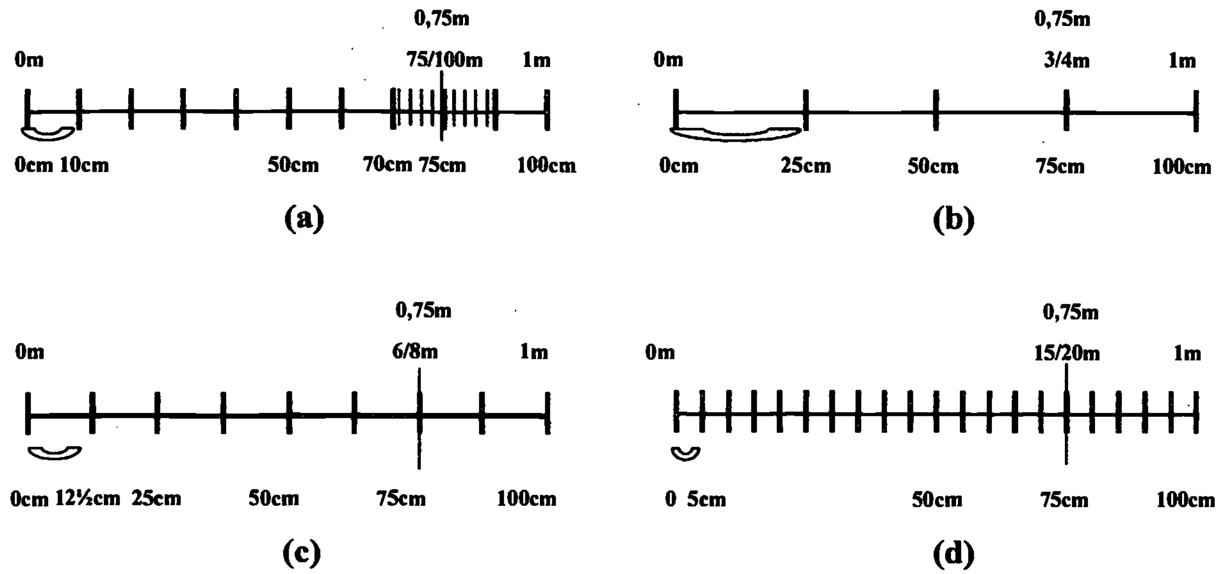

Figure 1: Place 75cm on the number line

The number line seems to have played a significant role in this episode. As our instructional intent is for students to reason with decimals independently of the imagery of the number line, inevitably the meaning of the number line will change: from students' using it with a context-specific meaning (measuring strength on the wheel or length with a meter stick) to using it in order to think about relations among units of a different rank. Therefore, analysis of subsequent episodes will be focused on students' interpretations of their activity as they reason with the number line.

After the first student's double decimating in order to place 75 cm on the number line, the teacher asked students to describe the result of his repeated decimating in meters. Apart from students' response 75/100m, the request for a decimal number in meters resulted in several answers: 75,0cm, 7,5m, 7,5cm, 0,75m. These answers became a topic of discussion and invalid answers were rejected. There were students however, who were unwilling to accept 0,75m as a legitimate way of notating. A possible explanation might be that in their prior experiences from mixed numbers "0" was not used to describe the absence of whole units. As students negotiated their different ways of symbolizing the results of repeated decimating, the first practice of connecting decimals to the records of decimally partitioning the number line seems to emerge. Subsequent episodes gave students additional opportunities to negotiate their ways of symbolizing the results of repeated decimating. For example, a few days later, the difference between 0,10 and 0,100 emerged as a topic of discussion. As students had not yet created the decimal number as a true symbol of decimating, the equivalence of these two decimals was not self-evident.

The "75cm episode" showed us that a possible source of students' difficulties was their familiarity with arbitrary partitioning. By including tasks that involved students in enlarging the unit of measurement for lengths not convenient for the use of different than decimal partitionings, we tried to support their attempts to create an operational meaning for decimals. As an example, 9cm were easily converted to a decimal after decimating the number line (see Figure 2).

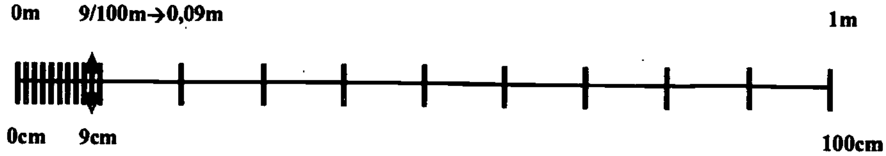

Figure 2: Place 9cm on the number line

Christos' activity on this number line was still dependent on the imagery of the meter stick. How else could he place 9cm on the number line? On the other hand, in contrast to other students, he consistently used decimal fractions before attempting to write down the equivalent decimal number, in this as well as in other similar tasks. Moreover, his behavior is representative of what some few other students were also doing in this type of activity. In accounting for these behaviors we conjectured that these students were applying the well-known trick of writing the numerator of the decimal fraction and then holding so many digits after the decimal point as the number of zeros in the denominator of the decimal fraction. Instead of coordinating the records of decimating with the digits of the decimal number, these students seemed to rely on numerical patterns. At the same time, their participation in the classroom mathematical practice of connecting decimals to the records of their decimating activity was unhindered. Their solutions based on their reasoning with decimal fractions were not challenged. In fact, it can be argued that these students were not in a position to see decimals in relation to repeated decimating, as they were not reasoning with the number line. In the Greek language the prefixes used for the subunits of the meter are identical to the fractional names used to describe their relationships to it. "Tenth-meter" is the word used for decimeter, "hundredth-meter" for centimeter, and so on. So they could well write the decimal fraction without paying any attention to what they had been doing on the number line. In such a case, the only purpose for carrying out the decimating process could be to specify the distance of a given length on the number line. Therefore, for these students understanding of place value would remain instrumental if further opportunities to reflect on the results of their decimating activity in connection to decimal numbers were not given. To this end, we involved students in the following activity.

Students were asked to place different decimal numbers with up to two decimal digits on schematized rulers. These rulers were already divided in centimeters or decimeters. Our main purpose in introducing this activity was for students to realize how repeated decimating was signified by the decimal digits. Students were encouraged to describe the decimal digits of the given decimals in terms of their actions of placing them on these rulers. The topics of conversation that emerged in the context of this activity provided opportunities for students who relied on tricks to reorganize their activity. Now, it seemed that the classroom's activity with the number line implied the numerical values of the decimal digits. In other words, it became a true model of measuring with units of different decimal rank.

With the next activities we tried to support students' reasoning with decimal numbers. We thought that if the values of the decimal digits would become independent of students' activity on the number line, this would allow students to focus on relationships among decimal numbers. Students' activity on the number line would evolve into a model for reasoning with decimal numbers. Gravemeijer (1997) notes that: "This transition from 'model-of' to 'model-for' implies a process of 'reification'. [...] what is reified is the process of acting with the model, not the means of symbolization itself". In our case, the activity of repeated decimating had to be reified. Signs for this reification emerged, in the context of several activities. The following episodes will clarify how this process took place in our classroom.

In one of the first lessons, the teacher asked students to find a length that would be very close to 3 meters. She drew a number line on the board (see Figure 3) and the first offer of 2,99 meters was placed on it. She then asked for a decimal number even closer to 3 meters. The number 3,0 was offered by a second student but it was rejected by other students as being equal to 3 meters. Another student suggested 2,999 meters. To the teacher's question of how far is 2,99m from 3m this student answered 1cm and when he was asked to place his number on the number line, he said: "I would divide it in 10 pieces, and it will be at the ninth small line". This episode closed with the specification of the distance that 2,999m had from 3m.

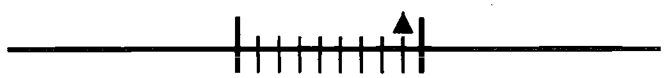

Figure 3: Find decimals close to 3m

The student's reasoning concerning the placement of 2,999m on this empty number line shows that for him the decimating process was already reified. The fact that other students did not ask any clarifying questions suggests that they could follow his argument. However, the focus of their comments indicates that they were not in a position to anticipate decimating activity. Many students objected that their classmate's drawing was not precise. As they said the small lines were not equidistant and the distance of 2,999 from 3 was longer than 1mm. The imagery of the meter stick was still too dominant. Therefore, reasoning in terms of repeated decimating was not yet a practice.

Later, students' work in the context of the activity with the schematized rulers might be considered as providing opportunities for many students to reflect on their decimating activity. So when for example, students had to place decimals with two decimal digits on the ruler with decimeters, they started anticipating their decimating activity. However, their anticipations did not become an explicit topic of discussion. Discussions were mostly focused on the results of their decimating activity and not on the students' intentions. An activity where students were asked to find decimal numbers in between other given numbers allowed students to make their reasoning in terms of repeated decimating explicit. When the teacher asked for a decimal at the midpoint of the distance between 8,6 and 8,7 students explained their solutions as follows: Stavroula said that the number will be 8,65 and then she explained her thinking by drawing a number line on which she had first placed 8,6 and 8,7. Then she divided the distance between the two numbers in ten parts and she showed 8,65 on the fifth line. Costas placed the two given numbers on a number line and then he explained that the wanted number will be at 865cm. In his mind, he had already transformed the given numbers in centimeters. Then he placed 8,65 at the midpoint of the distance separating the two given numbers. Helias tried to find 1/2 of 1dm. He said that it will be 5cm and then he added 0,05 to 8,6.

In all of the above solutions, students reasoned with the decimating process. Moreover, in the last two solutions the decimating process has already been concealed behind students' reasoning. The same finding concerns students' reasoning in converting fractions to decimals. Progressively they institutionalized a formalization of the decimating process in a long division scheme. For example, students jointly transformed 1/3m to 0,3333...m by making three successive divisions (100cm:3, 10mm:3, 10dmm:3). It was in this task that the notion of the infinite number of decimal places first struck them. However, the presence of the metric unit names in students' explanations might be considered as showing students' inability to escape from the imagery of measuring length. But the adaptability of their ways of acting as we posed tasks in different contexts, documents that their understanding of decimals was not context dependent. Without losing their ties to the phenomena that gave rise to them, decimal numbers have now become mathematical objects that students could reason with.

The practice of reasoning with decimal numbers seemed to be well established by the end of the decimal instructional sequence. Actually, students were able to reason not only with decimal numbers but at the same time to incorporate it into reasoning with fractions or whole numbers. Their sense of decimals is illustrated by students' reasoning on the task 4X0,75. The teacher had notated their solutions on the board as follows:

$$
\begin{array}{l}
\quad 0,75 \\
+\,0,75 \\
\hline
\quad 150\text{ cm} \rightarrow 1,5\text{m} \\
+150\text{ cm} \rightarrow 1,5\text{m} \\
\hline
\qquad\qquad 3\text{ m}
\end{array}
$$

$$
\begin{array}{c}
75\text{cm} \times 4 \\
/\backslash \\
(70+5) \times 4 \\
(70 \times 4)+(5 \times 4) \\
280\text{cm}+20\text{cm} \\
2,8\text{m}+2\text{dm} \\
\backslash\ / \\
3\text{m}
\end{array}
$$

$$
\begin{array}{l}
0,75=0,25+0,25+0,25 \\
\quad 1 \leftarrow 0,75 \\
\quad 1 \leftarrow 0,75 \\
+1 \leftarrow 0,75 \\
\hline
\quad 3
\end{array}
$$

75cm	0,75m=1m-0,25m	0,75=3/4
+75cm		
150cm	$4 \times 1=4m$	3/4+3/4=6/4
+150cm	$-4 \times 0,25m=1m$	$4 \times 0,75=6/4+6/4=12/4=3$
300cm=3m	3m	

Conclusion

In studying our classroom's mathematical development through an introductory course on decimal numbers, we identified three mathematical practices These practices represent a joint accomplishment of this particular classroom community. Individual students' participation in these practices was varied. For example, at the end of our teaching experiment, most of our students could reason with decimal numbers. However, there were students who were still reasoning in terms of repeated decimating. As the final episode shows, these students' participation in the classroom discourse was by no means insignificant.

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TEACHERS' QUESTIONS AND THEIR ROLE IN HELPING STUDENTS BUILD AN UNDERSTANDING OF DIVISION OF FRACTIONS

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The purpose of this paper is to share some of the results of a year-long teaching experiment in which fourth grade students were provided with opportunities to develop an understanding of fraction concepts before the introduction of formal algorithms. In particular, we will address the role of the teacher/researcher's questions in prompting students to build these ideas. Our results indicate that the students were successful in building models to explain division of fractions.

Introduction and Theoretical Framework:

This research examines one portion of a teaching experiment[1] that focused on the development of fraction knowledge in a class of fourth grade students. In the sections that follow, we will briefly discuss the role that the teacher can play in helping students deepen their understanding of this content, and several instances that document one teacher/researcher's skillful use of questions to help children overcome difficulties with a problem dealing with division of fractions.

Many students have experienced difficulty in solving problems involving fractions (Tzur, 1999; Davis, Hunting, and, Pearn, 1993; Davis, Alston, and Maher, 1991; Steffe, von Glasersfeld, Richards and Cobb, 1983; and Steffe, Cobb and von Glasersfeld, 1988). When considering these difficulties, Towers (1998) states that traditionally, the teacher has been seen as separated from the student, and that teaching and learning have been regarded as discrete entities. Tower's research examines the role of teacher interventions in the development of students' mathematical understanding, and her findings suggest that children can overcome some difficulties traditionally related to fractions, when appropriate conditions are in place. Other researchers have also documented how teachers can help in facilitating the development of ideas relating to fractions (c.f. Steencken, 2001; Steencken and Maher, in press; Ma, 1999; Cobb, Boufi, McClain and Whitenack, 1997).

The role of the teacher in helping children develop insight into ideas relating to fractions was a central topic of the year-long teaching experiment that is the focus of this paper. The fundamental premise was that in order to help students build a conceptual basis for considering these ideas, teachers must move away from more

[1] This work was supported by grant MDR 9053597 from the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the author(s) and do not necessarily reflect the views of the National Science Foundation, Rutgers University or Rider Univeristy.

teacher-centered instructional approaches which emphasize rote memorization and the execution of rules and procedures, and move toward instructional practices which are more student centered and provide them with the opportunity to build concepts and ideas as they are engaged in mathematical activities that promote understanding (Davis & Maher, 1997; Maher, 1998; Cobb, Wood, Yackel & McNeal,1993; NCTM, 2000; Klein and Tirosh, 2000; Schorr, 2000; Schorr and Lesh, 2001). A more student-centered classroom requires teachers who listen to the explanations of their students, probe them for justifications, encourage them to share their solutions with their peers as they work together to refine, revise and extend their solutions.

Teacher questioning is a critical component of a more student-centered classroom, and the topic of this paper. International studies have documented the effects of teacher questioning in helping children advance their mathematical thinking (Klinzing, Klinzing-Eurich and Teicher, 1985; Sullivan and Clarke, 1992; Martino and Maher, 1999). In each of these studies, it was found that asking more "open-ended questions aimed at conceptual knowledge and problem-solving strategies can contribute to the construction of more sophisticated mathematical knowledge by students" (p. 55, Martino and Maher, 1999). In addition, Cobb, Boufi, McClain & Whitenack (1997) propose that when students are engaged in discourse with peers, there is growth and development of mathematical ideas. In this study, the researcher used responsive questioning to elicit explanations, to help students develop appropriate justifications and to redirect them when they were engaged in faulty reasoning. In the sections that follow, we will describe how questions were used in this study, and the types of student responses that were elicited.

Methods and Procedures:

Background, Setting and Subjects: This research took place in a small suburban NJ district, over the course of a year, as part of a teaching experiment. The focus of this intervention was to investigate the development of children's mathematical ideas about fractions. A 4th grade class was chosen because these students had not been introduced to the formal algorithms usually associated with learning fractions. This particular class consisted of 25 students, 14 girls and 11 boys. Each of the approximately 50 classroom sessions lasted approximately sixty to ninety minutes. This paper reports on the 21st session of the study.

One of the goals of this teaching experiment was to create a classroom community in which student inquiry and discovery were of paramount importance. Teachers and researchers did not tell students that their work was correct or incorrect. Rather, students were questioned and encouraged to justify their solutions so that they could develop their own sense of accuracy, not based on the approval of an outside authority. The overarching perspective was that if students were invited to work together and conduct thoughtful investigations with appropriate materials, they would be able to build mathematical ideas relating to fractions (Maher, Martino, Davis, 1994). Throughout this experiment, the teacher/researchers worked to promote a classroom culture that supported children as they explained, explored, and reflected upon mathematical ideas. They were always invited to talk about their thinking, and they were challenged to defend and justify their ideas. The expectation was that each child, or group of children, would be able to build a model or representation of their idea, talk about their idea, and share their ideas with the teacher and their classmates. The children were always encouraged to build models of their solutions and share them with each other and the class. In discussing their solutions, children listened to each other and developed convincing arguments to support their ideas. They often raised questions that triggered further exploration. They drew pictures, labeled the pictures and eventually developed notations for their ideas. Steencken (2001) studied the earliest sessions of this project in depth, and documented the emergence of sophisticated mathematical thinking about fractions.

Data: All sessions were videotaped using two or three video cameras, one or two of which attempted to capture students' work and classroom interactions. An additional camera was frequently set up to record explanations by students, which were presented at the overhead projector. Students' written work and Teacher/Researchers' notes were also carefully collected. Transcriptions and detailed narratives of the data were recorded.

Coding: A coding scheme was designed to flag elements for study. The four classifications of codes used were intended to record teacher interventions, ideas expressed, representations used by students, and justification and reasoning by students. In all, there were twenty teacher intervention codes that fall into six categories. Codes relating to teacher's intervention included: T_I) giving information, T_R) rephrasing or requesting that another student rephrase what a student said, T_Q) asking questions, T_J) requesting justification, T_D) directing students to construct a representation, consider an existing representation, or to the work or ideas of others, to discussion with others or towards previously done work; and, T_N) seeming to do nothing. In this research, we will focus on these teacher codes as well as the outcome of these interventions in eliciting students' justification and reasoning.

Results and Discussion:

The episodes that are portrayed here occurred as students participated in an activity called, "Holiday Bows" (c.f. Bellesio, 1999), created to provide an opportunity for students to consider ideas relating to division of a natural number by a fraction. Questions that Teacher/Researchers asked were designed to be responsive to what students were doing and to help them to clarify and express their thinking by justifying their solutions.

In the first episode, the teacher intervention occurs when a Teacher/Researcher joins two boys who are working on a problem involving $2 \div 2/3$. The Teacher/Researcher employs various types of interventions: questions, reiterating what the students say, requesting justification and explanations. When the students do not know how to begin, she asks, "What do you think you know?" In the excerpt that follows, the teacher is working with one student, Andrew. Her previous question has invited him to look at the ribbon and begin to "imagine dividing" it into three pieces.

Andrew: I think it's three umm because when you divide thirds up it's...it's ummm... 1/3, 2/3, 3/3 *[places his finger on the ribbon which is lying on top of the meter stick indicating approximately where 1/3 m, 2/3 m and 3/3 m would be]* so it would be like this would be one ribbon if you made it into a ribbon *[places his finger at approximately the 1/3 m mark]*

Teacher/Researcher: uh huh

Andrew: this would be another ribbon *[indicating approximately 2/3 meter]*

Teacher/Researcher: uh huh

Andrew: and this would be another ribbon *[indicating end of meter stick]* so if you divide it into thirds, you would have 3 *[inaudible]* bows, but we want to make sure.

By reminding the students to use what "you think you know" the teacher has offered that the students consider a technique that they had used before in similar types of problems. When the Teacher/Researcher posed this question, the student, Andrew, spontaneously solved the problem of finding out how many bows, each 1/3 meter in length, could be made from one meter of ribbon.

The Teacher/Researcher then asked the students in the group to justify their solutions. Justification of solutions often drives students to examine the validity of their own ideas. In this excerpt, one student, James, believes that the solution to the problem is 4 bows. At this point, the Teacher/Researcher asks: "Why are you going to get 4 bows, cause I'm...I'm confused...convince me." As the children consider their justification, she requests that they clarify what appear to be conflicting statements and provide valid mathematical evidence for them. While working on the problem, how many bows, each 1/2 meter in length, can be made from a blue piece of ribbon which is two meters long, students confuse the idea of dividing two by one-half with taking one-half of two. Questioning by the Teacher/Researcher helped the students to examine their own and each other's ideas.

Teacher/Researcher: What's... What's...Ok, Let... Let me go back because I think you're telling me a lot of things and I just want to be sure I understand. Um...

If you have the blue ribbon and we want to figure out how many bows we could make and each one is 1/2 meter how did you figure out, first you told me it was 2 and then you said no, it's 4.

James: It's 4 because...

Andrew: It could be either way.

Teacher/Researcher: It could be either? What do you think about that, James? Could it be either?

At this point, the Teacher/Researcher uses the strategy of rephrasing in order to help James consider the ideas that Andrew is referring to. She attempts to refocus the discussion back to James, the student who has correctly stated that there would be four bows.

James: I don't know. [*Inaudible*] 1/2 of a meter.

Teacher/Researcher: Right,

James: Not 1/2 of two meters, so...

Andrew: He's saying 1/2 of two meters not 1/2 of one meter

It appears that James is now considering what Andrew has said and he verbalizes that there is a distinction between taking 1/2 of a meter and 1/2 of the entire ribbon, which is 2 meters long. Andrew states that he thinks that James is referring to 1/2 of the ribbon, not 1/2 of a meter, which would be the length of one of the bows.

Teacher/Researcher: Ok. I think what it's saying is the ribbon length of the bow is a 1/2 a meter so each ribbon is going to be a 1/2 a meter long; OK, and the question is how many bows can you get out of that blue ribbon if each bow is going to be a 1/2 a meter long?

James: Four 'cause this [*holding up blue ribbon*] is two meters long

Andrew: It's saying 1/2 of two meters

Teacher/Researcher: Ok, Let me see if I can understand what James is saying.

Andrew: It's saying it's a 1/2 of 2 meters because the blue ribbon here is 2 meters, is the whole

Here Andrew has indicated that he believes the problem is to find out what 1/2 of 2 meters would be. The Teacher/Researcher again directs the attention to James, who is interpreting the problem correctly.

Andrew: So then this is a 1/2 of 2 meters... that would only be 2

Teacher/Researcher: Ok. What does this [*points to the paper with the problem on it*] say to you? What is a 1/2 m?

Andrew: Half of this [*holds up white ribbon which is one meter long*]

Here, the Teacher/Researcher's question helps Andrew to focus on the fact that the 1/2 in the problem refers to 1/2 meter in length, which would be the length of each bow. Andrew indicates his understanding of this by holding up the white ribbon, which is one meter in length and saying that each bow would be half that length.

James: Yes. So that would be four (referring to the blue ribbon, which is 2 meters long) and this would be 6 (referring to 3 m gold ribbon in the next problem)

Andrew: And so this [the white ribbon] is 2 and both of these [white ribbons]...[*inaudible*] would be 2 halves and another one we'd put on then that would be 4.

Andrew uses the white ribbon as a metaphor for 2 bows. He communicates his representation of two bows being made from each one-meter piece of white ribbon. The teacher's interventions invite him to clarify the question and to formulate and explain his metaphor, the white ribbon, for 2 bows. Her manner of questioning encourages the boys to talk not only to her, but also to each other. When it appears that Andrew believes there are two possible solutions, she probes further to encourage him to state that both his answer and James's answer are correct but that they are answering different questions. Andrew ultimately explains the difference between finding out how many bows, each 1/2 m in length can be made from 2 m of ribbon and finding 1/2 of 2 m of ribbon.

Conclusions

The preceding episodes were intended to highlight how one teacher/researcher helped students build a solution to a task involving division of fractions. Her interventions invited the students to listen to each other, to consider each other's arguments, to express their ideas and to create a representation to help them solve the problem. In this case, the children did not rely exclusively upon the authority to supply affirmation for their work or to impart information which they were expected to acquire without understanding. Rather, the students built a solution that made sense and they did this by exchanging information with each other and with the teacher/researcher. The task of monitoring children's construction of ideas, and posing timely questions is a challenge to teachers. Polya (1985) states, "This task is not quite easy; it demands time, practice, devotion, and sound principles." (p.1). The above episodes are instances of how, in a research setting, attention can be directed to the students' thinking. The interventions and questions were designed to engage the students in working out a model that made sense to them. Questions can become a catalyst for urging learners to justify their ideas and explain them to each other. This, in turn, has the effect fostering deeper thinking about the ideas involved in the problem situations.

Davis advocated a learning environment for the teaching of mathematics, which fosters the connection between the representations in the mind of the teacher and the mind of the student. This implies that a teacher makes sense of the developing ideas of students by the representations they build, explanations they give, and arguments they make, and, as the late Bob Davis suggested, "taking their ideas seriously". Davis (1992, p.349)

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LACANIAN PSYCHOANALYSIS AND PEDAGOGICAL TRANSFER: AFFECT AND COGNITION

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ABSTRACT

We contend that some representative studies about affect, even if they seek support based on psychoanalytical theory, miss the interpretation of cognition in terms of transfer. We introduce Lacan's concepts of transference and affect which make it possible to touch the dimension of love. For the sake of the reader's understanding we add an example from our psychoanalysis-inspired learning experience. This is a theoretical paper with one example from practice to facilitate understanding.

INTRODUCTION

After gathering data showing that "student writings convey an overwhelming sense of fear and anxiety engendered by their encounters with mathematics", Breen [2000] points out that "the links between psychoanalysis and mathematics education seem to have largely been silent themes at PME with only a few discernible exceptions (...) and neither of these directly address the encountered dominance of fear in the mathematics classroom" [Breen, 2000:108].

Fear and anxiety are signifiers that fall under the general heading of *affect* in a vast research literature. McLeod's survey [McLeod, 1992] displays an impressive list of 219 references with none referring directly to psychoanalysis; Freud is mentioned *en passant* in the article. McLeod says that "the affective domain refers to a wide range of beliefs, feelings and moods that are generally regarded as going beyond the domain of cognition" [576], however "research on affect in mathematics education continues to reside on the periphery of the field" [575], "a major difficulty being that research on affect has not usually been grounded in a strong theoretical foundation" [590]. Indeed, from the survey we get the feeling that the affective domain is a union of its common sense connotations such as: *Aha!, anguish, anxiety, attitudes, autonomy, beliefs, confidence, curiosity, dislike, emotions, enthusiasm, fear, feelings, frustration, gender, hostility, interest, intuition, moods, panic, perseverance, sadness, satisfaction, self-concept, self-efficacy, suffering, tension, worry*. We note a remarkable absence: *love*. It is a small paradox that the alleged intellectual unemotional Lacan, as we intend to show, is the one whose theory allows the restoration of this omitted dimension.

Affect and its connotations have been the capture object of research reports in recent PMEs. Vinner [1996] reports on teachers' discussions about their professional lives brought about by their written answers to a set of questions. Analysis of videotapes led to interpretations under the headings of frustration, humiliation and hope connected to the cause and the way teachers coped with difficulties in teaching several content topics. DeBellis and Goldin [1997] use the term self-acknowledgment referring to a learner's willingness to acknowledge an insufficiency of mathematical understanding in a context of problem solving. Da Rocha Falcão and Hazin [2001] compare measures of self-esteem with mathematical performance and conclude that there is a need to take into account both affective and cognitive aspects in research about mathematical learning.

The two reports under the framework of psychoanalysis [Baldino and Cabral, 1998, 1999], as remarked by Breen [2000], do not directly address any of the affective connotations listed above. There are, however, some attempts in the literature to approach affect issues from the psychoanalytical perspective. The most consequential ones, Blanchard-Laville [1997], Wilson [1995] and Breen [2000], evoke the concept of *transfer*.

Blanchard-Laville [1997] postulates a certain *psychical reality* that could only be known through its effects, while remaining unknown in itself, like Kant's thing-in-itself. She acknowledges the paradox that, in order to study the human psyche, we must pass through another human psyche, namely, our own. The solution she proposes to the paradox consists of a direct appeal to the conscious ego: "A minimum of conscious intentions is required of the observer in order to perceive the unconscious dynamic of exchanges and its reflections and let oneself impress by the implicit aspect of messages among the participants of the didactical exchange" [Blanchard-Laville, 1997:158].

Wilson [1995] aims at studying his own feelings in his relationship with the students. He describes his method thus: "At the end of each day, or week, I sat quietly and allowed an incident from my teaching to enter my mind (...) writing (it) as objectively as I could" [Wilson, 1995:1]. To this introspective method he adds the concept or transference described in these terms: "The transference relationship describes distorted perceptions of counselors which arise because of clients' previous relationships" [Wilson, 1995, quoting C. Lago, Notes from Manchester Counseling Course, Sheffield University].

Breen [2000] reports on his teaching experience with adults who were seeking a primary school teaching diploma or who have blocks about doing mathematics and were given a second chance. Transfer is considered as "the imposition of an actual or imagined previous relationship onto a present one" [Breen 2000:110].

In spite of all appeals for the need to approximate research on Mathematics Education and affect, we could not find any learning experience organized according to, and interpreted in terms of, psychoanalytical theory. The foci of the studies are not on the direct interaction of student or teacher with the mathematical object that characterizes cognition studies. Whenever the interaction between the subject and a mathematical object is brought about, the focus is on the subject's feelings, described from the point of view of a superior conscious ego who relies on introspection to evaluate imaginary distortions from an exacting pattern. It seems researchers strongly believe that psychoanalytic theories and affective studies refer to what happens on the periphery of cognition and have nothing to say about cognition itself.

In all the above-mentioned works about affect or its connotations, using either a direct approach or via psychoanalysis, the researcher assumes an exterior position either as an observer and interpreter, as in Vinner [1996], Da Rocha Falcão and Hazin [2001] or assumes a conscious position of judge, as in DeBellis and Goldin [1997], Blanchard-Laville [1997], Wilson [1995]. In doing so, the researcher approaches the transfer via an "alliance with the healthful part of his own self" [Lacan, 1973 chap. X.4] which generally runs in an opposite direction to the unconscious reality that psychoanalysis is meant to actualize.

In addition, all the research on affect seems to be pervaded by a more or less explicit intention of improving practice. For instance, Breen [2000] concludes his paper appealing to "the need for further work to be done to identify and research contributions that a knowledge of psychoanalytic processes can make to understanding and improving the teaching of mathematics" [Breen, 2000:112]. If psychoanalysis were to be called for in such circumstances, it would start by asking: *Why do you think that the teaching of mathematics should be improved? Why is it not good as it stands?* These questions, as far as we know, have never been addressed in PME. In calling on psychoanalysis to solve the distressing problem of practice, the teacher becomes part of the problem to be solved, a not very comfortable position. We intend to show how Lacan's theory allows us to cope with this discomfort.

THE THEORETICAL BACKGROUND

The theory elaborated by Lacan in order to orient and explain the dialogic encounter with his patients may be characterized as the *dialectics of the subject and the other* [Lacan, 1973: 205, 239]. This theory models what happens whenever a human subject address an audience in a common language. In particular, it is fit both to explain and orient the learning/teaching experience in a mathematics classroom provided its concepts are conveniently specialized.

According to common-sense opinions about psychoanalysis, the transfer is a substance *transmitted* between subjects via communication such as the "transmissions psychiques" of Blanchard-Laville [1997], or a catharsis of unconscious elements displaying a distortion to be rectified by the counter transfer, as in Wilson [1995], or "the imposition of an actual or imagined previous relationship onto a present one" as in Breen [2000]. The transfer is also thought of as an affect that may be positive or negative. Positive transfer is identified with love. Lacan [1973] starts from this last common sense conception and takes it elsewhere. According to its meaning in the clinical experience, "the transfer is a phenomenon where both the subject and the psychoanalyst are included together" [210], hence two unconscious are simultaneously involved and it cannot be separated to "transfer" and "counter transfer"; "the transfer is the actualization (*mise en acte*) of the reality of the unconscious" [137] with all the implications the unconscious presupposes, including desire and sexuality. What has to be analyzed in the clinic is the transfer itself. In addition "according to its nature, the transfer is not the shadow of something that has been lived in the past. On the contrary, the subject as subjected to the analyst's desire, desires to cheat him about this subjection making himself *loved by the analyst*" [229, our emphasis].

The four fundamental concepts of psychoanalysis, the *repetition*, the *unconscious*, the *transfer* and the *triebe*[1], should not be treated separately. However, due to lack of space, we will only take up the transfer, and this only under the aspect of *imaginary and symbolic identifications*. What we lose in fidelity we hope to gain in clarity.

Specializing the transfer to the teaching/learning mathematics experience leads to the concept of *pedagogical transfer* [Cabral, 1998]. The pedagogical transfer implies that listening is not unrestricted, as it is in the clinic, but is *restricted to mathematical listening*, that is, restricted to the possibility of attributing mathematical meaning to what is heard. Mathematical listening is defined as occurring when the listener is able to *repeat the speaker's discourse* until the speaker agrees that it is exactly what he meant. Of course, the speaker can always disagree and say that something else was meant. Therefore the activation of restricted listening presupposes an agreement, tacitly established prior to the talking situation.

The pedagogical transfer occurs when 1) the student manages to adjust the image of himself that he sees in the mirror to his expectation of being loved by the teacher and 2) the teacher accepts this image as capable of being loved. This love is to be distrusted, since the student is only seeking the way to produce the right answer, so that the teacher and the parents will be satisfied, and he will be recognized as one who knows and will get credit for this. The identification process of the subject with the image that he supposes to be loved by the teacher is called the *imaginary identification* and denoted i(a), a "i" for "image" and "a" for the object of desire around which the image is built. Schematically, in pedagogical transfer, this object is represented by the ability displayed by the teacher in producing answers and deciding what is right or wrong, an ability offered to the student as one to be imitated.

Whereas the image produced by i(a) is perfect, the unconscious reality that commands the production of this image is not so nice. This is an obscure reality that the subject perceives as his own qualities and commitments. He needs to love what he perceives, so he fabricates a high representation of himself out of ideological commitments that the culture offers him. The process leading to these commitments is called the *symbolic identification*, or the *super-ego* and denoted I(A), a "I" for identification and "A" for the Other, approximately, the culture.

In the pedagogical transfer, the commitment is mingled with the ability to produce answers, leading generally to strong rote learning efforts. Lacan [1973] says that this mingling of *I* and *a* "is the safest structural definition of hypnosis that has ever been produced" [245]. Plainly: in transfer, the student is in a state of hypnotic trance, a

[1] This German word has tentatively been translated by *instinct*.

most delicate situation. However, the teacher's object of desire may well be represented by a student who produces the right answers *according to mathematical canons*, not by rote. She may be incapable of including among the representatives of her object of desire a nonsensical student-mathematics relation. Up to a certain point she may pretend that the student's rote answers are produced according to mathematical principles and accept them... up to a certain point. At this point she snaps and the student awakes from his trance without having been prepared for this moment by an adequate analysis of the transfer. It is not only the student's love message i(a) that is rejected, it is the student's unconscious commitments I(A) that supported i(a) that are impaired: his super ego crashes. Hence the emergence of fear and anxiety. If this does not happen as intensely with other subject matters, it is because rote learning is easier to disguise in them and, in certain cases, is compatible.

On the other hand, if the teachers desire is related to the analysis of transfer, than affect comes in: "What the psychoanalytical discourse leads to is that there is only one knowledge about affect: affect is the capture of the talking being in a discourse that determines him/her as object. (...) Of this object we know nothing except that it is the cause of desire". [Lacan, 1991:176-7].

This implies, first, that, in dealing with affect and its connotations, language is absolutely primary, not a "means of communication", since the constitution of the subject (students, teachers, researchers) depends on it. Second, if we want our practice to have anything to do with affect, that is, if we want the student to be captured as an object of desire in the discourse of our practices, we had better *stop talking and start listening to him*. Stopping talking requires the modification of our own commitments I(A) that support our cherished self image as good explainers i(a). However, we must assume the initial position that installs the pedagogic transfer, indicating to the student our position as the subject supposed-to-know (*sujet supposé savoir*). The student must believe that the teacher knows about his learning. It is from this initial position that we must provoke the student to work towards the production of his own knowledge. Like Blanchard-Laville [1997:168] we do not seek to eliminate suffering because it is what commands the moments of successive opening and closing of the unconscious whose reality transfer is expected to actualize, so that the subject can check his I(A) and assume it.

The capture of the nonsensical student-mathematics relation as an object of desire inverts the hypnotic situation. It is the teacher who must be hypnotized by the student's discourse, trying to make sense of it to the satisfaction of the student. In this way the teacher keeps himself at the largest possible distance from the object *a* around which the student seeks to constitute his imaginary identification [Lacan, 1973:245]. Let us see how this theory works in an example from practice.

THE SETTING AND THE EXAMPLE

The basic unit of our psychoanalysis-inspired learning experience is a weekly meeting assembling teachers and graduate students of a mathematics education program, undergraduate students of a mathematics teacher formation program, and mathematics teachers from the neighboring school district, totaling about 10 to 15 people. We call these meetings *integrated sessions*. They started several years ago when an undergraduate student asked for help in solving his special difficulties: he had failed all his freshmen courses. Gradually, other undergraduates joined in and took advantage of the teaching dispensed to him. Now these meetings provide course credits for undergraduates as well as research material for graduate dissertations and papers. They provided us with the experimental base to adapt Lacanian concepts to the classroom.

In these sessions, teacher and student are not labels attached to people, but positions of speech. Whoever is at the blackboard, generally an undergraduate student, is called "the student". The student is expected to work in order to produce some sort of knowledge in the connection of the mathematical object and his ignorance. "The teachers" are those who put restricted mathematical listening into practice and provide guidance. The teacher's position consists of sustaining the student's speech. We follow a lemma: "It is through speaking that one learns and through listening that one teaches" [Leal et al. 1996:243]. In the sessions the didactical, pedagogical and mathematical objects are treated simultaneously.

We shall report on the integrated session of August 24, 2001. In the dialogue, "teacher", of course, does not always refer to the same person. The student who volunteered, proposed a problem from his mathematical analysis course: *every interior point of a subset of R is an accumulation point*. Following the teachers' orientations, the student reproduced the definitions and concluded that, if x belongs to int(X), then x belongs to the derivate set X'. When everything seemed to be over and done with, the student expressed the following doubt:

Student: Alright, I have proved that if this x is in int(X) then it is in X', but this is not enough, since here, in X', I have ALL the accumulation points and I proved only for this particular x.

Notice that he did not say that "here in int(X) are all interior points and I only proved for this x". If it had been so, any of the teachers would have been able to help. Everybody offered suggestions but the student remained immovable:

Student: I understand that I have proved for this x which is any generic one, so that it is proved for all x. But there, *in X', are all the accumulation points, not only this x.*

What now? The discussion lasted for more than one hour. Everyone was eager to make a contribution. Voices grew louder, from student and teachers, denoting enthusiasm. But at each turn the student said:

Student: I know what you want me to say, but I am not convinced.

And he repeated his doubt. When the session time was over, one of the teachers asked permission to try her approach without interruption. In a low voice, she asked the student to repeat the whole reasoning. He summarized it while the teacher wrote on a clean black board what the student said, always asking him: – "Like this?" or "Is this right?" The teacher exhibited a genuine effort to understand every word the student said and did not write anything beyond what she heard. Finally the teacher expressed her doubt.

Teacher: Where do I write this "all"? Here, at int(X) or there at X'?

Student: At X'.

Teacher: This is the difficult point for us. If this "all" were here at int(X) we would know how to orient you. It is the old story about proving that every cat has a tail. It does not suffice to pick one cat. You have either to bring them all or to consider that something is nothing more than a cat, a generic cat, and so on. OK?

The student agreed and the teacher went on, always in a low voice.

Teacher: But since you put the "all" here, at X', we don't really get you. Can you explain?

Her mood was of attentive invitation. The student thought for a while and looked embarrassed.

Student: I do not know what I want to say, confessed the student.

Teacher: Try and say it, insisted the teacher.

The essential point is that this "*try and say it*" was not uttered in a mood such as to mean: *now do you understand?* or *do you see your nonsense now?* The meeting ended in a happy mismatch.

When the session was over, the ability of the teacher was praised, but she replied that she was really curious to see what was the student was thinking. She had assumed that there was indeed a meaning and she wanted to find it out. Since the student could no longer remember the content of his doubt, she confessed that she felt somewhat deceived. She had been playing the hypnotized party, the nonsense produced by the student playing the role of her *a*. The evaluation was made that in every previous attempt to orient the student, the teachers always presumed that they knew something about the student's difficulty (assumed the position of the subject supposed to know) or intended to reach some foreseen point (rectify the transfer). These evaluations were made live, in the presence of the student.

Later on, the authors conjectured that the student was mostly demanding some sort of affect, inserting himself as the object of desire of our discourse, which he knew was possible from our previous joint experience. Being recognized in his attempt he received a form of love that was not feigned. However, he was not spared his suffering: no one told him that what he was saying was nonsense, nor was he praised for abandoning his previous view. He had to take responsibility for his I(A).

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DIDACTICAL REFLECTIONS ABOUT SOME PROOFS OF THE PYTHAGOREAN PROPOSITION

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This theoretical paper presents some dimensions considered in the literature to analyze proof in the teaching and learning of mathematics. In order to show how different types of proofs can be used with students of different levels, we use these dimensions to analyze four proofs of the Pythagorean proposition.

This essay consists of three main parts. First, we show some classifications of previous work on mathematical proof according to dimensions that we consider useful in the teaching and learning of the subject. The second part presents mathematical proof as an important but problematica issue in mathematics education. This is shown in different contexts. In the classroom, on the one hand, students have difficulties with the notion of proof and, on the other, teachers do not know how to teach proof. Finally, we show how some research results can help teachers in their task in order to improve students' understanding of the Pythagorean proposition.

Classifications

Many studies on mathematical proof try to delimit, characterize, specify or define the term *proof*. We find different words with similar meanings such as *explanation*, *argument*, *justification*, *confirmation*, *verification* or *validation*. These meanings appear with no clear delimitation and sometimes even with quite confusing connotations. But, in spite of this and the differences between their theoretical frameworks, all of the studies consulted assign to mathematical proof a logical character that implies an unequivocal mathematical statement. Some researchers use the different terms to establish a rank between them. We have found two big groups of authors that use rigor to propose their classification. Rigor is a dimension, which conforms analytic dimensions that will help us in our later reflections. We have classified the meanings given to the notion of proof in some studies (see table 1).

In both groups we find different terms for the same kind of proof. They base their work on similar criteria to do their own classifications in terms of rigor. That is why heuristic argument, informal proof and so on are in the same stage. In the same sense, if we find a proof in which drawings or concrete numbers are involved and where inductive reasoning is predominant, all the authors who consider two levels would classify the proof in the less rigorous stage. On the other hand, as much deductive reasoning involves a proof, more formal it is considered.

Table 1

<table>
<tr><td rowspan="8">- R I G O R +</td><td colspan="4">TWO LEVELS</td><td colspan="4">THREE LEVELS</td></tr>
<tr><td>Jaffe & Quinn (1993)</td><td>Movshovitz -Hadar (1996)</td><td>Martínez (2000)</td><td>Gutiérrez (2001)</td><td>van Dormolen (1977)</td><td>Blum & Kirsch (1991)</td><td>van Asch (1993)</td><td>Balacheff (2000)</td></tr>
<tr><td rowspan="3">Heuristic Argument</td><td rowspan="3">Informal Proof</td><td rowspan="3">Informal Argument</td><td rowspan="3">Empirical Proof</td><td rowspan="2">Level 0</td><td rowspan="2">Proof</td><td rowspan="2">Proof with drawings or examples</td><td rowspan="2">Explanat.</td></tr>
<tr></tr>
<tr><td rowspan="2">Level 1</td><td rowspan="2">Pre-formal Proof</td><td rowspan="2">Pre-formal Proof</td><td rowspan="2">Proof</td></tr>
<tr><td rowspan="3">Rigorous Proof</td><td rowspan="3">Formal Proof</td><td rowspan="3">Mathem. Argument</td><td rowspan="3">Deductive Proof</td></tr>
<tr><td rowspan="2">Level 2</td><td rowspan="2">Formal Dem.</td><td rowspan="2">Formal Dem.</td><td rowspan="2">Dem.</td></tr>
<tr></tr>
</table>

Other researchers who also consider the rigor dimension find more than three big groups. Miyazaki (2000) establishes six levels of proof in lower secondary school from inductive proof to an algebraic demonstration basing on three axis: contents of proof, representation of proof and students´ thinking (see table 2). We shall apply them to some geometric cases.

Table 2

<table>
<tr><td colspan="3"></td><td colspan="2">PROOFS</td></tr>
<tr><td colspan="3"></td><td>Inductive Reasoning</td><td>Deductive Reasoning</td></tr>
<tr><td rowspan="4">Students´ thinking</td><td rowspan="2">Concrete Operations</td><td>Other languages</td><td>C</td><td>B</td></tr>
<tr><td>Functional language of demonstration</td><td>D</td><td></td></tr>
<tr><td rowspan="2">Formal Operations</td><td>Other languages</td><td></td><td>B</td></tr>
<tr><td>Functional language of demonstration</td><td>D</td><td>A</td></tr>
</table>

The previous classifications lead us to think about the uses of proof in different contexts, particularly in mathematics education and in mathematics research. An explanation that can be considered as a proof in mathematics education, maybe not be a proof in mathematics research. And even if we focus on the field of mathematics education, we find differences depending on the educational level. These levels are going to determine the role of proof in teaching. The function of the proof shall be a dimension in our analysis. In this regard, we consider the classification proposed by Hanna (2000) (who presents a compilation from Bell (1976) and de Villiers (1996)).

She presents the following functions that she considers useful for thinking and doing research on the topic: *verification*, *explanation*, *systematization*, *discovery*, *communication*, *construction* of an empirical theory, *exploration* of the meaning of a definition or the consequences of an assumption, and *incorporation* of a well-known fact into a new framework and thus viewing it from a fresh perspective. (p.8) De Villiers (1996) criticizes the almost exclusive function of verification that traditionally has been given to proof. He argues that this is an important function but not the only one. We shall support this idea with our reflections.

We can consider other useful criteria to classify proofs. For instance, Ibañes & Ortega (1996) concentrate on the reasoning done and the technique used. They distinguish between methods, styles and modes (see table 3).

Table 3

Method	Syllogism
	Cases
	Reductio ad absurdum
	Complete induction
	Constructivism
	Analogy
	Duality
Style	Geometric
	Algebraic
	Of the coordinates
	Vectorial
	Mathematical Analysis
	Probabilistic
	Topological
Mode	Synthetic or direct
	Analytic or indirect

Teaching and learning of proof

We would like to start from two assumptions related to the teaching-learning procedure:

- We argue that proof is an important and useful process in mathematics teaching at all levels.
- However, teachers do not need to present all the proofs of a concrete theorem to their students.

We find studies on proof according to educational level: involving students, from primary school to university levels; and involving teachers, ranging from prospective mathematics teachers to practicing teachers. Inductive reasoning is the most common type of proof used with the young students. Based on this idea and trying to develop deductive reasoning in some elementary school pupils, Lobo-Mesquita (1996) highlights the role of visualization in the learning of geometry. Some studies show how proof is an obstacle in the educational process. Ibañes (2001) shows that pupils in the third course of high school have problems with proof outlines, recognition of procedures, and the use of some expressions. Almeida (1996) suggests that, at the university level, it may be desirable to use semi-formal or preformal proofs as a bridge to formal proofs. Rico, González and Segovia (1999) study different heuristics used by pre-service mathematics teachers trying to prove a geometric proposition.

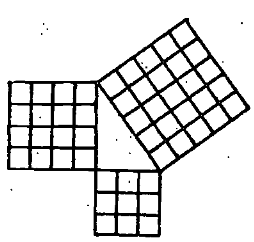
Proofs and the implied process is also a problem for teachers, who do not have a method to teach proof and sometimes don't know about the convenience of teaching a concrete proof (Cañadas, Nieto & Pizarro, 2001). Garnica (1996) analyzes answers given by mathematics teachers in terms of the modes of reading: technical and critical. Van Ash (1993; mentioned by Ibañes, 2001, p. 18) gives some ideas to help teachers to decide when to do a proof in class: to convince when we are not sure about a statement, to memorize a theorem, to learn useful algorithms, to end a search process, to expose a work method and to show meanings of definitions.

So proof seems to be an obstacle in the teaching-learning procedure. We are going to use the results of the mentioned studies to suggest ways to help both students and teachers in their daily task. In what follows, we try to connect the ideas related to proof mentioned before with a concrete case: the proof of the Pythagorean proposition.

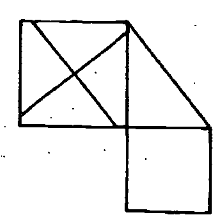
The case of the Pythagorean proposition

The Pythagorean proposition establishes a relationship in a right-angle triangle: *the square of the hypotenuse is the sum of the squares of the legs*. It has hundreds of different proofs. Scott (1972) classifies many of them in four groups: 1. those based upon linear relations; 2. those based upon comparison of areas; 3. those based upon vector operation; and 4. those based upon mass and velocity. We will focus on the geometrical ones due to their didactical interest and their accessibility to students of different levels. The geometrical formulation of the proposition is as follows: *the square described upon the hypotenuse of a right-angled triangle is equal to the sum of the squares described upon the other two sides*. The common objective of all the proofs is to *verify* this proposition. We shall present four geometrical proofs of the Pythagorean proposition and shall comment on the dimensions we have mentioned: rigor, contents, representation, students' thinking, method, style and mode.

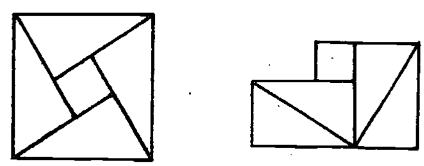
(1)



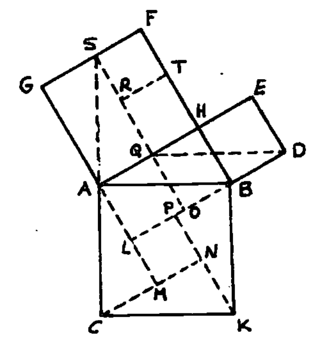
(2)



(3)



(4)



These proofs make evident the importance of visualization in geometry. Furthermore, they allow to *communicate* mathematical knowledge, to *construct* theory basing on empirical actions, to *explore* the meaning of the proposition and to *incorporate* a new knowledge into a framework.

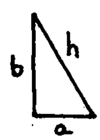
According to Ibañes & Ortega's classification, the proofs taken are direct proofs by cases, having a geometric style (although as we will see some of them involved other styles).

Focussing on rigor dimension (see table 1), most authors who consider two levels would situate these proofs as informal ones because they have been done with drawings and some of them do not need analytical reasoning. On the other hand, authors who consider three levels would situate proofs (2), (3) and (4) in the second rigor level because they are not the formal proofs we are used to in pure mathematics, but they are not just examples neither; and proof (1) would be classified in the lowest rigor level because is a concrete case that can be used as an example or an explanation.

We shall differentiate each proof from the others, pointing out differences and common points. Let us begin with proofs (1), (2) and (3), which follow the same strategy: to build the square upon the hypotenuse with the pieces given in the other squares.

In proofs (1) and (2) students just need to know the surface area concept. We have to avoid presenting the situation as a game and try to give meaning to the change of pieces. So we think that proofs (1) and (2) would be appropriate for primary school pupils, whose inductive reasoning is predominant, and for whom visualization and manipulation processes are very important. Let us remember that at this age, students' reasoning is mainly based in concrete and real objects. With the Ibañes & Ortega's classification, both proofs are purely geometric. We notice that there are differences between them. In proof (1), the pieces have the same area and the same shape, and they are situated in the same position regarding the big square. The proof follows easily from the comparison of the number of unit squares in the big squares. It consists of inductive reasoning and it is represented with numerals. So it is a "proof C" according to Miyazaki (2000) (see table 2). In proof (2), the pieces have different shapes with different positions as a consequence of having drawn parallel and perpendicular lines. This implies that students have to understand the meaning of the Pythagorean proposition instead of just solving a puzzle (Cañadas, 2001). According to contents and representation is a "proof B", according to Miyazaki (2000)) with concrete operations because it consists of consecutive actions on concrete objects and it includes deductive reasoning supporting these actions. So proof (1) could be used as an introduction to the proposition, inducing the *discovery* of the theorem. Proof (2) could be useful as an exercise of reasoning in order for pupils to show their understanding.

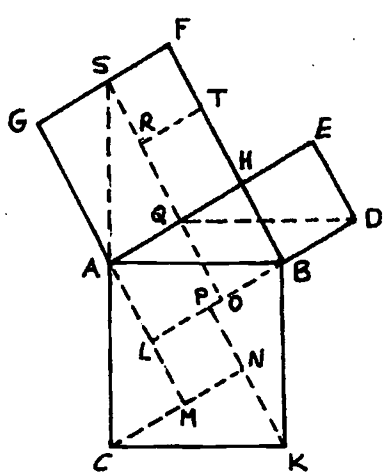
Proof (3) does not correspond to the geometrical formulation because we cannot visualize the right-angle triangle where we must build the three triangles to whose areas the proposition relates. In a Piagetian sense, this proof style follows "action strategies". This proof could be clearly situated in the second of the three rigor levels mentioned before. To finish the explanation process, we would have to draw and write as follows:



And comparing the areas of the two figures showed, we have that $h^2=a^2+b^2$.

It has an algebraic reasoning added to the general geometric position aforementioned. This is the characteristic that could differentiate "proof B with concrete operations" from "proof B with formal operations" (Miyazaki, 2000) in geometric proofs. So we think proof (3) is of the second kind. It is mainly the implicated reasoning with algebraic terms what allow to conclude that we should work with this proof at secondary level.

Proof (4) implies the highest level. The following reasoning is necessary to finish the proof:



Produce CA to S, draw SO parallel to FB, take HT = HB, draw TR parallel to HA.
Produce GA to M, making AM =GA. Produce DB to L, draw KP and CN parallel resp. to BH and AH. Draw QD.
Rectangle RH = rectangle UB.
Then: **Square AK** = triang. CKN (= triang. ASG) + triang. KBP (=triangle SAQ) + triangle BAL (=triangle DQO) + triang. ACM (=triang. QDE) + sq. LN (=sq. ST) = rectang. GQ + rectang. OE + sq. ST = rectang. GQ + sq. EB + rectang. QB + sq. ST = rectang. GQ + sq. EB + rectang. RH + sq. ST = **square BE + square GH.**

According to Ibañes & Ortega´s classification, it is a pure geometric proof. Lines drawing and reasoning bear concepts like parallelism, perpendicularity, similarity and congruence. We add *systematization* of some results into a deductive system to the mentioned functions for the other proofs. According to Miyazaki (2000), proof (4) could be classified as "proof A" because it represents the most advanced level in a geometric proof. It involves deductive reasoning with a functional language of demonstration. So the appropriate level for this proof would be the end of secondary school or even the university level. Students at these levels are better used to abstract concepts and their deductive reasoning is more developed.

Discussion

We have shown how we can use different proofs of the Pythagorean proposition at different educational levels. For this task, we have used different dimensions drawn from the literature. These dimensions are: rigor, contents, representation, students´ thinking, method, style and mode. This kind of analysis provides us with knowledge about which proof is more suitable for each educational level, keeping in mind students' characteristics in order to obtain better results in teaching and learning. The knowledge of the different proofs from this point of view help teachers to decide why (function), when (students´ thinking, content and style) and how (representation) to do a proof in class. In this sense, teachers could be better informed about the convenience of teaching a concrete proof. We feel that this analytical process (that we have put into practice for the Pythagorean theorem) can be used for other propositions for which presenting a proof might be relevant.

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Mathematical Thinking: Studying the Notion of 'Transfer'

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Abstract: These analyses form part of a three-year project looking at mathematical thinking as a socially organised activity. We revisit data from a University Calculus class using tools from two theoretical perspectives, used increasingly in mathematics education research: (1) semiotic mediation and (2) discursive practices. We highlight how different theory-driven analyses taking a sociocultural view of thinking and learning can offer insights into the conceptualisation of the 'transfer' of learning.*

Recent research on transfer

Mathematics educators have always maintained that the mathematical knowledge that students acquire in school should be able to be adapted and applied in workplace and everyday situations. It has also been recognised by the community that, far too often, it does not happen. Theoretical explanations for this failure depend on the author's views on the nature of the boundaries between the practices involved (see Muller & Taylor, 1995; Evans, 1999). Three main positions can be identified in the literature:

(1) The boundary between the everyday and school mathematics as permeable and theoretically unproblematic, if practically a considerable challenge for pedagogy. Inadequate instruction or inadequate learning can result in instrumental understanding (Skemp, 1976). Here, some authors emphasise the value to students' learning of the use of *authentic* contexts (Sullivan, Warren & White, 1999).

(2) Transfer is not possible, because of the impermeable boundaries between contexts and practices. These include the strongly situated view (often drawing on Lave (1988)) that meanings are produced and remain within specific social and cultural practices.

(3) Transfer is problematical, since boundaries exist, and are not automatically crossed, but still it is possible to enable something like 'transfer'. This includes the use of concepts such as 'consequential transitions' (Beach, 1999), 'translation' (Evans, 1999, 2000), and 'recontextualisation' (Cooper & Dunne, 1999). Amongst those who consider the boundary as problematic, Boaler (1998) argues that developing identities in communities of practice in which "students are enculturated and apprenticed into a system of knowing, thinking and doing" (Boaler, 1998, p. 118) might be helpful in enabling students to transfer their knowledge (see also Goodchild, 1999).

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* The project "Teaching and Learning – Mathematical Thinking" was supported by Fundação Ciência Tecnologia, Grant no. PRAXIS/P/CED/13015/98. It included, in addition to the authors, João Filipe Matos, Madalena Santos, Anna Tsatsaroni.

In our larger project, we have focused on three key issues – emotion (Matos & Evans, 2001), assessment (Morgan & Lerman, 2000) and transfer – as aspects of mathematical thinking, and we have deployed a range of theoretical perspectives to look at mathematical thinking as *socially organised*. We have taken data produced for other purposes and developed tools from those perspectives in order to read the data for when and how transfer occurs *as interpreted through those theoretical lenses*.

The context of the production of the data

In this case, the data were collected during the second semester of 1996 in a First Calculus Course for university students taking a Business degree. The work reported was developed at the University of Algarve, as part of a research project aimed at studying the process of meaning making in mathematics learning and, particularly, the influence of real world situations in the meanings produced (Carreira, 1998).

From a methodological point of view, the research design assumed the double character of teaching experiment and curriculum development program. The teacher had very firm intentions of innovating and questioning the traditional format that such courses tend to assume. One premise that became central to the structure of the course was privileging the activity of mathematisation over the mastering of mathematical techniques and proficiency on specific topics (Moreira & Carreira, 1998).

On such a basis, the guiding lines of the curriculum emphasised: (1) the development of connections between mathematics and reality; (2) students' co-operative work on applied problem situations; (3) students' oral and written presentations of the work developed; (4) whole-class discussions on critical mathematical models and applications.

At the beginning of the session, students were given some information on a particular model describing the 'utility' of wine and beer. Their subsequent activity evolved around open questions that guided their exploration and investigation of the model.

> *We say that two commodities are competitive when the consumer tries not to exclude the consumption of neither of them. Assuming that the consumer's satisfaction can be measured in some way, economists have created the notion of utility to describe the degree of the consumer's satisfaction. In particular, this concept may be used to describe the utility of both wine and beer to a certain consumer.*
>
> *Consider the utility function $U(x,y) = (x+1)(y+2)$ as representing the utility that a certain consumer gets from the consumption of x units of wine and y units of beer.*

Previously, the teacher had explained to the class what level curves are, in the context of multivariable functions. Afterwards students were asked to interpret the designation of indifference curves used in Economics for contexts like the given utility model.

33. Cristina: I have heard of level curves in cartography and I worked a lot with that.
34. Isabel: But that kind of curve is only used in economics.
35. Cristina: Excuse me! In cartography there are also level curves! If you want to represent the terrain elevation on a system of axes, you get something like this... You can mark the heights above the sea... for instance 200 meters, 300 meters.

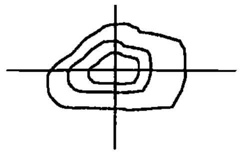

Figure 1. Cristina's representation of level curves

36. Isabel: No. In economics it's different, it's like this...

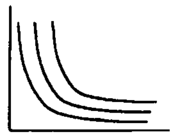

Figure 2. Isabel's representation of level curves

37. Cristina: In geography, level curves are usually defined with the help of a land survey, that is, a collection of data on the elevation of the land.
38. Eduardo: Those curves indicate different altitudes on a map.
39. Cristina: Well, in economics it must be something of the kind, I mean, in the same line of reasoning.
40. Isabel: No. What I think is that it must be more or less this... To have the same degree of satisfaction, we have to... follow one of these curves.
41. Cristina: What you're saying is that to get the same satisfaction, if the amount of beer increases, the amount of wine must decrease.
42. Isabel: Each of these curves is one utility...
43. Cristina: Yes, it shows that utility is always the same. And to have the utility unchanged, the variables have to change. If one increases, the other decreases.
44. Miguel: Each of the curves gives a different relationship between the variables x and y.
45. Cristina: Of course. It's like in geography. For instance, this utility is different from that one.
46. Miguel: But along each curve the utility doesn't change.
47. Cristina: It only changes from curve to curve.
48. Isabel: Each curve is a different level of utility.
49. Eduardo: But why are they called indifference curves?
50. Cristina: Because regardless of the values of the variables, the utility is indifferent.
51. Eduardo: I see. They're indifference curves because each curve has always the same level of utility. A curve is different only in relation to the other curves.
52. Cristina: The utility stays constant. Only the x and y change to make the utility unchanged.
53. Isabel: But what has that to do with cartography?
54. Cristina: It must have something. Look, I have this mountain. This inside area tells me the maximum height...
55. Eduardo: The point is that you have several levels. Except that here you have utility levels. Here the mountain is the utility. The consumers' satisfaction is also rising isn't it?

A semiotic mediation analysis

From a semiotic mediation point of view, transfer is seen as a process that is not automatically an outcome of learning. On the contrary, the conditions of learning are given a decisive role in facilitating the occurrence of transfer. Transfer would be conceived as something that can and needs to be taught if one of the aims of teaching is to help students to make connections between different semiotic and conceptual systems. To allow for transfer may be thought of as creating the opportunities for students to engage with different conceptual tools, and to work with them simultaneously as happens in the case of metaphorical thinking (Carreira 1997, 1998). In addressing the data from this perspective, we will focus on the following aspects:

(a) students' verbal interactions as instances of the production of interpretants (that is, what makes the sign mean something to a particular individual, in a particular context) for mathematical signs

(b) students' mathematical thinking as instances of metaphorical thinking

Chains of interpretants: The interweaving between mathematics and other conceptual domains. A significant semiotic chain can be traced from the way students thought about the concept of level curve. At first, two of the students showed different conceptions of level curve, one coming from the field of cartography [33] and the other from economics [34]. They tended to see them as completely independent things: there seemed to be no clear connection between their contrasting sketches of level curves.

In spite of the apparent disconnection, students struggled to find some way of bridging the two conceptual domains. It was a shared effort which highlights an on-going *social* process of introducing successive interpretants based on the articulation of diverse semiotic means supported by student's differently experienced ideas. Cristina, for instance, tried to add information on the process underlying the depiction of level curves (contour lines) in cartography [35], [37]. She talks about altitude, about a land survey and explains how the different curves indicate different heights on a map. Other students recognise that in economics, level curves outline points for which the utility remains constant. They were also able to understand these curves as formal mathematical representations of a relationship between the independent variables [44], [52]. From the sketch produced by Isabel, we see that they observe the fact that an increase of one variable corresponds to a decrease of the other.

The overall process of interpreting the concept of level curve can be mapped as a sequence of interpretants, each of them tied to a certain referential domain:

- *Representation of terrain elevation*: "cartography", "land survey", "elevation of the land", "different altitudes on a map".

- *Path defining a certain constant utility*: "economics", "the same degree of satisfaction", "follow the curve", "along each curve the utility doesn't change", "each curve is one utility".

- *Representation of the relationship between x and y, for a given value of U*: "the variables have to change", "if one increases, the other decreases", "a different relationship between the variables x and y", "only the x and y change to make the utility unchanged".

Transfer: Models as the surface of conceptual metaphors. One aspect that is quite central in the data is that students insisted on finding some way of bridging two apparently separate conceptual fields. They wondered what the possible link might be and in their search they eventually came up with a metaphor to describe it – "the mountain is the utility" [55]. This is an outline of the implicit mapping contained in the metaphor:

In a mountain there are several height levels / Consumers experience different levels of satisfaction

The mountain rises up / Utility increases with the increase of consumption

To walk on a level curve of a mountain is to keep a certain altitude, in spite of the change of position / To follow a level curve of utility is to preserve a certain satisfaction, in spite of the change of the amounts of wine and beer.

The anchoring in two situations, which is visible in this metaphor, namely by tying utility, indifference, and consumption to mountains, heights, altitudes, reveals how students brought into their reasoning their specific knowledge and familiarity with different meaningful contexts. This takes us to the claim that transfer is closely related to the production of meanings in mathematics classrooms and that such meanings are not independent from the pedagogical scenario where learning takes place.

From a semiotic mediation point of view, mathematical models of real situations are important mediating tools in uncovering powerful underlying metaphors and in fostering metaphorical thinking. If students engage in exploring the multiple facets of a model, they have the opportunity to come up with genuine mathematical thinking in light of other cultural and semiotic systems. This is a mathematical practice and it can also be a school mathematical practice if models are to be seen as revealing something rather than conveying some frozen and fixed school mathematical content.

A discursive practice analysis

From a discursive practice perspective, we conceptualise transfer as occurring when concepts originating in one discourse are linked to concepts of another discourse through a *chain of signification* (Evans, 2000). Such chains do not have an independent existence but arise for each participant (or, as in this case, for a group of participants) as they use their personal discursive resources, history and positionings to make meanings within a specific context (cf. Morgan, 1996). In addressing the data from this perspective, we need to identify the *resources* available to the participants and attempt to follow possible chains of signification through the developing conversation in the classroom. We also need to identify the available *positions* and their potentials for promoting or preventing shifts between discourses but, for reasons of space, we omit this part of the analysis.

The tasks for the analyst:

a) Identify the discourses available and the relevant concepts, values and relationships within these discourses.

b) In the transcript, look at the text as a whole and trace the discursive resources through the text. Identify whether there are key signifiers that play a role across discourses and attempt to follow the chains of signification. What is the contribution of each discourse to the solution of the problem and how are links between discourses constructed by and for the group as a whole?

The discourses available: The consumption of wine and beer may involve a number of 'everyday' discourses with which the students are familiar. At the same time, there are a number of 'esoteric' or academic discourses that are relevant to the problem as posed by the teacher and that the students may be able to draw on. The students are, of course, also participating in a classroom discourse with its own norms and available positions.

Everyday discourses: Consumption of wine and beer is generally located within particular patterns of social engagement – in various settings, with family or friends, at specific times. There is often strong regulation of the amount and type of consumption in a particular setting and the sort of behaviour that may accompany it and the meanings of consumption may vary considerably from one everyday practice to another. The everyday consumption of beer and wine may also involve a type of economic discourse. Unlike the esoteric economic discourse, however, this generally involves a *local* calculation – "Shall I buy beer or wine today, given my circumstances (my needs, desires, finances) today or this week?"

Esoteric discourses: The problem posed for the students and their identities in the context of this university class as students of Business Studies and of mathematics highlights certain discourses of academic subjects. In particular, resources from both economics and mathematics are to be seen in the form of the problem itself. As we see in the transcript, the students may also be able to draw on resources from other areas of their academic experience. The discourse of academic economics objectifies the experience of individuals and incorporates it into a *global* calculation, using terms such as 'indifference' that have meanings different from those associated with their use in everyday discourse. In everyday discourse the individual consumer is not indifferent to the specific make-up of a basket of purchases but will have preferences based on a range of non-financial criteria. It is only when 'the consumer' is conceived as an abstract generalised agent that 'indifference' occurs. Drawing on mathematical discourse creates a further abstraction. It is no longer important that the formula or the curve on the graph represents 'indifference' or that the purchase of beer or wine is involved. What is important is the algebraic and graphical representation of the problem, the relationships between these and the mathematical techniques for solving the problem that are facilitated by these representations.

Classroom discourses: The discourse of this classroom lays explicit value on making connections and on co-operation and communication between students. However, these students are likely also to be familiar with more traditional pedagogic discourses in which different subjects are strongly insulated and where individual work is more highly valued than group co-operation.

Tracing discursive resources in the transcript: Several aspects of the transcript might be analysed to give insight into transfer processes from a discursive perspective, for example: the ways the different students engage (or not) with the geographical discourse; the chain of signification formed by the shifting use of the terms *same*, *different*, *change*, *indifferent* and *indifference*; relationships between each student's positioning within the class and their use of resources from particular discourses. In the limited space available in this paper, however, we will focus on a brief section of the transcript in which may be seen a move from use of resources from everyday discourse and economics discourse to use of resources from mathematical discourse [lines 40-44]. The key terms are abstracted in the table below, showing three chains of signification.

[40 Isabel]	satisfaction (*everyday/economics*)		
[41 Cristina]	satisfaction	increase/decrease (*everyday/economics*)	in beer and wine (*everyday*)
[42 Isabel]	utility (*economics*)		
[43 Cristina]	utility	increase/decrease	in variables (*economics/mathematics*)
[44 Miguel]		relationship (*mathematics*)	between variables x and y (*mathematics*)

The move from *satisfaction* [40 & 41] to *utility* [42 & 43] is one that is provided by the wording of the given problem but this shift away from everyday to esoteric economic discourse is associated with a parallel move from the everyday discourse of beer and wine [41] to the esoteric mathematical discourse of variables [43]. A similar move is made from the everyday notion of *increase* and *decrease* [41 & 43] towards the mathematical *relationship* [44]. The final move is achieved by Miguel rather than by the two women who dominate the discussion, though Cristina, at least, seems to accept it as meaningful [45]. The students play different roles in the process of making links between the concepts arising in everyday, economics and mathematics discourses. Miguel, in particular, introduces the mathematical resources that are later picked up by Cristina [52]; this role seems compatible with other descriptions (Carreira, 1998) of his behaviour in the group but would need confirmation by further data. We see the group as a whole achieving success with each member contributing to the chains of signification leading from everyday concepts to mathematical ones. 'Transfer' arises as a product of the play among the resources each participant brings to the group interaction.

Conclusions and some methodological remarks

Both of our analyses are located in position (3) on transfer (see above): we see it as problematical, but capable of being supported by the conditions of learning and the chains of signification set up. These two approaches pay detailed attention to the (choices of) words used by the participants, emphasising the role of language both as a resource for structuring the individual's participation in the social practice(s) and methodologically as the primary means by which we as researchers construct meanings for the practices we observe. Further both analyses presuppose the social organisation of learning and transfer, and in particular, depend on ideas of practices and discourses and the boundaries between them.

Following the semiotic mediation position, it is possible to understand the term 'transfer' as a metaphor (Beach, 1999) for the processes occurring, when 'anything like transfer' takes place. Such would be the case, too, for the production of metaphorical meaning here described as a 'double-anchored meaning', in the sense that it reflects a connection between two objects (mountain, utility) and two interpretants (height, level) that are primarily tied to distinctive semiotic chains. Our analyses show that these processes – all them 'translations' or 'transitions' - are much more complicated than traditional views (see (1) above) suspect. They also suggest that mathematics educators can be more hopeful than the strongly insulationist positions ((2) above) allow.

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AN ALTERNATIVE MODEL FOR THE DESCRIPTION OF COMPUTATIONAL ESTIMATION STRATEGIES

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In this study we have analyzed the estimation strategies employed by preservice elementary teachers. It has been used an alternative model for the description of computational estimation strategies. In this model, strategies are integrated by approximation skills, mental computation algorithms, cognitive processes (reformulation, translation and compensation) and metacognitive processes (as the assessment of the outcome). The adoption of this model allows making the identification and characterization of estimation strategies and to complete a systematical classification of the strategies, attending to the estimation processes intertwined with them.

In the analysis of strategies used in computational estimation tasks, Reys, Bestgen, Rybolt, and Wyatt (1982) identified three high–level cognitive processes that are intertwined with these strategies: reformulation, translation and compensation. Reformulation is "the process of altering numerical data to produce a more mentally manageable form. This process leaves the structure of the problem intact" (p. 187); translation is "the process of changing the mathematical structure of the problem to a more mentally manageable form" (p. 188); it is considered that there is a change in the mathematical structure of the problem (and, in consequence, a translation process) when the substitution of the initial data produces a change in the computation algorithm employed to find the result; this change can be produced in the operations effectuated or in the order in which the operations are executed. Finally, compensation is manifested in the "adjustments made to reflect numerical variation that came about as a result of translation or reformulation of the problem" (p. 189).

In most of the investigations devoted to the identification and characterization of computational estimation strategies (Dowker, 1992; Hanson and Hogan, 2000; LeFevre, Greenham, and Waheed, 1993; Lemaire, Lecacheur, and Farioli, 2000; Levine, 1982; Reys and et al., 1982) it has been considered that the "specific strategies" are the approximation skills used to substitute the initial data by other more mentally manageable (rounding, truncation, substitution of a decimal number by a fraction, use of compatible numbers, etc.). However, they have also been considered "specific strategies" the computational algorithms ("proceeding algorithmically" and "incomplete partial products"), the properties of the operations (distributive strategy), the general estimation processes (intermediate compensation, final compensation), and even the performances of the students in which is manifested that they are not accomplishing an estimation (trying to make an exact mental computation, or the cited "proceeding algorithmically") or those in which there is not a genuine mathematical procedure (attempting to guess the result or refusing to solve). This absence of criterion when making a systematical classification of the estimation strategies leads to the fact that each author uses a different classification and causes that it is difficult to make a coherent synthesis with the results of these studies on strategies.

On the contrary, this study is different from the previous one in the sense that the term "strategy" is taken here in a broader sense. We consider the approximation skills as constitutive elements of the strategies, but within them it can also be included the estimation processes, the mental computation algorithms and the assessment of the outcome. Adopting this position enables us to include –within estimation strategies– metacognitive processes as those described by Sowder (1994). This authoress asserts that: "individuals who are considered to be successful at computational estimation are usually characterized as flexible, self–confident, tolerant of error in estimates and [...] seek the reasonableness in results" (p. 142). Thus, the subject that accomplishes an estimation must be capable of choosing in a flexible way a strategy adapted for the estimation problem and assessing the process (modifying it if it would be appropriate) as well as the result (examining its reasonableness). Sowder (1994) considers the flexible choice of strategies and the valuation of the process and the result as examples of "self–regulation" and "self–monitoring" that constitute metacognitive processes.

The present study analyzes the computational estimation strategies employed by preservice elementary teachers through a model described in Segovia, E. Castro, Rico and E. Castro (1989). In this model, all the estimation processes identified in previous studies are collected. In the approach adopted in this work, producing an estimation consists basically in: Substituting the original data by approximations that enable to reduce the complexity of the calculations, maintaining the necessary proximity to the exact result; applying a mental computation algorithm to these approximations; effectuating a compensation (previous or subsequent to the computation); and making a valuation of the obtained result. As has been exposed, it depends on the type of substitution that is made with the initial data, and if this implies (or not) a change in the computation algorithm, we will face a process of reformulation or one of translation.

The characteristics of this model for the estimation strategies can be summarized in figure 1, taken from Segovia et al. (p. 152). In previous pages some general strategies that stem from this model are presented in various schemas. An example of them is the following diagram:

$\boxed{\text{CEP}} \rightarrow \boxed{\text{Refor.}} \rightarrow \boxed{\text{Translation}} \rightarrow \boxed{\text{Compensation}} \rightarrow \boxed{\text{Computation}} \rightarrow \boxed{\text{Outcome}} \rightarrow \boxed{\text{Assessment}}$

The main objective of this research is to describe the general strategies employed by preservice elementary teachers in the accomplishment of computational estimation tasks without context, and proving if these strategies can be explained through the model described in the previous paragraph.

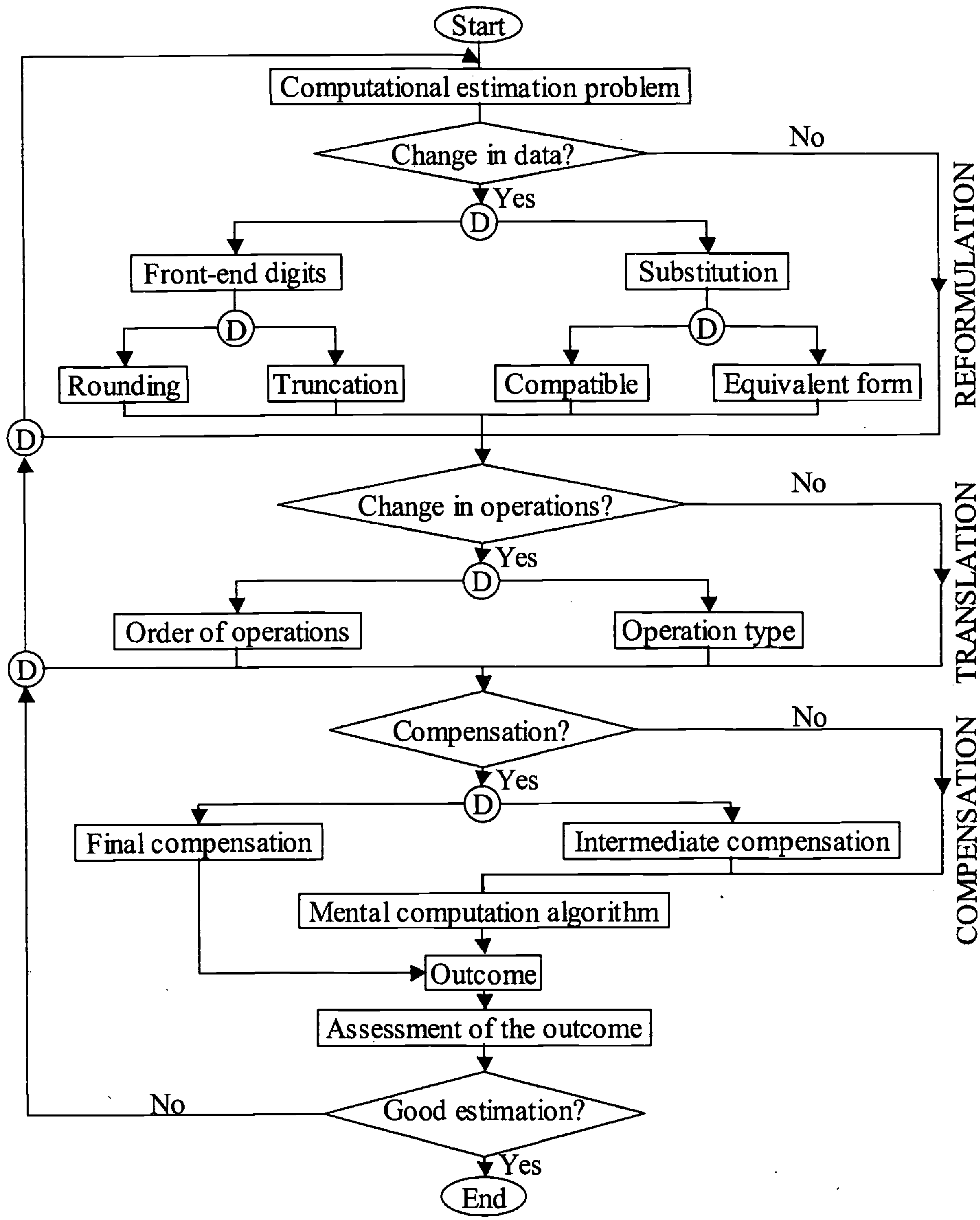

Figure 1. Flowchart of computational estimation strategies. The symbol D represents the points of the procedure where one must to take a decision

METHOD

This investigation belongs to a most general work (Castro, 2001) in which it is studied the influence of number type in computational estimation tasks. With this objective it has been used the test TEA (Test of Estimation Ability) (Levine, 1982) that consists of ten multiplications and ten divisions and combines different types of numbers: whole, decimal greater than one and decimal less than one. In this section, it is given the description of the methodological part corresponding to the strategy analysis.

Subjects

The sample used is constituted, in the general framework of the investigation, by 53 preservice elementary teachers of different first course specialties of the Escuela Universitaria La Salle, assigned to the Universidad Autónoma de Madrid. From this sample 10 subjects were selected for the strategies exploration. All the students had a period of instruction on computational estimation of 10 hours, during which they received teaching of computational estimation strategies and estimation was practiced in direct and applied computations.

Instruments and application

After the administration of the Test of Levine (1982) to all the participants, the results were used to select subjects for the phase of strategy analysis that is described in this work. The analysis of the strategies used by the subjects was accomplished through an interview in which it was requested to the subjects to give an estimation for a proposed calculation and, afterwards, they had to explain the procedure used to produce their estimation. In the interview they used the items of the test of Levine (1982) in which appear decimal numbers less than one (187.5×0.06; 64.5×0.16; 424×0.76; 0.47×0.26; $66 \div 0.86$; $943 \div 0.48$; $0.76 \div 0.89$). The subjects were selected for the interview when their estimations –for the cited items– were all compatible (or all incompatible) with the adequate knowledge of the relative effect of the operations. We consider that an estimation for the calculation 187.5×0.06 is compatible with an adequate knowledge of the relative effect of the operations, if it is within the interval (0.06, 187.5); in other words, if the student does not incur in misconceptions such as: "multiplication makes bigger" or "the division makes smaller". The interviews have been accomplished individually. They were registered using a recorder and transcribed to paper for their subsequent analysis.

RESULTS

The strategies produced by the subjects have been classified taking into account the estimation processes appearing in them. Thus, there are strategies in which only it is given a process of reformulation; in other strategies, the reformulation is produced together with a compensation process (intermediate or final); a third group of strategies is formed by those in which we can detect processes of reformulation and translation; and finally, in some strategies we found all the estimation processes (reformulation, translation and compensation) considered by Reys et al. (1982). Now, we are going to present some examples of analysis of strategies belonging to each one of the cited groups. The analysis of each strategy begins with a fragment of the transcription of the interviews accomplished by the participants of the study, that comes accompanied with a diagram of the strategy.

Reformulation

In some strategies, it has been only identified a process of reformulation. We propose the following example to illustrate this situation:

Interviewer: 0.47×0.26

Student 41: 0.3... [Rounding the second factor]. I have multiplied 50 by 30, 0.150.

Computational estimation problem (CEP)	→	Reformulation	→	Computation
0.47×0.26	→	$(50 \times 30) \div 10000$	→	0.150

In this case, the reformulation has been carried out through the rounding of both numbers. Taking into account that both roundings have been accomplished upward, it would be suitable to make a final compensation "downward". However, it has been observed that some subjects refuse to accomplish any type of compensation. Maybe because of the complexity of this process, some subjects prefer not using it instead of taking the risk of committing an error, even in clear situations as the previous ones.

Reformulation and compensation

In the following examples, besides a process of reformulation, we can found intermediate and final compensations accomplished for trying to "repair" the error produced in the substitution of the initial data in the reformulation.

Interviewer: 0.47×0.26

Student 2: 0.1. Rounding this [pointing 0.47] up and this [pointing 0.26] down. 0.5×0.2.

CEP	→	Reformulation + intermediate compensation	→	Computation
0.47×0.26	→	0.5×0.2	→	0.1

Interviewer: $66 \div 0.86$

Student 42: Let's see. When we are dividing, the number increases ... This number [pointing to 66], if we divide it by one it doesn't change. But it's smaller and then this number makes bigger. Then it would be... 0.86 [the student returns to check the size of the divisor for adjusting the compensation] ... it would be almost ... 75.

CEP	→	Reformulation	→	Computation	→	Final compensation
$66 \div 0.86$	→	$66 \div 1$	→	66	→	75

In the first case, 0.47 is rounded to 0.5 and, in order to compensate the increase in one of the factors, the other factor is rounded down. Thus, we have an intermediate compensation (previous to the calculation). In the second case the compensation is subsequent to the accomplishment of the calculation and is produced at the end of the process.

Reformulation and translation

In other strategies the processes of reformulation and translation are combined. These strategies usually have a greater conceptual wealth, manifested in the abundance of relationships that are established (especially those which emphasize connecting decimal numbers with fractions).

Interviewer: 0.47 × 0.26

Student 53: Multiplying by 0.25 is approximately multiplying by 1/4. Then, I divide 0.47 by 4, well… 0.12.

CEP	→	Reformulation	→	Translation	→	Reformulation	→	Computation
0.47 × 0.26	→	0.47 × ¼	→	0.47 ÷ 4	→	0.48 ÷ 4	→	0.12

This strategy begins rounding 0.26 to 0.25. Then 0.25 is substituted by ¼. This substitution supposes a change of operation (multiplication by division) that implies a translation process. After translation, it has been realized another reformulation to simplify the calculation, substituting the dividend by a multiple of the divisor (use of compatible numbers). The first reformulation carries implicitly an "intermediate compensation" that has not been indicated in the diagram because it does not seem intentioned. Another example of translation is the following:

Interviewer: 424 × 0.76

Student 53: Multiplying by 0.76 is approximately multiplying by 3/4. Then, this is approximately… First I divide by 4 and get 424, then it is 106 by 3, 318.

CEP	→	Reformulation	→	Translation	→	Computation
424 × 0.76	→	424 × ¾	→	(424 ÷ 4) × 3	→	106

In this case, instead of multiplying by 76 (or by 7 or 8 in the case of using only one significant digit) and then dividing by 100 (or by 10), 424 is divided by four and then multiplied by three. Thus, in addition to changing the numbers in the operations, there is a change in the order of the operations and, by so much, changes the mathematical structure of the problem.

Reformulation, translation and compensation

Only in some few cases, the three computational estimation processes –described in Reys et al. (1982)– have been combined in the same estimation strategy. In the following examples it is produced a substitution of a decimal by a fraction that thereinafter leads to a translation process (because in both cases is produced an operation change).

Interviewer: 64.6 × 0.16

Student 42: This is 16% of 64, which becomes the sixth part; a little more… 12 would be my estimate.

CEP	→	Reformulation	→	Reformulation	→	Translation	→	Computation	→	Compensation
64.6 × 0.16	→	16% of 64	→	1/6 of 64	→	64 ÷ 6	→	¿10?	→	12

Interviewer: 943 ÷ 0.48

Student 42: 943 divided by ... 1750. If we multiply this [pointing to 943] by 0.5 this becomes the double, it is like multiplying by two. And this [multiplied] by 2 is almost ... 1900, then would be a little less 1800 or 1750.

CEP	→	Reformulation	→	Translation	→	Computation	→	Compensation
943 ÷ 0.48	→	943 ÷ 0.5	→	950 × 2	→	1900	→	1750

Concerning the two final compensations, there have been errors in the direction given to them.

Metacognitive processes

Now we present examples in which it is manifested the presence of metacognitive processes as those described by Sowder (1994).

Interviewer: 64.6 × 0.16

Student 45: Here I make... 64.6... I put 65 and 0.16... I put 2 and then they are 170. Then I put 168... the point? ... No, because it isn't the same 65 for 2 than... [The student adjusts the decimal point and gives the estimation] 0.170.

CEP	→	Reformulation	→	Computation	→	Assessment	→	Adjustment
64.6 × 0.16	→	65 × 2	→	170	→	Decimal point?	→	0.170

There have also been estimation processes where the subject has made an assessment conducive to a change in the strategy:

Interviewer: 66 ÷ 0.86

Student 53: 66 [divided] by 0.86 is like multiplying by 85 hundredths. Then I divide by 100 and remains... Oh! It's very difficult.
I multiply this [pointing to 66] by... more or less… I'm going to approximate it to 0.9. Then 9×10^{-1}, it is a division, then this would remain me approximately 660 [divided] by 9 and will be... [9 multiplied] by 7... 63, is approximately 70.

CEP	→	Translation	→	Assessment	→
66 ÷ 0,86	→	66 × (85/100)	→	It is difficult	→

→	CEP	→	Reformulation	→	Translation	→	Reformulation	→	Computation
→	66 ÷ 0.86	→	$66 \div (9 \times 10^{-1})$	→	660 ÷ 9	→	630 ÷ 9	→	70

In the second attempt, the subject uses the substitution through the use of exponents and this supposes a translation process. This substitution is complemented –as in the two previous cases– with the use of compatible numbers, replacing 660 by 630 (that it is multiple of 9).

CONCLUSIONS AND IMPLICATIONS

In this work, we have used an alternative model based on a "global" conception of the estimation strategies. In this model, strategies are integrated by approximation skills, mental computation algorithms, cognitive processes (reformulation, translation and compensation) and metacognitive processes (as the assessment of the outcome). The adoption of this model has allowed us to accomplish a classification of the computational estimation strategies (attending to the cognitive processes appearing in them). We think that this approach enables us to make the description and the analysis of the strategies produced in the performance of computational estimation tasks, in a more systematical way than the approach provided by previous theoretical frameworks. The results obtained in this study have permitted to validate the flowchart of strategies analysis shown as a comprehensive theoretical model for the estimation processes accomplished by the subjects of the sample.

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INFLUENCE OF NUMBER TYPE AND ANALYSIS OF ERRORS IN COMPUTATIONAL ESTIMATION TASKS

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In this study we analyze the difficulty of computational estimation tasks –with operations without context– in function of the operation type –multiplication and division– and number type –whole, decimal greater than one and decimal less than one– that appears in them. Errors made in estimating with decimal numbers less than one are also analyzed. The research counts with the participation of 53 preservice elementary teachers. An estimation test is administered to the teachers and some of them are selected to accomplish interviews. The conclusion is that estimating with decimals less than one is more difficult than with whole numbers or decimals greater than one, and most of the errors –but not all– produced in estimation processes is due to teachers' misconceptions about operations of multiplication and division.

The influence of number type in the difficulty of computational estimation tasks has been studied in several investigations. Bestgen et al. (1980) found out that items with decimal numbers were more difficult than those in which only whole numbers appeared. Rubenstein (1985) attains in her study –in which 309 eighth grade students participate – the same result that Bestgen et al. (1980). However, Goodman (1991) obtains results in his investigation –with the participation of 46 preservice elementary teachers– that contradict the results of the investigations cited before. This author does not find significant differences of difficulty between items with decimal numbers and those in which there are only whole numbers. An important difference between these studies is that, whereas all the decimals appearing in the estimation test used by Goodman (1991) are greater than one, the estimation tests employed by Bestgen et al. (1980) and Rubenstein (1985) includes decimals greater and less than one. Nevertheless, that none of the investigators cited distinguishes explicitly if the decimals appearing in the items used in their tests are greater or less than one.

The distinction between decimals greater and less than one has been crucial in other studies in mathematical education. Thus, one of the aspects that has received more attention in the investigations on multiplicative structure problem solving has been the influence of number type (whole or decimal less than one) in the choice of an adequate operation for the resolution of the problem. Greer (1992) and De Corte & Verschaffel (1996) made a review of these investigations summarized in the following paragraphs:

a) Children have difficulties in the choice of the appropriate operation for solving multiplicative verbal problems with decimal numbers less than one.

b) These difficulties are due to children' misconceptions about the effect of multiplying or dividing by decimal numbers less than one. Many children believe that "multiplication always makes bigger", "division always makes smaller" and "we always divide a large number by a smaller number". These ideas, valid for certain types of numbers (whole numbers), do not work when they are extrapolated to decimal numbers less than one.

c) Misconceptions in multiplication and division are originated by the predominance in the teaching of operations of the repeated addition model for multiplication and the partitive model of division. In the interpretation of multiplication as repeated addition, the multiplier should be a whole number and the product greater than the multiplicand. On the other hand, if division is interpreted as "sharing out", the divisor should be a whole number and the divisor and the quotient less than the dividend.

These results have also been found in studies with preservice elementary teachers. Tirosh and Graeber (1989) detected that 10% of the teachers held explicitly that "multiplication makes bigger", whereas almost the majority (more than 50%) of the teachers held explicitly that in a division problem, the result should be less than the dividend. In the case of multiplication, the belief "multiplication makes bigger" is implicit for a majority. It appears in the solutions that students give to the problems but it is not maintained explicitly. Students show a very strong dependency on the operations with whole numbers and their procedural knowledge of the operations. They have also a strong dependency on the repeated addition model for multiplication and the partitive model of division.

On the other hand, Levine (1980) and Morgan (1990) have also found difficulties in estimation tasks with decimal numbers less than one. Both authors attribute these difficulties to the presence of misconceptions about the operations.

According to all these antecedents, the hypothesis outlined in this work is that the real difference of difficulty in computational estimation tasks appears when items with whole numbers or decimals greater than one are compared with those with decimal numbers less than one. Thus, the main objective of this study is to analyze the relative difficulty of computational estimation tasks –with operations without context– depending on the operation type –multiplication and division– and the number type –whole, decimals greater than one, and decimals less than one– that appears in them. Another objective is to analyze teachers' errors in estimating the results of operations with decimal numbers less than one. In this paper the term "error" is considered in a broad sense as a lacking and incomplete knowledge. For example, misconceptions about the operations –as the belief "multiplication always makes bigger"– reflect a knowledge that has a certain validity domain –operations with whole numbers– but constitute an error when it is attempted its extrapolation to operations with decimals less than one. The analysis of errors performed in this study has the purpose of determining if all the errors produced in the estimation processes –in tasks with decimals less than one– have their origin in misconceptions about the operations or if, on the contrary, there are other types of characteristic errors in these estimation tasks.

METHOD

Two independent variables have been considered: "operation type" –multiplication and division– and "number type" –whole, decimal greater than one, and decimal less than one–. The dependent variable is the score obtained by a subject when performing an estimate. When an estimation is performed, we can calculate its percentage of error. If it is greater than 30%, zero points are scored; if it is greater than 20% but no more than 30%, one point is scored; if it is greater than 10% but no more than 20%, two points are scored; and if the percent of error is no more than 10%, three points are scored. Furthermore, some variables have been controlled: considering the format of the questions, the test is constituted by estimation items with direct –not applied– computations; all the computations have been presented in horizontal format; and finally, only opened response items have been used.

Subjects

The research has counted with the participation of 53 preservice elementary teachers of different first course specialties of the Escuela Universitaria La Salle, assigned to the Universidad Autónoma de Madrid. The ages of the participants were from 18 to 24 years. Ten subjects were selected from this sample to accomplish –through an interview– the analysis of errors in computational estimation. All the students participated in a period of instruction about computational estimation of ten hours, during which they received explicit teaching about estimation strategies. Estimation was also practiced in both direct and applied computations. At the end of the period of instruction, the participants realized the estimation test.

Instruments and application

The Test of Estimation Ability (Levine, 1980) was administered to the students. This test consists on estimating mentally the results of ten multiplications and ten divisions. Items are of three types: (a) operations with whole numbers, (b) operations with a decimal number greater than one, and (c) operations with a decimal number less than one. Numbers in the test have zero, one, or two decimal digits. Furthermore, no number has more than five digits in total.

The test was administered in the computer science classroom during a period of class using PCs. The participants came into the classroom without carrying anything with them. These limitations were established to avoid the students' usage of any computational procedure that was not mental. There was no direct limitation of the response time for each item or for the whole the test. However, the students received indications to realize their estimates briefly.

The results of the estimation test have also been used to select subjects for the phase of the study of error analysis, realized through an interview. The subjects were requested for giving an estimate for a proposed calculation and then, they explained the procedure used to produce their estimate. In the interview we used the items from the test of Levine (1980) with decimal numbers less than one (187.5×0.06; 64.5×0.16; 424×0.76; 0.47×0.26; $66 \div 0.86$; $943 \div 0.48$; $0.76 \div 0.89$). The subjects selected for the interview were those whose estimates –for the cited items– were all compatible (or all incompatible) with an adequate knowledge of the relative effect of operations. For example, an estimation for 187.5×0.06 fulfils this condition if it is within the interval (0.06, 187.5). In other words, if the student does not incur in misconceptions as: "multiplication always makes bigger" or "division always makes smaller". The interviews have been done individually. They were registered using a recorder and transcribed for their subsequent analysis.

RESULTS

To study the possible effect of the independent variables in the dependent variable, a factorial design of two factors, with repeated measures in both factors has been accomplished. The factor "operation type" has two levels: multiplication and division; and the "number type" has three levels: whole numbers, decimals greater than one, and decimals less than one. The statistical analysis has been performed with the Statistical Package for the Social Sciences (SPSS) for Windows (version 9.0.1). Table 1 gives the mean scores and the standard deviations for the estimation test.

Table 1. Mean scores and standard deviations for estimation test

		Mean	SD
Operation type			
Multiplication		1.236	0.097
Division		1.121	0.079
Number type			
Whole		1.448	0.083
Decimal greater than one		1.261	0.122
Decimal less than one		0.825	0.087
Operation type × Number type			
Multiplication	Whole	1.453	0.107
	Decimal greater than one	1.547	0.152
	Decimal less than one	0.708	0.095
Division	Whole	1.443	0.107
	Decimal greater than one	0.975	0.134
	Decimal less than one	0.943	0.113

Results of the analysis of variance

There are statistically significant effects for the factor "type of number" ($F = 20.056$, $p = 0.000$) and for the interaction between the factors "type of operation" and "type of number" ($F = 8.895$, $p = 0.000$). On the contrary, there is no statistically significant effect for the factor "type of operation".

In order to know between which levels of the factor "type of number" the significant differences have been produced, we have used the test T with the Dunn–Bonferroni correction. We have been found statistically significant differences between level 3 –decimals less than one– and level 1 –whole numbers– ($T = 7.090$, $p = 0.000$) and between level 3 and level 2 –decimals greater than one– ($T = 3.991$, $p = 0.001$). However, there is no significant difference between the mean scores corresponding to levels 1 and 2 of the factor.

In spite of the fact that the mean scores for multiplication and division items are very similar, the study of the interaction allows to clarify that this equality is not maintained through all levels of the factor "number type". Table 1 gives the mean scores corresponding to all the combinations of levels of the factors in the design. We can observe in this table that the mean score for multiplication is much greater than for division when decimal numbers greater than one are operated, and smaller when decimals less than one appear. As we have seen before, the interaction between operation type and number type has a significant effect in the scores. To determine between which combinations of levels these significant differences have been produced, we have accomplished the within-subjects contrast test –simple contrasts option –. There is a significant interaction for levels 1 and 2 of the factor 1 and levels 1 and 2 of the factor 2 ($F = 9.463$, $p = 0.003$). The other significant interaction is given between levels 1 and 2 of the factor 1 and levels 2 and 3 of the factor 2 ($F = 16.739$, $p = 0.000$).

Results of the analysis of errors

To accomplish the error analysis we have used the diagrams for the description of estimation strategies proposed by Segovia, E. Castro, Rico and E. Castro (1989). The errors have been classified attending to the estimation processes –identified by Reys, Bestgen, Rybolt and Wyatt (1982)– in which they have been produced. Thus, errors may be found in reformulation, translation, and compensation processes.

Within some strategies, errors have been detected in the process of reformulation. We can find among them the following example:

Interviewer: 187.5×0.06

Subject: 1.2. I have rounded the first to 200, multiplied by 6 and then I have put the three decimals.

Computational estimation problem (CEP)	→	Reformulation	→	Computation
187.5×0.06	→	$(200 \times 6) \div 1000$	→	1.2

In this case the subject has reformulated the initial problem rounding one of the numbers. In this reformulation, an error has been produced. In fact, the subject operates decimal numbers as if they were whole numbers, ignoring the decimal points. Afterwards, he puts as many decimals on the result as there are in total in the multiplied numbers. However, the rounding of 187.5 (substituting this number in the calculation by 200) supposes the loss of a decimal figure, so that 187.5×0.06 should have been substituted in the reformulation by (200 × 6) ÷ 100 = 12. These errors in reformulation are caused by an inadequate combination of the rules of operating decimal numbers with the rules of rounding.

In other strategies, errors appear as a consequence of the application of students misconceptions about the operations with decimals and, particularly, due to an inadequate knowledge of the effect of multiplying or dividing a number by a decimal less than one. An example of this situation is the following fragment of the interview:

Interviewer: 943 ÷ 0.48

Subject: And that [pointing to 943] [divided] by half. Or [multiplied] by a half. 943, the half… and then… Four hundred and fifty something.

CEP → Reformulation → Translation → Translation → Reformulation → Computation

943 ÷ 0.48 → 943 ÷ ½ → 943 × ½ → 943 ÷ 2 → 900 ÷ 2 → 450

The subject decides to substitute 0.48 by 1/2. This substitution leads to a translation process (where the division is changed by a multiplication). In this first translation process the error is found. This error is caused by the mixture of the misconception "division always makes smaller" with the substitution of 0.48 by ½, and the interpretation of ½ as "making the half".

Finally, in the following examples, errors are located in the intermediate compensation or in the final compensation.

Interviewer: 424 × 0.76

Subject: 400. I have rounded down, 400, and this [pointing to 0.76] up, by 1.

CEP → Reformulation + Intermediate compensation → Computation

424 × 0.76 → 400 × 1 → 400

Here, the subject has employed the substitution by "powers of ten" changing 0.76 by 1 and, after, but previous to the calculation, he has been attempted to compensate that first substitution changing 424 by 400. This intermediate compensation has had an adequate direction but not enough "intensity". The same circumstance has occurred in the following example. The only difference is that the compensation in this case is accomplished at the end of the process (after the calculation).

Interviewer: 424 × 0.76

Student: That [Pointing to 0.76]... [Rounded] to 1. Well, it would be a little less than 424, 420.

CEP → Reformulation → Computation → Final compensation

424 × 0.76 → 424 × 1 → 424 → 420

As in the previous case, the size of the compensation is too small. It has been observed that sometimes, if the student does not know a better criterion –as would be in this case the use of the distributive property–, an intuitive compensation is effectuated. Many times, this intuitive compensation consists simply in a rounding in the adequate direction (as 424 rounded to 420).

In the following estimation, a final compensation has been accomplished with an error in its direction. It has been used as an approximation skill the substitution by a power of 10.

Interviewer: 66 ÷ 0.86

Subject: I would divide this [pointing to 66] by one. Well, but then I would subtract from it a little. It would be 60.

$\boxed{\text{CEP}} \rightarrow \boxed{\text{Reformulation}} \rightarrow \boxed{\text{Computation}} \rightarrow \boxed{\text{Final compensation}}$

$\boxed{66 \div 0.86} \rightarrow \boxed{66 \div 1} \rightarrow \boxed{66} \rightarrow \boxed{60}$

CONCLUSIONS

As we have seen in the review of literature about the problem, there was a disagreement between the results obtained in the studies of Bestgen et al. (1980) and Rubenstein (1985), and those of Goodman (1991) with respect to the relative difficulty of the computational estimation tasks depending on the number type. In the first two studies the authors arrived to the conclusion that it was more difficult estimating with decimals than with whole numbers. However, in the study of Goodman (1991), there were no significant differences of difficulty between these two types of items. This contradiction guided the elaboration of the main hypothesis of this research, after proving that in the study of Goodman (1991) –to the opposite that in the others– items with decimal numbers less than one did not appear in the estimation test. In this paper, items with decimal numbers less than one have been more difficult than those with whole numbers or decimals greater than one. This circumstance can explain the apparent contradiction derived from the results of Bestgen et al. (1980), Rubenstein (1985), and Goodman (1991).

In this research it has been crucial to make the distinction between the decimals greater than one and less than one. This distinction had not been made in the studies cited before and allows relating the results found in this work with the ones originated from other research fields in mathematical education. For example, as we have been seen in the introduction, in the field of problem solving, it has been found that there are students with misconceptions about multiplication and division that emerge when decimals less than one appear. The difficulty of the estimation tasks as well as the difficulty in choosing the appropriate operation for solving a problem seem to be different demonstrations of these misconceptions.

Misconceptions in the operations are also responsible for some of the errors produced in the estimation processes. Within this type of errors, there are those produced by giving an improper direction to the compensations and other errors in the translation processes –as the substitution of "dividing by 2" by "making the half"–. These types of errors had been already described by Levine (1980). However, the error found in the processes of reformulation –absence of coordination of rounding with the adjustment of the decimal point– had not been described before. It is a characteristic error in estimation processes that is not originated by misconceptions in the operations. Consequently, we think that it will be necessary to accomplish –in a future research– a more detailed analysis of students' errors in making estimations. This type of analysis can facilitate the determination of the factors that, together with the misconceptions on the operations, influence in the greater difficulty of the computational estimation tasks with decimal numbers less than one.

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DESIGNING TO EXPLOIT DYNAMIC-GEOMETRIC INTUITIONS TO MAKE SENSE OF FUNCTIONS AND GRAPHS

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ABSTRACT. We report on the results of two exploratory experiments which are part of a broader study seeking to find ways for students to express their knowledge of functions and graphs within dynamic geometry-based situations that do not explicitly involve algebraic representations. In the first experiment, the students displayed an intuitive appreciation of coordinates and graphs, understood geometrically without any algebraic representation. We attempted to capitalise on this intuition in a second experiment, designed to assist them in finding ways of expressing themselves algebraically. The findings suggest that this approach might be effective in activating new ways of viewing graphic representations and, from a research point of view, for helping to appreciate the expressivity of dynamic-geometrical media.

INTRODUCTION

This paper is based on an exploratory study in the area of using dynamic geometry (DG) to look at simple locus problems and conic sections. The overall aim is to link Euclidean geometry and analytic geometry, which tend to be regarded by students as separated subjects, often poorly understood. The aim of this phase of the work is to elaborate principles for design, and to study in some detail, the epistemological and cognitive opportunities and constraints of our approach.

It seems that students rarely see the geometrical sense underlying analytic geometry because tends to be dominated by algebraic formalism: as Mason (1997) points out, algebra is usually developed in the curriculum as a form of generalised arithmetic, with a "rush to symbolism" that is divorced from geometry. We are interested in developing an approach which preserves and extends geometric intuition and it turns out that there is an historical precedent for this: ideas for linking Euclidean and analytic geometry can be found in the history of mathematics, in particular in the ways that the ancient Greeks (Menaechmus, Apollonius, Pappus) developed an understanding of conic sections as loci and their 'equations' without any algebraic symbolism, using geometric constructions to express 'algebraic' relationships (Coolidge, 1940). This was a fundamental inspiration for the later inventors of analytic geometry and algebra (Vieta, Descartes, Fermat). What is notable about the Greek work is how the lack of an algebraic notation severely hampered their ability to appreciate the generality of what they had discovered. As Boyer notes:

> there appear to be no cases in ancient geometry in which a coordinate frame of reference was laid down a priori for purposes of graphical representation of an equation or relationship, whether symbolically or rhetorically expressed. Of Greek geometry we may say that equations are determined by curves, but not that curves were defined by

equations. Coordinates, variables, and equations were subsidiary notions from a specific geometrical situation (Boyer 1968, p.173).

BACKGROUND

The examples of locus first encountered by students are typically the circle, perpendicular bisector, and angle bisector. UK students are often simply taught to solve these simple locus problems in terms of a geometric construction, neglecting the 'pointwise' structure — i.e. they are seldom encouraged to express the properties common to points on the locus. Unsurprisingly, students often develop a superficial impression of the locus idea which can be summed up as: 'To solve this locus problem, construct this geometrical figure'. For example, to find the locus of points equidistant from two given points, you construct the perpendicular bisector. Or, as one student put it, "You just find two points and then join them up." This is a 'global' meaning for locus, seeing it as a whole shape, in contrast to a 'local' understanding, which means seeing the properties of individual points on the locus. The pointwise idea is fundamental in analytic geometry, beginning with an arbitrary point which satisfies given conditions and then generalising the point into an algebraic form so that the locus can be plotted in a Cartesian system.

METHODOLOGY

We carried out some experiments with several 14 year-old students, designed to compare students' understandings of the 'local' and 'global' structure of loci under the different cognitive and cultural influences of working with conventional tools (compass and straight-edge) or DG. In Experiment 1, the students were provided with worksheets that consisted of two tasks to be attempted, first using compass/straight edge and then with Cabri. For Experiment 2, working with Cabri only, we selected the two best students from those who participated in Experiment 1. The data shown here are based on audio transcripts of the students' conversations working with Cabri, their written responses to the worksheets, and their Cabri files.

EXPERIMENT 1: AN INTERESTING STRATEGY

For reasons of relevance and space, we report only the result of one part of this experiment. In this, we asked students to find the locus of points which have twice the distance from one given point as the distance from a second given point (a construction known as the Apollonius Circle). This is a difficult task for students who are not versed in geometry, and was certainly more complex than any simple standard loci question they had encountered.

The students needed to find points P where AP:BP is 2:1. The point P can be dragged by the mouse, and the two bars on the left represent the lengths of the segments AP and BP, dependent on the point P. Note that it was made easier for the students because they had to look for *equal* bars when the lengths are 2:1.

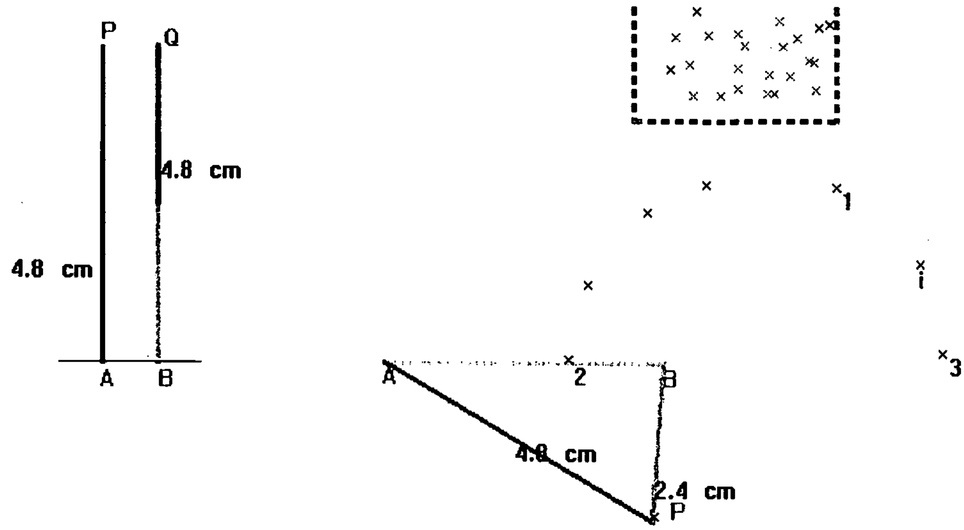

Figure 1: Students drag the point P so that the dynamic segments AP and BQ stays roughly the same length. Points can be dropped onto the screen by dragging them from the box at top right.

When the student finds a point P that has AP = 2BP she marks it with a cross (×), dragged from the box at top right (Arzarello et al, 1998, describe this kind of systematic dragging as "dummy locus", in which "dragging can act as a mediator between figures and concepts", ibid, p. 37). The points marked in Figure 1 indicate the order in which the students found them. Notice that they did not first find the 'internal' and 'external' points on AB, which one would normally start with in an analytic approach.

The locus is a (Apollonius) circle, and given that a circle is the most familiar curve, the students all identified it as such without further justification. However, a different Cabri situation provoked an interestingly different response.

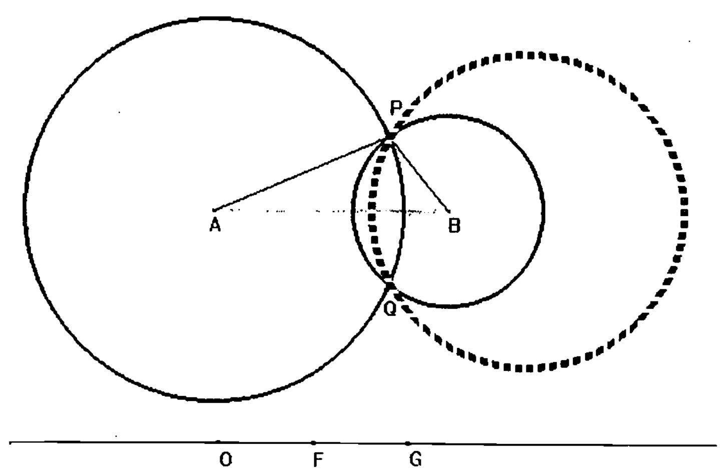

Figure 2: The Apollonius Circle using Cabri's Trace function

Given the set-up as shown in Figure 2, the students were firstly asked to make a guess about the shape of the locus as the control point F is dragged along OG (which determines the radii AP and BP). Before turning on *Trace* (used in preference to

Locus for reasons of simplicity), their first conjecture was that the locus might be a circle or an arc. But, suddenly, they shifted to say it is a line:

Alice: It's like an arc isn't it?

Jackie: It would be a circle. Can you move it [F] further?

Alice: (*repeatedly*) Always arc.

Jackie: They get, close it together. It's just part of a circle. It always is a circle.

Alice: It's a straight line. It's a straight line isn't it? It is between P and Q. *(keeps saying 'straight line')*.

Jackie: Agreed.

They turned on *Trace* and then they saw the circle. They said, "Uh… no its a circle". When they were asked why it's a circle, they replied as follows:

"The distance between P and Q is a straight line. But they [the circles] move, they are together and then they are separate [and then] together again"

Clearly, they are using some mental image of a line. Of course, they have a limited range of experience to explain and make sense of geometric images. For example, in their experience having two points is often a cue to join them up, and this becomes a prototype strategy for what they are supposed to do with two points. Also, they had recently worked on a task involving a perpendicular bisector where they had joined two intersection points P and Q to make a line. However, notice that they say "they are together and then they are separate [and then] together". They are, it seems, *imagining* the line joining up points P and Q as they drag the point F, and have described the locus in these terms (see Figure 3).

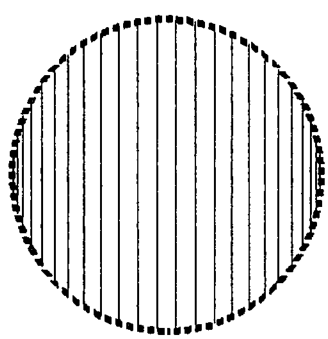

Figure 3: Visualising the locus as the end points of a changing line segment

We think that the students have hit upon a rather interesting way of making sense of the Apollonius circle. Seeing the dynamic image of the points P and Q (Figure 2) moving seems to have stimulated their geometric intuition in a novel way. Although it is not 'pointwise', it is a 'local' understanding. This kind of intuition, something like 'slicing a disc', can be found in the history of mathematics. It is very like the techniques of Apollonius and Dürer for constructing conic sections, which use a pre-Cartesian form of coordinates. And in the 17th century, Kepler used the idea of 'slicing' to find a formula for the area of an ellipse, which was a significant step in the development of the calculus (Boyer, 1968).

In the early 16th century, Dürer described a method to construct the conic sections of a right circular cone (Pedoe, 1976), based on Apollonius' study of the conic sections (Boyer, 1968, pp.164-5). Dürer's approach seems to be related to the strategy we observed with our students, in that it is a rather intuitive technique to draw a 'graph' without using algebraic representation. Figure 4 illustrates the method using Cabri. Suppose a cone is sliced by a plane. The triangle DEF shows the situation at the centre of the cone, where the plane cuts the cone at points A and A'. We make an object point M on the segment AA'. The distance A'M is the first quantity that we want to plot. For each value of A'M we need to know the width of the conic section at that point. Dürer found this by drawing the circle corresponding to the height of the cone at that point, and finds the width PQ by locating the intersection of a horizontal line through M with the circle. Finally, on the "axis" A'M we add the segment PQ perpendicular to A'M. And finally if we do *Locus*, we can see the locus of the conic section.

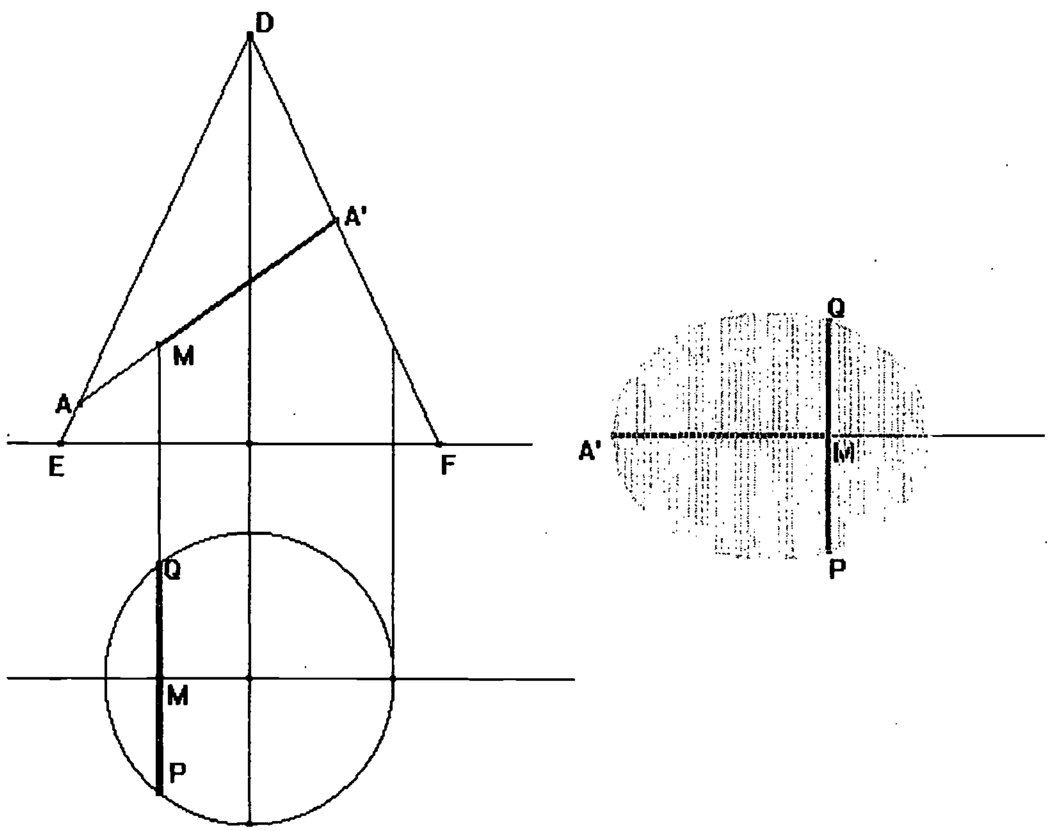

Figure 4: A Cabri version of Dürer's construction of the conic sections.

It seemed to us that 'Apollonian coordinates' in Cabri might provide some similar cue for our students, allowing us to build on their existing intuitions to extend, geometrically, their knowledge of functional relationships without necessary recourse to algebra.

EXPERIMENT 2: GRAPHING WITH 'GEOMETRICAL COORDINATES' WITHOUT ALGEBRA

We designed an exploratory sequence in which two students were invited to make geometric constructions and to investigate the relationships between different distances and areas in the construction, plotting the relationships on Cartesian axes using the *Measurement Transfer* function of Cabri. In these tasks we were interested to see how students interacted with and interpreted the Cabri situations, rather than (at this stage anyway) looking for learning outcomes.

One task was the familiar optimisation problem of finding the maximum area for a rectangle whose perimeter is fixed. As the construction was explained to the students, we discussed for some time what would be the shape of the graph of the area as the side lengths are varied by dragging a vertex. They easily sketched a parabola-like shape.

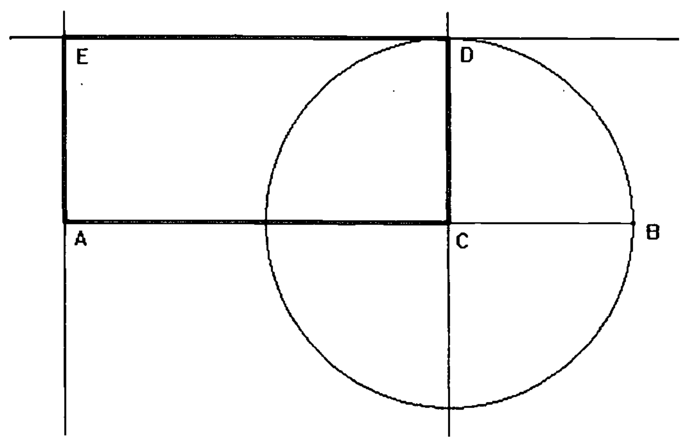

Figure 5: The rectangle area task

They constructed, by following instructions from a worksheet, the rectangle (Figure 5) and informally (without measurement) observed how dragging the point C affects the area and perimeter of the rectangle, and in particular, where the greatest area occurs. Afterwards we turned on the measurements for them to view, and by looking at the change of the numerical value for the area they noticed it had a maximum when the rectangle become a square. These students had had some experience with this kind of problem, and when we asked them about the relationships involved, they found it natural to talk about a formula although they could not precisely express it in symbols. They knew that a parabola-like shape was reasonable for the function, and they noticed that the area of the rectangle can have the same value at two different values of the side length.

After explaining to them how to plot a graph (using *Show Axes* and *Measurement Transfer*) they were able to construct a 'plot', based on *Locus*, for the graph of the rectangle area against side length (Figure 6).

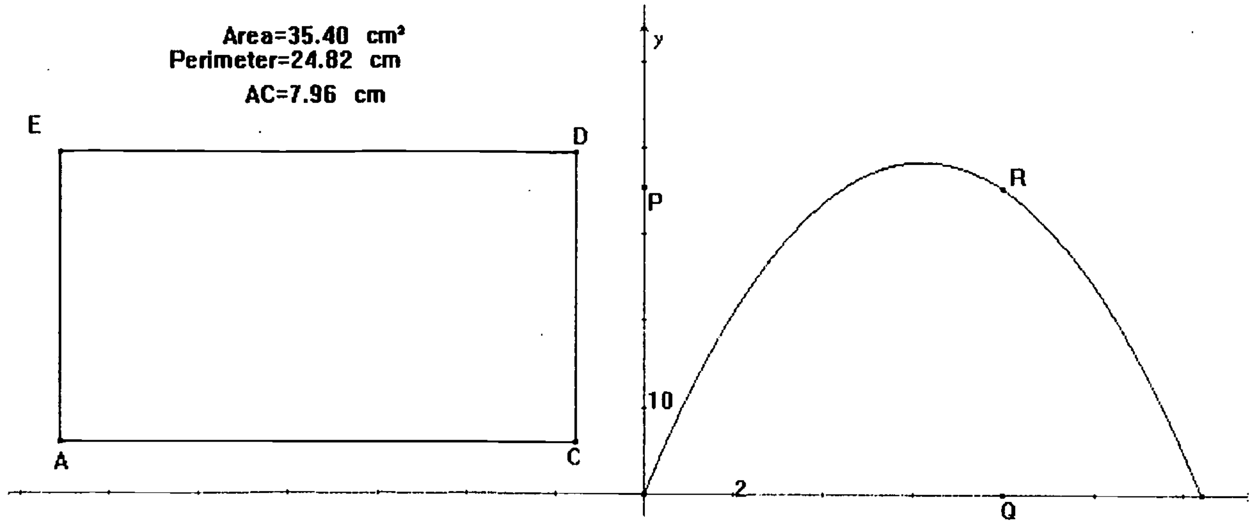

Figure 6: A graph of rectangle area against side length in Cabri

The final step was to introduce algebra into the problem: we asked them *If CD has length x and the perimeter has length P what is the area?*

In response to this question, they articulately expressed the structure of the algebraic expression of the area of the rectangle. We pushed them towards using the quantities x and P, and after some prompting to think about a specific numerical example, they eventually said:

A: ... oh, I know it's P take away 2x and then divide by 2, then that number times x.

However, they had great difficulties in *writing* the algebra, and were unable successfully to construct the expression for the area. The problem, we think, came from the students' limited ability to think with 'abstract' symbols, compared with thinking about symbols as labels on a concrete geometrical figure. They held onto the idea that a quadratic function is associated with a parabolic shape, but arrived mistakenly at the expression for area $S = P - x^2$ (although wrong, we carried on the discussion with it in order to probe the students' thinking). Surprisingly, Jackie thought this equation did *not* represent the graph that they plotted in Cabri, because she sketched *S* as a ∪-shape, reasoning that by substituting numbers at a few positive values (x = 1, 2, ...) the value of *S* is decreasing and must therefore reach a minimum at a later point — a reflection of how she was taught to plot graphs in the classroom by calculating a few points and then joining them up. Though there are no problems plotting a line from its equation using this approach, with higher degree equations it can be misleading. By contrast, the approach from geometry using loci gives students a sensible whole graph.

CONCLUDING REMARKS

In the experiments we have described, locus played a mediating role to afford an expression for generalising the relations between quantities. As one student put it:

"Oh, the Locus [is] where we can go! That's all the places they [the points or segments] can go"

This is a semi-abstract situation in advance of using algebraic expressions, in which quantities are 'numericalised' from the concrete geometric constructions. Given the tendency in the UK curriculum to substitute empirical experiment and measurement for the elaboration of structural relationships, we believe that we might have hit upon a way that students can be led naturally to algebraic representations and functional relationships. In our experiment, for example, from observing a geometric construction by dragging, the students elaborated their thinking about quadratic relationships, the symmetry of the situation and the existence of a maximum. More importantly they were able to express some of this knowledge geometrically. As another example, x=constant and y=constant graphs are among the most difficult equations to learn about algebraically (and these students struggled with pre-task questions we posed on this topic), but when the students 'accidentally' created an x=constant graph by plotting area against perimeter, they seemed to have no hesitation in reading off the vertical line graph in terms of its equation.

The exploratory studies we have undertaken so far have proved sufficiently interesting from cognitive and epistemological points of view, to provide grounds for cautious optimism. In future work, we intend to explore ways in which we can design didactical situations that afford students the opportunity to coordinate quantities by geometric construction in a DG environment, and further, to use this as a stepping stone towards the expression of functional relationships in algebraic terms.

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TOWARDS UNDERSTANDING TEACHERS' PHILOSOPHICAL BELIEFS ABOUT MATHEMATICS

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The aim of this study was to examine teachers' philosophical beliefs (PBs) about mathematics, the factors influencing the development of these beliefs, and their relation to teachers' beliefs and practices about teaching and learning mathematics. Data were collected through 229 questionnaires and five interviews. Analysis revealed a five-factor model, representing combinations of the three dimensions of a model proposed by Ernest (1991). Four homogeneous groups of teachers according to their perceived importance were identified. A relative consistency was also found between teachers' PBs and their beliefs regarding teaching and learning Mathematics. However, inconsistencies between PBs and teaching practices emerged, which could be partially attributed to the factors influencing the development of PBs. Implications for the development of teachers' training programs regarding the affective domain and suggestions for further research are drawn.

INTRODUCTION

Teachers' philosophical beliefs (PBs) are considered as the cornerstone of their teaching practises and their beliefs concerning teaching and learning. Thompson (1992) defines PBs as "teachers' conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (p.132). Hersh (1998) asserts that these beliefs affect teachers' conception of how mathematics should be presented, since "the issue is not what is the best way to teach mathematics but what is mathematics really all about" (p.13). Ernest (1991) concludes that the research literature indicates that mathematics teaching depends fundamentally on the teacher's belief system and particularly on his/her conceptions of the nature and meaning of mathematics.

The review of the literature (Raymond, 1997; Thompson, 1992; Roulet, 2000) suggests that many factors influence the development of PBs, and particularly teachers' school experiences as mathematics students, that constitute their early systematic contact with mathematics. However, PBs can be considered as a dynamic body of beliefs influenced by many other factors, including, early family experiences, teachers' education programs, social and educational characteristics and school environment. The socio-cultural environment is also considered to exert influence on the development of PBs.

Teachers' beliefs may develop into a coherent philosophical system that directly influences their overall classroom behaviour. A teacher's own philosophy is thought to function as a filter influencing decisions and actions made before, during and after instruction (Philippou & Christou, 1997). Class organization, the choice of learning activities, the questions posed by teachers, and the homework that teachers assign to students are likely to be influenced by teachers' PBs (Stipek, Givvin, Salmon, & Mac Gyvers, 2001). Moreover, PBs were found to have impact on students' achievement, teachers' attitudes about the effectiveness of various teaching methods, innovations, curricula, textbooks and software material (Ernest, 1991; Philippou & Christou, 1997; Roulet, 2000). A number of researches, however, point out that there are inconsistencies between expressed PBs and actual teaching practices (Raymond, 1997; Thompson, 1992). The subconscious character of PBs and the influence of school environment on the development of these beliefs can justify these inconsistencies (Ernest, 1991). Evidently, teachers' training programs should help prospective teachers develop and become aware of these beliefs.

Despite of the excessive research in this domain, no coherent way to measure PBs has been so far reported, due to the acknowledged complexity of PBs. The review of the relevant literature reveals a number of models that have been proposed in order to study these beliefs (e.g., Lerman's two-dimensional model, Ernest's three-dimensional model, Perry's four-dimensional model, and Raymond's five-dimensional model). We adopted Ernest's model, since it is based on three philosophical ideas, derive from the history of mathematics evolution. First, the platonist view considers mathematics as an a priori static unified body of knowledge which exists out there and is discovered, neither invented nor created. Second, the instrumentalist view regards mathematics as a set of unrelated but utilitarian rules and facts; a viewpoint that can be linked with formalism in mathematics (Roulet, 2000). Thirdly, there is the experimental–constructivist view that attributes a prominent role in problem solving and regards mathematics as a dynamic, continually expanding field of human creation and invention. Teachers holding this point of view consider mathematics as a cultural product the results of which remain open to revision.

However, we are not aware of any systematic attempt to verify whether the three aforementioned viewpoints can satisfactorily describe teachers' PBs. Thus, the main purpose of this study was to collect empirical data in order to examine the efficiency of Ernest' s model in describing teachers' PBs. The study also aimed to provide evidence regarding the factors influencing the development of PBs and the consistency between these beliefs and teachers' beliefs and practices related to teaching and learning mathematics.

METHODS

A three dimensional questionnaire was developed based on statements found in the literature (e.g. Ernest, 1991; Thompson, 1992). The first part included 17 five-point Likert-type items (1=Strongly disagree, 5=Strongly agree) reflecting teachers' PBs along the three dimensions of Ernest's model. The second part included four items regarding teachers' beliefs about teaching and learning mathematics, namely it included questions related to the characteristics of a good teacher, a good student and indications of learning and learning goals in mathematics. For each item, teachers were provided with six statements, which corresponded to the three levels of Ernest's model and were instructed to put them into a hierarchical order that best described their own beliefs about mathematics. The third part of the questionnaire included open-ended questions related to teachers' practices. Of the 345 Cypriot teachers approached, 229 responded (192 primary and 37 secondary teachers), a response rate of 66.4%. The reliability of the scale, which measured teachers' PBs, was calculated using Cronbach's Alpha (α=.69).

Semi-structured interviews with four primary teachers and one secondary teacher were also conducted. The purposive sampling technique was used. More specifically, the primary teachers belonged to each of the four cluster groups that emerged from analysing teachers' responses to the questionnaire, while the secondary teacher belonged to the second cluster group. Teachers were asked to reply to open-ended questions, which could help us identify their conceptions about mathematics, such as "*What is the first thought that comes to your mind when you think of mathematics?*" "*How would you define mathematics if you were asked to do so by one of your students*". Furthermore, teachers were prompted to mention factors influencing the development of their PBs as well as domains that could be influenced by these beliefs. Specifically, teachers were urged to mention their school and university experiences in mathematics and their daily experiences with students, colleagues and headmasters. They were also requested to describe teaching goals, how they introduce a new concept, the structure and the context of their tests and their favourite activities. The constant comparative method (Denzin & Lincoln, 1998) was used to analyse the qualitative data emerged from the interviews.

FINDINGS

Since, a large number of significant correlations among teachers' responses concerning their PBs were identified, factor analysis was used to identify underlying "factors" that explain these correlations. Moreover, direct oblimin rotation was used, since the literature review underlines the correlations among clusters of beliefs and thereby relations between the expected factors. Table 1 illustrates the five-factor solution, which explains 56.3% of the variance and can be considered as the most appropriate solution in isolating teachers' PBs since all loadings are relatively large and statistically significant. The first factor F_1 can be seen as a combination of platonist-formalist views of mathematics and explains 20.6% of the variance, while the second factor F_2 explains 13.1% of the variance and reflects the instrumental-formalist views, which are close to the second dimension of Ernest's model. The third factor F_3 refers to the experimental-formalist belief of mathematics and explains 9.1% of the variance, the fourth factor F_4 combines experimental-instrumental views of mathematics and explains 7.2% of the total variance, while the fifth factor F_5 that explains 6.3% of the variance, is a rather pure platonist approach.

Since the five factors which emerged from our study represented combinations of the three dimensions proposed by Ernest, it was considered important to examine teachers' philosophical profiles, that is groups of teachers according to their scores views in each of the above five factors.

No	Items related to PBs	FACTORS					
		I	II	III	IV	V	h^2
1	M consists of exact results and correct procedures	.77	.10	-.13	-.01	.11	.60
2	Mathematical ideas are eternally truth	.71	.07	-.12	0.6	.17	.54
3	M consists of well defined concepts and well structured procedures	.67	.23	.10	-.04	.11	.47
4	M is a set of rules and theorems	.59	.36	-.45	-.14	.13	.62
5	M exists irrespective of time and place	.47	.21	.42	-.26	.20	.50
6	M is important because it's useful for people	.20	.72	.29	.06	.06	.60
7	M derived from social needs	-.09	.65	-.16	.41	.13	.61
8	M promotes operations, skills and procedures useful for daily needs	.01	.65	.15	.26	.13	.47
9	M is a set of algorithms and procedures	.47	.64	-.19	-.16	.16	.62
10	M is a dynamic, continually expanding domain	-.04	.23	.68	.12	.11	.51
11	M is a formal way of representing real world	.03	.23	.50	.47	.32	.52
12	Mathematical ideas should be formally expressed	.38	.37	.49	.08	.22	.51
13	M is constructed through experimentation and research	-.02	.13	-.05	.80	.16	.66
14	M serves human needs	.23	.33	.18	.59	-.05	.51
15	Mathematical ideas pre-exist in human minds	-.12	.11	-.05	.08	.74	.63
16	M is discovered	.34	.04	.26	.24	.64	.58
17	Mathematical ideas exist irrespective of the learner	.48	.18	.10	-.17	.64	.59
	Eigenvalue	3.50	2.22	1.55	1.23	1.07	
	Percentage of Variance	20.56	13.06	9.10	7.21	6.31	
	Cumulative Percentage of Variance	20.56	33.61	42.72	49.93	56.24	

Viewpoints of mathematics: I: Platonist-Formalist, II: Instrumentalist-Formalist, III: Experimentalist-Formalist, IV: Experimentalist-Instrumentalist, V: Platonist

Table 1: Factor Loadings of the Five Factors Related to Teachers' PBs as concerns Mathematics (M) Derived From a Direct Oblimin Rotation Procedure.

Though factor scores can be obtained in a number of ways, the calculation of the sum of variables which load most highly on each factor can be seen as an appropriate method of estimating factor scores (Kline, 1994) since it correlates highly, in most cases, with more elaborated procedures in which multiple regressions of all the variables on to the factors are computed. However, the relatively high values of the standard deviations, which emerged from calculating factor scores, revealed a variation among Cypriot teachers about the extent to which each factor is seen as representative of their PBs. Given that the 17 items can be classified into five broader categories, teachers' PBs were examined further by using Ward's clustering method. Cluster analysis revealed four relatively homogeneous groups of teachers, representing four philosophical profiles. The four-cluster solution is justified since the Agglomeration schedule shows a fairly large increase in the value of the distance measure from a four-cluster to a three-cluster solution, and the standard deviations of the four groups are much smaller than those of the whole group of Cypriot teachers.

Cluster Groups	F_1		F_2		F_3		F_4		F_5	
	$\bar{x}$	SD	$\bar{x}$	SD	$\bar{x}$	SD	$\bar{x}$	SD	$\bar{x}$	SD
Group 1 (n=44)	3.14	.74	3.90	.44	3.48	.42	4.09	.55	2.58	.48
Group 2 (n=79)	4.13	.52	4.18	.54	3.94	.57	4.35	.64	4.03	.50
Group 3 (n=87)	3.30	.45	3.75	.53	3.64	.46	3.98	.42	3.79	.47
Group 4 (n=19)	3.37	.39	3.45	.69	3.38	.47	2.44	.60	3.46	.47

*1= Strongly disagree, 5= Strongly agree

Table 2: Means and Standard Deviations of the Factor Scores of the Four Groups Produced by Cluster Analysis on the Five Factors of Teachers' PBs.

Table 2 shows the means and the standard deviations of each group in each factor. It is clear that teachers of Group 1 (G_1) hold positive beliefs in four of the five factors. Taking into account the relatively high mean on F_4, and their negative attitudes on F_5, one can claim that G_1 teachers hold experimentalist-instrumentalist views. Group 2 (G_2) teachers were found to hold positive high views in all five factors and likewise the mean scores of Group 3 (G_3) in all factors were positive though less high. Thus, these two groups of teachers can be considered as holding a mixture of beliefs. Group 4 (G_4) teachers support the platonist-formalist views but disagree with the experimental-instrumental views. We can finally observe that the majority of teachers (166 out of 229) had complex PBs and only 63 can be positioned at the two ends of a spectrum describing PBs.

Kendall Coefficient of Concordance was calculated to examine the degree of consensus among teachers of each group in the ranking of the relative importance of the statements of the second part of the questionnaire. The analysis revealed a significant level of agreement ($p<.05$) amongst teachers of each group. Based on the mean ranks, a consistent pattern of responses was identified, in the sense that G_1 and G_3 teachers considered the statements reflecting experimental-constructivist ideas of teaching and learning mathematics as more important than platonist-formalist ideas. On the other hand G_2 and G_4 teachers responded on a reverse way, judging the platonist and formalist ideas as more important than the other ones. This pattern emerged for all the questions (i.e. characteristics of a good student and a good teacher of mathematics, evidence of learning and learning goals in mathematics).

The responses of these four groups on the third part of the questionnaire were found to be inconsistent to their PBs and teaching practices in three ways. First, a high percentage of teachers in all groups stated that they would like to teach every mathematical concept, to help students attend mathematics in the following years. They pointed out that parents, headteachers, inspectors and even colleagues exert pressure on teachers to cover all the prescribed content. Second, a high percentage of teachers, irrespective of the group, described a traditional approach in teaching mathematics (i.e., presenting the new concept on the blackboard and providing students with exercises for practice). However, there was a group of teachers who claimed that they use a dynamic investigative approach in teaching mathematics (experimentation, construction of meaning etc.) Though most of them belonged to G_1, there were teachers of the three other groups who taught mathematics in that way. Thirdly, teachers in all groups complained that the new book series does not provide for enough practice in the four basic operations, and most of them did not seem to recognize the scope of the innovation introduced. Moreover, teachers considered that the books are appropriate for the most able students.

Analysis of the interviews

The interviews produced additional evidence on some features of teachers' PBs, clarification of the factors influencing the development of PBs and how they can influence teaching beliefs and practices. First, it was clear that most teachers never took time to reflect on their PBs, as one of them put it "*I never thought of issues related to the nature of mathematics, although I teach mathematics for many years*". Second, teachers' answers revealed that their PBs were frequently inconsistent. For instance, while their responses concerning the nature of mathematics reflected the beliefs of the group they belonged to, their responses to questions regarding the evolution of the discipline and mathematical truth showed inconsistencies. Even though all the interviewees replied that mathematics derived of social-practical needs and it develops to serve social needs, almost all teachers, including those in G_1, pointed out that mathematical ideas and procedures are regarded as valid only if there is an acceptable proof for their validity.

Teachers' responses were also informative of the factors influencing their beliefs. All the participants mentioned the role played by their teachers and their school experiences on their mathematical beliefs. Some indicative extracts "*my attitude towards mathematics can be attributed to my mathematics teacher, who tried to teach in a way that mathematics lessons were never boring*" and another "*my past positive school experiences formed the way I am perceiving mathematics and learning in mathematics*". On the other side, the university tutors did not seem to make any real difference, since "*university lessons paid more attention to knowledge and ways of teaching mathematics than issues related to the affective domain*". The interviewees attributed a very restricted parental influence on their PBs, but they valued the interactions with colleagues. As one of them put it "*I had a very good group of colleagues and we exchanged ideas about mathematics and ways of teaching specific* *mathematical concepts*". These ideas and practices seemed to be more influential during their first years of employment.

The interviews seemed to reveal a marked inconsistency between teaching practices and PBs. All primary teachers, irrespective of their initial PBs, paid lip service to experimental approaches; they acknowledged the role of activities that offer students opportunities to construct knowledge. However, their favourite activities and their assessment practices echoed a rather traditional-formalist approach. Only the teacher of G_1 claimed that he developed activities that enable students to construct the meaning of mathematical ideas. On the other hand, the secondary teacher, although belonging to G_2, adopted a more formalistic approach in teaching mathematics; as she stated, "*it is important to provide students with ample opportunities for practicing the algorithms presented by the teacher*". Proofs were a basic feature in her teaching, while learning theorems and definitions were considered as very important. Thus, the secondary teacher mentioned that her tests are based on exercises that require the presentation of definitions of mathematical concepts.

DISCUSSION

The results support previous studies, which refer to the complexity of teachers' PBs and their subconscious character (Raymond, 1997; Stipek et al, 2001). The majority of the subjects in our study were classified into the two groups reflecting teachers with a mixture of PBs. The interviewees did not seem to have thought seriously about their conceptions about mathematics and its' teaching; they were unaware of their own philosophy. The complexity of beliefs was indicated by teachers' replies during the interviews concerning different issues related to the nature of mathematics and that is why the data failed to verify Ernest's model of PBs. However, the model that emerged from this study can be considered as representative of teachers' PBs, since it covers a wide spectrum of beliefs with teachers of G_1 and G_4 occupying the opposite poles of the scale. Teachers of G_1 can be considered as holding a more dynamic view of mathematics, also accepting its usefulness. On the other hand, teachers of G_4 regard mathematics as a static unified body of knowledge, consisting of facts, rules and skills that students have to acquire.

The interviews showed that that school experience plays the most prominent role in influencing the development of teachers' PBs. Some previous teachers are seen as models and influence both beliefs and practices; this and the collegiate effect accounts for the appearance of traditional beliefs, even among younger teachers. University education did not seem to influence significantly teachers' PBs. This finding is in alignment with Thompson (1992) claim that it is not possible to alter teachers' PBs during two or three university courses.

The results of this study indicate that PBs might influence teachers' beliefs about teaching and learning in mathematics as well as their teaching practices. Yet, the inconsistencies among beliefs and teaching practices found in this study show that the domain of beliefs is a complex one. Thus, it seems more appropriate to refer to intercorellations between the two variables instead of domains influenced by beliefs, since we found evidence of the reverse direction of influence i.e., previous practices influence teachers' PBs. Consequently, our study provides support to arguments about the dialectic relation between beliefs and teaching practices (Raymond, 1997; Thompson, 1992).

Finally, the present study underlines the importance of prompting teachers to reflect upon and examine their own beliefs systems. Since it is acknowledged that affective competencies can be learned and consequently be taught (Goldin, 1998), institutions that are responsible for teacher training, should also pay attention to the affective domain, providing teachers with opportunities to get aware of their own beliefs, the ways these beliefs are formed, and their dialectic relation to other domains e.g., their teaching practices. Further studies based on nonparticipant observations would be useful in shedding more light in the issues related to the inconsistencies witnessed between PBs and teaching practices.

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THE EVOLUTION OF A STUDENT TEACHER'S PEDAGOGICAL VIEWS ABOUT TEACHING MATHEMATICS PROOF

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Abstract. *This paper analyzed the change of a student teacher's pedagogical views about teaching mathematics proof during four years in a teacher education program. The changes identified in this study have shown a shift of her pedagogical views from convincing-formal to discursive explanatory. The changes of her related views about proof from formal to explanatory have been observed, too. Her change can be interpreted by Cooney's model of student teachers' belief change. A reason underlying her change is due to her participating in an action research in which she was becoming a reflective practitioner.*

INTRODUCTION

Issues such as the role and function of proof (Villiers, 1987), the view about proof (Lerman et al., 1993), teaching approaches of proof (Hoyles & Healy, 1999; Sekiguchi, 2000) and learning theory of proof (Duval, in press) have been concerned and studied. Villiers (1987) found that about 60% of pre-service mathematics teachers realized the function of proof only in terms of verification/justification/conviction, and were not able to distinguish any other functions of proof. Studies of student teacher's views about teaching proof indicate that pre-service teachers' interpretations of mathematics proof differed from what the mathematics community would consider as mathematically acceptable (Knuth & Elliott, 1997). Their expectation of proving by their students was limited and they have a different notion of what is an acceptable mathematical proof outside of the formal setting of their university mathematics classes. Wittmann (1992; 1996) distinguished demonstration from proof, the former often having a very concrete nature, and argued for introducing demonstrative reasoning as soon as the first grade. He suggested that operative proofs should be included in primary mathematics.

Those studies could help us on understanding different pedagogical views about teaching mathematics proof. But how did the student teachers' pedagogical views about teaching proof evolve during the teacher education program is still unclear. The relationship between the view about proof and the pedagogical view about teaching proof and what is the mechanism to promote the evolution of student teachers' pedagogical views about teaching proof needed to be clarified. Those problems are what we are studied in this paper.

METHODOLOGY

This study was part of the research-based pre-service Mathematics Teacher Education Project, an inquiry into the development of pedagogical power of pre-service secondary mathematics teachers. One of two principles about mathematics teaching posed by the educator in the teacher education program is teaching for making sense of mathematics (Lin, 1999). The courses, mathematics learning, mathematics teaching and evaluation, mathematics content and method, and teaching practice, were included in this program.

The program could be divided into three stages: entering the program during the sophomore year (stage 1), practicing teaching during the junior year and senior year (stage 2) and being a beginning teacher for one year (stage 3). In the course of mathematics learning (stage 1), student teachers were required to investigate students' understanding of some mathematics concepts and discuss strategies to help students' learning. In the courses of mathematics teaching and evaluation, mathematics content and method, and teaching practice (stage2), the main activities were creating student teachers' experiences on different teaching approaches such as guided discovery, group discussion, and investigative teaching, via activities of analyzing mathematics materials and designing learning activities, and provided opportunities to reflect on expert teachers and peers teaching. The aim through the program is to develop student teachers' pedagogical powers about teaching mathematics. At stage 3, the selected student teacher for the study taught mathematics in a secondary school and conducted an action research with us. There was a mentor in the school who should give suggestion to her but he did not do so.

The methods used in this study include questionnaire survey, classroom observations, and semi-structure interviews. A questionnaire designed to elicit student teacher's views about proof and pedagogical views about teaching mathematics proof was administered to the student teacher Echor at stage 1, 2 and 3 (1998.03-1999.05-2000.08). The questionnaire had two tasks. The first task was modified from Lerman et al., 's (1993) study: "the sum of the interior angles of a triangle is equal to **180°** ". The second task adopted Hanna's (1983) three ways to prove $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ and added four typical solutions collected from Taiwanese students. The student teacher, Echor, was asked to choose which way she preferred, give comment and to rate the quality of each solving method with scale of 1 to 5 for every item and choose which ways she preferred to use in her classroom.

Classroom observations and semi-structure interviews were conducted (stage 1--stage 3) and verbatim transcribed. In addition to those, we have collected her reflection notes and learning profiles during her pre-service education. All data were confirmed by Echor later and used as a means of gleaning additional evidences of the questionnaire results.

Hanna's (1990) and Hersh's (1993) taxonomy of proof: formal proof and explanatory proof was adopted as an initial scheme to characterize student teachers' views about proof. After the data of classroom teaching were analyzed, it was found that formal proof could be divided into two categories, formal view and convincing-formal view, and explanatory view could be divided into two categories, instructional explanatory and discursive explanatory, according to the teaching approach, that is teacher-centered or is student-centered respectively. Thus four pedagogical views about teaching proof are characterized as follows.

1.formal view (for convincing): teacher demonstrates well-organized deductive statement.
2.convincing- formal view (for convincing): teacher convinces students the truth by manipulation and special case, and follow with formal proof.
3.instructional explanatory view (for understanding): teacher demonstrates explanation.
4.discursive explanatory view (for understanding): the explanation is resulted from students' discourse.

Those four views distinguish how teacher understand the functions of proof and what are the approaches of teacher's teaching.

ECHOR'S PEDAGOGICAL VIEWS ABOUT TEACHING PROOF

This section reports the student teacher's responds on which ways she would use in her teaching with reasons. The following table categorizes the ways Echor chose on each task at different stages.

Tasks	Stage 1	Stage 2	Stage 3
Task 1: the sum of interior angles	1-2: Measuring 1-5: Arnauld's 1-3: Euclid's	1-1: Paper & Scissors 1-4:Logo-style	1-1: Paper & Scissors/ 1-2: Measuring 1-4: Logo-style
Task2: 1+2+3+...+n $=\frac{n(n+1)}{2}$	2-7: Mathematical induction 2-2: Induction 2-5:Triangular numbers 2-6: Staircase-shaped area	2-1:**modified** Gauss's	2-1: Gauss's 2-3:Arithmetic formula 2-5 and 2-6 are used conditionally

Fig.1 Categorizing the ways chosen at different stages

At stage 1

Refer to Fig.1, Echor chose multiple methods to prove the tasks. Her comment on measuring method was that it is inaccurate. She preferred formal proofs and convinced students the statement is true by manipulation. She said:

" *I must help students to make sense (1-2) first and then show them one* *or two formal proofs (1-5, 1-3, 2-7). After these procedures, I will give more explanations (2-5, 2-6) to them finally."*

Hence, her pedagogical view is toward to convincing- formal.

The above data was compared with her interview data at the beginning of this stage about teaching mathematics (Chin& Lin, 1998) .It was found that she emphasized students' needs should be considered first, and teacher's instruction ought be accommodated to meet halfway with their students. She said that students' learning should be the priority but she described her teaching design with teacher-centered approach. She chose formal proofs to teach the two tasks, and added teacher-centered empirical verification for making sense before proving. Thus, we classified her pedagogical view about teaching proof as **convincing- formal view** at stage 1.

At stage 2

Echor chose Paper & Scissors way as empirical verification for making sense. Logo-style was in the unique Taiwanese textbook and was popularly used in Taiwan mathematics classrooms. Gauss's method was the genetic method to prove task 2. The word "modified" means adding several numerical examples for testing. Her choice changed from multiple methods to a single one. All the methods of proving she chose were explanatory. Therefore, her pedagogical view was explanatory.

Data cited from our analysis of her teaching practice at stage 2 indicated that she taught proofs by lecturing without providing opportunities for her students to be engaged. She emphasized on making sense of mathematics for her students, and carried out her idea with teacher-centered way. For example, when she taught the theory of the sum of exterior angles in a triangle, she said everything by herself and just asked: "*Do you understand?*". She tried to implement the ideas she have learned from the teacher education program to her classroom, but her ideas did not come true in her teaching practices. Her pedagogical view at stage 2 was classified as **instructional explanatory**.

At stage 3

Refer to Fig.1, Gauss's method was used to explain Arithmetic formula. Her pedagogical view was explanatory. By contrasting her teaching at stage 3, we found that there were many activities for conjecturing, discussing, reporting, and interacting with the students. For example, when she taught polygonal pyramid, she said: "*how do you feel about this polygonal pyramid*?" "*Please make a conjecture: how many vertex, edges, and faces does a polygonal pyramid with a regular polygon base have*?" When her students make an error in constructing an identical line, she asked: "*why do you know they are identical?*" and "*when are they identical?*" Echor posed questions to guide students thinking and encouraged students to talk about their ideas. Her pedagogical view at stage 3 was classified as **discursive explanatory**.

THE CHANGE OF ECHOR'S PEDAGOGICAL VIEWS

From stage 1 to stage 2, the changes included that she changed her comment on induction, pointed out the function of the connection between algebra and geometry concretely, and was able to adopt strategies to fit individual difference. She agreed that using figural representation to explain identity has its positive meaning—it was intuitive and concrete. But she would choose numerical examples in her classroom teaching rather than the number-shape transformation under the consideration of its complexity and the rigorous of proof. Echor recognized mathematical induction was not easy for students to learn because of it was with little explanatory value. Mathematically, she was aware of the rigorous of mathematical induction, but she would not use it in her teaching. Her criterion was the simplest the best. The interview data about her change was as follows:

"*I think my big change is that I try to do my best to help students getting a lot of knowledge before. So I use many ways to prove one thing. But I think one method is enough now. For beginning learner, one simple method is good enough.* "

From stage 2 to stage 3, her change was unclear if we just analyzed data in Fig.1. All the ways she chose were explanatory proofs like she did at stage 2. We analyzed her classroom teaching and found that she have realized the function of proof for explanation, testing, and justification (Yackel, 1998; 2001). She changed her view from instructional explanatory into discursive explanatory and adopted different criteria of rigorous for professional mathematics and classroom context respectively.

Two reasons are underlying her change. The feedback from students during her teaching practice in school for about four weeks at stage 2 have stimulated her to reflect that rigorous is not the first thing for student to learn at secondary level. She said:

"*The experiences of teaching had an important influence on my view about teaching proof. When I responded on the questionnaire first time, I just considered which one had the most beautiful structure from my point of view. ...... After I have taught, I knew that if I teach following the beautiful ways in the questionnaire, only very few students could understand.*"

Another reason is her reflection on the processes of the action research. She said:

"*After this experience, I knew that the platform should not be occupied by me. Students need platform, too. Teaching could be implemented in different way and students could learn actively. I think...I should wear their shoes to teach mathematics from their views....* "

Her reflection on teaching in the action research at stage 3 helped her to be aware that student-teacher interaction may be a good approach for teaching and helped her to become a reflective practitioner.

ECHOR'S VIEWS ABOUT PROOF AND VIEWS ABOUT TEACHING PROOF

We analyzed Echor's responses on the questionnaire about proof as Fig.2 shown. The change of her views about mathematics proof at different stage was from formal view, to formal view, and then to instructional explanatory view.

Questionnaire Items	Stage 1	Stage 2	Stage 3
Problem 1-- the sum of interior angles			
1-1: Paper & Scissors	2	2	3
1-2: Measuring	2	1	3
1-3: Euclid's	5	5	4
1-4:Logo-style	4	4	5
1-5: Arnauld's	5	5	4
Problem2 -- $1+2+3+\ldots+n=\frac{n(n+1)}{2}$			
2-1: Gauss's	5	5	5
2-2: Induction	3	3	1
2-3:Arithmetic formula	4	5	5
2-4:Trapezoid formula	3	3	1
2-5:Triangular numbers	4	3	5
2-6: Staircase-shaped area	4	4	5
2-7:mathematical induction	5	5	5

Fig.2-- Different ratings given for different ways of proving the tasks at different stages

Refer to Fig.2, Echor's views about proof kept consistence from stage 1 to stage 2. She rated high scales to formal methods. Observing the flow of the scales from stage 1 to stage 3, it could be seen that the weight of explanatory proof was raised. The weight of formal proof became less and the weight of explanatory proof became higher. At the end of stage 3, Echor emphasized the function of explanation in a proof, understood the cognitive function of informal ways, considered the cognitive needs of the students and then gave up the formal way she preferred. We conjecture the change of her views about proof was resulted by the change of her pedagogical views about teaching proof.

DISCUSSION

Cooney et al., (1996) posit a scheme to conceptualize the professional development of pre-service secondary mathematics teacher. They regarded teacher's development as a means for categorizing teachers, i.e. four positions: isolationist, naive idealist, naive connectionist, and reflective connectionist. Those four positions reflect the extent to which teachers resisted or accommodated new teaching methods into their teaching schemes and exhibit a reflective orientation toward mathematics and teaching. Those four positions mentioned above could be used to describe Echor's development of pedagogical power in this study. At stage 1, she accepted the authority suggestion and wanted to use many ways to enhance students' understanding because she believed that every way of proof had its explanatory value and could contribute to students' learning. This showed she was a **naive idealist.** At stage 2, she just chose a simplest way for teaching proof. This decision was made pedagogically. She was aware that most of students in her class were confused with multiple methods for proving one statement. She can connect mathematics with her pedagogical mathematics. She also could identify the tensions resulted from the difference between her views of proof and about teaching proof. But she failed to see the significance of the connections and made no attempt to resolve the identified tensions. Thus she was a **naive connectionist**. At stage 3, she was becoming a **reflective connectionist** because she accommodated pedagogical ideas as her belief system that was restructured. The pedagogical view that was accommodated showed that she changed her view into explanatory, interactive, and discursive after her belief system was reconstructed. The change was not only the shift of her belief system about teaching mathematics proof but also the change of her belief about mathematics proof. The development of Echor's pedagogical power could be described as starting from a naive idealist, transferring to a naive connectionist, and finally growing to a reflective connectionist.

ACKNOWLEDGMENT

The paper is a partial result of a project funded by the National Science Council of Taiwan (NSC, 90-2521-S-003-016). We are grateful to NSC and the participating student teacher. The ideas and opinions expressed in this paper are the authours, not necessary reflect that of the NSC.

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PRE-SERVICE PRIMARY TEACHERS' ASSESSMENT OF AND PERFORMANCE ON MULTIPLICATIVE NUMERACY ITEMS

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Three groups of pre-service primary (elementary) teacher students responded to a questionnaire involving items on multiplication and division. Their responses to these items were analysed to compare the performance of the groups. In addition, students were asked to indicate the extent to which they felt that the items tested "numeracy" understanding. Students exhibited some difficulties on certain items, with ratio proving particularly difficult, but on average students indicated that they agreed that each question was related to numeracy. The variations in perception may be due to the content of the item, the context, or the background of the students.

INTRODUCTION

In recent years many countries have made explicit a concern for "numeracy standards" in the general populace, among students, and among teachers of mathematics. The fact that "numeracy" is gaining currency as a word somehow parallel to "literacy" has, perhaps, influenced the efforts taken to recognise, test for, and remedy perceived problems. One difficulty with the term "numeracy", however, is that it has a number of definitions. In Australia, one definition states that "To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life" (Australian Association of Mathematics Teachers, 1997, p.15), and numeracy is stated explicitly to include numerical, spatial, graphical, statistical, and algebraic skills. In the United Kingdom, in contrast, the definition of numeracy emphasises mainly numerical aspects of mathematics (Department for Education and Employment, 1998, p.11).

There is now a growing number of government programs being developed in Australia intended to improve the levels of students' numeracy (e.g., Department of Education, Employment and Training, 2001), and testing of numeracy performance now occurs in some states (e.g., at Grade 3, 5, and 7 levels in Victoria). Bearing in mind that it is known that teachers' beliefs affect their teaching practices (see, e.g., Thompson, 1992), there is a possibility that teachers' emphasis in mathematics may focus on things that they perceive to be more "numeracy related". It is thus of interest to determine how teachers perceive the numeracy content of different types of mathematical problems. It is presumed that they will make such judgements based on the mathematical content required and other aspects such as, perhaps, the context of the problem.

This study, therefore, reports on some pre-service teachers' assessment of the "numeracy value" of some questions involving multiplication and division, one of the key areas of subject matter knowledge for primary (elementary) teachers. There has been a long-standing concern about the levels of subject matter content knowledge of pre-service teachers (see, e.g., Clarkson, 1998). It is well-known that multiplication and division questions cause difficulties for learners, including pre-service teachers. Rowland, Martyn, Barber and Heal (2001), for example, report on the low competence exhibited by prospective primary teachers on problems involving scale factors and percentage increase. Klemer and Peled (1998) also point out a difficulty with ratio and proportion. This report, therefore, will also examine pre-service primary teachers' actual performance on the items for which they assessed "numeracy".

At the university where this study was conducted, pre-service primary teachers gain their teaching qualifications by one of two routes: either through a four-year Bachelor of Education (BEd) degree or by first completing a degree in some non-education discipline before doing a two-year Bachelor of Teaching (BTeach) course. Students in the BEd program complete six semesters of mathematics (content, curriculum, and teaching methodology), and BTeach students complete three semesters. Suitably qualified BEd students may elect to do an advanced mathematics subject in each of their final three years, which involves additional tutorial time and content. This study investigates whether or not there are any differences between the groups of students in (a) performance on items requiring multiplication or division, and (b) the extent to which they perceive such items as concerned with "numeracy". Any differences may be due to the students' backgrounds and differences in content covered in their courses.

METHODOLOGY

Three groups of students participated in this study: two groups—from the advanced class (N=19), and from the mainstream class (N=51)—from the fourth year of the BEd course, and a group of students from the second year of the BTeach course (N=89). All students were thus in their final year of teacher training.

During a 45-minute tutorial, near the beginning of the year, students were given a "numeracy questionnaire", which comprised 23 questions, some with sub-parts, covering a range of basic mathematical topics. The content was chosen to correspond with that of the primary and early secondary years of schooling. In what follows, "item" means a question or part thereof. Students were required to answer the items and, for each item, indicate on a 5-point Likert scale the extent to which they agreed that the item was a "numeracy question". The questionnaire included the following definition of numeracy, adapted partly from the Australian Association of Mathematics Teachers (1997) definition, to use as a basis for this judgement: "Mathematical knowledge and understanding that adults need and should be able to use in everyday life without specific revision".

Six of the questions and their parts involved some aspect of multiplicative thinking, and students' responses to these were analysed for this paper. Question 1 asked students to express $28 \div 3$ as (a) a whole number with remainder, (b) a mixed number (whole number with fraction), and (c) a decimal. Question 2 told of Amy who worked from 8.45am to 5.30pm and asked (a) how long she worked, and (b) what she earns if she is paid $10 an hour. Question 3 asked which is the better deal: a 375g can of beans for $2 or an 810g can for $5. Question 4 asked how much flour will be left over if there are 11 cups of flour, each batch requires 3/4 of a cup, and as many batches as possible are made. Question 5 gave the exchange rate of $1 Australian for 50 kemmits and asked (a) how many kemmits will you get for $3.50, and (b) how much is 210 kemmits worth in Australian dollars. Finally, Question 6 asked how much concentrate is used to make 200ml of cordial, if cordial is made by mixing concentrate and water in the ratio 1:4. These questions corresponded to questions 1, 5, 9, 11, 14, and 23 on the questionnaire respectively.

Each item was marked right or wrong; if students answered Question 2a incorrectly but carried out a correct calculation in Question 2b based on their incorrect data, then Question 2b was marked right. The numeracy rating, ranging from strongly agree through neutral to strongly disagree, was assigned a number between 1 and 5, with 1 signifying "strongly disagree", and 5 "strongly agree", with "neutral" given the value 3. Students who did not respond to an item or who did not give a numeracy rating for an item were excluded from the data set when calculating overall success rates or numeracy judgements for that item. This inflates the percentage of correct answers for each item, but all items except Question 6—the last on the questionnaire—were attempted by 90% or more of the students. Question 6 was omitted by 21 of the 89 BTeach students and 3 of the 51 BEd students, perhaps because of its placement at the end.

RESULTS AND DISCUSSION

Students' Performance on the Items

Figure 1 shows the percentage of students responding correctly to each of the items. As can be seen the most problematic item was Question 6, concerning the ratio 1:4, with only a quarter of those attempting it getting the correct answer of 40ml. Nearly half of the students (47%) responded with 50ml, presumably working with the fraction 1/4 instead of 1/5. A probability item on the questionnaire reported elsewhere (Chick & Hunt, 2001) also revealed that students often confuse ratios and fractions. An additional 20% of the students gave the answer 800mls to Question 6, perhaps because they misunderstood the difference between the cordial mixture and the concentrate.

Items 1b and 1c (requiring 28÷3 to be written as a mixed number and as a decimal respectively) also caused difficulty for students. A lack of understanding of how to treat the remainder when determining a fraction or decimal seemed to be the cause of most errors. In Question 1b, 12 of the 157 students wrote 9 1/28, confusing which of the divisor or dividend is the denominator of the fractional part. Of the 147 students who responded to Question 1c, 14 gave the answer 9.1, suggesting that the remainder of 1 obtained from the division was used as the number after the decimal point. A large number of students gave answers closer to the correct value but did not show that the decimal expansion of 1/3 involves recurrence, with 21 giving the answer 9.3, 19 giving 9.33, and a further four giving 9.333 or 9.3333.

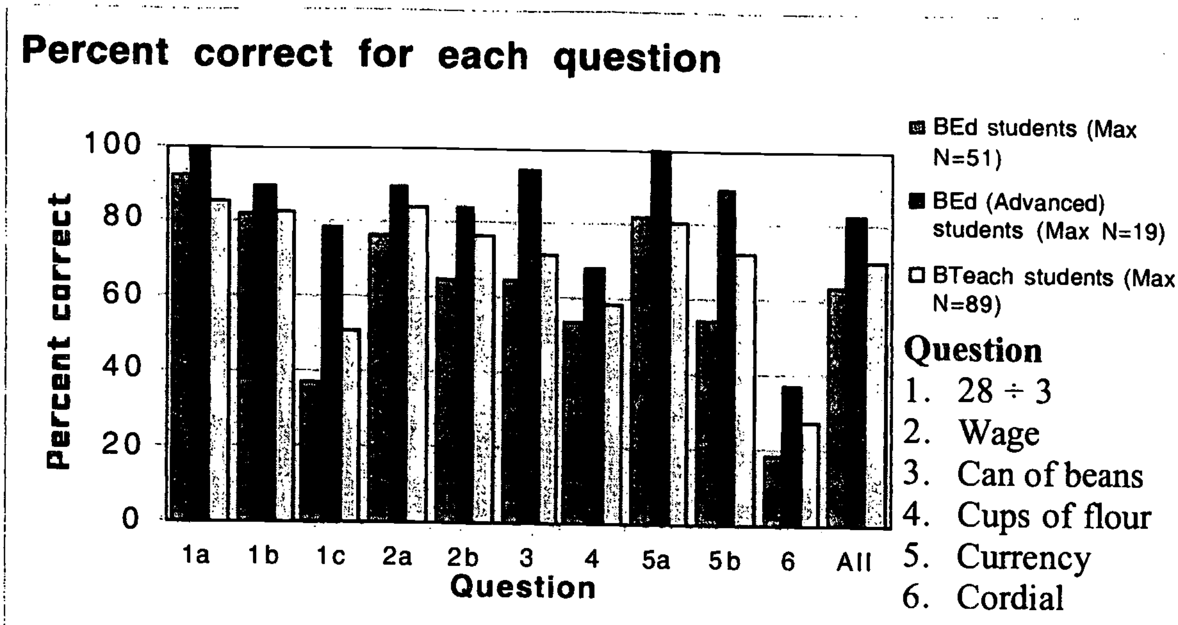

Figure 1. Percentage of students in each group answering questions correctly.

In Question 2b (working out the wage based on the number of hours worked) a common difficulty came from dealing with 45 minutes as a decimal or fractional part of an hour, with at least eight students using 0.45 to give a final answer of $84.50. Just as Question 2b required students to operate on 8 hours and 45 minutes, Question 5b could also be answered by treating the number to be operated on as two parts. Many students treated the 210 kemmits as 200 and 10 more, and successfully converted the 200 kemmits to $4 Australian. They then had difficulty with the remaining 10 kemmits, leading eight students to give the answer $4.10 and another five to halve the 10 (rather than double it) to give $4.05. This part of Question 5 was not done as well as Question 5a, suggesting that students find one direction of currency conversion easier than the other, depending on the rates.

Looking at the overall results, the BEd advanced group performed significantly better than the other two groups, with a mean score of 8.2 for the ten items compared with 6.2 for the mainstream BEd class and 6.6 for the BTeach class (one way ANOVA, $p<0.0001$). When the results of the BEd advanced class are combined with the mainstream BEd class, however, there is no significant difference in overall performance compared with the BTeach group ($p=0.67$), suggesting that the path to a teaching qualification has no effect on performance. Question 1c (decimal evaluation of 28÷3) was the only question for which the differences between the groups on individual questions approached significance (a χ^2 test with $df=2$ yielded $0.05<p<0.1$), with the BEd advanced class performing better than the other two groups. It would be expected that the advanced class perform better, as ability is the main criteria for eligibility for this class. Nevertheless students elect to participate in it, so it appears that the appropriate group is choosing to enrol.

Students' Perception of "Numeracy"

The average numeracy ranking given to the questions by each group is shown in Figure 2. Overall, the average numeracy ranking for the questions ranged from 3.5 (for the BEd students on Question 1c) to 4.8 (for the advanced class on Question 2a), suggesting that the students felt that the items came in the purview of "numeracy" to some degree. It will be interesting to compare these values with those obtained for the remaining items on the questionnaire. To illustrate the potential for differences, the three probability items on the questionnaire—requiring students to place events in order of likelihood, and then assign a word and a numerical value for the likelihood of each event—were given much lower values for numeracy by the BTeach cohort. The BTeach students gave values of 3.5 to 3.7 as the numeracy ranking for the probability items (see Chick & Hunt, 2001), whereas their values on the multiplicative questions reported here ranged from 4.0 to 4.5.

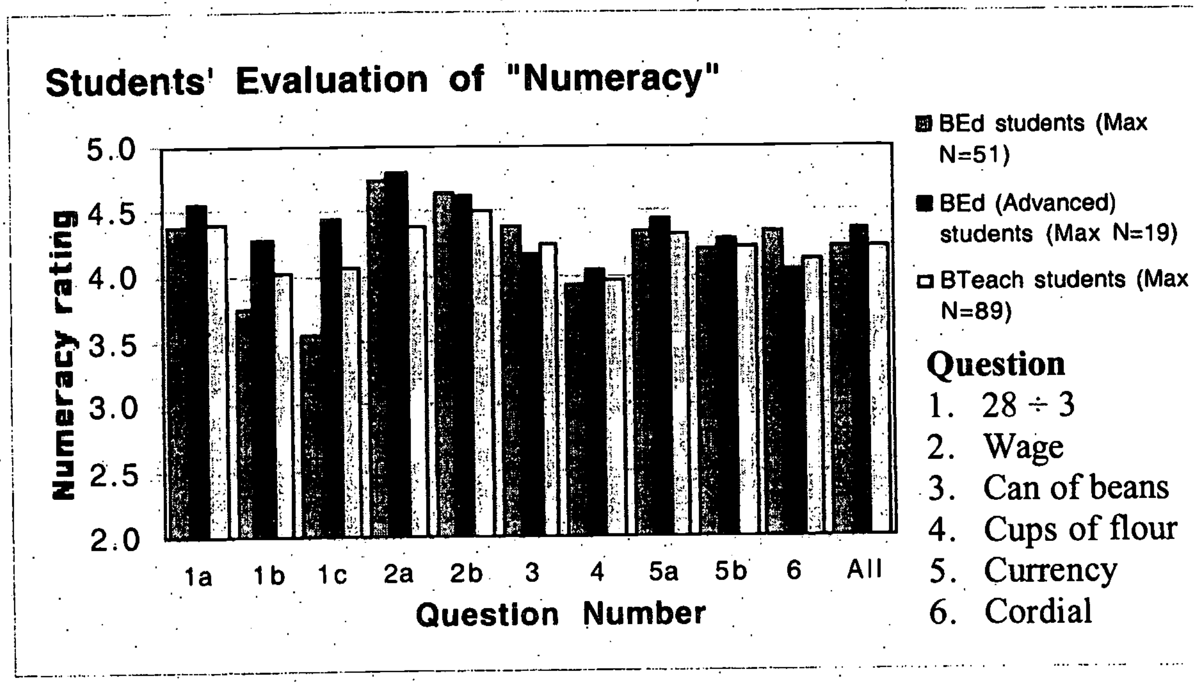

Figure 2. Average numeracy rating of the questions for each group.

When the cohorts are all combined, the questions most regarded by students as being numeracy items were Questions 2a and 2b (time and wage calculation): for both items over 60% of all the students strongly agreed that the items were numeracy-related, with at most 3 students in all disagreeing or strongly disagreeing. The items with the lowest numeracy rankings were Questions 1b, 1c, and 4. Items 1b and 1c had the greatest number of students on the disagreeing side of neutral (17 and 16 respectively), but the modal (and median) responses were still agreement in both cases. Question 4 (fractional cups of flour) had nine students disagreeing and two strongly disagreeing that the item concerned numeracy; nevertheless nearly half agreed and another 30% strongly agreed. It should be pointed out that of the 112 students who assigned numeracy rankings to *all* items, only 18 gave the same response for all questions. Six students did not give any numeracy evaluations at all.

Significantly different perceptions exist between the groups about the "numeracy content" on two of the questions. In Question 1c the BEd advanced class gave a higher numeracy ranking than the other groups (one-way ANOVA, $p<0.01$), with all but one student agreeing or strongly agreeing. All of the students who disagreed or strongly disagreed that the question concerned numeracy came from the other groups. It may be that the lack of context for this question (and also Question 1b) influenced the lower rankings, although the related and context-free Question 1a was more strongly regarded as numeracy. It is possible that students could more readily visualise a context for working out a remainder, than they could for determining fractions and decimals. In Question 2a (calculation of time difference) the BTeach class gave a lower numeracy ranking than the other groups (one-way ANOVA, $p<0.01$).

It was thought that there might be a relationship between students' success rate on questions and the numeracy ranking, perhaps because students might assign higher numeracy rankings to questions they find easier. Figure 3 plots, for each question and group, the percentage of students in the group responding correctly against the average numeracy ranking given to the question by the group. There is the suggestion of a relationship; if all the data is combined the correlation coefficient is $r = 0.45$, with the BEd group marginally more scattered than the other two. This outcome may be a reflection of students' confidence, given that other studies have established correlations between confidence and achievement (see, e.g., McLeod, 1992).

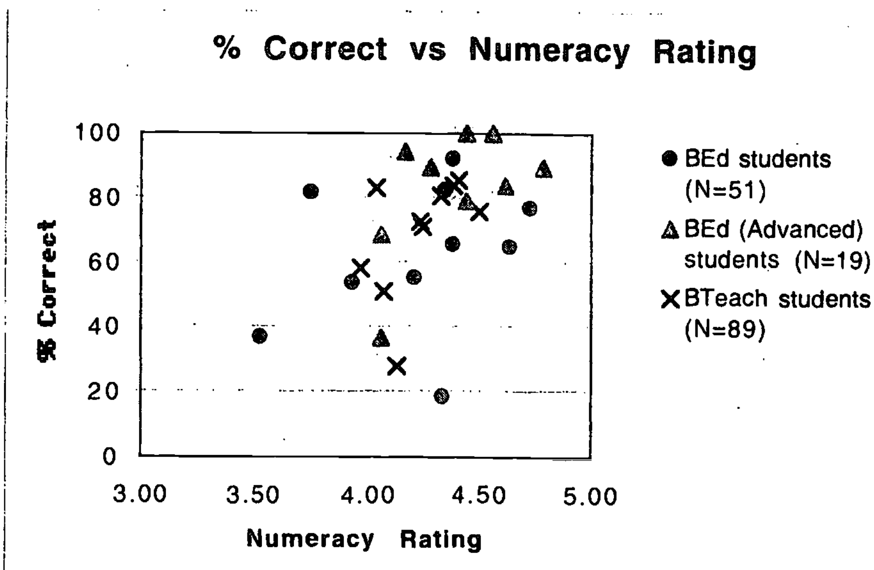

Figure 3. Relationship between numeracy rating and correctness for each of the groups.

Of interest, however, is the fact that at least one question—Question 6 about ratios of ingredients—was regarded as quite highly numeracy-related despite students' poor performance. In contrast, students performed quite well on Question 1b—which required $28 \div 3$ to be expressed as a mixed number—but regarded it as less numeracy-related than most other questions. It may be that the issue of context comes into play here, with students able to do Question 1b but not seeing it as something they "should be able to use in everyday life". Alternatively it may be the content, because Question 4 (which also involves fractions, with the cups of flour used to make batches of food) was given a reasonable context and yet students did not regard it highly as a numeracy question. The teaching and use of fractions has declined in Australia in recent years and this may have influenced students' evaluations; certainly the students found Question 4 one of the more difficult items.

CONCLUSION

Not surprisingly, given the criteria for eligibility, the BEd advanced class performed better on the questions overall, although there were no significant differences on any individual items. It seems that students' background prior to commencing a teaching degree and then the degree chosen makes little difference to the outcomes. While students do cover some mathematics content within their teacher training subjects, the content of questions used in this study was assumed to have been taught to students in their years of compulsory schooling. It would be interesting to confirm this by comparing an entering cohort with an exiting one, to determine whether or not students' performance changes in the course of a teacher training degree. Similarly, it would also be informative to study the performance of primary and secondary school students: to see what, if any, changes take place in understanding of basic mathematics once basic education has been completed.

All items were regarded by the pre-service teachers as "numeracy related" to a greater or lesser degree. There were minor differences in perceptions of the numeracy value of items, perhaps as a consequence of the background or ability of the group of students, or the topic or context involved. It will be informative to examine other, less numerical, items from the questionnaire to compare numeracy rankings.

With teachers being called upon to explicitly teach numeracy and account for students' numeracy performance, the question of what mathematical content is perceived by teachers as being numeracy is important. This is of particular concern where, as is happening already, education systems decree that a certain amount of teaching time be set aside for numeracy. With an overcrowded curriculum, it may be that some important aspects of mathematics and/or numeracy will be pushed aside if teachers do not perceive them as being strongly associated with numeracy. It is therefore of interest to continue investigating teachers' perceptions in this area.

ACKNOWLEDGEMENTS

Thanks to Darien Hunt for setting up and maintaining the database, and to Dr Jane Watson for helpful advice on an early draft of this paper.

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DEVELOPING STUDENT TEACHERS' PEDAGOGICAL VALUES: TWO INITIAL STATES AND A CO-LEARNING CYCLE

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To examine the possibility of introducing values to mathematics teachers, this paper describes the pedagogical values that a group of Taiwanese student teachers share, and the two competing states of, and the co-learning cycle for, values learning emerged from a study of three student teachers. Some reflections on the constraints, issues, and challenges of values education for mathematics teachers are discussed.

BACKGROUND TO THE RESEARCH

To resolve the questions of "How to help teachers become aware of and clarify their own values?" and "Through what process could teacher educators assist those teachers who are not aware of or not willing to modify their values?" concerning the issue of values education for mathematics teachers, raised in the PME 25 conference (Bishop, FitzSimon, Seah, & Clarkson, 2001; Chin & Lin, 2001a), this three-year follow-up PVIMTE (Pedagogical Values In Mathematics Teacher Education) project aimed to: examine the possibility of educating mathematics teachers about values; and develop the plausible curriculum of values education for mathematics teachers. This paper aims to describe the initial states of, and the co-learning cycle for, values learning emerged in the process of learning-to-teach from a case study of three student teachers.

Mathematics teachers may hold various beliefs that become values when enacted in their classroom teaching (Bishop et al., 2001; Chin & Lin, 2001a). For example, Rokeach (1973) suggests that values are prescriptive beliefs wherein some means or end of action is judged to be desirable or undesirable; and Allport (1961) contends that a value is a belief upon which a man acts by preference. For them, value is a preference, a desirable mode of conduct, or a desirable end-state of existence, concerning the conception of something that is importance and worthwhile of thinking and doing for the person. Thus, beliefs are more concerned with the nature of "propositions about phenomena", and values are more about the "key substances" underlying such propositions for people to think and act (Chin & Lin, 2001a). In this study, values were conceived as "teachers' pedagogical identities", referring to "their personal commitment to and action on a set of words, concerning the importance or worth of such words for thinking and practice of mathematics in the classrooms", for example the value of "individual thinking"; and beliefs as "their personal acceptance to a testable sentence, concerning the truthfulness or existence of such sentence", for instance the belief of "There is no learning if students do not think". This operational definition is in a way related to Seah and Bishop's (2000) contextualisation and de-contextualisation difference on beliefs and values, and it also echoes Aspin's (2000) distinction on the nature of beliefs and values for school contexts.

THEORETICAL CONSIDERATIONS OF THE RESEARCH

Two approaches seem to be helpful for developing teacher values (Chin, Leu, & Lin, 2001): (1) encouraging teachers to articulate the differences between one's intended and implemented, and the discrepancies between one's own and others, values; (2) developing values-related activities for teachers to model, justify, and reflect. Rokeach (1979) argues that social groups implicitly transmit, inculcate, and implement a certain cluster of specialized educational values among its members. This means that values are better developed in the process of group sharing and reflection. From a Socratic view (Ling & Stephenson, 1998), values education may involve such strategies as values clarification, critical thinking exercises and conversation in which individuals' values positions are articulated and critically examined. For Aristotle, the education of values includes debates and value examination activities; for Kant, moral reasoning within dilemma situations are useful for clarifying and developing values. Conceiving these philosophical ideas about values education, Raths' (1987) values clarification and Fraenkel's (1977) values analysis approaches were adopted in the study.

Findings in the literature of value change suggest that values may be re-considered by individuals through cognitive and affective incongruity or inconsistency of some kind. For instance, if persons are induced to behave in a manner incompatible with their values; or expected to new information, including evaluations, from significant others that is inconsistent with one or more central values; or exposed to information about inconsistencies already present among their values, then the persons' values are expected to be changed. It is the resulting conceptual and behavioural change from incompatibilities, dissonances, and incongruities that would enable individuals to re-assess, re-organize, and re-construct their values (Rokeach, 1973). There are in my views three major phases for developing pedagogical values with mathematics teachers (Bishop, 2001; Chin et al., 2001): (1) sensitising them to values issues through the analysis of and reflection on value-loaded teaching activities; (2) showing them examples of approaches to mathematics teaching at which differ markedly in the values aimed; (3) helping them clarify and modify their initial values, they might be in a better position to re-construct, re-organize, or re-assess a coherent values structure for their own classroom teaching. As a result, the provision of critical incidents and the creation of doubt; the clarification of one's own values positions; and the justification and criticism on value-related classroom teaching activities in a collaborative working-and-discussion team, were used to develop teachers' pedagogical values. The scheme addresses the socially shared and personally constructed nature of values development, in which the processes of values clarification, argumentation, identification, and action are central.

RESEARCH METHOD

The Teacher Participants

A questionnaire (Chin & Lin, 1998), concerning varying views of mathematics and mathematics teaching using 5-point Likert format, was used to select the participants from two in and pre service teacher groups. 42 secondary school mathematics teachers, enrolled in the Master of Teaching program in the Department of Mathematics, National Taiwan Normal University, played as the sample for in-service side. A class of 24 student teachers at the third year of teacher education program, joined in the author's "methods of mathematics teaching" course, acted as the cohort for pre-service side. Two in-service and three pre-service teachers were selected according to the resulting factorial structures of the item responses. The case study and action research methods (McNiff, Lomax, & Whitehead, 1996; Yin, 1994) were used as the major approaches of enquiry to explore the values learning of the five teacher participants. The author played as a collaborator with two teachers and as a coach with three student teachers. The major purposes were to examine the possibility of introducing a set of selected pedagogical values and developing the curriculum for them to learn.

A Framework for Developing Teachers' Pedagogical Values

A two-level learning cycle, consisting of the resources of and scaffolding for learning values, was developed in terms of the first year data. The value-loaded activities, the teacher's reflective journals, and self-descriptive written interview data developed in the former study (Chin & Lin, 2000, 2001a, 2001b), including the topics of mathematical induction, permutations, trigonometric functions, and equation of circles, were used as the materials for value dialogues within a co-learning team. Before the participant teachers taught the topics, videos of that topic provided by Ming (see Chin et al., 2001) were playing back, discussed, and criticized within the team, and later the written self-descriptive interview data were provided for discussion and reflection. In addition, a topic-related values questionnaire was administrated for examining their value preferences. Therefore, we have four whole topics of audio-video records and written data prepared, accompanied with the topic-specific value preferences surveys for group discussion and reflection. The working-and-discussion team, including one teacher educator (the author), one mentor (Ming), two experienced (T_1, T_2), three student teachers (ST_1, ST_2, ST_3), and one independent observer (O), was formed for the learning of values. Those ideas related to Schön's (1987) intelligent action, reflection-in-action and knowing-in-action, Vygotsky's (1978) social formation of individual concepts, and the three pillars - construction, narration, and reflection - suggested in Dutch Standards of mathematics teacher education (Goffree & Dolk, 1995), are all the alliances of this framework.

In the scaffolding, construction, reflection, narration, and diagnosis are the four major activities for learning-to-teach values. In reading, observing, and criticising the learning resources in such a co-learning team, the participants would be in a good position of using and constructing their own narratives about teaching. They could also re-consider the possibilities of rectifying their classroom values teaching. This self-and-collective regulation process, in which reflection and communication are two major activities of values clarification, and narration and metaphor are two indicators of a value communicator (Chin & Lin, 2001b), may create space for teachers to diagnose their values teaching. Two aspects about teacher intrinsic motives (awareness and willingness) derived from a comparison between Taiwanese and Australian teacher values research (see Chang, 2000; Chin & Lin, 2001b; Leu & Wu, 2000; and Bishop et al., 2001) play as two affective requirements of learning-to-teach values. One is concerned with teacher awareness of values in classroom mathematics teaching and the other is about their willingness to teacher that values. Moreover, thinking and action are two recursive levels of learning-to-teach values. It is very important to take the aspect of action into serious consideration, and to separate implicitly mental thinking from the enactive aspect of values practices. Thus, I used the procedure of values construction focusing strictly on the content of mathematics, from observation, simulation, micro teaching, to that of teaching practice and actual classroom teaching, to develop the three student teachers' content-specific pedagogical values.

The Research Instruments

Six questionnaires were developed for eliciting the participants' pedagogical values. One of the two general values surveys is to explore their degrees of agreement on the 25 propositions using 5-point Likert format. The second parts of the questionnaire ask teachers to select and rank 5 from the 25 statements according to the most/less importance for them, and describe reasons for the rankings. The second questionnaire asks teachers to rank the 14 values according to the degrees of importance for them in lesson planning and classroom teaching, for example the values of felt pleasure. The 4 remaining questionnaires address different values for each of the 4 teaching topics.

SOME PRELIMINARY RESULTS

An initial analysis of the questionnaire surveys, classroom observations, and interviews with the student teachers showed two initial states of, and a co-learning cycle for, values learning. The statistical procedure of factor analysis (SPSS, 1994) with principal components extraction and varimax rotation was used and tested on the item responses of the student teacher group. In comparison with the results produced by oblimin rotation, the factorial structure was almost identical. The resulting six-factor model, explaining 77.44% of total variance, was used to construe their shared beliefs. For example, the six items with highest loadings (>0.82) on the first factor suggest that the first belief proposition is "To teach mathematics, teachers should recognise beforehand student backgrounds/needs of learning". The remaining 5 beliefs can be found in Chin (2001).

Underlying pedagogical values nominated and taught by ST_1, ST_2, and ST_3

The values of mathematical forms and abstraction seemed to be the best stands for ST_1 and ST_2 to organise and teach mathematics. ST_1's first two choices out of 14 values in the second questionnaire were mathematical forms and contents; and the last three options were about the affective aspects of learning mathematics, for example, felt pleasure, felt happiness, and willingness to learn mathematics. He said, "No matter students would be pleasant or not, as a teacher, I should teach them some mathematical forms and also try to force them to learn the forms" and "We should also make very good use of some typical mathematical exemplars and to build student knowledge on that particular exemplars." ST_2 ranked hardworking, intellectual growth, and creating mathematical abilities, higher than the values of practical knowledge and felt interest to the knowledge. For him, "Forms and exercises are more practical for classroom realities of mathematics teaching." and he felt that "Students will be very happy if they feel that their teachers teach them more mathematical content than other classes." and "The affective factors are the most difficult things for me to control in teaching, therefore, it is better to teach the mathematical contents more." In the microteaching, ST_1 started with the example $\sum_{k=1}^{n}\frac{1}{k(k+1)} \equiv \frac{n}{n+1}$ to introduce the mathematical induction. The reason to do so was "It is better for them to learn new concepts from a well known example, otherwise, classroom situations would not be able to under my control." For him, classroom orderliness and the understanding of mathematical forms are more important than just playing some artificial mathematical games. He said, "There are not much value to introduce mathematical concepts using games or pseudo-artificial activities, I would rather spend my time on demonstrating as clearer as possible the mathematical forms and rules." As a result, his teaching relied much on forms and exercises. If students couldn't get the sense he would "Give them five seconds to think, and then I will solve the problem and show them the solution. It is no way for me to stop and wait there." ST_2 took over the lesson by introducing the formal proof of mathematical induction although this idea was criticized by T_1 and T_2, he still insisted to do so, because "It is absolutely necessary for students to understand logic necessity and existence within the system of mathematical knowledge." Therefore, the values of mathematical proofs, forms, and rules were addressed because "The concept is very rigorous and beauty in the sense of its structures."

The values of pleasure and practical knowledge were central for ST_3. ST_3's first two choices out of 14 values in the second questionnaire were practical knowledge and felt interest to the knowledge; and the last two options were about the formal aspects of mathematics learning, for example, logic reasoning and orderliness. He hoped that "Students are the focus of my classroom teaching, and I will try very hard using daily life examples and student practices in which I can talk to students and to initiate their thinking." One central tenet of teaching for him was "increasing students' mathematical abilities using practical and realistic knowledge from daily life." In the microteaching, ST_3 used a self-made teaching resource, a clock like plate, to introduce the definition of angle through visualisation. After that, he gave some questions for the students to solve. He said, "I really wanted to use some real staffs, helping students access to and make sense of the mathematical concepts. I hope that they can learn from the realistic aspects of mathematical knowledge through practices and reflections." Therefore, there are two sets of pedagogical values emerged from the questionnaire surveys and interviews. One set of values, as ST_1 and ST_2 exemplified, includes mathematical forms, rules, and proofs, and abstractness; the other set, as ST_3 showed, consists of felt pleasure and practical knowledge.

Two competing states of values learning portrayed by ST_1, ST_2, and ST_3

A shared understanding, emerged in the process of learning values, is the recognition of "felt easy to know but difficult to act". As they all hesitated to use felt pleasure in the classroom, because "These ideas although are very important for teaching and learning, but, it seems very difficult for me to teach such a value, for example, feeling pleasure. Don't you think?" There are two initial states of learning-to-teach pedagogical values, in which one values the formal and logic-structural aspects of mathematics and the other values the practical aspects of mathematics and intrinsic motivation for learning. The most important tasks for ST_1 and ST_2 were to learn about "How to get more insight into the formal and abstractive nature of mathematical system and help them create approaches for teaching students understanding of such system?" They hoped that T_1 and T_2 could help them construct the strategies for introducing mathematical concepts. Their present states show a limited extension of visions about values, although they feel that it is useful to consider values in mathematics teaching. This suggests that they are in a "felt difficult to act however unwilling to act" state. What most important for ST_3, was "How to collect more examples of designing mathematical activities for students to learn?" He expected that T_1 and T_2 could help him develop the situations to increase student pleasure in learning mathematics. Although these values positions were not observed in his classroom teaching practice, but he wanted to do so. In this case, his present state of values learning is in a position of creating value visions but with limited teaching experimentations. This suggests that he is in a "felt difficult to act and yet willing to act" state.

A co-learning cycle for learning-to-teach pedagogical values

To help them moving from the base of "felt easy to know but difficult to act", through the present states of "felt difficult to act however unwilling to act" and "felt difficult to act and yet willing to act", to "knowing is action", a co-learning cycle for learning-to-teach pedagogical values is developed as the following Figure 1 shows.

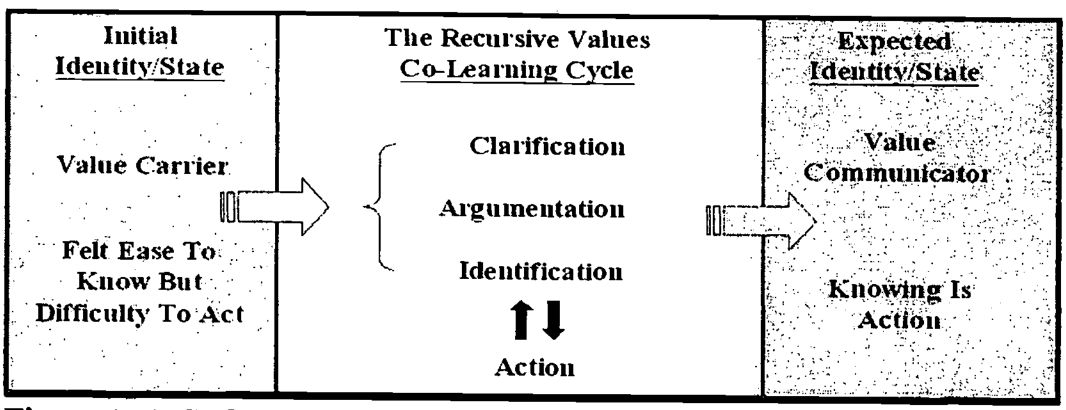

Figure 1: A Co-learning Cycle for Learning-to-Teach Pedagogical Values

In the values co-learning team, I was a participant observer and an action researcher; Ming acted as a values demonstrator; T_1, T_2, ST_1, ST_2, and ST_3 were the practitioners of values; and O played as an observer. To monitor them transferring from values carriers to values communicators (Chin & Lin, 2001b), the recursive learning cycle including the activities of clarification, argumentation, identification, and action was developed for thinking and practising of pedagogical values. Two substantial layers are underlined in the cycle: the thinking aspect in the process of learning-to-teach, and the enactive aspect of the values learning process. These two layers are implicitly connected and relatively informed each other in the co-learning process. Within the cycle, each member is a learner, learning about the concepts of values education and values teaching. The researcher learns "How to educate teachers about values teaching?", Ming learns "How to present his own values to outsiders?", the five participant teachers try to re-consider the values provided with their own. The levels within this cycle are in a way related to Jaworski's (2001) 3-level model of co-learning partnership for developing mathematics teaching for teachers, teacher-educators, and researchers. In the present study, the researcher and teacher-educator who develops one's recognition of values education within level 3a, level 3b, and level 2, focusing on the roles and activities that can be used to facilitate the participants' values learning. Ming and the five teacher participants are to enhance their abilities and recognitions about values teaching within level 2 and level 1. All the members were trying to understand other members' values and the differences that might be emerged in comparison of one's own and others' values during exchanges. To help student teachers reaching the goal of "knowing is action", all the members were trying to become value communicators through this developmental framework of values learning.

REFLECTION

To revisit the study aims, several aspects concerning the learning framework need to be re-considered. The group discussions seemed to be effective as there were different participants with different thoughts and backgrounds for student teachers to exchange, model, and reflect on their own thinking and action about values teaching. The value learning resources were effective and this might have contributed to the participants' value clarification processes. The co-learning cycle for values learning was also useful in the ways that it played in increasing participants' recognitions and awareness of thinking and practicing about values. Thus, the four main activities in the co-learning cycle seem to be useful for teacher educators to develop their student teachers' values. I would rather see it as a cyclic and recursive path for the teachers to reflect on and to learn about values, than as a static and single loop for them to go through.

The practices of educating mathematics teachers about pedagogical values that the present study describe, challenge much of our knowledge, beliefs, and values about mathematics teacher education from a very fundamental aspect, as the teachers are more seriously considered in their process of "becoming-a-teacher", related to his or her own processes of identities and values development. After all, classroom teachers are at the very hart of any curriculum reforms, not least of course in the recently launched Taiwanese New Mathematics Curriculum (ME, 2001) for the students under age of 15, in which the values of pleasure and practical knowledge are explicitly addressed. Is this a kind of "paradigm shift" for school mathematics curriculum and classroom teaching of mathematics? Or the challenge is more about the re-assessment and re-construction of teachers and educators' pedagogical beliefs and values?

Two issues concerning education of student teachers about values need to be further examined. One issue is about their willingness to teach those values in classrooms, and the other concerns the ability to teach them. Although student interest and motivation to learn were important but they resisted doing so, as ST_1 and ST_2 showed. There were also problems related to the abilities of teaching the intended values as shown by ST_3. We have to create intrinsic motives for student teachers to teach their intended values, and to provide learning arenas to empower their abilities to implement them.

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FACTORS AFFECTING THE IMPLEMENTATION OF REFORMS IN SCHOOL MATHEMATICS

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This paper describes on-going research that investigates the effects of an implementation project run in comprehensive schools in Israel. The research also examines teacher's self-esteem, and tries to discover its place in determining the teacher's willingness to adopt innovative practices. Results so far suggest that the implementation project is succeeding in promoting reforms in the schools' culture, but the complexity of the context in which teachers operate renders self-esteem nearly insignificant.

INTRODUCTION

The aim of the research reported in this paper is to examine the implementation of changes in mathematics teaching practices. It focuses on approaches to the implementation of new teaching-strategies for in-service high-school mathematics teachers, and on their feelings and reactions to this process. This topic has been high on the agenda of PME meetings for some time, and is still under scrutiny. My research also examines teacher's self-esteem, and tries to discover its place (amongst other factors like beliefs, norms, values and context) in determining the teacher's willingness to adopt innovative practices; at the same time, the effects of the reforms implementation on the teacher's self-esteem will be examined.

The research studies an implementation strategy operated by the author in 13 middle and high schools in Israel. The strategy is under constant review in order to improve the implementation methods, find ways to cope with resistance and promote renewal and regeneration of new ways in mathematics education.

THEORETICAL BACKGROUND

Change implementation processes have been widely examined in different contexts, including educational settings. This research looks at the problem from two perspectives: behavioural theories are used to examine beliefs, attitudes, norms and self-esteem; organisational/management theories are used to examine change implementation processes in organisations (schools) and teamwork (as part of the change); educational research literature ties them together.

Whitaker (1998) claims that "a key assumption upon which traditional orthodoxy in education has been built is that teaching in schools is concerned with the transmission of knowledge... that will remain valid throughout our lives." (p.15). This has been especially true for mathematics teaching (Weinzweig, 1999; Norton et al., 2000), and my observations in some 400 classrooms for this research suggest that this is still the case. Therefore, reform cannot occur without a change in the mathematics teachers' system of beliefs and conception of the nature of mathematics and theories of teaching and learning. These, in turn, may affect the image and self-esteem of the mathematics teacher, who may undergo a change from the sole authority and source of knowledge, to a co-learner with the students (Clarke, 1997). The teacher needs to become a moderator, stimulating students to question the assertions of their colleagues (Ponte, 2001).

Clarke (1997) identified 12 factors that appear to influence the process of changing teacher roles, amongst them the in-service program. The social context is also a powerful influence. This is a result of the interactions between teachers and pupils, parents, peers and superiors, and their expectations from the teachers (Ernest, 1994). The common myths concerning mathematics and mathematics education also affect the expectations from the teachers, and as a result, affect their performance (Lim and Ernest, 1999). For example: "mathematics is made up of rules and procedures" (Amit and Hillman, 1999, p. 21); "the teacher is the source of knowledge, the pupils – passive receivers" (Weinzweig, 1999, p. 26).

It is therefore apparent that effective professional-development models will need to take these aspects into consideration. The elements that need to be considered when designing such models include:

- the designs should never stop evolving and changing (Loucks-Horsley, 1998). Constant reflection is an essential part of an effective program;
- institutional support is crucial (Ernest, 1994);
- the design should be done through collaborative work (Whitaker, 1998; Valero & Jess, 2000);
- the possibility of a loss of confidence and self-esteem should be taken into consideration (Whitaker, 1998).

Reforms in teaching demand both vision and courage. There are tensions, challenges, doubts, failures, guilt and frustration involved. As Koch (1998) puts it: "Nowhere is reform deeper, more personal, or more threatening than with teachers" (p. 118). And even more so, I find, with mathematics teachers, who are always under public scrutiny. Therefore, mathematics teachers who face a decision on implementation of changes in their practices have to confront the dilemma: how much of their professional persona can they risk? (Sakonidis et al., 2001). Or as House says:

> The personal costs of trying new innovations are often high ... they [innovations] require that one believes that they will ultimately bear fruit and be worth the personal investment. (quoted in Fullan, 2001, p. 36).

Therefore, mathematics teachers' self-esteem seems closely connected to their willingness to attempt implementation of changes in their classroom practices. Many mathematics teachers display high self-esteem: they seem very confident both in their knowledge and in their didactics. Nevertheless, they might be reluctant to adopt new ideas and risk what they consider to be their 'achievements'. On the other hand, Yackel (1994) gives an example of a teacher who was thought of as an exceptional mathematics teacher and was aware of it; presumably, his self-esteem was high. Nevertheless, Yackel describes his efforts to reform his practice as "remarkable", and attributes this to his deep understanding of mathematics and his mathematical values.

Many psychologists believe that virtually all of human behaviour springs from two motives, the first of which is our desire for self-esteem (the second being our desire for sense pleasure) (Campbell, 1984). There is a wide range of books and articles on self-esteem and ways to measure it; the definitions and measurement of self-esteem are taken mainly from classical psychology (Rosenberg, 1965; Coopersmith, 1967). Blasovitch and Tomaka (1991) suggest an 'integrated' definition of self-esteem:

> Self-esteem is the extent to which one prizes, values, approves or likes oneself (p. 115).

To date, the most popular measurement tool of self-esteem for adults is still the Rosenberg Scale (1967), with the Coopersmith Inventory (1967) not far behind (Blasovitch & Tomaka, 1991).

THE CONTEXT OF THE RESEARCH

Since April 1999 I have been running a project for the implementation of contemporary teaching practices in 13 middle and high schools in Israel. Its aim is to encourage the on-going professional development of the teachers, to help change the teaching practices, to encourage the use of technology and to promote teamwork in mathematics teaching.

On-going support is given to the mathematics teaching staff by assigning each school a 'facilitator' for one day a week for three years. The school's head of mathematical studies (HoD) and the facilitator work together: they lead weekly workshops for the team, train them in up-to-date teaching methods, promote teamwork procedures, lead group learning sessions, and help to experiment with new teaching practices and with the use of technology in the classroom. The facilitators and department heads from the schools taking part in the project attend a monthly meeting in which there is an exchange of ideas, consultations, planning and evaluation of their work and progress (Chissick, 2000).

Regular feedback and reflections are exchanged between the project manager (who is also the researcher) and the facilitators and heads of department, and changes in the project's activities are made accordingly.

METHODS AND ANALYSIS

The research examines two interrelated concepts:

- effective implementation of contemporary mathematics teaching practices (through the above mentioned project);

- the self-esteem of the mathematics teacher, and its effect on his/her willingness to implement changes.

This is qualitative research using non-participant observational techniques (Glaser & Strauss, 1967), as well as questionnaires (both structured and semi-structured). The data accumulated from the project are analysed on a regular basis. As far as self-esteem and its effect on the adoption of reforms by teachers are concerned, the paradigm of grounded theory is adopted (Glaser & Strauss, 1967; Strauss & Corbin, 1998; Dey, 1999).

The analysis is inductive, based on a naturalistic paradigm as in Sanger (1996):

> "The label 'naturalistic' in educational inquiry would signal that the research has been conducted in an educational milieu and seeks to characterise participant activity within programs, projects or other settings" (p.11)

The research questions, the choice of the research tools and the way to use them, as well as the research population, are determined during the process and can be changed during and as a result of analysis. (Gibton, 2001)

In addition to the study of the data accumulated from the schools, case studies will be carried out in two schools; one will be discussed in the next section. The participants are three mathematics teachers (one of them the head of department) in each school. They are observed in their classrooms (three lessons each) and interviewed three times each during the school year. The interviews are semi-structured and concentrate on four themes:

- the teacher's beliefs and attitudes towards mathematics and mathematics teaching;
- the teacher's beliefs and attitude towards the change process;
- the teacher's personal history as a student of mathematics;
- the teacher's self esteem.

Questionnaires about the implementation project have been issued to teachers (at the end of the two previous school years), HoDs and school heads. Microanalysis, as in Strauss & Corbin (1998) has been used to analyse the responses. The codes were entered into a spreadsheet, and sorted twice for analysis: once in the order of Code-School-Document for an overall analysis, and then in the order of School-Code-Document for case studies.

A basic set of codes was built, and changed as deemed necessary while the coding process was going on. The process was flexible and creative: codes were added, discarded or changed along the way.

Non-participant, semi-structured observations were made both at staff meetings and in classrooms. The observation reports included informal, descriptive, narrative accounts (Wragg, 1999) and event-coding tables (Bakeman & Gottman, 1986; Robson, 1993). These were scrutinised for behavioural patterns that may indicate willingness or resistance to the implementation of reforms, and displays of high or low self-esteem.

In addition, regular weekly semi-structured reports from the project's facilitators, notes on informal conversation, and historical data accumulated in previous years are examined and analysed for indications of the main items mentioned above.

In the event, insufficient data has been collected on the teachers' self-esteem; its effect on the implementation of changes will be need to be studied more thoroughly in the future.

PRELIMINARY FINDINGS AND DISCUSSION - SCHOOL NO. 1

School No.1 is a comprehensive school (grades 7-12, pupils aged 12-18) situated in a development town, surrounded by affluent settlements. The town's population consists of 60% relatively new (residence 3-10 years) immigrants from the former USSR. This has an impact on the teachers (some of whom are new immigrants themselves) and on teaching methods, as the reigning learning paradigm amongst parents is a traditional one.

The implementation project is in its third and last year in the school. The team consists of 12 teachers and two HoDs: P. for grades 7-8, and A. for grades 9-12. They both co-operate willingly with the facilitator. Both are new to the task (first year for P., second year for A.) and are developing as leaders. The facilitator, D., has gained the teachers' trust and confidence by showing significant ability and knowledge, as well as empathy and understanding. This can be seen from the appreciation that the teachers express in the questionnaires and at meetings with the researcher; in addition, the relaxed and open relationship between D. and the teachers is notable.

In the analysis of the questionnaires (administered to the teachers, HoD and facilitator at the ends of the first two years of the project by the external assessor), open coding was used, with line-by-line analysis, as suggested by Strauss & Corbin (1998) and Dey (1999) for the first stages of the analytic process.

Some results can be seen in Table 1 below. In this table some of the codes were gathered into more general categories; for instance: use of technology, use of open-ended tasks, changes in assessment, and overall changes in classroom practices are all under 'changes in classroom practices'.

Further analysis of additional data (summaries of facilitators' meetings, notes from casual talks with D., A. and P.) shows that the facilitator was less convinced of the overall changes in classroom practices and the use of open-ended tasks than the teachers and the HoDs. As far as the use of technology is concerned, both the facilitator and the HoDs thought that some progress was made (although not as much as had been expected) while the teachers were divided on the issue: half of them did not see any progress, while the other half felt that some progress was made. This might be explained by the fact that each teacher responded from her/his personal viewpoint, whereas D., P. and A. saw the whole picture and responded accordingly. At one of the facilitators-HoDs meetings that are held regularly, A. reported:

> This year I was teaching the role of the parameters in the quadratic function on the shape and place of the graph, when I suddenly realised that this subject could be better taught with the graph-generator on the computer. I 'dared' it, and took the class to the computer lab. It was wonderful. They all understood and could generalise. (CT-371-2009: note on casual talk at a staff meeting, Sep. 2001)

It has to be noted that A. has attended quite a few courses on the use of computers in mathematics teaching in the past, and the benefits of using a computer had been well known to her for some time. It seems that the support and encouragement that D. and her peers gave her, as well as the realisation that she should set an example, gave her the strength to 'take the risk' and brave it. Her report (above) on her experience to the team, and her frankness about her doubts and fears served as a catalysing agent for her peers.

Except for one teacher, everybody felt that they have been going through a process of personal development. It remains to be seen whether this will result in the initiation of further studies or similar processes in the future.

As can be seen in Table 1, teamwork culture, which is one of the main aims of the project as an instrument for progress and reforms implementation, seems to be having an impact on the teachers. Most teachers reported active participation and peer-support.

Category	Dimension / properties	Score
Overall reform implementation	significant	6
	slight	6
	none	1
	negative	1
Changes in classroom practices	extensive	50
	few	19
	none	12
Head of Department status	strengthened a lot	3
	strengthened a little	2
	no change	1
Teacher's development	occurred	15
	did not occur	1
Teamwork culture	changed significantly for the better	63
	very little change	15
	no change	3

Table 1: Results of questionnaires' analysis - School No. 1

The effects of the implementation project on teaching practices and on teamwork can already be seen. The atmosphere in the department has changed: teachers consult each other, work in groups on subjects of common interest, produce open-ended assignments and worksheets and discuss assessment policies.

DISCUSSION

Analysis of the results from all the schools taking part in the research so far shows a significant change in teamwork culture in all the schools; some changes of classroom practices (excluding the use of technology) which includes the use of open-ended tasks and more pupil-centered teaching; and a general feeling amongst the teachers that they are undergoing a process of professional development. School No. 1 seems to represent the general results fairly accurately. These results may imply that the implementation project is successful in promoting the adoption of reform practices. Nevertheless, it remains to be seen if the effects of the project remain, and if the teams can carry on the momentum on their own.

Not enough data has yet been collected about the teachers' self-esteem; therefore, it is too early to come to conclusions about its effect on reform implementation. Although it is widely believed that self-esteem is a major factor that affects human behaviour (Campbell, 1984), it may be that the complexity of the context in which teachers operate renders this factor nearly ineffective. From close acquaintance with some of the teachers, it seems that there is no immediate connection between self-esteem and readiness for change; self-esteem may turn out to be just one of the many varied factors (for example: external exams, peer support, content knowledge, school environment, time limit and more) that affect the willingness of mathematics teachers to implement new teaching practices. It is expected that in-depth interviews, and the use of the self-esteem measurement scales (mentioned above) will give more insight on the counter-effects of self-esteem and reform implementation.

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THE EFFECT OF EFFICACY ON TEACHERS' CONCERNS WITH REGARD TO THE IMPLEMENTATION OF A NEW MATHEMATICS CURRICULUM

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The present study aims to examine the concerns of primary school teachers in Cyprus in relation to the recent implementation of a new mathematics curriculum. An adaptation of the Stages of Concern Questionnaire (SoCQ) based on the Concerns-Based Adoption Model (CBAM) and a teachers' efficacy questionnaire were administered to a representative sample of teachers. The findings suggest that teachers' concerns are largely affected by their efficacy beliefs. Furthermore, there were significant differences in teachers' self-concerns across years of involvement with the implementation. However, no significant differences were found in task and impact concerns across teachers in the initiation, implementation and refinement stages.

INTRODUCTION

Research results suggest that the perception of those involved in innovations are of major importance for the successful implementation of an innovation (Richardson, 1990; Sztajn, 1997). The significance attributed to the innovation by those involved in it is often considered as one of the most crucial factors for its successful implementation. However, the literature examining the relationships among the factors that contribute to the successful implementation of reforms, such as the experiences, concerns, and skills of the teachers involved in the process, appears to be inconclusive. This study aims to provide the framework of a model that may explain the relationships between teachers′ concerns and their teaching efficacy. Our work focused on the application of the CBAM model in Cyprus with teachers attempting to implement a new mathematics curriculum through the adoption of a new series of mathematics textbooks.

One of the main purposes of the new Cypriot mathematics textbooks is to engage students in thought-provoking, original problems that involve challenging themes. Students are expected to investigate a series of carefully sequenced activities aimed at hands-on discovery of mathematical concepts, with mathematics often integrated with other content areas. The textbooks also aim at supporting teachers' efforts in identifying and addressing students' prior knowledge and connecting it with new ideas at linking conceptual and procedural knowledge, and at relating mathematics to other fields of human endeavor. The adoption of the new textbooks and the subsequent changes in the teaching of mathematics is considered one of the most important innovations in primary education in the last ten years.

The new series of textbooks has been introduced gradually in all public primary schools in Cyprus since 1995: Grade 1 teachers were called to use the new series of textbooks in 1995, grade 2 teachers in 1996, and so on until 2000, at which point all teachers became involved in the adoption of the innovation. At present, there are teachers at different stages in the implementation of the reform. Given the importance of the innovation, it was considered necessary to investigate the degree to which Cypriot teachers had accepted the new curriculum suggested through the new mathematics textbooks and followed it in the classroom.

The proposed efficacy and concerns model (ECM) is an attempt to examine teachers' concerns about the implementation of the new mathematics textbooks and explain how teachers' efficacy influences their concerns.

THE PROPOSED MODEL AND THEORETICAL BACKGROUND

The ECM model integrates two basic ideas about the implementation of educational reforms - the concerns based stages and teachers' efficacy beliefs. Drawing upon relevant literature from each of the two domains, we have attempted to explain the proposed model, which is depicted on Figure 1.

Concerns-based Stages

Concerns refer to the feelings, thoughts, and reactions individuals have about an innovation that is relevant to their daily job (Hord, Rutherford, Huling-Austin & Hall, 1998). The adoption of an innovation requires a change that involves people, thus making it necessary to form an understanding of human concerns in order to help individuals move in the process of change. Concerns exert a powerful influence on the implementation of reforms and determine the assistance that teachers may need in the adoption process.

The CBAM helps educational leaders to this effect by showing them how the individuals most affected by change react to the implementation of innovations (Hord et al., 1998). The model can identify the special needs of individuals and enable administrators to provide the appropriate assistance. The CBAM includes three key tools used to collect relevant data: Stages of Concerns (SoC), Levels of Use (LoU), and Innovation Configurations (IC). The most important of the three tools is the SoC questionnaire, which is used to measure teachers' concerns about an innovation they are expected to implement (Hall & Loucks, 1978). The SoC questionnaire is structured around three dimensions of concerns: self, task, and impact, as depicted in the right side of Figure 1. In the self-concern dimension teachers are trying to learn about the innovation and understand what the innovation means to them. In the task-concern dimension, teachers express their concerns about how to cover objectives, provide and organize instruction. The third dimension, impact, focuses on teachers' concerns about the effectiveness of the innovation on students' performance and abilities.

Earlier research by McKinney, Sexton, & Meyerson (1999) indicated that implementation of innovations occurs in a sequential manner – initiation, implementation and refinement (Figure 1). The initiation phase is closely related to the development of self-concepts, the implementation phase with the task-concerns and the refinement with the impact-concerns of teachers.

Teachers' Efficacy

The conceptualization of teacher efficacy is based on the theoretical framework of self-efficacy developed by Bandura (1997). Bandura (1997) defined perceived self-efficacy as "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (p. 3). In the same sense, teaching efficacy, which is a form of self-efficacy beliefs, can be defined as teachers' beliefs in their abilities to organize effective teaching-learning environments and have positive effects on student learning.

Self-efficacy influences several aspects of behavior that are important to teaching and learning (Woolfolk & Hoy, 1990). Among these are the choices of activities that a teacher makes, the effort put forth and persistence in accomplishing a task (Bandura, 1997). Teachers' efficacy beliefs have also been related to student achievement, student motivation, teachers' adoption of innovations, and teachers' management strategies (Woolfolk & Hoy, 1990).

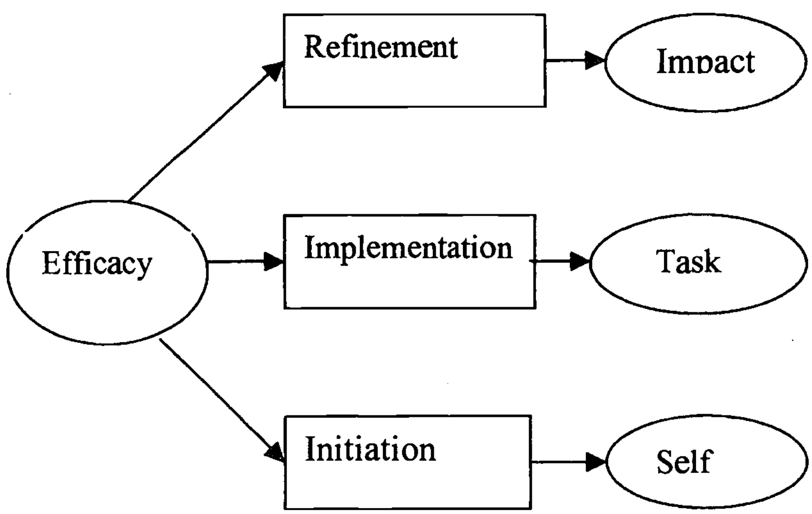

Figure 1: The proposed ECM model

Dempo and Gibson (1985) identified two dimensions of the construct. The "general teaching efficacy" which refers to the belief that teachers in general are able to influence students' learning, and the "personal teaching efficacy" that measures the individual's conviction in his/her own power to control students' motivation and achievement. More recently, Soodak and Podell (1996) supported a third dimension of teacher efficacy and proposed a different interpretation. They suggested that teacher efficacy is comprised of three factors: Personal Efficacy (PE), Outcome Efficacy (OE), and Teaching Efficacy (TE). More specifically, PE refers to teachers' beliefs that they have the skills to bring changes in students' behavior and performance, while OE refers to the belief that, when teachers implement those skills, they can achieve desirable outcomes. The TE factor refers to teachers' beliefs that teaching in general can lead to students' successful performance overcoming influences outside the classroom which affect learning, including children's home environment.

In this study we adopt Soodak and Podell's interpretation of teacher efficacy since it incorporates previous research results (Woolfolk & Hoy, 1990) and it is consistent with Bandura's (1997) differentiation between efficacy expectation and outcome expectations. However, in the present study we only use the PE dimension of teaching efficacy due to space constraints.

AIMS OF THE STUDY

The first aim of the present study was the validation of the ECM as a theoretical model of explaining the effect of teaching efficacy on the development of teachers concerns towards the implementation of mathematics reforms. The second aim is closely related to the first and it refers to the examination of how efficacy beliefs influence the concerns of teachers as they move through stages of implementation. Specifically, the model suggests that as teachers are involved in the teaching with the new mathematics textbooks, they move through the initiation to implementation and refinement stages. In the present study, we assumed that teachers who were involved with the innovation for 0-1 years belong to the initiation stage and thus they merely express self-concerns. Teachers involved with the innovation for 2-3 years belong to the implementation stage, which is related to the task-concerns. Finally, teachers with 4-6 years of involvement belong to the refinement stage and thus they are expected to express impact-concerns.

During the process of innovation, teachers express different concerns, which are influenced by their efficacy beliefs. To date, no other model has incorporated efficacy and teachers' concerns in a comprehensive manner involving causal relationships. The following two hypotheses reflect what we would expect if efficacy affects teachers' concerns about the implementation of new mathematics textbooks :

(a) The fitting of the ECM showing causal effect of efficacy on teachers' concerns to be acceptable, and

(b) A decrease in self-concerns with increases in task and impact-concerns when teachers' scores at the initiation stage are compared with those at implementation and refinement stages. Similarly, we would expect a decrease in task-concerns with increases in impact-concerns when teachers' scores at the implementation stage are compared with those at the refinement stage.

METHOD

Participants and Instrumentation

The participants in this study were 655 teachers (155 male and 500 female teachers) from 100 elementary schools in Cyprus. Schools were selected on the basis of size, location and demographic characteristics. The sample included three groups of teachers covering the initiation, implementation and refinement stages.

Two instruments were used as data sources. Both instruments were administered at the same time to all participants. The first instrument was an adaptation of the SoCQ as described by van den Berg & Ros (1999). The SoCQ questionnaire, which included 36 items, was used to measure the self, task and impact concerns. For the purposes of the present study teachers' rating to the 36 items of the questionnaire were made on a 9-point scale ranging form 1 (strongly disagree) to 9 (strongly agree): all responses were recoded so that higher numbers indicated greater agreement.

The measures of teaching efficacy were obtained through a questionnaire which was used in previous studies in Cyprus (see Christou, Philippou, & Dionysiou, 2001). Respondents used a six point agree/disagree scale to respond to 13 statements which measured TE, 11 which measured PE, and 10 which measured OE. The coding of the negatively stated items was inverted to ensure that high scores meant high efficacy on all items of the scale.

Data Analysis

The assessment of the proposed model was based on a multi-sample mean and covariance structures analysis, which is part of a more general class of approaches called structural equation modelling. One of the most widely used structural equation modelling computer programs, EQS 6.0, was used to test for model fitting in this study. In order to evaluate model fit, three fit indices were computed; the chi-square to its degree of freedom ratio (χ^2/df), the comparative fit index (CFI), and the root mean square error approximation (RMSEA). These three indices recognized that observed values for χ^2/df <2, values for CFI>.9, and RMSEA values close to 0 are needed to support model fit (Marcoulides & Hershberger, 1997).

RESULTS

To test the first hypothesis we posited a model with three factors (self, task and impact concerns), which are causally related to teaching efficacy (see Figure 1). Factor loadings, factor regressions and variable intercepts were constrained to equal across the three groups of teachers (teachers at the initiation, implementation and refinement stages). We tested the ability of a solution based on this structure to fit the data. As reflected by the iterative summary, the solution converged smoothly, and the goodness-of-fit statistics showed that the model had a very good fit to the three-group data (χ^2 = 364.174, df=236, χ^2/df=1.54, CFI=.96, RMSEA=.02). All items loaded strongly and distinctly on each of the factors shown in Figure 1. The standardized loadings of all measures were above .5, and were statistically significant in all groups. The regressions of efficacy on the self (.297), task (.880) and impact concerns (.461) were significant and equal in all groups, indicating that efficacy exerts an important effect on teachers' concerns (see Table 1).

To examine the second hypothesis of the study of whether the latent construct means of self, task, and impact concerns are significantly different in the three groups of teachers, we turned to the construct equations, which are presented in Table 1. The parameters of interest in this case were the factor intercepts that represent the latent mean values. Because the initiation teachers had their parameters fixed to zero for comparison purposes, we concentrated on estimates for the teachers in the implementation and refinement stages. The lower part of Table 1 shows the comparison between teachers in the implementation and refinement stages. To compare these two groups of teachers we fixed the factor means of the implementation group to zero.

Comparison between teachers in initiation and implementation stage
Self = .172V999* + .297Efficacy*
Task=-.218V999+.880Efficacy*
Impact=-.083V999 + .461Efficacy*
Comparison between teachers in initiation and refinement stage
Self = .254V999* + .297Efficacy*
Task=-.143V999 + .880Efficacy*
Impact=-.066V999 + .461Efficacy*
Comparison between teachers in implementation and refinement stage
Self = .297V999* + .297Efficacy*
Task=-.169V999 + .880Efficacy*
Impact=-.085V999 + .461Efficacy*

V999=Latent factor mean
*=statistically significant at .05 level

Table 1: Construct equations of teachers' concerns

Table 1 shows that self-concerns are increased significantly in teachers in the implementation (.172) and refinement stages (.254). This increment means that teachers have less self-concerns when their involvement in the innovation increases. On the contrary, teachers' task and impact concerns seem to increase as they proceed to higher stages of implementation. This is denoted by the negative sign of the latent factor means (-.143, -.169 for task, and -.066 and -.085 for the impact concerns). However, the latter effects were not significant by z-tests. These results replicate partially the results reached by van den Berg and Ross (1989).

DISCUSSION

The purpose of this study was twofold. First, to explore the effect of teaching efficacy on teachers' concerns regarding the implementation of a new mathematics curriculum and the use of new mathematics textbooks; second, to address the extent to which teachers at different stages in the implementation process of the innovation had different concerns in relation to the innovation.

Earlier research indicated that teachers' efficacy beliefs were related to student achievement, student motivation, teachers' adoption of innovations, and teachers' management strategies (Woolfolk, Rossoff, & Hoy, 1990). The data of the present study reinforced previous studies by validating a model in which the relationship between efficacy and concerns is a causal one. The model also showed that teachers' efficacy beliefs affect more the task concerns of teachers than the self or impact concerns (see Table 1).

The differences in concerns across different groups of teachers was the main topic of investigation under the second research question of the present study. In general, research evidence on the development of teaching concerns has yielded mixed results (Ghaith & Shaaban, 1999). The study provided evidence that years of teachers' involvement with the new mathematics curricula and mathematics textbooks explained to an extent the developmental nature of teaching concerns. Teachers′ concerns in the initiation stage appeared to be self-oriented, reaffirming van den Berg, & Ros (1999) conclusion that teachers during the early years of an innovation are absorbed with day-to-day difficulties and pay little attention to the newly encounter problems of the students. Teachers at the initiation stage in this study seemed to focus on the implications of the new mathematics curriculum and textbooks for themselves; they were largely interested in the changes that could occur in their personal work situations, and in the manner in which they could be required to prepare their daily work. In contrast, teachers at the implementation and refinement stages reported less interest in the consequences of the innovation for themselves but they did not report greatest interest in their students and did no have more ideas with regard to the adoption of the innovation in comparison to teachers in the innovation stage. Although this result seems to contradict the conclusions of other research studies (van den Berg & Ros, 1999), it can be explained by the fact that concerns in the present study were analysed in the light of the teaching efficacy effect.

The results highlight the importance of attending to the concerns and experiences of teachers with respect to the new mathematics curricula and textbooks. It is important for policy makers and school leaders to acknowledge and identify the concerns of teachers in order to increase the prospects of success for educational innovations. The differences found in this study between experienced and beginning teachers can be used to inform the planning and implementation of intervention program.

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UPPER ELEMENTARY TEACHERS' MATHEMATICS RELATED ANXIETIES AND THEIR EFFECTS ON THEIR TEACHING*

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This paper is related to a study of grade 5-8 teachers' math-related anxieties and attitudes, their impact on their teaching practices, and how these teachers can be helped in dealing with their anxieties and improving their teaching. Following initial teacher interviews, the teachers will participate in a series of "Math Empowerment" workshops that will utilize a holistic approach, including a lot of math work along with activities for emotional, psychological and social support. This paper reports the findings of the first phase of the study, consisting of initial teacher interviews. Teachers' various types of anxiety are described and illustrated, along with their own schooling experiences that gave rise to these anxieties, as well as their current teaching challenges and coping strategies in dealing with these challenges.

Introduction

It has been established that pre-service, and in-service elementary teachers frequently exhibit math dislike and/or anxiety (Bush, 1990; Hembree, 1990; Trice, A. D., & Ogden, E. D. (1986-1987). Furthermore, there have been numerous claims that these teachers inadvertently pass on their negative attitudes towards math to their students (e.g. Bulham & Young, 1982; Karp, 1991, Martinez, 1987, Widmer & Chavez, 1978) although these claims have not been supported by all studies (Bush, 1991; Swetman, 1994). Recent math reforms have added new dimensions in teachers' math-related fears: the need to teach a new curriculum, including new educational philosophies and approaches to learning and teaching math that teachers may have never been exposed to (Battista, 1992). Province-wide testing has been recently introduced in Ontario, and the threat of teacher testing is looming.

This paper deals with an ongoing project consisting of a qualitative study of a group of 12 upper elementary (grade 5-8) Ontario teachers who experience significant math-related anxieties and feelings of inadequacy. The purpose of the study is twofold: (a) to gain a better understanding of these teachers' math-related anxieties, attitudes and beliefs, and their effects on their math teaching, (b) through conducting a series of workshops for these teachers, to explore a holistic approach to helping these teachers in dealing with their anxieties and improving their math teaching.

This paper reports on some initial findings of the study based on teacher interviews conducted so far. Teachers' math-related anxieties and attitudes, classroom teaching challenges and coping strategies will be described. Since the data collection and initial analysis for the whole study will be completed by the early summer, some findings from the later stages of the project will also be included in the conference presentation.

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*This research was supported in part by a grant from the Imperial Oil Centre for Studies in Science, Mathematics and Technology Education, The Ontario Institute for Studies in Education of the University of Toronto (OISE/UT), Toronto.

Theoretical and Empirical Background

The problem of "math anxiety" came to the fore in the early 1970's (e.g, Fennema, 1974; Tobias, 1976; 1978). Earlier research on gender issues indicated that females are more likely to develop math anxiety than males (e.g. . Fennema & Sherman, 1976; Tobias, 1976; Zaslavsky, 1994). Moreover, math anxiety was shown to relate inversely to math performance and bound directly to avoidance of the subject (Hembree, 1990; Ho, et al., 2000. It was also shown that students who were exposed to traditional math instruction reported more math anxiety than those who were exposed to alternative emphasizing problem solving and discussion of students' own informal strategies (Newstead, 1995; 1998). Math anxiety was often considered as a subject-specific manifestation of the test anxiety construct. Since math anxiety lacked its own theoretical construct, researchers applied models of test anxiety theory to math anxiety (Humbree, 1990; Bandalos, Yates & Thorndike-Christ, 1995; Ho & al. 2000). More recently, however, researchers identified other components of math anxiety besides test anxiety, such as problem solving anxiety, abstraction anxiety, or fear of the public aspects of doing math in the classroom (Newstead, 1998; McLeod,1992; Hembrees, 1990; Brown & Gray, 1992).

There has also been much research on other aspects of affect, such as attitudes toward, and beliefs about math (e.g. Thompson, 1992; Battista, 1994; Ma and Kishor, 1997;a Gierl & Bisanz, 1994; Hannula, 1998; McLeod, 1992). A number of researchers have been exploring the whole array of affect in mathematics, including attitudes, beliefs and emotions (DeBellis & Goldin, 1997; McLeod, 1989). Particularly, in recent years there has been discussion of the unstable nature of some of these constructs and their dependence on the context in which they are observed. For instance, Tirosh (1993) uses the term: "specific affects" to describe students' emotions, attitudes and beliefs as they react to specific mathematical situations, activities or topics. With regards to attitudes, it has been suggested that they are unstable and unreliable and are strongly influenced by the specific conditions in which they are observed or perceived (Gellert, 2001, Martino & Zan, 2001; Ruffell, Mason & Allen, 1998). The approach used in this study is based on the above distinctions and we will also deal with particular manifestations of affect as they arise in specific situations.

Methodology

This is a qualitative case study of the group of teachers that will participate in the workshop series. We also hope to conduct individual case studies with two of the teacher participants. Data collection has already started with the teacher interviews, and will continue throughout the duration of the workshops and in the final teacher interviews following the workshops. Participants consist of 12 grade 5-8 classroom teachers who have self-identified as having math related anxieties.

Following the initial interviews (half which have already taken place), these teachers will participate in a series of seven "math empowerment" workshops during February and March of 2002. The workshops will utilize a holistic approach, including a focus on math learning and teaching approaches, combined with activities for emotional, psychological and social support. Constructivist principles (Confrey, 1990) and the NCTM Principles and Standards (2000) will guide the math learning component. Workshops will include small group work on various math tasks and problems, sharing and discussion, along with group reflection, journal writing, guided visualization and relaxation activities (Tobias, 1978; 1989; Eisenberg, 1992; Martinez, 1987; Zaslavsky, 1994). The workshops will provide a safe and supportive environment to help participants feel secure and free to take risks. Participants will be also able to interact via email or phone with each other and with the researcher at any time in-between workshops, so they can feel fully supported in-between workshops..

Data Collection: Besides the initial and final interviews, data will also be collected during the workshops and will include:(i) Samples of participants' work on math tasks during the workshops;(ii) Samples of participants' reflections written during the workshops; (iii) Transcripts of 5-6 recorded discussions involving participants working in small groups; (iv) Questionnaires to be administered three times during the workshop series; and (v) Field notes taken by the researcher and research assistant during workshops and interviews.

Initial Findings and Discussion

The findings reported here are based on transcripts of the interviews with six of the teachers. The interviews were semi-structured and lasted 40-65 minutes. The findings will be discussed according to the main themes that arose in the interviews.

Participants' Difficulties During Their Own Schooling: In one of the interview questions participants were asked to recall their elementary and secondary school experiences in math. Four of them experienced problems with math in elementary school, and all of them had problems in high-school. When asked about how she felt about math in elementary school, Sarah, a young grade 6 teacher, talked about the many tears she had shed about math during that period, and described her difficulties and how she came to hate math:

- Do you remember if you had any specific difficulties when learning math?
- *Um, Yeah, multiplication especially. It was really difficult for me, all through public school and even into high school.*
- When would that have started?
- *Grade 4! [big laugh from Sarah]. And I, I really remember it. And it wasn't until actually I started teaching multiplication when somebody told me it was repeated addition, I never understood what I was doing.*
- Sarah!
- *And, all the way through public school I remember I hated it. And I have memories of doing "mental math." I always hated that. And when teachers go around, and they ask you, you do it in order?*
- Yes.
- *And I would sit going, "Oh, please, what question?" and I would there and count backwards to see which one I was going to get.*

- Oh my!
- *And I remember I finally got one right one day, and my teacher went,* ***"Look everybody, Sarah FINALLY got one right!"*** *And it was that comment that threw me off for the rest of my life. I hated it since then.*

For Sarah, the above incident has made a lasting impact and set the stage for the development of her math anxiety.

Participants' Math Teachers' Teaching Styles: When asked about the teaching styles of their teachers in both elementary and secondary school, participants described them as very traditional and textbook-based. One participant talked about her frustration when she wanted to know *why* things were to be done in a certain way, but nobody could tell her:

I remember them placing more emphasis on skills and procedures. That's what I remember. If they did something other, you know, I don't want to take credit away from them because they may have, but I remember the procedures and the skills being very specific, our homework being very specific in that way – you have to do this and this in order to get that answer. And the problem that I had at the time when I had homework, I would often ask my brother to help me, I have an older brother, and the thing that I always asked was "Why?" I remember this very clearly, as I was asking him, "Well why do you do that?" And he would say "This then this, make this." "Why?" And nobody answered why and it bothered me. I wanted to know why because it made no sense to me. That this and this would equal, that.

Another young participant stated that the teachers' 'drill and kill' approach made math a boring and meaningless subject for her. This is how she kept herself busy during math lessons:

- *...But, especially Gr. 11 I wasn't with it at all. Gr. 10, I spaced out in class. I actually, I used to write songs about math. [laughter]*
- Do you have any of them still?
- *I do.*
- You'll have to bring them to the workshops.
- *I have this whole series – The Pythagorean Song Series.*
- Really?
- *Oh yeah, about like, "Pythagorus, I love your root hypotonous..." I made like whole songs and poems and stuff. [laughter] That's what I did in math.*

Not surprisingly, research has shown that teachers who identify their own math teachers as having stressed computation and drill are more likely to be math anxious than teachers whose math teachers had stressed understanding (Carroll, Widmer and Chavez, 1979; Hembree, 1990).

Participants' Teaching Styles and Classroom Challenges

The teachers had a lot to say about how their limited math knowledge impaired their teaching and expressed concern about not giving their students the right skills they would need later. A grade 7 teacher who is also a music specialist said:

- *...I just teach from the textbook and I don't do that in any other subject... but I just feel like.. I don't know it so I guess I just decided to do it, exactly following the prescription...*
- That's quite common.
- *Yeah...It's just like playing a musical instrument, I understand that you need to read the notes, you need your fingering, you need the sound of it in your head, so you need all these different components so then I can break it down and teach each of these components... With the math, I just don't know what to teach first, I don't know what the smallest stuff is.. so I just follow the book..*

Another participant reflects on the impact of her negative school experiences on her teaching:

- After having described your math history, can you sort of pinpoint where your anxiety comes from?
- *Yes, I almost think it's lack of support during my whole math career...because....I think it's not being able to extend my knowledge of math.... I have the basics but I can't extend it..*
- Do you feel you are stunted somehow?
- *In some ways, yeah, .. I mean, ..you know, even in my own classroom in teaching math I find it difficult to bring them beyond the basics....how do you extend that knowledge .... and some of these kids have some pretty interesting questions that I'm stuck because I have the basic knowledge but find it difficult to extend it...*

As is typical for math anxious people, the two teachers cited above are unable to experience any creativity when it comes to math. They seem to lack the relational (or conceptual) understanding required in order to be creative. (Skemp, 1978)

Teachers' Coping Strategies:

The same teacher cited above then went on to describe her ways of coping with her teaching challenges:

- *... I now have to do a lot of extra work for myself; in order to walk into the math class prepared I practically do the homework myself in order to make sure I know exactly ..*
- How much time you spend preparing for a math class?
- *Probably around a good hour for each math class, plus marking, ...so that I make sure that I can answer even their basic questions...and you know, I have some pretty bright students in my class.. but you know, I also depend on a lot of the bright students in my class because they often see, ...and are able to explain things sometimes better than I can...or in a different way, at the kids level...*

Interestingly, over-preparing for lessons and doing all the exercises ahead of time has been mentioned in the literature as a strategy that helped teachers compensate for their anxieties (Martinez, 1987).

All of the interviewees discussed their worries about being asked difficult questions in class that they were unable to answer, or being asked to solve a new problem unlike the ones in the text. In the excerpt above the teacher describes a common coping strategy for dealing with such situations: asking for help from the bright students in the class. A couple of other participants reported using similar strategies and were able to ask for help from their brightest students. But some of the participants had a problem with admitting to their class they did not know how to solve a problem. One of the teachers stated that "*I am the teacher, so I am expected to know the answer*". He would not admit to his grade 8 class that he did not know the solution to a problem and found other ways of getting by. Sometimes he would pretend he knew the solution but asked one or two of the bright students to solve the problem on the blackboard. At other times he would shift to another activity and seek help from his colleagues after class.

<u>Various Types of Math-Related Anxiety</u>:

As seen above, these teachers reported a number of different types of math-related anxieties, from the "classical" math anxiety (or "mathophobia" defined by the early researchers (Lazarus, 1974; Tobias, 1978) to more specific anxieties in relation to specific topics, activities or teaching situations. By far the most dreaded teaching situation for all participants was being asked a difficult question in class that they could not answer. Teachers devised a variety of coping strategies for dealing with such occurrences, as was discussed above.

When asked which math strands or topics they felt the most apprehensive about, the teachers varied quite a bit in their responses. Some mentioned geometry as a difficult topic, while others were pleased to report they found geometry easy because they could "see it". Measurement was considered "manageable" by most of them. Yet nearly all of the teachers mentioned algebra as a topic they were most apprehensive about, obviously because of its abstract nature, and had difficulty explaining it to their students. *Abstraction anxiety* has been a focus of some research (e.g. Brown & Gray, 1992).

Problem solving anxiety (McLeod & Adams, 1989) was by far the most dominant within this group and manifested in participants' fear of being asked to solve an unfamiliar problem in front of their class. It was also the reason why some participants reported solving all the relevant textbook problems prior to class. The last question in the interview was aimed at assessing teachers' degree of problem solving anxiety and whether they would exhibit resourcefulness and optimism, or feel helpless and hopeless, when faced with a non-routine math problem. The question asked:

- Suppose I were to give you an open-ended, non-traditional math problem to solve right now. What would be your immediate reaction? How would you feel about being given this problem to solve? – (Probe as necessary: identify emotion, identify bodily reactions) - Would you attempt to solve the problem?

Here is a grade 7 teacher's response to this question:

- Suppose I were to give you an open-ended, non-routine math problem… (teacher tightens up in her chair, looking very nervous)…. Just SUPPOSE I were to give you a math problem, I am NOT actually giving you one! - How would you react? - What would you do?
- *Just you saying that is making my stomach go in a big knot… Oh, I just hope she's not going to ask me a math problem…*
- Would you try to solve it?
- *No way!* (appears to be frightened)

This response demonstrates this teacher's *learned helplessness* when it comes to math problem solving. Learned helplessness happens when a person has experienced repeated failures in the past and believes that whatever their effort in the future, the outcome for them will not change. It's a feeling of loss of control. (Gentile and Monaco, 1986).

There was one more participant who exhibited learned helplessness when answering this question. The other four participants replied that they might feel somewhat uneasy but would go ahead and try to solve the problem. It could very well be that some of the more severe cases of math anxiety are indeed related to learned helplessness. This phenomenon deserves further study.

In conclusion, this was an initial report on some of the early findings in our project. We are looking forward to starting the workshop series where we hope to gain a much deeper and richer understanding of the participants' cognitive and affective responses to doing math tasks and their progress in trying to deal with their math anxieties.

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FUNCTION MACHINES & FLEXIBLE ALGEBRAIC THOUGHT

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We explore how college students understand ideas of functions, and which representations are productive for them in promoting their ability to work flexibly across representations. We use pre- and post-test scores, and triangulations via student self evaluations, to generate a hypothesis related to flexible thinking and success in algebra. We use confidence intervals to provide evidence for a highly significant change in student flexibility in algebraic thinking, and to assist in generating a plausible model of how the use of function machines in a developmental algebra course is instrumental in stimulating that flexibility.

INTRODUCTION

Students who present in developmental algebra, in universities and two-year colleges, commonly see mathematics as consisting solely of procedures and formulas. Unfortunately this vision of mathematics does not generally assist them in carrying out the procedures accurately, efficiently or appropriately, or in getting the formulas right in context. The unifying idea of a function is generally not something with which developmental algebra students are familiar or comfortable. Further their mathematical thinking, focussed principally on procedures and "correct" formulas, is largely inflexible and highly context specific. We relate the ideas of functions as a unifying concept in algebra, and flexibility of algebraic thought, through the use of function machines as a generic representation of functions with power to assist students in becoming more flexible and competent in their algebraic thinking.

Functions and function machines

Students often rely on intuitive and unreflective ideas of functional relationships. The subtlety of the function concept with its process-object duality and various representations proves to be highly complex. It is a concept, with wide-ranging powers and with widespread misunderstandings, that has been studied extensively in recent years (see, for example, Markovits, Eylon & Bruckheimer, 1986; Vinner & Dreyfus, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Harel & Dubinsky, 1992; Cooney & Wilson, 1993; Even, 1993; Cuoco, 1994; Thompson, 1994; Wilson & Krapfl, 1994; DeMarois & Tall, 1999; Lloyd & Wilson, 1998). Both teachers' and students' conceptions of functions are of enduring interest in mathematics education, because of the fundamental organizing and analytic role in mathematics played by modern reflective notions of functions. Various representations of function (table, graph, algebraic, for example) may be seen as ways of representing calculation of an input-output relationship. Functions viewed as input-output machines were studied in mathematics education as far back as 1965 by Peter Braunfeld. Tall, McGowen & DeMarois (2000) proposed the function machine box as a generic image that can act as a cognitive root, embodying the salient features of the idea of function, including process (input-output) and object, with various representations seen as methods of controlling input-output. The function machine embodies both an object-like status and the process aspect from input to output. Recent studies (McGowen, DeMarois & Tall, 2000; McGowen & Tall, 1999) indicate that the introduction of the function machine as an input-output box enables students to have a mental image of a box that can be used to describe and name various processes. This can take place often without the necessity of having an explicit process defined. Other forms of representation may be seen as mechanisms that allow an assignment to be made - by a table, by reading a graph, by using a formula, or by some other assignment procedure.

Flexibility

Krutetskii characterized flexible thinking as reversibility: the establishment of two-way relationships indicated by an ability to "make the transition from a 'direct' association to its corresponding 'reverse' association" (Krutetskii, 1969, p. 50). Gray and Tall (1994) characterize flexible thinking in terms of an ability to move between interpreting notation as a process to do something (procedural) and as an object to think with and about (conceptual), depending upon the context. In this article, flexibility of thought encompasses both Krutetskii's and Gray & Tall's ideas as facets of a broader notion of flexibility. In addition to reversible associations, and proceptual thinking, we are interested in connections between various representations of functions, including tables, graphs and algebraic syntax, which we refer to as conceptual. The proceptual divide (Gray & Tall, 1994) is, in a broader sense, part of a conceptual divide in which flexibility is compounded by student difficulties in using and translating among various representational forms.

The theoretical background on flexibility suggested we pair questions so that a student scoring correct on both of a pair was evidence of flexible thinking for that student. For example, for a simple function f we might ask: (a) what is f(5) ? and also ask (b) for what value or values of x is f(x) = 0? Essentially this is the difference between evaluating a function and solving for a value. 12 of a total of 31 questions were organized in pairs - 6 pairs in all. Question pairs addressed differing representations of functions: syntactic formulas, tables, graphs, and function machines. Success in both questions of a given pair was regarded as an indication of flexibility of algebraic thinking with respect to the representation for that question pair. Question pair 8 & 9 (Q8. Given $f(x) = x^2 - 5x + 3$, find f(-3); Q9. Given $f(x) = x^2 - 5x + 3$, find f(t-2)) relates to flexibility in that a student who can answer both questions correctly can substitute a number for a variable in an algebraic expression, and can also generalize the substitution from numbers to the situation of another algebraic expression in a different variable. Question pairs 10 & 11, 12 & 13, 14 & 15, and 16 & 17 are all concerned with evaluation versus solving. Success on both questions for any one of these question pairs indicates flexible algebraic thinking in the sense of Krutetskii. Details of question pairs 14 & 15 and 16 & 17 are given below in the results.

METHOD

Data were collected from a cohort of students taking a common 16 week course in developmental algebra at a two-year college. Data were of two sorts: pre-test and post-test scores (weeks 1 and 16 respectively; identical questions), and student written self-evaluations, turned in by week 16. Of the 135 students initially enrolled in the course, 87 students completed the course and were taught in 6 sections by 4 different instructors. The text for the course (DeMarois *et al*, 2001) emphasized the use of function machines, as well as syntactic algebraic formulas, tables, graphs and finite differences. However, function machines are not mentioned explicitly after chapter 3. Instructions given to students for their self-evaluations included: "What mathematics have you learned during this time? Write a summary of what you have learned." Students were asked to cite specific examples of mathematical growth, and to place their mathematical understanding, knowledge and skills competencies, in the context of Clarke, Helme & Kessel's (1996) criteria for meaningful learning in mathematics. They were not explicitly or specifically asked to address function machine or other representations for functions.

RESULTS AND ANALYSIS

Students' pre- and post-test responses were significantly different. We present results for the set of linked pairs of questions: 6 pairs, comprising 12 questions of the complete set of 31. The inter-item reliability (Cronbach's alpha) of the linked pairs of questions on the pre-test was 0.43 and on the post-test 0.88 (identical questions on the pre- and post-test). Since the questions did not change from pre-test to post-test, we conclude that this cohort of students saw the questions as having greater internal consistency at the end of the course than at the beginning. The correlation between pre-test and post-test scores for the linked pairs of questions was small and statistically significant ($r^2 = 0.04$, $p < 0.05$). This indicates no linear relationship between pre-test and post-test scores on the paired questions: students did not increase uniformly and proportionately from pre-test to post-test.

A confidence interval analysis for the paired two-sample difference in proportions, between the pre- and post-tests, of students who correctly answered both of a linked pair of questions yielded the results shown in table 1, below. Note that for the first linked pair (questions 8 & 9) one section comprising 12 students was not available for analysis, so the data there comes from 75, rather than 87, students.

The differences in proportions are all significant at the 99% level: no 99% confidence interval for the difference in means contains 0. In the context of the other question pairs, the least significant difference in proportions correct on both of a linked question pair is that for questions 14 & 15:

Questions 14 & 15: A function table copied from a TI-82 graphics calculator is shown:

Question 14: What are the output(s) if the input is -2?

Question 15: What are the input(s) if the output is -3?

The most significant difference in proportions correct on both of a linked question pair is that for the linked pair of questions 16 and 17, which are as follows:

Questions 16 & 17: Consider the equation $y = 3x - 7$.

Question 16. What are the output(s) if the input is 5?

Question 17. What are the input(s) if the output is 0?

Question pair	95% CI	99% CI
8 & 9	[0.26, 0.49]	[0.22, 0.52]
10 & 11	[0.21, 0.48]	[0.16, 0.52]
12 & 13	[0.20, 0.44]	[0.16, 0.47]
14 & 15	[0.05, 0.27]	[0.01, 0.31]
16 & 17	[0.57, 0.77]	[0.52, 0.71]
18 & 19	[0.09, 0.25]	[0.06, 0.29]

Table 1: 95% & 99% confidence intervals for differences in proportions pre- to post-test, of students getting both question pairs correct. Note that none of the confidence intervals contains the value 0 and that the confidence intervals show effect size.

The confidence intervals, at varying confidence levels, and the relative effect sizes can be seen in a visually striking way via confidence bands. These consist of a plot of the confidence interval, for the difference in proportions, versus the specified confidence level. In Figure 1, below, we illustrate these confidence bands, for confidence levels between 0.95 and 0.99, for the question pairs 14 & 15 and 16 & 17.

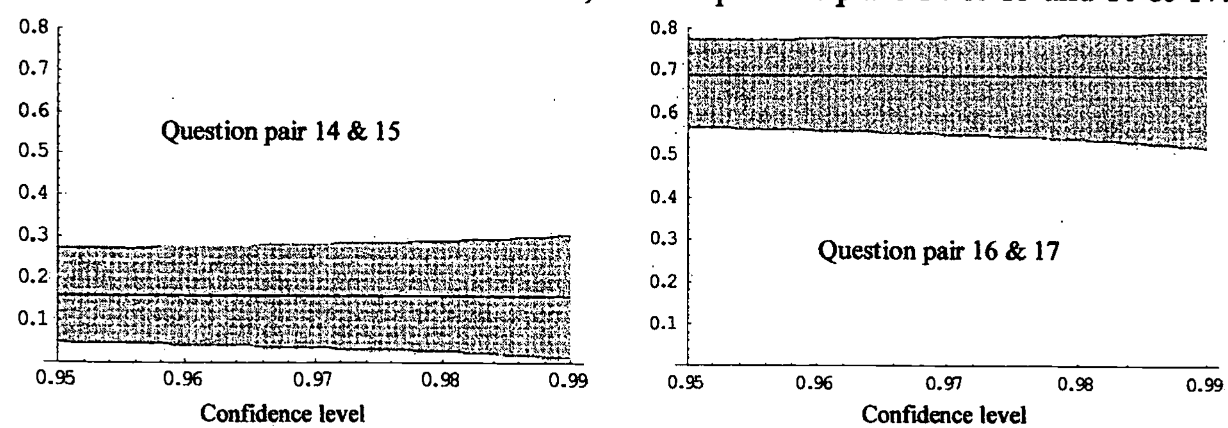

Figure 1. Confidence bands: confidence intervals for a difference in proportions, pre-test to post-test (vertical axes) versus confidence level (horizontal axes). The straight horizontal lines show the difference in proportions for the sample of 87 students

Student written self-evaluations

Student written self-evaluations indicated that many feel a function machine model assists them to make sense of notation, to organize their thinking, and to produce equations to describe data. References to input and output occurred in the work and interviews throughout the semester of students who were successful. They used the function machine notion to organize their thinking as they worked problems and interpreted notation. In contrast to the more successful students, the least successful students made very few references to function machines in their work or in the vocabulary they used. The least successful students demonstrated little or no improvement in their ability to think flexibly. Students were deemed "least successful" by a combination of post-test, final examination, and final portfolio grades - all of which form the components of a student's final grade for the course. Usually the top 15% and the bottom 15% of the class, after final course grades are determined, are examined to identify the most and least successful students, given the 15% criteria. Given class sizes, this is generally the top 3 to 4 and bottom 3 to 4 students in any given class.

Excerpts from self-evaluations for some of the more successful students are provided below.

> Student A: I know that function machines are good models for mathematical relationships when one wants to clearly identify and separate the input, output and process. I also can look at function notation, f(x), and understand what it is stating: That f is a function of x. I see the input (x), the output f(x), and the process, when in a relationship form like $f(x) = 3x + 10$.... I know now that 'solving' can refer to many different things including: 1) solving for an unknown variable; 2) solving a system of equations (where the given functions share the same input and output values); and 3) solving an equation (finding the input value(s) that produce an output of 0).

> Student B: Function machines are a great way to visualize the process of inputting, finding # processing, and finding the output.... Before I never knew there was a difference between evaluating, simplifying, and solving an equation. We EVALUATE the function to find the output of a function when the input is given. After we evaluate, we simplify using various mathematical properties such as associative, distributive, and identify. We use order of operations to obtain the output or the answer. We SOLVE to find the input, given the output.... Numerically, a linear relationship is the change between two outputs divided by the change between corresponding inputs that always produce a constant slope.

> Student C: I now have the skills to interpret and use mathematical notations appropriately, reflected in my work and ability to interpret a function machine; convert from one type of mathematical notation to another, convert a function machine to an equation/equation to function machine.... I have identified inputs and outputs, which was the key to answering the questions, appropriately and accurately in my tests on questions involving: recognizing the given slope to make a table, showing input/output;

identification of a given notation and breaking it down by input/output; naming input/output from a given function machine or equation; finding output if given input/finding input if given output.... I am knowledgeable in understanding the difference between evaluating and solving: solving for X; evaluating for Y.... I have learned more about equations and functions during the past 4 months than I have ever learned in my lifetime.

Student D: I had no idea what function is and what a relation is, but now I do. Relation is mathematical notation with at least two variables. Independent one is input. Dependent is output. I know that relation, which has only one output for each given input is a function. If I see f(8) = x + 7, I know I am given an input. f(x) = 8 gives me output. I can work with ordered pairs. In case (8, 45), I see 8 as input and 45 as output.

CONCLUSION

The major lesson to be learned from our analysis of the data collected on the pre- and post-tests and student self-evaluations is that some factor (or factors) across 6 classes, with 87 students and 4 instructors, is associated with a dramatic change in flexibility of algebraic thinking at the whole class level. An emphasis on functions as machines is the novel element that is likely to assist students to form mental images to assist them in the interpretation and use of syntactic algebra. How far and wide, and how strong across ability ranges, this effect might be we do not yet know. The evidence we have presented suggests that the representation of functions as function machines assists students' understanding of or development in algebra, and provides them with models that will require no significant re-shaping and modification in further mathematical studies.

The average increase in syntactic skills over the 16 week course was very significant. The linked question pair showing the highest gain consisted of syntactic algebra questions 16 & 17, asking about input and output for the function f(x) = 3x – 7. The mean gain, across the cohort of 87 students, for the average number of correct answers to both these questions, from pre-test to post-test, was a very high 0.82 (ref. Hake, 1998). This is very strong evidence that students were flexible, in the sense of Krutetskii, in interpreting this type of syntactic algebra question by the end of the course, compared with a general inflexibility at the beginning.

A function machine representation was used throughout the text (DeMarois *et al*, 2001) and the course as a means of helping students make sense of notation and to understand processes of "solving an equation" versus "evaluating a function". The pre-post tests show a statistically highly significant increase in students' ability to use algebraic notation. This, taken together with student self-evaluations, suggests strongly, but does not prove, that a function machine representation provides students with a way of organizing their thinking and with a more process-oriented view of mathematics.

This approach may not be appropriate for all students at all levels. However, for students who have had algebra previously and place into a developmental algebra courses, starting with a process notion of function and a function machine representation seems to help them make sense of notation. Such students believe they need to be able to manipulate symbols successfully and, as a consequence of this belief, focus on being able to "do algebra": that is, manipulate syntactic expressions. A function machine representation provides them with a potential means to organize their thinking so they can be more successful in dealing with notation than in their past experiences with algebra.

Student work and writings are replete with references to "input" and "output" in their concept maps, their explanations, and in their self-evaluations - indications that introduction and use of function machines impacted their thinking and provided an organizing tool or principle throughout the semester.

We hypothesize that use of a function machine representation of functions correlates with a student's ability to work flexibly across syntactic, graphical, and tabular representations. More specifically, we hypothesize that:

(1) a student's ability to form mental images of functions as machines;

(2) use of a graphing calculator as a concrete manifestation of a function machine; and

(3) use of finite differences and finite ratios to make sense of parameters and to help students have some understanding of where equations come from, given data;

are jointly highly and significantly correlated with their ability to utilize syntactic algebraic representations of functions and, to some extent, correlated with an ability to utilize graphical and tabular representations of functions. The pre- and post-test data we have presented, together with student self-evaluations, provides a strong foundation for this hypothesis. The class of students we studied dramatically increased syntactic algebraic skills over a period of 16 weeks. Many of the students stated explicitly this was due to understanding functions as processes, through a representation as function machines, particularly the ability to form images of function machines, and through the use of finite differences in conjunction with function machine representations.

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IMPLICATION AND MATHEMATICAL REASONING

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Implication is Omnipresent as a tool in mathematics. However this concept is neither clear nor easy. In this paper, we present a didactic analysis of implication under three points of view : sets, formal logic, deduction reasoning. For this study, our hypothesis is that most of the difficulties and mistakes, as well in the use of implication as in its understanding, are due to the lack of links in education between those three points of view. Then, we will show, thanks to the analysis of a problem from our experimentations, how the sets point of view can be implied in geometry, even with few knowledge.

INTRODUCTION

The implication seems to be a transverse mathematical object. Although it is in the heart of any mathematical activity, since it is essential for the formulation of proof, it does not have a definite place in French teaching, and is hardly ever taught.

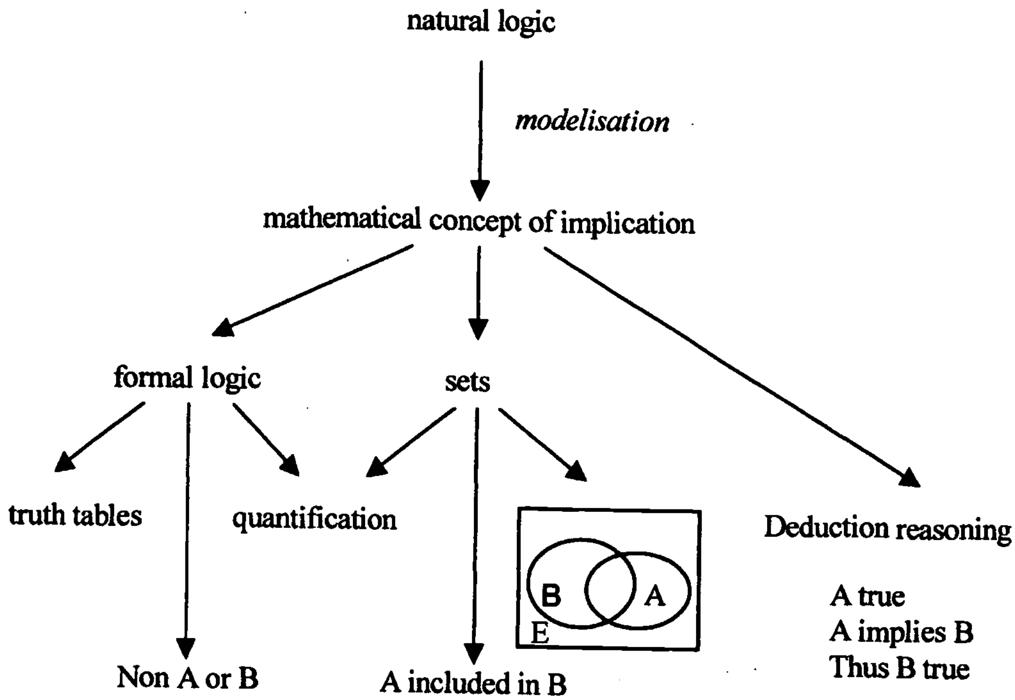
Moreover, the existence of the implication as an object of natural logic, leads to confuse it with the mathematical object. As a result, the implication seems to be a clear object. Yet, students have difficulties related to this concept until the end of university, especially with regard to necessary conditions and sufficient conditions.

Our theorical framework is placed in the theory of french didactics, in particular, we use the tools of Vergnaud's conceptuals fields theory and those of Brousseau's didactical situations theory. Our study is based on the work of V. Durand-Guerrier [Durand-Guerrier, 1999] on the one hand and of J. Rolland [Rolland, 1998] on the other hand. V. Durand-Guerrier shows, in particular, the importance of the contingent statements for the comprehension of the implication. J. Rolland, as for him, was interested in the distinction between sufficient condition and necessary condition.

We will present three points of view on the implication and their place in French teaching. Then we will show, on some examples, the effects of a causal conception of the implication. Lastly, we will study a problem of geometry taken from an activity tested on beginner teachers of mathematics during year 2001.

THREE POINTS OF VIEW ON THE IMPLICATION

The notion of implication does exist in natural logic as it is necessary to our everyday life. The mathematical implication then seems a model of the natural logic implication. Like any model, this mathematical concept is faithful from certain angles to that of natural logic but not from others. This distance between the mathematical concept and the object of our everyday life leads to obstacles in the use of the mathematical concept. An epistemological analysis [Deloustal, 2000] enabled us to distinguish three points of view on the implication : formal logic point of view, deductive reasoning point of view, sets point of view.

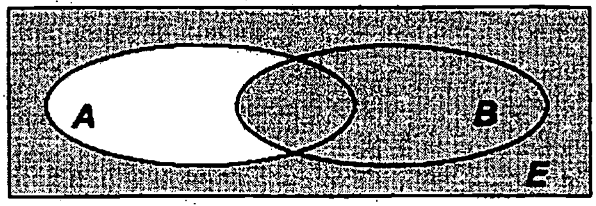


Of course, these three points of view are linked and their intersections are not empty. We will not develop here the formal logic point of view (for example truth tables or formal writing of the implication).

We call "deductive reasoning" the structure of an inference step : "A is true ; A implies B is true ; Thus B is true". Its ternary structure includes a premise "A is true", the reference to an established knowledge "$A \Rightarrow B$" and a conclusion "B is true" [Duval, 1993, p 44]. The reference statement may be a theorem, a property, a definition, etc. One thus builds a chain of inference steps : the proposition obtained as the conclusion of a given step is "recycled" as the entrance proposition of the following step. Therefore, in the deductive reasoning, the implication object is used only as a tool. However, in French secondary education, where this point of view is the only one, it often acts as a definition for the implication.

Generally speaking, having a sets point of view, means to consider that properties define sets of objects : to each property corresponds a set, the set of the objects which satisfy this property.

The sets point of view on the implication can then be expressed as follows : in the set $\boldsymbol{E}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are respectively the set of objects satisfying the property A and the set of objects satisfying the property B. Then, the implication of B by A (i.e. $A \Rightarrow B$) is satisfied by all the objects of the set *E* excluded those which are in *A* without being in *B*, i.e. by all the objects located in the area shaded below.



OBSERVATIONS IN FRENCH TEACHING

The definitions of the implication or of the associated terms[1] are hardly ever found in in school textbooks. They appear in some first years of university textbooks and in some new highschool textbooks (syllabus 2000).

There is a compartmentalization of these points of view in French school textbooks, no link is established. Whereas the ensemblist point of view is completely missing (only the sentence "A included in B if for every X, $X \in A \Rightarrow X \in B$" can be found in some university textbooks), and whereas the logical point of view appears only in some university textbooks, the deductive reasoning is dominating particularly in the secondary school where it acts as a definition.

Secondary school textbooks do not assume the definition of the implication, which is identified with the natural logic object.

> If... then: "standard" expression which tends to explain that if a property is satisfied, one can deduce from it that a second one is also. [in Mathématiques seconde, collection Pyramide, éd. Hachette éducation, 2000]

> An implication is a mathematical sentence indicating that a data (1) involves or implies a conclusion (2) [in Mathématiques seconde, Déclic, éd. Hachette, 2000]

Many definitions of the implication, within the register of the deductive reasoning, connote an idea of causality and even temporality[2]: "One has Q as soon as one has P" ; "If P is true then Q is true". This causal aspect is stengthened by the definition of the demonstration in school textbooks. Indeed, this one is presented like a succession of sentences, connected by theorems, properties or definitions, leading from the hypothesis to the conclusion.

> To prove that the statement "P implies Q" is true, is to prove that, on the basis of the hypothesis P is true, one reaches, by observing rules of calculation, theorems, definitions, the conclusion Q is true. [in Mathématiques seconde, IREM de Poitiers, éd. Bréal, 2000]

CAUSAL CONCEPTION OF THE IMPLICATION

A conception is "a set of rules, practices, knowledge which make it possible to solve a class of situations and problems in a satisfactory way, whereas there is another class of situations where this conception fails, either that it suggests false answers, or that the results are obtained with difficulty and under adverse conditions." [according to Brousseau, 1997]

We understand by "causal conception of the implication" all the rules, practices and knowledges related to the interpretation of the sentence "A implies B" by "A is the cause of B". This conception of the implication is obviously very close to natural logic and its validity field[3] is wide, it includes, in particular, all usual problems requiring a deductive reasoning. As we showed in the preceding paragraph, this conception is strengthened by teaching practices, but leads to inconsistencies in the use of the mathematical concept. Indeed, from this interpretation, one can easily derive the interpretation "A is the cause of B and only A" then the interpretation "A is the cause of B, A thus precedes B" which we will call "temporal conception of the implication". This last interpretation is reasonable within natural logic since in the physical universe, the cause precedes the effect ! However, it leads to a paradox in the use of the mathematical implication : if $A \Rightarrow B$ is translated by "A is the cause of B and thus A is before B", how to accept that B is a necessary condition for A ?

Here are some examples from our experimentations[4] which illustrate errors that one can explain by the causal conception. They are Sarah's answers, Sarah who studies for the competitive examination to be a teacher in secondary education.

Give the negation of $P \Rightarrow Q$[5]

Sarah: Q can exist without P existing

Are there implications between these expressions "M is a necessary condition for T" and "T is a sufficient condition for M"[6] ?

Sarah: There are no implications between those expressions because T requires M and having T is sufficient for having M. [...] For me "M necessary condition for T", that means that necessarily one must have M to have T. Thus M implies T. In the second one, having T is sufficient for having M, therefore T implies M.

The "causal conception of the implication" model may explain many errors, in particular those due to the implication "which is not in the right way". Experimentations at university showed that, contrary to a widespread idea, a logic lecture is not enough to get rid of this conception and of the errors which result from it.

RESEARCH HYPOTHESIS

The experiments carried out for three years, within the framework of our research, have shown that the implication was not a clear object even for beginner teachers and that the difficulties were primarily due to a causal conception of the implication. Yet this causal conception is not only present in natural logic, but it is also strengthened by French teaching practices. Lastly, this conception may "live" in spite of a logic lecture. Following these comments, we formulate the research hypothesis :

it is necessary to know and establish links between these three points of view on the implication for a good apprehension and a correct use of it.

In the following paragraph we show that a problem of geometry, using only easy properties, may question the reasoning in a non obvious way and allow a work on the implication under the sets point of view.

STUDY OF A QUESTION OF THE GEOMETRY-PROBLEM

Let ABCD be a quadrilateral with two opposite sides with the same length. What conditions must diagonals satisfy to have : two other parallel sides (P1)[7] ?

In an usual French secondary education problem, the implication is generally in the way "A $\Rightarrow$ B" where A and B are known. To solve the problem, one makes then use of a deductive reasoning in which A is considered as true. This implication usually takes place in a specific class of objects, for example quadrilaterals, triangles or parallelograms... One considers, in fact, the implication : "in ***H***, A $\Rightarrow$ B", but ***H*** is implicit when the corresponding class is "institutionalized", that is to say very well known and used, like in teaching. Indeed, if the implication is in the parallelograms class, to solve the problem one uses properties of the parallelograms implicitly (for example, convexity), without expliciting this restriction.

The search for sufficient conditions follows the model "in ***H***, (A?) $\Rightarrow$ B", i.e. what is the property A that is enough for the objects of the class ***H*** to satisfy, if they are to satisfy the property B as well. This is hardly ever practised in French teaching.

Our problem suggests yet another approach. The hypothesis "to have two same-lenghted opposite sides" does not, here, represent a class of "institutionalized" objects. This hypothesis must thus remain explicitly present during the resolution. Since the class ***H*** does not exist as such in teaching, it is necessary to come back to the associated property. There is thus, on the one hand, H which one knows and A that one does not know and, on the other hand, B that one knows. Between them, there is an implication whose direction is not given since we did not specify if the requested conditions were necessary or sufficient.

Let us present, now, three approaches which may induce different solving strategies.

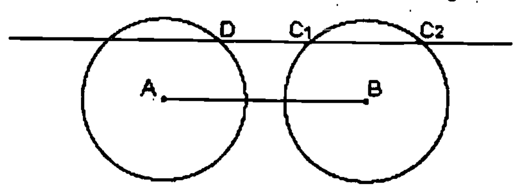
The first approach raises the question of sufficient conditions. One may list conditions on diagonals (same length, perpendicular) and then check if these conditions, added with the hypothesis H, imply the conclusion B. This approach puts back the problem within the deductive point of view. Then, the found conditions are known as sufficient, but this strategy is "expensive".

The second approach refers to known objects. Some quadrilaterals which satisfy both H and B are well known, for example squares, rectangles, parallelograms. Besides, the properties of their diagonals are also well known, and then one can work directly with equivalences. However, if some conditions may be cheaply found, this strategy does not give the exhaustiveness of the results, all the configurations are not a priori reached.

Lastly, the third approach raises the question of necessary conditions. Which objects satisfy both H and B ? Then, what properties have their diagonals ? This approach is basically related to sets point of view. Since the set ***A***, such as the set ***B*** contains the intersection of ***A*** with ***H*** is sought (in terms of properties : A such as (H and A) $\Rightarrow$ B), seeking fisrt the intersection of ***H*** with ***B*** (i.e. the objects satisfying both the properties H and B) seems natural. Then, there are two ways to study those objects which satisfy H and B, either to be in ***H*** and add the property B, or to be in ***B*** and add the property H. The first strategy is closer to the text of the problem but the second one looks easier. It seems, indeed, easier to draw two parallel sides than two same-lengthed opposite sides.

First sets point of view strategy : H then B (H : two equal opposite sides)

Once the points A and B placed in the plane, the hypothesis (H), AD=BC, means that the points C and D belong to two same-rayed circles respectively, one centred on B, the other centred on A. Once D placed, the property (B) "two other sides parallel", means that C is the intersection of the straight line parallel with (AB) containing D with the centred on B circle. There are two intersection points C1 and C2.



Two configurations are thus obtained : isosceles trapezium (ABC1D) and parallelogram (ABC2D) [fig.1]. But one must not forget that, once A fixed, one may still change the distance AB, the ray of the circles and the position of D (linked to that of C) on its circle. So, when the two circles intersect, there is a new configuration : a cross quadrilateral called CQ1 (ABC1D) [fig.2]

Fig.1

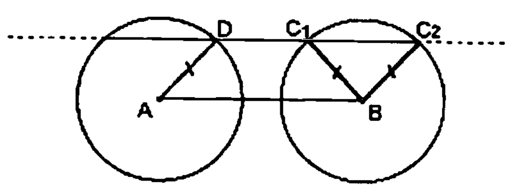
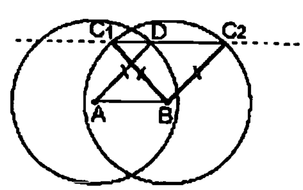


Fig.2



So there is the implication : (H and B) $\Rightarrow$ (parallelogram or isosceles trapezium or cross quadrilateral CQ1). We thus know the configurations which satisfy both H and B, it remains then to find the conditions on the diagonals.

However, for a quadrilateral, being a parallelogram is equivalent, to having diagonals which cross in their middle. This property is well-known by French pupils. Isosceles trapeziums and cross quadrilaterals CQ1 have same-lengthed diagonals. Now remains to see whether "to have same-lengthed diagonals" (A1) is a sufficient condition, i.e. if the implication, within the quadrilaterals, (H) and (A1) ⟹ (isosceles trapezium or cross quadrilateral (CQ1)) is true.

For that, the sets point of view is necessary again, we have to study the quadrilaterals which satisfy (H) and (A1).We will not detail the rest of the solving, but let us say that these two conditions bring obviously the isosceles trapezium and the cross quadrilateral CQ1 but also a cross quadrilateral CQ2 (cross quadrilateral linked to parallelogram) which does not satisfy the conclusion (B). The condition "having same-lengthed diagonals " is thus not sufficient and will have to be restricted to exclude CQ2. The final solving of this exercise is not the subject of this article, but we wanted to show how this problem, with easy objects, can question the implication.

Second sets point of view strategy : B then H (B: two parallel opposite sides)

To express (B), one draws two parallel straight lines, one including A and B, the other including C and D. A, B, D being fixed, there are two points C so that AD=BC (ie so that (H) is satisfied) : C1 and C2. Two configurations are then obtained : parallelogram (ABC1D) and isosceles trapezium (ABC2D) [fig.3]. But,again, A being fixed, B and D can move on their line. Thus, when [BC] crosses [AD], a new configuration is obtained : a cross quadrilateral CQ1 (ABC2D) [fig.4]. The rest of the solving is the same as in the previous strategy.

Fig.3

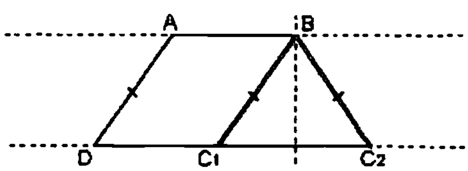
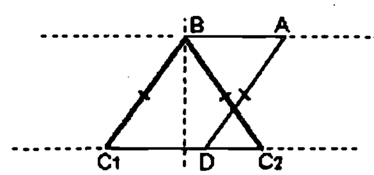


Fig.4



This second strategy, also based on the sets point of view, is not as far-reaching as the preceding one. Indeed, proving that the conditions are sufficient needs to come back to the hypothesis (H) and thus needs to use the previous strategy.

CONCLUSION

The analysis of the students' answers of the whole geometry problem is still in progress. However, we can already say that, although students first found this exercise very easy, its solving required a very long research in groups. The answers are incomplete and, in the end, the students declared this exercise complicated.

Robert: "It is an exercise which as a teacher, I would not give before university"

The exercise fulfiled its role, as for the work on the implication, since discussions about necessary and sufficient conditions took place in the groups in an explicit or implicit way. In addition, the exercise also fulfiled its role, at least partly, as for the work on the sets point of view. In particular, to have the exhaustiveness of the results, the groups, which had drawn apart the cross quadrilaterals, had to take them again into account.

These results are to be placed among others. Indeed, this problem of geometry forms part of a six hour experimentation including other stages of work, in particular, studies, in groups, of written proofs and of a problem of discrete mathematics. Moreover, this experimentation takes sense when one knows that it was preceded by two others, carried out in 1999 and 2000.This problem of geometry is, thus, to consider as part of a broader context. Now, remains to finish the analysis of these results and to connect them together, this is our goal for next year .

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1 We take into account the following expressions : P implies Q ; P brings to Q ; P thus Q ; Q is a consequence of P ; if P then Q ; P $\Rightarrow$ Q ; P is a sufficient condition for Q ; Q is a necessary condition for P.

2 This will be detailed in the following paragraph.

3 We call validity domain of a conception the goup of the situations wich may be correctly solved with the practicies associated to this conception.

4 These experimentations were carried out in june 1999 with four mathematics students, [Deloustal, 2000].

5 This expression is equivalent, in mathematics, to the expression " P and Non Q".

6 These two expressions are equivalent, in mathematics, to the expression : "T implies M"

7 There were two other questions : two 90 degrees angle (P2) ? ; two other same lengthed sides (P3) ?

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Teacher Behaviours that Influence Young Children's Reasoning

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Abstract: Children are expected to develop the habit of reasoning from their earliest years at school. However, there has been limited emphasis on strategies that teachers can use to support young children's reasoning. We report on a case study of four teachers who implemented mathematical investigations that provided reasoning opportunities for their classes of seven-to-eight year olds. An analysis of teacher behaviours and their students' responses suggests that reasoning is influenced by (1) teachers' expectations of reasoned actions and responses, (2) instruction in reasoning as systematised thinking, and (3) authentic opportunities for reasoning. These influences are discussed together with examples of how these teacher behaviours support or inhibit reasoning.

Reasoning is central to mathematics as a discipline (Steen, 1999) and underpins mathematical learning (Russell, 1999). As the National Council of Teachers of Mathematics (2000) emphasizes, the ability to reason is essential to mathematical understanding, and should be a primary goal in mathematics education: "By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and — with different expectations of sophistication — at all grade levels, students should see and expect that mathematics makes sense (p. 56). While the NCTM recommends that students from their earliest years at school should engage in reasoning activities, there are conflicting viewpoints about young children's capacity to reason. Some have proposed (e.g., Piaget & Inhelder, 1969) that there are developmental constraints to young children's reasoning. However these assumptions have been strongly challenged. For example, Tang and Ginsburg (1999) have argued that poor mathematical performance is a consequence of educational failure rather than children's inability to reason. Two alternative constraints to young children's reasoning have been proposed. First, young children's ability to reason is limited by their knowledge (Brown & Campione, 1994; Metz, 1997). Thus, reasoning is adversely affected by a weak knowledge base. Second, young children's reasoning can be inhibited by teacher competence (Brown & Campione, 1994). Metz (1997) argues persuasively that if the constraints to young children's knowledge and skill could be overcome then with judicious scaffolding, they should be able to engage in abstract reasoning. This hypothesis was supported in a study of young children's reasoning in mathematics and science that involved conjecture, argumentation, and evaluation of evidence (Watters & Diezmann, 1998).

Given that teacher competence can be a constraint to reasoning and the NCTM recommendations, the purpose of this paper is to identify the teacher behaviours that influence young children's reasoning. Teachers' competence in supporting young children's reasoning appears to be related to the nature of the instructional tasks, and the classroom culture.

Challenging Instructional Tasks

The teacher plays a crucial role in supporting the development of students' reasoning through the selection and implementation of cognitively challenging mathematics tasks (Henningsen & Stein, 1997). Hiebert et al. (1996) explain that the cognitive value of a task resides in the opportunity that it provides for students to explore and solve a problem. Teachers need to maintain the cognitive challenge in a task by proactively supporting students' cognitive activity without unnecessarily reducing the complexity of the task (Henningsen & Stein, 1997). While some tasks provide limited opportunities for reasoning, mathematical investigations are ideal tasks due to their scope for cognitive challenge (Boldt & Levine, 1999; Diezmann, Watters, & English, 2001). Investigations involve solving open-ended problems in which students make and test conjectures, engage in logical thinking, seek patterns and relationships, and explain and convince others of their viewpoints (e.g., Greenes, 1996).

Reasoning in the Classroom

Reasoning in the classroom is affected by teacher expectations, the classroom discourse, opportunities to make sense of mathematics through different types of reasoning and more general conditions such as a supportive climate. Gravemeijer, Cobb, Bowers, and Whitenack (2000) argue that in supportive classrooms: "(1), The students would explain and justify their thinking when contributing to whole-class discussions; (2) The students would listen to the contributions made by their classmates; (and) (3) The students would indicate when they did not understand a classmate's explanation or contribution and ask clarifying questions" (p. 251). These expectations encourage student participation, and a quest for understanding. They also highlight the importance of productive discourse in mathematics whereby students explain their ideas, build on others' ideas, and generalize beyond a specific example (Sherin, Mendez, & Louis, 2000). Mathematical investigations involve the exploration of a particular line of thinking in which students use resources to test conjectures, and hence, engage in transformational reasoning. According to Simon (1996) transformational reasoning is a dynamic process in which the conclusion is reached by running physical or thought experiments. Investigations also provide opportunities for inductive and deductive reasoning when students generalize from their experiments or explain the chain of thinking that led them to particular conclusions.

While sound reasoning is ultimately the goal, instances of flawed reasoning are inevitable in classrooms where students are novice reasoners. Russell (1999) argues that flawed reasoning plays a dual role in the classroom. First, students need to practice the various components of reasoning (e.g. conjecturing). They also need opportunities to justify and critique their own and others' reasoning, and respond to challenges to their reasoning. Second, flawed reasoning may also highlight mathematical issues that are relevant to the whole class and need to be addressed. For example, flawed reasoning may reveal that students hold the erroneous view that it is acceptable to reach different solutions in a convergent problem if different solution processes have been used. Thus, teachers should be alert to flawed reasoning and capitalise on the learning opportunities they it presents.

Design and Methods

This research adopts a case study design (Yin, 1994) in which a teaching experiment was conducted with the goal of supporting the development of investigatory abilities in young children. The study was implemented in four Year 3 classes with a total of 95 children (ages 7-8 years) in a parochial school in a relatively affluent outer suburb of a major city. Three of the teachers had in excess of ten years teaching experience while the fourth was a first-year, albeit mature age, teacher. Class sizes ranged from 18 to 26 students with the smallest class containing two special needs students. Students engaged in a 90-minute session each week for 10 weeks focussing on mathematical investigations (Diezmann et al., 2001) that was taught by their classroom teachers. The teachers were provided with ongoing professional development about investigatory approaches and reasoning, and teachers and researchers met regularly during the ten-week period to debrief, plan, and evaluate the program. The case study database comprised achievement tests, teacher and student interviews, student work samples, teacher and researcher notes, and photographs of teachers' and students' classroom work. Additionally, two researchers (CMD, JJW), who were non-participant observers, captured salient events on video. An assistant videoed the whole class. Data were analysed using constant comparative strategies (Glaser & Strauss, 1967) to identify emerging patterns, themes, and issues related to young children's reasoning and investigatory abilities while being sensitive to the existence of conflicting data to disconfirm the analysis. In this paper, we report on those data that inform us about how teachers can facilitate or inhibit reasoning behaviour in young children.

Results

Analysis of teacher behaviours revealed three patterns of interaction that influenced students' reasoning. First, by providing modelling and timely intervention, teachers developed the perception that reasoning involves justifiable actions. Second, teachers built on this perception by encouraging students to engage in systematic thinking using the language, and strategies associated with reasoning. Third, teachers capitalised on situations and conversation to create opportunities for reasoning.

1. Reasoned Actions and Responses

Teachers developed an expectation of sense making in mathematics through their patterns of discourse with students and by modelling sense making. These behaviours included asking students to justify their responses and actions; encouraging students to use "because" in sharing information; pressing students to explain unclear or incomplete responses; rephrasing students' words to enhance clarity; explicitly connecting ideas from different students; drawing students' attention to critical aspects of tasks; negotiating step-by-step guidelines for accomplishing challenging tasks; and by justifying their own actions — as if thinking aloud. The following examples illustrate how teachers influenced reasoned actions and responses from the students. In the first example, the teacher set up a clear expectation of reasoning to which the students responded. In contrast, in the second example, the teacher's failure to address flawed reasoning inhibited reasoning.

Smartie Can Example: Students were working in small groups to predict how many Smarties (sweets) were in small opaque sealed canisters. In one group, Melissa predicted that there were 27 Smarties. She reached this prediction by establishing that the canister had a length and circumference of three Smarties and nine Smarties respectively and computing the product of these figures. After the teacher instructed the class to justify their conclusions, Melissa repeatedly resorted to demonstration and explanation to convince her group that 27 was a sound prediction. While the group readily accepted the plausibility of Melissa's approach, they did not accept her prediction immediately. Melissa's persuasive argumentation and use of evidence led to a vigorous group discussion, which culminated in unanimous support for her conclusion. Thus, the teacher's clear expectation of reasoning resulted in this group striving to make sense of the task and each other's responses.

Card Game Example: During a place value card game, the teacher generally prompted students to justify their actions and encouraged them to monitor and challenge each other's actions. However, not all students' reasoning was subject to the same level of scrutiny. Michael, a mathematically gifted student, twice engaged in flawed reasoning, which went unchallenged by either teacher or peers. For example, small groups of students were dealt three cards to represent a three-digit number (e.g., $\boxed{2}\,\boxed{4}\,\boxed{3}$). They then took it in turns to draw a card from the pack (e.g., $\boxed{8}$) and used it to make the highest possible number by replacing an existing card in the hundred's, ten's or one's position (e.g., $\boxed{8}\,\boxed{4}\,\boxed{3}$). When all cards had been used the person with the highest three-digit number was declared the winner. Michael had seven turns during this game and demonstrated flawed reasoning on his second and sixth turns. The teacher queried his reasoning on the second turn but did not press Michael for an explanation: "Why? Why didn't you put it (the $\boxed{1}$) on the one's"? Michael shrugged and the teacher commented: "Now you have made it 30 smaller not just four smaller." Michael again made an error on his sixth turn but his peers did not query his placement of cards, though they queried each other's actions during the game.

(Dealt) $\boxed{3}\boxed{4}\boxed{5} \rightarrow \boxed{\mathbf{8}}\boxed{4}\boxed{5} \rightarrow \boxed{8}\boxed{\mathbf{1}}\boxed{5} \rightarrow \boxed{8}\boxed{\mathbf{3}}\boxed{5} \rightarrow \boxed{8}\boxed{\mathbf{6}}\boxed{5} \rightarrow \boxed{8}\boxed{6}\boxed{\mathbf{1}} \rightarrow \boxed{8}\boxed{6}\boxed{\mathbf{8}} \rightarrow \boxed{8}\boxed{6}\boxed{\mathbf{3}}$

Michael's exemption from justifying his actions is a concern. His flawed reasoning could be explained by disinterest if the game was insufficiently challenging. However, given that he was a keen participant in this game, was highly competitive, was playing with another capable student, and was observed to engage in flawed reasoning at other times, it seems more likely that Michael was either impulsive or did not fully understand the game. Thus, while the learning environment provided support for other students to account for their reasoning, Michael was not afforded this same opportunity. Other students' reasoning can also be inhibited if they unquestioningly accept responses from students, such as Michael, whom they somehow regard as infallible. Teachers also inhibited reasoned actions and responses through their use of recall-oriented questioning, limited wait time, and through inadequate responses to requests for assistance.

2. Reasoning as Systematised Thinking

These students were generally unfamiliar with the culture of reasoning. Teachers supported their enculturation into the practices of reasoning by developing relevant vocabulary; scaffolding students to make and test conjectures; challenging students' assumptions; focusing students' attention on the available evidence; promoting logical thinking; encouraging students to present their ideas as a chain of reasoning; and highlighting argumentation as a tool for evaluating alternative conclusions. The following examples illustrate the importance of discourse in developing students' understanding that reasoning is systematised thinking. In the first example, the teacher's interaction and use of mathematical language supported the students to think about the unanticipated result. In the second example, the teacher failed to adequately address the student's incorrect use of everyday language, thus limiting the development of his knowledge that reasoning is systematised thinking.

Speed Investigation: During a class sharing session, one group reported on an investigation in which they explored whether the speed at which a Smartie rolls down a slope is affected by its colour. The group had predicted that colour would have no effect on speed. However, after completing their investigation, this group recorded that "coulor efex speed. We though it would'nt efet but it did [Sic]". The class reaction to the group's conclusion about colour being a critical variable for speed was mixed. The teacher worked from the students' comments to draw attention to the ways in which the Smarties might have differed from each other apart from colour, thereby challenging the conclusion that colour alone affected speed. She also proposed how disagreements could be resolved through further investigation.

Mark	It might just be their weight or something.
Teacher	So you think actually that probably some of the (differently coloured) Smarties were **a little bigger** than some of the other Smarties. And you think that is a better reason (than colour alone) for them sliding faster?
Andrew	It could have been the way they were made.
Teacher	Yes. You think they had **different sizes and shapes**?
Andrew	We do think colour affected it as well.
Amanda	I don't think colour (alone) would affect it.
Teacher	Why don't you think colour would affect it?
Amanda	Because, it doesn't matter what colour it is. Colour is just to make it look better (on the surface).
Teacher	Look different? ...So that is a very interesting point isn't it. **We probably would have to investigate that a bit more before we come to some opinion that we all shared.**

Colour Example: The class had been investigating the frequency of particular colours in boxes of Smarties. Michael's report on his Smartie box was marked by idiosyncratic use of the terms "all" and "except".

Michael	We had **all** of them (colours) **except** green
Teacher	You had **every** colour **except** green? Where there **any** white ones?
Michael	No.
Teacher	Were there **any** black ones?
Michael	No.
Teacher	Oh, **any** browns?
Michael	Yes.

Though the teacher challenged Michael's statement twice, this was insufficient to prompt him to revise his statement that "We had all of them except green". Thus, there was a need for the teacher to pursue this inconsistency and encourage Michael to reflect on his reasoning. This example illustrates the need to emphasise the correct use of everyday words that indicate class inclusion and exclusion, such as "all", "every", "any", and "except". The student in this example, Michael, was the gifted student also featured in the card game example. Thus, although Michael had a heightened capacity for reasoning, there were shortcomings in his reasoning, which needed to be addressed.

3. Authentic Opportunities for Reasoning

The use of intrinsically interesting situations can enhance young children's motivation and commitment to reasoning. The following two examples illustrated how teachers promoted and limited opportunities for reasoning in these situations.

Story Example: One teacher promoted reasoning about quantity through her implementation of the story, "The Doorbell Rang" (Hutchins, 1986). This story involves the repeated occurrence of children sharing out a batch of 12 cookies but before they can eat the cookies the doorbell rings and they also have to share their cookies with the new arrivals. Thus, over time, each child receives fewer and fewer cookies. The teacher capitalised on the reasoning opportunities in this story through her organisation of and interaction with the students. She seated the class around a batch of cookies and assigned students to be the story characters. After the actors depicted the arrival of each new group of children in the story and shared the cookies out, the teacher asked questions, such as "Have they got the same amount of cookies? How do we know? How else could we know?" Through the enactment of the story and the teacher's encouragement to reflect on what was happening, the class developed a clear understanding of the inverse relationship between the number of children who were to share the cookies and the quantity they would receive. Later in the story, the students were also able to ascertain the number of cookies each child would receive after Grandma arrived at the door with another batch of 12 cookies. Some students continued to use this experience as a referent. For example, over 12 weeks later, Alice reasoned from cookies when the outcome of a division was not a whole number: "And if we had an odd number of cookies we had to do something ... like break them in half." Hence, the implementation of the story provided a memorable referent from which the students could reason about quantity.

Dissolving Smarties: Students were using Smarties to explore an aspect of mathematics. One group decided to explore what happened to Smarties after they had been dropped into hot water. This investigation provided opportunities for reasoning, about time, temperature, volume, and quantity. However during the investigation and in the subsequent sharing session, the teacher emphasised the "fun" element of the activity to the extent that the mathematical opportunities were overlooked. Thus, while it can be motivating for students to engage in interesting activities, the teacher needs to preserve and foster the mathematical value of the activities, and support students' reasoning within the particular context.

Conclusion

Young children's reasoning can be enhanced or inhibited by teachers' behaviour through their discourse, the type of support they provide for their class, and how they implement mathematical tasks. Reasoning was promoted when teachers clearly valued reasoning, modelled reasoned actions and responses, and held high expectations of the class. The children responded to high expectations with continual attempts to engage in sense making in their actions and discourse, and challenges for their peers to do likewise. While these children were generally unfamiliar with reasoning as a systematic way of thinking, with ongoing teacher guidance and scaffolding, most children began to utilise at least some of the language, rules, and strategies associated with reasoning. Learning experiences that provided rich opportunities for students to develop and practice this more "formal" reasoning were those where the teacher focussed strongly on the mathematics in the situation and used the task to create a forum for reasoning about topics of interest.

These insights provide the base for us to extend our studies in developing more comprehensive documentation of the teacher behaviours that support and inhibit reasoning. If reasoning mathematically is to become "a habit of mind" (NCTM, 2000, p. 56) from the earliest years of schooling, action needs to be taken at preservice and inservice levels to develop the appropriate pedagogical content knowledge.

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CONTEXT COMPLEXITY AND ARGUMENTATION

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This paper elaborates on the hypothesis (taken from psycholinguistics research) that the development of argumentative skills depends on contextual factors. With reference to Mathematics Education, the potential inherent in context complexity is investigated, distinguishing between different kinds of complexity. Some educational implications concerning the teacher's didactical choices are sketched.

The importance of argumentation [1] in Mathematics Education has been increasingly recognised in the last decade. In spite of the development of research concerning the 'why' of argumentation, a relatively small number of papers dealt with the 'how', in particular with the conditions suitable for the development of argumentative skills; only few dealt with the mathematical and extra-mathematical content of the problem situations as a relevant factor. However, there are good reasons to think about the relevance of these 'context' factors (for a discussion about the meaning of 'context' see the next Section). In psycholinguistics the hypothesis of context-dependence of argumentative skills was an object of widespread investigation during the 70s and the 80s (for surveys, see French, 1985; Sell, 1991). The aim of this paper is to investigate the potential of context complexity in the development of students' argumentative skills related to mathematics at primary school level. I will also draw attention to some conditions (as regards the teacher's choices) for the potential of context complexity to be exploited. The orientation of this paper is theoretical. However two teaching experiments, performed in the past with different purposes, will be shortly presented; they will provide the reader with some examples to ground the analysis.

THEORETICAL FRAMEWORK

The general hypothesis underlying the research reported in this paper is that the quality of students' argumentation strongly depends on the nature of the objects and situations which argumentation refers to, provided that suitable didactical choices (concerning the tasks and the educational environment) are performed by the teacher. This hypothesis draws on the context-dependence of peculiar forms of argumentation. Considering the contributions brought by *"several major developmental theorists"* (Brown, Donaldson, Gelman, and others) to research concerning contextual factors in cognitive development, L. French wrote: *"They are in basic agreement about the importance of contextual factors in both the acquisition and display of cognitive abilities. Their essential premise is that cognitive competence initially arises within, is embedded within, and is practiced within particular contexts"* (French, 1985, p. 321). Referring to specific psycholinguistic research concerning the acquisition and display of cognitive competencies related to the comprehension and use of relational terms ('because', 'if', 'but', 'or', etc.) she stressed the importance of *"the content of sentences in which relational terms are embedded"* (ibidem, p. 322).

As regards the meaning of the word 'context' in this paper, we can observe that, in Mathematics Education as well as in common language, the word 'context' may take different meanings (for a survey see Wedege, 1999, pp 206-207). It can refer to constraints, signs, situations that must be considered by the student in accomplishing a given task ('reference context'); to the formulation of the task (context as 'text'- Freudenthal, 1991); and to the environment – in particular, the educational environment - where the activity takes place (the 'situation context' - see Wedege, 1999). In our hypothesis all the different meanings of the word 'context' intervene, and 'reference context'-dependence is not seen as isolated from 'situation context'-dependence. In the next Section we will consider some aspects of the 'reference context' (simply named **'context'**), while some educational implications concerning the 'text' and the 'situation context' factors will be sketched in the last Section.

As concerns the argumentative skills considered in this paper, we will refer to the mastery of relational terms as well as more complex skills that intervene when contradictions are detected and made explicit, when coherence is questioned, etc. Tasks and argumentative skills will be considered in the perspective of mathematics education at primary school level, with an eye to more advanced mathematics.

In this paper, the potential of context complexity will be considered as regards both the number of elements to be co-ordinated in a problem situation (or in a set of tasks dealing with the same subject) and the quality of the links to be established (from simple juxtaposition to comparison, up to a comprehensive and coherent systematisation) (cf. Sell, 1991). This paper will elaborate on the specific educational hypothesis that primary school students' argumentative skills can be enhanced by exploiting context complexity in two ways: for some students, as an opportunity of exercising specific argumentative skills; for other students, as an opportunity of approaching these skills through the interaction with more competent peers and/or the direct mediation of the teacher, who suggests the appropriate linguistic forms to express complexity (cf 'Zone of Proximal Development': Vygotsky, 1978, Chapt. VI).

EXAMPLES

Concrete examples illustrating the ideas developed in Section 4 will be taken from two teaching experiments performed by the Genoa Group Project in the years 1997 and 1999. The first experiment was carried out in a grade-II class and concerned the task of measuring the height of wheat plants growing in a pot with a ruler having a one-centimetre distance between the edge and the beginning of the scale: *"How can we measure the height of the plants in our pot with our ruler, in order to account for their growth over time?"*. From the educational point of view, the task was aimed at constructing some operational invariants (Vergnaud, 1990) of the concept of measure, e.g. the translation invariance and the additivity of lengths. From the research point of view, the aim of the teaching experiment was to study the argumentative roots of the construction and refinement of those operational invariants up to an explicit and conscious mastery of them. The research was reported in Douek and Scali (2000).

The second experiment concerned the 'Stefano's problem' in a IVth-grade class:

"At the beginning of classwork on sun shadows, Stefano (a VI grade student) thinks that shadows are longer when the sun is higher and stronger. Other students think the contrary. In order to explain his hypothesis, Stefano produces the following drawing:

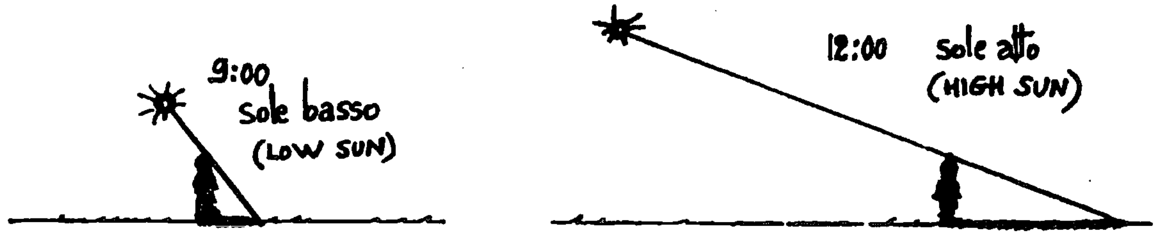

Fig. 1

and writes: "As we can see in the drawing, the sun makes a longer shadow when it is higher, that is at noon, when it is also stronger"

We know very well that shadows are longer when the sun is lower (early in the morning and late in the afternoon). So, in Stefano's reasoning there is something that does not work. What is wrong with Stefano's reasoning, and particularly with his drawing? Try to explain yourself clearly, so that Stefano can understand."

From the educational point of view, the aim of the task was to refine the model of the Sun shadows phenomenon (the 'shadow schema') already used by students in many activities. The task put into question the meaning of the word 'high' when referred to the 'height' of the Sun in the sky. A link with the concept of angle was implicit in the task. From the research point of view, the aim was to study the process of conceptualisation and the role of argumentation in it (see Douek, 1998 and 1999a).

In both experiments individual tasks were alternated with classroom discussions about individual productions, managed by the teacher. The experiments were performed in an educational context where verbal reporting of thinking processes (with no worry about their correctness) was an important element of the didactic contract.

SOME KINDS OF CONTEXT COMPLEXITY, AND ARGUMENTATION

Different kinds of context complexity will be considered in relation to their specific potential in the development of argumentative skills. We will move from complexity depending on the characteristics of the content of the problem situation, to complexity inherent in the tools needed to tackle the problem.

Time and Space Complexity

In mathematical modelling activities, objects and situations to be dealt with can belong to different space scales (micro-space, meso-space, macro-space: see Berthelot&Salin, 1992). They can also be involved with different time scales. We can observe that the passage from one space (or time) scale to the others entails relevant changes in cognitive processes. Complexity depends on the need for a comprehensive consideration of phenomena, which are to be analysed under different time and space scopes. The specific forms of argumentation which intervene in managing time and space complexity concern comparing arguments related to different space or time environments, or checking the validity of a statement according to different space or time scales, (etc.), particularly under the form of hypothetical reasoning.

In the specific case of 'Stefano's problem', space complexity intervenes when students must create links between the drawings (made in the micro-space of the sheet of paper), the observed phenomenon (happening in the meso-space of the courtyard experiment) and the phenomenon occurring in the macro-space of the sun system. During a classroom discussion concerning the problem of the 'command variables' of the geometric model of shadows, the teacher asked what would happen to his own shadow if he moved one step forward. He meant to put into question the relation between the virtual observation of his shadow if he moved forward and the schema drawn on the blackboard. A 'good' student, Federico, transferred to the macro-space what he could imagine in coherence with the schema on the blackboard, where the sun was drawn near the obstacle that cast the shadow. He said: *"If you move, then your shadow becomes longer, because the Sun stays there and you go farther"*. The conception of the relative positions of sun, earth and obstacle in the macro space that had been developed during the experimental work in the meso-space of the courtyard, was replaced by the direct observation of the micro-space schema. As a consequence, it was then necessary to rebuild coherence between the previous conception and the schema. The evident contradiction of Federico's statement with what many students knew (the invariance of the length of the shadows of a moving object) provoked high quality argumentation in the classroom. Some excerpts are reported below:

Simone: *"I agree that the shadow will become longer; if you move in that drawing, the Sun does not move and you can see that the shadow becomes longer and longer"*

Andrea: *"I do not agree with Simone and Federico, I can give you an example: if you are in a car and move... for instance, from Turin to... to Pinerolo... you see the sun...always with the same height in the sky... and the shadow does not change. The position of the sun changes only if a long time elapses! So there is something that does not work in this drawing."*

Mariella: *"In the real world, near to the Earth, sun rays are parallel, while in this drawing... using this drawing ... they are convergent"*.

Anna: *"Yes, the sun is not there... it is very, very far"*

Coherence was gradually reconstructed through argumentation by relying on different sources of arguments, as described in details in the next Section.

Complexity of Sources of Arguments

Arguments may originate from the classroom history, from empirical or theoretical evidence, from shared principles, etc. Complexity depends on the need for integrating different sources and making them coherent, when for example one source is not sufficient to get a satisfactory solution, or when some contradictions emerge. This need for integration and coherence requires specific forms of argumentation; the *"A, while B"*, *"A, but B"* and *"if A then B so C"* clauses intervene as necessary tools.

Let us go back to Federico's confusion. A contradiction appeared between his statement, based on the 'shadow schema' drawn on the blackboard, and other references shared by his schoolmates. First, it clashed with another geometrical model previously elaborated in order to solve the specific problem of the effect of changing the position of the obstacle in the courtyard on the cast shadow. In that model the sun was not drawn; a series of parallel rays on the sheet of paper showed that wherever the obstacle moved, the rays cast shadows with equal lengths (see Mariella's utterance). Second, it clashed with the content of a classroom discussion about the 'shadow schema' in which it was agreed (by moving the sun farther and farther) that, since the sun is very far from the Earth, therefore sun rays can be considered parallel, and not divergent from their source, when they reach the Earth (see Mariella's and Anna's utterance).

We can remark that the complexity of the situation was successfully faced by the multiplicity of voices (it would have been difficult to tackle it individually!), and this was necessary to bring the different pieces of knowledge together coherently.

Static-Dynamic Complexity

In applied mathematical problem solving, dynamic situations must be represented in a static way in order to elaborate the relations that connect relevant variables. For instance, in the case of Stefano's problem the dynamic evolution of the sun shadow phenomenon is usually represented (in the students' individual solving processes) through two or more drawings concerning specific states of the phenomenon itself. Complexity derives from the need for relating static representations to one another and to the evolution of the phenomenon in a coherent way. The argumentative skills involved concern the discussion of the coherence between representations and reality.

In particular, in the case of 'Stefano's problem', one student, Ambra, drew a pair of

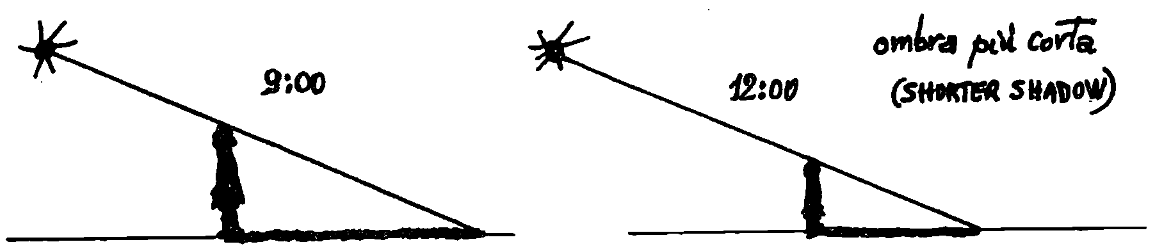

Fig. 2

schemas in which the position of the sun and the inclination of the sunrays remained unchanged; instead she changed the height of the obstacle casting the shadow in order to obtain a *shorter shadow at noon* (this was a shared knowledge in the classroom). Ambra did not relate consciously the different hours of the day to the different positions of the sun in the sky. At that moment she lacked arguments to choose another variable (the slope of the sunrays) and produce schemas coherent with her past experiences. The slope of the sunrays appeared as crucial during the following discussion, and various *connected* arguments were considered. In particular, the students recalled a past experience; with the help of the pictures taken in that occasion they represented the slope as the inclination of the arm pointing at the sun *"moving in the sky"* in different imagined positions. So a link with the static representation of the 'sun shadow schema' was established. Some excerpts are reported below:

Aurora's drawings do no work together, because the heights of the child are different, while in the real world the height does not change"

" It is the slope of our arms when we imagine ourselves pointing at the sun moving in the sky that changes"

"It is the slope of our arms in the pictures that changes, and so it is the slope of the rays in the sun shadow triangle that must change ".

Complexity Inherent in the Change of Frames

A problem situation can be considered within different frames (for 'frames' in Mathematics Education, see Arzarello et al, 1995). For instance, some difficult Euclidean geometry problems can easily be solved within the frame of analytic geometry. Complexity depends on the need for comparing different frames and moving through argumentation from one to the other.

In the case of the 'Wheat plants measuring problem' this kind of complexity intervened during the individual solution phase, in which the students interacted with the teacher. The first thing most students wanted to do was pushing the ruler down into the pot, or breaking the bit before the zero (a concrete, physical solution). Then the teacher argued against such solutions (because they would damage either the roots or the ruler), so the students gradually switched to *imagining* to push the ruler (or to break it). The teacher drew their attention to the number they could read when the ruler was just beside the plant and the one they would read if the ruler was pushed down (or broken). The students had to consider both what they read and what they would read. Many of them developed solutions: either they imagined the numbers were all sliding, one after the other ('translation solution'), or they imagined to stick the little broken piece at the top of the plant ('additive solution'). They arrived to a solution within a material physical frame, consisting of virtual actions on real objects. When they came to writing their solutions or giving examples, in order to explain their solutions, they often moved to the numerical frame, and said *"It is the following number"* or *"I add one to the number I can read"*. The physical solution was a necessary step to gradually get to a 'translation' or an 'additive' solution within the numerical frame. Afterwards, the solutions were shared in the classroom (see next Section) and a discussion followed. At the end, the students developed complex argumentation in order to account for the validity of each solution and compare the two solutions:

"In our case Rita's reasoning works, because 1 on the scale goes to 0 on the ground, and then 2 goes to 1, 3 goes to 2, and finally I can read the exact measure. But this shows that it works well because the initial piece of the ruler is 1 cm long. Alessia's reasoning always works because it is as if I broke it and add the initial piece, and I perform the same thing with numbers. And I can always do the same and add the number accounting for the measure of the initial piece, whatever it is".

Complexity of External Representations

A plurality of systems of signs (verbal, iconic or geometric, symbolic) may intervene in many mathematical activities (such as modelling). Complexity depends on the necessity of co-ordinating such signs and dealing with them in a functional way (cf Duval, 1995):

for instance, language as a command and reflective tool, geometric signs as models (as in Stefano' problem), algebraic symbolism as computational devices, etc. In collective discussions this kind of complexity involves different games of interpretation and related argumentative skills, especially concerning the control of solution strategies.

In the 'Wheat plant measuring' activity, after the individual problem solving (in interaction with the teacher) and presentation of the 'additive' and 'translation' solutions, the students were provided with two drawings (to stick in their copy books) representing a wheat plant in its pot, and two paper rulers. They were first asked to express clearly (in written words) the 'translation' solution, and they all did it within the physical frame. Then they had to represent the translation of the numbers on the ruler with a sequence of arrows; and then to write the calculations that represented symbolically in the numerical frame what they got in the physical frame. They repeated this kind of representation for the 'additive solution'. This activity allowed intensive argumentation in interaction with the teacher, because the teacher systematically asked to justify the arrows and the calculations with reference to the imagined virtual solutions within the physical frame. These solutions played the role of justifying the choice of the calculations that solved the problem. Here is an example of an individual text (after the interaction with the teacher):

"In words I can say that I move the ruler, so each number slides, it is as if each number vanished and became the following one: 0 becomes 1, and 1 becomes 2, and so on. If I have to show it I must draw arrows from 0 to 1, from 1 to 2. But to get the measure only the last step is important: I read 22 on the ruler, then 22+1 is the measure".

The same complexity in dealing with various representations can be found in the discussion about Ambra's drawing in the 'Stefano's problem'.

SOME REMARKS ABOUT THE ROLE OF THE TEACHER

The exploitation of the potential inherent in the contextual elements described in the previous section relies on appropriate didactical choices made by the teacher.

First of all, we must consider the importance of the choice and formulation of tasks, and the choice of the right moment to perform them in the classroom, in order to avoid that complexity is reduced. Concerning the choice of tasks: A problem like the 'Stefano's problem' involves different kinds of complexity, with an increased potential for argumentation. Concerning the text: In some experiments performed within the Genoa Group Project we have seen that the 'Wheat plants measuring' activity looses most of its potential if the following text is given to the students: *"Keeping into account that plants cannot be extracted from the ground, that the ruler cannot be put into the ground and that the ruler has a one-centimeter distance between the edge and the scale, establish how to measure the height of wheat plants in the pot"*. We saw that the percentage of good solutions decreased and the quality of the classroom discussion was lower. Probably, the formulation of the task guided the solution process and prevented students from detecting and managing the constraints of the problem situation. Moreover it is obvious that the argumentative potential of the problem situation cannot be exploited if the task is proposed too late.

Second, we must consider the management of classroom situations. As regards Stefano's problem, another teacher tried to exploit the same problem in her class. As a reaction to students' initial difficulties, she decided (before the classroom discussion about students' individual productions) to suggest the use of the concept of angle, saying that *"the mistake of Stefano was to consider the height of the sun as a distance from the ground, not as the angle formed by the sunray with the ground"*. The consequence was that no real debate was developed during the following discussion and the students simply wrote (after the discussion) a more or less precise paraphrase of what the teacher had said.

NOTES

[1] In this paper, the word **argumentation** indicates two things. First, it denotes the individual or collective process that produces a logically connected, but not necessarily deductive, discourse about a given subject. This agrees with the first definition for 'Argumentation' provided by the Webster Dictionary: *" 1. The act of forming reasons, making induction, drawing conclusions, and applying them to the case under discussion"*. Second, it points at the text produced through that process, in accordance with the meaning the Webster dictionary presents: *" 3. Writing or speaking that argues"*. The linguistic context will allow the reader to select the appropriate meaning, whenever this word is used. In this paper, the word '**argument**' is used with the meaning of *"A reason or reasons offered for or against a proposition, opinion or measure"* (Webster) . So, an 'argumentation' consists of one or more logically connected 'arguments'.

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ENCOUNTERS WITH INDEPENDENT GRADUATE STUDY: CHANGES IN LEARNING STYLE

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This paper considers students whose undergraduate study was characterised for them by either natural or alien learning and examines the ways they evolve or change their style of learning when they encounter independent graduate study in mathematics. We explore the changes they undergo and, towards the end of the paper, fit those movements into a developing theory grounded in the data from interviews with all the PhD students at one medium sized mathematics department in the UK.

Introduction

What happens when you find yourself learning on your own? While the notion of a transition is fundamental to many areas of mathematics education, the transition we examine in this paper is perhaps the one that we might genuinely call 'becoming a mathematician': the transition to independent graduate study.

The transitions in understanding concepts, social transitions that accompany changes between schooling levels and transitions between levels of abstraction/generalisation have all been the subject of significant research programmes within the mathematics education community. Often these transitions may co-occur: new concepts or different levels of abstract thinking can come just as students move to a new level of schooling. In each case, the new situation requires the learner to develop both new mathematical ideas *and* new ideas about the learning of mathematics and the didactic contract which must be negotiated between student and teacher. In this paper, we consider the situation in which, while there may still be a teacher (in the form of a supervisor), the relationship between the learner, the knowledge and this teacher is radically different from all those at previous levels.

We consider some of these aspects in relation to the transition between undergraduate study and independent graduate study (which, in the UK context, we take to be a substantial doctoral study leading to an original thesis in some field of mathematics). This paper examines the form this transition took for 13 PhD mathematics students – the entire available PhD population, who were at various stages in their work – at one

UK university and, in particular, the ways in which the students' learning styles appeared to change in response to the new learning situation.

Background and research method

The existing literature on learning styles is quite extensive, but has been built predominately from studies at school or undergraduate level. The theories which have been developed have been useful in analysing the difficulties students have in learning new knowledge they encounter. Many of these are built around a basic dichotomy between students who seek to connect new knowledge with old and those who do not. Relational and instrumental understanding (Skemp, 1976), deep and surface learning (Marton & Saljo, 1976), procedural and conceptual knowledge (Hiebert & Lefevre, 1986) etc. all place this split at the centre of their theory.

In our own work (Duffin and Simpson, 1993), we have developed a theory of the ways in which learners might respond to new knowledge by absorbing it into existing structures (a *natural* response), limiting or modifying existing structures to deal with it (a *conflicting* response) or by building a new, initially separate structure to house the new knowledge (an *alien* response).

Following Pinto's work with our theory (Pinto, 1998) we have extended this theory about responses to learning encounters to develop a split between learners. On the one hand we have those who attempt to make analogical connections between new knowledge and existing ideas and are loathe to accept knowledge which does not fit easily into existing structures (who we call *natural* learners). On the other we have those who are happy to develop a separate schema to cope with new experiences, perhaps only later using conflicts to connect pieces of knowledge together (who we call alien learners).

While student responses to the new learning environment they face in independent graduate study have been the subject of some research, it tends to be general, focussing across a large number of subject boundaries. The large, ESRC-funded projects reported in Burgess et al (1995), Pole et al (1997) while focussing significantly on the relationship between students and supervisors, suggested a split between students as 'technicians' (focussing on the skills required for research) and students as 'scholars' (focussing on the substantive knowledge uncovered). Similarly, Ford (1985) examined a split between holist and serialist graduate learners, albeit with students at masters, rather than independent doctoral level.

However, in this paper, we will see that when we focus more tightly on the specific topic of mathematics, the straightforward dichotomies of natural/alien, scholar/technician and holist/serialist need to be expanded.

We draw upon a larger paper based on semi-structured interviews focussing on two main areas of interest. We asked students about their personal learning histories from primary school onwards, but focussing more often on recent, accessible memories from their undergraduate study. We also asked them to consider their current learning style, specifically about how they would respond to being asked to learn a new area of mathematics. In addition, all of the students had some contact with undergraduates, normally through tutorial teaching and we gained some insight into their understanding of their own learning from their discussions of their perceptions of others.

The interviews were conducted in the style of a 'conversation with a purpose' (Burgess, 1988), they were independently coded and initially analysed by each author and, while we inevitably had the existing learning style dichotomies in mind, we were careful about imposing the theory on the new situation so that any theory concerning how students respond to independent graduate study was grounded in the data. As we will show, we were right to take such care as this investigation caused us to expand and refine the theory in unexpected ways.

While we did not intend to gather data representative of a postgraduate mathematics population, the sample consisted of all the available doctoral students from a medium sized mathematics department in the UK and may be seen to provide access to the key themes which shape students' renegotiations of their learning style in response to independent graduate study. The number of issues raised by the data from this sample suggests the validity of this transition as an area of study that we plan to investigate across a much larger group.

Movements

Clearly, the interviews were rich in issues to be investigated. In particular we encountered on student who showed flexibility in his learning, able to choose a style depending on both context and the value he placed on the knowledge to be gained. Another retained a strongly alien mode of learning, becoming an arch-formalist. Yet another defied expectation by appearing to move from a natural to an alien learning style.

However, in a paper of this length, we will concentrate on only the general sense of movement (in terms of the change of learning style) which the sample as a whole students revealed to us as the students described how they came to terms with their own changed circumstances as postgraduates.

Our independent coding clearly split the students on their undergraduate (and school) style of learning into natural and alien learners as described above. What we discovered, however, was that while the natural learners evolved their learning style to cope well with independent study, the alien learners had to adopt new styles – in two quite different ways.

Natural Learners

Six of the students were classified as showing aspects of natural learning in their undergraduate style. The movement to graduate study seemed to be quite smooth for them and we get little sense of change of learning style. A typical example is Brian, who described his attempts to learn new mathematics as a graduate student in exactly the same ways as he had as an undergraduate:

> When I do try to look at the definitions of a new subject, I do try to think how it relates to different things I've done before, that does make me feel a lot more sure about it. For instance, like if you first look and it said the inner product on a Hilbert Space, I see that as a generalisation in multiplication and then it becomes more sort of meaningful.
>
> (Brian)

However one notable aspect of the movement to graduate study for natural learners was that they viewed it with some relief: it gave validity to their way of working, in contrast to their undergraduate experience where they appeared to feel that they achieved less well than did those students who saw the mathematics as 'just something to be learned by rote'.

Thus Oliver, thinking of himself when he was an undergraduate observing fellow students, spoke of how he would tussle with a problem: 'I would interrogate it to death ... it took a long time for me to get it into my thick skull' while others `just seemed to recite the mantra and see it as something to be learned'. His perception was that these students often appeared to achieve better than he did: there was, to him, a clear disadvantage to his learning style:

> ... for exams yes, which in some sense is a little silly and is perhaps reflected by the fact I didn't get a first. With some subjects analysis for example standard questions prove whatever theorem, and so there are some people who will just develop short term memory tricks just rather like an actor learning lines. ... I suppose in some sense I emerged with a greater understanding for having done so. But the downfall of that I suppose is that I would spend too long on learning certain proofs and therefore not learn enough proofs and would then invariably be left in the dark on certain questions.
>
> (Oliver)

In graduate study, however, the "downfall" is less obvious as the sense of a frenetic pace of learning is replaced by a sense of confidence:

> ... it's just a question of, well occasionally talking to different people in the coffee room assimilating different examples, examples are important. Very, very, very slowly absorbing principally I guess through osmosis
>
> (Oliver)

Not all the natural learners had completely left the frustration behind. Christos described his own learning as being "like a big building where I start all the way from the ground until the top of it". Although he felt he had been a natural learner throughout most of his school and undergraduate learning, he did undergo some changes on becoming a postgraduate student. He said that before graduating he limited himself largely to learning from his lecture notes and contrasted this with his postgraduate work because he used books more at that level. However, he found that there still wasn't really time to do as he liked to do 'to get to understand every detail' saying "I know we don't have all the time in the world and life is short so I just try to understand as much as possible".

So natural learners seem to take their encounters with independent graduate study in their stride. The frustrations of the speed of undergraduate learning (or appearing to be surrounded by those who learn faster – if only by rote) are much lessened. Their fundamental learning style is rewarded by the freedoms of graduate study.

Alien Learners

In contrast, alien learners, accustomed to the comparative safety of undergraduate learning where they had a set text to follow and procedural methods to use to achieve success, found that their accustomed way of working was no longer adequate to their new circumstances. It required them to be able to work independently and some tended to feel less secure and confident than hitherto. Lucy, for example, says

> There's no set syllabus. I find that a lot harder ... It's a bit difficult if you are trying to muddle through by yourself. I like to be told. Once I've been told I can then understand. I don't really like finding things out for myself very much.
>
> (Lucy)

Lucy was more articulate about her position than some of the others but her words sum up the general position for these students.

Our study found that these students who clearly showed that they were alien learners at the undergraduate level and before, tended to move in one of two directions. Some moved towards a more natural learning style, but others seemed to respond by developing a learning style which did not fit into our existing natural/alien split (or indeed, the general connected/disconnected learning styles of other theories outlined earlier).

While natural learners, or those who had been alien learners as undergraduates who moved towards natural learners as postgraduates, were interested in making analogical links between new mathematics and existing knowledge, some of the alien learners subverted their learning style to a new one (which we came to call 'coherence' learning). In response to independent graduate study, these previously alien learners appeared to search for a structure within the new mathematics, while still basically keeping it initially separate from their existing knowledge.

Alien → Natural Learning

Alan spoke about his own way of learning as an undergraduate and earlier, indicating that he had merely tried to be a good student learning what he was being taught without giving much thought to whether he understood it or not though at the time he had thought he did understand it because he did quite well. `I learned a lot of proofs just by learning them and regurgitating them'. But now he thinks much more about what the mathematics is about:

> I prefer it if he [his supervisor] first talks with me about it to give me an intuitive idea ......just on an intuitive level and then I try to make the mathematics fit to what my intuitive ideas are … some proofs are intuitive ....but if the proofs are not intuitive ..... [I like] to work through it and think about how I worked through it ... I've matured a lot as a mathematician; I'm better as a mathematician but not as good as I thought I was before'.
>
> (Alan)

This reflection, intimating the confidence and security engendered by competent alien learning, shows a growing recognition that understanding is not merely about successful reproduction of techniques and proofs. Instead, he is beginning to see the need for a 'fit' between the mathematics and 'what my intuitive ideas are' – a sentiment which sets him clearly on the path to a natural learning style.

Lucy, who so clearly enunciated the difficulties faced by alien learners in encountering graduate study, when asked how she viewed her current way of learning said

> ... instead of learning things by rote I'm having to go much more into the background and start with things I already know and learn by analogy or build them up; I found that a lot harder.
>
> (Lucy)

Lucy and Alan both seem to be in the early stages of the transition but appear to be moving towards a new perception of what 'to understand' means in which they seek analogies and a fit between the new mathematics they are learning and their existing knowledge. They are in the process of moving more towards a natural way of learning.

Alien → Coherence Learning

Not every alien student moved towards natural learning in response to the new learning environment. Indeed, some students were causing us some difficulty since their descriptions of their current, postgraduate learning styles did not appear to fit the basic natural/alien dichotomy (nor the relational/instrumental, deep/surface, conceptual/procedural theories discussed earlier). Instead they seemed to develop in a different direction. Yes, they were seeking structure, but not the structure *around* the mathematics which the natural learners and natural converts appeared to seek, but structure *within* the new mathematics they were learning

Rebecca came across as our archetypal coherence convert. She was highly articulate and had reflected on both her own learning and that of the undergraduates she taught as a Graduate Assistant.

> I always want to know where it's going so the first thing would be to see where it's going to and to get there I've got to do each of the steps one after the other.
>
> (Rebecca)

There remains a sense of the procedural, and no sense of the need for the support of an analogy, but while for her younger self, understanding lay in "getting it finished" and "I wasn't bothered about why I was doing it", now understanding comes gradually:

> So I think I'm probably ... looking at each of the steps on their own to start with, to work out this bit and then the next bit, and then the next bit, until I've got that, and then eventually it all comes together
>
> (Rebecca)

Sudeep, also a strongly alien learner as an undergraduate, really saw himself as a physicist rather than a mathematician but he too had come to have a strong sense of the internal structure of what he was learning as he came to his postgraduate work. He still saw himself as one who liked to become proficient in techniques and felt that understanding for him came through that proficiency, a characteristic of many alien learners, but this is increasingly mixed with this sense of structure within the mathematics:

> I would to start with, I would try to find a certain structure in the piece of mathematics that I'm looking at, try to find some logical procedure that's involved and try to understand how and why the result is obtained.
>
> (Sudeep)

So, for these learners, the encounter with independent graduate study has led them to adapt their learning style to accommodate to the new learning environment. Separate pieces of knowledge will no longer suffice. However these students do not move to the wholly cohesive sense of natural learning (as Alan and Lucy were beginning to do), instead they develop a learning style in which they need to sense the coherence of the new mathematics as a structure.

Summary

Encountering independent graduate study is, then, a quite different experience for natural and alien learners. For the former, it is met with open arms, a vindication of a learning style which has been a struggle to maintain in other pedagogical environments with fellow students appearing to succeed with seemingly easier learning styles. For the alien learners, however, there is a need to change – the independence which the natural learners see as liberating is bewildering and even threatening to the alien learners. The procedures which could be followed and the pre-digested ideas which could be readily absorbed are missing and they must face the prospect of developing new knowledge for themselves. Some are able to do this by adopting a more natural-like learning style – looking for analogies which help them make sense of new ideas. However, most excitingly for us in developing our theory of learning, some developed a different way of learning: seeking not analogies, but internal structure, which allows them to retain some aspects of alien learning (the initial separateness of the new knowledge) while enabling them to progress by building new knowledge into frameworks.

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LEARNING MATHEMATICS COLLABORATIVELY - LEARNING THE SKILLS

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The research in this study explores the learning of mathematics through collaborative activity in two pedagogically identical environments but with distinct and different skill-learning agendas. Evidence is provided through analysis of small group discussion in the naturalistic emancipatory and socioconstructivist setting of both classes. Audio tapes are used as evidence for both qualitative and quantitative analysis. Findings indicate that groups which explored mathematical problems collaboratively sought solutions which reflected the input of the whole group. A greater degree of 'helping' activity was exhibited amongst groups of students who were overtly taught collaborative skills. Results suggest that a coconstructed group environment is more conducive to learning mathematics than an imposed structure.

TALK IN THE CLASSROOM

The study of talk in the classroom involves three disciplines: linguistics, psychology, and sociology. Whilst overlaps in interests are represented by, for example, psycholinguistics and sociolinguistics, each discipline contributes a component to the overall picture. Educationists have, more recently, tended to a sociological interpretation of talk and its relation to learning in classrooms, though the psycholinguistic analysis of classroom talk has done much to move thinking forward and validate the use of peer talk in the classroom. In a review of small group talk, Good, Mulryan and McCaslin (1992) describe "clear and compelling evidence that small group work can facilitate student achievement as well as more favourable attitudes towards peers and subject matter" (p167). This evidence validates further research into small group talk in the classroom.

Research in the area of psychology has offered a relation between talk amongst peers and metacognitive activity. Peterson and Swing (1985) investigated students' cognitions as mediators of the effectiveness of small group work. They use the quality of explanations as a measure of metacognitive activity and relate achievement by outcome to the level of these explanations, both given and received. Their study found that small groups which engaged in high order explanations achieved higher scores in individual work.

Similarly, Larson *et al* (1985) use metacognition as a focus for their study. Their stance is that knowledge about one's own cognition leads to effective self-monitoring activity and thus to more purposive cooperative learning with effective transfer of this learning to the individual. They conclude that metacognitive activity facilitates cooperative learning and that elaborative activity facilitates transfer to individual learning. They summarise a need to "tailor cooperative learning strategies to instructional goals". This involves the use of "cooperative learning with a focus on metacognitive activity" and "elaborative activity within cooperative learning emphasised".

There are difficulties inherent in attempting to assimilate findings such as those above into a study which is essentially naturalistic, though comparative. Firstly, very few studies have undertaken similar psychological research using small group work in mathematics classrooms. The control of variables is such that there are serious implications for the validity of these research findings in naturalistic settings. Furthermore, these studies were especially designed research situations and undertaken in a matter of hours. One of the factors implicit in effective small groups is the time involved in working together (Laborde 1994). This is not accounted for in psychological studies because it is a social factor, though Peterson and Swing acknowledge the possible impact of the nature of the group on cognition.

COLLABORATIVE AND COOPERATIVE SMALL GROUPS

Cooperative and collaborative learning is perhaps the greatest change to be seen in mathematics education in recent years. Classroom discussion, often involving cooperative or collaborative group work, has become an increasing focus for research. Debate still rages about whether discussion *per se* is effective in the mathematics classroom (see, for example, Sfard, Nesher, Streefland, Cobb and Mason, 1998). The consensus is that there are positive effects but questions remain about the nature of discussion which actually generates a positive outcome on mathematical learning.

A good deal is known about cooperative small group learning (for reviews, see Good *et al ibid*, or Cohen 1994). Discussion of cooperative learning in these reviews begins from the assumption that research has already established it as a legitimate means of teaching and learning. For example, peer collaboration is effective for mathematical tasks which require reasoning but not for tasks which require rote learning. Both the above reviews advocate a future focus for research on the socially situated learning which occurs in small groups. They argue that research on small groups has gone beyond a need to justify its benefits through improved learning outcomes. They emphasise the need for work on the factors which affect *discourse processes* rather than factors which affect *achievement outcomes*. There is suggestion of a paucity of process data because of a lack of classroom observations and student interviews which would help to describe what happens during small group interactions and provide information about these processes. There is also a perceived need for cooperative group participants to develop interpersonal skills relevant to group work. Team-building activities conducive to prosocial behaviours are recommended *before* cooperative group work takes place.

Much less is known about collaborative small group work than cooperative small group work (Lyle 1996). This may be due to the divide between US and UK approaches to both cooperative and collaborative learning. The US work tends to test hypotheses in experimental situations whereas the UK work tends to be more ethnographic. As a result, little has been reported about a range of issues such as how the composition and dynamics of collaborative groups affect their ability to function effectively (for a recent report, see Barnes 1998), or whether the students themselves find it an effective way of working. One study which does address this issue is reported by Edwards and Jones (1999). The research discussed above suggests a need for more research to be undertaken on small group work in mathematics classrooms relating to the influence of the task, processes of group interaction, the status of group participants and the need to overtly teach skills for cooperation, the focus of the study described here.

COLLABORATIVE AND COOPERATIVE LEARNING

In an overview of cooperative learning research by Davidson and Kroll (1991) one definition of cooperative learning is "learning that takes place in an environment where students in small groups share ideas and work collaboratively to complete academic tasks" (p362). However, they recognise that this definition encompasses a broad range of practices and meanings applied to the same terms. Most definitions of cooperative learning do not encompass the essence of collaborative learning. This demonstrates the need for a careful definition of terms.

Damon and Phelps (1989) consider it imperative that distinctions are made between types of peer interaction. They describe cooperative learning as utilising distinct principles and practices such as specific role assignments in a group, division of tasks and goal-related accountability of both individuals and the group. It is defined by Damon and Phelps as high on equality and variable on mutuality. Peer collaboration, in contrast, involves groups of novices working together to solve challenging problems which none of the participants could do prior to the collaboration. Relationships in peer collaboration are described as high on equality and high on mutuality.

Much of the research into cooperative learning has not made the necessary distinction between cooperative and collaborative; indeed many studies interchange the terms. Most such studies have centred on outcome objectives as a measure of success. Few have studied the internal dynamics of small group work and the relation of this aspect to cognition. More recent studies that do so include those reported by Webb (1991) and Cobb and Bauersfeld (1995).

For the purposes of this study, I define collaborative learning as that which is constructed amongst student peers working together in self-selected groups (defined in this study as coconstruction). The process involved in mathematical endeavour is as important a focus to the group as the end outcome. Though the aim is to provide a solution to the activity, the lack of an outcome is not seen as failure, as the process is an end in itself because it is viewed as valid mathematical activity.

Some studies of small group work explore ways of supporting such mathematical reasoning by students through problem-solving (for example, Gravemeijer, McClain and Stephan 1998). In a response to Gravemeijer *et al*, Groves (1998 p210) outlines three critical aspects in developing students' powerful problem-solving: "the role of the teacher, the 'design and enactment' of sequences of instructional tasks, and the development of a classroom culture which supports students in explaining and justifying their thinking".

Jaworski (1994 p56), furthermore, describes four aspects of mathematics learning which characterise an investigative approach to learning. These descriptions by Groves and by Jaworski point to the need to develop an environment (or culture or ethos) conducive to the processes of mathematical activity.

THE PEDAGOGIC APPROACH

The pedagogic approach in the study reported here can be described as an emancipatory socioconstructivist model in which a variety of open-ended situations are used as learning contexts. It is emancipatory, firstly, because the teacher is constantly involved in reflective action research (Jaworski, 1992). Secondly, the direction of classroom mathematical activity is directed by students within small groups and assessment outcomes are negotiated between the class and the teacher. The role of the teacher is that of a facilitator of ideas, challenger of decisions, and scaffolder for content learning. The influence of direct teaching is minimal, carefully timed, and designed so as not to affect the power relations established in the classroom.

The classroom approach is socioconstructivist, based on the teacher's underlying belief that mathematics learning cannot occur in isolation of the learner's culture, gender, politics, environment and social history and that these influences will affect the way in which learners construct learning and how they interpret learning situations (see, for example, Ernest 1991). The use of collaborative classroom activity amongst students supports the belief in such an approach.

Research has suggested that the direct teaching of cooperative or collaborative skills may be beneficial to learning (see, for example, Sharan 1990). Some studies in cooperative mathematics learning in small groups call for the direct teaching of collaborative skills or team-building skills to enable groups to work effectively (see Good *et al ibid*, Cohen *ibid*). The study described in this report addresses this call within collaborative groups

DATA COLLECTION AND ANALYSIS

Students in this study attend an inner-city comprehensive secondary (11-16) school. A Year 8 group (12-13 year olds) whom the teacher had taught for the previous two terms received the 'normal' pedagogy. Any references to collaboration were made within the context of the mathematics or mathematics learning. A Year 7 group (11-12 year olds) provided the comparative group. This class received direct instruction (after Mercer *et al* 1999) on collaborative learning skills as a means of learning mathematics. Reference to collaborative organisation was made during lessons where necessary to reinforce this initial introduction to collaboration.

The whole class session for Year 7 was audio-recorded during the 'teaching' of collaborative skills which took the form of a class discussion to establish shared meanings for collaboration. No recording was made for Year 8 at this time as a parallel session did not take place. Throughout the ensuing 10 teaching weeks, pilot sessions of single groups were audio-recorded to accustom students to being recorded. These tapes were not considered as data for this study because of their possible contamination of outcomes in an experimental situation. After 10 weeks all small groups were audio-recorded in each class for three parallel lessons during one week.

To fit with the sociolinguistic approach to the analysis, two methods of analysis are used: constant comparative analysis and categorisation/typology. Data from audio-tapes and field notes was categorised in sections of various lengths, to reflect the nature of the talk. Such sections were defined as 'episodes'. Results were recorded as numerical data to indicate frequency of occurrence with a proviso that each occurrence may not be equivalent to another in terms of representative time. This is a task for further analysis of the data. One aspect of group work that is intimately related to talk which is not exposed by the use of audio tapes is the skill of listening to others. While there may be some evidence in the form of responses to each other that group members are listening to each other, it is a feature which is not directly measurable.

FINDINGS

There were 30 categories of talk identified, some of which were common to both classes, some of which were particular to Year 7 and some to Year 8. These categories are summarised in Appendix 1 with a brief description of each.

There are distinct trends in the results which suggest that the Year 8 class tends to focus on the mathematical skills involved in group work, rather than the organisational skills. Categories 1 and 2 (the organisational categories) are negligible in Year 8. For year 8, category 5 (talking aloud) is a feature which may function also as category 6 (checking with each other). It may serve as a means of keeping members jointly informed of the direction of thinking within the group. This is borne out in a decrease in necessity for category 6 as the amount of talking aloud increases. It is also reflected in a decrease in the number of periods of silence indicating individual work.

Helping behaviours, prevalent in cooperative groups, (categories 7, 8 and 9) were only evident in the Year 7 class. The degree to which assertions were made (category 18) was much lower in Year 8. This is juxtaposed against the tendency for this class to use more questioning, explaining and challenging modes of communication (categories 10, 11 and 12). Categories involving more sophisticated group talk, such as justifying decisions, accepting another's reasoning, confirming behaviours, argumentation and making collective decisions (categories 13, 14, 15, 19 and 22) were also only represented amongst the Year 8 class. Predicting or hypothesising about the work (category 21) and demonstrating some pleasure in the work (categories 23 and 24) were almost equally represented between the classes. However, of 50 instances of 'negative' behaviours (categories 27, 28, 29 and 30) only 4 were exhibited by Year 8.

SUMMARY

The Year 8 class whose learning is implicit through mathematics exhibited many more features of positive mathematical activity. Results therefore indicate that the coconstructed group environment is more conducive to learning mathematics together than an imposed structure. Previous discussion has suggested that the direct teaching of group skills is a lengthy process. A mathematics teacher needs to consider the most efficient *mathematical* use of time in the classroom. The findings from this research suggest that a more efficient use of mathematics lesson time is a focus on the nature of mathematical processes, mathematical actions and the building of mathematical knowledge as a means of promoting effective group skills rather than on the group skills themselves. This is evident in the extent to which the Year 7 class, taught directly, exhibited negative and 'off task' behaviours compared with the Year 8 class.

There is evidence that when group skills are taught directly, it is these skills that remain overt in group practice. The social activity is emphasised more than the mathematical activity in this situation. The directly taught groups referred to group processes during the mathematical activity. Similarly, this class were able to give specific examples of instructions to encourage group activity. In contrast, the class who learned group skills through their mathematical practices, could not think of explicit examples, though indicated that they were aware of these skills being part of their mathematical activity. This suggests a greater focus on the mathematical activity.

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Appendix 1

Categories of talk

1: *Organising (non-work)*: of equipment, paper and other requirements
2: *Organising (work)*: administrative activity related to the functioning of the group;
3: *Comparing data*: occurred when students were generating initial data for analysis;
4: *Assigning tasks*: allocation of specific areas of the activity to each of the members;
5: *Talking aloud*: a 'stream of consciousness' activity by one or more members;
6: *Checking with each other*: ensuring the group were working in the same direction;
7 : *Requesting help*: indicated a request for a peer tutoring arrangement;
8 : *Receiving help*: a member of the group responded positively to a request for help;
9 : *Receiving no help*: a member of the group asking for help was ignored;
10: *Questioning to compare*: assertive roles sometimes requiring explanations;
11: *Explaining*: involved the information directly available to group members;
12: *Challenging*: involved a request to explain or justify findings or decisions;
13: *Justifying*: use of information beyond that worked with at the time;
14: *Accepting reasons*; verbal indication of understanding and/or acceptance;
15: *Confirming*: an offer of a finding with an explanation and a request for agreement;
16: *Agreeing*: a verbal indication that a result or decision was agreed upon;
17: *Seeking agreement without explanation*: an offering made without justification;
18: *Asserting*: a statement made which was not responded to by any other member;
19: *Argumentation*: an argument with a positive direction and a positive outcome in the form of agreement, though not structured as a challenging argument;
20: *Collective act*: two or more members of the group are in verbal unison about evidence or decisions;
21: *Predicting/Hypothesising*: the group is working towards a solution for the activity
22: *Making decisions*: related to outcomes of the mathematical activity and involved agreement amongst members of the group;
23: *Excitement*: reflected by an indication of joy in a finding or decision;
24: *Enjoyment*: reflected by an overt statement to this effect;
25: *Silence*: a period of time when members of the group were working individually;
26: *Comment on group work*: awareness of whether the group was collaborating;
27: *Arguing*: negative interactions, often about assertions about results;
28: *Boasting*: about achieving a result that others in the group had not;
29: *Correcting*: another group member's suggestion without offering an explanation;
30: *Off task work*: chatter about anything other than the task in hand;

UNDERGRADUATE STUDENTS' VERIFICATION STRATEGIES OF SOLUTIONS TO COMBINATORIAL PROBLEMS

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This study is part of a larger study focusing on one of the difficulties in solving combinatorial problems, namely, on the verification of a solution. Our study aimed at identifying verification strategies students employ when solving combinatorial problems, and at evaluating their level of efficiency in terms of contribution to reaching a correct solution. Forteen undergraduate students participated in this part of the study, of which 8 were interviewed individually and 6 in pairs. During the interviews the participants were asked to solve 10 combinatorial problems, thinking aloud, and to try to verify their solutions. Five verification strategies were identified, two of which were rather frequent and more helpful than others. The most frequently used strategy was the least helpful.

BACKGROUND

Combinatorics is one of the important areas of discrete mathematics, which is "an active branch of contemporary mathematics that is widely used in business and industry" (NCTM, 2000, p. 31). It is significant to people's everyday experience as well as their professional practice, and is connected to various strands of mathematics and other disciplines, (e.g., computer science, communication, genetics, and statistics).

Combinatorics is a topic that can, and furthermore should, be an integral part of the mathematics curriculum, from early elementary grades through senior high school (English, 1993; NCTM, 2000). First, because the fundamental principles of Combinatorics are rather easy to comprehend, and rest on very little factual knowledge and memorization. Secondly, many combinatorial problems can be solved intuitively, yet yield themselves to the development of systematic exploration, argumentation and proof (Maher and Martino, 1996). Third, combinatorics provides a rich source for challenging mathematics. Most combinatorial problems echo the spirit of the NCTM (2000) and qualify as "good problems" (ibid, p. 52), as they have the potential of integrating multiple topics, involving significant mathematics, making connections to the learner's personal experiences, fostering the use of different solution strategies, linking various representations, and forming different levels of difficulty.

In spite of the above, Combinartorics is considered one of the more difficult mathematical topics to teach and to learn. Most problems do not have readily solution strategies, and create much uncertainty regarding how to approach them and what strategy to employ. Moreover, verifying an answer to a combinatorial problem is a particularly difficult task, because there are no guaranteed ways to ensure the detection of an error. There are numerous examples in which different solutions to the same problem, resulting in different answers may seem equally convincing. In addition, the detection of an error does not necessarily yield a correct the solution. Several studies support the assertion that students encounter many difficulties in solving combinatorial problems and shed light on some factors contributing to these difficulties (Batanero et al, 1997a, 1997b; English, 1991; Fischbein & Gazit, 1988; Kahneman & Tversky, 1973). Although verification plays a critical role in problem solving (Polya, 1957; Schoenfeld, 1984; Silver, 1987), we have not found any studies that have investigated the ways in which students and teachers cope with the difficulties associated with the verification of combinatorial problems. Our study is a step towards understanding students' approaches to verifying solutions to combinatorial problems.

THE STUDY

Goals: The main goal of the study reported hereof was to identify and characterize the ways in which students deal with the need to verify their solutions to combinatorial problems and the verification strategies they employ.

Participants: The participants in the study consisted of 14 undergraduate students all of which had completed a basic course in combinatorics prior to the study.

Method: In order to be able to follow, describe, and understand the problem solving processes of the participants a qualitative method was employed. Data collection was done by audio taped interviews and field noted observations.

Borrowing from Schoenfeld (1985), the participants were encouraged to work in pairs when engaged in the problem solving tasks. Those who felt comfortable to work with a peer were grouped in pairs. Thus, there were 3 pairs of students, each of which worked collaboratively on the combinatorial problems they received. The remaining 8 students worked individually and were prompted by the researcher to think aloud. Altogether, 11 interviews were conducted with the students. All interviews were transcribed and coded.

Every problem, for each individual or pair, at each stage of the interview, was coded according to a number of parameters, including its degree of correctness (correct, partially correct, or incorrect), the number of attempts to verify it, and when applicable, the verification strategies employed. For most of the analysis we combined the partially correct and the incorrect solutions in one category named "Incorrect Solutions".

Research Instruments: The interviews were semi structured and consisted of two parts: In the first part of the interview participants were asked questions related to their personal background, as well as their experience with and attitudes towards combinatorics. The second and main part of the interview focused on problem solving and verification of solutions to ten combinatorial problems.

The main interview included a set of 10 combinatorial problems specially designed for the study. The problems varied with respect to their underlying model, according to Dubois (1984): a selection model, a distribution model, or a partition model. All problems required only basic combinatorial tools for solution. However, in order to create some uncertainty that would foster the need to verify the solutions, each problem required the use of some blend of principles and operations (combinations, arrangements, and permutations) for its solution. In addition, 2 of the 10 problems were very similar problems, and were given to the participants consecutively. This was done in order to see whether the participants, first, identified the connections between the two problems and secondly, built their solution and verification of the second problem on the first one.

For each problem there were 1-2 occasions when the interviewer prompted the student (or pair of students) to verify the solution. Accordingly, for each problem the interview consisted of 2-3 stages: the first stage V_1, in which the student(s) solved the problem with no prompting at all, the second stage V_2, in which they were prompted to verify their solution, independently of its correctness, and the third stage V_3, in which only those that held an incorrect solution were prompted to verify their solution once again, disclosing the fact that their solution was incorrect.

FINDINGS

Through the 11 interviews that were conducted with the students (3 with pairs of students and 8 with individual students) we received altogether the total of 108 solutions.

The findings include a description of the different types of verification strategies employed by the participants and an analysis of their usefulness.

The Verification Strategies

Altogether, 219 attempts were made by students to verify their solutions at Stages V_1, V_2, and/or V_3. All attempts to verify a solution were analyzed and classified into 5 main categories of verification strategies:

1. Verification by *reexamining* the solution;
2. Verification by *adding justifications* to the solution;
3. Verification by *evaluating the reasonability* of the answer;
4. Verification by *modifying some components* of the solution;
5. Verification by *using a different solution method.*

We turn to a description of the verification strategies.

Strategy 1: Verification by *reexamining* the solution

The participants who used this strategy reexamined their solution by going over and checking all or parts of it a second time, without adding any substantial justifications to their solution. This kind of checking focused on various aspects of the solution, such as rechecking their calculations or the extent to which their original plan for solution was carried out. In several cases, this strategy served as a springboard for a more profound strategy of verification.

Strategy 2: Verification by *adding justifications* to the solution

The participants who used this strategy added justifications to their solution to support it. The justifications referred to either a particular step in the solution or to a more global aspect of the solution. Generally, the justifications were of three mutually related types:

One type was directed to clarify and support some (or all) specific parts of the solution. A second type aimed at justifying in a more global way the model (or formula) used to solve the problem, by showing how the conditions and nature of the problem match the model. A third type of justification was based on an analogy to another, more familiar or previously solved problem, the solution of which was known to the participant. This strategy was particularly helpful for improving partially correct solutions, in which the general solution strategy was appropriate, however, there were some steps in which an error occurred in applying it.

Strategy 3: Verification by *evaluating the reasonability* of the answer

The participants who used this strategy looked at the final result they had obtained and tried to examine its reasonability either by an intuitive estimate, or more commonly, by calculating the size of the outcome-space. In a number of cases, the participant noticed that the result s/he had reached was larger than the outcome-space, which did not make sense. In some cases this strategy led to the identification of wrong answers, however, it was not helpful in locating the specific erroneous considerations and steps in an incorrect solution.

Strategy 4: Verification by *modifying some components* of the solution

The participants who used this strategy made one of the following modifications to their original solution: They either altered the representation they had used in their solution or tried to apply the same solution method by using smaller numbers. Those who used the former approach tried to represent the situation of the problem in a different way, mostly by using some visual symbols (e.g., circles, squares, blocks etc.) to represent the different components of the problem. Unfortunately, this kind of attempt did not prove helpful for them, because they repeated the same considerations and arguments as in their original solution, failing to identify any faulty step. One student used the latter approach, that is, used smaller numbers. This enabled him to detect an error and identify where he went wrong in his solution.

Strategy 5: Verification by *using a different solution method*

The participants who used this strategy employed a completely different solution method for the problem. In these cases, the new solution method led to either the same result that they had reached or to a different one. Table 1 presents the different cases according to the correctness of the first and second solution methods, the difference between the results obtained in each way, and consequently, the decisions the participants made within this strategy of verification.

As seen in Table 1, Strategy 5 was used altogether in 57 cases, of which, 43 cases were with an incorrect solution, and 14 cases with a correct solution. Of the 43 cases with an incorrect solution, in 12 cases this strategy proved helpful in correcting their solutions. In one case a student managed to reach a correct solution in the second time, however, made a wrong choice, thus remained with his first incorrect solution. Interestingly, there were two pairs of students and one individual student who reached the same incorrect result in two different solution methods, thus their answers remained incorrect leaving them more confident of their solutions.

First Method	Second Method	Comparison of Results	Choice of Solution	Individual Students (8)	Pairs of Students (3 pairs)	Total
Incorrect (N=43)	Correct (N=13)	Different	Second	11	1	12
			First	1	-	1
	Incorrect (N=30)	Same	Same	1	2	3
		Different	Second	15	2	17
			First	9	-	9
			None	-	1	1
Correct (N=14)	Correct (N=12)	Same	Same	3	9	12
	Incorrect (N=2)	Different	Second	2	-	2

Table 1: Decisions made by participants employing verification Strategy 5

Of the 14 cases with a correct solution, 12 cases remained correct in both solution methods. However, this strategy led to two unfortunate decisions (in stage V_2), where a wrong result was obtained the second time, causing a switch from a correct to an incorrect solution. Luckily, in the following stage (V_3) they both corrected their solution.

Some Comparisons between the Verification Strategies

Altogether, there were 219 verification attempts made, in various stages, to incorrect as well as correct solutions. One hundred and sixty six attempts were made by students working individually, and 53 by students working in pairs. Table 2 presents the distribution of the number of verification attempts by the different verification strategies. It should be noted that the distribution of the verification attempts of only the incorrect solutions was similar to the one in Table 2.

It should be noted that there were no conspicuous differences between the students who worked individually and those who worked in pairs, in the distribution of use of the different verification strategies. The most frequently used strategies were Strategy 1 (38%), Strategy 2 (26%), and Strategy 5 (26%).

Types of Verification Strategies	Individual Students (8)	Pairs of Students (3 pairs)	Total
1.Reexamining the solution	66	17	83
2.Adding justification to the solution	42	15	57
3.Evaluating the reasonability of the answer	10	1	11
4.Modifying some components of the solution	6	5	11
5.Using a different solution method	42	15	57
Total	166	53	219

Table 2: Distribution of attempts to verify a solution by the type of verification strategy employed, in percents

A further analysis of the use of the different verification strategies focused on the extent to which these strategies were helpful in leading the participants to improving their solutions. In this analysis we considered an improvement as a change from an incorrect solution to a partially or completely correct solution, or from a partially correct to a completely correct solution. Altogether, there were 27 improved solutions as a result of employing a verification strategy (note that some improvements are attributed to more than one verification strategy).

Table 3 presents the distribution of the improved cases by the different verification strategy and by group. As shown in Table 3, the most helpful verification strategies were the 2nd (in 12 cases) and the 5th (in 13 cases). Note, that there were 5 cases in which a combination of verification strategies led to an improved solution. For example, one student used Strategy 3 to detect that an error had been made, then moved to Strategy 4 through which he identified the kind of error that he had made, and finally used Strategy 5 to correct his solution.

Types of Verification Strategies	Individual Students (8)	Pairs of Students (3 pairs)	Total
1. Reexamining the solution	2	0	2
2. Adding justification to the solution	8	4	12
3. Evaluating the reasonability of the answer	3	0	3
4. Modifying some components of the solution	1	0	1
5. Using a different solution method	11	2	13
Total	25	6	31

Table 3: Distribution of cases in which the use of a verification strategy was helpful in the process of improving a solution, by type of verification strategy and group

A close look at the number of cases, for which the use of a verification strategy was helpful, points to a considerable difference in favor of the individual students compared to the pairs. However, this is due to the fact that the students who worked in pairs were significantly better in reaching correct solutions to begin with.

Another measure of efficiency of the verification of a solution was obtained, for each stage separately, by the percentage of the number of cases in which the use of a verification strategy was helpful out of the total number of incorrect solutions that were verified at the respective stage. Thus, 13% of the incorrect solutions in stage V_1 were improved, 12% of the incorrect solutions in stage V_2 were improved and 30% of the incorrect solutions in stage V_3 were improved. Clearly, there was an increase in the efficiency of the verification attempts in stage V_3, when the students became aware that their solution was incorrect.

DISCUSSION

Our findings support the merit of encouraging students to verify their solutions to combinatorial problems. Apparently, when pushed to it, students are capable, to a certain extent, of finding efficient ways to verify their solutions. We suggest that through such experiences students are likely to become aware of the potential of verifying their solutions, and hopefully, will be motivated to verify their solutions out of their own initiative. The differences that were found with respect to the efficiency of the various strategies may serve as a springboard for teaching problem solving in combinatorics with a focus on metacognitive processes, in general, and on the use of more efficient verification strategies, in particular.

As shown above, Strategy 1 was the most frequently used, however, it turned out one of the least efficient strategies, in terms of helping students shift towards an improved solution. The low efficiency level of Strategy 1 is in accordance with the view that merely going over a solution of a mathematical problem is insufficient for verifying it (Polya, 1957; Schoenfeld, 1985). For those who applied Strategy 2, which turned out considerably helpful, what proved useful was mainly the need to add justifications to the various steps in a solution. This was helpful particularly in detecting minor errors. Unlike Strategy 2, Strategy 3 was not very frequent. This is mainly because estimating an expected outcome in a combinatorial problem is extremely hard to do. Fischbein and Grossman (1997) found that when asked to estimate such results, students usually gave lower estimates than the actual number. Thus, Strategy 3 was helpful in detecting that an error had occurred only in cases when the answer that was obtained was larger than the size of the outcome-space and when the student(s) applying this strategy compared these two numbers. Strategy 4 was also the least frequent. We speculate that most of the students were not familiar with this strategy and did not know how to apply it. In another part of our study, in which expert mathematicians were interviewed about useful verification strategies to combinatorial problems, they recommended Strategy 4 as one of the most helpful methods, provided the solver does it carefully and is aware of some underlying subtleties of using smaller numbers. Finally, we turn to Strategy 5, that was both frequent and rather helpful, and point to its limitation. There were three cases in which students reached the same incorrect answer in two different solutions strategies. This raised their degree of confidence in their incorrect solution.

To conclude, we bring a quote from a student, conveying the limitation of Strategy 5: A pair of students who reached stage V_3 was told that their solution was incorrect. In response, one of them suggested solving the problem in a different way. Before they started to carry out his plan, he turned to his partner and said: "if we reach the same result in a different way then the answer is right and the question is wrong".

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DEVELOPMENT OF 10-YEAR-OLDS' MATHEMATICAL MODELLING

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This paper addresses the developments of a class of fifth-grade children as they worked modelling problems during the first year of a 3-year longitudinal study. In contrast to usual classroom problems where students find a brief answer to a particular question, modelling activities involve students in authentic case studies that require them to create a system of relationships that is generalisable and re-usable. The present study shows how 10-year-olds, who had not experienced modelling before, used their existing informal mathematical knowledge to generate new ideas and relationships, and how these developments were fuelled by significant social interactions within small group settings.

INTRODUCTION

Our ever-changing global market is making increased demands for workers who possess more flexible, creative, and future-oriented mathematical and technological skills (Clayton, 1999). Of importance here is the ability to make sense of complex systems (or models), examples of which appear regularly in the media (e.g., sophisticated buying, leasing, and loan plans). Being able to interpret and work with such systems involves important mathematical processes that are under-represented in the mathematics curriculum, such as constructing, describing, explaining, predicting, and representing, together with quantifying, coordinating, and organising data. Dealing with systems also requires the ability to work collaboratively on multi-component projects in which planning, monitoring, and communicating results are essential to success (Lesh & Byrne, in press).

Given these societal and workplace requirements, it is imperative that we rethink the nature of the mathematical problem experiences we provide our students— in terms of content covered, approaches to learning, ways of assessing learning, and ways of increasing students' access to quality learning. This paper reports on one approach to addressing this issue within the primary school curriculum, namely, through mathematical modelling activities. Although these activities provide all students with rich learning opportunities, their use with younger children has received limited attention.

MATHEMATICAL MODELLING FOR CHILDREN

Problem solving over the past couple of decades has typically engaged children in problems where the "givens," the "goals," and the "legal" solution steps have been specified clearly; that is, the interpretation processes for the student have been minimalized or eliminated. The difficulty for the solver is simply working out how to get from the given state to the goal state. The solutions to these problems are usually brief answers obtained from applying a previously taught solution strategy, such as "guess and check," or "draw a diagram." Furthermore, although these problems may refer to real-life situations, the mathematics involved in solving them is often not real world and rarely do the problems provide explicit opportunities for learners to generalize and re-apply their learning (English & Lesh, in press). While not denying the importance of these problem experiences, they do not address adequately the knowledge, processes, and social developments that students require in dealing with the increasingly sophisticated systems of our society. Mathematical modelling activities, in the form of meaningful case studies for children, provide one way in which we can overcome this inadequacy.

As used here, models are systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other experienced system (Doerr & English, 2001). The modelling activities of the present study engage small groups of children in challenging but meaningful problem situations that encourage multiple solution approaches and multifaceted products. Key mathematical constructs are embedded within the problem context (which takes the form of a case study) and are elicited by the children as they work the problem. In contrast to typical classroom problems, these case studies require children not only to work out how to reach the goal state but also to interpret the goal and the given information along with permissible solution steps. Each of these components might be incomplete, ambiguous, or undefined; furthermore, there might be too much data, or too little data, and visual representations might be difficult to interpret. When presented with information of this nature, children might make unwarranted assumptions or might impose inappropriate constraints on the products they are to develop. This is where the input from group members plays a powerful role.

Unlike traditional non-routine problems, modelling activities are inherently social experiences (Zawojewski, Lesh, & English, in press). Their design demands the work of small teams of students, who must develop a product that is explicitly sharable. Team efforts are required to generate the multifaceted products that involve descriptions, explanations, justifications, various mathematical representations, and frequently, media presentations. Numerous questions, issues, conflicts, revisions, and resolutions arise as students develop, assess, and prepare to communicate their products. Because the products are to be shared with and used by others, they must hold up under the scrutiny of the team (Zawojewski, Lesh, & English, in press).

DESCRIPTION OF THE STUDY

The present study involves a class of 30 ten-year-olds and their teachers, who are participating in a 3-year longitudinal study of children's developments in mathematical modelling. The children are from a co-educational private school that caters for preschool through to Year 12. Drawing upon the multitiered teaching experiments of Lesh and Kelly (2000), the study has adopted a four-tiered collaborative model that addresses the simultaneous development of researchers, classroom teachers, preservice teachers and classroom students (English, submitted). This paper, however, is confined to the developments of the children.

The data reported here are from the first year of the study (2001) and are drawn from two of the activities the children completed towards the end of the program. The program commenced in June 2001 and continued until November 2001, with a month's break between September and October. The children had not experienced modelling activities prior to the program. The classroom teacher implemented each of the activities. Preparatory meetings and feedback sessions were conducted with the classroom teacher and with the preservice teachers. The children usually worked the modelling activities twice a week, with each session lasting around 80 minutes.

Description of the Modelling Activities

The program commenced with preparatory experiences where the children expressed their feelings about, and their perspectives on, mathematics and mathematical problem solving and posing as they occurred in and out of school. The children also explored a range of non-routine problems, where they analyzed the mathematical structures, identified similar structures, discussed ways they would approach the problems, and shared their solutions. These experiences were followed by introductory modelling activities (from the "Packets Investigators"; ETS, 1997), with more advanced modelling being introduced next, as indicated below.

The activities that elicited the present data are the *Aussie Lawn Mowing Problem* and the *Christmas Holidays Jobs Problem* (adapted from Hjalmarson, 2000, and Lesh & Lehrer, 2000). Both problems involve interpreting and dealing with multiple tables of data, exploring relationships among data, using proportional reasoning and the notion of rate, and representing findings in visual and written forms. The Aussie Lawn Mowing Problem involves three components: (a) a warm-up task comprising a mathematically rich "newspaper article" designed to familiarize the children with the context of the modelling activity, (b) questions to be answered about the article, and (c) the model-eliciting activity. Excerpts from the third component appear as an appendix. The Christmas Holidays Job serves as a model application activity where children apply their learning within a new context (selecting part-time and full-time vendors for an amusement park, based on their performance in the previous Summer). The children spent two sessions in completing each of the problems. After the groups had developed their models, they presented their work to the class for questioning and constructive feedback. Next, a class discussion focused on the key mathematical ideas and relationships that developed.

Data Sources and Analysis

The data sources included audio- and video-tapes of the children's responses to the problem activities, together with their work sheets and final reports detailing their models and how they developed them. Field notes were also taken. The data were analyzed for evidence of children's mathematical and social developments over the course of the activities. In the next section, consideration is given to the progress of groups of children across the two problems. There were six groups of mixed achievement levels. The children's developments are reported in terms of cycles of increasing sophistication of mathematical thinking, with each cycle representing a shift in thinking (Doerr & English, 2001). The cycles also display significant social interactions that impacted on the children's modelling developments.

RESULTS

Cycle 1: Focusing on Subsets of Information

Each of the groups commenced the Aussie Lawn Mowing Problem by scanning the tables of data to find employees who scored highly in one or more of the categories (i.e., hrs worked, no. of lawns mowed etc.). Limited mathematical thinking was displayed in this unsystematic approach, as evident in Gavin's (Group 1) comments: "*Also, I think Jonathon is good because he works top hours and doesn't drive much. Also mows quite a lot of lawns and makes a bit of money....*"

While most groups initially used this approach, Group 4 decided to choose employees "with different specialities" and remained with this decision in developing their model: "*We'll get Travis to work at the shop selling fertilizers and all that—from 8 to 5—that's about 9 hours. He earns the most money from the info we've got, so if we have our best worker at the shop they make the most money.... What's Matthew's speciality? He loves big lawns. Matthew could work all the time from 9 to 5 because mowing is a lot...what's Jonathon like? Jonathon likes small lawns. He could do the small lawns.*" Because the Lawn Mowing Problem lacked some information, the groups frequently brought in additional ideas and assumptions based on their real-world knowledge (e.g., hours the garden shop should open; how much customers should be charged; how much the employees should be paid).

With the exception of one group, none of the groups commenced the problem by considering whether some items of information were more important than others or whether some information might be irrelevant. The children did, however, engage in heated debates over how to interpret "kilometres driven" and whether more kilometres driven indicated a more desirable employee. Again, the children used their informal knowledge to make a number of conjectures and justify their claims:

Tim:	No, wait a second. We're looking at how many kilometres you drive **in a truck that's owned by them; that's bad.**
Samantha:	No, it's good if you drive a lot because that means that you're not a slacker, not lazy, and you're willing to go and drive over to someone's.
Ben:	Isn't it social, like they're just going out to buy some beer or something.
Tim:	**No, Company truck. It costs a lot of money to have company trucks.**

Because most of the groups did not use any systematic approach to tackle the problem initially, they frequently argued over which employees should be chosen. This led them to see the need to mathematize, in some way, their employee selection. The groups began to use two main mathematical operations to aggregate the data for each employee, namely, (a) simply totalling the amounts in each category (hours worked, kilometres driven etc.), and (b) finding the average for each category.

Cycle 2: Using Mathematical Operations

The need to mathematize their procedures was initiated in Group 1 by Joanne, who challenged Gavin over his unsystematic approach: "*Gavin, not necessarily he [Jonathon] mows the most lawns..... How about we work out the hours they work. How about we work out their average.*" She justified her decision by explaining, "*Well, it's kind of difficult working out how much they worked each month. Sometimes they worked less and sometimes more.*" Gavin and Alison remained unconvinced, however, so Joanne, along with Mindy, proceeded to work out the average number of hours worked, lawns mowed (treating, 'big,' 'medium,' and 'small' separately), kilometres driven, and money from products sold for each employee listed in the tables. At the same time, substantial discussion and argumentation took place when the group members tried to convince Alison that she was misinterpreting the table of money from products sold (interpreting it as the amount of money the company *paid to the employees*.) Despite repeated explanations (e.g., "*Average money per week from **products sold**. They **don't give** them this.*"), Alison would not accept their arguments. In fact the group became quite bogged down as Gavin and Alison continued to argue over the meaning of the table.

Joanne and Mindy, however, remained very much on task and frequently had to remind Gavin and Alison that they had to apply a systematic method: "*You have to work it out properly. Like, you can't just get someone you like and say, 'O.K., I like this one'...I don't think that works. You have to work out like Joanne's doing—working out the average —and then understanding what they are and everything.*"

Because Alison (and at times, Gavin) insisted on using an unsystematic approach, Joanne and Mindy frequently reminded them of the problem goal and clashed over their interpretations of "best people." For example, when Gavin and Alison were insisting that they include a person "who isn't *that* bad," Joanne and Mindy insisted, "*But there are four better people. We need the **best** people.*" Alison retorted, "*It* [the problem statement] *doesn't say **good** employees.*"

The group continued to be divided over their approaches to solving the problem. While Joanne was working out all the averages, Mindy was repeatedly reminding Alison and Gavin of the need for a method: "*Some structurable thing.*" Mindy also challenged the selections of Gavin and Alison:

Mindy: Alison, how do you know that? (that Mathew is one of the best)

Gavin: Look at all the stats!

Mindy: So, just on that information, you can't just say, "O.K., we want him." This is pulling people out of the hat again.

Alison: We're not pulling people out of the hat. We're just compromising.

Mindy: Where's the structure for that?

Once all the averages had been found, the group did not progress further. They selected those employees who scored high averages across all categories, explaining in their report: "*Well, we worked out the average for average money per week from the products sold and looked for the 4 highest and did the same for the hrs worked.*"

Cycle 3: Identifying Trends and Relationships

Two of the groups progressed to looking for trends and relationships to help them choose the employees for the Lawn Mowing Problem. Group 4, for example, explored trends within categories (e.g., "*Kim is always gaining... 200, 250, 256*" [in the money category]). This led the group to compare trends across categories: "*So Travis should be our first guy. He may have done 5 less hours than Jonathon, but he did more jobs.*" Group 4 did not progress to the notion of rate, however, in part because they kept conjecturing about why the trends occurred (e.g., "*With the lawns mowed, they hand them out maybe, but then if they hand them out, he* [Aaron] *might not have been able to get them because someone else got them*").

On the applied problem (*Christmas Holidays Job*) all of the groups transferred their learning and frequently referred to what they had done in the previous problem (which they had solved 5 weeks earlier). Furthermore, 4 of the 6 groups extended their understanding by exploring relationships between hours worked (by vendors in a previous Summer) and money collected (for busy, steady, and slow periods). The groups used these relationships as their basis for deciding whom to employ full and part time in the next Summer. In Group 2, for example, Marianne was recapping on what she thought was the group's decision: "*I thought the person who works the less or the least but makes the most money we were going to employ because that would work.*" To counter some disagreement and misunderstanding that followed (e.g., "*No, because if they work tons of time they earn tons of money*"), Marianne provided a concrete example: "*Because, like someone might work for say, 10 days and make $5000, and somebody might work for 5 days and they get $10,000.*"

Likewise, Group 5 spent considerable time debating the relationship between hours worked and money earned, which led to observations such as, "*Chad works 20 hours. Will works 19. Chad earns $1031. Will earns $1034. Will works one hour less, and he makes $3 more.*" Earlier, Group 5 had been working out averages, but once they had explored these relationships, a couple of the group members noted, "*Why did we have to use averages? Why in the world did we use averages? That doesn't make sense!*" Using their understanding of kilometres per hour, two of the groups progressed to calculating rates (money per hour) in developing their model.

CONCLUDING POINTS

The first year of this study has shown how one class of 10-year-olds was able to work successfully with mathematical modelling problems when presented as meaningful, real-world case studies. On the problems addressed here, the children progressed from focusing on isolated subsets of information to applying mathematical operations that helped them aggregate the given data. Moreover, some children displayed an explicit awareness that they needed to adopt a structured approach to developing their final model. While some groups remained with averages, other groups moved on to discover relationships and trends in the data, and applied this learning to the second problem. These developments took place in the absence of any formal instruction, and involved the children in describing, constructing, explaining, justifying, checking, and communicating their ideas. Of significance in these developments are the social interactions that took place naturally within the groups. These interactions engaged the children in planning and revising courses of action, challenging one another's assumptions and claims, asking for clarification and justification, monitoring progress, and ensuring the group worked as a team. Few traditional problems generate learning of this nature.

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APPENDIX

Aussie Lawn Mowing Problem: Green Thumbs Garden to Open Soon

Part (c): The model-eliciting activity

Background Information: At Green Thumb Gardens, James Sullivan will provide lawn-mowing service for his customers. Another local landscaping service has closed, so he has offered to hire 4 of their former employees in addition to taking on some of their former clients. He has received information from the other landscaping business about the employee schedules during December, January, and February of last year. The employees were responsible for mowing lawns and selling other yard products like fertilizer, weed killer, and bug spray. The other business recorded how many hours each employee worked each month, the number of lawns each employee mowed, and how much money they made selling other products. The lawns mowed are divided into big, medium, and small jobs. Big jobs may have larger lawns or additional work than medium or small jobs. Some lawns may be small, but may have many obstacles for the mower to get around or there may be different kinds of edging or trimming to be done which determine the size of the job. They also recorded the kilometres driven to clients in one of the green company trucks during each month.

Problem: James needs to decide which four employees he wants to hire from the old business for this summer. Using the information provided, help him decide which four people he should hire. Write him a letter explaining the method you used to make your decision so that he can use your method for hiring new employees each summer. (The following **tables** were supplied [data for 5 of the children have been omitted here].

Hours Worked			
Employee	Dec.	Jan.	February
Jonathan	80	80	80
Cynthia	75	65	70
Jack	66	64	63
Kayla	45	50	55
Tim	67	70	79

Kilometres Driven			
Employee	Dec.	Jan.	Feb.
Jonathan	198	200	201
Cynthia	199	201	198
Jack	197	199	198
Kayla	201	203	199
Tim	200	199	200

Total Number of Lawns Mowed									
	December			January			February		
Employee	Big	Medium	Small	Big	Medium	Small	Big	Medium	Small
Jonathan	15	12	30	16	14	34	16	15	35
Cynthia	18	10	35	19	12	35	14	16	36
Jack	14	16	22	15	16	22	13	16	22
Kayla	15	13	15	14	13	17	15	12	18
Tim	20	12	14	22	14	16	20	13	25

Average Money Per Week from Products Sold			
Employee	December	January	February
Jonathan	$150	$175	$170
Cynthia	$75	$80	$80
Jack	$125	$150	$150
Kayla	$80.	$72	$65
Tim	$135	$130	$125

IMPLICATIONS OF COMPETING INTERPRETATIONS OF PRACTICE TO RESEARCH AND THEORY IN MATHEMATICS EDUCATION

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In this article we explore the issue of interdependency of theory and research findings in the context of research on the practice of mathematics teaching and learning at school. We exemplify how analyses of a lesson by using two different theoretical perspectives lead to different interpretations and understandings of the same lesson, and discuss the implications of competing interpretations of practice to research and theory in mathematics education.

INTRODUCTION

In their chapter on 'Competing paradigms in qualitative research', Guba and Lincoln (1994) point to an interesting, sometimes unforeseen, connection between the theoretical framework used by the researcher and the findings of the research. They claim that "theories and facts are quite *interdependent* – that is, that facts are facts only within some theoretical framework" (p. 107). Moreover, they assert, "Not only are facts determined by the theory window through which one looks for them, but different theory windows might be equally well supported by the same set of 'facts'." (p. 107). In this article, we explore this issue of interdependency of theory and research findings in the context of research on the practice of mathematics teaching and learning at school. We exemplify how analyses of a piece of practice – a lesson – by using two different theoretical perspectives lead to different interpretations and understandings of the same lesson, and we discuss the implications of competing interpretations of practice to research and theory in mathematics education.

BACKGROUND

The lesson to be analyzed was part of an innovative yearlong introductory course on functions for ninth-grade. The curriculum developers aimed for the students to investigate problem situations with computerized tools, raise hypotheses, collaborate on problem solving, explain and discuss their solutions, and reflect on their learning in individual or collective written reports. After a successful implementation of the course in high-achieving classes (A-level), the function course is tried in lower-achieving classes (B-level).

The teacher of this trial has been a central member of the curriculum development team. She has decades of experience in the dual role of high-school mathematics teacher and curriculum designer. In both roles she regularly sought for innovations in content and ways of teaching, and systematically reflected on her own teaching and the learning processes of her students. The year preceding the present experiment, she successfully taught the function course to an A-level class in the high school where she regularly teaches mathematics – a religious all-girl academic oriented school. The experimental B-level class belonged to the same school.

The lesson we analyse in this article took place rather early in the school year. It centred on the "Fence Problem", displayed in Figure 1.

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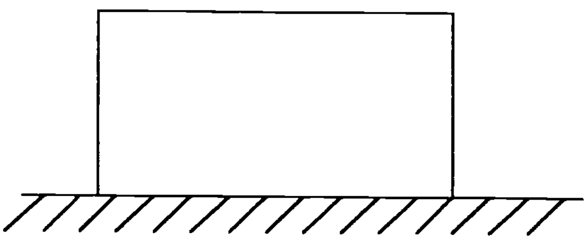
The Fence Problem

Oranim school received a 30m long wire to fence a rectangular vegetable garden lot. The lot is adjacent to the school wall, so that the fence has three sides only.



a. Find four possible dimensions for the lot and their corresponding areas.

b. For which dimensions does the lot have the largest area?

c. If one of the dimensions is 11m, what is the area of the lot? Can you find another lot with the same area? If you can, find its dimensions; if not, explain.

d. How many lots with the following areas are there: $80m^2$? $150m^2$?

****

Figure 1. The Fence Problem

The work on the Fence Problem was spread over several lessons. One of them is the "Windows lesson" which is the focus of this study. The Windows lesson was designed to enhance student learning of the concept of function, their understanding about different representations of functions and how to use the graphic calculator to solve problems that require the passage from one representation to another (mainly, between algebraic and graphic representations).

The following two sections present analyses of the 50-minute Windows lesson from two different theoretical perspectives. We first analyse the lesson from a cognitive perspective and then from a socio-cultural perspective.

VERBAL ANALYSIS: A CLASSIC COGNITIVE APPROACH

Cognitive Science focuses on processes and representations. The research question of the cognitive component of the study concerns with the extent to which students, during the Windows lesson, conceive the passage to a new (especially graphical) representation of a function as a problem solving *method*. We use verbal analysis (Chi,

1997) as a representative of Cognitive Science approach. The data for the verbal analysis is the protocol derived from the video recording of the three whole-group discussions that comprised about one-half of the class time (the rest of the lesson comprised of small-group student work). The whole-group discussions provided rich and coherent verbal data, enabling us to consider the whole group of students as an entity interacting with the teacher.

Segmenting the protocol

In line with the research question focus, we chose the passage from one representation to another, as well as the passage within the same representation – between referring to it as a display or an action representation (Kaput, 1992) – as a natural boundary for segments. The different representations of function in the Windows lesson were algebraic, tabular, graphic, and verbal. We identified 19 segments in the reduced protocol.

Coding

For each segment we identify who initiated the passage to a new representation (the teacher or the students), who was the trigger for this passage, and what was the nature of the interlocutor's response (if any). We define five kinds of responsiveness. *Accompanying* talk refers to talk in which the interlocutor attends to the other's talk without elaboration, typically acknowledging that the interlocutor follows the other's talk. *Elaborating* talk refers to talk in which the interlocutor elaborates utterances and expresses deeper cognitive involvement. *Opposition* refers to talk in which the interlocutor explicitly expresses disagreement and objection. *Puzzlement* points to talk expressing confusion, perplexity or bewildering. Finally, *non-responsiveness* refers to the absence of talk subsequent to the initiator's talk. We also distinguish between a passage to a different representation that is embedded in the context of the Fence problem situation and a passage to a different representation that is disconnected from a situational context. Moreover, as we assume that the nature of the class discourse is related to student understanding, we characterise the types of utterances at each segment according to the following categories: Presentation (Pr), Short Questions (SQ), Extended Questions (EQ), Short Answers (SA), Extended Answers (EA), Rephrasing/re-voicing (R), Objection (Ob).

Depicting the coded data

Depicting the coded data is conducted in two ways. One way is displayed in Figure 2. The segments of the lesson are marked in the first row (segments I, S and V indicate the small-group work parts of the lesson which are not part of the analysis). Each of the following four rows designate one of the four representations to which the class talk refers when the context concerns the Fence situation. Sit designates segments in which participants focus specifically on objects and events of the Fence task; Gr, Tab, and Al designate segments in which the talk refers to graph, table and algebra/formula (respectively). The last three rows similarly designate the representational systems to which the talk refers, but here, with no connection to the Fence situational context but rather related to a formal task.

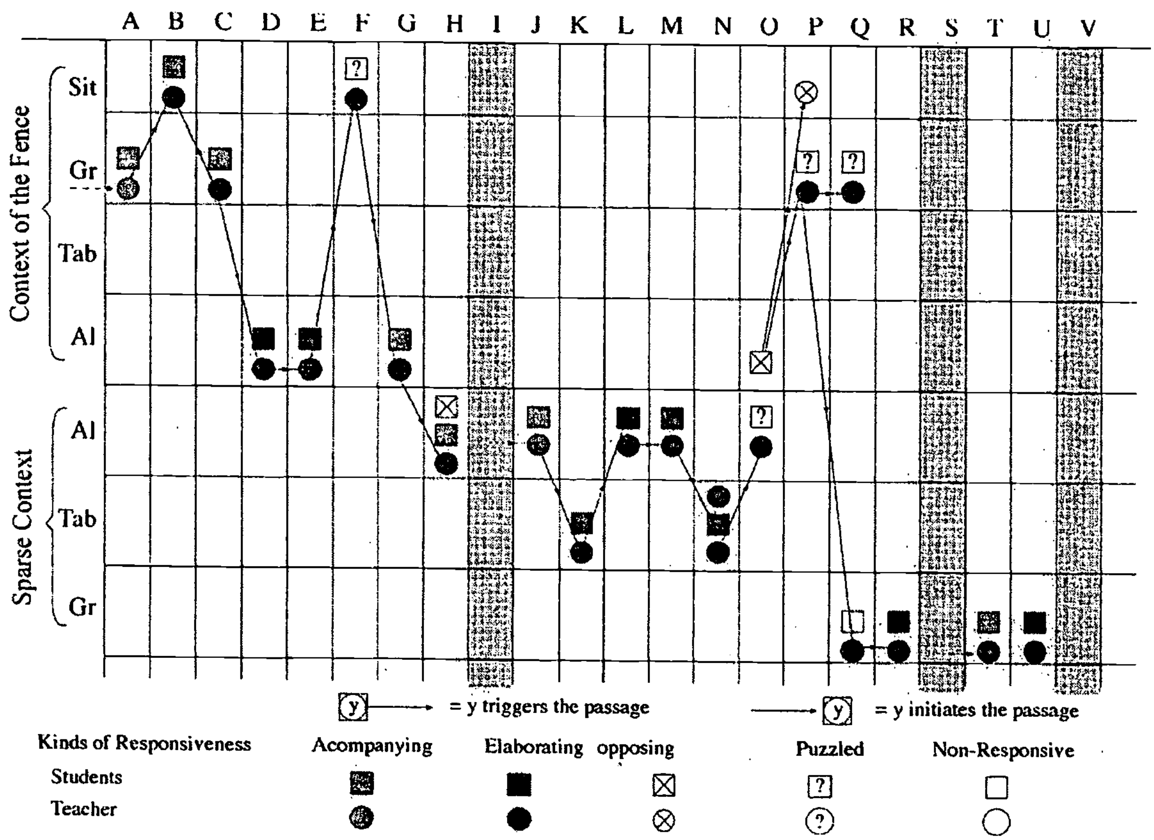


Figure 2. Schematic description of the Windows lesson

Circles represent the teacher and squares, the students. For each segment we mark who initiates the passage to a new representation or the change in the way the representation is used (display or action) by an arrow targeted to the initiator. The trigger for the passage to a different representation is marked at the beginning of the arrow. The nature of responsiveness is marked as following: gray represents *accompanying* talk, black – *elaborating* talk, inscribed question mark – *puzzlement*, crossed – *opposition*, and white – *non-responsiveness.*

Figure 2 displays a comprehensive picture about initiatives and the nature of responsiveness regarding the passing from one representation to another. But it does not show information on the nature of the utterances that comprised the talk. Another method we use to depict the coded data focuses on the categories of utterances. Table 1 presents the percentages of the following kinds of utterance: Presentation (Pr), Short questions (SQ), extended questions (EQ), short answers (SA), extended answers (EA), rephrasing or re-voicing (R), and Objection (Ob).

Table 1. Percentages of utterance kinds

	Pr	SQ	EQ	SA	EA	R	Ob
Teacher (68%)	27%	42%	7%	4%	2%	19%	
Students (32%)		9%	6%	76%	6%		3%

Seeking patterns and coherence in the depicted data

Figure 2 shows that the teacher has a major role in initiating the passage to a new representation throughout the lesson while the students generally only respond to the teacher's initiative. An examination of the protocol indicates that even when students react to the teacher's initiatives with elaborating utterances it is often only few specific students who do it.

Table 1 that shows the nature of all the utterances during class talk (and not only those related to the passage between representations) confirms the central role played by the teacher. About two-thirds of the utterances are the teacher's and she is the only one to make Presentations. When students do participate in the class talk, they typically respond with Short Answers to teacher's Short Questions. Students almost never ask questions. Throughout the lesson both students' and teacher's extended Questions as well as Extended Answers are rare.

Interpreting the data

A reasonable interpretation of the findings is that for the students to use multiple representations to solve problems is a cognitive obstacle. Detailed analysis of different parts of the Windows lesson supports and verifies this interpretation. This conclusion fits with findings of other cognitive studies where it was found that interpreting information from one representation to another is cognitively difficult (e.g., Even, 1998; Schwarz & Dreyfus, 1995). While it is frequently argued that the multiplicity of representations facilitates the learning of concepts as it helps in the integration of perspectives, the multiplicity of representations could also be a *cognitive load* that hinder learning processes.

ACTIVITY THEORY APPROACH

Activity theory (Leont'ev, 1981) takes into account social origins of cognitive processes. The unit of analysis is the human *activity*. Need is always an essential part of an activity. Activities are chains of actions related by the same *object* and *motive*. *Actions* are the basic "components" of human activities that translate them into reality and can be understood only within the activity in which they are embedded. The actions that constitute an activity are energized by its motive, and are directed toward conscious *goals*. *Operations* are the means by which an action is carried out. They depend directly on the conditions under which a concrete goal is attained.

The research question guiding the analysis from an activity theory perspective is: What is the nature of the activity in which the teacher and the students participate during the Windows lesson? Data sources for the analysis include the video recording of the lesson, classroom observation, the protocol used for the cognitive analysis, the two ways of depicting the coded data (Figure 2 and Table 1), and numerous formal and informal discussions with the teacher and the curriculum development team.

The Teacher

Examination of the data for identification of the teacher's motive, actions, goals, and operations indicates that the teacher's overall motive when teaching mathematics is that her students understand and learn mathematics in meaningful ways. More specifically in the Windows lesson her motive is that her students learn about different representations of functions and understand how to use the graphic calculator to solve problems that require the passage from one representation to another. The teacher's actions during the lesson aim at helping the students learn and understand the above and at creating a need to use the graphic calculator as a problem solving tool and to choose cleverly a Range that would enable them to see the relevant part (window) of the graph. Throughout the lesson, the teacher's operations reflect her desire to engage her students in the activity while being attentive to their understandings. The operations are aimed to help the students become involved, understand, and develop a sense of shared ownership of the activity's motive as well as the actions' goals. The teacher's actions and operations suit the teacher's general approach to learning and her belief that students learn by constructing their knowledge through active participation in mathematical investigations.

The Students

Analysis of the lesson observations and of the video recording indicates that many students in the class appear uninvolved and uninterested in the intellectual challenges presented by the teacher to the whole class. Only a small number of students participate in the whole class discussions that the teacher strives to conduct. Even then, as Table 1 and Figure 2 also show, their participation is rather shallow and of low-level, characterized by short answers to teacher's short questions. The majority of the students become involved in the mathematics tasks only during the small group work parts. But also then, most of the students are not involved in exploration, problem solving, debates among themselves, and the like, as was common to the A-level students the teacher has taught in a previous year. Rather, many of the current students tend to yell for teacher help after minimal attempts to solve the tasks, asking her to tell them what the correct answers are. In general, students seem satisfied when they quickly reach the correct final answers and frustrated when they do not. These students, who have a history of learning difficulties and low achievement in years of traditional mathematics teaching, behave according to common beliefs about school mathematics, adapting social and socio-mathematical norms prevalent in traditional mathematics classes. Thus we may conclude that the motive of many students is *surviving* the lesson.

Same Lesson But Different Activities

The analysis of the lesson from an activity theory perspective suggests that the teacher and the students in the Windows lesson participate indeed in the same lesson but in different activities. Their motives are different and consequently the different actions that constitute the activity and are energized by its motive are different. The teacher's actions during the lesson are derived from her overall and specific motives. Her actions are well connected, one leading to another, creating a complete whole. However, beauty is in the eye of the beholder – what the teacher sees is not necessarily seen by the other lesson participants – the students. Many students do not make connections between the different parts of the lesson. Students' goals are different from those of the teacher's – theirs usually centre on obtaining correct final answers to the problems assigned by the teacher, not on making sense of the mathematical situation nor on using connections between different representations of functions as a tool to solve problems. As such, the nature of the students' participation in the activity is very different from what the teacher had wished for and does not contribute to the development of the activity as designed by the teacher. While this may seem odd when the teacher's design of the activity is considered, it makes perfectly sense when *surviving* is taken as the student's motive.

CONCLUSION

Both the cognitive verbal analysis and the activity theory based analysis of the Windows lesson indicate that things are not going smoothly in this lesson. Both analyses show that students do not behave mathematically as desired by the teacher. However, the two approaches suggest different interpretations of the situation and of the sources of the problems observed. The verbal analysis points to students' cognitive difficulties taken as independent of context. The activity theory based analysis suggests that the teacher and the students participate in the same lesson but in different activities, where different motives, goals, beliefs and norms regarding school mathematics drive and guide them. As a result, students' ways of participation in the lesson are different from what the teacher had wished for.

The discrepancy between the two interpretations is disturbing, as it seems to suggest that different methodologies yield different explanations of the same phenomenon. An immediate response is that the two analyses are guided by two different research questions. Naturally, answers to different questions may not coincide. Still, another question emerges: Are the two answers provided by the two perspectives compatible?

For several decades mathematics education research used to focus on cognitive development of mathematical concepts. Recently, the focus of research in mathematics education has extended from the individual student's cognition and knowledge to contextual, socio-cultural and situated aspects of mathematics learning and knowing.

The practices and culture of the classroom community (e.g., the nature of social engagements and norms) have become an important factor in studying learning processes, and mathematics education researchers started to incorporate the two perspectives – cognitive and socio-cultural – into a complex view of mathematics learning. This new focus signals a shift from examining human mental functioning in isolation to considering cultural, social, institutional and historical factors. The mathematics education community increasingly embraces the view that cultural and social processes are integral components of mathematics learning and knowing. In line with the current trend in the field of research in mathematics education we respond by proposing that a lesson, which is part of the practice of teaching and learning mathematics, is too complex to be understood by only one perspective. Consequently, a more complete understanding of the complicated practice of teaching and learning mathematics requires the use of both cognitive and socio-cultural perspectives.

This suggestion settles what at first seemed to be two conflicting interpretations of the same phenomenon. However, such resolution raises an epistemological issue that is crucial to the advancement of research in general and to our domain of mathematics education. The two interpretations of the Windows lesson and their harmonisation fit well with Guba and Lincoln's (1994) reflections on the relations between theory and research findings mentioned at the beginning. To a large extent, research findings could be explained by the theories they rely on, and confirm what the theories already could have predicted. Theory and research are then trapped in a vicious circle, which might not be productive. The effort to harmonise the two interpretations is in itself laudable, but is it legitimate?

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MATHEMATICAL SYBOLISM : A FEATURE RESPONSIBLE FOR SUPERFICIAL APPROACHES ?

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Abstract

In arithmetical problem solving process, young pupils' competencies were stressed on through analyses of their informal solving strategies. Other research works have shown pupils have significant difficulties to use the mathematical symbolism. Symbols seem to be meaningless for some of them and that feature could be responsible for the starting of some superficial approaches. Through individual interviews, we analysed the approaches developed by first graders who were not able to solve some problems. These solving errors are mainly due to conceptual difficulties, whereas the number sentences-writing task seems to develop superficial approaches.

BACKGROUND

Today's research concerns mainly focus on studies about the links between mathematics and everyday life situations. Those links can be considered according to a double research aspect, depending on whether we are interested in the use of school knowledge in situations outside of the school, or whether the issue is considered as the integration of real life knowledge into the school (Verschaffel et al., 2000). Research pointed out a gap between both types of knowledge. One of the reasons of that gap could be found within mathematics themselves: *at school, children learn rules and symbols but tend to lose the relationship to what these symbols represent* (Asman and Markovits, 2001, p. 66).

In early school years, the link between mathematics and real life situations should appear through arithmetical problem solving. More specifically, the word problems should help to establish some relationship between, on the one hand, the mathematical symbolism used to represent additions and subtractions, and on the other hand, the actions on the objects or the relationships mentioned through described situations.

Some reflections about the first mathematical learnings could bring some light on the general issue mentioned here above. Several studies (see Fuson, 1992, for a review of the literature) focused on the analysis of informal solving strategies developed by young pupils. The categories elaborated by the authors helped to enlighten a large variety of the strategies aiming at modelling the actions or the relationships described in the problem. Hiebert and Lefevre (1986) think that at that level, pupils generally do not develop any superficial strategies and that only the problems that are understood can be solved.

Other researches (Carey, 1991 ; Carpenter et al., 1983, 1988 ; De Corte and Verschaffel, 1985 ; Fagnant, 2002) focused on the analysis of the mathematical symbolisation abilities shown by young pupils. Most of these studies are based on individual interviews and help to analyse the relationships between the informal solving strategies and the number sentences writing task. These studies pointed out specific difficulties at the stage of symbolisation: pupils are not always able to produce a correct number sentence when they are confronted with a situation, even if they solved it correctly.

Furthermore, various studies have shown that young pupils' informal strategies, based on the word problem analysis as well as on a modelling of the involved actions or relationships, are often replaced by superficial strategies in which the pupils only wonder which operation they are going to perform (Carpenter et al., 1983 ; Verschaffel and De Corte, 1997).

Why do pupils develop superficial strategies in which mathematics consist in applying operations without trying to build some sense related to the suggested situations? Given the difficulties enlightened by the studies focused on mathematical symbolisation, it is possible that *part of the answer may lie in the transition from informal knowledge of addition and subtraction to the formal symbolic system of arithmetic* (Carpenter et al., 1983, p. 55). Generally, nothing is done to connect number sentences to the informal strategies of young pupils, they are not linked to concrete referents and we can therefore wonder which meaning they acquire. *Many students seem to learn symbols as meaningless marks on paper... After instruction on writing number sentences (equations) many first graders still not see the connection between the story problems and the number sentences that represent them* (Hiebert and Lefevre, 1986, pp. 21-22). That lack of meaning attributed to symbolism at its very introduction could be a feature responsible for superficial strategies.

This study aims at clarifying this issue through a quite particular perspective. Through the data collected during the individual interviews, we will focus particularly on the results of pupils who were unable to solve and to symbolise some problem situations. Most current errors made by young pupils consist in answering with on of the given numbers. On the contrary, wrong-operation errors are the most current among pupils with several months of experience in formal arithmetic in general and in problem solving in particular (Carpenter et al., 1983). The first types of error are generally due to conceptual difficulties (misunderstanding of some terms, lack at the semantic schemata,... - Lewis and Mayer, 1987 ; Stern, 1993 ; Riley et al., 1983). Wrong-operation errors can be explained in various manners; they can notably result from superficial strategies. It is generally considered (Verschaffel and De Corte, 1997) that the reason of these approaches lie in the stereotyped nature of the problems and in some focalisation of teaching on the final product, to the detriment of the process (not enough attention is paid to the building of the representation). The introduction of the mathematical symbolism could also be an explaining feature; that's what we are going to investigate now.

METHODOLOGY

This study focuses on 25 first graders who were selected randomly from 6 distinct classes of 4 different schools. They did not receive any teaching related to problem solving. Mathematical symbolism was introduced very early in the year but the word problems are only used in grade 2, in order to illustrate the applicability of formal operations. They are not used in grade 1 *to promote a thorough understanding of the basic arithmetic operations* (Verschaffel and De Corte, 1997, p. 69). The methodology consisted in individual interviews about the 14 problems of the Riley et al.'s classification (1983). These interviews were performed in the middle and at the end of grade 1. It was an approach almost similar to that used by De Corte and Verschaffel (1985) : the interviewer reads loudly the problem, the child retell it and tries to solve it with the help of physical objects, he is invited to write down a number sentence which corresponds to the story and/or to his solving strategy, the interviewer repeats the question to the child and asks him to tick the answer within his number sentence, the interviewer finally asks the child whether he could propose another number sentence (flexibility – Carey, 1991). The task requiring to tick the answer within the number sentence is essential for the future analysis. One of the advantages of the interview technique is that it helps to check whether the child tick really the number he identifies as the solution to the problem (the answer to the asked question). When this requirement is just proposed as a paper/pencil test, children tend to tick rather systematically the number located just after the equal sign (Fagnant, 2002), because they consider it as signifying "becomes" or "results in" (Fuson, 1992). Another advantage of that procedure is that it helps to analyse the solving strategy developed by the child before he produces his number sentences and, therefore, the link (or the absence of link) between informal and formal solving strategy.

We considered that the problem solving process was correct when it helped to end up in the expected answer. For number sentences, several criteria were taken into account: the number sentences must present a correct structure and be built from the expected number triple. Moreover, the answer must be correctly identified in the number sentence. For the present analysis, we chose to focus on the pupils who were neither able to solve nor to symbolise the analysed problem(s). We thus try to analyse the solving errors themselves and those in the number sentences. That could help us to perceive how the first ones can be explained by conceptual difficulties, whereas the second ones would rather be due to superficial approaches. And that is how we will try to support the hypothesis according which the introduction of mathematical symbolism could be a feature responsible for superficial approaches.

RESULTS

The analyses are mainly focused on 3 problems of the classification of Riley et al. (1983) : compare 1 (*Peter has 5 candies. Ann has 11 candies. How many more candies does Peter than Ann have?*), compare 3 (*Peter has 3 apples. Ann has 9 more apples than Peter. How many apples does Ann have ?*) and compare 5 (*Ann has 11 books. Ann has 6 more books than Peter. How many books does Peter have ?*).

The errors at the level of the informal solving strategies

For the three problems, we can notice that most errors at the solving level are given-number errors (the number to which it is referred in the relational sentence: respectively 11, 9 and 6 for the three problems). To explain the error committed in solving compare 3 problem, we can consider that the child interprets the relational sentence (*Ann has 9 more apples than Peter*) as a belonging sentence (*Ann has 9 apples*) or as two distinct sentences (*Ann has 9 apples. She has more apples than Peter* – Riley et al., 1983). We list 27 errors of that type (19 in January and 8 at the end of the school year). For compare 1 problem, we can extend this explanation: "*How many more candies does Peter than Ann have?*" becomes "*How many candies does Ann have*?". We list 18 errors of this type (12 in January and 6 at the end of the school year). According to the *Consistency hypothesis* (Lewis and Mayer – 1987), compare 5 problem would be presented in a structure that does not correspond to the child's "preferences". It is the reason why the child would try to modify the relational sentence : "*Ann has 6 more books than Peter*" would *become "Peter has 6 less books than Ann*". An incomplete modification would then make him think that "*Peter has 6 more books than Ann*" and would lead him to propose "17" as the solution (6 more than 11). We only list 2 cases of this type at the end of the school year. Furthermore, considering the relative sentence as a belonging sentence could lead the child to answer "6" *(Peter has 6 books)*, whether he made the complete modification (*Peter has 6 less books than Ann*) or not (*Peter has 6 more books than Ann*). We list 11 errors of this type (6 in January and 5 at the end of the school year).

It is also interesting to observe that a superficial key-word strategy (based on "*more than*") would bring to produce the answer inverse to the expected solution for compare 1 and 5 (for compare 3, that brings to the expected solution). No case of this type has appeared for compare 1 (no pupil answered 16) and only two cases of this type emerged for compare 5 (they can also be explained thanks to the *Consistency Hypothesis*). The remaining errors are either omissions, or errors difficult to explain but which are not an indicator of superficial approaches.

In relation to our hypothesis, errors at the solving level could be explained by conceptual difficulties rather than by superficial strategies. Stern (1993) had end up to similar conclusions through several studies aimed at compare problems.

Errors at the level of number sentences production

Now, what does happen when pupils are requested to produce a number sentence whereas they solved the problem by proposing a data from the problem text (given-number errors)?

About half of them do not propose any number sentence, which is quite a "logical" behaviour in relation to their wrong solving process. Some pupils propose a wrong operation, quite difficult to interpret. These number sentences are detached from the informal solving and indicate some type of gap between both tasks. Finally, the remaining pupils propose number sentences corresponding to the sum of the data from the text (the brackets indicate the number identified by the child as the solution of the problem) : 4+(9)=13 for compare 3 (11 number sentences) ; 5+(11)=16 for compare 1 (3 number sentences) and 11+(6)=17 for compare 5 (3 number sentences).

About compare 3, the number sentences could had been considered as correct if the solution identification would not have been taken into account. For compare 1 and 5, number sentences lead to produce the operation inverse to the expected solution, but the answer identified by the child is a data from the problem text. For the three problems, it can be supposed that the number sentence is the result of a superficial strategy based on the key words (*more*) (or an « *extract-and-add* » strategy – Bebout, 1990). These number sentences are detached from the pupil's problem solving approach and the final answer is not considered as the problem solution. Some pupils interpret the number corresponding to the data sum as being what the children (of the problems) have together. Other pupils are not able to give a concrete meaning to that number. In any case, the children are able to explain their number sentence in relation to their understanding of the situation.

If we had tried to interpret the number sentences only, we would have been led to consider that the number sentence « 4+9=13 » was correct in relation to compare 3 problem and that the number sentences « 5+11=16 » and « 11+6=17 » resulted from superficial strategies in relation to compare 1 and 5 problems. The implementation of the interview procedure (solving before writing the number sentence and task requiring to tick the answer) helped us to observe the gap between the solving process and the number sentence. It is thus really the symbolisation task (and not the solving process) that seems to be responsible for the superficial strategies of some children.

Though that type of error does not occur very often, we have to admit that it appears systematically for most problems of the Riley et al.'s classification (1983). For instance, for combine 2 problem (*Peter has 5 candies. Ann has also some candies. Together Peter and Ann have 11 candies. How many candies does Ann have?*), a current error at the solving level is to answer by proposing the great number (the whole). This error can be interpreted as a conceptual one (Riley et al., 1983). We do not list any answer resulting from an inverse operation (which could be the mark of a superficial approach) but the number sentence production task leads some pupils to propose number sentences of that type (5+11=16). For combine 1 problem (*Peter has 4 apples. Ann has 9 apples. How many apples do Peter and Ann have together?*), some pupils answer proposing both numbers of the terms (4 and 9 – confusion in words "together" = "each one"). The number sentence production task leads them to propose the number sentence (4)+(9)=13 ; considered as incorrect, since "13" is not identified as the solution (pupils do generally not give it any concrete meaning).

CONCLUSIONS AND DISCUSSIONS

The results presented here above focus on a few cases only. However, it is important to note that this type of analysis turns out to be relevant for most proposed problems : if pupils do not seem to develop superficial approaches at the level of the problem solving process, the task requiring to produce a number sentence seems, on the contrary, to generate the development of superficial strategies, of the " *extract-an-add*" type (Bebout, 1990). Faced with addition problems, that strategy leads to produce a number sentence which would had been considered as correct if the solving identification criteria had not been taken into account. Faced with subtraction problems, that leads to produce the number sentence inverse to the expected one. The gap between the solution and the number sentence is important : the child generally does not consider the number resulting from the number sentence as the problem solution.

To strengthen those observations bringing out a tendency to develop a superficial approach at the very moment of the number sentence production, we can briefly evoke the results concerning children who have been enable to symbolise the problems they had yet correctly solved. The wrong number sentence analysis (that we have no opportunity to detail here) shows also a strong tendency to develop superficial approaches of the " *extract-and-add*" type. Furthermore, even the results of the pupils who had correctly solved and symbolised the problems reflect that tendency to detach completely the symbolisation stage from the solving one. The change 2 problem (*Peter had 12 cherries. Peter gave 7 cherries to Ann. How many cherries does he get now*?) helps us to illustrate both cases. The child correctly solves the problem by taking 12 blocks, by removing 7 blocks, then by counting it remains 5 blocks (5 is thus proposed as the solution of the problem). He may then produce either the incorrect number sentence "12+7=19" or the correct one "7+(5)=12" (this type of number sentence is produced by about half of the pupils who propose a correct number sentence faced to that problem).

The various results presented here show the detachment between the solution process and the number sentence. Those difficulties can partly be explained by the way the symbolism is introduced : *Because the operations are initially learned outside the context of verbal problems and children are simply told that addition and* *subtraction can be used to solve these problems, they have no basis for using their natural intuition to relate the problem structure to the operations they have learned* (Carpenter et al., 1981). That way to proceed places the pupils in a difficult position : not only they do not attribute easily any concrete meaning to the mathematical symbolism, but they are too "constrained" to develop superficial approaches in order to perform tasks which are not much connected to their intuitive understanding of the situations.

Faced with those observations, we are induced to think that the first mathematical learnings are already partly responsible for the behaviour of « *suspension of sense making* » (Verschaffel et al., 2000), noticed among older children. There is indeed a risk that the pupils consider mathematics as meaningless and consisting in applying the rules taught by the teacher and that they do finally not understand a lot (Schoenfeld, 1992). In other words, it can be said that at the early school years, children have some opportunities to make them think that mathematics are quite distinct from real life.

In the classrooms we worked in, the problem solving teaching is coming later on, in grade 2, with the objective of showing the applicability of mathematics in everyday life situations. The results we got make us think it would be very more productive to build learning of problem solving on basis of pupils' informal solving strategies, rather than to choose to be supported by a basic expertise of operation techniques (wrongly considered as a pre-requisite). If problem solving teaching could begin earlier, we would get more chances to be able to build on the pupils' experience (*i.e.*, on their informal approaches), to give more meaning to the mathematical symbolism, to "avoid" the development of superficial approaches and then, perhaps, to create from the very beginning a less important gap between the mathematics and the real world…

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DEVELOPING LANGUAGE THROUGH COMMUNICATION AND CONVERSION OF SEMIOTIC SYSTEMS

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This paper investigates the use of language by 2 classes of 7th graders in a context involving linguistic exchanges and conversions between representation systems in mathematical setting. The pupils were given tasks requiring the verbal description of a figure by means of a written text with no drawings for one class, and the interpretation of the text for the other. Both the tasks have been dealt with through discussion. The constraints involved in the experiment (communication with people not sharing the same context of situation; conversion between figures, written and spoken texts) have been designed in order to promote the acquisition of linguistic skills suitable for the learning of mathematics.

INTRODUCTION

This study is concerned with the role of language in mathematics learning. Recently various frameworks have been proposed that underline the role of languages in the learning of mathematics. For example, Sfard (2000) interprets thinking as communication and regards languages not just as carriers of pre-existing meanings, but as builders of the meanings themselves. So, under this perspective, language heavily influences thinking. On the other hand, there is evidence that a good share of students' troubles in mathematics, at any school level, including undegraduates, can be ascribed to the improper use of verbal language. More precisely, students often produce or interpret mathematical texts[1] according to linguistic patterns appropriate to everyday-life contexts rather than to mathematical ones (for examples see Ferrari, 1999). The difference is not just a matter of vocabulary, grammar or symbols, but heavy involves the organization of verbal texts, their functions and relationships with the context they are produced within. In other words, as appropriately claimed by Morgan (1998), the distinctive features of mathematical language are not confined to the symbolic component but involve the verbal one as well.

Through the paper I am interpreting mathematical language from the functional linguistic perspective. This means that I am focusing on the functions that language plays in the various contexts. I will use the idea of *register* (defined as a linguistic variety based on use, i.e. the linguistic means developed to express meanings related to some context and goals) to link contexts, texts and the functions that the latter are expected to play. For examples see Morgan (1998), and Ferrari *et al.* (2001). Even if, as suggested by Morgan (1998, p.3) the "unity across mathematical texts that makes

[1] Through the paper by 'text' I mean any written or spoken instantiation of language, possibly longer than a sentence, not necessarily a textbook.

it possible to talk of 'the language of mathematics' ... is an illusion", there are nevertheless some features of 'institutional' mathematics registers (e.g., those adopted in research, academic communication or mathematics textbooks, at almost any school level) that are common to a wide range of literate registers. Most of these features, at least in principle, are functionally appropriate for the needs of mathematical practice (but not necessarily for those of learning). Among them, in this paper I will focus on the relationships between a text and the context of situation[2] it is produced within; in most written mathematical registers, texts do not depend upon the context of situation, whereas in everyday-life ones they heavily do. The dependence of a written text on the context of situation may lie, for example, in the reference to space and time (use of expressions like 'this', 'here', 'now', ...) or to participants ('you', 'she', ...) also as physical subjects. The progressive construction of the ability to write and interpret texts not depending on the context of situation is a necessary step for both linguistic and mathematics education. Unfortunately, in mathematical practice and teaching, mathematical registers are often used or taught with little reference to their functions. Students at any level and often teachers are hardly aware of the reasons why certain constructions and styles are adopted and are induced to use them as formal[3] patterns detached from their functions. This is one of the factors that may induce students to deal with language in mathematics without taking care of meanings. It is worth mentioning that in mathematics education students (and teachers) are exposed to continual shift of registers, as it is necessary to adopt both conversational forms to effectively communicate and other forms taken from mathematical registers to express mathematical relationships. Thus a goal of mathematics education as concerns language is promoting flexibility in the use of registers and awareness of the relationships between linguistic forms and contexts and purposes. This is by no means spontaneous but must be carefully promoted[4]. This suggests to work within contexts that underline the functions and the meanings of texts rather than just their linguistic form. This means to design learning situations that force a more and more refined use of language as an answer to specific requirements and constraints explicitly posed by the context. The relative autonomy of the texts from the context of situation, for example, is to be regarded as an answer to explicit (and recognizable) communicative needs (such as communicating with people not sharing the same context of situation) rather than as conformity to some

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[2] The context of situation refers to specific aspects of the situation including also those related to the physical environment such as space, time, participants as physical subjects, and so on. Independence from the context of situation does not imply independence from any level of context.

[3] I am aware that 'formal' is used in literature on mathematics education with a variety of meanings often incompatible each other, such as: formal as 'symbolic'; as 'devoided of content'; as 'institutional'. According to the use prevalent in linguistics, here I mean 'related to linguistic form', in opposition to 'functional'.

[4] This is one of the reasons why I am not using the expression 'natural language' to denote any variety of verbal language in opposition to symbolic one.

linguistic pattern. Methods based on the teaching of grammar and lexicon detached from use can hardly achieve the goal proposed. Most often students exposed to these methods achieve some knowledge of linguistic structures, but can hardly apply this knowledge to scientific contexts.

THE EXPERIMENT

The experiment involved 2 middle-school classes (namely, A and B) located in different places (both in rural areas in North-Western Italy). Class A had to solve a measurement problem involving the ground floor of their school and to propose the same problem to the other class, communicating the data in verbal written form with the help of no drawings. The task of class B was to reconstruct a drawing, based on the verbal description only and with no knowledge of the building, to solve the problem and to submit their answers. Class A was then requested to evaluate the other pupils' solution and point out both ambiguities in their own verbal text and improperties in the drawing or in the solution, if any. All the tasks were dealt with by the whole classes and the answers were adopted through discussions which were guided by a teacher and a graduate prospective teacher, leaving the pupils completely free to decide the formulation of the problem or its interpretation.

A priori analysis

The situation involves 2 kinds of constraints both relevant from the semiotic perspective. First, the requirement of communicating (in written form) with unknown pupils (even if of the same age and from a similar socio-cultural environment) asks for an effort (mostly by class A) to make explicit a portion of the knowledge that usually is left implicit in spoken conversations, as writers and addressees did not share the same context of situation. Second, the requirements of converting[5] a geometrical figure into a text or a text into a figure asks for some explicitation and selection of the relationships embodied in both the representations. Both the requirements should prevent pupils from exploiting the context of situation and force them to convey all the information in explicit form and to reflect on the pieces of knowledge they could assume as shared by the other pupils. The situation essentially involves figural, verbal spoken and verbal written expressions. Written and spoken productions may play different cognitive functions (see Duval, 2000, for a convincing account); it may be expected that written texts and drawings should be used as relatively stable products to reflect upon, whereas spoken language could be used to explore provisional, unsettled ideas, to compare different opinions (exploiting interpersonal relationships and processes) and to focus on the aspects regarded important, temporarily neglecting the others.

All the sessions have been audio-recorded. To illustrate the arguments adopted by pupils I include some transcripts and try to analyse them. A deeper linguistic analysis

[5] The cognitive role of conversion of semiotic systems has been widely discussed by Duval (1995).

of the transcripts requires to deal with the original texts, as the english translation, although it may properly convey basic aspects of the text, may fail in preserving some features such as register and focus.

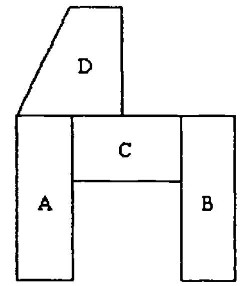
OUTCOMES

Constructing a text

Pupils of class A chose to base their text on the drawing in Fig.1. They agreed to describe the problem situation by means of the text reported below.

Notice that they chose to give measures referred to the drawing, rather than to the real building. I have tried to translate the text as literally as possible. To make reference easier, the sentences are numbered.

Fig.1



(1) Our school is much alike a cradle seen in profile.

(2) Our building consists of 3 rectangles, 2 of which are placed vertically and one horizontally, which joins them in the upper part.

(3) Let us name the 2 vertical rectangles A and B and the horizontal one C.

(4) Trapezium D (which is our gymnasium) is right-angled and is placed on rectangle A and part of rectangle C, with its oblique side adjacent to the altitude of rectangle A.

(5) The two rect.A and B are equal.

(6) Now we give you measures: the base of rect.A (thus of B too) is 11 cm long, and the altitude 21 cm.

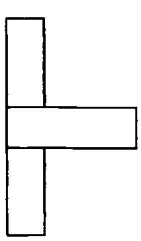
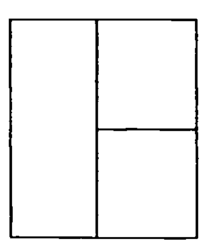
(7) The base of rect.C is 22 cm long and its altitude is equal to the altitude of rectangle A minus a 10 cm recess.

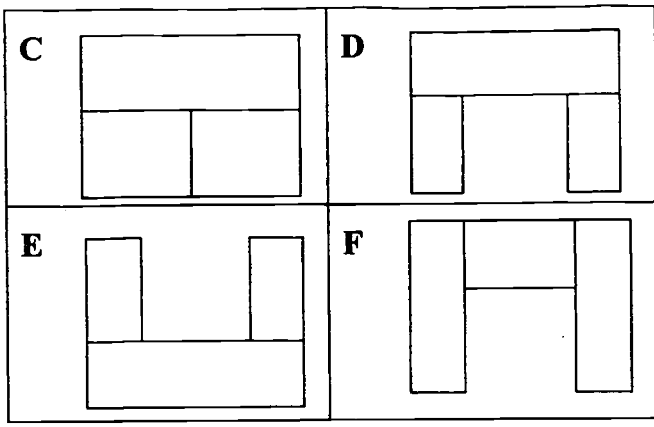
(8) In trapezium D the greater base laid on the 2 rect.A and C is 18 cm long and the smaller one 16 cm, altitude is 19 cm long.

Interpreting the text

All the pupils of class B have been given a printed copy of the text which has been also read aloud by the teacher. Pupils immediately produced a number of drawings based on the first two sentences only. They are reproduced in figures 2.A-F, with on the right some of the arguments produced to discard some of them.

Fig.2

A 	**B** 	A: "The horizontal rectangle doesn't join both in the upper part, but one in the upper and one in the lower." B: "There are two horizontal rectangles, the text speaks of just one."



C: "It is not like a cradle seen in profile, maybe it is a front view."

"Schools aren't like that."

E: "The horizontal rectangle joins the other two in the lower, not in the upper part."

F: "The horizontal rectangle joins them sideways, not in the upper part."

After some discussions, all pupils agreed to discard drawings A, B, C, E. The choice between D and F elicited further discussions, focused on the interpretation of "... *which joins them in the upper part*". Here is an excerpt from an exchange between Alessandro, a boy supporting drawing F and Barbara, a girl supporting drawing D.

Barbara: "The first drawing doesn't work because it doesn't join them in the upper part, it joins them sideways"

Alessandro: "The triangle below ..."

Barbara: "It's a rectangle!"

Alessandro: "Ok, the rectangle, it's the same, is joined in the upper part, but the triangle, joins them in the lower part."

Barbara: "There is no measure that tells us which is the upper part."

Through the whole discussion Alessandro often used 'triangle' in place of 'rectangle'.

When the teacher asked him "Where is your triangle?" Barbara replied for him: "Oh, he means 'rectangle', you know"

After reading the text up to sentence (4), all pupils agreed in choosing the drawing in fig.3. Immediately after 2 questions arose: the interpretation of 'on' ("...placed on rectangle A ...") and of 'adjacent'.

Fig.3

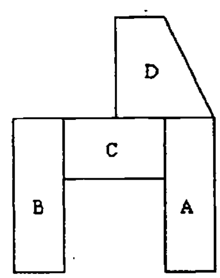
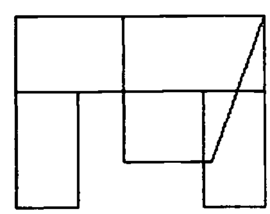


Fig.4



As regards the first, some pupils proposed drawings like that in fig.4, which were discarded at once by arguments like "It isn't like a cradle anymore" or "Where you put your trapezium there are the classrooms"

As regards the second question, pupils were puzzled by 'adjacent'. Barbara actually searched a textbook and found out that 'adjacent segments' are collinear ones. After some discussion, the contradiction between the definition of 'adjacent' and the other information was disregarded and pupils decided to adopt the drawing in fig.3 and to solve the problem accordingly.

Rearranging the text

Pupils of class A were keenly interested by the report of the interpretation of their text by class B. The discussion focused on the fact that the drawing produced by class B was not congruent to their own, even if the values of area found by both classes was the same and on the interpretation of 'adjacent'. In both cases they conceded that the interpretations produced by class B were legitimate. The text was revised by modifying sentences (3) and (4) as follows:

(3) "Let us name A the vertical rectangle on the right, B the one on the left and C the horizontal one."

(4) Trapezium D (which is our gymnasium) is right-angled and laid[6] on rectangle A and part of rectangle C, with its oblique side consecutive to the altitude of rectangle A."

Some doubts have been raised about the correctness of referring to 'left' and 'right' ("They aren't mathematical words")[7]. The whole class approved the following final remark by Barbara: "Next time we should throw four or five of us out of the classroom, then write down the text, let them in and see if they can draw the figure."

DISCUSSION

The experiment has provided a wide range of suggestions and ideas worth to be developed. Here I am focusing on some of them only.

Pupils of class A adopted various means to describe the figure, from the metaphoric reference to a cradle to the introduction of letters to name the basic figures involved. The need for communicating (along with the feedback received) forced pupils to reflect on the meaning of some of the expressions they were using (e.g.: 'upper part', 'adjacent', 'placed/laid'). The outcome has not been the adoption of a register nearer to those adopted by textbooks, but rather the exploitation of all the linguistic and cultural resources available in the context they were acting within. The use of letters to name figures, for example, is different from standard school notation, according to which capitals would represent points; the same holds for their use of 'adjacent' or the replacement of 'placed' by 'laid'.

As regards class B, at least two points of the interpretation process are worth mentioning: the interplay between the text and the drawings produced, and the spoken register adopted. When interpreting the text, pupils proposed a number of drawings immediately after reading sentences (1) and (2), and ruled most of them out after discussion or after reading sentences (3) and (4). Some of these drawings were clearly inconsistent with the first two sentences. Pupils may have paid more attention

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[6] The original italian words were 'posto' (placed) in the first version, 'appoggiato' (laid) in the revised one. 'appoggiato' suggests the idea of physical contact (including weight) much more than 'posto' (which is a more formal word common in standard geometry vocabulary).

[7] In the same context, 'top' and 'down' did not raise any doubts.

to the reading aloud of the text, rather than to their printed copies, and to some isolated words (not necessarily the same for all) rather than to the text as a whole. Anyway, this behavior may be explained by the need for an explicit representation as a common ground for reflections and discussions that may help in taking into account the meanings of the text neglected at first. Generally expert people neglect at once clearly inconsistent interpretations and rule them out through inference (or other methods) with no need for explicitly representing them. Most likely, they can deal with a text as a whole and can select the features that are relevant directly from the text; these behaviors are all typically associated to the use of literate registers. For pupils, on the contrary, reflection on explicit models and discussion and comparison of different opinions (with the activation of interpersonal processes based on their mutual relationships) seem to play a relevant function in the interpretation process. Also the final choice of drawings sometimes has proved inconsistent with the interpretation of some parts of the text. When interpreting sentence (4), no pupil objected against Barbara's interpretation of 'adjacent', which was inconsistent with the drawing adopted; nonetheless this was not rejected, maybe because the reference to a cradle or their knowledge of what schools are like were regarded (more or less consciously) as more reliable than the interpretation of a single word, taken from a textbook. Processes like those described here are quite common in everyday-life interpretation processes, whereas in mathematical registers consistency and the meaning of each word are more important.

As regards the spoken register adopted, it may appear that in some exchanges pupils have been quite inaccurate; Alessandro, for example, employs the word 'triangle' in place of 'rectangle' but the others nonetheless do understand what he is saying. Some other pupils discussing fig.2B speaks of 'two horizontal rectangles', referring to a drawing where all the rectangles have altitudes longer than their bases. These behaviors are not uncommon in spoken interactions; they might even be regarded as good features of everyday-life registers, as they allow to build sentences focusing on some aspects only (for Alessandro, the mutual positions of the figures rather than their classification) without paying too much attention to the others. So Alessandro's 'triangle' is not meant to express the whole of the meanings usually attached to it in mathematical registers but rather works as an indexical (such as 'this one' etc.), maybe associated with some gesture of hands, and is strictly related to the context of situation: his classmates understand him quite well whereas pupils of class A most likely would not understand a written transcription of his argument. Mathematically speaking, these behaviors are errors, but it is not deniable that in everyday-life registers they are standard ways of constructing and interpreting texts.

Teaching implications

The results of the study point out that 7th graders may successfully use verbal language to represent mathematical ideas and communicate them to peers not sharing the same context of situation. It seems that the methods they use are quite different from those adopted in institutional mathematical registers (in particular, textbooks).

This paper has presented examples of linguistic behaviors that are inconsistent with the organization of standard mathematical registers (e.g. 'inaccurate' use of some words; lack of consideration for inconsistency) but may be regarded as effective for the purposes of communication and thinking. It seems that pupils attempt to use all the resources available at the level of context they recognize as suitable to carry out communication successfully. For example, pupils of class A when writing the text do not rely on the physical environment of their school, but do use other pieces of knowledge they expect to be shared with the other pupils (reference to a cradle, to geometry vocabulary and so on). All this means that one cannot expect that pupils (and even students at almost any level) can recognize standard mathematical registers as the answer to actual representation and communication needs, as the methods they would adopt to deal with the same needs are quite different. It is necessary from the one hand to avoid any use of mathematical language (not just lexicon but also construction and interpretation processes) that does not correspond to recognizable representative and communicative needs, from the other hand to design activities (like the one I have described) that force pupils to produce and interpret texts accurately without asking them to conform to the patterns proposed by institutional mathematical registers.

In my opinion, further research should focus on the long-term outcomes of activities like those I have described in the paper.

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ROBOTICS ACTIVITIES AND CONSTRUCTED PROBLEM SOLVING: CREATING SPACES FOR LEARNING/DOING

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The case of students solving complex tasks for autonomous robots is explored as an example of the dynamic of learning/doing. Related to the interdependence, emergence and form of classroom dynamics, as discussed by Davis et al. (2000), and to the complexity of relationship, systems and meaning dynamics as described in Fleener (2002), learning/doing extends traditional notions of constructivist learning by considering the social and cognitive spaces of learning and doing as a complex system of relationship, meaning, and activity.

FOCUS OF THE PAPER

In Engaging Minds, Davis et al. (2000) discuss the complexity of the interdependence, emergence, and form of classroom dynamics, including the web of relationships among teaching, learning, and curriculum. We will offer a glimpse of their vision of a dynamic learning environment by describing how students, engaged in robotics activities, create spaces of meaning for mathematics learning and discourse. Focusing on learning rather than teaching, as do Davis et al. (2000), while emphasizing meaning-making as a hermeneutical and social process of relationship, meaning and system, we will consider how rich activities may provide opportunities for "recreating heart" – reviving schools as places of learning/doing.

THEORETICAL FRAMEWORK: LEARNING/DOING DYNAMIC

Foucault (1986) developed the notion of knowledge/power as the inextricable relationship between knowledge and power. According to Foucault, "power can never be the property of an individual" (Appelbaum, 1995, p. 39), and knowledge, as well as power, are aspects of the social dimension. Knowledge/power, then, is something separate from either individually and is the dynamic between the two. Similarly, neither learning nor doing can be separated from the social dimension or from one another. Learning/doing, like Foucault's knowledge/power relationship, is a structural dimension of the social context. This relationship extends the constructivist notions of learning beyond the cognitive realm to include the active embodiment of knowledge as a complex of relationships.

The logic of relationship can be found in Dewey's logic as inquiry (Dewey, 1938), and Whitehead's process philosophy (Whitehead, 1929). A logic of relationship understands that all that is and that we know is in interdependent, dynamic relationship. Nothing exists, posits Whitehead, that is not in relationship. The logic of systems challenges fragmented, piecemeal approaches to meaning and purpose, considering the form of systems per se. Inextricable relationships within systems cannot be studied apart from the complex web of relationships that give the system its identity and purpose. Chaos, complexity, and complex adaptive systems approaches explore system dynamics from this perspective of interdependence and connectivity. Finally, a logic of meaning, as first explored by Wittgenstein (1953), suggests that meaning structures themselves do not exist and cannot change until we change the language we use to describe our world.

We cannot re-vision schooling while maintaining our traditional ideas about teaching, learning, and knowing. Until we change the ways we talk about what it means to know mathematics, we cannot hope to recreate classrooms where the very soul of mathematical inquiry, as a process of discovery and invention rather than consumption or transmission of inert facts, is possible. The learning/doing dynamic captures the complexity of learning as active, social, and contextual processes.

This paper presents a description of a team of students engaged in creating an autonomous robot. The nature of their problem solving efforts, both in programming their robot to solve particular tasks and in their creation of tasks for their robot to solve, will be explored using case study methodology. From the perspectives of meaning, relationship and systems, the students' construction of an autonomous robot will exhibit the interdependence, emergence, and form of a learning environment that engages the minds (Davis et al., 2000) and souls (Fleener, 2002) of these students as dimensions of learning/doing.

RELATED LITERATURE: OPPORTUNITIES FOR LEARNING/DOING

Seymour Papert (1980) was among the first to envision the potential of students' programming robots to transform schools. His Logo environment, he argued, supported Piaget's notions of students' needs to actively construct mental structures through experimentation, experience, and activity. He hoped that by programming turtles in his virtual Logo environment, students would themselves become mathematically empowered by gaining confidence in their abilities to do mathematics, deriving new understandings of and appreciations for mathematical relationships, and engaging in problem solving and reasoning activities to achieve complex tasks.

The early Logo virtual environment was extended with the invention of the floor "Turtle" in the 1980's. Mirroring the moves of the Logo turtle on the screen, the floor turtle was linked to the computer so students could solve problems in three-dimensional space. While the move from two-dimensional virtual space to three-dimensional "meso" space (Berthelot & Salin, 1994) greatly enhanced the opportunity for students to experience the movement of the turtle as an "object to think with" (Papert. 1980, p.11), the tasks to be solved and the inability to instill in the Turtle it's own abilities to "think," limited the impact of Logo in either its virtual or three-dimensional applications.

In the last decade, Papert's vision has been revisited as computers have become more powerful and accessible. While research on the impact of students' programming, per se, has not been definitive, current research (see, e.g. Kaput, 1992) re-examines how general-purpose programming may be valuable as tools for problem solving. Yelland (1995), for example, suggests solving problems through programming may promote higher order thinking skills, develop flexible and creative thinkers, and strengthen problem-solving abilities.

Current research on brain dynamics supports approaches to learning that include rich contextual problems allowing for multiple levels of and approaches to understanding. Richardson's book (2000) The Making of Intelligence suggests traditional constructs of intelligence that reduce it to either genetic endowment or environmental factors fail to capture the complexity of the dynamics of active engagement and multiple levels of representations that occur as we interact in rich environments.

Still unexamined, however, is how students' efforts to program autonomous robots capable of interacting with their environments, may challenge traditional notions of problem solving and extend ideas about learning. No longer programming the turtle to solve a particular task, students must explore how to make their robots "think" – how to program a robot to interact with its world in order to solve tasks. More authentic, the challenge becomes not how to set the robot in motion but, rather, how to instil in the robot the ability to take in sensory data as information about its environment and react and respond in ways that still allow the robot to accomplish a particular task. It may be that by moving between spaces of self and robot, thinking and doing, students are afforded opportunities for engaging in what Foucault describes as technologies of the self: "models proposed for setting up and developing relationships with the self, for self-reflection, self-knowledge, self-examination, for deciphering the self by oneself, for the transformation one seeks to accomplish with oneself as object" (Peters, 1999, p. 12). Beyond the cognitive dimension alone, working with robots that are capable of sensing light, touch, and other environmental dynamics, students are afforded opportunities to engage in learning/doing.

METHODOLOGY

The students described in this paper participated in an after-school robotics club, and were members of a middle school team that competed in the Spring 2001 Botball competition. Botball is a six-week K-12, national robotics competition sponsored by the KISS Institute of Practical Robotics (www.kipr.org). Teams that enter receive two small processors (a Handyboard and a Lego Mindstorms RCX processor), software (Not Quite C for the RCX and Interactive C for the Handyboard), various sensors for both processors, and Lego parts for constructing their robots. The teams must design, build and program robots using the C computer language to compete against other teams' robots in a competition arena.

Our case study explored how these students came together and approached creating and solving problems for their robots as they learned about robot construction and programming. We used categories described by Davis et al. (2000) as our interpretive framework for exploring the complexity of learning/doing afforded by the robotics activities of this group. Thus, interdependence, emergence, and form (see Davis et al., 2000) were the multiple lenses through which we examine the problem solving activities of this group of students.

RESULTS: TEACHING ROBOTS TO THINK

Our Botball team organized into three groups: one robot team for each of the two processors and a web design team. Each group further divided into teams of builders and programmers. The builders constructed the mechanical structure of the robot. They had to figure out how to integrate the processor, motors and sensors into their designs while working with the programmers to determine design features that were necessary to interface with program objectives. The programmers wrote the computer code to coordinate the robot's actions. Recognizing and anticipating environmental characteristics were aspects of the problem solving space that required discussion among both builders and programming groups. After the Botball competition, the BOTS club continued to meet, creating and solving their own robotics problems.

The robotics club met every week after school for the remainder of the semester. All but one student remained active after the competition while several new students joined the club. The students determined procedures for deciding what tasks they wanted their robots to accomplish. Club sponsors, including two of the co-authors of this paper, facilitated club activities. Student sense-making activities included problem posing, anticipation of environmental and design features, and explaining, listening, justifying, facilitating and probing each other's ideas. The adult facilitators enabled sense-making activities rather than providing solution strategies or setting parameters for problem posing and/or solutions (Wood, 1999). This approach is consistent with Cobb and Steffe's (1983) teaching experiment where the teachers assume the role of participant observers, engaged in student activities through their own listening, querying, and interactions as students work toward problem definitions and solutions. This research approach is particularly relevant in robotics activities as a case where the teachers seldom have a single "best" approach in mind. Creativity as an aspect of solution strategy is supported by the richness of the potential of robotics activities.

As the students in the robotic club learned to work with their robots and use them to accomplish particular tasks, it became clear that there were multiple dimensions to their problem solving. The complexity of the activities in which they engaged can be captured by examining their efforts from the perspectives of the structure of problem solving efforts, interdependence of activities, and emergence of meaning as aspects of learning/doing.

Structure of Problem Solving Approaches. Initially, determining the task for the robots to accomplish was the responsibility of the organizers of the competition and the facilitators of the club, especially during team competition. As students became more adept at working with the robotics components, they began to take responsibility for defining their own tasks. Whether the tasks were defined by the facilitators or by the students, however, there were specific components of the problem solving process that were identifiable.

For example, whether the task was simply to get the robot to move forward and knock down an object, follow a straight line using sensors, or negotiate a maze, students had to design their robot for movement and appropriate sensory inputs, program the computer to use the sensory data the robot received through the various sensory input devices, and write a program that allowed the computer to use this data and take the appropriate actions. Decomposing the problem into these structural characteristics was just one aspect of successfully programming their robot to complete its tasks, however. Anticipating environmental complexity was a problem solving feature beyond the structural dimension of task completion.

Interdependence of Activities. The mechanical or structural features of programming their robots to accomplish tasks were not separate from learning to respond to and anticipate aspects of the environment in which the robot acted. The interdependence of learning to interface with and respond to the complexity of the learning environment, including learning to respond to and anticipate obstacles to task completion, were seamless with learning the mechanics of programming, robot design, or sensory interfacing. Thus, an important feature of the richness of the robotics learning environment seemed to be the complexity of tasks necessary for problem solutions. The students couldn't first concentrate on the mechanics, then focus on the problem for the robot to solve. Linear problem solving approaches (Polya's approach, for example) were not viable for solving the tasks these students were tackling. Neither "top-down" nor "bottom-up" strategies seemed to work as students' problem solving efforts fluidly oscillated between the mechanics of problem solving and the strategies of problem solving.

Learning to navigate was a complex activity that exemplifies this dance between mechanics and inspiration, knowledge and creativity. Students initially approached the problem of robotic navigation by programming the drive motors to propel the robot for a specified time. Backward or turning motions were conceived as reversing or differentially turning motors at each wheel. The students typically started with dead reckoning to determine the length of time necessary for motors to run. Gear slippage and loss of power in the batteries, however, created drift and affected the distance and direction travelled, resulting in missed targets and errant robot motion. Students learned to anticipate these problems and developed strategies that reflected an understanding of motion as an interaction with the environment and not simply movement through space and time. Thus, for example, students typically solved this problem by using the encoder sensor to determine wheel revolutions allowing for more precise determination of movement distances. Later approaches incorporated the range sensor allowing the robot to determine relatively precise distances from objects. This later approach extends robot motion from a calculation of distance to an intelligent response to its environment.

Emergence. While attending to the larger problem solving task, students learned to communicate with one another as well as with the robot. The interdependence of mechanical and inspirational aspects of problem solving is extended as new meanings and skills are gained. As problems with solution strategies became apparent, opportunities for accommodating these difficulties qualitatively extended the learning opportunity. These emergent understandings were facilitated by the learning/doing interface.

Learning/doing became apparent to us as students alternated among skill development, creative problem solution, brainstorming, mechanical engineering, and programming. Within each problem context, there were multiple events of emergence accompanied by routine activity. Without emergence, however, problem solving efforts were stymied or reached a plateau.

IMPLICATIONS: THE SOUL OF LEARNING

Davis et al. (2000) suggest "teaching is about affecting perception" rather than "helping students to know what they don't know" (p. 26). The importance of the robotics activities was not in solving particular tasks, or even in the process of solving the tasks. Instead it was in changing how the students looked at their world, problems, and communication. By trying to anticipate how the robot would experience its world and creating a program that would allow the robot to respond appropriately, there were multiple opportunities for the learning/doing complex to unfold.

Just as Whitehead (1929) rejected Newton's idea that the most basic, fundamental reality is entities in space and time, elevating relationship as the essence of all being, so this robotics case illustrates that learning is not about "knowing," "thinking," or "communicating" but is about knowing-relationships, thinking-relationships, and communication-relationships. Learning/doing captures the dynamic of these multiple relationships as our students' problem solving efforts changed when they stopped looking for solution (things) but instead started seeing problems as rich environment-interpretation interactions.

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COMPUTERS FOR LEARNING MATHEMATICS: GENDERED BELIEFS

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The fields of mathematics and computing have been stereotyped as 'male domains'. Efforts to challenge the stereotype within mathematics appear to have had some measure of success. Computers are now common in schools and it is widely believed that using them for mathematics will enhance student learning. However, not much is known about students' beliefs about using computers for the learning of mathematics. In this paper, findings from a large scale survey that included questions tapping attitudes towards mathematics, computers, and computers for mathematics learning are presented. The results appear to confirm recently reported changes in beliefs about the gender stereotyping of mathematics, but lend some support to the view that computers for the learning of mathematics may be more suited to boys.

INTRODUCTION

Historically mathematics was viewed as a *male domain*, that is it was considered a discipline more suited to males than to females. Research on affective dimensions and gender issues in mathematics education is extensive (see Leder, 1992). A range of affective variables was included in models explaining gender differences found to favour males in mathematics learning outcomes – achievement and participation rates (Leder, 1992). In general, males have been found to have more functional (likely to lead to future success) patterns of beliefs and attitudes associated with these affective dimensions. More recently, it has been reported that students' gendered patterns of beliefs associated with the stereotyping of mathematics as a male domain appear to be changing, at least in some countries (see, for example, Forgasz, 2001a, 2001b).

The same male stereotype has been attached to the field of computing. The variables examined and the research findings on secondary school students' attitudes towards computers are similar to those in the mathematics education literature. Compared to males, females are generally reported to be less positive about computers, like them less, perceive them as less useful, fear them more, feel more helpless around them, view themselves as having less aptitude with them, and show less interest in learning about and using computers; females are also less likely than males to stereotype computing as a male domain, to have received parental encouragement, to use computers out of school or to own one (e.g., Busch, 1995; Colley, Gale, & Harris, 1994; Durndell, Glissov & Siann,1995; Levin & Gordon, 1989; Makrakis & Sawada, 1996). Shashaani (1993) concluded that gender differences were influenced by socialisation and, as a result, females "have low expectations for success in computing" (p.179). Loyd, Loyd and Gressard's (1987) findings were at variance with those reported by others. They found that grade 7 and 8 female students' computer anxiety levels were lower and their liking of computers was higher than males' and suggested that it may be possible to compensate for females' less positive attitudes towards mathematics "by using computers more extensively in mathematics curricula at the middle school level" (p.18).

Galbraith, Haines and Pemberton (1999) developed a computer attitudes instrument and found that their computer-mathematics subscale correlated more strongly with computer confidence and computer motivation than with the equivalent mathematical scales. They claimed that the consistent and strong relationship often reported between mathematics confidence and performance meant "that the implications of a nexus between technology and mathematics needs specific research attention" (p.216). Gender should also be included as a variable in such research. Hoyles (1998) claimed that introducing computers into mathematics classrooms might widen the gap between males and females, typically those with less confidence or prior experience with technology.

Computers are now commonly found in mathematics classrooms and there is much pressure to use them. It is crucial to know whether using computers for mathematics learning exacerbates or challenges previously identified gender differences in mathematics education. Of interest in this study were students' gendered perceptions of mathematics, of computers, and of computers for the learning of mathematics.

THE STUDY

Aims

The findings reported in this paper are based on data gathered in the first year of a three year study [1]. The main aims of the entire study are: (i) to determine the effects on students' affective and cognitive learning outcomes of using computers for mathematics learning, (ii) to identify factors which may contribute to inequities in these learning outcomes, and (iii) to monitor how computers are being used for the learning of mathematics in grades 7-10. Students' attitudes and beliefs about using computers for the learning of mathematics were gathered in the first year of the study.

Sample, instrument and methods

Students in grades 7-10 from 28 co-educational schools in Victoria (Australia) participated in the study. There were 15 metropolitan and 13 rural schools from across the three educational sectors — government (17), Catholic (4), and Independent (7). The total sample size was 2140 (F=1015, M=1111, ?=14).

A survey questionnaire was administered to the students in semester two of the 2001 academic year. Included in the survey were three sets of ten items tapping students' perceptions of the gender stereotyping of mathematics (*Who & mathematics*), of computers (*Who & computers*), and of computers for learning mathematics (*Who & computers for mathematics*) — see Table 1 for the three sets of ten items.

Nine of the ten *Who & mathematics* items were drawn from the 30 item instrument described in more detail elsewhere by Forgasz (2001a, 2001b) and Leder (2001). The tenth item — *Tease kids who are good at mathematics* — was a combination of two of the 30 items — *Tease girls who are good at mathematics* and *Tease boys who are* *good at mathematics*. The ten items reflected several dimensions previously identified as associated with the gender stereotyping of mathematics: ability, general attitude, future careers, parents, teachers, and classroom factors (see Leder, 2001). The ten *Who & computers* and the ten *Who & computers for mathematics* items were developed to replicate the same dimensions of gender stereotyping — see Table 1.

For each item in each of the three sets of ten items, students were required to consider the wording of the item and then to select one of the following responses with respect to the behaviour or belief represented by the item:

BD boys definitely more likely than girls
BP boys probably more likely than girls
ND no difference between boys and girls
GP girls probably more likely than boys
GD girls definitely more likely than boys

Analyses, results and discussion

In order to determine an average directional response to each item, mean scores were calculated based on assigning scores to each response as follows:

BD = 1 BP = 2 ND = 3 GP = 4 GD = 5.

Mean scores less than 3 thus indicate that, on average, respondents believe that "boys are more likely than girls" to reflect the behaviour or belief encompassed by the item; means greater than 3 that they believe that "girls are more likely than boys" to do so. For mean scores close to 3 (no difference between boys and girls), one-sample t-tests were used to determine if the mean score obtained was significantly different from 3. Response directions for each item are shown in Table 1:

F = "girls are more likely than boys to..."
M = "boys are more likely than girls to..."
nd = mean not different from 3 ie. "no difference between girls and boys".

The mean scores are shown graphically in Figure 1. It should be noted that the vertical axis passes through 3, the score indicating a belief that there is "no difference between girls and boys". Bars to the right of the axis therefore reflect means > 3; those to the left, means < 3. The length of the bars shows the extent of deviation from 3, thus revealing the relative strength of students' beliefs with respect to each item.

Interesting patterns emerged when responses to the three sets of items were examined individually and then compared. The data were also analysed by gender to explore for differences in the response patterns of male and female students.

Who and mathematics

The directional responses to the ten items replicated those reported by Forgasz (2001a, 2001b) for a different sample of grade 7-10 Australian students. The findings appear to challenge the stereotype of mathematics as a *male domain*. The results indicate, for example, that students believe that girls are more likely than boys to say that mathematics is their favourite (item 2), to find mathematics interesting (7) and easy (1), and to get on with their work in class (10). Boys are believed to be more likely than girls to give up when a problem is too difficult (item 6), to need more help with mathematics (4) and to tease students who are good at mathematics (3).

Table 1. *Who & mathematics* (10 items), *Who & computers* (10 items), *Who & computers for mathematics* (10 items) and response directions

Who & mathematics		*Who & computers*		*Who & computers for mathematics*	
1. Find mathematics easy (Ability)	F	11. Like using computers (General attitude)	M	21. Are good at using computers for learning mathematics (Ability)	M
2. Mathematics is their favourite subject (General attitude)	F	12. Are good at fixing software problems (Ability)	M	22. Mathematics teacher gives them more help when using computers in class (Teacher)	F
3. Tease kids who are good at mathematics (Classroom)	M	13. Need more help with computer activities (Ability)	F	23. Think it important for their future jobs to be able to use computers for mathematics learning (Career)	M
4. Need more help in mathematics (Ability)	M	14. Teachers expect them to be able to use computers (Teacher)	M	24. Find using computers for mathematics to be boring (General attitude)	M
5. Think mathematics will be important in their adult life (Career)	F	15. Tease kids if they are good with computers (Classroom)	M	25. Do not like using computers for doing mathematics (General attitude)	nd
6. Give up when they find a mathematics problem is too difficult (Ability)	M	16. Give up when something goes wrong with the computer (Ability)	F	26. Tease kids who are good at using computers for their mathematics work (Classroom)	M
7. Think mathematics is interesting (General attitude)	F	17. Need to be able to use computers well to get a good job when they leave school (Career)	M	27. Give up when they find using computers for mathematics to be difficult (Ability)	nd
8. Mathematics teachers spend more time with them (Teacher)	F	18. Parents encourage them to use computers (Parents)	M	28. Like to take control of the computer when students work together in mathematics classes (Classroom)	M
9. Parents think it is important for them to study mathematics (Parents)	nd	19. Take control of the computer when working with others (Classroom)	M	29. Distract others as they work on computers in mathematics classes (Classroom)	M
10. Get on with their work in class (Classroom)	F	20. Are easily distracted when using computers (Classroom)	M	30. Parents think it is important for them to use computers for learning mathematics (Parents)	nd

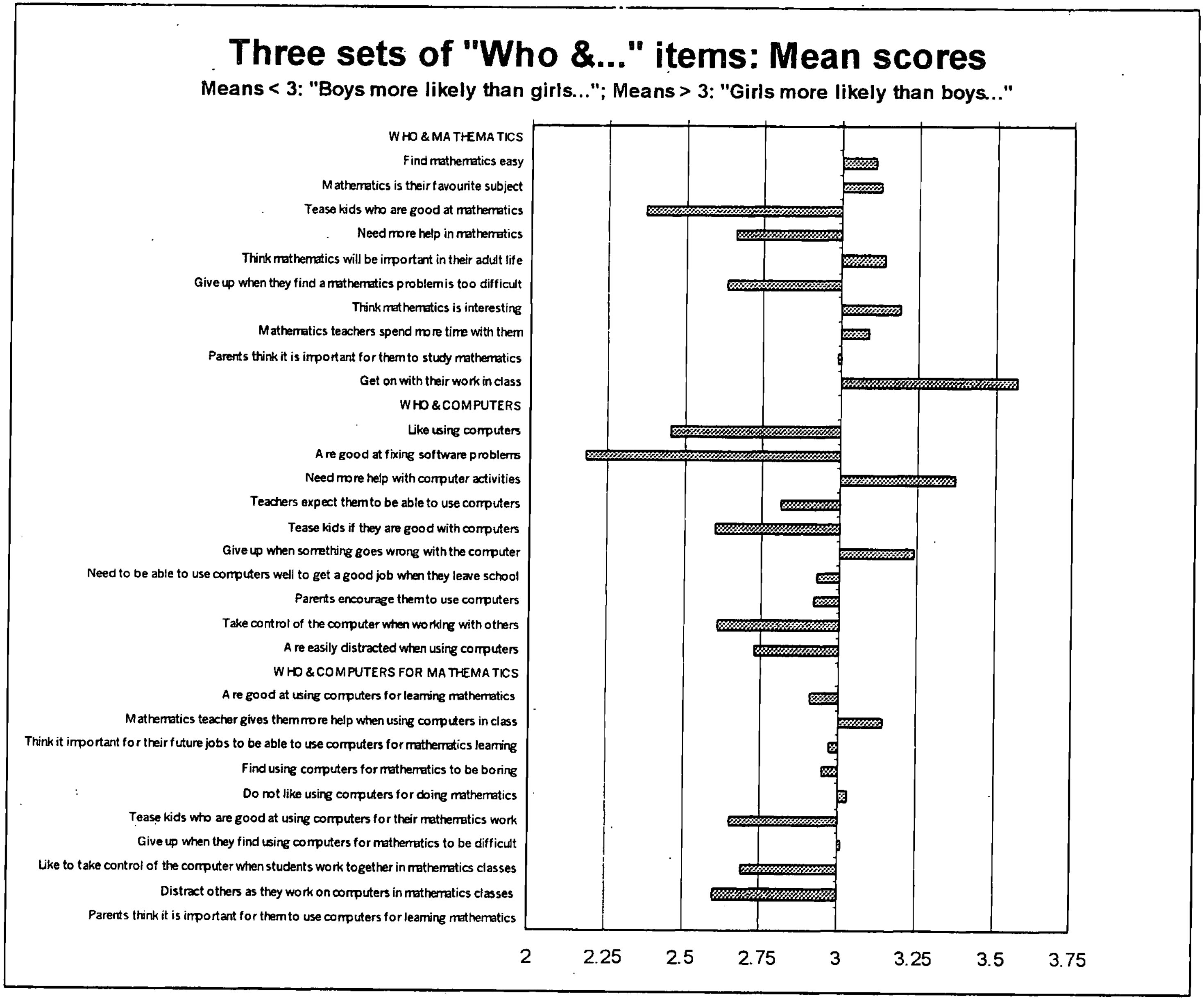

Figure 1. The three sets of "Who & ..." items: Mean scores (Australian grade 7-10 students)

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Who and computers

For these ten items, students' beliefs appear to reflect the stereotyping of the field of computing as a *male domain*. The results indicate that students believe boys like using computers (item 11), are good at fixing software problems (12), are encouraged by parents (18), and that teachers expect them to be able to use computers (15). Boys are considered more likely than girls to take control of computers in the classroom (19), tease students who are good with computers (15), be easily distracted when using computers (20), and be able to use computers for future jobs (17). Girls, on the other hand, are believed to need more help with computer activities (item13) and to give up when something goes wrong with the computer (16).

Who and computers for mathematics

The extent of stereotyping seems less marked on these ten items than for the *Who and computers* items and more consistent with the *Who and mathematics* items. Students believed, for example, that there was no difference between girls and boys in not enjoying using computers for mathematics (item 25), giving up when using computers for a mathematics problem is difficult (27), and regarding who parents think it is important use computers for mathematics (30). Students considered that boys were more likely than girls to be good at using computers for mathematics (item 21). Although it was boys who were also believed to think that using computers for mathematics was boring (item 24) but important for their future job prospects (23), these beliefs were not strongly held (mean scores just less than 3). Boys were strongly considered to be the teasers of students who were good at using computers for mathematics (26), as well as the ones to distract others (29), and to take control of the computer when working with others (28). Students believed girls received more help from the teacher (item 22).

Gender differences

For each item, independent samples t-tests were conducted to determine if there were statistically significant differences by gender. The results of the t-tests including mean scores by gender and significance levels for the 30 items are shown in Table 2 [See Table 1 for wording of items].

A close inspection of the means in Table 2 reveals that for the vast majority of items males and females responded in the same direction, that is both means were either >3 or both were <3. [Some means were not significantly different from 3 indicating a belief that there was "no difference between girls and boys" and are shown in *Italics*.]

As can be seen in Table 2 there were many items with statistically significantly different means by gender. Among those for which males and females responded in the same direction, there is no consistent pattern of either males or females holding the stronger view. Interestingly, there were only four items with significantly different means for which males and females held beliefs that were in opposite directions: items 8, 9, 24 and 25. For item 25, for example, females believed it was girls who were more likely than boys "not to like using computers for doing mathematics"; males believed that boys were more likely to do so.

Table 2. Results of t-tests by gender on items from the three sets of ***Who &...*** items

Who & mathematics				*Who & computers*				*Who & computers for mathematics*			
Item	F	M	p-level	**Item**	F	M	p-level	**Item**	F	M	p
1	3.19	*3.04*	***	**11**	2.56	2.36	***	**21**	*2.98*	2.85	***
2	3.08	3.19	***	**12**	2.23	2.13	**	**22**	3.06	3.21	***
3	2.28	2.45	***	**13**	3.30	3.44	***	**23**	*3.03*	2.92	***
4	2.69	2.63		**14**	2.83	2.79		**24**	3.03	2.84	***
5	3.27	*3.01*	***	**15**	2.56	2.63		**25**	3.11	2.95	***
6	2.64	2.63		**16**	3.23	3.26		**26**	2.59	2.70	**
7	3.11	3.24	***	**17**	*3.00*	2.88	***	**27**	*3.03*	*2.99*	
8	2.91	3.25	***	**18**	*2.99*	2.86	***	**28**	2.68	2.71	
9	3.06	2.93	***	**19**	2.63	2.60		**29**	2.56	2.64	*
10	3.67	3.48	***	**20**	2.73	2.73		**30**	3.02	*2.98*	*

NB. p-levels: * = <.05 **=<.01 ***=<.001

Means shown in *Italics* – not significantly different from 3.

CONCLUDING COMMENTS

It would appear that Australian grade 7-10 students no longer stereotype mathematics as a *male domain*, at least along the affective dimensions previously reported in the mathematics education research literature and tapped in this study. The same cannot be said about the discipline of computing. The students' beliefs seem consistent with the traditional gendered perception of male competence and female incompetence with the technology. Interestingly, however, when computers are associated with the learning of mathematics the students appear a little more ambivalent. Their views appear to sit somewhere between their beliefs about the stereotyping of mathematics and of computing. Within the constraints of the affective dimensions included in the three sets of items examined here, the data revealed that the views of male and female students were remarkably similar, at least with respect to the gendered directions of their responses to individual items. Although numerous statistically significant differences in mean scores were noted, there was also no clearly apparent pattern that one group held consistently stronger views than the other.

Further work is needed. The study needs to be replicated in different contexts. When other variables such as school type, socio-economic status, and ethnicity are considered, more clearly discernible patterns of differing stereotyped perceptions may emerge.

ENDNOTES

1. This study is being funded through the Australian Research Council's Large Grant scheme.

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STUDENTS' CLASSROOM NOTEBOOKS AND THE BLURRING OF PUBLIC AND PRIVATE DOMAINS

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Students' notebooks are frequently the source of data concerning students' mathematical thinking and practice. But the part the notebook itself plays in the mathematics classroom and the students' mathematical life has rarely been the subject of research in mathematics education. This paper presents some observations showing how notebooks tend to be of an exclusively public nature, and it explores theoretical implications of this tendency. A brief description is also given of the Learners Perspective Study in which these observations were made. That study is an ongoing international project concerned with understanding school mathematics practice from the students' point of view.

INTRODUCTION

Among the accoutrements of mathematics learners, hardly any could be more universal than notebooks. Yet, while student notebooks are a never neglected source of data about students' mathematical work and thinking, the place of the notebook itself in students' school mathematical life seems to have been little studied. In considering the notebook and its significance for the student, one question that arises is to what extent the notebook is a private or public object. In this paper, we shall explore this question and present some observations made in an eighth grade algebra class. These observations indicate that, in the classroom, the mathematics notebook takes on an utterly public character, a character unchallenged by the student and unwittingly encouraged by the teacher. We believe that this has practical and theoretical implications which touch on such issues as the importance of communication in the mathematics classroom (as emphasized in NCTM, 2000), of reflection, and of writing in the development of mathematical thinking.

We shall proceed in a somewhat unorthodox fashion, beginning with the research setting in which we made our observations about notebooks, namely, the Learners' Perspective Study, going on from there to the observations themselves, and only ending with the theoretical background and implications. In this way, we believe, the paper will better reflect the process by which we came to ask our questions, make our observations, and draw our conclusions concerning student notebooks.

THE SETTING: THE LEARNERS' PERSPECTIVE PROJECT

Our observations concerning students' notebooks in the mathematics classroom were made in the course of a more extensive, and still ongoing, study of the students' point of view, called the *Learners' Perspective Study* (Clarke, 1998, 2000). This project, which involves eight countries including Australia, Germany, U.S.A, Hong Kong, Japan, Israel, Sweden, and South Africa, seeks to explore a number of questions concerning the way students conceive mathematics classroom practice and mathematics learning. The project arose out of the Third International Mathematics and Science Study (TIMSS). As is well known, the TIMSS study not only established national profiles of student achievement, but also sought to identify national norms for teaching practice that might account for "poor" or "high" achievement scores by videotaping and analyzing a statistically representative sample of eighth-grade mathematics classes in Japan, Germany and the USA (Fernandez, C. et al., 1997). Although this component of the TIMSS study was impressive and unprecedented in international comparative studies, the validity of the "effective scripts" discerned in the TIMSS videos was widely debated and not universally accepted (e.g. Keitel & Kilpatrick, 1999; Stigler & Hiebert, 1997, 1998, 1999). Among the major objections was that the TIMSS Video Study focused exclusively on the teacher and ignored the important role students have in the learning process. The present project, accordingly, expands on the work done in the TIMSS study by focusing on *student actions* within the context of whole-class mathematics practice and by adopting a methodology whereby student reconstructions and reflections are considered in a substantial number of videotaped mathematics lessons.

The particular case on which we focus in this paper was a sequence of 15 lessons on systems of linear equations taught by a dedicated and experienced teacher, whom we shall call Danit. Danit's 8th grade class is heterogeneous regarding level; it consists of 38 students, mostly native born Israelis, but also new immigrants from the former Soviet Union and one new immigrant from Ethiopia. As specified in Clark (2000), classroom sessions were videotaped using an integrated system of three video cameras, one viewing the class as a whole, one on the teacher, and one on a "focus group" of two or three students. Following each lesson, the students in the focus group were interviewed, and their notebooks, containing the notes for that particular lesson, were photocopied. Moreover, once a week Danit herself was interviewed. Although we had a basic set of questions for both the student interviews and the teacher interview, we allowed the interview protocol to remain flexible so that we could freely pursue particular classroom events; in this respect, our interview methodology was along the lines of Ginsburg (1997). An important aspect of the interviews was that in the course of them the students could view and react to the videotape of the lesson for which they were the focus group. Needless to say, the interviews themselves were also videotaped.

DANIT'S LESSONS AND NOTEBOOKS IN DANIT'S CLASSROOM

The sequence of lessons viewed in Danit's classroom began with the idea of an equation in two variables, went on to the graphical representation of linear equations, to the graphical solution of a system of two linear equations, and, finally, to the algebraic solution of such systems. The lessons followed a fairly consistent pattern: a frontal lesson for roughly 10 minutes, then independent work on exercises by the students for the remaining 35 minutes of the class period. On days when there were two consecutive lessons, the work period was often extended through the second class period. The students worked in groups of two or three while Danit went from group to group, checking the students' work, asking questions, and helping with difficulties.

In each meeting, we noted that all of the students opened their notebooks only after Danit finished the frontal lesson; at the beginning of the class period, when she presented new material, gave examples, went over exercises from the previous lesson, the students' notebooks lay closed on their desks (there was one meeting that deviated from this pattern; it is the first incident recounted below). We found this curious: did the students not find Danit's examples and lessons worth recording? Did Danit not think that the students ought to write down her remarks and examples so that they would have them to think about and to refer to later? In several of the interviews, we asked the students about this. They replied that Danit allows them to choose whether they use their notebooks or not. In fact, however, the use of notebooks among the students bore no individual stamp; it was determined by Danit, as the two following incidents show. The first incident occurred when Danit began her lessons on the graphical representation of equations. In that lesson, she not only told the students to take out their notebooks and copy the coordinate system, but she also told them what heading to write, and she also told them to use a ruler in setting out the axes of the coordinate system. Afterwards, in the lessons involving graphs, the students produced coordinate systems just as they had done in this particular lesson. The second incident occurred in the meeting following the one above. In that meeting, Danit began immediately with exercises and opened the lesson by saying "Take out your notebooks—I want to see that you're doing the work correctly." The students showed no expression of surprise; this was not an odd request.

Both these incidents, which, we remark, are not at all unusual in Israeli schools, show Danit's control over what the students write in their notebooks and when. The second, however, was striking because it was in such blatant contradiction with the students' claim that Danit lets them choose whether or not they write in their notebooks. Indeed, if they know that Danit will eventually inspect their work, surely they must weigh how Danit might react if nothing were written, or if what was written was written badly, or wrongly. Far from being a matter of choice, then, three basic rules of notebook use seem to be followed in Danit's class:

1) During frontal lessons, students only listen; their notebooks are closed.

2) When given exercises, notebooks are to be opened immediately, and all the exercises are to be recorded neatly (the students should use rulers, for example) in them.

3) The teacher will occasionally look at the notebooks, so these rules must be followed.

The knowledge that the students' notebooks are to be open for inspection, in particular, means that they are a public matter—they are not to be a record of the students private thoughts about what they are learning: desultory reflections; false starts; mistaken conclusions and their, perhaps embarrassing, corrections. In fact, to their chagrin, Israeli teachers commonly find their students carrying out preliminary work for exercises, calculations for example, not in their notebooks, but on their desks! What students seem to learn is that the notebook must contain finished work that can be confidently held open to view (see the example reproduced in the appendix). The notebook is, in this respect, a sort of rehearsal for that part of school life in which the students' work is inspected most closely, the examination. Thus, students emphasize the use of notebooks to record the sort of exercises that will appear on examinations rather than the explanations behind them:

Interviewer [referring to examples Danit had written on the blackboard]: Did you write these exercises in your notebook?

Moshe/Sharon: Yes.

Interviewer: Why did you write them?

Sharon: So we won't be confused or something…for example, I try to solve…

Moshe: She also told us to write in our notebooks.

Interviewer: And if she didn't tell you?

Moshe: We write.

Interviewer: Write…

Moshe [without letting the interviewer finish her remark]: This is all material for the exam, so it can help.

Interviewer: Do you also write Danit's explanations?

Moshe: You mean her words of explanation? No…I don't anyway.

These students, and others in the class, say that Danit explains her lessons well, but these explanations are not written in the notebooks, only what the teacher might see on the students' examinations sheets. In this connection, it is worth noting that one of the only signs of individuality we could discern in Danits' students' algebra notebooks was the decoration that often adorned the headings, decoration that one often finds adorning their exam sheets as well; but are not such decorations also an expression of the notebooks' public nature, as something to be seen by others?

The public character of the students' notebooks is, interestingly enough, mirrored in the character of Danit's own lesson book. During our first interview with her, we asked if we could see her own notebook. She said that she does not have a detailed lesson book for her 8th grade class since, having taught that grade often, she no longer needs a lesson book, however, she let us see her lesson book for her 9th grade class. The lessons on geometry consisted of pages of solved problems and those on algebra, pages of numbered exercises. We could see no stated goal, no exposition of the material, no distinctions between major and minor examples, and no lesson structure. She also told us that she brings her lesson book to class and allows, even encourages, the students to compare their notes with hers (as we later confirmed in the lessons following this interview); this way, she said, the students can check and see if they have missed anything. We asked her if she hopes their notebooks will look like hers. She said yes. Thus, there is a notable consistency here between teacher and student practice, namely, that 1) a notebook is the place to record exercises only and 2) a notebook may be inspected at any time, that a notebook is always public.

PUBLIC AND PRIVATE FUNCTIONS OF NOTEBOOKS AND WRITING

To gain some insight on the significance of the observation concerning the public nature of notebooks in Danit's classroom, we need to remember that questions about the importance and function of notebooks in students' mathematical life are closely, if not intimately, related to questions about writing. Considerations of writing, on the other hand, suggest that notebooks, in principle, have a place in *both* private and public domains, and not, as we observed in the use of notebooks in Danit's classroom, in the latter domain alone. In fact, it can be said that writing mediates between the two domains. Take its most perspicuous public function, that of being a form of communication.

Though one is tempted to say that, as a means of communication, one writes only to be read, it is clear that the act of writing is not completely separate from the thinking that goes on before it. That this is true of creative mathematicians was long ago recorded by Hadamard in his classic work on mathematical invention (Hadamard, 1949, chap. V). But it is no less true for young students of mathematics. Thus, in the communication standard for grades 6 to 8 of the NCTM Principles and Standards, it is stated that "To help students reflect on their learning, teachers can ask them to write commentaries on what they learned in lesson or a series of lessons and on what remains unclear to them" (NCTM, 2000, p.271). Carpenter and Lehrer (1999) are even more explicit in connecting internal mental activity and mathematical understanding to external expression. They write:

> Articulation involves the communication of one's knowledge, either verbally, in writing or through some other means like pictures, diagrams, or models. Articulation requires reflection in that it involves lifting out the critical ideas of an activity so that the essence of the activity can be communicated. In the process, the activity becomes an object of thought. In other words, in order to articulate our ideas, we must reflect on them in order to identity and describe critical elements. Articulation requires reflection, and, in fact, articulation can be thought of as a public form of reflection" (Carpenter & Lehrer, 1999, p.22).

The argument has been extended by some relatively new research by Lillie R. Albert (Albert, 2000). Albert highlights the mediating function of writing by seeing it as a bridge between Vygotsky's socially dependent zone of proximate development (ZPD) and a zone of learning in which the student is independent and self-regulating, a zone of learning she calls the zone of proximate practice (ZPP). The question Albert addresses is, of course, the one Vygotsky himself saw as "an important concern of psychological research," namely, "to show how external knowledge and abilities in children become internalized" (Vygotsky, 1978, p.91). For Albert, the ZPP "is the result of the students' transformation from the interpsychological to the intrapsychological plane of functioning," and adds that "A basic assumption that underpins the ZPP construct is that the writing students do, as shaped by the collective practice in which they engage, determines how they independently think about mathematical ideas or concepts" (Albert, 2000, p.117).

Whether it is understood as an activity of clarification in the course of articulation or of reflection and internalization following social interaction and collaborative work, writing is seen to come between communication, which is fundamentally public, and reflection, which belongs to a private domain. To the extent that it is connected to communication, surely, writing must be clear and organized, and to the extent that it is connected to reflection, it must be free and exploratory. As an illustration of the latter, consider this description by Thomas Mann concerning his work habits:

> For a longer book I usual have a heap of preliminary papers close at hand during the writing; scribbled notes, memory props, in part purely objective—external details, colorful odds and ends—or else psychological formulations, fragmentary inspirations, which I use in their proper place (quoted in John-Steiner, 1985, p.76).

Since students' notebooks are their *own* notebooks, their own possession, it would seem that they are the proper loci for writing serving reflection, that is, writing in which thoughts are worked out and developed, and not only writing in which expected and taught solutions to given problems are recorded. In any case, since the notebook is the chief place in which students do their writing, the research by Albert and others alluded to above suggests strongly that notebooks lose an important function if they do not contain such writing serving reflection.

CONCLUSION

In Danit's classroom, student notebooks are exclusively public: they contain finished work only and may be inspected at any time. They contain no exploratory work, no false starts, alternative strategies, or random reflections on material being taught to them. Since their notebooks contain no writing of this kind, the ability for writing to mediate between the students' learning on a interpersonal plane and on a intrapersonal plane becomes seriously limited.

We would like to conjecture that a lack of a private domain in learning—and, surely, the notebook is the most natural place for such a domain—influences students' very ability to grasp that mathematical ideas are the sort of thing requiring reflection. Our reason for making this conjecture is that in the course of our interviews with students, we found that they had great difficulty understanding and appreciating the connection between "doing mathematics in your head" and writing mathematics down; on the one hand, "doing mathematics in your head" lacked legitimacy, and, on the other hand, writing mathematics down is done only to show the teacher that "you understand the material, that you didn't just guess." What we are conjecturing is if students' notebooks were allowed to be a private domain for the students, a place for written reflection, they might have the opportunity to see how writing mathematics down and "doing it in your head" can be complementary and mutually enriching.

If this conjecture is true, it would support the practical suggestion, such as that in the NCTM Principles and Standards (NCTM 2000), that students keep a mathematical journal together with an exercise notebook. We would stress, however, that in order for the benefits of a mathematical journal to be fully realized it ought to remain strictly within the student's private domain; their journal must be the place for truly *uninhibited* reflection. For this reason, we differ with the NCTM's recommendation that the journal play a part in assessment. But, be that as it may, we believe that, in general, our observations about notebooks point to a need for educators to recognize where the lines are drawn in mathematics classrooms between private and public domains, and to recognize that finding a balance between these domains may have serious implications for the students ability to reflect on mathematical ideas.

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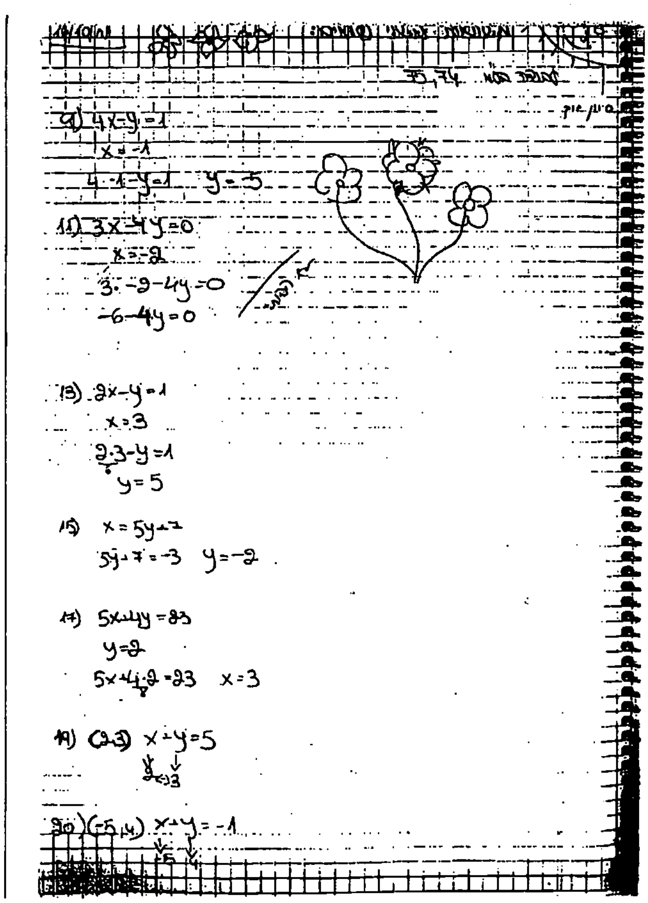

THE BRIDGE BETWEEN PRACTICAL AND DEDUCTIVE GEOMETRY: DEVELOPING THE 'GEOMETRICAL EYE'

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The dual nature of geometry, as a theoretical domain and an area of practical experience, presents mathematics teachers with the opportunity to link theory with the everyday knowledge of their pupils. Very often, however, learners find the dual nature of geometry a chasm that is very difficult to bridge. With research continuing to focus on understanding the nature of this problem, with a view to developing better pedagogical techniques, this paper reports an analysis of innovative geometry teaching methods that were developed in the early part of the 20th Century, a time when significant efforts were being made to improve the teaching and learning of geometry. The analysis suggests that the notion of the geometrical eye, *the ability to see geometrical properties detach themselves from a figure, might be a potent tool for building effectively on geometrical intuition.*

INTRODUCTION

The teaching and learning of geometry remains a major problem for mathematics education. As Villani (1998, p321-2) observes, in the conclusion to the ICMI study, "to build a [geometry] curriculum is a very difficult and demanding task" yet "teaching methods [in geometry] are even more important than content. And it is also more difficult to improve them". In a similar vein, the recent UK study of geometry teaching (Royal Society, 2001) concludes that "the most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources" (p19).

One of the major characteristics of geometry, as each the aforementioned reports acknowledges, is its dual nature. Geometry is both a theoretical domain and perhaps the most concrete, reality-linked part of mathematics. This dual nature has dual consequences for the teaching and learning of geometry. While, hypothetically, the dual nature of geometry should help teachers to link mathematical theory to pupils' lived experience, in practice for many pupils the dual nature is experienced as a gap that they find very difficult to bridge. Thus, research continues to focus on the difficulties that pupils have in developing an understanding of geometrical theory and making the transition to formal proofs in geometry (see, for example, Arzarello *et al*, 1998; Malara and Iadorosa, 1997; Miyazaki, 2000).

While the use of software tools, such as dynamic geometry, is proving to be helpful (for recent research evidence, see the special issue of *Educational Studies in Mathematics* edited by Jones *et al*, 2000), there is an urgent need to develop a more effective pedagogical theory for geometry so that such tools can be integrated more successfully in mathematics classrooms. With a view to informing the development of better pedagogical models, this paper reports some of the findings from a study of forms of innovative geometry teaching that were developed in the early part of the 20th Century, a time when significant efforts were being made to improve the teaching and learning of geometry. The analysis of curriculum materials and associated teaching methods undertaken as part of this study focuses, in part, on ways of bridging the gap between practical and deductive geometry. The analysis suggests that much promise lies in the notion of the *geometrical eye*, a term coined by one of the major movers behind the reform of the geometry teaching in the early 20th Century, Charles Godfrey[1] (1910). Godfrey defined the *geometrical eye* as "the power of seeing geometrical properties detach themselves from a figure" (*ibid*, p197). This paper argues that this notion might be a potent tool for building effectively on geometrical intuition.

THEORETICAL CONSIDERATIONS

Of the range of theoretical work concerned with the learning of geometrical ideas, that of Piaget (and colleagues) and of the van Hiele's is probably the most well-known. In the Piagetian work (see, Piaget, Inhelder and Szeminska, 1960), one of the major themes is that a learner's mental representation of space is not a perceptual 'reading off' of what is around them. Rather, learners build up mental representation of the world through progressively reorganising their prior active manipulation of that environment. The van Hiele model also suggests that learners advance through levels of thought in geometry, characterised as visual, descriptive, abstract/relational, and formal deduction (see, van Hiele 1986). Both the Piagetian approach and the van Hiele model have been subject to critical review that is beyond the scope of this paper. Suffice to say that much additional research is needed on the relations between intuitive, inductive and deductive approaches to geometrical objects, the role and impact of practical experiments, and the age at which geometrical concepts should be introduced.

Geometry is an area of mathematics in which intuition is frequently mentioned. Views vary, however, about the role and nature of geometrical intuition, and how it might help or hinder the learning of geometry (and other areas of mathematics). Piaget, for instance, appears to suggest a hierarchy when he equates intuition to what he calls non-formalised operational thought:

> Although effective at all stages and remaining fundamental from the point of view of invention, the cognitive role of intuition diminishes (in a relative sense) during development. .... there then results an internal tendency towards formalisation which, without ever being able to cut itself off entirely from its intuitive roots, progressively limits the field of intuition (in the sense of non-formalised operational thought).
>
> Piaget 1966, p225

Van Hiele similarly gives intuition a relatively minor role in the latter stages of learning. In contrast, Fischbein suggests either a plurality or a dialectic when he writes that:

> The interactions and conflicts between the formal, the algorithmic, and the intuitive components of a mathematical activity are very complex and usually not easily identified or understood.
>
> Fischbein 1994, p244

Geometers, nevertheless, tend to recognise the importance of geometrical intuition. In his classic text on geometry and the imagination, Hilbert wrote:

> In mathematics ... we find two tendencies present. On the one hand, the tendency towards abstraction seeks to crystallise the logical relations inherent in the maze of materials ... being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency towards intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations.
>
> Hibert 1932, piii

More recently, Atiyah writes:

> spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool...
>
> Atiyah, 2001

Yet not all mathematicians share this view. For many, intuition, even geometrical intuition, is not to be relied upon. As Tall (2000, p20) reminds use, the influential *Bourbaki* approach rejected any notion of geometrical intuition as being untrustworthy. Such a view remains fairly prevalent amongst many mathematicians.

The question that these considerations highlight is how to resolve these apparently opposing positions, if this is possible. To address this question, and attempt to illuminate the relationship between practical and deductive geometry, we examine a time in mathematics education when these issues were being seriously tackled.

METHOD

In the UK at the beginning of the 20th Century, the teaching of geometry was in the spotlight. The suitability of the formal teaching of deductive geometry in the form of Euclid's Elements was being seriously questioned. The form of theorems and the order in which they should be introduced was the subject of great debate (see, Howson, 1982).

As part of a wider study of the process of change in geometry teaching in the early 20th Century, the historic sources examined in this paper are the geometry textbooks and educational writings of two influential scholars of the time, Charles Godfrey and Arthur Siddons. The pivotal role played by Godfrey and Siddons and the lasting importance of their textbooks in the history of the teaching of geometry is confirmed by Howson (1982) and Quadling (1996). The methodological approach is documentary analysis (Jupp and Norris, 1993) enhanced by the methodology for textbook analysis proposed by Schubring (1987).

The documents analysed include the two main textbooks by Godfrey and Siddons, *Elementary Geometry* (Godfrey and Siddons, 1903) and *A Shorter Geometry* (Godfrey and Siddons, 1912), together with many of their articles published in the journal *Mathematical Gazette*, a collection of educational writing published in the 1930s (Godfrey and Siddons, 1931) and a memoir published in the 1950s (Siddons, 1952). The analysis conducted for the study focused on the design and various roles of practical tasks in the textbooks and on their pedagogical purpose gleaned from documents written by Godfrey and Siddons. Through this analysis we consider Godfrey and Siddons' approach to the relationship between practical and deductive geometry, and propose some insights for future research.

PRACTICAL TASKS IN THE TEXTBOOKS BY GODFREY AND SIDDONS

In both *Elementary Geometry* (*op cit*) and *A Shorter Geometry* (*op cit*), and unlike that recorded in Euclid's *Elements*, the design was such that the early stages of each book comprised practical activities while the latter stages were devoted mainly deductive proof of various geometrical theorems. Throughout both of the Godfrey and Siddons texts, experimental tasks can be seen.

For example, through the exercise below, students would learn how to measure angles (Godfrey and Siddons, 1903, p. 12);

> Exercise 37. Measure the angles of your set square (i) directly, (ii) by making a copy on paper and measuring the copy.

An example of an exercise (Godfrey and Siddons; 1903, p. 28) that would lead students to discover a geometrical truth that is proved deductively in a later section of the book is as follows;

> Exercise 123. Cut out a paper triangle; mark its angles; tear off the corners and fit them together with their vertices at one point. What relation between the angles of a triangle is suggested by this experiment?

In some of the practical exercises, students apply chosen theorems in a practical way. For example, the following exercise would be undertaken after the students have learn that 'there is one circle, and one only, which passes through three given points not in a straight line'.

> Exercise 1162. (using graph paper.) Draw a circle to pass through the points (0, 3), (2, 0), (-1, 0), and measure its radius. Does this circle pass through (i) (0, -3), (ii) (1, 3), (iii) (0, -2/3)?

In summary (for more details of the analysis see, Fujita, 2001a and 2001b), the roles of these tasks can be categorised as follows:

a) Making students familiar with geometrical instruments and figures;

b) Leading students to discover geometrical facts; and

c) Applying the theorems to practical problems.

In addition, some of the exercises are included to justify geometrical facts through work on experimental tasks. Godfrey discusses the importance of this from a general educational point of view in one of his articles (see, Godfrey and Siddons; 1931).

THE RELATIONSHIP BETWEEN PRACTICAL AND DEDUCTIVE GEOMETRY

It is clear from the writings of Godfrey and Siddons that they considered that a disconnection between experiment and deductive geometry would be inappropriate in the teaching of geometry (Godfrey and Siddons; 1931, p21, Siddons; 1952, p9). Godfrey, for example, was explicit that mathematics cannot be undertaken by logic alone (Godfrey; 1910, p. 197). He wrote that another important 'power' is necessary for solving mathematical problems. This he called *geometrical power*, defined as "the power we exercise when we solve a rider [a difficult geometrical problem requiring the use of several pieces of theoretical knowledge]" (Godfrey; 1910, p. 197). To develop this *geometrical power*, Godfrey suggested that it was essential to train what he called the *geometrical eye*. This, Godfrey defines as 'the power of seeing geometrical properties detach themselves from a figure' (Godfrey; 1910, p. 197).

To illustrate what Godfrey means by the *geometrical eye*, consider an example:

> If A, B are the mid-points of the equal sides XY, XZ of an isosceles triangle, prove that AZ=BY' (Godfrey and Siddons; 1903, p. 94).

When we consider this problem, we would not be able to prove this statement unless we could 'see', first of all, that, for example, triangle AYZ and triangle BZY are likely to be congruent. A study, carried out in Japan, showed that some students could not 'see' which triangles, in the problem above, were likely to be congruent. Nakanishi (1987), who tried to identify the difficulties that students experience when they solve geometrical problems, gave 87 Japanese students aged 14-15 the isosceles triangle question quoted above. Even though 65 students could prove this problem correctly, nine of them were not sure what to do because they could not see any congruent triangles, and four of them could not focus on an adequate pair of congruent triangles (*op cit* p71).

Godfrey stated that this kind of 'power' would be essential to solve geometrical problems, and it was experimental tasks that would make possible to train the 'geometrical eye' at any stages in geometry:

> There must be a good foundation of practical work, and recourse to practical and experimental illustration wherever this can be introduced naturally into the later theoretical course. Only in this way can the average boy [*sic*] develop what I will call the 'geometrical eye'.
>
> Godfrey; 1910, p197

When we analyse Godfrey and Siddons' textbooks from this point of view, we can find that practical exercises were placed in various places in the deductive stages in *Elementary Geometry* or *A Shorter Geometry*. For example, before the theorem 'a straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord', the following exercises were included in *A Shorter Geometry* (Godfrey and Siddons, 1912, pp. 151-2),

> Exercise 877. Draw a circle of about 3 in. radius, draw freehand a set of parallel chords (about 6), bisect each chord by eye. What is the locus of the mid-points of the chord?
>
> Exercise 878. Draw a circle and a diameter. This is an axis of symmetry. Mark four pairs of corresponding points. Is there any case in which a pair of corresponding points coincide? (*Freehand.*)
>
> Exercise 879. What axes of symmetry has (i) a sector, (ii) a segment, (iii) an arc, of a circle?

These exercises would make students aware of the symmetry of the circle as well as leading them to discover the theorem (notice that a symbol in the textbook against each exercise denoted that each required classroom discussion between the teacher and students). Also, to prove the theorem, the fact that the triangles OAD and OBD (see the figure 1 below) are congruent needs to be shown. Our analysis suggests that the precursor exercises are designed to help the students to 'see' the congruency of the triangles. That is, that the exercises are purposefully included to develop the students' *geometrical eye*.

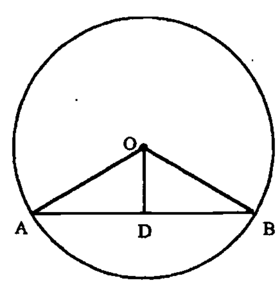

Fig. 1

CONCLUDING COMMENTS

In a number of countries, the early stages of geometry in schools comprise practical activities such as the drawing and measurement of geometrical figures. Later stages of schooling are then devoted to deductive geometry. The specification for geometry in the Japanese 'National Course of Study' (Japan Society of Mathematics Education, 2000), for example, takes such an approach. While this is somewhat in line with the van Hiele (1986) model of learning in geometry, the relationship between practical and deductive geometry remains unclear, and, in particular, the transition between them is one of the major concerns in the study of the teaching of geometry.

A major improvement in geometry pedagogy would be to improve on calls to develop geometrical intuition by linking more directly with geometrical theory. This would entail developing pedagogical methods that mean that a deductive and an intuitive approach are mutually reinforcing when solving geometrical problems (see, Jones 1998).

This paper argues that Godfrey's notion of the *geometrical eye* might be a potent tool for building effectively on geometrical intuition. As we have shown through our analysis, Godfrey and Siddons considered that practical and deductive geometry should be combined in the latter in the teaching of geometry. Godfrey considered that the *geometrical eye* would be essential for successfully solving geometrical problems, and that it should be trained by practical tasks at all stages of geometry. It is illuminating that innovative teachers 100 years ago mentioned the importance and roles of visual images in geometry, and it is worth considering the following issues in the future research for improvement of the teaching of geometry in primary and secondary schools. Future research could examine whether it would be possible to define more clearly the notion of the *geometrical eye*, what the relationships are between difficulties of proof in geometry and the *geometrical eye*, and how (or whether) it would be possible to train students' *geometrical eye* though practical tasks.

NOTES

1. Godfrey's use of the term *geometrical eye* predates that of Wittgenstein (see Wittgenstein, L.J.J.: 1958, *The Blue and Brown Books*, Oxford, Blackwell, pp. 73-74) and has a different sense.

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DEFINING WITHIN A DYNAMIC GEOMETRY ENVIRONMENT: NOTES FROM THE CLASSROOM

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ABSTRACT. *This paper concerns the activity of defining. We report about an experiment in which we studied students' behavior in constructing and classifying quadrilaterals within a dynamic geometry environment (Cabri-Géomètre). In particular, we considered the problem of the consistence of certain definitions with the constructions made with Cabri, i.e. we used the microworld to make students reflect on the adequacy (within the microworld) of the definition they use. The findings show that there are kinds of thinking that are developed as a result of the interaction with the tool and suggest considerations on the problem of providing students with a meaningful and active approach to theoretical thinking.*

INTRODUCTION AND THEORETICAL FRAMEWORK

In this paper we discuss the problem of defining that we have faced elsewhere, see (Furinghetti & Paola, 1999 and 2000). The reason why we pay such a great attention to the theme of definition is that it is the first gate to enter a theory. Thus a way of defining not suitable to students' mind may affect the entire path in the construction of a theory. This theme is set in our general concern of providing students with means to approach theoretical thinking with *awareness*. Other authors have faced the issue of awareness in doing mathematics, e.g. Mason (1987). In using the term "awareness" we means that students must be active participants in the process of constructing a theory and have to grasp the meaning of what they are doing. This view is based both on recent literature in mathematics education and on evidence emerging from the past. At the beginning of the twentieth century, when the foundation movement stressed the importance of the axiomatic method and logic, mathematical programs (and, as a consequence, textbooks and teaching style) became oriented to follow a pattern based on axioms-theorems-deduction. This "axiomatic style" affected mathematics teaching in many countries. Even at the beginning, however, this approach was questioned by teachers, as evidenced by papers appearing in early journals oriented to mathematics teaching, and also by important mathematicians. Felix Klein claimed that in school, as well as in research, the phase of formalization must be preceded by a phase of exploration based on intuition. He wrote: "I maintain that mathematical intuition [...] is always far in advance of logical reasoning and covers a wider field". (Klein, 1896, p.246). We find a similar statement in the introduction of a school geometry book by a famous Italian mathematician, Francesco Severi. He wrote (1930, Vallecchi, Florence, p.IX; our translation): "Who is aware of the value of foundation theories, does not make the dangerous mistake of giving to the elementary teaching a critical and excessively abstract style." Giovanni Vailati (1907), an Italian secondary teacher and researcher in logic, supported a method of teaching in which exploration, experimental mathematics, drawing with rule and compass have to precede deduction.

In the 1920s the idea of a "genetic" principle took shape and interesting treatises were published[1]. For example, Gusev and Safuanov (2000) refer about a school geometry book by N. A. Izvolsky, in which it was advocated that teachers explain to students the origin of geometrical theorems. According to the author, when this is done, students see geometry in a different way. The idea of a genetic approach later took a definite form in the book by Otto Toeplitz (1963) on infinitesimal calculus. This author was aware that in this domain the notion that learning mathematics takes place in a sequence predetermined by mathematical logic has shown its pedagogical limitations. Indeed, when organized around their logical kernel, the definitions of the main concepts of calculus (integrals, limits, derivatives) come out of the blue and the burden of formal rules and of theorems makes it difficult for students to grasp the meaning of what they are doing.

Our historical outline stresses the widespread (over the years and across the countries) concern about providing students with an active and meaningful approach to theoretical thinking and about the search for mediators/environments to realize this approach. Some of the authors we have mentioned look at history as a mediator effective for this purpose. Others of them consider different mediators: in some passage of Izvolsky's writings we find an embryo of classroom discussion, Vailati's program is centered on the use of exploration and of mathematical instruments.

In the present paper we have chosen the dynamic geometry software *Cabri-Géomètre* as a mediator to make students aware in defining. Our aim is to study which strategies students apply and how this environment affects their behavior. To realize it we have studied the work on the definition of quadrilaterals of 21 students of junior high school (15 years old). Quadrilaterals (which, in theory, were a topic known by our students) have been chosen because the focus of the described activity is not on the mathematical object to be defined, but just on definitions considered as mathematical objects which need of careful reflection. We think one masters and reifies a concept when this concept is used as an object. For example, one understands the concept of function when the function becomes an element of the domain in which he/she works. Thus one starts to understand what a definition is when definition itself becomes an object of study[2].

The family of quadrilaterals has been often considered for the purpose of studying the problem of defining (both as for students and for teachers). For example, among the articles and departments (about 700 and 1000 respectively) published in the years 1990-2000 in the journal *Mathematics teacher* we have singled out only 26

[1] A pioneer example of the application of this method is offered by the book *Éléments de géométrie* by the mathematician Alexis-Claude Clairaut first published in 1741 (Paris: David fils).

[2] For a wider description of our theoretical framework on definition see (Furinghetti & Paola, 1999 and 2000; Shir & Zaslavsky, 2001).

contributions that may be considered belonging to the domain of definitions. Of them, 6 treat quadrilaterals. The reason of the popularity of quadrilaterals as object of definition is that there is a great experience (from Euclid onwards) about this subject both from the theoretical and from the teaching side. As shown in the papers (De Villiers, 1994 and 1998; Shir & Zaslavsky, 2001) this subject allows to work on hierarchical classification and application of notions about sets, it may be treated through verbalization or visualization, and it is a good field in which one may test the notion of equivalent definitions, see (Leikin & Winicki-Landman, 2000). Moreover this subject is suitable to exploit the intrinsic cognitive character of the dynamic geometry environment, see (Jones, 1998 and 2000; Mariotti, 2000). In particular, we deal with the problem of the consistence of certain definitions with the constructions that can be done with Cabri. We use the microworld to make students reflect on the adequacy (within the microworld) of the definitions they use. Our study has not the pretense to give final answers to the issues in question, but we hope that it can be a further tessera in the mosaic that researchers are constructing about these issues.

METHODOLOGY

The experiment involved 21 Italian 10-grade students with previous experience of using Cabri for solving geometrical problems. They worked in pairs (one PC per pair), except one group of three students. The teacher and a researcher acting as observer (he was one of the authors) were present. The observer was not passive, but talked with students and addressed their activity to Cabri in order to obtain more information about the interaction student-Cabri. We know from other experiments that if the teacher use Cabri as a blackboard students do not enter completely in the "logic" of software, see (Arzarello et al., 1998).

The first day of the experiment the students received the following instructions:

- Give a classification of quadrilaterals.
- You can use what you know from the previous school years, and, if you think it is useful, Cabri.
- Please keep a diary of the sessions (the final classification, discussions with the mates working with you, ideas that have not been completely developed).
- Remember that a construction with Cabri is validated only if it is kept by the dragging test.

All students used Cabri; after two sessions of work they wrote a report, as it was requested. In the next session the teacher orchestrated a classroom discussion on the students' work, and wrote a report. Since she had the impression that the less extroverted students were not able to express their thoughts in front of all mates, she asked students to answer some written questions prepared by her. Our reflections are based on the notes by the observer, students' reports, the report of the teacher, and the written interviews.

DIRECT OBSERVATION OF THE WORK WITHIN THE DYNAMIC GEOMETRY ENVIRONMENT

Few days before the experiment students had classified triangles (only through paper and pencil), nevertheless they were quite worried about the task of classifying quadrilaterals. The reason of that was the **anxiety for the technical problems linked**

to Cabri[3]. The presence of an external person, however, has motivated students and they worked with good will. The groups acted collaboratively. Who used the mouse was the leader in the activity, but the low number of students in each group allowed to use the mouse alternatively (at least this happened in the most collaborative groups), so that the risk of passivity in the group was avoided.

All groups began **with the construction of a square**. The most common sequence was "to draw **a circle** and a square inscribed or circumscribed". The pair Dalila-Chiara drew a circle and two points on it, then the square circumscribed to the circle. At the beginning they wrote the macro construction before checking if their construction passed the dragging test. They had four times the message "Macro is not consistent". They never checked the geometrical construction, but wrote again the macro until the computer said that it was correct (the construction, indeed, was correct, while the macro was wrong).

This behavior was widespread among students working with the dynamic software. Also the pairs Elisa-Michela and Paolo-Carlotta did not think to validate their construction when found that their rectangles did not pass the dragging test. This means that, even if they were able to construct macro and had sufficient abilities to use the software, they did not enter the logic of the tool. They worked only at the level Laborde (1998) terms "spatio-graphical level": thus they were not able to use Cabri as a mediator to pass from drawing to the geometrical theory.

The other groups (except one) made **drawings which were figural (i.e. not obeying geometrical rules)**. After having drawn a parallelogram as a mere drawing, Luca **asked himself** "How can I decide that this is really a parallelogram?" This is a question that would not have arisen without Cabri. It was spontaneous to **discuss on which conditions** the figure was really a parallelogram. It was important to have an expert at disposal (the teacher or the observer) who listened to the questions and answered in real time, not after some time when the questions are no more important.

Paolo and Carlotta drew a circle and after drew the symmetric C of a point B respect to the center O, then constructed the perpendicular line to the line BC. The intersection of this perpendicular line with the circle gave two points A and D. $ABCD$ was the resulting square. After that they used (correctly) symmetries to obtain a rhombus. They used **symmetries** and **definitions which are construction-oriented** (i.e. rich of hints for constructing). They succeeded in doing figure, even if they were not able to state the **characteristic properties** of the figures they have correctly drawn. When the researcher asked a definition of the rhombus they repeated the construction learnt in junior secondary school based on drawing two perpendicular lines and taking congruent segments on them.

[3] We have emphasized with the boldface type the features we consider interesting in our experiment.

Elisa and Michela constructed a rectangle by combining two right-angled triangles. They tried with translations, with 180° degrees rotation. This pair was very collaborative, students tried to explain how it is possible to obtain rectangles from right-angled triangles, but the figures they produced did not pass the dragging test. Thon and Federica were much worse: they drew objects and erased them without the project necessary to perform the task in Cabri.

The attitude of combining blocks of elementary figures to obtain sophisticated figures, as it is done with the *Lego* blocks, was even more evident in the second session dedicated to trapezia. Students looked at the figure as a mere drawing and act as a scanner in reproducing the stereotyped figures of their textbook. The conceptual aspect was absent. In this behavior we may see a certain analogy with the behavior in algebra were signs are manipulated without meaning. Cabri may be useful to change this behavior, since emphasizes the construction-oriented aspects to the detriment of the pure reproductive ones and thus it calls for meaning.

From this first session we may draw some general conclusions that have been corroborated in the second session on trapezia:

- Students have in their mind figures (not the properties of figures).
- They desperately explore Cabri looking for a way to reproduce figures and use strange means and tricks on the Cabri screen. Figures they have in mind are static. They, indeed, recognize only the figures, which are in particular configurations. We'll find these observations also in the teacher's report. For students the construction is not validated by the fact that a certain property is kept by dragging, but by the drawing obtained in a particular position. Cabri has made visible the students' remoteness from the theory they should study.
- Students do not use the potentialities of the tool in a rational way.
- Only one pair uses symmetry. This is done to construct the figure, but not to define the figure.
- Some behavior observed is fostered by the current teaching style: since many years drawing with rule and compass has been abandoned in school.

THE TEACHER'S REPORT ON THE DISCUSSION

The teacher's report reflects her concern about the practical teaching problems she has to face. The report is based on the discussion in classroom. As an example of what happened during this session we give the following description of the discussion about "dragging".

Chiara began by stating that if the construction is correct, the shape of the figure is kept.

> Elisa: "No! Because sides shorten or lengthen!"
> Patrizia: "But what do you mean by shape of a figure? Even if measures change a square is always a square!"

This clam addressed the discussion on what the "shape" of figure is. For someone it was "the set of sides", for others "the construction one has made", "the aspect with which the figure appears", "enlargement or reduction (scaling off)". Ambra stated that perimeter and area of figures are kept by dragging. The teacher asked whether the dragging is an isometry; Ambra was puzzled and changed her mind. The teacher writes in her report that, even if isometries have been treated in the previous school year, students' reactions show that isometries remained very abstract. The movement originates confusion and students have difficulties in singling out invariants. This difficulty emerged also in the written interviews. Students were more able in recognizing the symmetric of a figure in respect to a point or a straight line and in recognizing symmetries of figures. Some student have noticed that the "little hand", which allows to drag, appears only in some positions.

ANSWERS TO THE WRITTEN INTERVIEW

The questions of the written interview have been prepared by the teacher on the ground of the crucial points emerged in the discussion. They concerned:

- which quadrilateral has been the starting figure
- what is kept and what is left by dragging
- how the passage from the construction to the definition happens: 1. Listing the actions in the menu? 2. Enunciating the properties used in the construction? 3. Enunciating only the properties sufficient to construct the figure?
- why it is required to drag the figure one has constructed.

Students answered individually on the ground of the work at the computer and the discussion in classroom. Protocols give interesting insights on kinds of thinking brought to light by the activity with Cabri. All students confirmed that they started from the square. A paradigmatic explanation of this fact given by a student is:

> I started from the square because it is the quadrilateral with more properties and because I can imagine it more easily than the other quadrilaterals.

Other students say that "The square is the easiest quadrilateral". The generic is more difficult to be conceived than the particular and figures with regularities (i.e. specified figures) are more easily perceived.

To have started from a square has reflections on the classification of quadrilaterals. Even if this subject had been developed before in classroom, 11 students did not answer. Six students drew a figure with Venn diagrams showing the hierarchical classification presented in their the textbook (may be reproduced in a ritual way). Four students chose a classification going from squares to trapezia, which is inspired by the sequence they used in constructing quadrilaterals with Cabri. Two of them simply listed quadrilaterals, one explicitly wrote that she did not agree with her textbook classification. One student drew the figure below ("quadrato" means "square"). The dynamic geometry environment, indeed, has oriented to a different criterion of classification, which we may term "by default". It is a kind of reverse hierarchy: one starts from the more specified figure (the square) having the greatest number of properties and goes on by leaving some properties.

CONCLUDING REMARKS

Coming back to the issue discussed in the introduction (what may be the suitable sequence for introducing students with awareness to theoretical thinking) we have now further elements of reflections. From many examples provided by ethnomathematics of the present and of the past we already know that each environment gives different stimuli and opportunities to mathematical thinking. As it is reported in (Gerdes, 1988) Mozambicans peasants conceive and use "spontaneously" properties of parallelograms, which in the teaching sequence of our schools are theorems that must be proved. In teaching spatial geometry we have observed that students have difficulties in dealing with some polyhedra (e.g. dodecahedron), but this does not mean that these polyhedra are difficult to be conceived in an absolute way. Artmann (2001) reports that not only dodecahedra appear naturally (this is one of the three shapes in which the mineral of pyrite crystallizes), but also bronze dodecahedra were popular craft objects in Roman Imperial times. In the same way the dynamic geometry environment, as a new kind of ethnomathematical feature, brings to light forms of approaching mathematical situations different from those emerging when working with paper and pencil (with or without rule and compass). In our experiment the dynamic geometry software enriched students' experience in certain fields, by providing new situations. Also it revealed itself a powerful environment to detect students' behaviors and difficulties. In addition, the discussion orchestrated by the teacher brought to light interesting teaching implications: the crossing of metric and non-metric properties, the fuzziness of the concept of transformation, the relation between perimeter, area and shape, the difficulty of invariants.

In our experiment we have observed a kind of "computer anxiety", i.e. students' main concern was to exploit the facilities of Cabri rather than to design a project for performing the task (constructing and classifying quadrilaterals). The choice of the circle as a figure on which to work for constructing, and of the square instead of a generic quadrilateral as a starting figure for classifying quadrilaterals is conditioned by the dynamic geometry software.

The environment of Cabri addresses students towards definitions construction-oriented. A student constructed rhombuses starting from the properties of diagonals. And, when he attempted to inscribe a given rhombus in the circle that he had drawn at the beginning, he discovered that this is possible on particular conditions. Thus a statement to be proved may be spontaneously generated by the activity with Cabri. Also the environment fosters the use (even limited) of symmetries, which are object of teaching, but are rarely used by students when working with paper and pencil.

We point out that the persistence of the figural conceptions and the reluctance to use the dragging test show the lack of links with theory. Sometimes students seem to consider the construction of figures as a mere practical activity separated from a mathematical theory. Educational research provides teachers with means to perceive the existence of this gap and to fill it with appropriate strategies.

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SITUATIONS BY THEORY OF PERCEPTION

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Abstract

In this paper we suggest an analysis of 'elementary' difficulties and PLS (problematic learning situations) encountered in geometry classes in junior high school using theories of perception and perception-based knowledge representation. Our purpose is to illustrate a neglected aspect of the difficulties encountered in geometry studies. The demonstrations in this paper are based on and taken from a longitudinal research study using methodology described in previous works (e.g. Gal & Linchevski, 2000).

FOREWORD

Over an extended period of time we observed student teachers that taught geometry to slow and average junior high school students. The focus was on locating, identifying, and analyzing Problematic Learning Situations - PLS. (We use the terminology of PLS for situations arising in the course of teaching where teachers find it difficult to help students with problems in learning. See also Gal & Linchevski, 2000). We were interested in both the student's and teacher's point of view in an attempt to explain difficulties encountered (e.g. Gal & Vinner, 1997, Gal, 1998, Gal & Linchevski, 2000).

While analyzing the PLS we found that perception and perception-based knowledge representations, among other cognitive theories, provide explanations for many of the difficulties. Therefore, introducing these theories to teachers and showing their relevancy help teachers uncover and understand student difficulties that they were not previously aware of, and propose coping strategies.

As part of our research, we have planned a yearly academic course that aims to enhance teacher understanding of the ways that students think. Teachers participating in this course are presented with theory (general topics - e.g. concept and concept image - and specific topics included mainly topics relevant to understanding geometrical concepts - e.g. visual perception, Van- Hiele theory etc.) and actual student difficulties in geometry (using video clips of PLS, reports of class discussions, etc.) We found that teachers' ability to analyze the reasons for student and teacher difficulties increased significantly during the course. Later in this paper we will present some examples of difficulties that were recognized and explained by these teachers. Describing the method used and results of the involvement are beyond the scope of this paper.

In this paper we wish to focus on theories of perception and demonstrate their use to explain and analyze difficulties in geometry class.

THEORIES ABOUT PERCEPTION AND PERCEPTION-BASED KNOWLEDGE REPRESENTATION

Visual perception

Visual perception involves processing information that comes through our eyes (e.g. Anderson, 1995). There are two phases:

1. Registering the sensory information - visual information processing.
2. Interpreting the identified shapes and objects - visual pattern recognition.

Visual information processing

In the first stage of visual perception, shapes and objects are extracted from the visual scene. In order to do this, we need to know "what goes with what" to form the object. The set of principles according to which objects are organized into groups is called the Gestalt principles of organization (e.g. Anderson, 1995).

a. Principle of proximity: elements close together tend to organize into units.
b. Principle of similarity: objects that look alike tend to be grouped together.
c. Principle of continuation: we perceive lines with continuous turns better than lines with a sharp turn.
d. Principle of closure and good form: we tend to see shapes as closed rather than open, and with a regular shape rather than an irregular one.

Visual pattern recognition

In the second stage of visual perception, shapes and objects are recognized. Recognition is the result of feature analysis, in which the object is segmented into a set of sub-objects, the output of early visual processing (first stage). Each sub-object is classified, and when the pieces out of which the object is composed and their configuration are determined, the object is recognized as a pattern composed of these pieces (Anderson, 1995; Barsalou, 1992).

Recognition can be the result of either 'bottom-up' or 'top-down' processing (Barsalou, 1992). Bottom-up processing uses information from the sensory physical stimulus for pattern recognition. When context or general world knowledge guides perception, we refer to the processing as top-down processing, because high-level general knowledge contributes to the interpretation of the low-level perceptual units (Anderson, 1995). The context has an important role in pattern recognition, tuning a specific interpretation in top-down processing.

Attention is another parameter that affects information processing. There are automatic processes that require no attention or conscious control, and there are controlled processes.

Perception-based knowledge representation

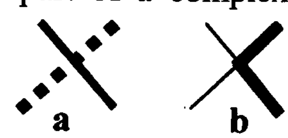
There are two ways for visual information to be represented in the cognitive system (Anderson, 1995): perception-based knowledge representation and meaning-based knowledge representation (which is beyond the scope of this paper). In perception-based knowledge representation, there are separate representations for verbal and visual information. Some visual information, such as the shape of geometric objects, is stored according to spatial position, while words are stored in linear order. Moreover, memory for pictorial material is superior to memory for verbal material.

Mental objects are dealt with like physical objects. Mental comparison of visual properties involves difficulties similar to those involved in visual perception. Complex figures composed of hierarchical decompositions (Anderson, 1995).

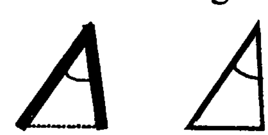
IMPLICATIONS FOR THE ANALYSIS OF STUDENTS' DIFFICULTIES IN GEOMETRY

Analysis by the Gestalt principles

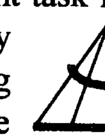
Identifying an angle formed between two straight lines. Many students, when talking about perpendicular lines, cannot point out where the right angle lies. They point at the area near the point of intersection. Or if the perpendiculars are part of a complex shape, such as the diagonals of a square, the student sometimes points at other right angles (see Gal & Vinner, 1997). This difficulty can easily be explained by the principle of continuation (third Gestalt principle mentioned before): the tendency to perceive lines which continue in the same direction rather than lines with a sharp turn causes the student to see two straight lines (figure a) instead of two rays emanating from one point, forming an angle (figure b).



Mistaking an angle for a triangle. There are students who draw a triangle whenever they are asked to draw an angle. Why is that so? Generally, students encounter angles as parts of triangles and not as 'basic angles' i.e. two rays (or ray segments) emanating from one point. In such a case, separating the angle rays from the triangle's third side, in order to identify the basic angle, contradicts the principle of closure and good form. It also explains why the angle is frequently interpreted as a closed shape (triangle).



Identifying common parts (segments or angles) in complex shapes. A frequent task is pointing out shared parts of two (or more) triangles with a common side. Very often, students point out the wrong parts as shared parts: the side intersecting with the shared side may be identified as a shared side, or a combination of the two angles may be pointed out as a shared angle (see the darkened side/angle in the drawn figure). In order to find the right answer, one needs to separate the configuration into two triangles and check their parts. But the need for separation contradicts the principle of proximity as well as the principle of closure (of the "big" triangle).

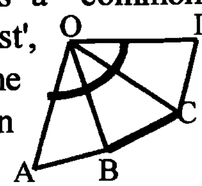


A junior high school teacher reported [1] problems with the drawing shown here, where students indicated AB as a common side for the two triangles. The teacher also explained that this was due to the principle of continuation.

Using the principle of similarity. The principle of similarity can be used to overcome difficulties based on the principle of proximity or the principle of continuation. Coloring parts we want to focus on can make use of the tendency to group together objects that look alike. (We can use a transparency placed on the figure, and color the relevant parts). To further demonstrate, in the first three examples presented above, we actually used this principle: the relevant parts of the figures were widened, darkened or drawn as a dotted line to group them together through similarity.

Analysis by top-down and bottom-up processing

1. Using the Gestalt principles to explain difficulties in *identifying common parts in complex shapes* we actually presumed a bottom-up processing which uses sensory information. Another explanation considers top-down processing, i.e. when a wrong context impairs recognition. The segment BC can be considered as a "common" segment to all three "small" triangles by means of 'in between', 'amongst', as in the case of a common yard or premises of several buildings. The same for the angle AOD, which can be considered as "common" in context of 'every one contribute a part' to create 'altogether'.

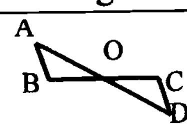


2. It is possible to recognize a right angle by bottom-up processing using sensory stimulus, which thus recognizes horizontal and vertical segments as a pattern of a (prototype – e.g. Rosch, 1978) right angle or a (prototype) right triangle:

Alternatively, recognizing a right angle by top-down processing occurs when using the conventional sign of a small right angle (instead of an arc). In this case, the special sign context is used to supplement feature information in recognition of the figure.

3. The following task requires the solver to identify two triangles in the configuration (in order to prove that they are congruent). This can be done by decomposing the figure into two triangles.

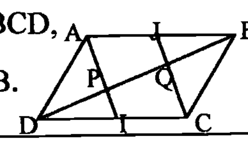
Given that:
AO=OD, BO=OC,
Prove that AB=CD



Such decomposition fits the principle of closure and good form. Therefore, information based on the stimulus is sufficient to enable the perceiver to recognize the triangles. This is an example of how bottom-up processing occurs.

A different task is presented by Duval (Duval, 1998, p. 41). In this case, bottom-up processing is generally not enough to get to the solution. That is because many sub-configurations can be seen. The Gestalt principles encourage decomposing into two "big" triangles (ABD, DBC), or six "small" polygons (APD,

Given a parallelogram ABCD,
I, J - midpoints of CD, AB.



AJQP, JBQ, etc.) In order to find the relevant decomposition (ABP, DQC) one needs to be aware of the context of *middles* explicitly recalling its theorem. This helps in finding the correct decomposition of the configuration. This is top-down processing.

Analysis by automatic and controlled processes

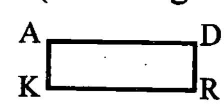
Many controlled processes are problematic for geometry students. Typical examples of a concept (prototypes) are easier to recognize (Rosch, 1978). The occurrence of an automatic process can explain this. A controlled process is needed to recognize less prototypical members of the category, e.g. recognizing a right angle in various orientations, an angle in a composite configuration, the height of triangle (oriented towards a not horizontal base), a square as a rectangle etc. Turning recognition into an automatic process can be part of the solution. Repeatedly exposing the learner to a wide set of examples can contribute to automation.

Another difficulty results because the organization of the prevailing field determines perception of its components (i.e. "field dependent/independence", Witkin et al., 1977). Again, turning recognition of geometrical figures in "conflicting" surrounding frameworks into an automatic process can help to overcome the difficulty.

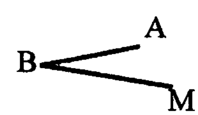
Analysis by perception-based knowledge representation

Some difficulties may be explained by the fact that we have different representations for verbal and visual information:

Naming polygons by indicating their vertex names in a clockwise fashion (according to the spacial path) is an accepted practice. For example, this rectangle is named ADRK. Unfortunately, students sometimes label such a rectangle according to its verbal representation. Therefore, instead of naming it ADRK it will mistakenly be called ADKR because of the linear representation of verbal information (reading from left to right, from top to bottom).

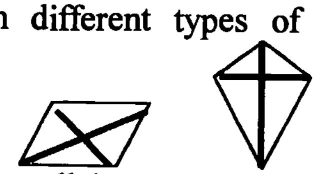


The three-letter notation for angles causes difficulties for many students. Many of them mistakenly call the angle in the figure ∠BAM. However, it may also be explained by the fact that the students consider the letters as conveying verbal information, and so they read them from left to right, and from top to bottom, as they would read English.



The following example demonstrates the mental comparison of visual properties:

A student who needs to use the properties of diagonals in different types of quadrilateral will probably check a mental image of the prototype of the specific quadrilateral.

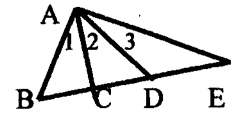


In such a case, it may be easier to mentally analyze the kite's (rhombus) diagonals' properties rather than analyzing that of the parallelogram's. This is because of the greater difference in the size of the diagonals in the prototype of the kite than in the parallelogram.

The following is one of the examples brought by the teachers during the course (see box). It demonstrates the hierarchical decomposition of complex figures. The teacher reported that the students could not see that the angles were equal.

Given: AC, AD bisectors of angles in ABD, ACE.

<BAE=66°



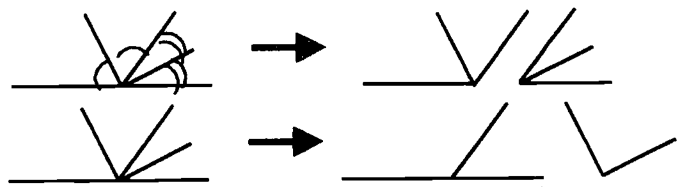
Find: <A1, <A2, <A3

Another teacher, one of the course participants, explained this as resulting from the hierarchical structure of the drawing, which prevents the students from seeing angle A2 as part of both BAD and CAE, simultaneously.

One more teacher reported difficulties in proving that the bisectors of adjacent angles form a right angle. She proposed two possible hierarchies for the configuration explaining, the difference between the desirable decomposition (needed for the proof) and the actual decomposition of the configuration:

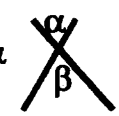
Actual decomposition:

Desirable decomposition:



Finally, one teacher was asked if she can apply the different representations of visual and verbal information, and the superiority of memorizing pictorial material in comparison to that of verbal material to geometry instruction:

> Of course these things can be applied to geometry instruction. In tests, before proving problems, I ask for definitions and phrasing of theorems. Though I'm doing it for long time, it's the first time I understand why most of the pupils draw a picture before the verbal definition, if they add the verbal definition at all.
>
> For example, if I ask what are vertex angles, the common answer is to draw: $\beta=\alpha$



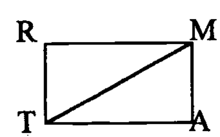
ANALYZING PLS

Here is an example of a protocol of a PLS. We shall give a detailed analysis involving theories of perception.

During a lesson on quadrilaterals in a ninth-grade class, the students were solving in class a problem they had been given for homework.

Given: a quadrilateral RMAT
RM=TA, RT=MA

Prove that sides are parallel



The students show that the triangles are congruent. Some claim that they can use the fact that there is a right angle because they see it. They find it difficult to explain why the sides are parallel.

The teacher asks if they remember what they have learned about parallel lines.

Their answers compel her return to parallel lines. She draws two lines and a third one that intersects them, reminding them about the different types of angles.

Student A: But angle A has nothing to do with that thing!

The teacher extends RM and TA in both directions and also extends TM.

Now the students are unsuccessfully trying to figure the alternate and corresponding angles in the original drawing.

Student A suggests erasing RT and MA so that it will be possible to see the connection with the teacher's drawing of the parallel and intersecting lines.

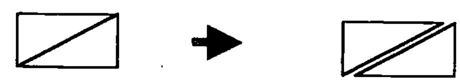
The teacher accepts the idea, praising the student. The discussion goes on until they conclude that the sides are parallel.

Analysis

This PLS involves two kinds of difficulties: Difficulties of visual pattern recognition (feature analysis and object recognition), and difficulties concerning Van Hiele's theory. In this paper we discuss only the first one.

The configuration appearing in the assignment has two possible "natural" decompositions in accordance with the principle of closure and good form:

or

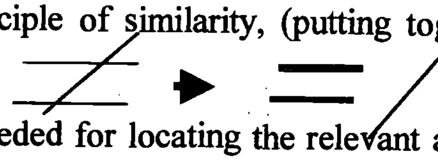


In order to recognize the applicability of the parallel theorem to this problem, however, it is necessary to decompose the configuration as follows:

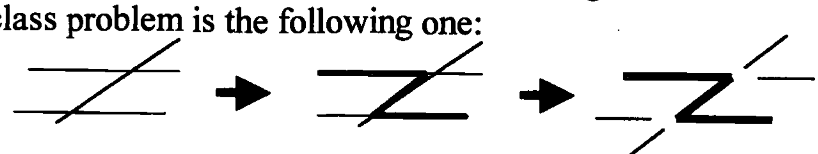
(1)

Such decomposition contradicts the principle of closure and good form.

Moreover, the drawing demonstrating the parallel theorem is decomposed in accordance with the principle of similarity, (putting together the two parallels with same orientation):



But the decomposition needed for locating the relevant angles in the above discussed class problem is the following one:



(2)

This decomposition contradicts the similarity principle. And even if the proper decomposition is found, the phase of pattern recognition remains: the two patterns, (1) and (2) need to be recognized as being the same.

SUMMARY

Theories of perception, though not new, are rarely introduced to teachers in general, or in the context of geometry teaching in particular. These theories can be a powerful tool for explaining a wide range of difficulties. In this paper, we have presented a few of the examples we encountered in order to exemplify the potential of these theories for teaching geometry. Being familiar with these theories can help teachers cope with Problematic Learning Situations by helping them to plan their instruction and making decisions during instruction. However, presenting this knowledge to teachers is not enough. Making it relevant, presenting actual class situations and inviting teachers to look carefully at their own teaching experiences, would likely improve their ability to recognize, analyze and cope with PLS. These methods were implemented in the course we suggested. Our findings regarding these intervention methods are the subject of a future paper.

[1] We include examples reported by teachers attending the above-described course in order to enrich ours. These examples may give an idea about the awareness and capability of the teachers participating the course to analyze PLS. Describing the changes in teachers' ability to identify and analyze difficulties and cope with PLS during the course is beyond of the scope of this paper.

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INTERIORISATION OF *FORMS* OF ARGUMENTATION: A CASE STUDY

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This report deals with the interiorisation of that form of argumentation which consists in the dialogic elaboration on, and transformation of, arguments within a given theoretical framework, up to getting a contradiction. A teaching experiment (carried out in a Grade VIII classroom) concerning the mathematical modelling of the elongation of a spring, and conceived in the perspective of the 'Voices and echoes game', will be analysed in order to better understand the mechanisms of interiorisation and the potential inherent in Galileo's dialogic voice.

The development of argumentative skills is a relevant issue in Mathematics Education. Argumentative skills that are important in mathematical activities include the mastery of sophisticated forms of organisation of the discourse, as for instance hypothetical reasoning, reductio ad absurdum, etc. How can students approach and develop such skills? In an early Piagetian perspective, their roots are embedded in appropriate social peer interactions (see Piaget, 1924, chapter II). In a Vygotskian perspective, the more such skills are far from 'common' forms of reasoning, the more they need the mediation of an adult (or a more competent peer). Imitation in the Zone of Proximal Development is seen as a possible way to perform the necessary mediation. (see Vygotsky, 1978, chapter VI). A reflection about these general hypotheses brought us to consider the potential of the "Voices and echoes game" (see Boero et al, 1997) as a possible effective educational methodology (based on active imitation) to convey some *forms* of organisation of argumentation which are inherent in the scientific debate.

Galileo's dialogues offer examples of high level organisation of scientific discourse. Empirical evidence is not so frequently used (due to reasons inherent in the counter-intuitive character of the hypotheses proposed by Galileo in his dialogues). A typical *form* of Galileo's argumentation consists of the elaboration on, and transformation of, the adverse arguments *within the frame of the adverse theoretical position*, up to getting an "evident" contradiction. Mental experiments, reduction to the limit, etc. are frequently used for this purpose. We can recognise this *form* of organisation of the scientific discourse as an inner dialogue in a number of important personal mathematical activities related to checking mathematical conjectures and scientific hypotheses in order to validate them, for instance in applied mathematics, when a mathematical model is put into question: by transforming an algebraic formula, proposed as a model, it is sometimes possible to draw conclusions, which are in contradiction with some known properties of the modelled situation. In the search for counter-examples of a mathematical conjecture, examples that work for it may be transformed in such a way to hold the coherence with the constraints inherent in the hypotheses and, at the same time, to contradict some aspects of the thesis.

It is out of the scope of this report to discuss the (controversial) historical and personal sources of the *form* of Galileo's argumentation that we are considering. It is more important for us to remark that, according to our observations, this *form* of argumentation is not produced spontaneously by VIII-grade students neither in situations of interaction, nor in individual performances concerning mathematical modelling situations. But students of that age do elaborate on arguments and transform them in everyday life argumentative situations. In mathematical and scientific activities, the main difficulties in approaching the *form* of Galileo's argumentation seem to consist for students (while trying to elaborate on, and transform, the arguments) in *holding the coherence with a given theoretical framework* and also in the *complexity* of the elaboration and transformations needed.

This report will elaborate on the hypothesis that the use of Galileo's dialogues in an active imitation perspective (like the one of the "Voices and echoes game" – see the next Section) can help students to appropriate the *form* of argumentation that we are considering, and use it in mathematical and scientific activities. We will present some essential features of a classroom activity concerning the problem of the elongation of a double length spring in relation to the elongation of one single spring hanging the same weight. Students produced two hypotheses, then discussed them and discovered the right one. Then they were asked to produce, as an 'echo', a Galilean dialogue about the two hypotheses, following the model of the Galileo' dialogue about the falling body phenomenon. The discussion of collected data will lead us to consider the potential inherent in the *virtual* character of Galileo's Dialogues.

THE 'VOICES AND ECHOES GAME' (VEG)

What is the VEG? Some verbal and non-verbal expressions (especially those produced by scientists in the past) represent important steps in the evolution of mathematics and science in a rich and communicative way. We called these expressions *'voices'* (cf. Wertsch, 1991). We called *VEG* an educational situation aimed to make students produce echoes to a voice through specific tasks, for instance: *"How might Aristotle have interpreted the fact that a feather falls down at a slower speed than a stone?"*.

What are the aims of the VEG? Our general *initial* hypothesis was that the VEG might broaden students' cultural horizon, by embracing some elements of the theoretical knowledge that are difficult to construct in a constructivist approach and difficult to mediate through a traditional approach (see Boero & al, 1997). The need to exploit the potentialities that emerged in the first series of teaching experiments led us to try to *find a better characterisation for the elements of the theoretical knowledge* to be mediated through the VEG, in order to better organise and analyse how students interiorise them (see Boero & al, 1998). The research reported in Garuti et al (1999) concerned another potential of the VEG, namely the possibility of developing skills related to detecting conceptual mistakes and overcoming them by general explanation.

THE TEACHING EXPERIMENT

The teaching experiment involved one VIII-grade class with 17 students. The mathematics and science teacher has been the same since grade VI. The didactical contract included (in particular) the production of exhaustive individual verbal reports about personal problem solving strategies, the comparison of solutions, argumentation about hypotheses discussed in the classroom, etc.. Students' individual texts as well as transcripts of classroom discussions were systematically collected. This class had already been involved in two preceding teaching experiments concerning the VEG. In grade VII, the students had been asked to produce an echo to Plato's 'Menon' dialogue concerning the length of the side of a square of double area of a given square. The echo concerned a common conceptual mistake recognised and discussed in the classroom (see Garuti et al, 1999, and the end of the preceding Section). In Grade VIII, they had had to produce echoes to Aristotle's and Galileo's theories about the 'falling bodies' phenomenon, with tasks concerning the *content* of those theories (such as *"How could Aristotle have interpreted the fact that ... ")*.

The Choice of the 'Double length spring' Problem as a Target Problem for This Study

The 'Double length spring' problem consists in the following task (the weight of the spring is supposed to be very small in relation to the weight hung up on it):

> *"Imagine to know the elongation of a spring under a given weight and to take another spring of the same material, diameter, etc. having a double length. What can you say about its elongation? Why?".*

In previous teaching experiments concerning mathematical modelling, the 'double length spring problem' emerged as a very interesting elementary mathematical modelling problem because of the following reasons (see Boero & Garuti, 1994): it is a challenging problem, even for cultured adults; the "wrong" hypotheses rely on some principles which work very well in other situations; the "valid" hypothesis cannot be easily detected on the basis of immediate everyday life experience; the "wrong" hypotheses can be demolished through an argumentation which exploits accessible arguments in a suitable way. Our a priori analysis established some links with the falling body phenomenon. In the same way, in this case the 'wrong' hypothesis agrees with some principles, which work rather well in many other situations, and the immediate experimental evidence is not in favour of the 'good' hypothesis, but the "wrong" hypothesis can be put into question by an appropriate elaboration of accessible arguments.

Before this teaching experiment concerning the 'double length spring' problem, the students had already performed activities concerning the elementary mathematical model of the elongation of a spring. They had arrived, under the teacher's guidance, to discover that the formula $L= L_0 + KP$ is a good model for the length of a spring of initial length L_0 under the weight P, provided that the coefficient K is well chosen for that specific spring and that P takes values in a suitable interval (not too extended on the right). The activity had an experimental counterpart, allowing students to discover (in particular) that K is smaller if the spring offers a stronger resistance to the elongation. Also the proportionality $L-L_0=KP$ had been discussed.

Afterwards the 'double length spring' problem was posed. The discussion of the "equal elongation" and "double elongation" hypotheses (produced during the individual solution phase) led students (even before the experimental testing) to share the correct hypothesis of the "double elongation". In this process the students brought forward different arguments, in particular: cutting the double length, and imagining what happens under the weight, then adding these effects; or thinking about each coil, and imagining the effect of the weight on it, then the global (additive) effect on the single length spring and on the double length spring. Then some parts of Galileo's Dialogue were read and discussed. Attention was paid to the role of the three interlocutors and to some Salviati's dialogic strategies (in particular, the use and transformation of Simplicio's arguments within the frame of Aristotle's theory in order to get an evident contradiction with his own premises). Finally, the following individual task was given:

> *"Imagine to be Galileo writing a dialogue about the problem of the double length spring. The characters are: Salviati, who represents you, trying to convince Simplicio (and the reader) that the double length spring elongates the double, and to explain why; Simplicio, who supports the hypothesis that the double length spring elongates the same length, because the material is the same and the coils have the same diameter; Sagredo, the moderator".*

The students worked individually for approximately two hours. Their individual texts were analysed by us according to the criteria listed at the end of the next Section.

From Galileo's Dialogues to Some Criteria to Analyse Students' Dialogues

Here I will try to make a summary of a crucial part of Galileo's dialogue concerning the 'falling bodies' phenomenon. This part was read and discussed in the classroom with the help and under the guidance of the teacher:

> Simplicio illustrates Aristotle's theory in general and with an example. Salviati elaborates on Aristotle's theory and puts it into question by using Simplicio's example. Simplicio tries to contrast Salviati's doubt by providing an interpretation of Aristotle's words, while Sagredo takes a position that relies on experimental evidence. But the core of the debate is not the experimental evidence! Salviati wants to provide Simplicio (and the reader) with a theoretical proof within Aristotle's framework. Galileo involves Simplicio in a mental experiment which in three steps leads to a contradiction with Aristotle's theory, by argumenting within Aristotle's framework. Then Simplicio reacts by making reference to intuition, and presenting an example. Galileo answers providing another example (which takes to the extreme the kind of example proposed by Simplicio).

The analysis of this part of Galileo's 'Dialogues' suggested some criteria to analyse students' dialogues and evaluate their quality:

- existence of a real dialogic structure, in the sense that the direct questions Salviati pose to Simplicio play the role of involving him in the argumentation and getting his consensus on crucial arguments and steps of reasoning
- elaboration on, and transformation of, the adverse hypotheses through examples or analogies;

- evidence is explicitly shown for the contradiction;
- Salviati wins through the logical strength of his argumentation;
- concrete experiments deserve only the function of putting into question Aristotle's theory and justifying doubts about it;
- the final "passage to the limit" is used to win Simplicio's last resistance based on physical intuition.

During the discussion about Galileo's dialogue these criteria surfaced (under the teacher's guidance) as main characteristics of its organisation.

Some Excerpts from Students' 'Dialogues'

Sara:

Simplicio: *Two springs made of the same material and with the same coil diameter, but different initial length (one is the double of the other) elongate the same because the initial length is not influent. For instance* {HERE AND AFTER: A SUMMARY OF OMITTED PARTS: the example of a 10 cm spring and a 20 cm spring follows; the L=H+KP formula is evoked to say that K is the same}

Salviati: *I am against your supposition; I say that the initial length is influent and K varies; if your double length spring elongates 2 cm, the other one elongates 1 cm.*

Sagredo: *I am afraid, Simplicio, I made the experiment and saw that the double length spring elongates the double.*

Salviati: *We can prove it.*

Simplicio: *They must elongate the same because the weight is divided according to the number of coils. For instance let us suppose that the weight is 20 grams, if the number of coils of the shortest spring is 10, each of them elongates 20:10=2, while each of the 20 coils of the longest spring elongates 20:20=1. Summing up, we obtain the same elongation.*

Salviati: *You made a mistake, because if you take a double length spring and you divide it into two equal parts, each of them elongates the same under a given weight, so if I join them again, the elongation is double.*

Sagredo: *Salviati is right, because* {Sagredo provides an example, showing that each part of the double length spring supports the same weight}

Salviati: *Dear Simplicio, do you think that the weight hung up to the longer spring is different from the weight hung up to the first and the second **spring?***

Simplicio: *I got it!*

Salviati: *As a conclusion* {he says the valid hypothesis}

Francesco:

Simplicio:	{he presents the erroneous hypothesis: general statement first, then an example, like Sara}
Salviati:	*According to what you have said, it is as if you performed the experiment, but I have doubts about it*
Sagredo:	*I performed the experiment and I can tell you that the double length spring elongates the double*
Salviati:	*In accordance with Sagredo who performed the experiment, I can prove that a double length spring elongates the double. Do you share, Simplicio, the idea that all springs with the same length and made of the same material do elongate the same?*
Simplicio:	*Certainly* {he provides an explanation for it}
Salviati:	*And do you think, Simplicio, that two springs with the same length, same material, same coils when connected together elongate the same as one of them?*
Simplicio:	*It seems logical to me*
Salviati:	*Let us suppose that what you think is true. Let us consider two springs with the same initial length; let us hung up the same weight on each of them, then let us join the two springs keeping them elongated with the same weight: we get a double elongation, and this conclusion is against your hypothesis.*
Simplicio:	*I am confused; it seems to me that if I join the two springs together, I get one spring, then I get the same elongation*
Salviati:	*Here is your mistake: it is not true that if you join the two springs the elongation of one of them disappears* {still Simplicio resists; and then Salviati suggests the comparison with two springs in parallel; finally, reacting to Simplicio's scepticism, Salviati brings to the extreme Simplicio's example with a 20 cm spring and a 100 m spring}.

An Overall View of Students' Productions

8 students out of 17 produced a "dialogue" like the one presented by Sara. They show they understood some crucial theoretical arguments in favour of the double elongation hypothesis. They also keep into account the result of the experiment. But their 'dialogues' do not fit the first, third and fourth criterion (and indeed in some cases, like in Sara's dialogue, Sagredo must intervene to explain the right hypothesis). The second criterion is only partially satisfied: the dialogues refer to Simplicio's hypothesis, but Salviati's reasoning follows his own path, and there is no real interaction with Simplicio's arguments. It is as if Salviati (or Salviati and Sagredo together) illustrated his (their) own theory against Simplicio's theory. The students were able to perform a satisfactory 'content' echo to the voices, of the classroom

discussion, supporting the valid hypothesis, but not to Galileo's dialogue; in particular, their dialogues do not contain its typical *form* of argumentation (see Section 1). 9 students out of 17 produced "dialogues" with the same quality as Francesco's (or even superior). They share all the crucial characteristics of Galileo's dialogue. It is interesting to observe that there was no intermediate performance: the dialogues that did not contain Galileo's typical *form* of argumentation were the same ones that do not meet the other main requirements of Galileo's dialogue. In particular, they were either parallel developments of two monologues, or dialogues with no real interactions between the arguments brought in by Salviati and Simplicio.

DISCUSSION

Our research hypothesis was that the use of Galileo's dialogues in an active imitation perspective (like the one of the VEG) could help students appropriate the *form* of Galileo's argumentation, which consists of the elaboration on, and transformation of, the adverse arguments *within the frame of the adverse theoretical position*, up to getting an "evident" contradiction. On the basis of the available data it is not possible to prove that students learned to do it in situations that are very far from the 'falling bodies' example. And it would be even more difficult to prove that they interiorised this method in the perspective of a real individual inquiry situation that requires questioning a personal hypothesis and demolishing it by elaborating and transforming some arguments supporting it. The available data only show that about one half of the students were able to use the *form* of Galileo's argumentation in another similar situation. The analysis of students' individual productions and recordings of classroom discussions in previous problem solving situations shows that this *form* of argumentation was far from the organisation of students' argumentative performances.

In our opinion, more interesting outcomes of the reported study concern the mechanism and the conditions of the interiorisation process in relation to the *virtual* character of Galileo's dialogue. Indeed Galileo's dialogues are not real interpersonal dialogues; they were produced by a scientist who wanted to present his theory and convince the reader through a virtual debate.. As such, Galileo's dialogues are close to the inner dialogue that we, as adults, produce when we get ready for contrasting an interlocutor in a public debate (when we imagine his arguments and think about arguments which would convince him and the other listeners). They represent the inner, individual (or intra-personal) counterpart of interpersonal practices. This feature of the dialogic voices (with the inherent differences between a real debate and a virtual debate) might represent an important opportunity for the interiorisation of high level *forms* of argumentation. This specific issue is related to some ongoing research about the process of interiorisation (from interpersonal construction to intrapersonal development) (Engestrom, 1991; David'ov, 1991). In particular, considering the *"interiorisation process"*, i.e. *"the process of individual activity formation on the basis of collective activity"* Davydov wrote: *"Numerous versions of the theory... notice the fact that the structures of these two forms of activity are to a certain degree similar, but pay very little attention to their difference. But exactly the characteristics of this very difference and dissimilarity form a particular problem of* *activity theory"*. And then, referring to the need for revealing "structure and functions of the specific character of each activity form", he wrote: *"It is vital to give a more exact, certain and comprehensive description of various stages of this process* [interiorisation*] and emphasise the specific importance of the conditions of its realisation"*. A reasonable outcome of the case study reported in this paper is the hypothesis that the active imitation of a virtual dialogic voice through the VEG creates the conditions for the interiorisation of those forms of organisation of the scientific discourse, which are inherent in the voice; this is due to the fact that the voice itself represents an interiorised activity. Another related indication that emerges from the analysis of the teaching experiment is that in future experiments attention should be paid not so much to the scientific debate in the classroom about a given subject as a preliminary, interpersonal construction in the perspective of the subsequent interiorisation, as to the interpersonal reflective practices on the functioning of Galileo's dialogue before the 'echo' task. Those practices might represent a crucial condition for the development of the process of interiorisation. Indeed, the students who failed to construct a dialogue bearing that form of argumentation, which was the goal of the teaching experiment, were the same who were unable to produce a real dialogue. It was as if the lack of control on the dialogic structure of their text prevented them from producing the required argumentative form (cf. Engestroem: *"The new activity structure... requires reflective appropriation of existing culturally advanced models and tools")*.

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DIFFERENTIAL EFFECTS OF DYNAMIC GEOMETRY SOFTWARE ON THE ACQUISITION OF BASIC GEOMETRIC NOTIONS

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Introduction

The various capabilities of Dynamic Geometry (DG) have caused an ever-growing interest during the last decade. But many publications in this area might as well bear the generic title „The Impact of Computer Technology (on the Mathematics Classroom)“, as they focus mainly on *technology* and its really promising potential, but neglect somewhat the *impact* caused by it (and all too often only allude to the *classroom*).[1]

Nevertheless, the ever-growing availability of computers in the classroom and the public expectations concerning their use entail administrative measures towards regular use of DGS, as nowadays can be readily noticed in Germany. Therefore it seems appropriate to investigate in *regular* classrooms the impact of DGS on students' achievements, conceptions and attitudes. In this paper, we consider mainly differential effects on students' achievement.

Related research

Only few concepts of computer application in the mathematics classroom have been evaluated on a scientific basis. Moreover, a detailed meta-analysis by Ruthven (1997) concluded that only a small number of studies (regarding CAS) had an acceptable design (i.e. experimental and control groups, pre- and post-tests). Concerning DGS, Hölzl (1996, 2000) meticulously performed a diversity of qualitative case studies revealing epistemological shiftings and increased cognitive challenges as unwanted, but perhaps unavoidable side effects of applying DGS.

> His results show that the interactive aspects of the medium motivate the students to develop individual interests and stamina – But there are also subtle interactions between the implementation of geometry by the software and the students' understanding of geometry. This shows clearly the double-edged character of computer application: on the one hand the heuristic potential yields a – mathematically extremely attractive – extension of scope, but on the other hand interactivity entails also the danger of "degoaling“ (Hoyles und Sutherland 1989).

Thus a "computational transposition“ (Balacheff 1993) seems inevitable also in the realm of DGS. In order to minimize its effect, a thoroughly and didactically reflected teaching concept is required.

Also, from an instructional psychologist's point of view, the prospective value of DGS has to be appraised retentively, since it is usually taken for granted that the choice of medium is far less important than the nature of the treatment[2]. But

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[1] Hölzl (2000)

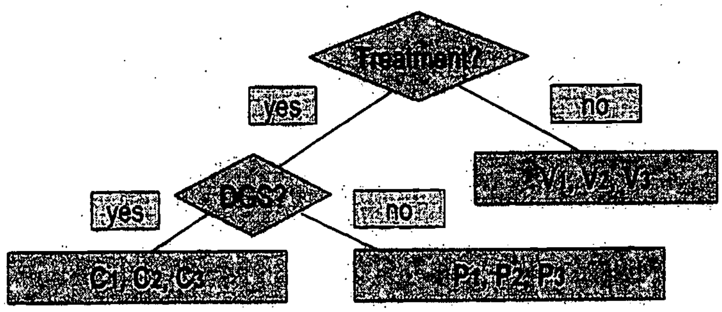
[2] Salomon (1978) and Clark (1983)

since the genuine possibilities of DGS cannot be reduced to just changing the medium, a comparative study seems nonetheless appropriate.

There are only few studies on the impact of *highly interactive* computer usage. Lewalter (1997) investigated cognitive information processing when learners utilized animated vs. static computer illustrations. Though the possibilities to interact with the software are far more restricted than with DGS, Lewalter's results shed some light on the core problem: even the restricted possibilities of the dynamic presentation are not really used by learners for the purpose of elaboration. Rather, her analysis reveals that dynamic visualizations of kinematical processes seemed so easily understandable to the learners that they all to early got the false impression to have grasped the whole issue. This result seems to be in accordance with the "degoaling" phenomena described by Hölzl (1996).

Method

Procedure The study was performed in nine grade 7 classes (N=214) of three senior high schools. At each school, three classes were assigned to the **C**(omputer), **P**(aper) und **V**(control) group yielding the following hierarchical design:



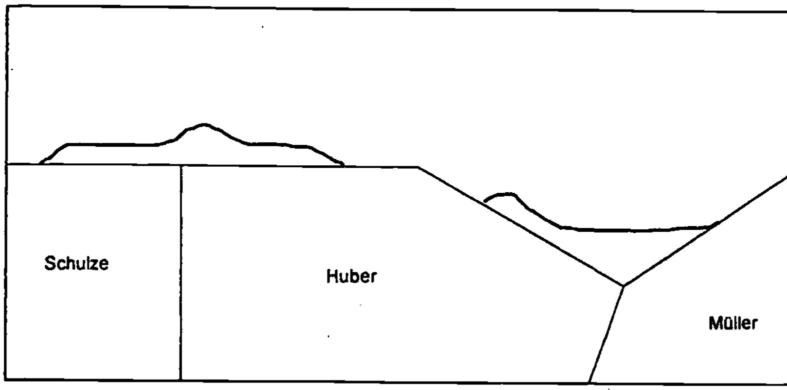
Pre-and post-tests were performed in all classes (the pre-test surveyed the prior geometric knowledge of the students as well as their attainment levels). Meanwhile, the treatment was implemented in the experimental groups using the respective learning environment (i.e. paper and pencil in P resp. DGS in C). For the 12 lessons given, teacher students that had in advance been trained intensively on our courseware replaced the usual teachers. At every school, the C and P classes were taught by the same teacher student. In the V classes, ordinary geometry lessons took place. After six months, a follow-up post-test was performed in all classes.

Treatment The treatment covered perpendicular and angular bisector, the circumcentre and the circumcircle, the incentre and the incircle. In C, we used interactive electronic work sheets[3] that focus on *investigating given* figures rather than *constructing new* ones. As an extra dimension, we added embeddings of geometric problems into problem contexts: for instance the distribution of soil was posed as a problem (compare fig.1) to introduce a small modelling process

[3] compare Elschenbroich (2001)

ending up with the necessity to find a construction for bisecting angles – thus demonstrating the utilizability of mathematical tools in "real life".

Measurement The tests can be divided into procedural and explanatory parts, so the pre-post-difference D splits into a procedural component D_p and an explanatory component D_e : $D = D_p + D_e$.



Find out how the new land can be distributed to the three peasants according to the "Roman law" (remember: new land belongs to the one, whose lands

Fig. 1

Hypotheses

1) The problem-based approach influences significantly the achievement[4].
2) The use of DGS does not influence significantly the achievement[5].

Results

The box plot of the pre-post-difference D and its explanatory component D_e in fig. 2 reveals:

- Overall outcomes were about equal in P and C and considerably higher than in V, with smaller variance in P.
- In the explanatory parts, students in P did somewhat better than in C and V, with considerably greater variance in V.
- All in all, C lay consequently a little behind (regarding both measures).

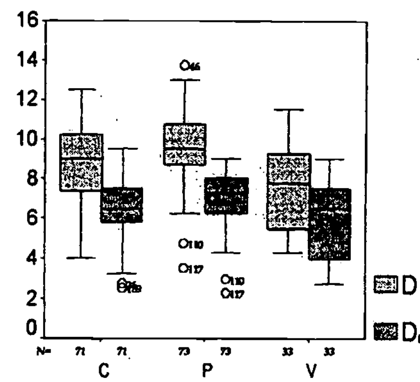


Fig. 2

Inference statistics yields somewhat heterogeneous results: In the whole population we find for D that μ (P) >> μ (C) $\approx$ μ (V), μ (P $\cup$ C) >> μ (V), so 1) can be confirmed, 2) has to be rejected – but this result is slightly obscured by the fact that the means of P and C differ only by .75. This difference is *significant* (α=.05), but is less than the *insignificant* difference 1.00 of C and V. So one might suspect a statistical artefact due to the diverging group sizes (the drop-out rate was –not uncommonly– significantly higher in V).

An explorative data analysis is thus in order. For practical purposes, we restricted ourselves to factors that are easily accessible to teachers: Besides gender, we distinguished higher and lower achievers by splitting at the median

[4] measured as pre-post-difference, which were checked against mean posttest achievements adjusted by pretest.

[5] but the chosen strategies!

(that has also statistical advantages). Here, we have to restrict to display some tendencies in the data (fig. 3-5)[6]:

- except for girls at public schools, in C classes higher achiever profit more than lower achievers, vice versa in P,
- while in public schools P and C scores are about equal, for girls in the private schools P was (significantly) superior to C,
- for girls at public schools, C is slightly more effective than P, irrespective of achievement level.

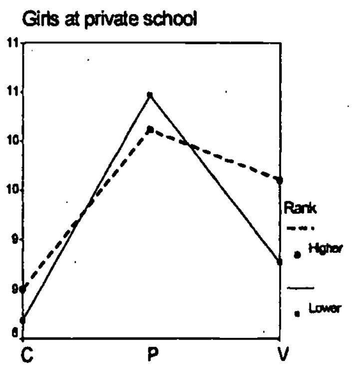


Fig. 3

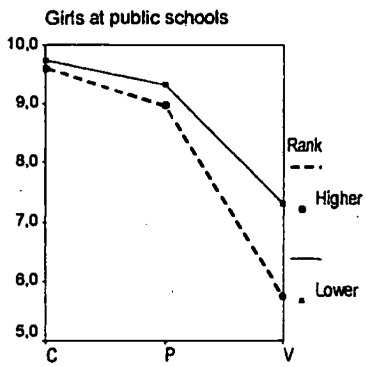


Fig. 4

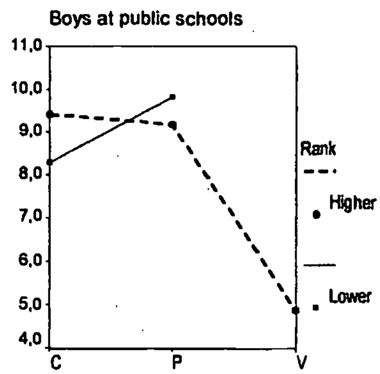


Fig. 5

Also, differences in achievements and strategies between P and C occurred with "dynamic" problems[7]: Namely, the post-test contained one especially „dynamical“ task – the students were given fig. 6 and had to answer the following questions:

1. This is a campground from above. The friends Anton, Bert and Charley want to have breakfast together. Can you place a table on the campground in equal distance to their tents?
2. What happens if Charley moves his tent along the sea to find the best fish ground? Why?
3. If he chooses a certain place for the tents, a problem will occur to the three friends. Can you figure it out?

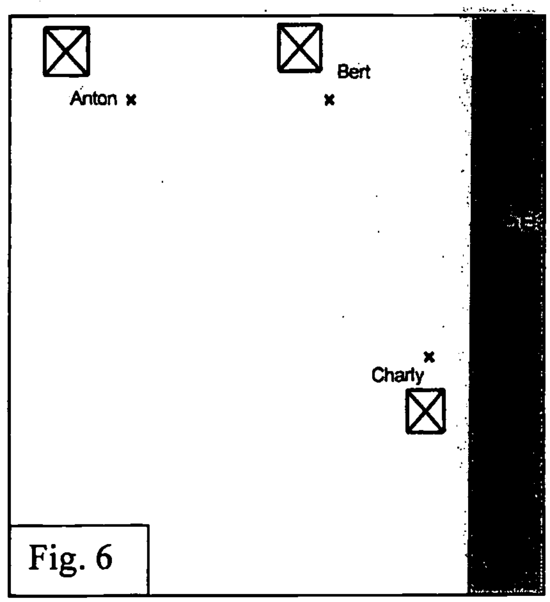


Fig. 6

1. involves of course the utilization of the circumcentre in a problem context. And to see, whether DGS was helpful in developing dynamic mental images they were asked 2. This made it also possible to check by 3. whether students

[6] See Gawlick (2001a, 2002) for more details.

[7] but surprisingly also with some "static" ones, compare Gawlick (2002).

realized the limitations of such a mathematical modelization. All in all, the "dynamic task" 2. is certainly apt to show differences in outcome if DGS is really more than "just another medium". But again the results are rather ambiguous: The total score for the "dynamic task" does not show any significant differences. But if you consider only, their hypothesis what happens to the table when Charley moves his tent, C is clearly ahead of P – but vice versa if you consider their reasons, *why* this will happen. These adverse effects nearly cancel each other. A possible interpretation would be: DGS is helpful to generate hypotheses, but not to find arguments supporting them. This would fit neatly to the .„degoaling"–effects observed by Hölzl (1996).

On the other hand, one of the rather procedural tasks was to explain the construction of the middle perpendicular. But in this rather "static" task the C groups did astonishingly somewhat better than the P groups. This may be some evidence for a reinforcement of mental images by DGS that can even strengthen students' ability to scrutinize the rationale of the said construction.

Discussion

Of course, the findings should not be overestimated – they are based on a relatively small sample and difficult to generalize due to their heterogeneity. The inconclusive statistical results are in accordance with media and ATI research. But the interaction of environmental and gender factors seems to be new in the context of DGS and – though statistically insignificant – deserves further attention[8]. It seems that in coeducative classes, girls profit from DGS treatment, but possibly at the cost of lower achieving boys. Therefore, special care should be taken of these in order to prevent them from using DGS as a plaything.

Hopefully the evaluation of the follow-up test (work in progress) will shed some light on these and eventually additional other differential factors.[9] Also, the analysis of students' strategies in the "dynamic" task may yield some further evidence how to ameliorate the disappointing outcome in this task.

Finally, it should also be noted that the preliminary evaluation of the follow-up data gives some evidence toward the conjecture that *in the long run* DGS as an additional tool for the acquisition of geometric notions may be superior to the paper-and-pencil-only approach.[10]

Concluding Remarks

It seems at present that when dealing with fairly standard examples, the benefits of dynamic exploration can even in a carefully designed course far too easily be outweighed by the extra costs of DGS. So we strongly confirm that *dynamics is not a didactical advantage per se* (Hölzl 2000) – the use of DGS should

[8] The deviating results in the private school seems difficult to interpret, but should be kept in mind when discussing the issue of coeducation concerning DGS.

9 Eventually, evidence on this will be presented orally at the conference.

[10] Again we have to refer the reader to the oral presentation and/or electronically obtainable information.

therefore be preceded by thorough consideration. It will be most favorable when an objective requirement for the tool meets an advanced mathematical experience. Therefore, the design of teaching units and learning environments that make the most out of the computer's heuristic and computational capabilities continues to be one of the most challenging objectives of nowadays mathematics education.

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FROM UNIVERSITY TO SCHOOL: A LONGITUDINAL STUDY OF A TEACHER'S PROFESSIONAL DEVELOPMENT IN MATHEMATICS TEACHING

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This paper reports findings from a longitudinal study of a newly qualified elementary school teacher from the last year of her university studies up to the third year of teaching in a primary school. The development of this teacher's teaching has been investigated in three strands, which have been distinguished in terms of the kind of support provided to this teacher. In particular, the teacher's characteristic view of mathematics teaching as a process of children's involvement in activities acquires a different meaning in her teaching actions as she experiences the various contexts.

Introduction

A number of research programmes on initial mathematics teacher education have been developed focusing on the integration of mathematics, pedagogy and children's ways of thinking (Cooney, 1994; Even et al, 1996). Most of these programmes are characterized by a constructivist view of learning, emphasising the role of research in the mathematics teacher's development (Simon, 1995a, 1995b, Steffe and D' Ambrosio, 1995) and viewing the teacher as researcher and reflective practitioner (Rice, 1991; Joffili & Watts, 1995; Potari, 1997). Although research studies indicate positive results from these programmes, questions remain on whether and how the experience that these prospective teachers gained through their initial education could influence their professional life. There is also a body of research that investigates mathematics teachers' development while teaching in their own classrooms. In particular, a number of in-service education programmes are conceptualised as contexts for integrating theory and practice by involving teachers in undertaking action research in their own classrooms (Cooney & Krainer, 1996; Irwin & Britt, 1999; Jaworski, 1998; Krainer, 1999). However, there is little research on how the teacher deals with the transition between pre-service education and actual teaching. This research is not very encouraging for the mathematics teacher educators as it reveals the difficulties beginning teachers find in making connections with their university experiences (Cooney, 1985; Raymond, 1997). However, these difficulties are mainly attributed to the "reality shock" that the teacher feels during his/her first year of teaching (Weinstein, 1988). Therefore, to study a teacher's development only during the first year of his/her professional life limits our understanding of his/her potential development.

This study aims to investigate the development of the teaching of a primary school teacher while finishing her pre-service education and during her first three years of teaching. More specifically, we focus on the influence of the particular contexts that this teacher experienced in her teaching practices throughout this time period. We attribute a personal character to the meaning of "context", considering it as interactively constructed by the teacher through discussion and reflection on action (Jones, 1997).

Methodology

This is a longitudinal case study of a newly qualified teacher, Christina, who has been working in a primary school for three years. The research started when Christina was a student teacher in the last year of her university studies, and continued over her next three years of teaching in the school. We distinguish three strands: the period at the university; the first year of teaching in the school, and the second and third year of teaching. These strands were formed in relation to the kind of support provided to her for the duration of the study. During her university studies, Christina took two courses concerning mathematics and mathematics teaching and learning. In the last year of her studies she did her teaching practice in schools, and participated in an optional course characterised by two general perspectives: the teacher as researcher of children's thinking and the integration of mathematics and pedagogy (Potari, 1997). In that period the two authors, one being the teacher educator in the course and the other the mentor, supported Christina through discussions during the various activities of the course and while she was planning and reflecting on her teaching. During the first year of her teaching in the school, Christina's support came mostly from the school; for example, her colleagues and the principal. However, she continued informal discussions about her new experiences with one of the authors and her peer student teacher. In the second and third year, Christina participated in a research project organised by one of the authors (Georgiadou-Kabouridis, 2000). The other author acted as a participant observer in some of the group meetings, a model which has been suggested by Simon (2000). In this case, support was provided mostly through a number of meetings with her colleagues-participants, where the researcher aimed to link theoretical issues about teaching and learning of mathematics to the teachers' experiences in their classrooms. Support was also given to Christina through discussions with the researcher about her teaching.

Research data consisted of transcripts of videotaped sessions from Christina's teaching in the first and third strand, transcripts of audio recorded informal discussions and semi-structured interviews between Christina and the researchers in the period of the three strands, Christina's lesson plans, worksheets and her written comments on the group meetings. Analysis of the data is realised in two dimensions. One concerns what Christina discusses about her teaching and the other what she is actually doing in her classroom. More specifically, in each strand we analyse certain classroom incidents, which are typical of Christina's interactions with the children. The latter dimension identifies Christina's teaching approach and reveals possible teaching development. By linking the two dimensions we explore Christina's awareness of her actions and we investigate how the context affects her teaching.

Christina's typical teaching approaches throughout the three strands

Christina's beliefs about mathematics and mathematics teaching have been investigated in a previous study (Georgiadou & Potari, 1999). In that study Christina seemed to have developed a rather broader view about mathematics than it being "just numbers, rules, exercises", viewing it as "logic, thinking, and a particular way of seeing things", during the last year of her university studies. Moreover, she seemed to have developed an inquiry approach to teaching while interacting with children in the lower grades where she was feeling mathematically confident. Christina viewed mathematics teaching as a process of actual involvement of the children in a variety of activities, and she acknowledged the role of multiple representations in understanding mathematical concepts. This view seems to be consistent throughout the period of the research but it acquired a different meaning in Christina's teaching actions as she experiences the various contexts. In the following analysis we attempt to illustrate the above assumption.

Exploratory teaching

Since being a student teacher Christina has believed in the power and efficacy of activities in mathematics teaching. This view had been expressed in various discussions between her and the researchers but was also evident in her actual teaching practice during her university studies. The following example shows the practical character that she attributed to such activities. In an interview at the end of the course, Christina discussed the following problem that a peer who had taken a third year class, experienced: "A window is 4m long and a door is 2m long. Which one is longer and how much longer is it?" Most children added the two numbers to get the result, although the student teacher read the problem to them many times. Christina judged his approach as inappropriate: "He (the student-teacher) did not realise that the children could not understand the words of the problem. I suggested that he asks the children to represent it. Well, he did so...four children were placed along the window and two children along the door. Then the children gave a correct answer. The problem is that they (her peers) do not interpret children's thinking."

Here, Christina recognises the importance of interpreting children's thinking and their needs in planning appropriate activities. Thus she considered that making the problem concrete for the children could possibly help them to tackle it. Moreover, in her teaching practice in that period, she attempted to get the children involved in activities where her main aim was to encourage them to explore mathematical ideas. The activity acted as the starting point for exploration, which was encouraged by her rather "open" questions. Christina's teaching actions seemed to be influenced mainly by the course where she was expected to act as a researcher. The mathematics classroom was more a place for experimentation rather than a place where the children had to acquire certain mathematical knowledge. Constraints like the mathematics curriculum; the classroom teachers' goals and time pressures did not seem to be an obstacle to her teaching decisions.

It is somewhat unrealistic to attempt to find "cause-effect" relations between Christina's university experiences and her teaching actions. However, we can infer that this particular context had challenged existing traditional approaches that she had experienced as a student at school, and helped her at the time of her studies to develop tools for her future professional development. As she admitted: "If I hadn't attended the course I would have continued to teach in a mechanistic way as I was accustomed to in the previous year, where my teaching was based on the textbook and the children could not understand".

Exploratory versus instructive teaching

Christina's first year experience has been studied in terms of the way that she conceptualised her teaching role in the school environment (Georgiadou & Potari, 2000). In that study we identified specific elements of the general school environment such as the educational policy of the school, and the important role her colleagues and her former peer student teachers played in Christina's process of adaptation. In this paper, we will focus on the way that she adapted her view concerning the importance of children's involvement in activities in this new context. Christina claimed that during that year she was doing a lot of activities in the playground where the children played games with numbers. However, she did not organise classroom instruction around these kinds of activities. From her teaching planning it appeared that she used activities that aimed to inquire into her pupils' ways of understanding, by getting them involved mostly in written work.

It seems that Christina worked on two planes: the exploratory and the instructive. In the first plane she "transferred" experiences from the university while, in the second, her actions were affected by the reality of the school environment and, in particular, by the restrains of the national curriculum. So, she was striving to see immediate results in the children's learning, although she was still interested in developing their thinking. In the interview, Christina claimed that she only had 5 to 10 minutes per day for oral activities. She also had a difficult experience at the beginning of the school year: she devoted the whole of the first week to activities and games on preliminary mathematical notions, such as orientation. At the end of the week the principal admonished Christina because parents had complained that she had not done any work because nothing had appeared in their children's homework notebooks. In this difficult year Christina managed to survive and, at the same time, to keep alive her view on how children learn mathematics. This was unconsciously achieved by separating these two planes of work; in some cases, she balanced these two different perspectives of the role of the activities in teaching. For example, while working in the second plane, she realised that the children faced difficulties in understanding the pairs of numbers adding to ten, from their responses to the written work she had planned based on the textbook. She overcame this difficulty by developing the 'Box' task described in Hughes (1986), which she had used in the context of her university studies.

Discovery teaching

In the following year Christina felt well established in school, as she had gained the approval of the head, other colleagues, the children and the parents, and recognition of her effectiveness as a teacher of Year 1. Thus she was appointed to teach Year 1 for the next two years of the study. At the same time she participated in the research programme aiming at teacher's professional development as described in the methodology section.

During the first year of the third strand, Christina developed independently a large number of activities to support children to understand specific mathematical concepts. In that period the exploratory plane was more a resource for ideas in the planning of her teaching. She realised her instruction through the use of multiple representations and a variety of contexts for the embodiment of the mathematical concepts. The following game that she organised in her attempt to lead the children to discover addition and subtraction as opposite operations demonstrates this approach:

> *"From this moment the dumb king starts. Nobody speaks. Not even me. We are going to play a game. You need to be very careful. Your eyes open! When I hit somebody's back, she speaks. Now, we start the game. I want you to tell me, using a lot of words, what I am doing."*

Using the abacus, she shows ten balls, ads five and then takes away five again. This was quickly repeated several times. She then asked the children to identify the mathematical model, which described this action. Her teaching approach can be seen as "discovery" learning where she expects children to work towards her aims. Talking about her teaching in that year she believed that, "I am better this year than last year in the classroom". She justified this claim by arguing that her children had obtained better results that year although she believed that she was asking them more difficult questions. An explanation for this was that, "The children in this year worked more on written tasks so they were prepared for the tests".

It seems that Christina does not exploit children's thinking in the same way that she used to in the previous strands. Her teaching aims are more result-oriented and the activities she developed were closely related to those aims. Christina's teaching priorities were possibly driven by her feelings of success and the acceptance that she experienced in the broader school environment. However, her view about children's active involvement in the learning process was still apparent.

Reconsidering existed teaching practices

Through her engagement in the group meetings and the individual discussions with the researchers, she started to challenge the effectiveness of her approach. To illustrate this, we present below an extract from a discussion with one of the researchers towards the middle of the second year of the third strand. In this discussion, Christina asked for help to resolve a conflict she faced when her children could not reply to the following sentences:

$\square=1+5$ and $6=1+\square$; although they could solve the sentences $1+5=\boxed{6}$ and $1+\boxed{5}=6$.

In the dialogue below, Ch stands for Christina and R stands for the researcher.

1. Ch: What is happening? And why?
2. R: What are we doing in cases like these? We probably try smaller numbers, change the problem...
3. Ch: You will laugh. I was thinking again of a visual solution, visual representation, you know, as usual.
4. R: Why didn't you try?
5. Ch: Because I wanted to try something different.

Line 3 indicates that Christina is aware of her tendency to use a variety of representations in her teaching. In line 5 she probably wanted to change her approach because it was inadequate for resolving the conflict. This point is further supported later on in the discussion, when she admitted that the following year she would teach that specific content differently. Four days later she came up with three different approaches to the problem. In the first she started with the equality 6=6, and she planned to ask the children to substitute the numbers with appropriate sums. Another idea was to use two bags, each containing 6 items. Then the children had to choose from a pile of cards with numbers and symbols, and construct sentences to represent the equality of the number of items in each bag. The third idea was to ask children to create a story about 5 + 1=6 but she rejected this immediately because she was worried that the children would give a trivial word problem from the textbook. In this proposed planning she again acknowledged the role of different representations but it seems that she allowed more space for the children to act. Christina had probably arrived at a certain point where she conceptualised the activity not merely as a means for discovery but as a starting point for mathematical constructions.

Promoting the continuing development of a teacher's practice

The above analysis reveals that Christina formed certain views about teaching and learning mathematics during her university studies. She started building a vision of mathematics teaching different from that which she had experienced as a school student. The latter context supported her in viewing teaching as a process for exploring children's thinking, and provided her with the opportunity to experiment with this perspective. Normal classroom constraints that a schoolteacher faces were not Christina's main concern. So, her desire for "the children to learn" could be fulfilled in this "ideal" situation.

By becoming a schoolteacher she started to experience the disparity between vision and daily classroom reality (Goldsmith and Schifter, 1997). In this new context she felt responsible for children's learning; she wanted to be approved by the head of the school, the parents and her colleagues. At the same time she wanted to develop further her university experience, although this experience was in contrast to the culture of the broader school environment. A way to satisfy all these expectations was to derive elements from both contexts and move in parallel directions. On the one side, she adapted to "established" school teaching practices but, on the other, she was choosing open activities from time to time, just to get children involved and explore their thinking. However, we did not discern many moments in her teaching behaviour where she could integrate these two practices. It is possible that Christina could have found her own way to modify her teaching behaviour in order to reach an acceptable level of harmony with her view about teaching, which was developed at the university. This assumption has been supported in a way by the work of Chapman (2001) who demonstrated that teachers' primary beliefs about mathematics remained unchanged while teaching behaviour was adjusted to those beliefs. However, Cooney (1985) found that a beginning teacher lost his belief that he could try a problem-solving approach in his teaching.

It is difficult to foresee what Christina would have done if she had been operating in the same school context without any further connection to her pre-service mathematics education. In the second and third year of teaching, Christina experienced situations similar to those she had had at the university such as group meetings, reading of research papers, and discussions on the use of materials and about classroom incidents, with her colleagues and with one of the researchers. In these situations the issues discussed emerged from the actual mathematics teaching classroom, yet in some cases, Christina recalled events from the university course that she found to be relevant to those issues. In that period, it seems that Christina reformulated her teaching practice by viewing learning as a process whereby the children discover certain mathematical relations. In this way, she managed to associate the exploratory teaching approach with the instructive but she limited the opportunities for the children to develop their mathematical thinking. The mutual rapport that was developed between her and one of the researchers possibly helped her to reflect on her action and to identify the limitations of her teaching approach.

Concluding Remarks

The seeds for change can and should be planted early in teachers' professional lives, taking advantage of the social contexts in which teachers learn to teach (Cooney and Shealy, 1997, p.92).

The above quotation implies that, especially for the beginning teacher, there is a need for support and feedback in order to foster the continuing development of teaching practices. In Greece, the school context does not promote approaches other than those characterised as "traditional". So, there is a big gap between what the teacher has experienced at University and what he/she faces in the school environment (Potari, 2001). The school environment constrains rather than supports the beginning teacher to act in accordance with his/her views. The first findings of this study indicate how long and strenuous the process of a teacher's development is and, more specifically, how her view of children's learning was expressed in her teaching behaviour while working in different contexts. Further analysis of the data could reveal other aspects of this process of adaptation.

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ON LIMITED VIEWS OF THE MEAN AS POINT OF BALANCE

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It is clear to every high school student that the mean of two values is half their sum. Is it also clear that the mean is their point of balance? Not quite. In the course of studying pattern exploration disciplines of high school computer science majors, we noticed that a non-negligible number of students lack a clear view of the mean as the point of balance. The students were asked to design a computer program that inputs N, a positive integer, and outputs all the positive integer pairs <x,y> which average N/2. The majority of the students demonstrated limited orientation with patterns of the mean. In particular, a considerable number of them designed programs that "search for each x the y's which average N/2 with x". The student solutions, together with representative interviews, reflect diverse levels of mathematical insight and pattern recognition.

INTRODUCTION

One of the fundamental concepts in basic statistics, as well as everyday life, is that of the mean (average). The mean is introduced in primary school and invoked in numerous occasions thereafter, in mathematics, science, and other domains. The computation procedure of the mean is almost as fluent as the fundamental arithmetic operations of addition and division. However, the conceptual understanding of the mean is far less assimilated. In this paper we display a study on the limited conceptual understanding of the mean with respect to its notion as the center point of balance.

The lack of conceptual understanding of the mean was studied during the last two decades in several respects and for different group ages. Pollatsek, Lima, and Well (1981) reported on difficulties in conceptual understanding of the weighted mean among college students. Mevarech (1983) expanded their study and showed that non-mathematically oriented students "misconceive a set of given means under simple mean computation as … satisfying the four properties of closure, associativity, identity, and inverse" (1983, p. 425).

Strauss and Bichler (1988) studied development aspects of understanding seven mean properties among young students of the ages 8-14. They found that the property of 'average as representative' was difficult for students to see. Mokros and Russell (1995) focused in mathematical representativness of the mean, and identified five basic approaches among 4^{th}, 6^{th}, and 8^{th} graders, and named them from the less insightful to the more insightful - average as 'mode', 'algorithm', 'reasonable', 'midpoint', and 'mathematical point of balance'. Cai (1998) noticed that a significant number of 6th graders were unable to correctly solve a contextualized average problem. Those who used algebraic representations performed better than others.

These studies demonstrated the gap between procedural and conceptual knowledge of the mean. While students are easily able to calculate the mean by straightforward sum-and-divide, many lack understanding of its meaning and have difficulties in performing insightful mean-related tasks. The gap between procedural and conceptual knowledge is apparent in a variety of mathematical domains, as discussed by Hiebert and Lefevre (1986), Bin-Ali and Tall (1996), and others. The lack of conceptual knowledge is strongly related to indications of the need for elaborating pattern exploration in mathematics (e.g., Schoenfeld 1992).

This paper extends the view on conceptual understanding of the mean, together with the formulation of arithmetic expressions. In addition, it illustrates the important role of pattern exploration and exploitation. Different from previous studies, in which students where questioned on particular values of elements, our study focuses on general formulation of tuples (pairs) of elements with the same mean. Its core emphasis is on conceptual understanding and arithmetic expression of the mean as the center point of balance. The participants were high school students, rather than college or elementary school students in the earlier studies.

The study was part of a broader inquiry into mathematical pattern exploration of high school computer science majors in solving algorithmic tasks. The broader study involved a series of short algorithmic tasks with emphasis on efficient algorithmic solutions. Although we expected some inefficient student solutions, we were surprised to see the extent of inefficiency and the lack of pattern recognition regarding the concept of the mean. As a result, we decided to carefully examine, analyze, and display the solutions.

In the next section we present the methodology used, involving both qualitative and quantitative evaluations. In the section that follows we display the results and focus on several solution categories that reflect gradual levels of mathematical insight. In the last section we discuss the results, relate them to previous work, and note on tying statistical concepts to pattern exploration and exploitation.

METHODOLOGY

Participants

The study involved 82 computer science majors (36 10th graders and 46 11th graders), with sound mathematical background. The students were selected from four different schools in different geographic locations in the country. All students had basic programming background, and were particularly aware of the fundamental concepts of correctness and efficiency of algorithms.

Task

In the broad study, the students were asked to design algorithmic solutions to four algorithmic tasks aimed at examining their mathematical pattern exploration. The students were explicitly told that they should write efficient solutions based on mathematical patterns they identify. We let each student write the solutions in his/her most convenient language – either algorithmic pseudo-code, or one of the programming languages Pascal or C. The mean-related algorithmic task was:

> **Same-Average-Pairs**:
>
> *Design* an algorithm/computer-program which inputs a positive integer N and outputs all the pairs <x,y> of positive integers which average N/2.
> *Write* a short explanation of your underlying idea.
>
> Note: As our focus was on mathematical pattern exploration, we told students that each of them can choose, according to his/her programming convenience, whether to consider pairs of the form <i,j> and <j,i> distinct. They could choose to output either both pairs or only one of them.

Our purpose in posing this task was to examine whether students view the mean as the center of balance of all pairs with the same average, and whether they can express it arithmetically.

Procedure

The students were given up to 25 minutes to solve the above task (as part of 90 minutes provided for all four tasks). One source of our data was the 82 solution sheets we obtained from the students, with algorithms and descriptions of their underlying ideas. In addition, nine of the students were asked to elaborate on their solutions in individual interviews.

RESULTS

The solutions to the task varied considerably. Only one third of the students demonstrated a clear view of the mean as the center point of balance. The rest of the students showed limited degrees of insight. We refer to the different degrees of insight through five patterns that characterize the task output:

- *element range* – the range of the output elements is 0..N.
- *unique pairing* – no range element has more than one pairing element.
- *pair symmetry* – the two elements of an output pair are located on two different 'sides' of the mean, at the same distance from it.
- *pair arithmetic* – one way of arithmetically expressing pair values in the task is: <N/2-d,N/2+d>. Another way is: <d,N-d>. Since we are interested in integer values, the second expression is more convenient to utilize.

- *pair adjacency* – adjacency in the sequence of 'same-mean pairs' corresponds to adjacency on the line of natural numbers.

We divided the student solutions to four different categories based on their insight of the above patterns.

- **Clear-View**: Solutions that reflect clear view of the five patterns.
- **Incomplete-View**: Solutions that reflect incomplete view of the *pair arithmetic* and the *pair adjacency* patterns.
- **Search-View**: Solutions that reflect no clear view of the *unique pairing*, *pair symmetry*, *pair arithmetic*, and *pair adjacency* patterns. These solutions are based on inefficient search "for the y's that match each x".
- **Dim-View**: Solutions that do not reflect any clear pattern. Some of these solutions only include a search for pairs of the form <x,x> or <x,x+1>.

In what follows we describe each of the above solution categories.

Clear-View Solutions

34% of the students (28 of the 82) provided algorithmic solutions that reflect clear view of the mean as the center point of balance of the output pairs. The code typical for these solutions was:

```
for i from 0 to N do
   write (i, N-i)
```

Those students who chose not to output both <x,y> and <y,x> bounded the "for" loop by $\lfloor N/2 \rfloor$ (rather than N). This loop outputs the pairs starting from the range ends towards the mean. A couple of students output the pairs in the opposite order – from the mean towards the ends. Below are two student explanations.

Omer: The mean of two numbers is exactly their middle, thus if we increase the smaller by 1 and decrease the bigger by 1 we will preserve the same middle point, which is the mean.

Nimrod: Every two numbers have the same mean as the mean of the lowest number (of the range) plus a constant and the highest number (of the range) minus that constant; and that constant may be from one of the (range) ends until half the difference between them.

Both students explicitly expressed the patterns of *pair symmetry* and *pair adjacency*. Their algorithmic solutions expressed the additional patterns.

Incomplete-View Solutions

18% of the students (15 of the 82) provided algorithmic solutions that reflect partial view of the mean as the center point of balance. They all noticed the *element range* and the *unique pairing* patterns. Many also had 'a picture' of *pair symmetry*. But their notions of *pair arithmetic* and *pair adjacency* were incomplete. They noticed that each element within the range 0..N may be in at most one output pair, but they were not clearly convinced that each element will indeed be in the output. Their programs generated the relevant symmetric pairs, but checked for each pair whether its mean indeed equals N/2. One representative solution was:

```
for i from 0 to N do
   if (i+(N-i))/2=N/2 then write (i, N-i)
```

We wondered how is it that the students who provided this solution did not see that the condition in the **if** statement is completely unnecessary. The interview below sheds some light on that:

Interviewer: Amit, what is the idea underlying your solution?

Amit: When you take a number from the beginning and a number from the end, 0 and N, and then add and reduce 1, you get the potential pairs.

Interviewer: What do you mean by 'potential pairs'?

Amit: 0 and N have the mean N/2, also 1 and N-1, ... as **i** grows you get more pairs ...[pause]... I am not sure about some of the numbers further on ...

...<some additional interaction>...

Interviewer: Can you explain the condition (i+(N-i))/2=N/2 in your code?

Amit: For example, for i=0 it will be true ... then, for 1 also ... then ...???

Amit did not immediately notice the algebraic equality in the condition that he himself wrote. While he did have some notion of symmetry (<0,N>, <1,N-1>), he had a difficulty in expressing it with an arithmetic expression. He also had a vague view of the complete sequence of output pairs.

Search-View Solutions

32% of the students (26 of the 82) provided solutions which reflect a view that does not adhere to any of the patterns: *unique pairing*, *pair symmetry*, *pair arithmetic*, or *pair adjacency*. Many simply generated all the <x,y> pairs in the range 0..N (a total of $(N+1)^2$ pairs). Some let x be in the range 0..N/2 and y in the range N/2..N. The only pattern to which these student solutions clearly referred was *element range*. A common solution in this category was:

```
for i from 0 to N do
   for j from 0 to N do
      if (i+j)/2=N/2 then write (i,j)
```

From interviews with some of the students we learned that they did not have a clear picture of the output pairs, and quite a few were not sure of the *unique pairing* pattern. The following interviews with Kfir and Anat illustrate that.

In his solution, Kfir enumerated both **i** and **j** from 0 to N.

Interviewer: Kfir, why did you write the nested loops with **i** and **j**?

Kfir: I needed to find for each element the elements that can be paired with it.

Interviewer: Can you say something about the number of elements that may pair with a particular element?

Kfir: Well … 1 has only one match, but 2 may have more …

Interviewer: How many? For example let's say that N=100.

Kfir: Then, 1 has one match – 99; but 2 may have many …?

Interviewer: ???

Kfir: Eh, … 98 is one match …<trying further>… maybe only 98? … Yes …

He gradually started realizing *unique pairing*, *pair symmetry*, and *pair adjacency*.

In a different class we interviewed Anat. Her solution included the enumeration of: **i** for 0 up to N/2 and **j** from N down to N/2+1.

Interviewer: Anat, what is the rational underlying your solution?

Anat: I divided the numbers into two groups – those from 0 to N/2 and those from N/2 to N; for each number in the first group there is a search for a match in the second group.

Interviewer: Why did you do that?

Anat: When we deal with average you cannot immediately tell both numbers; I need to loop through the 'upper' group in the order to search for corresponding matching …

Later in the interview Anat conjectured the *unique pairing* and *pair symmetry* patterns, but she had a difficulty to formulate a proper arithmetic expression.

Dim-View Solutions

16% of the students (13 of the 82) provided solutions that reflect no clear recognition of any of the five patterns. From their solutions it was even unclear whether they noticed that the range of element pairs is 0..N. Some designed solutions that output a single pair: <N/2,N/2>, or <N/2-1,N/2+1>. The latter case shows some 'picture' of symmetry, but only with respect to one output pair. Some provided solutions in which all the pairs of the form <i,i+1> were generated and checked for the possibility of having the mean N/2.

Interviews with two students who provided solutions in this category revealed that they searched for some time to find pair examples. They mentioned rather quickly the output pair <N/2,N/2>, for the case that N is even. The case of odd N was more subtle. We posed the task of generating the pairs which to N. This assisted one of them, who noticed the similarity to the original task. Shortly after, she managed to write down relevant pairs, mentioning that "the sum question is much easier". We got this response from students of other categories as well (see next section).

DISCUSSION

The results of this study showed diverse levels of recognition and exploitation of patterns of the mean. The task posed to the students involved the formulation of a general scheme, based on the patterns common for all pairs that have the same mean. As in all the previous studies of conceptual understanding of the mean, here too, the task focused on a perspective more involved than the simple computation of sum-and-divide. The new contribution of this study stems from examining the formulation of a general expression rather than calculation of values for particular examples. In addition, the participants were high school students (rather than 4^{th} to 8^{th} graders or college students).

In order to properly answer the task, students had to identify and utilize the five patterns *element range*, *unique pairing*, *pair symmetry*, *pair arithmetic*, and *pair adjacency*. The pattern of *element range* is related to the property of extreme points examined by Strauss and Bichler (1988). The pattern of *pair symmetry* corresponds to Mokros and Russell's midpoint representation. The pattern of *pair arithmetic* is correlated with Cai's (1998) observation of algebraic representation.

Our division of the diverse solutions into different categories characterizes four different levels of mathematical insight with respect to conceiving the mean as the center of balance. The first, Clear-View category, includes the students who realized and efficiently exploited all the five patterns. Only one third of the students were in this category. The second, Incomplete-View category, includes about a fifth of the students, who realized the first three patterns but did not clearly see the latter two. Their solutions were still rather efficient. The third, Search-View category, includes almost a third of the students, who exploited the first pattern, but not any of the other four. Their solutions were correct, but very inefficient. The fourth, Dim-View category, includes the students with very vague insight.

We posed to the interviewed students the task of calculating all the pairs of elements which sum to N (rather than average N/2). The first, Clear-View category students instantly noticed the identity between the two tasks. Students from the other categories noticed it more gradually, and indicated that the sum task is easier (to some - "much easier") then the average task. One explanation of this phenomenon is the gap in familiarity between the notions of the sum and the mean. Apparently, conceptual understanding of the sum is more developed, even for high school students, than conceptual understanding of the mean.

Shaughenessy (1992) indicated, in his review on reflections in probability and statistics, the limited perspectives presented to students. One study that elaborates perspective diversity and conceptual understanding of the mean is that by Meyer, Browning, and Channell (1995). Their study introduces four activities for elementary and middle school students, involving concrete values. Our study suggests the need for further insight elaboration of the mean, for more mature students. General formulation, based on recognition and exploitation of patterns of the mean, as in the task presented here, should enhance conceptual understanding as well as competence in arithmetic and algebraic representation.

The results in this study add not only to the study of statistical conceptions but also to the research on limited pattern exploration. In particular, it illuminates the need for elaborating mathematical insight in algorithmic problem solving. It is our hope that this study will encourage further elaboration of students' pattern exploration both in statistics and algorithmics.

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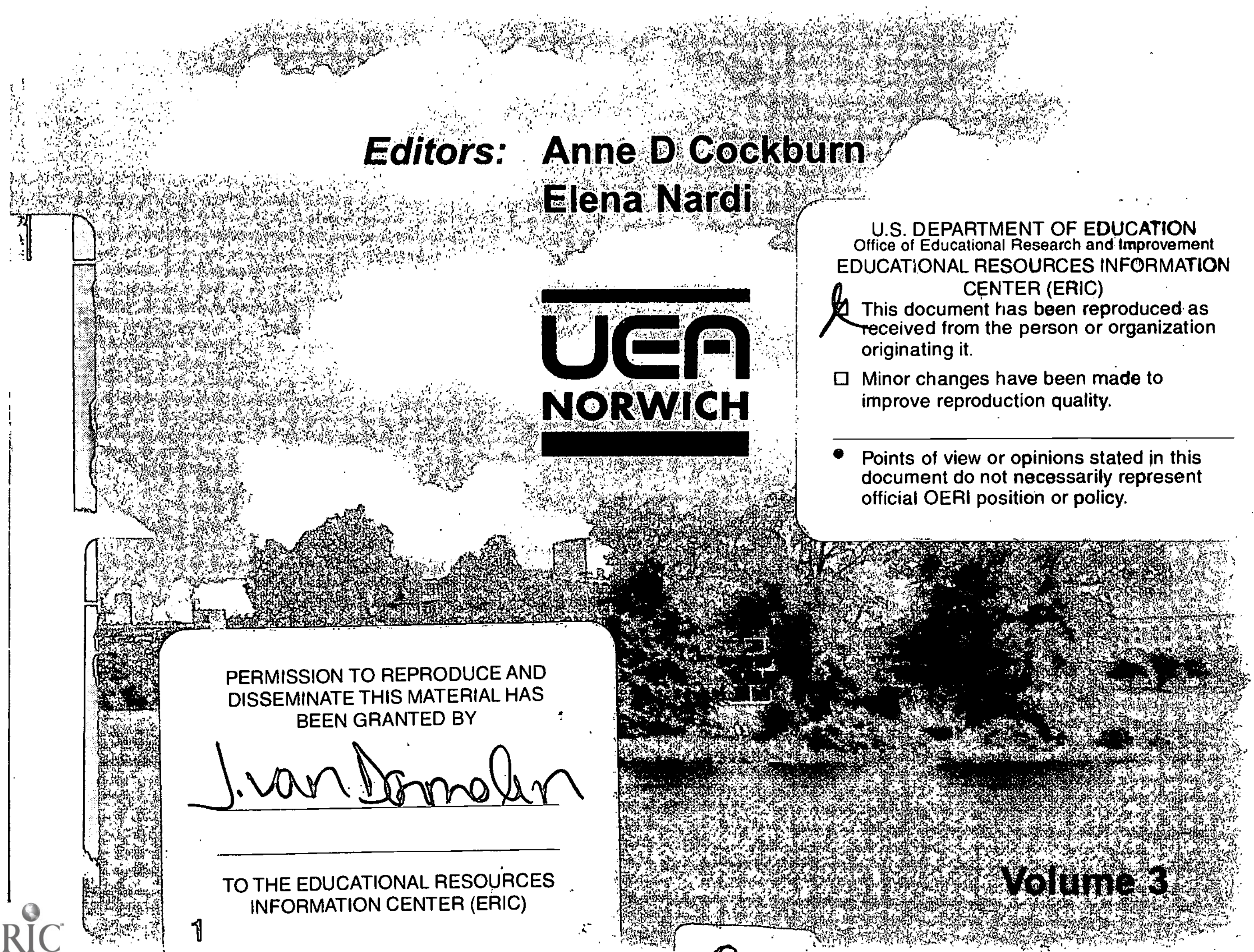

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Volume 3

INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

PME26
UNIVERSITY OF EAST ANGLIA • UK

21 - 26 July 2002
Norwich, UK

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Proceedings of the 26th Conference of the International group for the Psychology of Mathematics Education

Volume 3

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Fax: + 44 1603 593446
Website: www.uea.ac.uk/edu
The Proceedings are also available on cd-rom

ISBN number 0-9539983-6-3

Design of the PME26 logo: Steve Giles

Printed by the University of East Anglia, Norwich, UK

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RESEARCH REPORTS

COMPARING REPRESENTATIONS AND REASONING IN YOUNG CHILDREN WITH TWO-YEAR COLLEGE STUDENTS

Barbara Glass
Rutgers University

Carolyn Maher
Rutgers University

Problem solving and justification of a diversified group of two-year college students was compared with approaches of younger pre-college students working on the same task. The students in this study were engaged in thoughtful mathematics. Both groups found patterns, justified that their patterns were reasonable and, utilized similar strategies for their solution and methods of justification. The findings support the importance of introducing rich problems to pre-college and college students, under particular conditions.

Students enrolled in college level mathematics are expected to have developed effective reasoning skills. Unfortunately this is often not the case. This may be explained, in part, by a history of mathematics instruction in settings that devalue thinking and focus on procedural learning. From a perspective of conceptualizing reasoning in terms of solving open-ended problems, it was of interest to learn whether two-year college students enrolled in a liberal-arts college mathematics course could be successful in providing arguments to support their reasoning in a problem-solving based curriculum.

Considerable data have been collected showing pre-college students' success in solving open ended problems, over time, under conditions that encouraged critical thinking and building arguments to support their solutions (Maher & Martino, 1996, 1997, 1998; Muter, 1999; Maher, 1998; Maher & Speiser, 1997; Kiczek & Maher, 1998; Muter & Maher, 1998). These studies with younger students raise the question if similar reasoning and justifications are achievable by liberal-arts college students within a well-implemented curriculum that includes a strand of connected problems to be solved over the course of the semester. Specifically, this paper reports on one aspect of a larger study of two-year college students enrolled in liberal arts mathematics. It will describe, in the context of combinatorics, (1) how college students solve non-routine mathematical investigations and (2) how college students' representations and level of reasoning contrast with those of younger children from a longitudinal study engaged in the same investigations.

THEORETICAL FRAMEWORK

The growth of mathematical knowledge is the process whereby a student constructs internal representations and connects these representations to each other. Understanding is the process of making connections between different pieces of knowledge that are already internally represented or between existing internal connections and new knowledge. (Hiebert and Carpenter, 1992) Students build their understanding of concepts by building upon previous experience, not by imitating the actions of a teacher or being told what to do. (Maher, Davis & Alston, 1991) Learners who first learn procedures without attaching meaning to them are less likely to develop well connected conceptual knowledge. When students encounter new problems they are more likely to retrieve prior knowledge that is well connected than to retrieve loosely connected information. (Hiebert and Carpenter, 1992). The students in this study were encouraged to think about their solutions to the problems that they were given, develop understanding of the mathematics and justify their answers.

BACKGROUND FOR PRE-COLLEGE STUDENT RESULTS

Researchers have documented children's thinking as they investigate problems in the area of combinatorics to determine how they think about the problems and justify their solutions (Maher & Martino, 1996, 1998, 2000; Muter, 1999; Maher, 1998; Maher & Speiser, 1997; Kiczek & Maher, 1998; Muter & Maher, 1998; Martino, 1992). One of these problems is the Towers Problem, which invites a student to determine how many different towers of a specified height can be built when selecting from two different color cubes, and to justify that all possibilities have been found. In a Rutgers University longitudinal study, students first encountered the Towers Problem in the third grade when they worked on towers that were four tall. Researchers found that these pre-college students were able to invent strategies that they used to solve the problems. Most of the pre-college students started the Towers Problem by using guess and check methods to create towers, and check for duplicates. They then became more organized and created local organization strategies This resulted in the use of patterns such as "opposites", two towers with the colors reversed, "cousins", two towers that were vertical inversions, staircase patterns, and elevator patterns. They later developed global organization strategies as they realized that their local organization strategies created duplicates and could not be used to account for all possibilities (Maher & Martino, 1999, 2000; Muter, 1999; Martino, 1992). These students were also able to create convincing arguments that they had accounted for all possible combinations (Maher & Martino, 1996, 1998, 1999, 2000; Muter, 1999; Muter & Maher, 1998; Martino, 1992). The initial method of justification of many of the third and fourth grade pre-college students was to state that they could not find any more towers (Martino, 1992; Maher & Martino, 1996). For example, fourth grade Milin justified that he and his partner had found all possibilities because they had gone a long period of time without finding another one (Alston & Maher, 1993). Some of the children developed a proof by cases and others developed an inductive argument. For example, Milin started to build a proof by cases, but then developed an inductive argument when he looked at simpler cases of the Towers Problem (one-tall, two-tall and three-tall) and noticed the doubling pattern. He was able to explain how he could build up the larger towers from the smaller towers by placing either one or the other of the available colors on the top of the smaller towers

(Alston & Maher, 1993). Within this same time period, fourth grade Stephanie was able to present a proof by cases for the towers that were five tall. During a small group assessment, Stephanie presented a proof by cases for towers that were three tall and Milin presented his argument by induction (Maher& Martino, 1996; Alston & Maher, 1993). About a year later, in grade five, Stephanie also developed an inductive argument (Maher & Martino, 1996).

COLLEGE STUDENTS

Data Collection and Analysis

Nine classes ranging in size from 6 to 25 students were studied from 1998 – 2000. Two groups from each class were videotaped as they worked on the Towers Problem. Following the class sessions each student was required to submit a write-up of the problem. In addition, videotaped, task based, interviews were conducted.

Eleven students, representative of the larger population were selected for a case study analysis. The criteria included student willingness to be videotaped, fully participate in regular class problem investigation sessions and participate in follow-up interviews. Videotapes from class sessions and interviews were transcribed, coded, and analyzed for methods of problem solving and justification. The written work of students was coded and analyzed. The coding schemes for problem solving strategies and methods of justification are in table 1 and table 2.

S_r – random checking	S_p – looked for patterns	S_b – worked backwards
S_a – thought of a similar problem	S_s – thought of a simpler problem	S_{al} – used an algebraic equation or formula
S_i – used inductive method	S_c – conjectured	S_v – controlled for variables
S_{sp}– divided the problem into sub-problems	S_f – applied a previously learned procedure	

Table 1: Code of Problem Solving Strategies

A_o – none	A_c – cases approach	A_i – inductive approach
A_a – checked with another member of the class	A_e – everyone in the group agreed	A_f – can't find any more
A_s – same answer as a previous problem	A_l – applied previously learned formula or procedure	A_n – numerical argument
A_v – created a visually appealing pattern	A_{ci} – challenges instructor to show missing combination	A_g – geometric argument
A_p – discovered a pattern	A_{pe} – explained why pattern worked	

Table 2: Code of Methods of Justification

Setting

The study was conducted at a small community college in a mathematics course for liberal arts majors. Sections of the course met for two seventy-five minute classes each week or three fifty minute classes each week for fifteen weeks. The students spent approximately half of all class time working on various non-routine problems in a small group setting. The Towers Problem was given during the ninth week of the semester. Two groups from each class were videotaped as they worked on the problem. The students began by working on towers that were four cubes tall. They then were asked to consider towers that were five cubes tall. Some groups also worked with towers where three different color cubes were available.

Results

Most of the college students used patterns or some other form of local organization immediately and some immediately imposed a global organization scheme. One group started the problem by randomly generating towers using a build and check method. A student in the group, Jeff, soon suggested, however, that they should organize the towers that they had built by cases so that they could check more easily for missing combinations. After another pair of students, Stephanie and Tracy, built all sixteen towers using opposite pairs, they stated that they had to impose some type of organization to convince themselves that they had found all combinations. They first used staircase patterns, but rejected this because it did not account for all towers. They then arranged the towers by cases (Glass, 2001). While working on the five-tall Towers problem one student, Wesley, built his towers five tall by adding a red cube to the top of each of his towers of four and then building the opposites of these towers to find all towers with a yellow cube on top. Another student, Errol used an inductive argument that he had developed while working with towers four tall to predict that there would be thirty-two towers of five. He produced a list of all towers five tall on his written assignment using an inductive method (Glass, 2001).

Six of the profiled college students justified that they had found all possible towers four tall by using a cases approach. Each of these students used an elevator pattern to justify the towers with three cubes of one color and one cube of the other color. The students used a variety of methods to justify that they had found all towers with two cubes of each color. (Glass, 2001). Two groups, Melinda's group and Donna's group justified that they had found all towers because they couldn't find anymore. As these groups spoke to the instructor they began to organize their towers and moved toward a proof by cases. Both groups still justified that they had all towers with two of each color because they could not find anymore.

Five of the profiled college students did a proof by cases for the five-tall towers. Each of these proofs by cases referred to opposites. They also all used an elevator pattern to account for the towers with one cube of one color and four cubes of the other color. They used a variety of methods to justify that they had all towers with two cubes of one color and three cubes of the other color (Glass, 2001). One student, Wesley built his towers five tall by adding a red cube to the top of each of his towers of four. He then built the opposites of these towers to find all towers with a yellow cube on top. He justified that he had found all towers five tall with an inductive argument. He was unable at this time to extend this reasoning to predict how many towers that he would get six tall. During an interview seven weeks later, Wesley correctly extended the doubling pattern beyond the case that went from four tall to five tall. Another student, Jeff applied the fundamental counting principle to predict that there would be thirty-two towers that were five tall. After Jeff's group had produced the thirty-two towers he used an inductive argument to show that they had found all possible towers by pairing each of the five-tall towers with the corresponding towers that were four tall. Errol, who did not wish to build the five-tall towers in class, used an inductive argument on his written assignment to justify that he had listed all the possibilities. Penny, a student in Errol's group who was absent the day that the class worked on the Towers Problem, completed the problem at home. She invented a tree diagram strategy to produce an inductive argument for the towers four and five tall. Tim used a binary coding system to justify that he had found all possible combinations. This is the same method that tenth grade Michael from the Rutgers longitudinal study used to justify that there were thirty-two different pizzas with five available toppings (Muter, 1999).

Several groups also had time to work on the Towers Problem with three available colors. Mike's group and Rob's group worked on four-tall towers, while Jeff's group worked on three-tall towers. Rob and his partners applied the inductive method that they had developed for towers with two colors to quickly solve the problem. Jeff used the fundamental counting principle to calculate the number of towers, but did not use an inductive method to build the towers. Mike's group divided the problem into the three problems of finding towers with two available colors and the problem of finding towers that contained all three colors. They then built all the towers in a systematic fashion.

CONCLUSIONS, IMPLICATIONS AND LIMITATIONS

The college students used many of the same strategies for solution that the pre-college students from the longitudinal study had used when they solved the same problems. However, unlike the third and fourth grade children they did not generally rely on random checking as a primary strategy when they began to solve the problems. In general, the college students were able to solve the problem more quickly than the third and fourth grade children working on the same tasks. All were able to solve the four-tall Towers Problem and start the five-tall Towers Problem within a fifty-minute or a seventy-five-minute period. Most also finished the five-tall problem within the same period. Several students also had time to work on extensions of the problem within the class period. The college students also showed less inclination to think about problems for extended periods of time. Many stopped thinking about the problem after they had arrived at an answer, even when the instructor asked them to think more about the problem in order to reveal their thinking.

The college students also demonstrated methods of justification that were similar to those of the pre-college students. The method used by one of the college students, Robert, to show that he had found all towers with two cubes of one color and three cubes of the other color was the same as that used by Michael and Ankur from the Rutgers longitudinal study in one of their tenth grade after school sessions (Muter, 1999). Two of the profiled college students, Stephanie and Lisa used a method in which they fixed one of the two cubes on the top of the tower and moved the second cube into all possible positions. They then fixed one cube on the bottom and moved the other cube into all possible positions. This is similar to what fourth grade Stephanie from the Rutgers longitudinal study had done with towers six tall. Lisa and Stephanie from this study accounted for towers that did not have one of the two cubes on either the top or on the bottom of the tower while fourth grade Stephanie did not pursue this idea (Maher & Martino, 1996; Glass, 2001).

After another student, Rob and his group had built their towers by organizing them by cases they noticed the doubling pattern and developed a proof by induction, which Rob used in his written assignment. This is similar to what fourth grade Milin had done. College-aged Mike also noticed that the number of towers was doubling, but was unable to think of a reason for the doubling pattern (Glass, 2001). It is interesting to note that both Stephanie and Milin from the Rutgers longitudinal study had noticed the doubling pattern in the Towers Problem as fourth graders. Stephanie's progression from pattern recognition to development of an inductive argument took about eight months while

Milin's understanding developed more quickly (Alston & Maher, 1993; Maher & Martino, 2000). Perhaps college-aged Mike would also have recognized the reason for the doubling pattern if, like fourth grade Stephanie, he had been given an extended time frame in which to develop his ideas. Mike, however, was limited to about eight weeks to develop his ideas about the problem.

Although some of the conditions of the Rutgers University longitudinal study such as extended classroom sessions and revisiting the same problem several times within an extended time frame, could not be replicated because of time constraints within a college classroom, many of the conditions that enabled the pre-college students to become thoughtful problem solvers were duplicated. Both groups were given rich mathematical tasks and were encouraged to explain their reasoning and methods of solution and justify their solutions to the problems. Both groups of students were engaged in thoughtful mathematics. They found patterns, justified that their patterns were reasonable, and developed methods of proof. While it cannot be disputed that the students in the Rutgers longitudinal study benefited from exposure to rich mathematical experiences over an extended period of time, the students in this study, who had previously experienced a variety of traditional mathematics instruction, demonstrated that it is not too late to introduce rich mathematical experiences in a collegiate level mathematics class. The level of reasoning that these students demonstrated provides evidence that it is possible to experience thoughtful mathematics within a traditional fifteen-week semester. The findings support the importance of introducing rich problems to college students and giving them opportunities to work together toward a solution.

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THE STRUGGLES OF A COMMUNITY OF MATHEMATICS TEACHERS: DEVELOPING A COMMUNITY OF PRACTICE [1]

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The purpose of this study was to understand how participants in a professional development program initiated, developed, and supported a learning community focused on students' understanding. The analysis focuses on teachers' activity as they participated in a community of practice and how that participation promoted (or restricted) their learning and therefore theirs students' learning. The components that characterize a community of practice are 1) mutual engagement, 2) a joint enterprise, and 3) a shared repertoire. The study took place in a bilingual (English-Spanish) high school in a large district in the Midwest, with a high percentage of Latino/a students, and situated in an urban setting. The study shows how the community was transformed by the practices and resources available to the members.

INTRODUCTION

Mathematics teachers' participation in communities of practice has been identified as a promising strategy for professional development. Studies focused on school organization have found strong relationships among teachers' participation in professional communities, innovative practices, and student learning (Adajian, 1995; Lee, Smith, & Croninger, 1997; Louis, Marks & Kruse, 1996). Mathematics teachers who participate in communities of practice are more willing to change their practices and their learning becomes generative (Franke, Carpenter, Levi, & Fennema, 1988; Stein & Brown, 1997).

While studies have found that teacher' participation in communities of practice is one significant element for supporting teacher change, we do not know how those communities are constructed and supported. In many studies, the communities were already organized or, in some cases, the communities emerged from the reform process in ways that are not clear. Regardless, communities are important in supporting both mathematics teachers' and their students' learning.

OVERVIEW OF THE STUDY

This study was part of a larger study that took place during a two-year period. The goal of the larger study was to examine how the focus on student understanding affects different systems at school. The hypothesis was that when teachers have knowledge about how students reason, they are able to interpret patterns in students' thinking and they are responsive to student understanding in the classroom (Carpenter, Fennema, Perterson, & Carey, 1998). This knowledge would also help teachers to select and construct meaningful tasks that highlight the development of student' algebraic thinking. At the same time, this knowledge would give teachers tools to promote discourse in which students are allowed to express their opinions, defend their solutions, question other's ideas, and reorganize their own thinking.

The purpose of this study was to understand how participants in a professional development program initiate, develop, and support a learning community focused on students' understanding. The analysis focuses on teachers' activities as they participate in a community of practice (Wenger, 1998) and how that participation promotes (or restricts) their learning and therefore their students' learning. The three dimensions characterizing a community of practice are 1) mutual engagement, 2) negotiation of a joint enterprise, and 3) development of a shared repertoire. This study seeks to describe how a specific design for professional development serves the different requirements for learning defined on a framework in which communities of practice and participation in those communities are the fundamental unit of analysis. Such a conceptual framework emphasizes the negotiation of meaning and the building of identities, which take place during the development of practice (Wenger, 1998).

THE CONTEXT

The concern for minority students' achievement in schools in the United States is well known. Many Latino students attend overcrowded schools, live in economically distressed areas, and face prejudice and stereotypes in their daily lives (Secada, et al., 1998). The city where this study took place had been making efforts for improving minority student opportunities. The school, Urban High School, faced strong pressure from the school district to improve student achievement and to get more students to pass the graduation test required by the school district. The school district agreed five years before the beginning of the study, to require algebra for all ninth-graders.

Urban High School offers a bilingual program for its 57% of Latino and Latina students. Close to 66% of students received free lunch (1997-1998). The school had 1500 students, and it was considered the most diverse school in the district with 19% of African-American, 10% white, and 7% Asian. Like any other inner city high school in the United States, Urban High School had many problems to confront. Teen pregnancy, truancy, and gangs were the most prominent.

The mathematics department at Urban High School had twelve teachers where five of them were bilingual. Four bilingual teachers volunteered to participate in the program. They were looking for mechanisms to get subject-colleagues support and to get a common understanding of the challenges they faced due to the changes in the school structure. The school was in the process of reconstruction due to poor student achievement and failing performance. The idea of "schools within schools" (or families) was being utilized since 1996-1997 when the new principal was hired. The family structure encouraged teachers to change their academic organization. Instead of traditional departments, the school was organized into families by areas—business, art, geography, science, and computers. Students belonged to a family and took most of their classes together. The mathematics department still existed but its function was not clear. While many teachers belonged to families and shared experienced with these teachers, others did not belong to a family and did not have an opportunity to talk to other teachers. The department was mostly a remnant from the previous structure and its function was mainly informative.

Participants

The teachers are Fernanda, Celina, Gustavo and Armando. They all identified themselves as Latino; three of them were Mexican-American and one Spanish-American. They lived in the neighborhood and participated in activities with the community. They were bilingual and received bachelor's degrees in mathematics or mathematics education. Urban High is the only school they had taught. Armando and Celina belonged to the arts and science family respectively. Fernanda and Gustavo did not belong to a family.

During the first year we observed five classrooms, two from Celina and one from each one of the other teachers. During the second year we observed two classrooms, one from Gustavo and one from Armando. One hundred and sixty students attended these classes during these two years. The demographic characteristics of these classes were very similar. In Celina' and Armando's classes, the students were mostly 9^{th} and 10^{th} graders. In Gustavo's and Fernanda's, they were mostly 11^{th} and 12^{th} graders. The age varied from 14 to 18 years old and the time living in the United States varied from one month to all their lives. Most of the students were bilingual. Ten percent of the students were born in the United States, 60% were born in Mexico, 27% were born in Puerto Rico and the remaining 3% of the students were born in other Central or South American countries. Sixty-two percent of the students in these classes were female and 38% male.

THE COMMUNITY OF PRACTICE

Different researchers have shown the importance of communities, usually in the form of academic departments, in secondary schools (Gutierrez, 1996; Adajain, 1995; Siskin, 1994). But, what happens when teachers do not find places and opportunities in the school for creating these communities? With efforts from different directions in changing school structure and improving students' learning, it is difficult for teachers to come together and find the cohesiveness necessary to maintain groups together. At Urban High School neither the families nor the department had the structure for community. These teachers spent most of their time in their rooms, where they had lunch, tutored students, had parent conferences, and planned their classes. These teachers fit what Hargreaves (1993) described as "isolation, individualism, and privatism" as model of teachers' work. Different from what Siskin (1994) said, at Urban High School the department structure had not given teachers opportunities for collaborative efforts, for creating a strong identity, or for having norms and shared values. The family structure had not given them that support either.

The community engaged in two kinds of practices—the general meetings and the reflection meetings—aligning efforts to understand students' thinking and creating images to communicate and share knowledge. The next section describes how this community defined the three components of the framework during the two years.

Mutual engagement

Because engagement refers to the ability to take part on meaningful activities of the community, in interactions with other participants, in the production of tools and artifacts and in the whole process of community building, it is necessary to look at the planned activities, the way those activities allowed different participants to be involved in, and the way the activities were structure to guarantee an uninterrupted succession of events in which the participants would be engaged. For this group of people the general meetings were opportunities to talk, interact, and engage in doing things together. Sharing their experiences in the classroom, designing a test, and critiquing the school curriculum gave every participant opportunity for mutual engagement. The reflection meetings provided the opportunity to reflect about specific aspects of each teacher's practice. Using observations of students' strategies and responses in class, the researchers focused the conversations around the meaning of those observations and the way this information could be used by the teachers to plan following classes. These conversations with critical friends were also opportunities to get to know the teachers in a more personal way.

One of the first activities the community engaged in was the sharing activity. It was proposed by the teachers during the second meeting as a way to maintain contact with others and keep some coherence between mathematics courses at Urban High School. During the fourth meeting three of the teachers presented some of the experiences they had designed for their classes. Another activity the group engaged in was the discussion about curriculum. Because the group of teachers had decided it was important to have some coherence in the curriculum at the school, the teachers decided to discuss what the purpose of algebra would be and what would be the best way to organize the courses at school to reach the goals they had defined. Fernanda expressed her concerns about coherence in the curriculum at school and the effect of this lack of consistency in students' achievement. She promoted the idea of discussing the coherence of the curriculum at Urban High School in order to help her understand what content was important for her courses and how much emphasis was adequate on certain topics for getting students ready for the next course.

The design of the test was an opportunity for the teachers to participate in the research process. The researchers used the classroom observations as a way to get some knowledge about what the teachers considered important for the content of the algebra course and using these ideas to design a first draft for the test that was shared and discussed with the teachers. Engaging in looking at student work and analyzing possible student responses in different situations proved to be a successful task for this group. Teachers' knowledge of their students and researchers' knowledge about theories of learning provided opportunities for complementary contributions. The results from the test gave teachers information about what their students knew. The gap between teachers' expectations and test results created an opportunity to reflect about the main ideas of the algebra class. The teachers saw the need to make changes in their practices and give students more opportunities to explain their thinking. Having the possibility of talking about students' different strategies and levels of development of understanding gave teachers a new insight into students' learning. Teachers began to focus on the big ideas of algebra and on developing strategies to help students understand those ideas.

One the one hand, the activities the community was engaged in during the study offered to all participants opportunities for using their knowledge about their students, for making decisions about curricular issues, for evaluating the situation of bilingual students at Urban igh, and for creating tools, specifically the test, to get a better understanding of student's inking. On the other hand, the teachers took responsibility for selecting the activities they onsidered worthwhile. Even though the researchers were interested in carrying out tivities more related with looking into student understanding, the teachers were concerned bout other issues they needed to deal with before, e.g. the discussion about curriculum.

egotiation of a joint enterprise

he negotiation of a joint enterprise refers to the ability to coordinate perspectives and tions to a common goal. It is necessary to understand the enterprise of the community, to ke responsibility, and to contribute to the process of pursuing it. The practice of this group teachers and researchers reflected an attempt to create a context in which to advance in eir understanding of student thinking. The teachers struggled to satisfy the restrictions om the school and the district, and they worried about their students' future. The teachers egotiated the enterprise with the other member of the community. The members were igaged together in the joint enterprise of understanding student's thinking. This does not ean all participants agreed on everything, had the same dilemmas, or found the same iswers. But they found a way of working together, negotiating the goals, and recognizing e effects of it in their practice.

his process of negotiation also entailed the development of relations of mutual countability. In this case, accountability included not only trying things out in the assrooms, but also sharing experiences, expressing concerns, and making sense of their actices. The need for coordinating a perspective on what all participants meant by udent' understanding, for example, made the processes of coordination and ommunication dynamic, open-ended, and generative. This community developed in a rger context, which shaped and influenced its practice. The practice of this community as part of an historical development and it was influenced by the institutions to which the embers were part of. Resources and constrains, from the school as well as from the iiversity, were negotiated by the community in the process of defining its enterprise.

ther aspects related to coordination include the research methods used by the group and e ways for communicating that were followed during the study. Most of the processes for ithering data were defined by the researchers and agreed to by the teachers. Some of those ocesses included giving tests to students, participating in interviews, and collecting ermission forms. Some other issues were negotiated with the teachers. Setting the terviews with the students is one example of how the teachers helped the researchers to ake use of the resources available and even proposed alternative solutions.

ne of the conflicts the group had from the beginning was the lack of time for the teachers meet and do things together. Other than time during the general meetings, there were very w opportunities for teachers to even see each other during the day at school. This fact stricted opportunities to engage in discussions or even in planning activities for their ourses and to reach their goal of getting some coherence in the curriculum. This conflict ould not be resolved during the first year. Even though the group found different ternatives like meeting on Saturdays, using the banking days, and meeting after school, there were always other issues that conflicted with the schedule and made difficult to find time to meet.

Development of a shared repertoire

The development of a shared repertoire refers to the creative process of generating new ways of looking at the activities the community is engaged in. In order to belong to a community of practice, participants must be engaged in activities and have aligned processes. Participants must interpret events, locate themselves in the world, and explore other possibilities. The process of pursuing the enterprise creates resources for negotiating meaning. In this case, the use of specific terms such us "students understanding" or "informal assessment", routines such as discussing scheduling issues, tools such as the test, stories such as those shared by the teachers, and actions such as discussing the school curriculum were examples of the repertoire of the community. This repertoire was developed during the process of mutual engagement. Each one had its own history, its well-established interpretation, and it could be re-used in new situations.

Another important piece of the repertoire was the shared positive beliefs and high expectations about Latino students. All participants expressed an honest concern for their students. They believed their students' Latino heritage was worthy of respect, and they believed all students were capable of success. This set of beliefs was critical for getting the group together and, even when other meanings were negotiated, they were almost "underlying principles" for the community.

During the joint process of pursuing the enterprise the community creates resources for negotiating meaning. In this community, certain terms, routines, stories, and concepts took on specific usage. Two clear examples were the concept of "student understanding" and the "rate of change test" as an artifact. During the first year, the ambiguity of the term "student understanding" made the process of communication difficult. The teachers were confident their students had complete understanding of the algebraic concepts. For them, being able to complete homework or not asking questions during the class were evidence of understanding. At the same time, they believed covering all the topics from the book would give students an understanding of all the concepts necessary to understand future topics. During the general meetings the concept of understanding as being able to see things related or connected to other things was brought by the researchers and adopted by the community after a process of negotiation.

The rate of change test was one of the artifacts that emerged as part of the engagement of the participants and allowed the convergence and stabilization of meanings. The test was developed based on researchers observation of teachers' classrooms and using teachers' insights about its relevance to their practice. Looking at strategies students used to solve these problems, instead of looking at performance, served as a resource to talk about understanding. Based on the test results from the first year, the teachers wanted to talk about the big ideas and ways to promote student participation in the classroom during the second year. The gap between teachers' expectations and the evidence from the test created a conflict the teachers struggled to resolve.

The community focused its attention and efforts to the development of meanings that mattered to the participants. For the researchers it was important to understand the impact of talking about students' understanding with these teachers. For the teachers it was important to look at their practice from a new perspective and pay attention to students thinking.

IMPLICATIONS

Membership in the community benefited the participant teachers in different and personal ways. From a social perspective, learning is fundamentally experiential and social. It depends on the opportunities to contribute to the practices of the community (engagement), on the connection to the frameworks that determine the social effectiveness of the actions (alignment), and on the processes of orientation, reflection, and exploration of identities and practices (imagination). In this way, the learning in the community presented here took place because of the practices, norms, and images the teachers and researchers constructed together over the two years.

In this group Isabel could be considered as an experienced teacher, while Fernanda and Armando were just passing the critical initial stage of their professional lives, and Gustavo was a new teacher with just one and one-half year of experience at the beginning of the study. On one hand, the two teachers in the extremes, Isabel and Gustavo, were the most engaged participants and benefited the most from the community by taking into their classroom what they learned within the community. On the other hand, the other two teachers, Fernanda and Armando, represent the larger group of traditional teachers who find the ideas of reform difficult and even threatening. Many teachers will listen and participate marginally in the activities with some attempting some changes despite finding the experiences intimidating and risky while others neglect to translate the experiences to their classrooms entirely.

The study contributes to research in professional development in two ways, (1) understanding how a community of practice emerges and (2) validating the relevance of the content on professional development programs. The framework used helped to describe the different stages of the community and the way the group matured over time. Looking at the facilities available for each component during the process of community formation has given a detail description of what activities the participants were engaged in, what goals they negotiated, and what tools they produced during these activities for reaching the goals. The description brings to the front the importance of the content in the development of the community.

The second contribution to the research literature is the relevance of the content in professional development programs. The changes observed from the first year to the second year, show how important is the content of the conversation in a professional development program. Only when student understanding was the real focus of conversations and the community had tools to talk about it, did the teachers engage in more meaningful activities. The focus of the discussion was always highly related with the mathematical content, specifically with student development of algebraic reasoning.

NOTES

1. The research reported herein was supported in part by a grant from the U.S. Department of Education, Office of Educational Research and Improvement, to the NCISLA in Mathematics and Science (R305A60007-98). The opinions expressed herein do not necessarily reflect the position, policy, or endorsement of the supporting agencies.

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LEARNING MATHEMATICS FOR TEACHING: DEVELOPING CONTENT KNOWLEDGE AND PEDAGOGY IN A MATHEMATICS COURSE FOR INTENDING TEACHERS

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Abstract. *This study is situated within the context of a mathematics course for undergraduate students intending to apply to the teacher education program. The course is problem-based and designed to develop students' mathematical knowledge and address students' fears and apprehensions in studying mathematics. Analysis of data collected through student autobiographies, journals, and written feedback indicate that: 1) students developed attitudes toward learning that included an inclination to seek conceptual understanding and alternative ways of understanding and solving problems, and 2) students reflected not only on their understanding of math but also on their roles as future teachers. This study builds on our understanding of mathematics for teaching and how a mathematics content course can provide opportunities for learning content and pedagogy.*

INTRODUCTION.

Over the past decade much has been written about the mathematical preparation of teachers. There are many studies that document teachers' adequate or more often inadequate knowledge of mathematics (Ball, 1990; Borko et. al, 1992; Leinhardt & Smith, 1985; Koirala, 1998; Ma, 1999; Heaton, 2000). Moving beyond a counting of the number or type of mathematics course taken or the kind of degrees obtained as a way of defining mathematical preparation for teaching, current research attempts to understand more about the kind of mathematical knowledge teachers need in order to teach well. Questions of what mathematics content teachers need to know, how they should develop that knowledge, and how their understandings of mathematics relate to teaching practices are not easily answered. Research on professional development provided to practicing teachers which includes opportunities for teachers to examine students' thinking (Franke, Carpenter, Fennema, 2001; Vacc, Bowman, & Bright, 2000), reflect on teaching practices (Schifter, 1996), develop or discuss teaching cases (Barnett & Tyson, 1999), or participate in teacher study groups (Stigler and Heibert, 1999), indicates promise toward enhancing teacher content and pedagogical knowledge. However, we know far less about how we might support beginning or intending teachers' understandings of mathematics for teaching. This paper examines issues around the kind of mathematical content and pedagogical structures needed to help pre-service teachers' develop their knowledge of mathematics. Our study is situated within the context of an undergraduate mathematics content course and we ask how such a course might provide opportunities for learning mathematics for teaching.

THEORETICAL FRAMEWORK

Researchers have developed various frameworks for describing the mathematics needed for teaching well. Ball (1991) suggests that teachers need not only knowledge of mathematical concepts, topics, and procedures but also about the nature and discourse of mathematical inquiry. Using her own teaching practice as a site for research, Ball concludes that teachers' subject-matter knowledge needs to be correct, connected, and meaningful. Ma (1999) extends Ball's work in her analysis of Chinese teachers' development of profound understanding of fundamental mathematics. A "Profound understanding of fundamental mathematics," suggests Ma, "goes beyond being able to compute correctly and to give a rationale for computational algorithms" (p. xxiv). With a profound understanding of mathematics, a teacher, according to Ma, "is not only aware of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics, but is able to teach them to students" (p. xxiv). Ma explains that a profound understanding of mathematics is an understanding that is deep, broad, and thorough. Deep understanding is evident when teachers make connections to topics that have further conceptual power. A broad understanding, states Ma, involves teachers in connecting topics that have similar conceptual power. Thoroughness, as an aspect of profound understanding, involves interweaving both deep and vast understandings into a coherent whole. For example, a teacher who is able to connect the topic of subtraction with regrouping to rates of composing and decomposing (depth), can connect this topic to addition or to subtraction without regrouping (breadth), and can interweave these into a coherent whole (thoroughness) would have profound understanding of fundamental mathematics and be in a position to connect student understanding to the curriculum.

Ma found that Chinese teachers, even with less post-secondary education than American teachers, developed their profound understanding of fundamental mathematics during their years of teaching. Comparing this to American teachers and using data collected in Ball's (1990) research, Ma found that neither experienced nor beginning American teachers tended to develop such profound understanding. The Chinese teachers in Ma's study also displayed various mathematical attitudes. Chinese teachers sought to "know how to carry out an algorithm and to know why it makes sense mathematically" (Ma, 1999, p. 108). Unlike American teachers they shared a disposition to ask why and to explore the mathematical reasoning underlying mathematical procedures. Other basic attitudes of Chinese teachers include the expectation that claims be justified with mathematical arguments and that problems be approached in multiple ways. These basic attitudes play an important role in teaching. A teacher may have conceptual and procedural understanding of mathematics but may not see the importance of conveying that understanding to students. On the other hand, a teacher may come to teaching with an expectation that learning mathematics involves a focus on proficiency with mathematical procedures over a focus on why they work. As a result teachers may not be in a position to hear or attend to the possibilities for learning or teaching displayed in student responses, problems, or solutions. Teachers' attitude, as an aspect of their understanding of the discipline, influences what mathematics they know, how they know it, and how they share their understanding with students.

An important aspect of Ma's research is the intertwining of content and pedagogy. The Chinese teachers in Ma's study developed their profound understanding of mathematics over time during their teaching careers. Teachers learn mathematics through multiple sources: examining curriculum materials while teaching, working with colleagues, learning mathematics from students, and doing mathematics themselves. Learning mathematics for teaching involves an interplay of learning mathematics content and pedagogy. One does not necessarily precede the other. Teaching practices provide opportunities for learning mathematics. Likewise undergraduate and teacher education programs are contexts for learning both content and pedagogy. However, what teachers take from their undergraduate mathematics coursework is not often of use to them in classroom practice. How we might offer mathematics courses to intending teachers, so that teachers are in a better position to develop the kind of understanding and attitudes Ma speaks of, is an important question.

RESEARCH CONTEXT AND DESIGN

The Course

The context for this study is an undergraduate mathematics course taught through the University of British Columbia's mathematics department. A condition for admittance into the university's elementary teacher education program is the successful completion of any undergraduate mathematics course. One particular course was designed by the math department to meet the unique needs of intending elementary teachers. Taught differently from most undergraduate math courses in the department, this course focuses on developing students' skill and confidence in doing mathematics and engaging in mathematical inquiry. The course is problem based; lecturing is replaced by cooperative work in small groups and whole class discussions, and reflective math journal writing is expected. The content is connected to the school curriculum and is composed of three sections: geometry, arithmetic, and combinatorics (with probability). These sections are organized around themes or conceptual anchors such as the Pythagorean theorem and scaling, exponential growth, and binomial probabilities. Moving within the themes and sections the historical development of mathematical ideas can be addressed.

Participants

Forty-six of 54 students enrolled in the mathematics content course during the summer of 2001 participated in this study. The class met everyday for two hours over a six-week period. All students enrolled in the course did so as a requirement for entry into the elementary teacher education program. Students were either nearing the completion of their undergraduate degrees or had extensive work experience and were considering a career change. Thus, the demographic blend of the students varied; the youngest were in their early 20's and the oldest in their mid 50's. Participants therefore came with varied educational backgrounds and work experiences, as well as different expectations and visions about the teaching profession. There were 11 males and 43 females in the class.

Data Collection and Analysis

For the purposes of this paper we draw upon data collected through the teaching of this mathematics course. Zahra Gooya was the instructor for the course. Data sources include: participants' autobiographies collected during the first day of the class; participants' math journal entries written throughout the course; students' written evaluations of the course collected at the end of the course; participants' responses to a course reflective feedback survey collected at the end of the course; and the instructors' reflective field notes. Student autobiographies, student daily journal entries, and their responses to the course were analyzed for common issues and themes that were raised by students in terms of their mathematical backgrounds, their initial attitudes toward mathematics, learning and teaching mathematics, and how these attitudes developed or changed by the end of the course. Using Ma's (1999) description and framing of basic attitudes for teaching mathematics our analysis involved both direct interpretation of individual student responses across the course and the aggregation of particular instances (Case, 1995).

RESULTS AND DISCUSSION

Changes in Attitudes

Student autobiographies and journal entries written at the beginning of the course indicate that most students were apprehensive about taking a mathematics course and successfully completing the course as a requirement for application to the teacher education program. More than three-quarters of the students in the class expressed negative feelings toward mathematics, writing that they were "terrified", "intimidated", "anxious" and/or "stressed" about studying it. They described their previous experiences learning mathematics as "difficult", "overwhelming" and "frustrating." Although a few students mentioned they liked the predictable and sequenced nature of solving problems, many wrote about their loss of interest in mathematics as it began to make less sense or became less meaningful to them over their years of school study. This student's comments are representative of others: "I actually liked math as a kid. It was always challenging and I liked that challenge. But

I gradually liked it less as I was not able to apply the higher level math to practical life". With apprehension in successfully completing a university level mathematics course many students expressed a desire for the course to be taught in a familiar and traditional manner. As one student wrote, "we need lecturing, solving examples, doing lots of drill and practice, and telling us what and how to do it."

The course, however, was not taught in this traditional manner. Students worked in small groups on rich problems [1]. The instructor emphasized conceptual understanding through problem solving and encouraged students to communicate their thinking orally to their peers in whole class discussion and in writing in their journals. As the course progressed, the content of students' journal entries and reflections of their work with mathematics changed. Students' journal entries and their course evaluations written toward the end of the course indicate that many students not only developed a conceptual understanding of the math content taught but also some basic attitudes they felt necessary for learning math. Many students wrote that "the course encouraged me to be more curious about math and greatly diminished my anxiety towards the subject." In addition to changes in feelings toward mathematics, students wrote about the importance of explaining mathematical ideas, about seeking understanding rather than memorizing, and about searching for different ways of solving problems. For example, this student's comments are representative of others: "I feel this course did an excellent job of easing me back into math. It made me realize that it is not sufficient just to get the answer, it is necessary to know where and how it came to be, and how the formulas are created." Other students wrote expressions similar to this student's: "I've developed not only a clearer understanding of the basic concepts of mathematics, but also an appreciation for the many different methods there are to explain one concept." Still others, such as this student, wrote about their participation in creating mathematics, "I really enjoyed learning how to make formulas and proofs, instead of being given them."

These comments indicate how students' attitudes developed throughout the course. However, such a change did not occur for all students. Eight students of the 54 in the class wrote about how the course did not meet their needs. Throughout the course during class discussions and through journal writing, these students expressed a preference for a more traditional teacher-centred approach to teaching and were angry when this approach was not adopted. Although the course instructor offered opportunities for individual instruction outside of class time to these and all students, few of these dissatisfied students accepted this as an opportunity to meet their needs.

Learning Teaching While Learning Mathematics

Although this course was a content course taught to students who were intending to but not yet enrolled in a teacher education program, students made explicit connections between their learning mathematics and their roles as future teachers. Analysis of students' journal entries and reflective course feedback indicate that students were learning about pedagogy while they were learning about content. Many, such as this student, wrote about the importance of developing conceptual understanding of mathematics, "It isn't sufficient just to get the answer, it is necessary to know where and how it came to be, ... and how the formulas are created." They reflected on how they might engage students in learning mathematics. This student's comments are representative of others when she wrote, "I've learned to work from where students are in terms of their level of learning, learned to ask what they know and how they understand what they know." A focus on teaching strategies was not an explicit focus of the course, yet students in this course drew upon their experiences as students in the class and upon their analysis of the instruction to inform their ideas about teaching. Their comments focus on both themselves as teachers (e.g. "I understand now that we as future teachers need to understand the underlying principles of math and also how and why, ... this course is preparing us"), and on their future students (e.g. "Children will ask 'why, why is math important, why do we do that, why is it important to learn math?' and we have to be prepared to respond to that"). Using their own experiences in the course as a place of reflection, the course provided beginning opportunities for students to learn content and pedagogy.

CONCLUSION

The results of this study point to the possibilities offered in mathematics content courses for developing productive attitudes and dispositions toward learning and teaching and for developing understandings of content and pedagogy. Studies of experienced and prospective teachers' subject-matter knowledge indicate that teachers require a rich and connected understanding of the mathematics they will be teaching in order to teach well. Ma's (1999) research indicates that teachers can develop their knowledge of mathematics for teaching over their careers of teaching and that these teachers display attitudes which include an inclination to pursue a conceptual understanding of a concept, to seek alternative solutions to problems, and require mathematical reasoning to justify claims. Most students in our study, as a result of their experiences in the course, developed the desire to pursue an understanding of mathematical concepts. They also wrote about the excitement of seeing and the challenge of understanding alternative solutions to problems. Although students wrote about these attitudes and their own changes in attitudes, how students use these in their roles as beginning teachers is a question for further research.

It must be stated that not all students wrote about a desire to learn and understand the conceptual underpinnings of various mathematical principles. These students sought a more traditional form of instruction, one that provided sample problems, clear steps to follow, and problems that allowed students to practice the application of the procedures. These students, eight in total, did not feel as if the course were designed around their learning needs. This raises questions around how we might provide instruction that meets the needs of all our students.

For a mathematics course to offer opportunities for students to learn mathematics and pedagogy it requires a structure and content unlike typical mathematics courses offered to undergraduate students. Two issues need to be considered, one issue focuses on the mathematics content and the other on the nature of teaching a mathematics course. Students in the course spoke about the need for the content they were learning to be connected to the content they will be teaching. Students who wrote and reflected on the course content, mentioned how they felt they could now engage their own students in a similar investigation of mathematics. Yet, the content for this course was only loosely structured around the elementary school math topics of arithmetic, geometry, and probability, and more closely fit the secondary school curriculum. Interestingly, those students who were dissatisfied with the course spoke of the need for the content to the same as that which elementary students would be learning. This raises an important point as to what mathematics intending teachers should study. Our study indicates that content which is aligned with school mathematics, is familiar to students, but is not necessarily the same as it can be a context for learning mathematics and engaging mathematical inquiry. A second issue focuses on pedagogy. Students need opportunities to learn mathematics with their peers, they need a chance to communicate their thinking to each other and to the instructor, they need to experience mathematical inquiry around rich problems, and their fears about studying mathematics need to be addressed. It is the modelling of good teaching that provides students with possibilities for how students might experience mathematics and how they might consider teaching it. Developing mathematics courses where students can develop their subject-matter knowledge and pedagogical knowledge requires the co-operative efforts of university mathematics and education departments, of mathematicians and mathematics educators.

This study shows that a mathematics content course can have a profound influence on students' learning in general, on their attitudes toward mathematics, in particular, and on their developing ideas for teaching. The study challenges the traditional separation of content and pedagogy, in which content is typically taught through a mathematics course and pedagogy through a methods course. Current studies suggest that methods courses can be places to learn mathematics (e.g. Tirosh, 2000), our study suggests that a content course can also be a place for learning about teaching. This study emphasizes the interconnectedness of content and pedagogy and points to the possibilities well-designed content courses can offer in helping develop teachers' mathematical knowledge, attitudes, and beginning ideas about pedagogy.

NOTES

1. An example of a problem posed to students early in the course involved squares and their roots. Moving historically from Meno to Pythagoras and one of the first recorded math lessons where Socrates helps a boy

discover how to double a square geometrically, students in the class are asked to generalize this method to find ways of producing squares equal (in area) to 5, 10, 13, 17 times a given original. In terms of side-length, this means geometrically constructing certain square roots. For example, re-arranging the area left over when four (equal) corner triangles are cut from big square. This area can form a single mid-size square or two smaller ones. This leads to the theorem of Pythagoras – merging two squares area-wise into a single one.

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DEVELOPING CONCEPTUAL UNDERSTANDING: THE ROLE OF THE TASK IN COMMUNITIES OF MATHEMATICAL INQUIRY

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As part of a study of mathematics classrooms functioning as communities of inquiry, video and other data were collected in ten randomly selected Victorian grade 3 and 4 classrooms and two classes at the Japanese School of Melbourne. In this paper, two lessons — one Victorian and one Japanese — are analysed in terms of the conceptual focus and cognitive demands of the instructional tasks and the opportunities these afford for advancing students' conceptual understanding. The stark contrast between the two lessons suggests that, in Australia at least, insufficient attention is being paid to the critical role of the development of conceptually focussed, robust tasks which can be used to support the development of sophisticated mathematical thinking.

INTRODUCTION

> A community of inquiry is not just ... a certain kind of social, interpersonal or ethical environment. It is an environment in which inquiry occurs, by which is meant, among other things, a focus on key concepts and complex thinking. (Splitter, 2000, p. 14)

The *Mathematics classrooms functioning as communities of inquiry: Models of primary practice*[1] project was based on the notion of communities of inquiry, which underpins the *Philosophy for Children* movement (see, for example, Splitter & Sharp, 1995; Splitter, 2000). One of Splitter and Sharp's (1995) conditions for dialogue, which is the cornerstone of a community of inquiry, is that "conversation is structured by being focused on a topic or question which is problematic or contestable" (p. 34). Thus, in a community of inquiry, participants are engaged in confronting problematic situations and participating in dialogue and argumentation (in the sense of Krummheuer, 1995).

Classroom discourse and the sociomathematical norms associated with achieving quality dialogue have received considerable attention (see, for example, Yackel & Cobb, 1996; Kazemi, 1998; Groves & Doig, 1998). However, frameworks for effective teaching to support children's conceptual understanding also emphasise the need for tasks which are mathematically challenging and significant (Askew, Brown, Denvir & Rhodes, 2000; Fraivillig, 2001). In particular, the critical role of the design and enactment of instructional tasks in the development of students' increasingly

[1] *Mathematics classrooms functioning as communities of inquiry: Models of primary practice* was funded by the Australian Research Council. The Chief Investigators were Susie Groves (Deakin University), Brian Doig and Laurance Splitter (Australian Council for Educational Research).

sophisticated mathematical reasoning and understanding is a key component of much of work related to mathematical communities of inquiry (see, for example, Cobb, Yackel & Wood, 1995; Gravemeijer, McClain & Stephan, 1998).

The purpose of our project was to examine current models of Australian mathematics practice in order to investigate the extent to which these support or hinder mathematics classrooms functioning as communities of inquiry, and to determine the extent to which the wider local education community endorse the goal of mathematics classrooms functioning as communities of inquiry. Analysis of focus group responses from principals, teachers and mathematics educators showed a high level of support for mathematics classrooms functioning as communities of inquiry, together with a realisation that current Australian practice falls far short of this goal (Groves, Doig & Splitter, 2000). Principals and mathematics educators rated the cognitive demands of typical lessons as low to very low and not challenging children, while teachers saw the cognitive demand as being determined by the tasks (Doig, Groves & Splitter, 2001).

This paper compares two videotaped lessons from the project in terms of the conceptual focus and cognitive demands of the instructional tasks and the opportunities afforded for advancing students' conceptual understanding.

METHODOLOGY

One mathematics lesson of approximately one hour's duration was videotaped in a stratified random sample of ten year 3 and 4 classrooms in the state of Victoria. A lesson observation schedule was also used to record detailed field-notes of each lesson. Teachers' lesson plans, as well as copies of any work-sheets used by the children, were also collected. In addition, each teacher was asked to complete a written questionnaire providing background information, information related to the specific lesson (e.g. the main topic of the lesson, the aims, resources used, the purpose of any group work and whole class discussion, and the most important part for children's learning), together with information on their mathematics teaching in general. In keeping with Hiebert and Stigler's (2000) view that one way to imagine alternative models is to "step outside the ... culture and look at how others handle similar issues" and that "Japan provides an eye-opening contrast" (p. 10), similar data were collected from the year 3 and the year 4 classes at the Japanese School of Melbourne.

An analysis of the videotapes was carried out, using a framework based on that developed by Schmidt *et al* (1996), who use the term "characteristic pedagogical flow" to describe recurrent patterns of observable characteristics in a set of lessons. Each lesson was analysed in terms of its structure, organisation, interactions, cognitive demand, and teacher actions. Based on this analysis and the written questionnaires, data from two lessons — one Victorian and one Japanese — were selected for a more detailed re-analysis in terms of the conceptual focus and demands of the tasks and the opportunities afforded for advancing students' conceptual understanding. These lessons are described and discussed below.

THE CHANCE AND DATA LESSON

This lesson, which occurred in a grade 4 class of 30 children, in a middle class suburb of Melbourne, focussed on aspects of Chance and Data and lasted for about one hour. Children were involved in three different tasks: using sales catalogue prices to spend $1 million while satisfying certain constraints; tossing three coins to investigate the outcomes; and estimating the amount of money in a jar of coins.

In her responses to the written questionnaire and the lesson plan provided, the teacher, Ms E, identified the main mathematics topic of the lesson as being "The chance of winning one million dollars" and listed the above three tasks.

During the lesson, children worked in two groups — one for each of the first two tasks — with the money estimation happening simultaneously as children took the jar of coins around the room and collected estimates. Ms E stated that the two groups would swap tasks in the following lesson.

For reasons of space and the emphasis in this paper on conceptual focus, only the coin tossing task will be discussed here. Ms E said that the aims of the lesson were stated in her lesson plan, where she described the coin tossing task as follows:

> Three coins to be tossed in the air at the same time. What are the possible outcomes? (These are to be listed.) Look at the pattern after 5 throws. BEFORE the 6th throw predict. RECORD what you think the outcome will be. Toss. Record. Repeat activity — compare results. What do you conclude?

This outline closely follows an activity from the *Mathematics: Course advice* (Directorate of School Education, 1996) on which Ms E based her lesson. The Course Advice version includes questions such as: "Did anyone predict the correct answer for any of the throws?" and "Is it possible that all coins could land on the same side?" It also suggests that results should be combined with those of another class leading to "Which results were most common?" and "How do these compare with predictions?"

Ms E identified the outcomes from this activity as being ones related to Level 3 of the *Curriculum and Standards Framework (CSF)* (Board of Studies, 1995), namely that children will be able to record and identify all possible outcomes arising from simple chance experiments; identify outcomes as being equally likely; and make predictions. (In fact, the Level 3 *CSF* learning outcomes state "identify *some* outcomes as being equally likely" [emphasis added] and the only mention of predictions is at Level 4 "make simple predictive statements about everyday events ...", which suggests a quite different context from the one in this lesson.)

In the lesson, children were told to use any three plastic coins from a collection of coins of different denominations and devise their own method of recording. Some children used three coins of the same denomination and others two or three different denominations. Children recorded their throws typically as either HTH or 2H 1T, but it was never clear whether or not HTH was considered different from HHT and there was no discussion of this at any stage.

While the children worked on their tasks, Ms E moved from group to group, discussing progress and generally praising children's efforts. Typical comments from Ms E include:

> What do you think the next [toss] will be? Have a guess. ... Have you got your estimate?
>
> Compare your results with those of ... same or totally different? [Two girls carefully compare their results line by line, with obvious enjoyment but without any comment or follow-up.]

After 17 minutes of coin tossing, Ms E called the group together to ask what they had found out. Children's comments included: "More heads than tails", "Tails, heads, tails the most — five times", "Three heads most", "Normally get two tails and a head". The discussion lasted a total of five minutes and continued as follows:

Ms E: Can you tell me how many different combinations you can have and what they are?

John: Five.

Ms E: Who agrees? [Three children raise hands.] What are they?

John: TTH, THH, TTT, HHH, [inaudible].

Ms E: Oh right. Has anybody got something to say about that?

Mary: I had tails.

Ms E: What is your most common combination? ... And how many did you have altogether Neil?

Neil: Two.

Ms E: Two! ... You just had two for all those throws!

Sally: I got the same.

Jim: I had five.

Ms E: And John had five. Did anyone else have five? [No hands.] Who had four combinations? How many did you have Helen?

Helen: Two.

Ms E: Two! Oh goodness! ... [You had] quite a few throws and still had only two different combinations every time, Goodness gracious! I think you can put these in your maths folders now.

According to Ms E's questionnaire responses, the most important part of the lesson from the perspective of children's learning was:

> Being aware of the price of homes, cars and holidays. The exchange of views between individuals. Thinking it would be easy to spend one million and finding the opposite.

A number of things are clear from this lesson. Firstly, the coin tossing task is severely flawed — both as recommended and as implemented. Is it intended that there are eight possible (equally likely) outcomes (i.e. are the coins intended to be treated as different) or only four? Secondly, what concepts are intended to be developed and what possible place does the prediction (far less "estimation") of outcomes have in such a development? Thirdly, there clearly needs to be some agreement about the possible outcomes and how these are recorded before any comparison can be made of the results of the experiment. But perhaps most importantly, the discussion at the end of the lesson segment proceeded no further than eliciting results, with questions all being of the form "Who got this?", "Who agrees?" or "Who got something different?". Unfortunately in our experience this type of discussion is not uncommon. What is clearly lacking in this lesson is any serious consideration of the mathematical concepts that might be developed through the use of the activity, the types of responses which children might give and how the discussion could be used to focus on the conceptual aspects of the task.

THE CIRCLE LESSON

This lesson occurred in a grade 3 class of 8 children at the Japanese School of Melbourne. The school teaches the Japanese curriculum, in Japanese, to children whose parents are in Australia for periods of one or two years, with teachers usually coming to Australia for three years. An interpreter was present throughout the lesson and assisted with completion of the written questionnaire. According to the teacher, Mr J, the main mathematics topic for the lesson was "the concept of a circle".

The lesson began with Mr J producing a pole for a game of quoits. Mr J placed the pole in the centre of an open space in the classroom and asked three children (referred to here as B1, G1 and B2) to stand at three marked places on a red line along one side of the room (see Figure 1). Children expressed concern that the game would be unfair, but mainly focussed on the distance between children on the red line. After a discussion on how to measure distances and children using a metre ruler to measure children's distances from the pole, Mr J put out pre-cut coloured strips of paper, as shown in Figure 1, and held them up to show that the distances were different. He then defined the problem as being: "How can we make the game fair?" It is only after five more minutes, during which children continued to try to find points to stand on the red line, that B2 came up with a way that two people can be the same distance from the pole — he moved the yellow strip so that it went from the pole to the point B2* on Figure 1. Mr J then gave all the

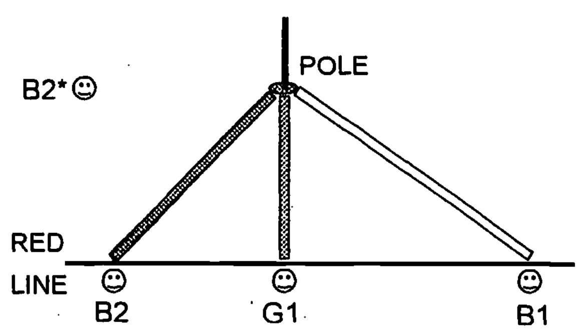

Figure 1: B2's solution for making the quoits game fair

children a yellow strip and asked them to "think for themselves" and find somewhere to stand so that everyone was the same distance from the pole. Children were excited that now would be at a disadvantage. This segment took 20 minutes in the 45 minute lesson.

Mr J then reproduced the situation on a large sheet of paper on the board. He stuck a miniature pole on the paper and asks children to use sticky yellow paper strips and dots to represent their positions (see Figure 2).

Mr J: Look at the different positions — what do you notice?

G2: It's like a round circle [makes circle shape with hands].

G3: No — it's like a flower.

G1: If you follow the end of each yellow strip it will become a circle [traces large circle on the desk with her finger].

Mr J: What if every student in the school took part? [adds more strips] ...

B3: If there are many students standing round, maybe it's a circle.

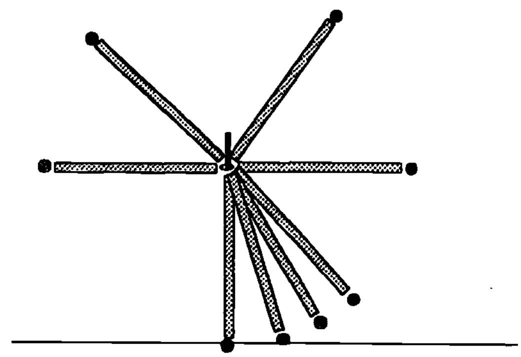

Figure 2: Paper representation of children's solution

Mr J removed the pole and put another sheet of paper over the first with a circle drawn where the dots were and asked "How many yellow points would we need? 100? 1 000 000?" He then put the word circle on the paper and elicited names for the centre, radius and diameter from the children.

The remaining 15 minutes of the lesson were taken up with the children working in pairs drawing circles. Initially many children chose to use a compass, even though Mr J told them that they had not yet learnt how to use one and encouraged one girl, who said that she could use a yellow strip of paper or a plastic circle to trace round, to show him how. After about 7 minutes, Mr J asked children to find a way to draw a circle without a compass. A few minutes later Mr J said "Now everyone is tracing — is there another way?" Children tried various ways, while Mr J pivoted one of the yellow strips of paper around an end held by his finger. B2 excitedly cried out that he could do it and demonstrated drawing a circle by holding the middle of one end of his pencil case and tracing a circle with his finger in the hole at the other end. Children applauded and Mr J demonstrated B2's method at the front of the class. The lesson finished with a few minutes of suggestions from children how to fix one end, culminating in the use of a drawing pin. Mr J summed up by saying: "As you suggested, there are other ways of drawing a circle than just using a compass".

In contrast to Ms E in the chance and data lesson, Mr J highlighted the conceptual aspects, both in the lesson and in his responses to the questionnaire. He stated that his aims were that "children have the *concept* of a circle and find *real* circular objects" [emphasis in original]. According to Mr J, the most important aspect of the lesson in terms of children's learning was that children understand that the circle is a locus. The purpose of working in groups (in this case pairs) was "to facilitate discussions while working", while the purpose of the whole class discussion was for children to "share ideas and strategies for solutions [demonstrating that] there are many different ways of thinking which reach the same conclusions". Mr J further described his mathematics lessons as follows:

> Introductory lessons [to a topic] use materials. So this was typical. The introduction is very important and takes a lot of time. After that there is much practice, then we go to calculations — a series of 3 or 4 lessons [per topic].

Mr J concluded his questionnaire with the comment that "Mathematics should be part of children's daily lives". In the 20 minute quoit activity, Mr J embedded the concept of a circle in a rich, intriguing, intrinsically motivating, problematic framework, by asking: "How can we make the game fair?"

CONCLUSION

The two lessons above provide a stark contrast in terms of the potential offered by the tasks on which they were based to advance children's conceptual understanding.

Kazemi (1998) argues that in his two classroom examples, where teachers use the same tasks and establish similar social norms, high press for conceptual thinking was created by the establishment of appropriate sociomathematical norms. While we would not wish to disagree with this, we contend that, in general, insufficient attention is being paid to the critical role of the development of conceptually focussed, robust tasks which can be used to support the development of sophisticated mathematical thinking. Moreover, teachers themselves need to engage in the detailed planning of how such tasks can be implemented in their lessons and this requires a deep engagement with the concepts involved, as well as an anticipation of likely student responses. The focus for teachers needs to be on all three of the key components of communities of inquiry: the social norms, the sociomathematical norms, and the conceptually complex, intriguing and problematic tasks.

While Mr J attributed the source for his lesson as being himself and the textbook, we believe it is no accident that while at ICME9 in Japan, we saw photos and descriptions of a number of similar lessons on the concept of a circle at a display by a local teachers' association. We concur with Hiebert and Stigler's (2000) view that the Japanese process of lesson study, where groups of teachers engage in an iterative process of refining a small number of lessons, warrants consideration as a procedure for reform of teaching. As they argue (p. 16), "Improving teaching does not depend on eventually perfecting 182 lessons but rather on engaging intensively with the issues involved in teaching any lesson".

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EXPLORATION OF AN ART VENUE FOR THE LEARNING OF MATHEMATICS: THE CASE OF SYMMETRY

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This study examines an art venue for abstract mathematics thinking, specifically, students' understanding of symmetry groups in the context of a course for art students. The findings suggest that when working with repetitious geometric art designs students' approach of various symmetries often go beyond the one-to-one correspondence between specific pieces of art and particular transformations, but they also develop a correspondence between types of art designs and abstract types of symmetries. Further, familiarity with symmetry groups allows students to use abstract mathematical thinking as a tool in their artistic creations.

The idea of viewing mathematics through an art lens can be appealing at many levels. People who find joy in doing mathematics advocate the natural connections between mathematics and art (e.g., Huntley, 1970; Kappraff, 1991; Linn, 1974). Convincing examples are provided by the complexity of Islamic mosaics in the Alhambra, the Greek canons of harmony and beauty, and the magnificence of many architectural designs. Renaissance artists like Leonardo da Vinci, Michaelangelo, Durer, and Piero della Francesca were not only master painters, but skilled geometers as well. In later years, mathematicians such as Kepler, Euler, Gauss and Fermat, studied problems related to art, including tilings, and 3-D solids. More contemporary designs, for example, the works of M C. Escher and A. Fomenco, are coming from the perspective of artists drawing inspiration from complex mathematics rules, and mathematicians who use art as a means to express certain mathematical abstractions, respectively, and they illustrate the strong bond between mathematics and art. An exploration of the natural connection of math and art is now being proposed in the service of education. The idea of "learning mathematics by making art" comes to advance our thinking on mathematics and on pedagogy. Mathematics educators suggest that the integration of arts with mathematics may liven our classes and may also be motivating to some students for whom the "traditional" curriculum and pedagogy has not proven fruitful. This vision is also presented by the National Council of Teachers of Mathematics (2000) who calls for the introduction of extended projects, group work, and discussions among students, as well as the integration of mathematics across the curriculum – elements of a learning environment that seems more similar to a studio art course rather than to a traditional mathematics class.

In recent years, there have been a few examples of interventions that demonstrate the power of the vision that integrates art and mathematics. Yet, little of this work has been done at advanced levels; the bulk of the research in this area has been conducted in elementary and secondary school classrooms. For example, in a quantitative study at the elementary school level, Willett (1992) demonstrated that mathematics learning can be very effective in the context of arts-based lessons. Similarly, Loeb's visual mathematics curriculum (Loeb, 1993), though it has not been studied formally, provides substantial anecdotal evidence which supports the study of the formal mathematics of symmetry through a design studio as an effective learning environment for undergraduate students. The "Escher World" project (Shaffer, 1997) explored successfully an open learning environment created by combining mathematics and design activities.

Here, we explore one more example of a learning environment created by combining mathematics and art activities in an art studio-like environment. We designed a course for students interested in arts and design, with the emphasis on understanding mathematical concepts by self-explorations in art (Grzegorczyk, 2000). Instead of introducing theoretical concepts, the course created a vast reservoir of art-related examples and hands-on experiences. The course contained a wide variety of accessible yet challenging problems, and served as an introduction to systematic and complex mathematical thinking, experiences and discovery used in contemporary and classic arts. One of the main themes in the course was symmetry in mathematics.

The focus of this article is to examine one specific aspect of this art venue for mathematics learning, specifically to document students' understanding of symmetry groups. While our broader goal is to study the process by which mathematics learning takes place in environments where it is approached through art, and to further develop a model for this type of teaching, here we focus on the learning of symmetry and the implications that this may have on the overall mathematics learning and attitudes of students towards mathematics. Specifically, our goal was to examine the ways, strategies, and approaches college students use art as a gateway in their attempts to develop an understanding of design classification and symmetry (through crystallographic groups).

METHOD

Participants and course activities: Twenty-five undergraduate students who major in art studies participated in a one-semester mathematics course. Students were asked to work on short problem investigations, or extended projects with increasing sophistication and complexity. These included either the study of existing artwork to develop mathematical insights, or the use of mathematical ideas to develop original art and design works. Students worked on these projects and then presented their work to their peers for discussion, questions, comments. The class was equipped with commercial and shareware drawing and image-manipulation programs as well as geometry and symmetry software.

Mathematics of the course: Central to this course was the development of the concept of various types of two and three-dimensional symmetries, a topic that is significant both in art and in nature. This was motivated by discussions on designs produced by individual artists and traditional designs from various cultures around the world. Students entered the course with a vague understanding of the concept of symmetry, one that was often disconnected from mathematics. As their discussions on works of art and the complexity of their designs progressed, they were encouraged to use precise language and to clarify the term "symmetric." It became apparent that the class would benefit from a systematic and deeper study of this broad topic. Students were invited to study symmetries as *rigid motions of the plane*. They identified the symmetry motions of small (finite) designs using polygons as a main example, i.e. they studied various types of rotations and reflections, and their compositions – a result of one motion followed by another. The classification of designs with various symmetries and the development of symmetry group tables, led to the underlying definition of the *dihedral groups* and *cyclic groups*, and, finally, to the definition of an *abstract group*. Students verified the associative, identity, and inverse properties of the transformations for various symmetry groups, and re-visited the art designs that initially motivated the study of group symmetry and discussed how their new understanding of group theory may assist them in gaining new insights in art. They generated their own designs using symmetric transformations, and discussed the utility of group theory with respect to the generation of repetitious designs.

The symmetries of strip patterns were analyzed next to discover that all seven symmetry groups for repetitious (periodic) strip patterns. The natural next step was to study plane patterns such as wallpaper and quilting patterns, that is, designs where the repetition of the theme occurs in more than one direction. It was noted that there are exactly 17 plane symmetry groups, which correspond to 17 types of patterns and could be analyzed or generated using various symmetries. The sophistication of the students and complexity of the patterns studied was increasing. The notion of the fundamental region for a pattern lead to the study of tilings – when is it possible to cover the plane with a single tile or combination of tiles. The most intriguing and interesting tilings of Escher especially lent itself for symmetry investigations on tiling. Finally, the study of symmetric groups was extended to three-dimensional solids (mostly Platonic solids) and their tiling.

Data collection and analysis: For this study we collected students' sketches and designs from the workshops for review and analysis. Further, students were administered a pre- and a post-test that focused explicitly on their understanding of various symmetries and were asked to complete a survey which focused on the students' attitudes towards mathematics and the class.

Responses to each problem were first coded for mathematical correctness. When responses included images, then these were coded for the presence of mathematical principles and their correctness. When responses included verbal descriptions, these, too, were coded for the extent to which mathematical ideas and concepts were used and, subsequently, for the correct use of mathematics. In this case, we differentiated between explicit, formal use of mathematics concepts and language and general, or vague references to mathematics. We also differentiated between different strategies that students employed when they approached tasks or designs that involved the use of symmetry in various forms. Finally, when coding was complete, frequencies were tallied for each code and pre- and post-interview totals were compared.

RESULTS

Recognizing and using symmetry. Our first coding scheme concerned the overall recognition of the existence of symmetry in designs and students' subsequent attempts (or lack thereof) to utilize symmetry principles. To code students' understanding of symmetry we used the criteria suggested by Gardner, i.e. the "ability to use ideas in appropriate contexts, to apply ideas to new situations, to explain ideas, and to extend ideas by finding new examples" (Gardner, 1993; Shaffer, 1997). Using this definition, our results indicate that all students were able to make designs using mirror symmetries and rotational symmetries (Table 1). In fact, students reported that they regularly used symmetry in their designs, both in and out of the course. By contrast, early in the of the course (pre-test), when asked to make a design which incorporates mathematical ideas, only 2 students used symmetry. The remaining 23 students used, or made references to mostly measurement concepts (dividing areas in parts, measuring) and, in a few cases, attempted to construct polygons.

Students also developed an ability to apply the concept of symmetry to their analysis and discussions of various designs and art works. Before the course, students made references to symmetry an average of 0.36 times (total of 9 references made by 25 students) while describing two images (see Figure 1). Note that all references to symmetry were made with respect to mirror symmetry only; no mention was made to either translations, rotations or glides. As the course progressed, all students mentioned symmetry, and the mean rose to 3.16 references over the same 2 images (79 references). These included references to mirror reflections (59 references) and rotational symmetries (16 references), as well as to the lack of symmetries (see Table 1). But, perhaps, more striking is the qualitative change in students' responses to the designs they were discussing. Students used a richer, more formal and mathematical vocabulary to describe images. In fact, 4 students made direct references to symmetry groups (i.e., dihedral and/or cyclic groups). Indicative of this qualitative change in students' discussions are the comments of one student, Julie, whose pre- and post-course discussion of the designs appears below:

> Pre-test comments: "You can fold this design either up or down in the middle and each side is the same, like a mirror. There's lots of symmetry."
>
> Post-test comments: "This has 2 symmetry lines; they divide it in 4 regions. There are four pieces in the square. When you divide it in half you can rotate [each piece] by 180^0. Or each piece, each fourth, can be rotated 90^0. It's like a D_{11} group [Dihedral 2]. All dihedral groups have both symmetry lines and rotations. The other one has 11 regions. No symmetries. It's a cyclic group. C_{11}. All cyclic groups just have rotations. No mirror symmetries…"(note that Julie used the word "symmetry" to denote mirror symmetry)

Julie's first mention of symmetry was made in general terms – "there's lots of symmetry". At the end of the semester though, Julie made specific references to both mirror and rotational symmetries and connected these to a more formal understanding of symmetry using group theory terms.

Symmetry use	Pre-test	Post-test
Use of symmetry in doing art		
- mirror reflections	2	25
- rotations	--	25
Use of symmetry in analyzing art		
- reference to mirror symmetry	9	59
- reference to rotation symmetry	--	16
- reference to formal symmetry concepts	--	4

Table 1: Frequency in symmetry use

Approaching Symmetry via Art	Approaching Art via Symmetry
(a) Use of a local, one-to-one correspondence between one specific piece of art and an instance of a mathematical concept - recalling design examples when asked to discuss the mathematics - e.g., "Cyclic group C_4 is like *that* four-leaf rosette"	*(a) Use of local mathematical concepts related to symmetry to understand art designs* - recalling a collection of symmetry concepts when trying to understand the structure of an art design, or to create one. - e.g., Escher in *Pegasus* made use of glides only, so the entire print looks like it is made of strips
(b) Use of a global correspondence among types of objects - connecting broader math concepts to broad art concepts - e.g., "cyclic groups are like rosettes"	*(b) Use of global, abstract concepts related to symmetry to understand and create art* - deriving specific facts and concepts from abstract symmetry concepts to aid in art creation.

Table 2: Types of symmetry use

Task 1: Discuss the mathematics of the two designs shown below.

Task 2: Fill in the circle with a design and explain the mathematics of your design.

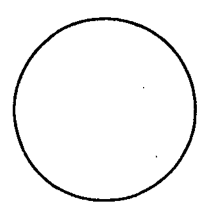

Figure 1: Two of the pre- and post-test tasks: Analyzing and creating symmetric designs

Towards a formal understanding of symmetry. All students developed their understanding of symmetry groups in the sense of recognition of symmetry in designs and in their ability to use symmetry to generate their own designs. Our coding suggested that students approached symmetry in two distinct ways: A number of students approached the mathematics of symmetry using art. Others, however, gave clear indications that, after a certain point, they used abstract mathematical objects as a gateway to art.

The former approach, math-through-art, did not surprise us; it was a clear expectation that students in this class engage in mathematical thinking and problem solving at a certain level, and art could be used as a tangible tool to help explain or apply the abstract concepts of mathematics. Since all of our students were well-versed in the language of art, we helped them build bridges to mathematics via art, so that underlying mathematics also becomes tangible and real. For example, students were expected to understand that cyclic groups have the characteristics of an asymmetric rosette – no mirror symmetries, but with a rotation symmetry, while the angle of the smallest rotation determines the type of the underlying cyclic group. This math-through art approach was manifested in two ways:

(a) recalling design examples when asked to discuss the mathematics (e.g., "Cyclic group C_4 is like *that* four-leaf rosette") – a local one-to-one correspondence between a specific piece of art and an instance of a mathematical concept, and

(b) connecting broader mathematical concepts to broad art concepts – a global correspondence among types of objects in art and in mathematics (e.g., "cyclic groups are like rosettes").

The latter approach, however, that is, using abstract mathematical concepts to understand art, deserves further attention. In this case, students used their understanding of a cyclic group to convince themselves that, once a design is identified as equivalent to a cyclic group, there is no need to look any further for mirror symmetries – they do not exist. Students were able to use *abstract* mathematical thinking in generating their own designs and to make *short cuts* in designing repetitious patterns using certain group properties. The art-through-math approach was also manifested in either a local or global perspective:

(a) Use of a collection of mathematical concepts related to symmetry to understand and create art designs – a local use of mathematics as a way to approach art. For example, a student may look at a given art design and look for the existence of mirror reflections, translations or rotations and attempt to understand the overall "behavior" of the art work, or attempt to reproduce the design using, partially, these symmetry principles. We often observed this behavior when students were exposed to the work of M. C. Escher – students examined the tessellation designs by looking for symmetries and, based on that understanding, attempted to reproduce Escher's work, or to create their own tessellation designs.

(b) Use of abstract concepts related to symmetry to understand and create art designs – a more global way to approach art via mathematics. In this case students derived information regarding a design by thinking of symmetry as a collection of organized and relatively abstract objects. This was a process of deduction. Students who developed an understanding of the structure of dihedral or cyclic groups systematically derived the characteristics of a group that were of use to them in their designs, or helped them justify their claims. The effect of this approach to the understanding or creating designs was obvious when asked to produce wallpaper patterns; a few students attempted to make use of their understanding of dihedral and cyclic groups – some students recalled that cyclic groups have no mirror symmetries and used this fact to produce wallpapers with rotational themes.

DISCUSSION

One of our goals for this study was to examine the potential of a learning environment created by combining mathematics and art activities, and, specifically to document students' understanding of symmetry. Furthermore, we aimed to examine the ways in which students come to understand symmetry via the use of art, their strategies and approaches. Willett (1992), Loeb (1993), Gura (1996) and Shaffer (1997) among others argued that an art studio can facilitate the learning of mathematics and the mathematics of symmetry can be a meaningful organizing principle when teaching a course in this setting. Indeed, we found that our students over the course of the semester developed their ability to detect the existence of symmetry in its various forms in art designs, and to classify various designs using familiar properties (in this case properties of symmetry groups). Furthermore, students regularly incorporated symmetry in their own designs. This involved not only the use of relatively elementary concepts such as mirror lines, but they also gave evidence of a thoughtful use of specific types of symmetries in order to achieve certain visual effects (e.g., students often explicitly mentioned that they *chose* to use a cyclic or rotational design to avoid having mirror images). Students' overall behavior suggests that they learned to appreciate an abstract approach to symmetry as a way to create designs and had begun to appreciate abstraction in mathematics in general.

Our second goal was to examine the *ways* in which undergraduate students use art as a gateway in their attempts to develop an understanding of symmetry. The findings suggested that students developed the ability to approach symmetry beyond the one-to-one correspondence between a specific piece of art and a mathematical concept, but they also develop a correspondence between types of art designs and abstract types of symmetry concepts. Further, familiarity with symmetry concepts allowed students to use abstract mathematical concepts as a tool in their art creation.

Finally, we focused on the learning of symmetry and the implications that this may have on the overall mathematics learning and attitudes of students towards mathematics. Students' comments suggested that they developed a positive attitude and, often for the first time, they could see a role for mathematics in their lives and their creations.

Overall, students who participated in our study learned about the mathematical idea of symmetry and discovered a new meaning for mathematics. An important question to ask is what facilitated the learning of mathematics in an art studio. Shaffer (1997) in a study involving middle school students in a similar environment suggested two venues to explore as factors in students' learning: the issues of control, that is the freedom to make decisions in one's own learning (Dewey, 1938) and expression (Parker, 1984). Shaffer suggested that expressive arts-based activities put students in control of their own learning. Observational evidence suggested that in our course, the freedom for students to choose the means in which to apply the concepts they find most useful in the context of art with which they feel familiar, also facilitated the learning of mathematics. Students often talked about using ideas of symmetry out of class in their own project and the transfer of ideas of symmetry to their art projects. Our results suggest a framework for thinking about the teaching of mathematics in the context of other courses. We believe symmetry to be only one of the many possible principles that can be used as powerful vehicles to build mathematical bridges for students that are often difficult to reach.

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THE ROLE OF CALCULATORS IN INSTRUMENTAL GENESIS: THE CASE OF NICOLAS AND FACTORS AND DIVISORS

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We present in this paper a case study of instrumental genesis involving a 14-year-old pupil, the multi-line-screened TI-73 calculator, and a family of numerical tasks related to notions of factors and divisors. This study sheds further light on the interaction among technique, task, and theory in the instrumentation process by showing the crucial importance of the role played by the pupil's need for control of the mathematical situation.

INTRODUCTION

In research on the use of technology in the teaching and learning of mathematics, increasing mention is being made of two distinct ways of viewing tools, according to a theory elaborated by Verillon and Rabardel (1995), that is, as artefacts and as instruments. For example, Mariotti (in press, p. 10) states:

> It is useful to make the following distinction; on the one hand there is the *artefact*, that is, the particular object with its intrinsic characteristics, designed and realised for the purpose of accomplishing a particular task, and on the other hand there is an *instrument*, that is, the artefact and the modalities of its use, as are elaborated by a particular user.

In accord with this theoretical approach, remark that it is the subject who transforms the artefact into an instrument by means of precise and well-organized actions, identified as "utilization schemes." Thus, a finer definition of instrument can be formulated:

> An *instrument* is an internal construction, the development of which is a long-term process; that means that at different moments, different *instruments* are concerned, although the same artefact is actually used. (*ibid*)

According to Verillon and Rabardel (1995), the transformation of artefact to instrument is a complex construction:

> A machine or a technical system does not immediately constitute a tool for the subject. Even explicitly constructed as a tool, it is not, as such, an instrument for the subject. It becomes so when the subject has been able to appropriate it for himself--has been able to subordinate it as a means to his end--and, in this respect, has integrated it with his activity. (p. 85)

Trouche (2000, p. 242) has described instrumental genesis as a combination of two processes: "A process of instrumentation by which the subject adapts self to the tool and a process of instrumentalization by which the subject adapts the tool to self." In the former process, "instrumental genesis is directed towards the subject, and leads to the development or appropriation of schemes of instrumented action which progressively constitute into techniques which permit an effective response to given tasks" (Artigue, 2001, p. 5). Recently, researchers (e.g., Guin & Trouche, 1999; Lagrange, 2000) have attempted to focus on related factors involved in the instrumentation process by which techniques are constituted. They point, for example, to the role played by technique and the task itself in the development of mathematical theory by the student. In fact, according to Lagrange (in press, p. 2), it is helpful to view technique as a link between tasks and conceptual reflection:

> A technique is generally a mixture of routine and reflection. It plays a pragmatic role when the important thing is to do the task or when the task is a routine part of another task. It plays also an epistemic role contributing to an understanding of the objects that it handles particularly during its elaboration. It offers also an object for conceptual reflection when comparing with other techniques or discussing its consistency.

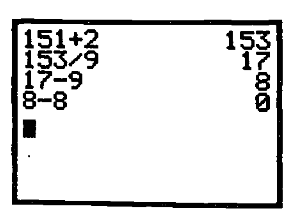
Within the framework of the above research studies, we present in this paper a case study of instrumental genesis involving a 14-year-old pupil, the multi-line-screened TI-73 calculator, and a family of numerical tasks related to notions of factors and divisors. This study sheds further light on the interaction among technique, task, and theory in the instrumentation process by showing the crucial importance of the role played by the pupil's need for control of the mathematical situation.

THE RESEARCH STUDY

GENERAL BACKGROUND

The research project consisted of two phases--the first in 2000 and the second in 2001. In each phase, six classes of Secondary 1, 2, and 3 students (12 to 15 years of age) participated, from schools in Cuernavaca and Montreal. The same basic problem situation was used in both phases, and the technological tools were similar (the TI-83 Plus in the first phase and the TI-73 in the second). The general aim of the research project was to investigate the nature of the mathematical strategies that emerged, as well as their evolution, in the context of particular calculator activities both within and across grade levels. The experience of the first phase led to three main adjustments in the second phase: a modification of some of the tasks; a refinement of the theoretical framework, along the lines described above; and the introduction of the classroom view-screen as a methodological tool that provided a trace of all the calculator attempts of certain students for a given task. The basic problem situation, "Five steps to zero," was as follows:

"Take any whole number from 1 to 999 and try to get it down to zero in five steps or less, using only the whole numbers 1 to 9 and the four basic operations +, -, x, ÷. You may use the same number in your operations more than once." (based on Williams & Stephens, 1992)



A set of 10 worksheets were developed to accompany the basic problem, with questions such as, "Here is a way of getting 432 to zero (a six-step method was given); show how you would do it in fewer steps; explain your strategy"; "The number 266 has as divisors 2, 7, and 19; what would you say is the best strategy for getting 266 to zero in the fewest number of steps; why?"; "Pick a number (less than 1000) that you think would be a hard one for the other group to get to zero in five steps or less; say why you think it is a hard number." Students were also given a brief pretest inquiring into their: (i) use of and attitudes toward calculators; (ii) knowledge of divisors, multiples, and primes. An individual interview with four students from each class was carried out at the end of the week-long activities.

The research was carried out in each class during five consecutive mathematics periods (50 minutes each). The students worked in groups, each student having her/his own calculator. While students were busy with their worksheets, there was always one student at the front of the class doing his/her work with a calculator hooked up to the view-screen that was video-taped (Phase 2). As well, the ensuing classroom discussions and student demonstrations of their strategies were videotaped.

The analysis of student strategies that emerged at each grade level and the way in which they evolved over the course of the first phase of the study was reported at PME last year. For the second phase of the study, rather than focusing on global trends, we decided to concentrate our analysis on the development in the thinking of individual students in the participating classes and to situate that analysis within the theory of instrumentation. Of the series of case studies emerging from this phase of the research, we present in this paper the story of Nicolas--a rather bright, Secondary 2 student, who seemed to enjoy mathematics.

THE CASE STUDY

Nicolas's initial strategy and its evolution

From the beginning, Nicolas's strategy was to find the largest divisor possible for the given number, and if it was not immediately divisible to then use trial-and-error until he obtained a number that would be divisible. However, by the third worksheet, this method was proving itself to be rather unsatisfactory to him (see Figure 1). The given number was 732. The way in which his first two attempts are crossed out--with large Xs and scribbling out of some of the steps--suggests the frustration that might have been mounting. Even though Nicolas's third attempt with 732 was successful in the required five steps or less, we notice that he continued in his efforts. In the fourth attempt, there seems to be no trial-and-error, as would be evidenced by mid-step additions/subtractions, and he arrives at zero in only four steps. One wonders whether his strategy is changing, but nothing to that effect has been made explicit.

It was not until the sixth worksheet where the technique that was hinted at, on the fourth attempt with 732 of Worksheet 3, was stated explicitly. On Worksheet 6, when he was asked explicitly for his strategy at getting 731 down to zero in fewer than 5 steps, he wrote (translated from the Spanish text): "First I multiplied 9x9x9 and I obtained 729. After that I noticed that 731-2 was 729. So I did the same operation, but in reverse."

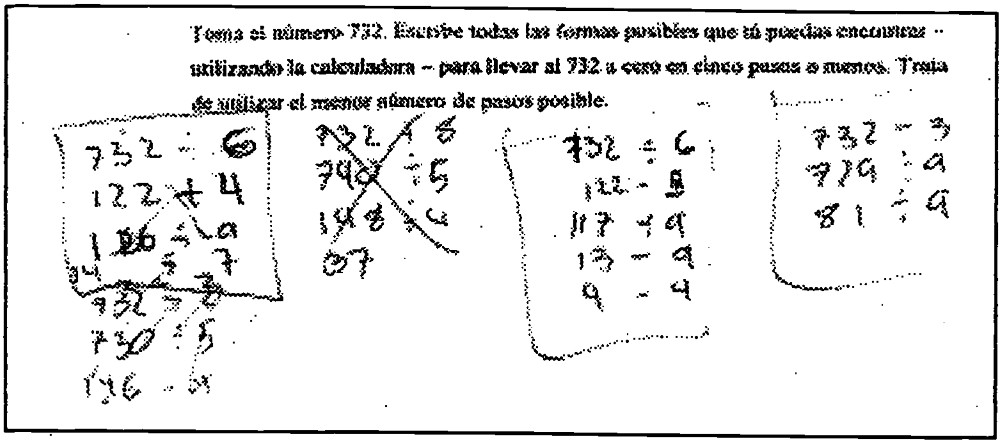


Figure 1: Worksheet 3 of Nicolas

The interview with Nicolas during the following week

It was during the individual interview with Nicolas that we were able to explore further Nicolas's use of his new, instrumented scheme, which had progressively become constituted as the product-of-factors technique. At a certain moment during the early part of the interview, Nicolas was asked what he would do if the given number was not divisible by a number between 2 and 9 on the first step (I = Interviewer; N = Nicolas), to which he said that he would add or subtract:

Verbatim	Comments
31. I: How do you know if you have to do an addition?	
32. N: Because ... Well, I also have a "technique" that I use. First, I do a multiplication, say, 9x9x3 or something like that to arrive at another number, and I look at that number.	He doesn't really answer the question; rather, he gives another way to attack the problem, using the word "technique."
33. I: Let's see, repeat that for me one more time.	Just to be sure about what he has just said
34. N: For example, if I have the number 571 and I multiply 9x9, it gives 81	Here he has chosen his own starting number.
35. I: Let us say that I give you the number 431	The interviewer wishes to offer one for which Nicolas may not have a ready-made solution.
36. N: OK, so I go: L1: 9×9 81 L2: ans×3 243 L3: 9×9×4 324 ~~L4: 9~~ L5: 9×9×5 405 So, like that, I arrive more quickly L6: 9×9×5 405	We notice here that Nicolas generates three potential factors. Then he systematically adjusts one of them. Legend for calculator screen transcription: Ln--refers to line on the screen (note that the TI-73 can display seven lines at a time, and then scrolls); A screen-line that is crossed out is one that the student has deleted from the screen.

37. I: But I said 431. With this strategy that you have just described, how do you begin?	An attempt to get him back to the given number 431.
38. N: First, 9×9×something, no? Until arriving close to the number. For example, L7: 9×8×6 432	He now controls the last two factors at the same time. The 9 is reduced to 8 while the 5 is increased to 6. Quite astonishing!
39. I: Yes, I told you 431	Again, to bring the task to its conclusion
44. N: So, 431 plus 1, divided by 6, divided by 8, and so on.	An oral solution, dividing first by the last factor of 9x8x6, and then moving toward the right, inversing each operation, until arriving at 9 which is handled by a final subtraction.
45. I: Let's see.	An invitation to show it on the calculator
46. N : L8: 431 + 1 432 L9: 432/6 72 L10: ans/8 9 L11: ans – 9 0 And there it is!	He enters his solution into the calculator.

Our analysis of the technique developed by Nicolas suggests that, in lines L2, L3, and L5, he may have been thinking about 81 as the basic unit and realized that 4 groups of 81 (9x9x4) would yield a product that is 81 units larger than 9x9x3. This would have given him rapid approximations of the increasing order of magnitude of the successive products. However, it is not at all obvious when he reached 9x9x5 (yielding 405) why he decided to decrease the second 9 to 8 and increase the 5 to 6 as a means of reaching a number close to 431. Did he "see" that the difference between 9x5 groups of 9 and 8x6 groups of 9 was 3 groups of 9, thus increasing the product of 405 by 27? However, he never did elaborate on how he was able to arrive so quickly at the appropriate factors.

As well, we never witnessed him using this technique with numbers larger than 738, which is 9 more than the product 9x9x9, and thus solvable in four steps. We are of the opinion that, while he could control three factors, four were beyond his range. Thus, by the end of the study, Nicolas appeared to possess the following hierarchical array of instrumented techniques for the tasks of the given family of situations: i) See if the given number is immediately divisible by a largish number, such as 9,8,7,or 6--the division technique; ii) If not, and if it is less that 739, use the product-of-factors technique that he had seemed to generate within the course of the research study; iii) Resort to various trial-and-error combinations of addition/subtraction and division.

DISCUSSION

USE OF SEVERAL INSTRUMENTS FOR THE SAME ARTEFACT IN THE COURSE OF A GIVEN SITUATION

Within the context of this study, there are two issues that arise in the process of instrumentation. One is related to the development of new instrumentation schemes, and the other to their use in given situations. Treating the latter first, we refer to a comment made earlier: "At different moments, different instruments are concerned, although the same artefact is actually used" (Mariotti, in press, p. 10). The Instrumented Activity Situations model as developed by Verillon and Rabardel "does not cover all the characteristics of situations where activity is instrumented: for instance, the fact that a same subject may use several different instruments in the course of a complex action" (1995, p. 85). Thus, a question of interest in our analysis was whether the subject was aware of his/her use of several instruments within a given task and what might be the mechanism driving their differentiated use.

As will be seen from the following protocol extract, Nicolas was quite aware of the two schemes of instrumented action that he was using (see especially the episode of line 64, along with the follow-up in line 66), as well as his reasons for trying the division technique before going on to use the product-of-factors technique. It was simply a question of speed and efficiency (line 48).

Verbatim	Comments
47. I: If the given number is divisible by a number between 2 and 9, do you always begin with a division?	
48. N: Yes, because I think it is faster if you do a division than if you solve it with another technique. So, first divide, and if that is not possible, then add or subtract.	Being able to solve a problem by the quickest method possible seems important for him.
54. N: Yes, add or subtract to obtain another number that can be divided.	
55. I: By what number?	
56. N: Say 8 or 9. The thing is to look for the largest divisor. Since you can add or subtract 9, you obtain many numbers.	His technique is aimed at finding the largest divisors possible.
63. I: Good, how about 362?	
64. N: Alright L1: 36 L2: 362/9 40.2222 L3: 362/8 45.25 L4: 362/7 51.7142 L5: 9×9×4 324 L6: 9×9×5 405 L7: 9×8×5 360 ~~L1, L2, L3, L4, L5, L6, L7~~ L8: 362 – 2 360 L9: 360/9 40 L10: 40/8 5 L11: 5 – 5 0	He begins with his first strategy (L2-L4), that is, see if the numbers 9, 8, or 7 are divisors of 362. He clearly does not know the rule for divisibility by 9. From L5, he uses his product-of-factors technique. In L6 and L7, he controls one factor at a time, but notice the shift from the last factor to the second-last factor in L7. He thus finds the factors of a number in the immediate vicinity of 362. He seems quite aware of the two schemes of instrumented action that he has used. From L8 to L11, after clearing the screen of his calculator, he applies his technique in reverse, using the largest factors, so as to provide a solution to the problem.
65. I: Very good. Now do the same with 323.	
66. N: OK L1: 9×9×4 324 L2: 323 + 1 324 L3: ans/9 36 L4: 9 L5: ans/9 4 L6: 4 – 4 0	Drawing on the information that his product-of-factors technique had produced for the previous problem (L5 of 64), he does a quick check (L1). From L2, he applies the results of the technique.

NEED FOR MATHEMATICAL CONTROL AS A MOTIVATION FOR THE DEVELOPMENT OF NEW INSTRUMENTATION SCHEMES

The driving force behind Nicolas's development of new schemes of instrumented action is conjectured to be the need to use the calculator artefact in the search for problem solutions in the most efficient manner possible. The scribbled-out work of his third worksheet, included above in Figure 1, suggested a distaste for his initial technique, one that combined division and repeated trial-and-error. This technique did not always allow him to advance in a clear way towards a solution. Quite quickly (by Worksheet 3), he seemed to tire of the trial-and-error component and want to have more control of the problem situation. We postulate that this mathematical need was the basic mechanism underlying the generation of his new instrumented scheme, a scheme oriented toward working in the opposite direction by means of multiplication of factors.

Obviously, Nicolas's technique of manipulating sequences of three factors was presently limited, within the context of the given family of tasks, to whole numbers less than 739. We have little doubt that, had the research activities continued over a longer period of time, his mathematical needs would have pushed him to extend the utilization scheme that he had developed so as to handle larger numbers. "An instrument is rarely definitively constructed; schemes evolve" (Trouche, 2000,p243).

ROLE OF THE TOOL IN THE CONSTITUTION OF TECHNIQUES

A thornier issue concerns a subtler question, that of the role played by the technological tool, in this case the multi-line calculator, in the development of the product-of-factors instrumentation scheme. When we consider the triad, technique-theory-task, it seems likely that, for Nicolas, the technique was constituted in response to the task and that the technique contributed to his further understanding (the theory) of the mathematical object--in this case, the interplay between divisors and factors. What is less evident is the extent to which the constitution of the technique was provoked by the affordances and constraints of the artefact. Clearly, the task was one that lent itself to the use of the calculator artefact. And perhaps the speed with which the calculator is able to carry out the demands of the user leads him/her to think about developing new approaches that would not have occurred in a paper-and-pencil environment. However, this factor is much harder to access. Certainly, Nicolas was NOT of the opinion that the calculator "helped him to think." At the end of the interview, he was asked what role the calculator had played for him in these problems:

N: Well, for the operations, I did them more rapidly and I knew if the numbers were divisible or not. The other way [i.e., with paper and pencil] I would have to do each operation: divide like we do normally and that would take me much more time in everything that we have just done.

I: Do you believe that the calculator helped you to think?

N: No

I: When you told me: 'First I look at the number, I do multiplications by 9, and I see if I am approaching the number.' Doesn't that mean that it helps you to think?

N: In my opinion, no. Because I think first about a procedure and how I could do it, and only afterwards I enter the numbers into the calculator and see the results.

Despite his claim that he thinks first before using the calculator, there were rarely any noticeable pauses on his part while using the calculator during his interview--even when he was manipulating various factor combinations. Even though the result presented on the calculator screen served as a basis for his next thought, Nicolas was unwilling to give it any more status than the result he might have generated by paper and pencil. Thus, the use of the calculator as a "tool to think with", in tasks of a numerical nature, is a concept that certain calculator users seem unwilling to entertain.

It was pointed out earlier that, according to Varillon and Rabardel (1995, p. 85), a tool becomes an instrument for the subject when "the subject has been able to appropriate it for himself--has been able to subordinate it as a means to his end--and, in this respect, has integrated it with his activity." Nicolas had certainly subordinated the calculator as a means to his end, and it had thereby become an instrument for him. However, the instrumented scheme he developed, which became constituted as the product-of-factors technique, did not--in his opinion--include explicit ties to the artefact, despite the fact that the tasks were calculator-based to begin with. Nicolas seemed to view his <u>need to control the mathematics of the situation</u> (the *task*) <u>in a 'thinking' way</u> (the *theory*) as the basis of his new *technique*.

ACKNOWLEDGMENTS

We thank the Social Sciences and Humanities Research Council of Canada (Grant # 410-99-1515) and CONACYT of Mexico (Grant # I32810-S) for their support of the research in this paper.

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ACTIVITY ANALYSES AT THE SERVICE OF TASK DESIGN

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The goal of this paper is to exemplify a model of design-oriented research, based on four different analyses of an inquiry activity, concerning proving in geometry. The designers' intention in this activity, was to create geometrical situations in which students will confront contradictions between their conjectures and findings, face uncertainties concerning the right conclusions, and thus search for explanations that will settle the uncertainties. In this way normative explanations in geometry might stem from students' need for justification. The first two analyses reflect the beliefs, hypotheses and intentions of the designer-researcher concerning the potential of appropriate situations to lead to a meaningful activity. The third and the forth analyses are experimental: students' conjectures and explanations serve as the database against which the task potential is validated.

INTRODUCTION

In this paper we exemplify a model of design oriented research, based on four different analyses of an inquiry activity, concerning proving in geometry. This model is based on the hypothesis, that it is possible to design geometrical inquiry situations, in which students are confronted with contradictions between their conjectures and the findings of their investigations in a Dynamic Geometry (DG) learning environment. These contradictions might lead students to uncertainties concerning the right conclusion, and thus push them to search for explanations, which may settle the contradictions. Since the only satisfactory explanations for this kind of situation are deductive (satisfactory in the sense that they can settle the mathematical contradictions), students might naturally appeal to such explanations.

For investigating this hypothesis, a few inquiry activities were designed in a DG environment in a process of *design-research-design,* as described in Hershkowitz et al. (in press). The design of these activities, intended to lead students towards contradictions in two ways, (see Figure 1).

Design of activities with potential for contradictions between conjectures and conclusions in two different ways.

(1) The findings contradict the conjecture. → Explanation for the surprising conclusion.

(2) The findings do not lead to a definite conclusion and thus, create uncertainty. → An explanation that points to the right conclusion, and provides conviction.

Figure 1: Two ways to contradiction

Our approach to the design of meaningful ways for teaching/learning to prove in geometry, is in agreement with the wide consensus of mathematics educators and researchers that proving activity should involve different actions, like discovery and reinvention, conjecturing followed by confirmation or refutation, including confrontation with situations of uncertainty (Chazan & Yerushalmy, 1998; Hoyles, 1997; Goldenberg, Cuoco & Mark, 1998; de Villiers, 1998).

THE TASK AND THE RESEARCH POPULATION

In this paper we will relate to one activity and exemplify the four analyses mentioned above.

The task:

Divide the side AC of a dynamic triangle into 3 equal segments. Connect the division points to the vertex B (Figure 2).

Make conjectures about the 3 varying angles (<ABD, <DBE, and <EBC).

Use the software to decide when (if ever) the three angles are equal. You may change your conjecture and check.

Describe your investigation and explain your conclusion.

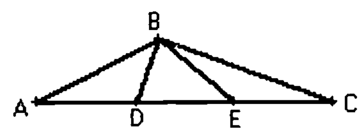

Figure 2: The three angles task

The above task is the product of a first stage of the *design-research-design process,* described in Hadas & Hershkowitz (1998), and has served as the "research tool" for the model mentioned above, and which will be described in detail below. The task was first used in semi-structured interviews with three pairs of students and then given as a written activity to all students in three classes, which were invited to work collaboratively in pairs. The students were all ninth graders, and had already had a year-long course in Euclidean geometry. Altogether 32 reports were collected (three from the interviews and 29 from the three classes in two different schools).

The interviews enabled us to qualitatively trace the changes in students' conjecturing and explanation processes. The transcriptions of the interviews and the written reports of students working in pairs in classrooms were collected and analysed. The results of these analyses enabled us to investigate quantitatively and qualitatively students' conjectures and explanations, and hence to clarify to what extent students faced contradictions as well as to make categories of students' explanations.

THE FOUR ANALYSES

Analysis no. 1: Epistemological analysis of the activity

The epistemological analysis demonstrates all possible investigation paths in the designed activity, without giving preference to any of them. This analysis is summarised in Figure 3.

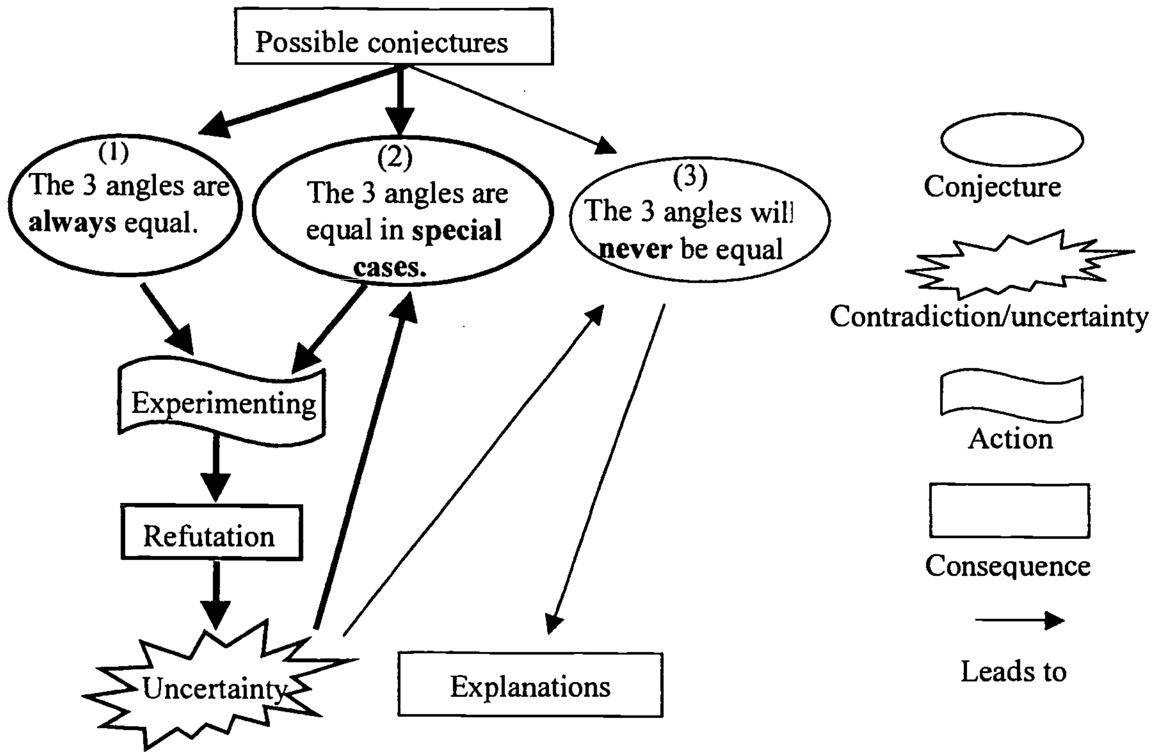

Figure 3: Diagram of the possible investigation paths

Starting with the conjecture in ellipse 1, 'the three angles are **always** equal', may lead students towards experimentation, which refutes this conjecture. This path exemplifies the first way of contradictions, (see Figure 1). After refuting this conjecture students may omit the **'always'**, and join those who conjecture 'equality in **special cases**' (ellipse 2). For checking this conjecture students may move from example to example, checking with the software and refuting every new conjecture concerning a special case in which equality seems possible (left and middle 'bold paths'). Thus, students are led to a situation of uncertainty where they don't know whether an example of equal angles exists. This may lead them either to raise a different conjecture about equality or to the conjecture that 'the three angles will **never** be equal' (ellipse 3) followed by an explanation (represented in the path on the right side of Figure 1).

Analysis no. 2: Didactical characteristics of the activity

This analysis is based on the previous one, and reflects our intention as designers, to create favourable conditions for contradictions and uncertainties. In this design we took into consideration students' common belief that opposite equal segments in a triangle, there are equal angles, and the power of DG to check if there are 'equal angles situations' ('bold paths' starting at ellipses 1 or 2 in Figure 3.) As the angles' equality situation is impossible, it was hypothesise that students will confront repeating refutations ('bold middle path in Figure 3). Thus the investigation involves uncertainty, namely not knowing whether one must look for an existence example or for an explanation why such examples do not exist (ellipse 3). The role of the DG software in this activity is not to provide an answer but to enable students to

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experiment (i.e. to manipulate geometrical entities involved, to refine their conjectures, then to refute them). The DG environment may also influence students' explanations.

The following third and fourth analyses describe the data concerning the students' real actions (their ways of investigating the task), and study their meaning.

Analysis no. 3: Conjectures raised by students

Here we describe students' initial conjectures and investigate to what extent they contradict the findings. We also describe how these conjectures are changed during the students' work. The analysis of the conjectures is based on the first analysis of the possible investigation paths (Analysis no. 1), and will be described here on the basis of Figure 3.

From the 32 reports that were collected, we had altogether 35 conjectures (the students in the three interviews conjectured individually).

Stage I: In 18 conjectures it was claimed that the three angles 'would **always** be equal'. The other 17 conjectures specified that the angles 'would be equal in **special cases**' like in isosceles or at least in equilateral triangles. At this stage, no one raised the conjecture, represented in ellipse 3, that 'the three angles would **never** be equal'. Constructing, measuring and dragging actions, using the software, immediately refuted the conjecture in ellipse 1, and convinced the 18 students that the angles 'are **not always** equal'. After this conjecture was refuted these students joined the 17 others and dragged the figure, trying to find cases of equality. This first stage of conjecturing is demonstrated in Figure 4 (Stage I).

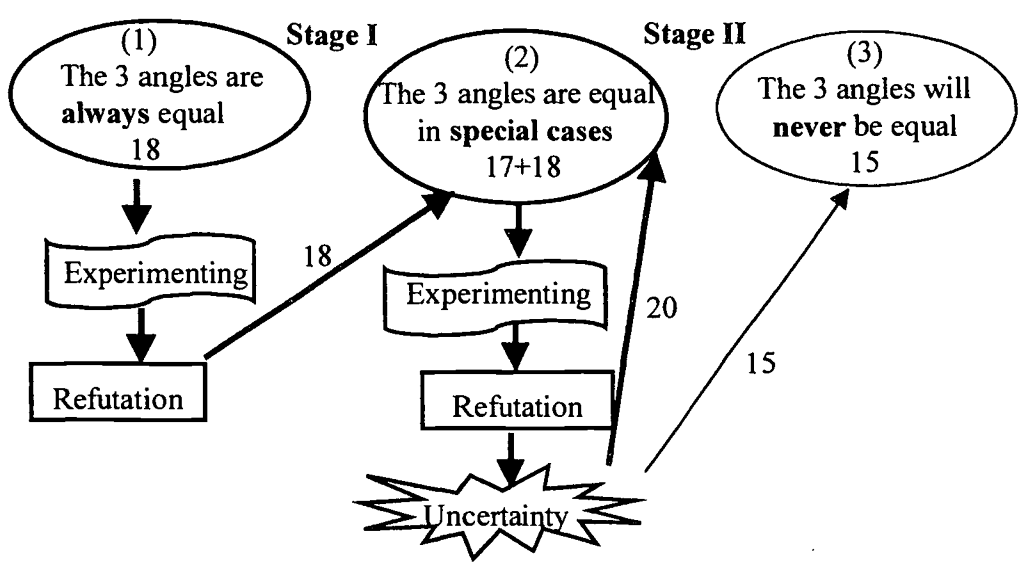

Figure 4: Stages I and II of conjecturing

Stage II: In 15 out of the 35 reports, after dragging the figure and measuring, students moved towards the conjecture that 'the angles will **never** be equal'. The remaining twenty moved in circles, from the search for examples to new conjecture. This second stage of conjecturing is also demonstrated in Figure 4 (Stage II).

Stage III: This process, of moving in circles, continued until 19 of the pairs joined the previous 15, in conjecturing that the three angles 'might **never** be equal' (ellipse 3). These 34 pairs of students began to search for a satisfactory explanation for the impossibility. The 35th pair removed vertex B so far from edge AC that the angles became small and numerically equal on the computer. They then concluded that there is a case for which the angles are equal. This Third stage of conjecturing is demonstrated in Figure 5.

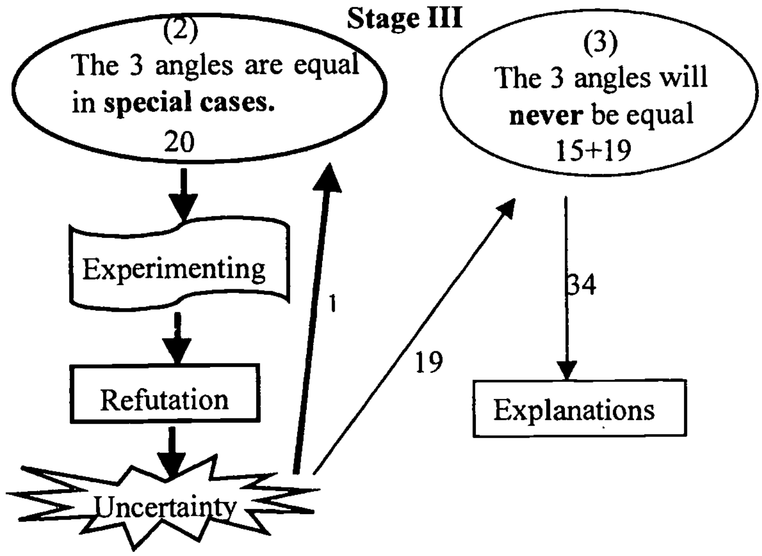

Figure 5: Third stage of conjecturing

In summary: All students began their work with conjectures, which contradict their final conclusion. Following the stages demonstrated in Figures 4 & 5, we conclude that all students choose investigation paths that lead to refutations and confrontation with uncertainty, and thus acted out the didactical intention described in Analysis no. 2.

Analysis no. 4: Explanations

In the previous analysis we concluded that all but one of the pairs followed at some stage the path starting in ellipse 3, which led them to search for an explanation why the three angles will never be equal (see Figure 5). We collected 34 explanations, which were categorised by the first author and two experienced teachers with agreement among all three in 88% of the cases. When full agreement was not attained, explanations were categorised according to agreement between two of them.

The goal of the categorisation is to find qualitatively and quantitatively (1) whether and how students use deductive considerations in their explanations; (2) whether and how students are influenced by the DG environment in their explanations (whether they use examples created on the screen to explain the findings inductively, or whether they make use of the dynamic variation of the figures on the screen, to explain their findings). Thus, we defined five categories emphasising these aspects.

In the following we will first describe each of the five categories with examples. (Examples from two other activities of the research are described in Hadas, Hershkowitz & Schwarz, 2000.) Then some quantitative information, concerning the number of explanations classified in each category will be given.

Categories of explanations

<u>* No explanation</u>

This category includes responses without any argument, and responses that were mainly tautological (in which students rephrased their conclusion or the results obtained on the screen).

Example 1: One pair explained why the three angles cannot be equal as follows:

> *Such a case cannot be obtained, as there is always something that messed it up.*

<u>* Inductive explanation</u>

This category includes responses in which students based their explanations on one or more examples. Sometimes students' examples were taken from what appeared on the screen and sometimes students drew their own examples on paper.

Example 2:

> *There is no possibility that the three angles will be equal because we checked with the computer several triangles (equilateral, isosceles etc.)*

<u>* Partial deductive explanation</u>

This category includes responses in which students constructed a chain of deductive arguments but at least one link is missing or wrong.

Example 3:

> *There is no situation in which, the three angles will be equal because if there was one, the four segments from B would be equal too (BA=BD=BE=BC), but this is impossible since only two segments from B to line AC can be equal.*

Students in this category constructed a short chain of deductive arguments. It is partial because some arguments were neglected; for instance, they didn't explain the connection between their conclusion concerning the equality of the four segments from B, and the assumption of the angles' equality.

<u>* Visual – variation explanation</u>

In this category we include explanations in which students made use of visual variation, as a result of the dragging action or of their imagery. As such this category is typical for explanations in a DG environment.

Example 4: After trying to find a case of equality for the three angles and concluding that such a case doesn't exist, one girl in an interview explained while dragging the triangle on the screen:

> *I thought that maybe if I drag B up and angle B will become small, the angles might be equal, but then it* [the figure] *doesn't look like a triangle anymore, it will be more like a line. So it is impossible to get the 3 angles to be equal.*

She accompanied her explanation with moving her hands up showing how the triangle becomes thinner.

* Deductive explanation

In this category we include explanations that consist of a complete chain of logical arguments.

Example 5:

> *Let us assume that the 3 angles are equal* (see Figure 2 above). *EC=ED and thus ΔBDC is an isosceles triangle as the median and the angle bisector coincide. ΔBEA is also isosceles as the median* [BD] *coincides with the angle bisector. Thus BE and BD must both be altitudes and we have two different perpendicular segments from point B to AC which is impossible.*

The quantitative data (number of explanations and percentages in each category) are given in Table 1.

Table 1: Classification of students' explanations

No explanation	Inductive	Partial deductive	Visual-variation	Deductive	Total
4 (12%)	8 (23%)	14 (41%)	4 (12%)	4 (12%)	34

It is important to note that: (1) The examples of partial deductive explanations (Category 3), described here and in Hadas, Hershkowitz & Schwarz (2000) demonstrate that in spite of the fact that these explanations do not comply with the criteria for normative proofs, students tend to rely on their geometrical knowledge and use deductive strategies to explain their conclusions. Thus, in eighteen out of the 34 explanations, students used deductive arguments (fourteen partial deductive explanations and four deductive). (2) We found that twelve explanations were based on experimenting with the DG software and resulted either in inductive generalisation (eight explanations), or in a visual explanation based on the variations of the figure caused by dragging (four explanations).

CONCLUDING REMARKS

This work exemplifies two issues: The first is the 'four analyses model' of design oriented research, in which the planning work and the intentions of the designer are the basis for experimental research, and the research findings concerning students' responses serve as the database against which this planning work and these intentions are validated. The second issue relates to the specific content, in which the model was demonstrated, namely proving in geometry. Through the four analyses of the above model we demonstrated our approach to teaching/learning this controversial topic.

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TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE: GRAPHS, FROM A COGNITIVIST TO A SITUATED PERSPECTIVE

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This report builds on our previous work on graphical conceptions using a diagnostic tool specially constructed to elicit graphical misconceptions, but also designed to function as a questionnaire for assessing teachers' pedagogical content knowledge (PCK). In this study we investigated 12 teachers' judgements of the difficulty of the items, their proposed learning sequences and their awareness of errors and misconceptions. We present a teachers' perception of difficulty hierarchy and contrast it with the learners' difficulty hierarchy. Results showed that teachers' judgement of what is difficult is structured by their curriculum and also their knowledge to be highly sensitive to the methodology adopted to collect it. This provokes us to develop a situated, social practice perspective on teacher knowledge, in which tools and instruments mediate teacher knowledge and its impact on practice.

INTRODUCTION AND BACKGROUND

This paper extends previous work on teachers' awareness of their pupils' errors and misconceptions, in the context of graphs, which Shulman (1986) classified as Pedagogical Content Knowledge (PCK). Shulman refers to PCK as knowledge 'which goes beyond knowledge of subject matter per se to the dimension of subject-matter knowledge for teaching' (p.9), which includes 'the ways of representing and formatting the subject that make it comprehensible to others' (p.9) and hence relies on an appreciation of learners difficulties and misconceptions. This categorisation was designed to draw attention to the traditional (but not historic!) underemphasis on PCK as opposed to other forms of teacher knowledge such as subject matter knowledge. He proposes that such knowledge is required in the triple forms of propositional, case-based and strategic knowledge. These might include research knowledge transformed for teaching, e.g. empirically-based *propositions* organised theoretically or conceptually, but rich in examples of memorable, prototypical *cases* (the analogy with case study seems applicable) and the *strategic* judgment to use the knowledge effectively in practice. In this paper we discuss teachers propositional and case knowledge of their learners' graphicacy.

Leinhardt et al (1990) reviewed the literature on functions and graphs and said, "*of the many articles we reviewed almost 75% had an obligatory section at the end called something like 'Implications for teaching' but few dealt directly with research on the study of teaching these topics*" (p. 45). A little later, Norman (1993) characterised research on teachers' knowledge on functions and graphs as insufficient or even non existent. He also stresses that 'there is little in the research literature documenting either what teachers know or the nature of their knowledge' (p.180). Williams (1993) also argues that 'the study of functions and graphs with an eye towards informing teaching and learning is in its infancy' (p.314).

We would add that the 'teaching implications' drawn from research on the psychology of learning mathematics are in any case in general problematic: for many reasons these rarely impact on practice. In particular teachers need to know at which stages of their development pupils are likely to exhibit the researched misconceptions and errors and where in the curriculum they are relevant. Williams and Ryan (2000, 2001) produced such data for errors scattered across the curriculum. Hadjidemetriou & Williams (2001) developed similar work by focussing in depth on the area of 'graphical understanding and interpretation' relevant to years 9 and 10 of the mathematics curriculum, using a diagnostic instrument based on previous cognitive research in the field. In this study we have developed this instrument to function as a questionnaire for assessing teachers' awareness of the difficulties and errors this diagnostic instrument reveals.

METHOD

The development of the original pupils' diagnostic assessment instrument is described in Hadjidemetriou and Williams (2001). Briefly, we mention that it involved the tuning of, or the development of, diagnostic items from the research literature on graphicacy which related appropriately to the following errors: slope-height confusion; linearity-smooth prototypes (Leinhardt et al, 1990); 'y=x' prototype; the 'origin' prototype; graph as 'picture' (Clement, 1985); reversing or misreading co-ordinates; misreading the scale (Williams and Ryan, 2000). The test can be seen in full on the web at http://www.education.man.ac.uk/lta/ch/.

In this study the diagnostic test was then given as a questionnaire to the teachers (N=12) with instructions that they should answer all the items and:

- predict how difficult their children would find the items (on a five-point scale starting form Very Easy, Easy, Moderate, Difficult, Very Difficult)
- suggest likely errors and misconceptions the children would make and
- suggest methods/ideas they would use to help pupils overcome these difficulties

Teacher's predictions of the difficulty of the items were subjected to a rating scale analysis (Wright and Masters, 1982). This provides an item-perception difficulty measure and consequently items can then be ranked according to their difficulty estimate. Their pupils' test results (N=425) were analysed using the Rasch model in order to get a pupil difficulty estimate for each item.

This data is used to explore the state of the subject matter and pedagogical content knowledge of this small group of teachers. The teachers were chosen as being thought to be knowledgable, leading teachers involved in training and management. We ask: 'What do teachers know and what can they recall about their students' problems/difficulties in graphicacy'? The teachers were also interviewed using a semi-structured format based on the way they introduce graphs to their classrooms, problems and difficulties students have in graphicacy, and how they teach graphs. These themes were used also in categorisation during transcript analysis.

RESULTS

The questionnaire data were subjected to a rating scale analysis and the item-perception-difficulty measures that resulted were correlated with the pupils 'actual' difficulty as estimated by the test analysis (rho=0.395). However, the teachers' estimates were significantly awry on seven items. When the seven worse items (having an absolute difference more than 2) were excluded from the analysis the correlation increased to rho=0.65. Their mis-estimation of their (relative) difficulty could be explained by one of the following reasons:

1. in at least three items the teachers underestimated the difficulty for the children because they apparently misunderstood the actual question themselves, i.e. they had the misconception the item was designed to elicit, or they had a limited understanding that did not receive full credit. We interpret this as subject knowledge
2. on two items the teachers' overestimated the difficulty because they did not realise the children could answer the question without a sophisticated understanding of gradient. This was interpreted as pedagogical content knowledge.

'Story 3', was the name of the most discrepant item with teachers underestimating its difficulty. It required pupils to draw the 'Height of a person' from Birth up to late thirties. A closer look at some of the teachers' graphs below illustrates the problem. Prototypes such as the 'Origin' (pupils', and here teachers', tendency to draw all their graphs through the origin) and 'Linearity' are dominant.

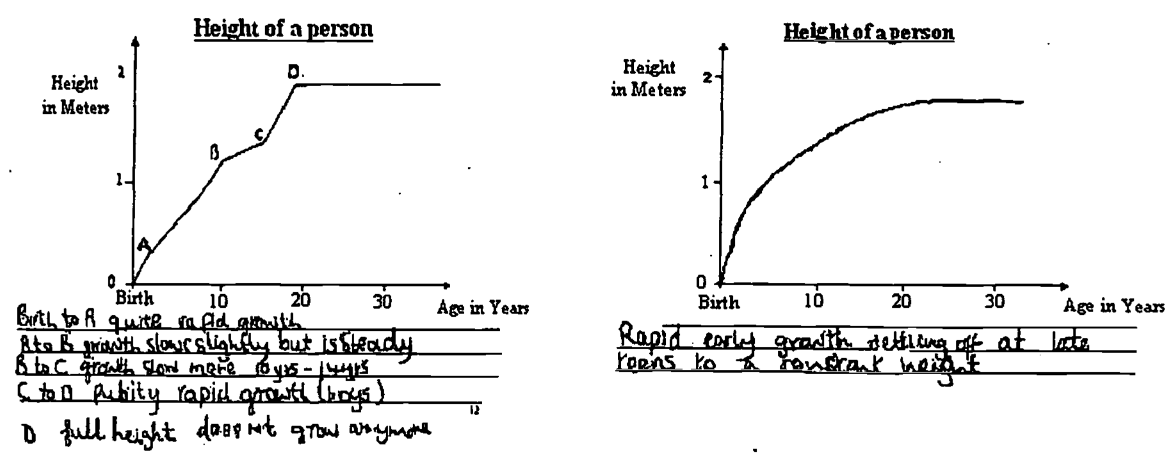

Figure 1: Two teachers' graphs for the 'Story 3' Item

'Transport 1' (shown below) was an easy question according to pupils' answers but some teachers seem to have given quite high difficulty ratings. These teachers believed that pupils had to be aware that the slope of the distance-time graph represents the speed of each transport. Pupils' transcripts verify that they could find the answer by looking at the time taken for each transport to travel to school:

INT: How can you see that it (A) is quick and that D is slower?

Sara: Because ...

Andrew: It takes more time.

Sara: Yes it takes more time. It takes more time to get to the same part. It takes 40 minutes to get to school and the others it takes 15, 10...

The graph shows journeys by four different means of transport from home to school, a distance of two kilometers: Bus, Car, Walking, Bicycle. Match each line with the appropriate transport.

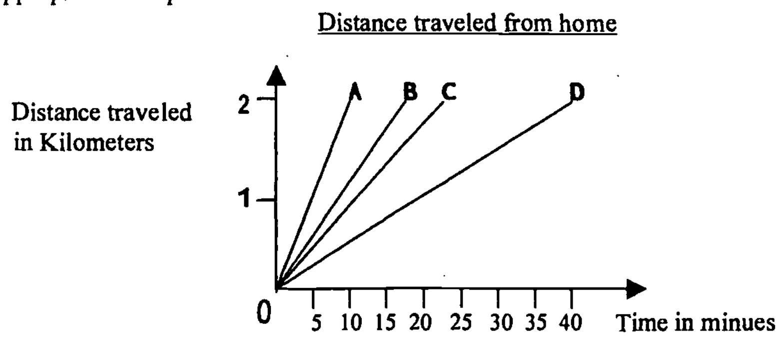

Table 1 shows teachers' proposed difficulty sequence described in 5 levels. Compared to pupils' hierarchy (Hadidemetriou and Williams, 2001) it yields significantly different results.

The table suggests that teachers have ranked the items in a 'curriculum' hierarchy. The bottom of the teachers' difficulty sequence is centred around pointwise reading. Further on teachers rate as slightly harder, items involving scales, parallel graphs and calculating the gradient. Interpretation of simple travel graphs is also included in this level.

In addition, this hierarchy matches the sequence of teaching evident in 6 teachers' descriptions of their curriculum during the interviews. They usually neglected the qualitative/interpretative perspective of graphs at the beginning of their teaching sequences and were preoccupied with abstract and algebraic aspects of graphs. For example:

SW: Starts of co-ordinates, study of co-ordinates. Exercising and using co-ordinates as positioning on a grid relative to a given plane, the origin. This is lower down the school, Year 7 Year 8. And then from there taken on to ordered pairs connections between x and y, mapping diagrams giving ordered pairs and then from that trying to take it on to equations. Straight lines and then on to curves.

INT: Straight line and then curves.

SW: And then when they've done that the use of obviously, em, apply to everyday sort of situations as well.

Levels	Brief Description	Description of Teachers' Difficulty Sequence
5	The idea of slope in the context of the 'rate of change'	Understand rate of change in an interval and instantaneous rate of change. Harder interpretation of 'constant rate' graphs.
4	Slope is adequately mastered and applied to situations involving linear graphs or curves.	**Parallel graphs have the same gradient, speeds interpreted as slopes: same speeds are drawn parallel on graphs [Hard]** Distinguish slope from height. Understanding no change or steady change. Understanding the 'covariance' of a graph. Interpreting discontinuous graphs.
3	Introduction of curves. Harder interpretation of linear graphs.	Understand varying slope of a curve (e.g. y=x-squared). Harder interpolation on y=x-squared. Overcoming the 'graph as picture' misconception by pointwise interpretation.
2	More complex reading and the introduction of slope (calculation and simple interpretation).	**Understanding calculation of gradient of a graph (y=4x) [Easy]** Use of scales in graph reading. Understanding of varying slope (linear graphs). **Interpretation of simple travel graphs [Hard]**
1	Pointwise reading of graphs, extracting information from points.	Use of unfamiliar co-ordinates. Compares y-ordinates of two graphs in context. **Interpreting the meaning of (0,0) in context [Easy]** Reading co-ordinates off a graph by interpolation and extrapolation. Reading co-ordinates off a graph.

Table 1: Teachers' Perception of Scale of the items of the Diagnostic Questionnaire
(in bold are major teachers' misjudgements i.e. relative misordering compared to pupils' hierarchy and in brackets, whether the items became Harder or Easier)

In the first two levels then, construction/algebraic related items are dominant. Teachers have only included here one contextualized task (the item called 'Transport 1' described above) and this resulted in the algebraically related items shifting downwards from their actual difficulty. For example calculating the gradient was an item located at the top of the pupils' difficulty sequence whereas teachers have awarded it a rather moderate difficulty.

Another underestimated item (Interpreting the meaning of (0,0) in context), located at the bottom of the teachers' difficulty scale (together with the rest of the coordinate-related items) was an item asking 'Why does the graph pass through the origin?'

Teachers' failure to consider possible terminology problems (pupils who were not aware of the meaning of the word 'origin') shifted this item downwards in difficulty.

Teachers' attention to the abstract perspective of graphs also explains why the difficulty of several qualitative tasks involving interpretation (the slope height confusion) and sketching graphs to tell a story were overestimated.

This bias also had an impact on the errors and misconceptions they mentioned during the interview. The transcripts refer mainly to errors/difficulties such as: reversing the x and y, mapping an equation to the graph, substituting negative numbers, inaccuracy in plotting, calculating the gradient as 'x over y', generating points from equations and misreading the axes. However in the questionnaire the teachers were also encouraged to list the misconceptions that children might exhibit. Here we summarise the misconceptions they mentioned during the interview or in the questionnaire:

TEACHER MISCONCEPTION	1*	2*	3 i	4 i	5 q	6 i	7*	8*	9*	10*	11*	12*
Slope height					4q				4q	4q	4q	4q
Linearity						4i						
Y=X prototype											4q	4q
Origin prototype											4q	4q
Picture as graph			4i	4i			4q	4q	4q	4q	4q	4q
Co-ordinates			4i	4i			4iq	4i		4i	4iq	4q
Scale	4i	4i			4q	4i	4i	4i	4i	4iq	4iq	4iq

Table 2: Misconceptions mentioned by 12 teachers in interview or in the questionnaire

q/i/qi: indicate whether the misconception/error was mentioned in the Questionnaire (q), Interview (i) or both (iq)

*: indicates the teachers who were both interviewed and answered the questionnaire

In summary we found that the 12 teachers had (a) most difficulty in identifying the linearity (1/12 teachers) 'origin' and 'y=x' prototype (2/12 each) conceptions, (b) moderate difficulty in identifying the slope-height confusion, and the problem with order of co-ordinates (5/12 and 7/12 teachers), and (c) least difficulty in identifying the picture-as-graph and misreading of scale problems (8/12 and 10/12 teachers respectively). However, teacher knowledge of the seven different misconceptions varies dramatically, with half the teachers mentioning only one or two of them, and two of the teachers mentioning all but one of them.

Most strikingly, indications of teachers' knowledge seem highly sensitive to whether the data comes from the Questionnaire or the Interview. The different data sources suggest that much of the teacher's knowledge is tacit, and elicited when provoked by an example question. Only explicit propositional knowledge is usually suggested spontaneously in interview without the questionnaire prompt.

CONCLUSIONS AND DISCUSSION

A diagnostic test designed from the graphicacy research literature and calibrated for pupils diagnostic errors was here further developed as a tool to investigate teachers' knowledge about their learners. This was supported by semi-structured interviews.

The conclusion is that:

(a) some teachers harbour misconceptions themselves, revealing some weaknesses in subject knowledge: eg the linearity prototype

(b) very few of the common errors or misconceptions are called up spontaneously in questioning at interview, and these generally concern the technical and algebraic aspects, not the 'interpretative' misconceptions, but

(c) a much wider range of errors are offered in response to the 'diagnostic' questionnaire, and

(d) there is some mismatch of the teachers' perception of difficulty and the students actual difficulty, with teachers underestimating technical aspects of graphing and overestimating the difficulty of the interpretative.

Empirical evidence suggests that teachers' knowledge as elicited from the interviews was structured around their curriculum descriptions, was rich in algebraic and abstract elements of graphs but lacking in the interpretational. However, the questionnaire acted as a tool that bridged the gap, apparently bringing to the surface tacit awareness of their children. This is consistent with a theoretical approach which insists that knowledge is situated, even distributed (Hutchins, 1990), and confirms our belief that researchers can have an impact on teaching through the development of such pedagogical tools and instruments. As Engestrom (1987) puts it, one of the roles of R&D activity on an activity system is to develop more advanced tools and artifacts of various kinds which mediate the practice of the system. In our case we see a well-designed diagnostic assessment as just such an instrument for advancing practice.

However, we are less sanguine about the nature of the knowledge that these teachers were able to evidence with the diagnostic tool. We draw a distinction between an error, i.e. erroneous responses to a question, and a misconception which may be a faulty cognitive structure that lies behind, explains or justifies the error. (Some errors may be symptomatic of a misconception, while others may not). If teachers can predict their pupils' errors, does this mean they can diagnose them? This diagnosis is an essential link between 'case knowledge' (in the sense of knowledge about typical errors based on classroom practical experience) and 'propositional knowledge' (in this case knowledge of students conceptual development and misconceptions).We suggest that the link between 'case knowledge' and 'propositional knowledge' is generally best conceptualised not just as a cognitive one, but one which is socially structured, and in particular it can be mediated by tools-in-practice. We will develop this theoretical analysis of cognitive and situated perspectives on Shulman's categorisation further in the presentation.

The purpose of the paper was not to generalise empirically from these teachers' pedagogical content knowledge but to suggest a methodology for evaluating and maybe developing this knowledge. There seems to be a gap between pupils' difficulties and teachers' perception of these difficulties. Our concern is to provide research findings and propose a methodology that will help to bridge this gap.

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DEVELOPING TAKEN AS SHARED MEANINGS IN MATHEMATICS: LESSONS FROM CLASSROOMS IN PAKISTAN

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This is a study of the role of social interactions in students' learning of mathematics. The study was based in two classrooms in Karachi, Pakistan. A small group of students (10-12 yr.) doing mathematics was observed in each classroom.

Methodology used was qualitative in nature. Participant observation was the primary mode of conducting the research. To follow up on questions emerging from ongoing observations students were interviewed under stimulated recall. Analysis was through grounded theory procedures (Strauss & Corbin, 1998).

The teachers had initiated a change in the social and socio-mathematical practices in the classroom. Findings showed that the purpose of changed mathematical practices introduced by the teacher did not necessarily come to be taken as shared by the students. Developing a taken as shared meaning of the purpose of classroom practices involved mutual negotiation between the teacher and the students.

This paper reports on the findings from my doctoral research[1]. I examined the role of social interactions in students' learning of mathematics. My study was located in two secondary schools (10-15yrs.) in Karachi, Pakistan. It involved observing two small groups of students, each engaged in working at mathematics tasks set for them by their teacher.

METHODOLOGY

Methodology was qualitative in nature. Participant observation was the primary mode of conducting the research. Questions emerging from ongoing observations were followed up in stimulated recall interviews with students. Meetings were scheduled with the teachers who were seen as a significant part of the socio-cultural context. Observations were recorded on videotapes and interviews were audiotaped. Analysis was through grounded theory procedures (Strauss & Corbin, 1998).

THEORETICAL PERSPECTIVE

To study learning in the social context necessitates taking account of those shared understandings and invisible meanings that establish a classroom culture, and that provide meaning to the interpretations being made by the participants of the classroom i.e. the students and the teacher. A sociological construct used to describe

[1] Tentative findings of this work while it was in progress were presented as a short oral presentation in PME 25.

the shared understandings of a culture is 'norm' which refers to understandings or interpretations that become normative or taken-as-shared by the group (Yackel, 2001). She goes on to elaborate that these norms are inferred by identifying regularities in patterns of social interaction in the classroom. Those classroom participation structures that are stable facilitate in inferring the social norms prevalent, while the notion of taken-as-shared implies that individual interpretations fit with the purpose at hand.

The teacher necessarily represents the discipline of mathematics and the culture of the mathematics classroom, therefore, the teacher is a significant force in initiating and stabilising the social norms in the classroom (Wood, 1994). However, understandings do not become taken-as shared or normative through teacher's initiation alone. Wood (1994) suggests that one way to establish classroom norms, would be to for teachers and students to negotiate mutual expectations and obligations that are constituted in the classroom. Wood's position implies that in a classroom activity, participants in the activity would have expectations from others, and obligations from self which guide their participation. The collaboration among the various participants would then take place as a consequence of fulfilling these expectations and obligations.

From the discussion so far I deduce social norms to mean normative understanding of the ways of collaborating with others in the classroom. For example, when learning in small group settings, the understanding that students are expected to share their thinking with others in the group would be a social norm. As such, the social norms are shared understandings that could be found in any classroom irrespective of the specific discipline, and are not unique to mathematics. For the analyses of students' learning of mathematics, the construct of socio-mathematical norms is taken to mean normative aspects of mathematical discussions that are specific to students' mathematical activity (Yackel, 2001). Hence, a socio-mathematical norm would be a shared criterion of what is valuable, in a mathematical explanation or solution.

RESULTS & DISCUSSION

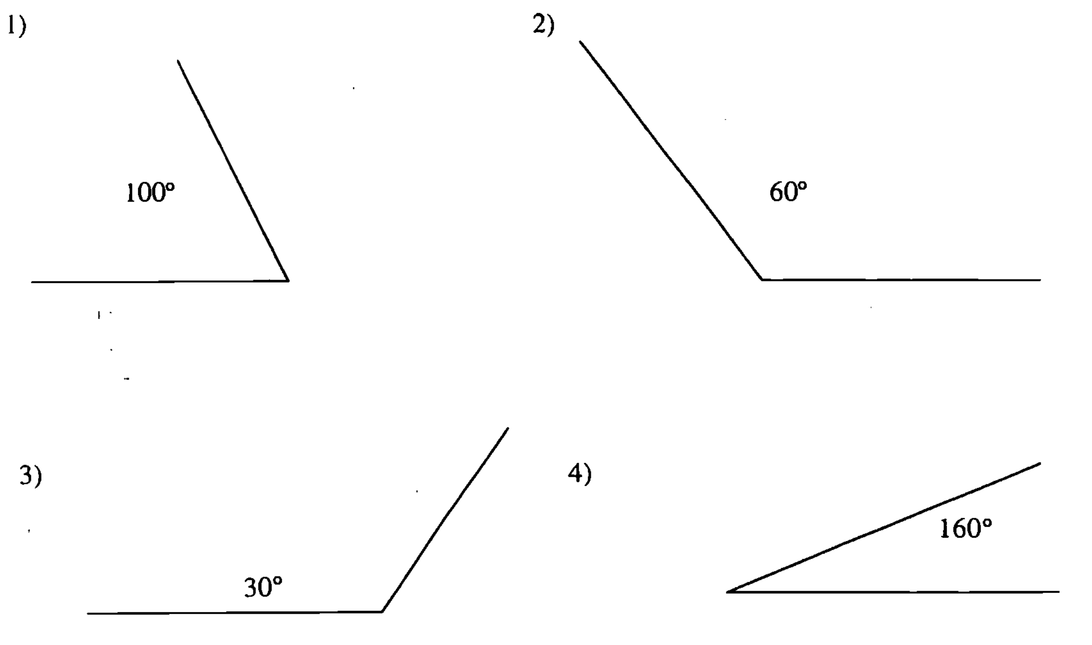
In this paper I discuss my findings from class VI (10-11 yrs) where I observed four girls, Maheen, Shabnum, Samina and Naima. The teacher Mohammad Aslam had initiated a change in the social and socio-mathematical practices in the classroom. He taught lessons that required students to work together in small groups at specially designed problem tasks. At the end of each group session the small groups were expected to report back their work to the whole class. This was a change from his previous social setting of the class. The problem tasks prepared by Mohammad Aslam were different from those that he said he had used previously. The difference was that the tasks that he prepared often had questions that were open ended in nature. For example, questions like, "can you decide the reasons for the mistakes shown?" opened up the possibility of more than one solution to be put forward.

Furthermore, students were free to use their own strategies to solve problems and share their solutions with the whole class using their own words.

From the above I inferred the teacher to have a goal that students should learn mathematics meaningfully i.e. students would rationalise their solutions and/or propositions, and at the same time the solutions and/or propositions should be acceptable mathematically. For example, enabling students to interact frequently with each other in small groups, use their own solution strategies, and explain their thinking to others meant that students had opportunities to explore their own interpretation of the mathematics embedded in the problem tasks. While, setting up the groups to report in the whole class ensured that the teacher could monitor those interpretations. However, this goal was implicit in the teacher's practices. There was very little evidence of explicit negotiation of the purpose of changed practices between the teacher and the students.

Episode: Alina's Mistake

Alina has drawn some angles in her maths book, but has measured them wrongly. Can you decide what she has done wrong?



To locate the discussion of findings in the classroom context I present an episode from my lesson observations. The students in this episode worked in groups at the task shown in the box above. Different groups shared their work with the whole class. Through this episode I illustrate the issues that arose for students' learning as they worked in the changed social setting of the classroom.

<u>Task Interpretation</u>: A recurring feature in the findings was that students interpreted the purpose of task based on their prior experiences in the classroom. In this case, the

students approach to the work on the task above suggested that they had interpreted the purpose of the task to measure angles. Once all angles were measured they considered their work finished and Samina read out the answers in the microphone left on their table for my data collection. This reading of answers stood out as a separate activity illustrating a pattern where one person would suggest reading the answers and the others would comply by reading the answers in turn. Moreover, every time they finished a task or a portion of a task they raised their hands to indicate to the teacher that work had finished. This implied an implicit belief in the answer as a valued product of the mathematics problem tasks, and a dependency on the teacher.

A possible reason for students' interpretation of task, as angles to be measured, could be that they were interpreting the task from their previous experiences of participating in classroom mathematics practices. The geometry curriculum in Pakistani primary and secondary schools has a heavy emphasis on Euclidean plane geometry. A consequence is that such cultural artefacts as protractors, compasses and work sheets and/or textbooks with diagrams of angles and other figures in plane geometry are common in the classroom milieu. Among other things, students are expected to measure and/or construct angles.

However, in this case the question statement "can you decide what she has done wrong?" is not clear in its intent because Alina's mistake (that she has 'measured the angles wrongly') has already been identified in the statement "had measured them wrongly". That the task requires a reason for Alina's mistake is difficult to infer from the question statement as it is phrased. Hence, the reason for student's interpretation of task as one of measuring angles could be because of the way the problem statement was phrased.

Mathematics being Learnt: The mathematics topic that was being addressed by the students was "acute angles and obtuse angles". The conversation being reported below is when Shabnum had placed a protractor on the second angle in the box above and the other three girls looked on. The numbers 60 and 120 refer to the readings on the protractor. The word 'degrees' is not used by the students it is implied

1 Shabnum: (reading the protractor) err sixty
2 Maheen: 120 120
3 Naima: (refers to the position of the protractor) bring it down, bring it down
4 Maheen and Naima: 120
5 Samina: 120 or 60?
6 Maheen/ Naima: 120
7 Maheen: 120 because our base line starts from here (points to the baseline on the protractor). So it is starting from here so 60 will come here (traces her finger on the protractor from 0 to where 60 degree angle comes).

8 Samina: No because 90 is coming in between (60 and 120) and at 90 the base line ends after which the numbers on top start. (refers to the readings on the protractor)

9 Naima: No

10 Shabnum: (with emphasis) No

11 Maheen: No this is obtuse. It is obtuse

12 Naima: 120

13 Maheen: (to Samina) accept it accept it. This is 120

14 Samina: (puts the protractor on the third angle) Data item: 1

The above conversation suggests that Shabnum (line 1) thought the angle that Samina measured was of 60 degrees. Samina (line 5) was not clear whether the angle was 120 degrees or 60 degrees and raised the question. Maheen and Naima (line 4) provided their answer and Maheen (line 7) went further to elaborate why the angle was 120 degrees and not 60 degrees. Samina's subsequent response (line 8) exposed her confusion not only about acute and obtuse angles, but also about what constituted the base line in an angle and how to read the protractor. From her statement in line 8, it seemed that in measuring an obtuse angle she first read the protractor from 0 to 90 and from 90 onwards she read the readings given on the top line of the protractor. However, Maheen (line 13) urged Samina to accept her (Maheen's) answer. This urging on Maheen's part had the effect that the group moved on to the next task while Samina was left with confusions regarding the difference between acute and obtuse angles. Questions also remained about Shabnum's understanding of acute and obtuse angles. It was not clear to me whom she addressed in line 10? Was she refusing to accept Maheen's correct explanation or was she addressing Samina? The former would imply that her knowledge of acute and obtuse angles was weak.

<u>Different Answers in Small Group & Whole Class</u>: A pattern was that students provided a different answer to the same task in the small group and in the whole class presentation. The responses in the whole class presentation were richer in terms of quality of thinking, longer, more detailed, with examples. For example, the conversation reported below is when Maheen reported the work her group had done.

1 Maheen: (points at the vertex of the acute angle already drawn on the black board) This is, this is, when we, err the base line is here so we show the base line here. From here we go this way

(with movement of hands shows a turn from the base line of the angle on the blackboard in an anti clockwise direction)

2 Teacher: Inaudible

(Maheen picked up the large black board protractor that was used by the teacher in the geometry lessons, and placed it on the angle already drawn on the blackboard by Shabnum).

3 Teacher: Yes that is right. Should I hold it? (comes forward and holds the protractor leaving Maheen free to give the explanation)

4 Maheen: First of all this is 100 and this is coming inwards.(points to the acute angle already there on the blackboard) This is an acute angle. An acute angle never goes above 90 (paused and looked at the class) and our base line is--- Data item: 2

(A disturbance in the classroom. The teacher asked Maheen to repeat herself)

Maheen's answer to the task was qualitatively different from the answer she had given in lines 2 & 4 (data item 1), during group work. Why was this expectation, not just to perceive why Alina was wrong, but to be able to explain and justify her mistake, not made evident in the group? Did the teacher's request in the plenary session to give the reason for Alina's mistake provide Maheen with the push to articulate the reason that led to Alina's mistake? Yackel (1995, p.148) identified similar occasions in the 'classroom teaching experiment' where children proposed solutions using methods during class discussions that were markedly different from those they developed during small group work. She goes on to show that this difference is due to a reconceptualisation of the problem. She suggests that the reconceptualisation occurred because the whole class session was more than a report back session. It involved discussion and questioning when students did not understand an explanation or a solution method. In the case of my research, students reported back their solutions to the whole class. However, it was rare for other children to challenge or question them (as in Yackel's work). Here, the teacher played the role of challenger and questioner. For example, in the above task 'Alina's mistake' the teacher asked the question 'what was the reason for Alina's mistake?' The question apparently led to Maheen, reconceptualising the question where she elaborated on her earlier answer in line 2 (data item 1), to a more detailed answer in line 1 & 4 (data item 2). An interpretation could be that Maheen articulated differently what she had conceptualised earlier because in line 7 (data item 1) she did give the reason for Alina's mistake. However, this explanation in line 7 was in terms of the mistake Alina might have made in selecting the wrong starting point on the base line. Whereas, in the whole class she extended her explanation of starting the reading from a wrong end of the base line to link it with her knowledge of acute and obtuse angles to explain the reason for Alina's mistake. Yet another interpretation could be that it was the social setting of whole class presentation as opposed to that of small group work that influenced students' responses. It could be that the students did not expect their peers to ask them questions and so did not feel obliged to answer them by providing a rationale for their arguments.

To get a deeper appreciation of the questions raised by these different responses, I followed this issue further in the stimulated recall interview with the students. In the following segment from the audiotape transcript Maheen reports on what she believed were her reasons for the difference in the work in the small group and whole class sessions.

1 Maheen: When I went in front of the whole class so there I had to explain all. I had to explain the mistakes in detail so it came to my mind there

the details but not so in the group?

3 Maheen :	It is in group also, but I thought that in the group they must have understood
4 AH:	Okay
5 Maheen :	When all (questions) were answered so it was obvious that they must have understood.
6 AH :	Okay so you thought they must have understood. Would any of the group members like to say something about this point? Yes Naima?

Data item: 3

Maheen's statement (line 1) indicates that she believed that in the whole class an explanation of her answer was required. It could be that the teacher's request to provide the reason made explicit the expectation in the problem task that the students were required to give the reason for Alina's mistake and not just point out her mistakes. Maheen's subsequent remark (line 5) implied that she did not provide the explanation in group because she assumed that her peers in the group had understood the task in the worksheet. So, a consideration guiding her responses were the needs of the participants in the social interaction. She judged the need to have been met once all answers had been given. Implicit in Maheen's statement is the message that being a student she is not used to thinking critically about her peers' answers and to taking responsibility for whether the other students have understood. Hence, once she had understood and once all answers had been given she assumed that her peers had also understood. Naima on the other hand gave a slightly different perspective.

1 Naima:	err I think she(Maheen) did not tell us (the explanation) because we were thinking that we only had to measure the angles-----
2 AH:	Okay
3 Naima:	------and we had not thought that sir could ask us these questions. When sir asked these questions, strange possibilities started coming to our mind, that it could be like this, or like this. Like she told.
4 AH:	Okay

Data item: 4

Naima indicated (line 1 and 3) that it was the teacher's question that led to the re-interpretation of the problem so that different possibilities were raised for the solution. Naima's remark confirmed my own interpretation that it was the teacher's questions that led to a change in thinking about the demands of the question. Naima's remark indicated a sophisticated level of thinking as regards the dawning realisation that the ways of working in the classroom were changing. Her statement in line 3, about "these questions" leading to "strange possibilities" suggested that she recognised the teacher's questions as not being ordinary routine questions that she might have been used to encountering in her class. Rather, these were questions that led to strange possibilities, probably, those of a non-numerical verbal answer or a variety of answers to the same task. This was a significant appreciation on Naima's the classroom participation were changing.

CONCLUDING REFLECTIONS

As students worked in the new social setting what appeared to be missing was some evidence of how the changed practices linked to the teachers' goal of learning mathematics meaningfully. Students appeared to work in groups through interpretations made from the perspective of old norms prevalent in the classroom. Hence, while ways of working in groups came to be taken as shared to some extent, mathematics rationalisation in the group did not. For example, working at specially designed problem tasks in groups and later presenting this work to the whole class appeared to be an accepted part of the classroom practice. Besides these new practices in the classroom that students appeared to regard as normative there were certain old understandings that also appeared to be taken as normative by the students. For example, a norm prevalent in the classroom was that a purpose of the mathematics problem task was to find the right answer. Hence, there appeared to be agreement among the students regarding this purpose when they engaged in mathematics tasks in the classroom. Although, wide experience of prevailing norms in Pakistani classrooms supports this interpretation I refer to these as 'old norms' because I did not have direct evidence of them. My interpretation is that two main factors led to students working through old ways in new settings. First, the teachers' communication of the purpose of change was largely implicit in his practice so that the students were expected to infer the purpose implicit in the change. Second, students' perceptions of mathematics and the purpose of classroom tasks did not support critical reasoning of mathematical ideas. Hence, classroom evidence led me to conclude that students and teacher are both participants in the classroom culture, so that meanings do not come to be taken as shared as a consequence of teacher initiation alone.

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GOAL REGULATION: NEEDS, BELIEFS, AND EMOTIONS

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Using the theoretical framework of self-regulated learning as a starting point, this article will elaborate students' self-regulation of goals. Three aspects of goal regulation will be discussed within the context of mathematics classrooms. 1) Goals are seen as elements of a needs-goals structure, and goal choices may be derived from needs values. 2) Self-efficacy beliefs are interpreted as beliefs about accessibility of goals. Goal accessibility belief is seen as a necessary, but not sufficient condition for adoption of a new goal. 3) Automated emotional reactions are seen as a possible inertia force. Some qualitative data of a three-year longitudinal study will be presented to illustrate the presented conclusions.

INTRODUCTION

To understand student behaviour we need to know their motivations. In the literature (e.g. Ryan & Deci, 2000) one important approach to motivation has been to distinguish between intrinsic and extrinsic motivation. Another approach to motivation has been to distinguish three motivational orientations in educational settings: learning (or mastery) goals, performance (or self-enhancing) goals, and ego defensive (avoidance) goals (e.g. Linnenbring & Pintrich, 2000; Lemos, 1999). When further elaborated, motivation can be conceptualised through a structure of needs, goals and means (Shah & Kruglanski, 2000). Such construct of goals and goal structures is an important part of the theory of self-regulated learning (SRL) (e.g. Boekaerts, 1999).

The importance of human needs as motivator for mathematical behaviour has been addressed, for example, by Vinner (2000) and Lerman (1998, p. 69). However, in a sample of five PME proceedings, there were surprisingly few papers explicitly on motivation or goals. Yates (1998, 2000) uses motivation (operationalised as task involvement and ego orientation goals) as one variable in a longitudinal survey, and Bikner-Ahsbahs (2001) and Moyer (1999) write about intrinsic motivation (interest) towards mathematics.

This paper will focus on students' goal regulation in mathematics and we shall discuss three aspects that influence students' goal choices. The first aspect is the students' needs. Students' different goal choices can be derived from different needs. The second aspect is students' beliefs – more specifically beliefs about accessibility of different goals. The third aspect is emotions that may function as inertia forces that restrict goal choice changes. Some data of a three-year longitudinal qualitative study will be presented to illustrate the theory.

The main theoretical framework for this paper is self-regulated learning. Boekaerts (1999) presented a three-layer model of self-regulated learning (SRL). Inner layers include choice of cognitive strategies, and use of metacognitive knowledge and skills to direct one's learning. The focus of this report is in the outermost layer, which includes choice of goals and choice of resources. This least developed area of SRL is essential in understanding student behaviour in classrooms:

> information about ... the goals [students] set for themselves ... provides an indication of *why* students are prepared to do what they do and *why* they are not inclined to do what is expected of them. (Boekaerts, 1999, p. 451)

Although SRL is the main theoretical framework for this paper, the standpoint needs to be specified. For example, Lemos (1999) writes about "internalisation of goals", "non-goal oriented behaviour", and that "strengths of self-directed behaviour lie in its flexibility". Instead of perceiving self-regulation as an advanced learning style, it is seen as a general psychological process that is part of every action. Thus, present approach assumes that student behaviour in classroom is always goal-directed and self-regulated. Students' goals are always self-chosen and internal, and behaviour is always goal oriented. However, the goals may differ from the learning goals set by the teacher and some students may be more flexible in their goal directed behaviour than others. Furthermore, it should be noted that present approach assumes that a lot of self-regulation is automatic and not conscious.

GOAL REGULATION: NEEDS, BELIEFS, AND EMOTIONS

Goals are part of a structure of needs, goals and means. The structure is personal and dynamic in time. There are individual differences in structure dynamics: some may pursue multiple goals simultaneously and elegantly navigate between them, while others put their goals in serial position and pursue one goal at a time. Students may also decide not to pursue learning goals when they feel that one or more of their psychological needs (autonomy, competence, social belonging) are thwarted. (Boekaerts, 1999)

For the purposes of the present paper we shall distinguish within mathematical behaviour two individualistic needs (autonomy and competence) and two social needs (belonging and status). Autonomy is the need to have a control over own actions and to feel self-determining. Competence is the need to be able to comprehend and influence own environment. Social belonging is the need to be part of a social group, and social status is a social equivalence for competence – a need to have influence within a social group.

As a starting point for goals we take the three motivational orientations: learning, performance, and avoidance. However, we do not regard them as alternative orientations, but as goals that may be pursued simultaneously (the case study below will illustrate this). Thus, when a student is given a mathematical task, he or she might adopt a goal to master the topic, to demonstrate high ability, and/or to avoid public failure. Furthermore, we can use the empirical results by Lemos (1999), who observed and interviewed Portugese sixth grade students. She concluded that students' activities in class could be classified under following goals: working goals, evaluation goals, learning goals, complying goals, interpersonal relationship goals, enjoyment goals, and discipline goals (in order of decreasing frequency). Some of these goals are on a more specific level, but evaluation goal roughly equals performance goal and learning goal equals mastery goal. Enjoyment, however, should not be accepted as a goal. Emotions regulate goal directed behaviour, and enjoyment is a general reaction that may be related to a variety of different goals (for further elaboration, see Hannula, in print).

Within the goal structure we can distinguish two kinds of relations (Fig.1). Firstly, there are personal beliefs about how goals are related to other goals and different needs. One may perceive a single goal to satisfy multiple needs and a need to be satisfied through multiple goals. Goals may also be seen as contradictory in a sense that reaching one goal might prevent achieving another goal. For example, it is not possible to show high performance without taking a risk of failure. Another kind of relations are the values of needs and goals (seen as comparative evaluations). Needs values are relatively stable characteristics of the personality although when a need is fulfilled at a moment it may be temporarily given lower priority. Goal values are partly derived from respective needs values and partly from the beliefs about how reaching a particular goal will affect different needs and other goals. In the given example (which shall be further elaborated later) the student gives higher value for social status compared to competence. In classroom context this leads to lower derived value for understanding (learning goal) compared to performance and failure avoidance goals.

A second aspect behind goal choices are beliefs about accessibility of goals. This aspect is usually discussed under the term 'self-efficacy beliefs' (e.g. Philippou & Christou, 1999, Risnes, 1998). Here, I only stress the importance of goals in relationship with beliefs. Beliefs as obstacles for an educational change have been discussed by Pehkonen (1999). Furthermore, Lemos (1999, p. 482) pointed that "in the absence of valued personal goals, individual's beliefs do not seem to play a helping role in overcoming stressful situations." It seems, that in order for change to take place two conditions must be met. Firstly, there needs to be a goal that motivates the change and, secondly, one's beliefs must support the change.

Earlier at PME (Hannula, 1998a) the author has reported a case study of a radical change in student beliefs and behaviour that includes these two aspects. Using the present terminology of goals, we may say that the case student had self-defensive goals dominating her behaviour in the beginning ('You don't need math in life'). However, this was later replaced by performance goals ('I will raise my math number'). Behind this change there was a new awareness of the importance of school success in general (change in goal values) together with more positive self-efficacy beliefs (success is possible).

The third aspect to be discussed here are automated emotional reactions as an inertia force to students' goal choices. There are two fundamentally different ways how emotional state may be changed (Power & Dalgleish, 1997). One way is the (possibly unconscious) cognitive analysis of the situation with respect to one's goals. Another

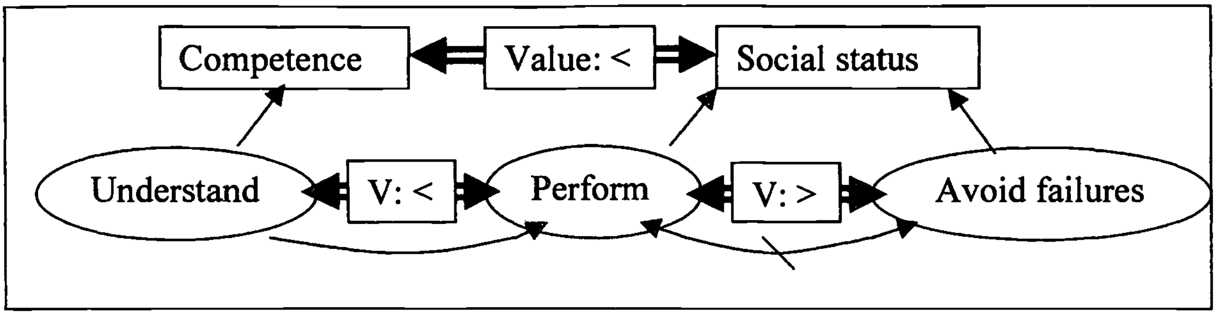

Figure 1. A part of a needs-goals structure. In top row are two needs and their relative value (competence is more important). In bottom row are goals and their values. Thin arrows represent beliefs about the relationships between goals and need.

route to change emotional state is through association to one element of the situation. Emotional associations are learned via classical conditioning and they are the core of attitude as an emotional disposition (Hannula, submitted). Although they allow shorter reaction times to possible threats, they lack flexibility and are an inertia force of behavioural changes. Once formed, these associations are difficult to change. During school years students usually develop some emotional disposition to different mathematical actions and goals. Therefore emotional associations may function as an inertia force against change, even when change would be 'rational'.

METHODOLOGY

There is a serious methodological problem with research on such mental constructs as beliefs and goals. We can't directly access student's all beliefs and goals. Some of the goals and beliefs are always hidden even to the student him/herself and they need to be reconstructed through interpretation of the observable. Even if we succeed in explaining all the utterances and actions of the student, we have constructed only one possible mental configuration behind the observable.

In present study the solution to overcome these methodological problems has been to collect a large and varied data (classroom observations, individual and group interviews, interviewing parents and teachers) on a small number of students. The study was longitudinal (three years) and the researcher interacted a lot with the students as their teacher and thus gained tacit knowledge that has guided the interpretations. Furthermore, the use of multiple frameworks to analyse students' beliefs and attitudes has enriched the understanding of students (Hannula, 1998a; 1998b; 2001; submitted). However, using a broad spectrum of analytical frameworks has its inevitable cost in lack of depth.

SOME DATA

Laura, performance through mastery

Laura was the student whose goal structure was used as an example above (Fig 1). She had been a successful student in elementary school. There she never had needed to prepare for mathematics tests, and it took some time (and unsuccessful tests) before she realised that in secondary school she needed to start working. She thought that studying mathematics was boring at times, but that it was nice in the class when she was able.

One ground to claim that performance goal was more important to Laura than learning goal was that understanding alone was not enough for her. She also wanted to get praises for her good performance.

> "If you have been thinking yourself crazy and if you have got them right, so that makes you feel real good except, if ... you have been thinking really hard, and ... the teacher does not say 'Good!' either."

Furthermore, her best memories in mathematics were when she could outperform the others at school.

> [The nicest thing in elementary school in math was to] "learn addition the first day ... because I could do them all and it was real fun."

There was consistent evidence that the social status was an important need for her. In classroom context she did not get to a leading position and the conclusion adds up from minor events, as a tacit knowledge gained through three years of observation. More explicitly, she expressed her pride and happiness for gaining a leading position in her hobby and her relationship with her younger brother also reflects an enjoyment of having power over others.

> "Maria asked, the other day, advice for what to tell her younger brother, who always is depressing her somehow, saying things like 'I'm better in math than you'. And Maria asks what she can do. I told her to grab him by his shirt tightly and yell: 'I am you elder sister!' [] Maria maybe has not enough charisma to influence him."

Although Laura's main goal was performance, she also had a mastery goal to really understand mathematics, and this goal she approached often with her father.

> ... all the interesting discussions that I have with my father, that why 4*(-4) is not, for example, + 16 instead of -16. And about what is to power of zero, such really interesting issues that I do not comprehend."

In this context she could achieve both need of competence and need of social belonging. This example illustrates how goal choices depend on context and situation.

Above, I have only presented a selected sample of Laura's interviews. That data alone would, of course, be open to several different interpretations. However, the interpretation that I have presented is supported by further data that can't be presented due to space limitations.

Maria, competence through performance

Maria was another student in the same class, who was mentioned by Laura above. She was a high achiever, and she wanted to be perfect in everything. At primary school she had felt that it had been difficult to keep up the fast tempo that some of her classmates had kept. She also had felt that it had been difficult to avoid mistakes, even though she had understood what to do. She had been bored by calculating long lists of routine tasks, and preferred doing word problems. At grade six she had started to understand mathematics better, had achieved higher, and had started to like mathematics more.

Maria had clearly a performance goal in mathematics, as she admitted in an interview:

> "But usually I like tests, I have always liked. ... Some say that I am the kind of person who likes so much to compete. ... Usually it's nice to show it, when you are good at something."

She did not like group work, because she felt that the others didn't work as hard as she did. Furthermore, when she worked alone, she would get all honour for the result herself. However, Maria did not boast with her success in the class.

Maria had also a mastery goal. She was challenged by more difficult problems even when nobody would know about her performance. She was driven by a will to overcome the challenges and she enjoyed especially tasks where she could see their applicability.

> "I do not know if that is allowed, but I do sometimes look the more difficult tasks" [while others check homework]"

> "[If a task is not solved] I can not go peacefully to sleep, because you still think how it would go."

> "I like [equations], because it feels natural and purposeful when, for example, with world problems you need to think and apply, so it is not only that you move figures, but there is a purpose. Such problem could exist in real life and so it is not just calculations."

My understanding of Maria is that she was, deep inside, uncertain of herself. Therefore she had a strong need to feel competent. Her goal in the math class was to learn and convince to herself that she is intelligent and competent. As a sub-goal she wanted to monitor her own success. Tests and challenging tasks were her way to convince herself that she is doing well enough.

SOME CONCLUSIONS

Altogether eight students' goal structures have been analysed. As a general finding it should be noted that there is great variation in goal structures and they do not provide easy means for classifying students. As it became evident in cases of Laura and Maria, performance and mastery are not contradictory goals. There also seems to be a developmental trend towards mastery goals. However, we do not know if this is a general developmental trend or due to teacher's efforts to promote such orientation. This development towards mastery goals seems to co-evolve with a view of mathematics as a sense-making activity. As an unsurprising finding we see that avoidance goals occur together with a belief of self as untalented in mathematics.

Three aspects of goal regulation were specified in the theory: deriving goals from needs, the influence of goal accessibility beliefs, and emotions as an inertia force. Within the empirical data it was possible to identify examples of all three aspects. The cases of Laura and Maria were presented as examples of the first aspect.

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„ ... BECAUSE A SQUARE IS NOT A RECTANGLE" – STUDENTS' KNOWLEDGE OF SIMPLE GEOMETRICAL CONCEPTS WHEN STARTING TO LEARN PROOF

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The knowledge of concepts is essential for students when they start to learn proof. Empirical findings of a study with 106 grade 8 students show that there are deficits in students' concept understanding scheme for quadrangles. These deficits are particularly based on a different idea of classification of quadrangles and difficulties in the understanding of the mathematical language and thinking - problems which will cause difficulties regarding learning and teaching proof.

INTRODUCTION

The mathematical definition is one of the basic terms which together with axioms, theorems, proofs, corollaries, lemmas and propositions constitutes the skeleton of the mathematical theory. Though it is clear that a definition has to satisfy certain necessary properties like noncircular, non-contradictory, there is little agreement which properties are sufficient to constitute a good and elegant definition (cf. Shir & Zalavsky, 2001). However, elegant or not, mathematical definitions are prerequisites for the formulation of theorems and proofs and, therefore, they are essential for the development of mathematics as a deductive theory.

Teaching definitions in mathematics classroom is a task which depends on more than only mathematical requirements. In particular, in geometry classroom on the primary level students do not learn geometry as a deductive theory. Based on pedagogical and psychological reasons, geometry is introduced to young students as a theory of the visual space. It is obvious that in this stage the concept formation and the recognition of simple mathematical "theorems" is not undertaken by teaching formal definitions but mainly by examples and visual representations. The transition from everyday life thinking in the theory of the visual space to a scientific thinking in deductive geometry in higher grades is afflicted with many problems. These problems cause students' difficulties in advanced mathematical thinking as it is required in problem solving, reasoning and proof. One of the basic deficits can be found in the students' personal knowledge of concepts and its usage in a mathematical context.

THEORETICAL FRAMEWORK

Scientific thinking skills and prerequisites for learning reasoning and proof

The last decades many researchers from cognition psychology and mathematics education contributed to the description of the development of students' thinking skills. Research in this area shows that there are several restrictions in the preadolescent scientific thinking (e.g., Kuhn, 1989). For example, an empirical-inductive reasoning is typical for students on the concrete-operational stage, whereas a more hypothetical-deductive reasoning is typical for the formal-operational stage (Flavell, 1977). This hypotheses is supported by empirical findings of studies with grade 8 and grade 10 students' ability in mathematical reasoning and proof (Healy & Hoyles, 1998; Reiss & Thomas, 2001).

In a former study with students in upper secondary level we identified different prerequisites for the understanding of proofs in the geometry classroom. As described in Reiss, Klieme & Heinze (2001) geometrical competence is specifically influenced by methodological knowledge, declarative knowledge, metacognition and spatial reasoning. For the evaluation of students' declarative knowledge we chose the concept "congruence", a central concept of school geometry. The students were asked to give a definition, an example, a visual portrayal of the word "congruence" and to name a mathematical theorem in which the concept features. Our analyses have revealed considerable deficits in declarative knowledge. It emerged that even students at the end of secondary level often have only a vague intuitive understanding of concepts such as "congruence", that this understanding is restricted to examples, and that they have no exact mathematical knowledge of the respective definitions and theorems.

The last point corresponds to findings of several other studies: students' ideas about a geometrical concept and the definition of this concept are frequently inconsistent (e.g., Hershkowitz & Vinner, 1982; Wilson, 1990). The use of examples as an interpretation of definitions is often restricted to certain prototypes which, in addition, contain irrelevant characteristics. Distinguishing between these irrelevant characteristic properties and the relevant definitional properties is difficult for students (e.g., Burger & Shaughnessy, 1986; Wilson, 1990). Other problems related to definitions are the students' understanding of necessary and sufficient conditions in definitions and their imprecise use of words if they give definitions (Burger & Shaughnessy, 1985, 1986; Wilson, 1990). These deficits in students' ideas about geometrical concepts are strongly affected by the representation of these concepts in mathematics classroom and textbooks (e.g., Burger & Shaughnessy, 1985).

Concept definition, concept image and concept usage

As mentioned above we have to distinguish between the mathematical definition of a concept and the personal image of this concept in student's mind. To describe this fact Vinner and others introduced the theoretical model of the personal *concept image* (Vinner, 1991) which was used to identify students' ideas of different mathematical concepts like function, limit etc. (e.g., Tall & Vinner, 1981). The concept image is described as

> "... something non-verbal associated with the concept name. It can be a visual representation of the concept in case the concept has visual representations; it can be a collection of impressions or experiences" (p. 68, Vinner, 1991).

The concept image is evoked in the memory by the concept name. It is specific for an individual. The existence of a concept image is a necessary condition for the understanding of a concept. "To acquire a concept means to form a concept image" (p. 69, Vinner, 1991). The knowledge of a concept definition may be independent from the formation of a concept image: to know a concept definition does not imply to understand the concept.

Moore (1994) extended the ideas of Vinner to the *concept-understanding scheme*. The concept-understanding scheme contains three aspects of concept understanding: the concept definition, the concept image and, as a third aspect, the *concept usage*. The concept usage "refers to the ways one operates with the concept in generating or using examples or doing proofs" (p. 252, Moore, 1994).

Both, Moore (1994) and Vinner (1991), described the main problem for students using concepts in their mathematical activities. If a student is confronted with a mathematical problem in which a certain concept appears, then in his or her mind the associated concept image is evoked. To get a correct solution of the problem the student has to check, if the operation done with the evoked personal concept image is compatible with the concept definition. Here a difference between every day thinking and scientific thinking appears: in every day thinking, in general, it is not necessary to check the concept definition (in many cases such a concept definition even does not exist, e.g., for the concept "tree"). In a mathematical context, in general, it is indispensable to compare the personal concept image with the concept definition. The ability to "switch" between the personal concept image and the concept definition is essential for the solving processes in a mathematical context. It can also be observed in the research process of mathematicians: mathematicians do not retrieve definitions and theorems from their memory to construct logical deductions. On the contrary, first they do not pay attention to each detail in the process but consider the line of argumentation in broad terms and recognise important properties and connections. Finally, if they know how to argue they will construct a mathematical proof using formal definitions and theorems (cf. Koedinger & Anderson, 1990).

Partitional and hierarchical classification

When discussing the concept understanding scheme of geometrical concepts like triangles or quadrangles, we must also consider the classification of these concepts. As described in de Villiers (1994) there are two main classification types: the hierarchical and the partitional classification. *Hierarchical classification* means the classification of a class of concepts in such a manner that the more particular concepts form subclasses of the more general concepts (class inclusions). In contrast, in a *partitional classification* the various subclasses of concepts are considered to be disjoint from one another. For example, in the first case we can define squares as special rectangles and rectangles as special parallelograms. In the second case a square is not a rectangle and a rectangle is not a parallelogram. Since in mathematics classifications and the associated definitions are arbitrary in a certain sense, the choice for a hierarchical or partitional classification is a question of convenience and economical and personal reasons. In general, mathematicians prefer a hierarchical classification for triangles and quadrangles.

There are several studies which show that students in lower secondary level still tend to a partitional classification in the case of quadrangles (e.g., Burger & Shaughnessy, 1985; de Villiers, 1994, 1998). Moreover, de Villiers (1994) showed that even students who exhibit excellent competence in logical reasoningstill prefer to define quadrangles in partitions, if given the opportunity. He suggests to treat the classification of concepts like triangles and quadrangles in such a way that a meaningful discussion is possible. As described in deVilliers (1998) by comparison of advantages and disadvantages the students will then realise that hierarchical classifications are more economical than the partitional ones.

RESEARCH QUESTION AND DESIGN OF THE STUDY

According to the theoretical framework described above we investigated aspects of the concept understanding scheme of students who started to learn proof. As concept we chose quadrangles, in particular, squares and rectangles which are well known to these students. The research question to be addressed in this paper is the following:

- Are the students' concept understanding schemes of (special) quadrangles sufficient for solving problems in different situations like recognising equivalent descriptions, finding counterexamples and distinguishing between sufficient and necessary conditions?

For this research question we considered three items of a study which was carried out for the investigation of informal prerequisites for informal proofs. In this study five mathematical principles (definitions, equivalent descriptions, arguments and proof, logical implication and counterexamples) were represented by ten items in a paper and pencil test. This test was administered by a teacher in four classes of grade 8 in a German Realschule (Realschule means lower secondary school for students with an average proficiency level). Altogether the sample comprises of 106 students (50 female, 53 male, no data 3). The students were asked to answer the test in 45 minutes. A detailed description of this study is given in Heinze (to appear).

The three items related to the research question are the following:

1. The recognition of equivalent descriptions of a square: Six descriptions of quadrangles were given and the students had to mark which of them describe squares (multiple choice).
2. The finding of a counterexample: We presented the following problem: "Klaus considers squares and rectangles. He says: 'In each quadrangle each angle is 90°.' Karin says: 'This is not true'. How can you show, that Karin is right?".

3. The distinction between necessary and sufficient conditions: Here we asked: "If a quadrangle is a rectangle, then the opposite sides are parallel. Consequently: If the opposite sides of a quadrangle are parallel, then it is a rectangle. Is this true?"

RESULTS

Table 1 shows the results of the first item (Which quadrangles are squares?):

	Quadrangles with ...	Frequency	Percentage
(a)	four sides of same length and all angles of 90°	88	83,0 %
(b)	four angles of 90 °	48	45,3 %
(c)	three angles of 90° and two neighbouring sides of same length	6	5,7 %
(d)	all sides of same length	34	32,1 %
(e)	opposite sides are parallel	63	59,4 %
(f)	four sides of same length and one angle of 90°	19	17,1 %

Table 1: Correct answers for the recognition of a square as a special quadrangle

If we consider the alternative descriptions of a square (answers (a), (c) and (f)), we see that 83 % of the students know that quadrangles with four sides of same length and four angles of 90° are squares. Answer (f) (four sides of same length and one right angle) is recognised by 17,1 % and statement (c) (three angles of 90° and two neighbouring sides of same length) is accepted by only 5,7%. Conversely, more than half of the students think that each quadrangle with four right angles is a square (b) and more than two thirds believe that each equilateral quadrangle is a square (d). The fact that, in general, a parallelogram is not a square (e) is known by nearly 60 % of the students. It is interesting to see that about two thirds of the students think that an equilateral quadrangle is a square (d) but only 17,1 % gave the answer that an equilateral quadrangle with one angle of 90° is a square (f).

The total number of correct answers for each student is presented in Table 2:

corr.	Frequency	Percentage
0	0	0 %
1	21	19,8 %
2	35	33,0 %
3	34	32,1 %
4	15	14,2 %
5	1	0,9%
6	0	0 %

Table 2: Total number of correct answers for each student

Furthermore, if we consider only the correct descriptions in the first item (answers (a), (c) and (f)), then 71,7% of the students recognised one of these, 13,2% two and only 2,8% (three students) recognised all three correct descriptions of a square among the six given answers.

A deeper analysis of the students' answers for this first item shows that there is indeed a certain ordering of the different descriptions by difficulty: the easiest cases are (a), (e) and

(b) (in this ordering), i.e., students with three or more correct answers mostly have these three cases correct.

Table 3 gives the results for the second item (In each quadrangle each angle is 90°):

	Frequency	Percentage
correct	68	64,2 %
false	22	20,7 %
no response	16	15,1 %

Table 3: Results for the second item

More than 60 % of the students gave a correct answer, nearly all of them gave a counterexample. About 20% gave a false answer, half of these students said that it is true that each angle in each quadrangle is 90°.

Table 4 presents the answers for the third item (necessary and sufficient condition):

	Frequency	Percentage
correct with reasons	2	1,9 %
correct without reasons	8	7,5 %
correct with false reasons	30	28,3 %
false	24	23,1 %
no response	42	39,6 %

Table 4: Results for the third item

About 37 % gave a right answer, whereby many students (28,3 %) gave false reasons for their response. More than half of these cases with false reasons (16%) was based on a partitional classification of quadrangles (without class inclusions). Typical answers were "The opposite sides of a square are also parallel, but a square is not a rectangle." In addition to this, we noticed that many students do not distinguish between the concepts of square resp. rhombus and quadrangle. Furthermore, it is remarkable that nearly 40 % of the students gave no (mathematical) response to this item. Often they wrote, that this item is too complicated or that there is no logic in this question.

DISCUSSION

For an interpretation of the described results it is necessary to analyse the different items and their requirements. In particular, the six problems in the first item require different kinds of thinking. For example, for the cases (c) and (f) it is not possible to find a correct answer without an analytical process of thinking. For these cases we get only a small number of correct answers which indicate that the students mainly did not use analytical approaches. The results for the two cases (d) and (f) (Is an equilateral quadrangle resp. equilateral quadrangle with one 90° angle a square?) supports this fact: 61 students (57,5%) said that an equilateral quadrangle *is* a square and an equilateral quadrangle with one 90° angle *is not* a square. The problems in this case may be based on the students' difficulties with sufficient and necessary conditions and the understanding of the "mathematical language" ("one 90° angle" means "only one 90° angle").

Restrictions in the understanding of sufficient and necessary conditions and the language resp. thinking in mathematics can be also identified in the third item (If the opposite sides of a quadrangle are parallel, then it is a rectangle). Though this item is similar to case (e) in the first item, it is more difficult for the students (37,7% to 59,4% correct answers). This may be caused by the explicit question if the *necessary* condition "opposite sides are parallel" of a rectangle is also *sufficient*. A fact that is also influenced by deficits in mathematical language is the students' preference of the partitional classification of quadrangles. Nearly half of the answers which were accepted as correct (16% of 37,7%) were based on this classification. The frequently given reason "a square is not a rectangle" may also be caused by the interpretation of the word "is" as "is equal to" (cf. de Villiers, 1994).

The best results the students obtained for the second item ("In each quadrangle each angle is 90°."). Here, about 64% gave a correct answer. Nevertheless, 15% of the students gave no response and 20% gave a false response. For 12% of the students with false response we identified a restricted concept image of quadrangles (In a quadrangle each angle is 90°). As one can expect, these students also achieved poor results for the other items.

The results show that many students have deficits in the concept understanding schemes for the discussed quadrangles. In particular, if they have to use the concepts for certain problems they remain on using their personal concept image and ignore the concept definition. In addition, the findings support the results of de Villiers (1994) that many students prefer a partitional classification for quadrangles. This seems to be related to a wrong understanding of the mathematical language and thinking.

The difficulties described above are problematic in a stage where the teaching and learning of reasoning and proof begins. In particular, the fact that in mathematics classroom teachers and a part of the students have a different understanding of the classification of concepts and of the mathematical language and thinking cannot be considered as a basis for first steps in advanced mathematics. It is essential for a successful instruction in advanced mathematics that teachers and students agree on a common view on the basic mathematics.

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FLEXIBLE MENTAL COMPUTATION: WHAT ABOUT ACCURACY?

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Flexibility in strategy choice in mental computation is considered to be a component of number sense. This paper reports on an investigation into cognitive, metacognitive, and affective factors that support both flexibility and accuracy in mental addition and subtraction in Year 3 students. While some factors appeared to be essential for flexibility, additional factors were necessary for accurate employment of strategies. Further, there were qualitative differences between the mental strategies employed by the students who were accurate and those who were inaccurate.

While standard computational algorithms (both written and mental) are still taught in many classrooms around the world, there is reduced emphasis on the importance of these algorithms and increased emphasis on "number sense" (e.g., National Council of Teachers of Mathematics, 1989). Several interpretations exist for number sense; however, "flexibility" and "inventiveness" seem appropriate ones in this discussion (Anghileri, 2001). It has been recognised that the development of flexible mental computation strategies are a component of number sense (e.g., Klein & Beishuizen, 1994; McIntosh, 1998; Reys, Reys, Nohda, & Emori, 1995), and that when children are encouraged to formulate their own mental computation strategies, they learn how numbers work, develop number sense and develop confidence in their ability to make sense of number operations (Kamii & Dominick, 1998).

It would appear that the purpose of the inclusion of mental computation in any mathematics curriculum would be to develop flexible computational strategies, and thus promote number sense. Some teaching experiments have focused on the successful development of students' flexible computational strategies (e.g., Buzeika, 1999; Kamii, 1989). Other literature reports that children have the ability to develop their own efficient mental strategies, even without instruction (e.g., Heirdsfield, 1999). While flexibility might be one of the foci of this research, proficiency (defined here as both flexibility *and* accuracy) would appear to be important as well. It is posited that the study of proficiency in mental computation extends beyond the development of flexible mental strategies, but also encompasses the study of associated factors that might contribute to both flexibility and accuracy.

Associated factors have been reported elsewhere (e.g., Heirdsfield, 1996; Kamii, Lewis, & Jones, 1991; Sowder, 1994; Van der Heijden, 1994). These include (a) number sense (number fact knowledge, estimation, numeration, effects of operation on number), (b) affective factors (beliefs, attributions, attitudes, self efficacy, social context), (c) metacognitive processes (strategies, beliefs, knowledge), and (d) memory (short term memory, long term memory – knowledge base).

This paper reports on a study of Year 3 children's mental computation (addition and subtraction), which was conducted in three classrooms (2 schools, A and B). In all of these classrooms, students were taught traditional pen and paper algorithms (mental computation is not mentioned in the existing Queensland Years 1-10 mathematics curriculum document and is not treated consistently in Queensland schools). Mental computation appeared to only refer to number facts and extended number facts (e.g., 30+40=70, because 3+4=7). However, in one of the classrooms (School B), students were also encouraged to consider alternative strategies, and would have been permitted to use them. Of interest here is those students who were identified as being flexible in their mental addition and subtraction strategies. While it appears that the students developed their own flexible mental strategies, not all were successful (i.e., accurate) in applying them. It could also be argued that not all were "efficient" either.

THE STUDY

Participants. Seven flexible Year 3 students were selected from three classes in two Brisbane Independent schools (Schools A and B) that served high and middle socioeconomic areas. The students were selected on the basis of their responses to a structured addition and subtraction mental computation selection interview. All students were identified as being flexible in their employment of mental strategies, 4 were accurate (2 from School A and 2 from School B) and 3 were inaccurate (School A). Accurate computers were those who were more than eighty percent correct in their responses on both the addition and subtraction selection items. Inaccurate students generally attained between thirty and eighty percent accuracy on either the addition and subtraction items (more errors were made on the subtraction items).

Procedure and instruments. All students from the three classes were withdrawn from class and interviewed individually in a structured mental computation interview. Students who employed a variety of strategies were selected to participate in further interviews. These interviews constituted a series of videotaped semi-structured clinical interviews in a quiet room in the school. The interviews addressed mental computation strategies, number facts, computational estimation, numeration, effect of operation on number, metacognition, affect, and memory. These have been described in more depth elsewhere (Heirdsfield, 2001; Heirdsfield & Cooper, 1997).

Analysis. For the purposes of identifying flexibility in mental computation, mental computation strategies were identified using the categorisation scheme (based on Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995; Thompson & Smith, 1999) that divided the strategies into the following categories: (1) *separation* (e.g., 38+17: 30+10=40, 8+7=15=10+5, 40+10+5=55); (2) *aggregation* (e.g., 38+17: 38+10=48, 48+7 = 55); (3) *wholistic* (e.g., 38+17 = 40+17-2 = 57-2 = 55); and (4) *mental image of pen and paper algorithm* – following an image of the formal setting out of the written algorithm (taught to almost automaticity in the schools the students attended).

Mental computation responses were analysed for strategy choice, flexibility, accuracy, and understanding of the effects of operation on number, numeration, computational estimation, and number facts. Analysis of the interviews investigating these individual factors was also undertaken, with the intention of exploring connections with mental computation. Students' responses were also analysed for metacognition and affects, and scores and strategies were recorded for the memory tasks. Each student's results for aspects of number sense, metacognition, affects and memory were summarised. These summaries were combined for each of the computation types: accurate/flexible and inaccurate/flexible, so that comparisons could be made between the two types. The knowledge shown by the students of each type were analysed for commonalities and these commonalities were used to develop a *composite* picture of a typical student of that type. The two resulting knowledge structures, one for accurate and one for inaccurate, were depicted by networks.

RESULTS

Both the accurate and inaccurate students spontaneously employed a variety of strategies (*separation*, *aggregation*, and *wholistic*) although the inaccurate students tended to have less variety, using predominantly *separation* strategies. When encouraged to access different strategies, both accurate and inaccurate students were able to do so but with different outcomes. The accurate students were successful in their use of the new strategies while the inaccurate experienced difficulties in completing the strategies (although they had sufficient understanding to access the strategy).

Accurate students. Although both accurate and inaccurate students were identified as flexible, there was little in common between the two groups. The students who were accurate showed in their responses to the interviews that they possessed well-integrated knowledge bases. The composite picture of their knowledge is depicted as a network in Figure 1.

As can be seen in the figure, the accurate mental computers were fast and accurate with their number facts, used efficient number facts strategies (e.g., 8+6=14, because double 6 and 2 more make 14) when facts were not known by *recall*, and had extended their number facts strategies to efficient mental computation strategies (e.g., 9+6=10+6-1=15 is similar to 246+99=246+100-1). Although it might have been expected that estimation would contribute to mental computation, only one of the accurate students exhibited proficiency in estimation. This student also employed estimation in mental computation to get a feel for the answer and check the solutions.

The accurate students used good numeration understanding (particularly canonical, noncanonical, multiplicative, and proximity of number) and some understanding of the effect of operation on number to support their efficient use of a variety of strategies. This was particularly so for the *wholistic compensation* strategy (e.g., 246+99=246+100-1) for which numeration understanding (particularly proximity of number) and understanding of the effect of operation on number appeared to be essential. The accurate students had accurate perceptions of their ability to solve the mental computation tasks (metacognitive beliefs), and they used metacognitive strategies (e.g., monitoring, reflecting, regulating, and evaluation). Beliefs in self seemed to be associated with a belief about the place of the teacher in the student's learning; for instance, accurate students tended to have confidence in self-initiated strategies (c.f., teacher-taught strategies). Although there was not always evidence of the belief that mathematics makes sense, that belief was strong in one student.

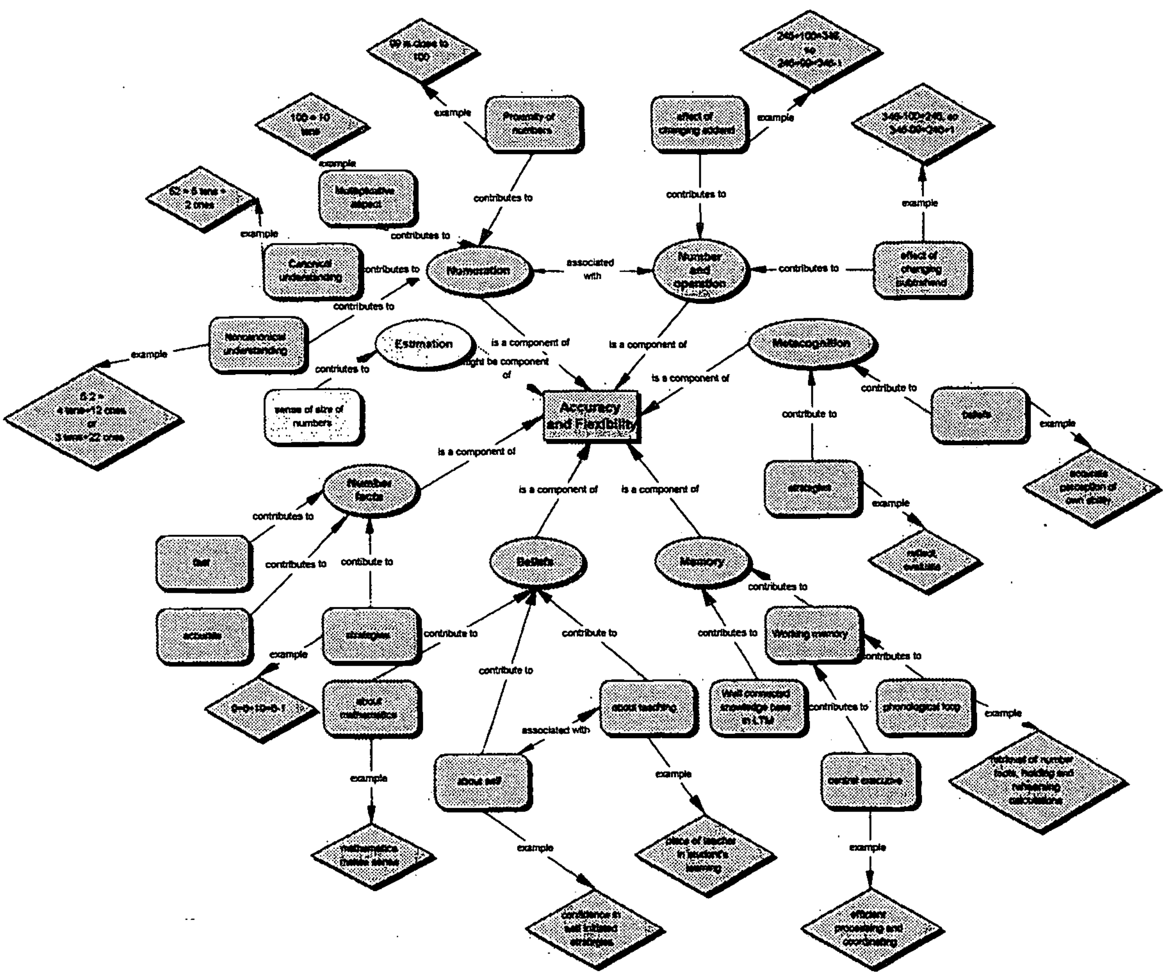

Figure 1. Network showing knowledge for accurate/flexible mental computation (shaded - present, speckled - partially present)

Accurate students showed reasonable short-term memory (STM) and central executive functioning. This would have provided them with efficient retrieval of number facts from long-term memory (LTM), effective holding and rehearsal of interim calculations, and efficient processing and coordinating of strategies. However, the study also showed that STM was not as important as might be

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predicted as efficient mental strategies place less demand on STM and require fewer interim calculations.

In summary, the composite accurate/flexible mental computer was shown to have a rich integrated network of cognitive, metacognitive and affective components.

Inaccurate students. Although the inaccurate students in this study were categorised as flexible, they did not exhibit the same degree of flexibility as the accurate students. They did employ a variety of strategies, but they tended not to be high-level strategies (e.g., *wholistic*), and there was very little in common between the two groups. The composite picture of their knowledge is depicted as a network in Figure 2. It shows that the inaccurate students had much less knowledge and fewer connections between factors than the accurate students.

All knowledge exhibited by inaccurate students seemed to be at a threshold level, rather than at an optimum. The inaccurate students exhibited some flexibility and efficiency (although not always speed and accuracy) in number fact strategies. However, these strategies did not always support interim calculations in mental computation, as the students often calculated interim calculations by counting, rather than by employing more efficient *derived facts strategies,* which they used in the number facts test. Similarly, numeration understanding was evident at a threshold level, particularly, canonical and noncanonical. A further attribute of numeration, proximity of number appeared to be at a threshold level, as the students attempted to use this when accessing the *wholistic compensation* strategy. However, their knowledge of the effect of operation on number did not support high-level strategies and their estimation was poor. There was evidence of some metacognitive strategies, such as reflection, evaluation, and checking solutions. However, unlike the accurate students, metacognitive beliefs were poor.

Beliefs, in general, were difficult to elicit from the inaccurate students, and when elicited, were inconsistent. There might have been several reasons this. These students might not have held any strong beliefs about themselves, about mathematics (e.g., whether mathematics should make sense), or about teaching. Also, they might have been unaccustomed to verbalising their beliefs.

Finally, as with the accurate students, the inaccurate students had reasonable STM and central executive functioning (e.g., planning and allocation of attention). However, these abilities were little help to the students because number facts were not sufficiently well known to be retrieved by STM, interim calculations were completed so slowly that they placed a heavy load on STM, and the students' knowledge base was so poor that the central executive could not successfully retrieve information.

The question remains: Why were the inaccurate students flexible? The answer might lie in what appeared to be the lack of understanding of taught procedures. When these students were unable to use these procedures, they compensated by inventing strategies. These strategies tended to be lower level (i.e., *separation*) and their use unsuccessful, as the inaccurate students' knowledge (particularly of numeration and effect of operation on number) was insufficient to enable higher-level strategies to be attempted and any calculation to be completed accurately.

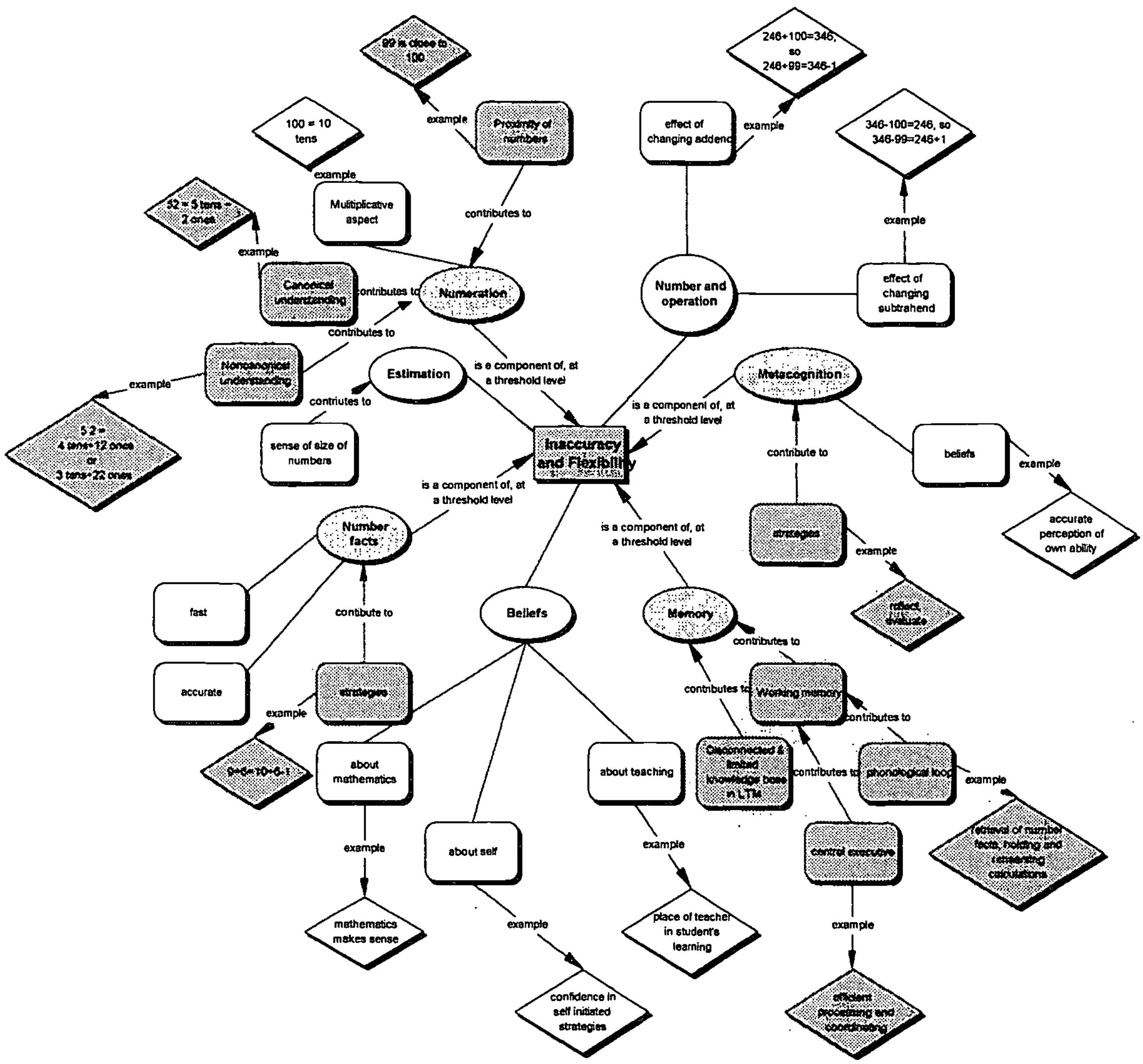

Figure 2. Network showing knowledge for inaccurate/flexible mental computation (shaded - present, speckled - partially present, clear - absent)

In summary, the composite inaccurate/flexible mental computer was shown to have knowledge at a threshold level that was insufficient for employment of advanced mental strategies and accurate use of other strategies.

DISCUSSION

Although flexible use of mental computation strategies is an important component of the development of number sense, this study shows that it is not sufficient for accurate computation. A well-connected knowledge base (where number facts, numeration, number and operation, and to less extent, estimation were part of that knowledge base), metacognitive strategies and beliefs, and an efficient central executive (to coordinate retrieval from LTM and allocation of strategies and facts for short-term storage and manipulation of numbers) supported accuracy and flexibility in mental computation. Without this knowledge base, students were inaccurate.

It seems that with a strong connected knowledge, accurate students had more options available for mental strategies. With a less connected and weaker knowledge base, inaccurate students' use of strategies was an attempt to compensate for their lack of knowledge. The inaccurate students compensated by inventing strategies when the teacher-taught strategies could not be followed. However, although STM was sufficient, these students' knowledge base was so minimal and disconnected that the use of the strategies was not efficient, and resulted in errors. Further, the knowledge base did not support high-level strategies.

This demonstrates the need for teaching practices to go beyond developing flexible use of strategies in mental computation. The practices should not focus on the strategies in isolation; they have to focus on the development of an extensive and integrated knowledge base to support the strategy use. This means covering number facts, numeration, effect of operation on number, and estimation. Other factors that need to be addressed are metacognition and affects.

Students can and do formulate their own strategies and this should be encouraged because of the learning that results with respect to number sense (e.g., Reys, Reys, Nohda, & Emori, 1995). However, if accuracy in mental computation is one of the aims of computation, more has to be done than encouraging students to formulate their own strategies. While research (e.g., Buzeika, 1999; Kamii, 1989) has reported success with teaching experiments that encourage students to formulate their own strategies, it is obvious that other cognitive, metacognitive and affective factors come into play. In this study, it was shown that accurate (and flexible) mental computation was supported by a complex interaction of cognitive, metacognitive and affective factors. Further research is warranted as to teaching practices that can develop flexibility and the supporting knowledge necessary for accuracy and flexibility, possibly following the lines of Cognitive Guided Instruction (Carpenter, Fennema, Franke, Levi, & Empson, 1999), but including children's affects and metacognition.

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CHANGE IN PUPILS' APPROACH TO PROPORTION PROBLEMS THROUGH CLASSROOM TEACHING: FROM THE VIEWPOINT OF MEDIATION BY SYMBOLS

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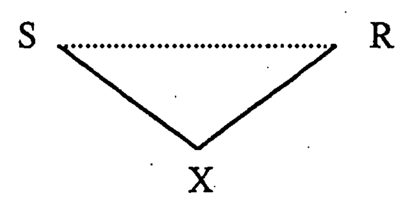
This article is part of a larger study that examines the development of pupils' proportional reasoning through classroom teaching. In a fifth-grade classroom, five pupils with varying proportional reasoning ability were studied by using qualitative research method. In this article, the pupils' approaches to the same proportion problems before and after a series of 11 lessons on "quantity-per-unit" are compared. Results show that the major change after the lessons was their more active use of notations in dealing with the aspect of ratio and proportion. Especially there are notations that lead one to make aware of his/her own thinking, i.e., make it articulated, systematic, conscious and intentional. These significant notations were observed clearly in four pupils. Still, specificity of the notations varied depending on the pupils' conception of invariant relation between two quantities.

BACKGROUND OF THE STUDY

The purpose of this study is to examine how the development of pupils' proportional reasoning takes places in the mathematics classroom. A great deal of studies on proportional reasoning has shown different strategies children use and relationships of them with task characteristics (e.g., Vergnaud, 1993; Kaput & West, 1994). Along with the result of investigation in everyday life and in workplaces, researchers began to discuss the inextricable links of people's thinking strategies with contexts (e.g., Nunes et al., 1993; Hoyles et al., 2001). In this study, classroom is also recognized as a unique place where students meet with different culture of mathematical ideas, feelings and discourse, which they will not know in their daily livings (Nagano, 1997). Indeed, in the classroom pupils' meet different ideas of ratio and proportion, novel notations and specific ways of attending and using the symbols. Here, it is essential not only identify different strategies but also to look closely at pupils' use of notations and associated meanings they give to them.

Hino (e.g., 1996, 1997) illustrated that pupils' use of notations, which is introduced in class, is changing in nature. To discern the change, Hino (1999) proposed three-phase model of evolution of pupils' use of introduced notations: "label use," "positive use" and "effective use." Case studies were done in different classrooms where notations such as "a÷b" ("a" and "b" belong to different kinds of quantities) or "y=mx" were introduced. They showed that many pupils stayed at the incipient "label use" in spite of the fact that the teacher was hurry in proceeding to the "effective use." Then what notations are productive and contribute to their evolution of use and eventually their development of proportional reasoning?

To delve into the question, the perspective of mediated action by symbols is taken (Vygotsky, 1978). As shown in the figure, elementary form of behavior is the direct reaction to the tasks in front of one's eyes (S-R). On the other hand, structure of sign operations requires another form of behavior, in which indirect link between stimulus and response is made and function as a secondary stimulus (S-X-R). This is called



"mediated action by symbols." The intermediate link not only improves previous thought operation, but also makes such operation qualitatively new and enables one to control his/her own action with the help of extrinsic stimuli. Vygotsky says that base of higher psychological processes is one's ability to create and use these artificial stimuli intentionally.

Studies on mediated action in the context of teaching and learning of ratio and proportion are not many. Lo & Watanabe (1997) traced one pupil's change in ratio and proportion schemes in the laboratory setting. The pupil initially found certain correspondence between two quantities by trial and error approach. As the sessions unfolded, he began to express his strategy by way of table. The table gradually contributed to shaping his direction of problem solving process; and later he came to curtail the table process by using multiplication. This process indicates that his thinking became to be mediated by tables ("X" in the figure above). The study shows the potentiality of analyzing pupils' change and role of notations in approaching proportion problems from the perspective of mediation by symbols. Still, there are unanswered questions, e.g., "what and how 'X' is chosen," "whether pupils choose introduced notations in class as 'X'," or "there are some relationships between the chosen 'X' and his/her proportional reasoning ability." In this article, five fifth-graders' approaches to the same proportion problems before and after a series of 11 lessons are compared. Specifically, the following questions are pursued:

- In what way can pupils' changes in approaching proportion problems be viewed by paying attention to the mediation by symbols?
- What kinds of symbols are chosen as the mediator and in what way do they function in pupils' thinking?

METHOD

In this study, data were collected in a fifth-grade classroom in a public elementary school, rural area of west Japan. There were 9 boys and 11 girls in the classroom. The author and a graduate student observed the lesson everyday while the teacher taught the chapter on "quantity-per-unit." We recorded each lesson by two video cameras and audio tape recorders, and by our field notes. We also conducted written tests for the entire pupils before and after the lessons. Moreover, as for the focused pupils, we interviewed them before, during and after the lessons. In the interviews, we asked the pupils to explain their approaches to some of the problems in the written test, and also to solve additional proportion problems to look closely at their approaches.

In the present article, the five focused pupils' approaches toward proportion problems before and after the lessons (abbreviations "B-L" "A-L" are used below) are compared on the individual basis. Such comparison is possible since many of the problems used in B-L and A-L are the same. The problems are composed of different situations and different ratio complexities. Their ratio complexities are classified into 4 levels according to Hart (1981):

Level 1: No rate needed or rate given.
(A car goes 200 km in 5 hours. If it keeps the same speed, how much time will it take to go 800 km?)

Level 2: Rate easy to find or answer can be obtained by taking an amount then half as much again.
(15 candies cost 240 yen, then how much will we pay for 20 of the same candies?)

Level 3: Rate must be found and is harder to find than above.

(A car goes 200 km in 5 hours. If it keeps the same speed, how much time will it take to go 75 km?)

Level 4: Must recognize that ratio is needed, the questions are complex in either numbers needed or setting.

(A rectangle (length : width = 6 : 8) is enlarged by copy machine. When the width of new rectangle is 12, how will the length be?)

Note: Examples are taken from the problems used in this study.

In the written test, four Level 2 problems and three Level 4 problems were prepared. In the interview, problems were given from the lower level to higher levels by fixing the situations as "speed" and "fertilizer." For example, in the speed situation, under the condition of driving a car at the speed of 200 kilometers in 5 hours, distance (time) was varied and corresponding time (distance) was asked. For each situation, two Level 1 problems, one Level 2 problem and one Level 3 problem were prepared. In the actual interviewing process, according to the response by each pupil, the interviewer judged whether to give problems with higher levels. Therefore, numbers of problems given to the pupils were not equal.

The focused pupils were chosen based on their performances, learning styles, seat locations and cooperativeness. Three boys (Yoshi, Himu and Honda) and two girls (Kawa and Ueda) were finally chosen (all names are pseudonym).In B-L, Yoshi, Kawa and Ueda had difficulty for approaching both the Level 1 and 2 problems. Himu was okay for the Level 1 but had difficulty for the Level 2 problems. Honda had no problem for the Level 2, did almost correct for the Level 3, but failed for the Level 4 problems.

During the 11 lessons, the teacher introduced quantity-per-unit (intensive quantity) to the pupils. By following the textbook, she dealt mainly with population density and velocity. The series of lessons began with discussing how to decide which of the two places is more crowded. Among different opinions, some pupils proposed to coordinate one of two quantities into the same amount and compare the amounts for the other quantity. The teacher pointed out the important idea of coordination and introduced the quantity-per-unit (number of people per 1 m^2). In the sixth and seventh lessons, the pupils were engaged in the activity of measuring speed. For the purpose of deciding the fastest in class, they again used the quantity-per-unit and developed a formula of speed ($v=d \div t$). In the subsequent three lessons, the teacher introduced the double number lines to represent the relationship among four amounts in two quantities.

RESULTS

Pupils' Performances across Problems with Different Levels of Ratio Complexity

Table on the next page shows percentages of correct answers for the five pupils in B-L and A-L for all problems given in the written tests and in interview sessions. The percentages were calculated according to levels of ratio complexity. The table shows that their performance for Level 2 problems increased in A-L. Kawa's percentage for Level 1 problems also increased. On the other hand, Honda's percentages did not show any change even though he showed highest performance among the five pupils in B-L.

Changes in Pupils' Approaches to the Same Problems

Then, how did the pupils' approaches to the same problems differ between B-L and A-L? For this purpose, their approaches to the same problems were put in order. In total, there were 51 such cases. Among them, it was possible to identify some differences in 26 cases. Almost all cases for problems in Level 4 were discarded at this point because no differences were seen in the pupils' approaches; they kept using 'incorrect addition strategy.' Honda's cases were also excluded due to the same reason. As for the 26 cases, the differences in the pupils' approaches in B-L and A-L were compared and contrasted, which led to four categories. In this section, they are described briefly with some illustrations.

Level	Yoshi		Kawa		Ueda	
	B-L(%)	A-L(%)	B-L(%)	A-L(%)	B-L(%)	A-L(%)
1	2/4(50)	1/2(50)	1/3(33)	2/2(100)	2/3(66)	1/2(50)
2	3/5(60)	5/6(83)	4/5(80)	6/6(100)	2/6(33)	6/6(100)
3	0/1(0)	1/1(100)	-	0/1(0)	-	-
4	0/3(0)	0/3(0)	0/3(0)	0/3(0)	0/3(0)	0/3(0)

Level	Himu		Honda	
	B-L(%)	A-L(%)	B-L(%)	A-L(%)
1	4/4(100)	-	-	-
2	2/6(33)	5/6(83)	4/4(100)	4/4(100)
3	0/2(0)	0/2(0)	1.5/2(75)	1.5/2(75)
4	0/3(0)	1/3(33)	0/3(0)	0/3(0)

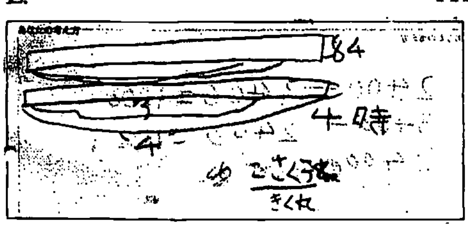
Note 1: "a/b" designates that the pupil got correct answers for "a" problems out of "b" problems given to the pupil.
Note 2: The shadowed cell designates that the percentage in A-L is higher than that in B-L.

Making one's thinking process explicit by numerical expressions. Compared with B-L, in A-L some pupils came to denote their process of thinking on the sheet of paper by using numerical expressions. Although the number of such cases was small (2 cases in Yoshi and Himu), the difference was remarkable in the eyes of observers.

The case by Himu is described here. He made his process of thinking explicit by verbalizing and denoting it on paper by both sentence and numerical expression. As the figure below illustrates, in B-L he tried but failed to express the idea of coordinating quantities to make two speeds comparable by using tape diagram. When asked how he did it in the subsequent interview, he again failed to make the idea explicit and showed insecurity in his answer.

Problem: (A table is shown. It shows that Sakura-maru goes 84 km in 4 hours and Kiku-maru goes 66 km in 3 hours.) Which ship goes faster, Sakura-maru or Kiku-maru?

B-L:



A-L:

(writing on paper)
"Let me make both [are going in] four hours,
66÷3=22 22×4=88. Answer Kiku-Maru"

In A-L, in order to make the two speeds comparable, he transformed the information "66 km in 3 hours" to "88 km in 4 hours" by keeping the rate constant. The idea of coordination was shown clearly in his writing. It is remarkable to know the difference of his notations (diagram vs. numerical expressions).

Becoming conscious of the quantity-per-unit. All the five pupils were observed to use the unitary strategy in B-L. In using the strategy, pupils need to find the quantity-per-unit and to make use of it to solve the problem. In B-L, even though the pupils used the strategy, they tended to be less conscious of what they try to find and how they make use of it. It was noticed that the consciousness of the quantity-per-unit was enhanced in A-L. Eleven of such cases were discerned, especially in the approaches by Yoshi, Himu and Ueda.

The pupils' growing consciousness of the quantity-per-unit accompanied the expression of rate orally or in written form. Yoshi came to describe the quantity-per-unit in words, to interpret the quantity and use the information in making decisions, and to apply the quantity to the relevant problems. For example, in B-L he calculated "66÷3=22" and "84÷4=21" for the ship problem above. However, he was not able to describe what "22" and "21" refer to. In A-L, he said clearly that both numbers refer to "the distance that the ship would go in 1 hour." Becoming conscious of the quantity-per-unit and being able to use it better was his most notable change.

The growing consciousness was seen in various places in their thinking processes. Himu was unclear in finding the quantity-per-unit in B-L. He seemed to exclusively rely on his guesswork in head. In A-L, he came to find the quantity-per-unit at some time by using division, and at other time by using multiplication (e.g., "はてな掛ける4は20" (meaning "?×4=20")). Here, his guesswork was replaced by more systematic and reliable means. Compared with the others, Ueda showed scalar operator strategy more in B-L, though she was clever in using numerical patterns rather than making sense of the strategy. In A-L, her use of the unitary strategy increased. Thinking of quantity-per-unit enabled her to overcome difficulty where the scalar operator strategy induced remainder:

<u>Problem</u>: This wire weighs 390 grams in 6 meters. How much does it weigh in 10 meters?
<u>B-L</u>: Ueda tried the division 10÷6 but got confused because it would not divide evenly. So, she did "10-6 = 4 390×4 = 1560 (g)."
<u>A-L</u>: Her reasoning is seen from her work on paper, "390÷6 = 65 65 = weight in 1 meter." It continued such as "6 meters = 390. Still 4 meter less. 65×4 = 260 390+260 = 650 (g)."

These two cases show that the pupils took the idea flexibly and made use of it in the specific problematic situations they were facing with.

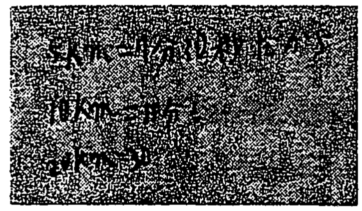
Emergence of notations that assist thinking. One notable feature of pupils' behaviors in A-L is that they put some notations down on paper when they had difficulty in proceeding their thinking. There were 4 such cases. It occurred when the level of ratio complexity was raised to Level 3. In such situations, they were either degenerated into the incorrect addition or quit their thinking in B-L. The pupils' restorations by the help of notations were seen in Yoshi, Himu and Kawa.

Both Yoshi and Himu wrote down the notations that designate correspondence between two quantities (e.g., "1 hour = 40 km"). Both pupils used the notations in the context of burden to proceed to their thinking. The case of Yoshi in A-L is illustrated below. He was approaching one of the speed problems (Level 3) that was totally out of his reach in B-L.

<u>Problem</u>: When a car goes at the speed of 200 km in 5 hours, how much time does it take to go 75 km?
Yoshi: 75 km? It sounds hard... (Hino reads the problem again.) Oh, 75... (he catches glimpses of the wall clock.) for 30 minutes it's 20 km (he is thinking with his eyes to the clock) Well, well... it is difficult (he looks bitter.) well... to go 75 km, how many tens of minutes? mm... what is the answer, answer, answer (he wrote

"Answer" and "1 hour") well… 7 minutes 30 seconds, oh, wait a second (he wrote "since '5 km = 7 minutes 30 seconds'.)

Hino: You don't need to get the answer. Would you tell me how you thought about it?



Yoshi: How about the remaining 30 km? well, let me put this, since 2 (meaning 20), km is for 30 minutes, so, as for 10 km, let's make both just a half, well for 15 minutes, 15 minutes, it's 10 km (he writes "10 km = 15 minutes".) and, for 20 km it's 30 (writing), since [the problem asks] 75, for 1 hour, it's 40 km, for 1 hour, it's 3, no, no, no… 35 remain, so let me add these two, 45, 45 (he adds 15 and 30), and add 7 minutes (he developed written calculation of 45+7=52.) it's 52, so, 1 hour 52 minutes 30 seconds (writing).

His inscriptions are shown in the figure. In his writing, the correspondence between the two was taken for granted and even stressed by the symbol "=." He was also aware of the distinction between the two quantities (which is on the left and on the right) and kept balance between the two (if you multiply one, you must multiply the other). This type of notations must have assisted him greatly to sort out the different amounts and quantities at hand.

In the case of Kawa, not numerical expression but drawing served as a clue in recognizing the correspondence. In B-L, she approached one of fertilizer problems by guesswork and eventually paid attention to the difference (minus 0.5 m^2 of land). In A-L, she was appealed by the difference again. However, this time she modified her thinking toward building up strategy. Here, Kawa relied on the picture she had just drawn (see the figure at right). She recognized that one small rectangle (1 m^2) corresponds to half package and therefore she needs to minus 1 m^2 to find the area for one and a half packages of fertilizer ("then this part is here, this part is here, and this part is here…, and this part is missing…."). She had written the same drawings in B-L, however, they were not functioned as a clue to modify her approach.

Attempting to use introduced notation. In A-L, uses of introduced notation, double number lines, were observed in 3 cases of Kawa and Ueda's thinking. However, neither used the notation in the way that the teacher taught. Even when they successfully represented the four amounts on the two lines, it was another hurdle for them to reason from the two lines. In addition, there were 4 cases that show the pupils' confusions for meaning of division. Using the number lines requests the pupils to identify and connect four amounts in two different quantities and to relate them with quotative division. This hurdle was high enough for them who had already possessed misconceptions on division in B-L.

DISCUSSION AND CONCLUSION

The pupils' performances on different levels of ratio complexity did not show steady change between B-L and A-L. Since the quantity-per-unit was taught in class, problems with Levels 2 and 3 were more to be targeted. Accordingly, the pupils' correct responses to Level 2 problems increased. Still, their development was not clearly seen across levels. Nor were their performances stable. In A-L, they still had difficulty in lower level problems.

Nevertheless, the results show that major change was seen in their more active use of notations in dealing with the aspect of rate and proportion. Notations that appeared in A-L varied, including different numerical expressions, drawings, expressions of correspondence between two quantities and the double number lines. They mediated the pupils' thinking by modifying the stimulus situation as a part of the process of responding to it (Vygotsky, 1978, p14). Such notations were recalled or generated according to the pupils' needs. They may be classified into three types by the extent to which they contribute to the pupils' thinking:

- Notations borrowed (e.g., Kawa & Ueda's use of the double number lines).
- Notations making aware of one's thinking (e.g., Yoshi's use of expression of correspondence, Kawa's use of drawing)
- Notations inseparable to thinking (e.g., Honda (described later)).

An important observation of this article is that there are notations in the second type, or the notations that lead one to make aware of his/her own thinking. The categories outlined supports this argument. The pupils became more articulated in their reasoning both orally and in written form, which enabled them to reason beyond guess and ambiguous image. Their uses of the quantity-per-unit became more conscious. They came to find the quantity-per-unit systematically, to make decision based on information of the quantity, modify their thinking, and use the quantity-per-unit in order to complement their insufficient parts of reasoning. The pupils who previously depended only on memory also came to proceed by controlling their own thinking with the help of notations. Vygotsky & Luria (1994) regard that changing relationship between speech and action is characteristic to the process of emergence of higher-order mental functioning. According to them, child speech is at first accompanied by activity and simply reflects on it, but it gradually shifts to the starting point of the process of activity, precedes the action and comes to possess the function of directing the activity itself. The observations of using notations in more articulated, systematic, conscious and intentional ways made in this article do indicate that this crucial change is seen in their approaches between B-L and A-L.

Notations that came to play these significant roles in the pupils' approaches to proportion problems also varied. Importantly, the notations that express correspondence between two quantities are widely used. Not only in written form, they were also heard from the pupils.

These significant notations did not exist uniformly out there but depended delicately on the conceptions of ratio and proportion that the pupils had developed earlier. The pupils were building their use of notations on their own strategies. This is shown, for example, in the cases of Ueda and Himu. Both made sense of the quantity-per-unit in A-L. However they began to use it differently. For Ueda, it served as a clue in overcoming the difficulty of division failure. On the other hand, for Himu it gave a reliable way in place of guesswork to find the quantity.

Another observation is the pupils' difficulty in the use of double number lines. For those pupils, this introduced notation did not necessarily become significant notation. It again suggests its dependency on the pupils' conception of ratio and proportion, especially invariant relation between two quantities. In developing the expression of correspondence, it is speculated that they attend only statically to the two quantities (c.f., "ratio" by Thompson, 1994). On the other hand, the double number lines request them to conceive the quantities as varying each other while they maintain the rate invariant. This gap must be profound for the pupils. Honda seems to be the only pupil who had mastered the double number lines. He solved them in the following way.

Problem: This wire weighs 390 grams in 6 meters. How much does it weigh in 10 meters?

B-L:	$6\times5=30\div3=10$	A-L:	$6\div3=2$
	$390\times5=1950\div3=650$		$390\div3=130$
	650 g		$2\times5=10$
			$130\times5=650$ 650 g

Honda's approach can be said to be an interiorized version of the use of double number lines. His conception of invariant relation was stable and also flexible. For Honda, parallel expressions with multiplication/division by the same numbers were inseparable to his thinking, or functioned as more than significant notations.

TASKS FOR THE FUTURE

The analysis is still going on. It is now analyzed what had occurred in the course of learning of the five pupils during the series of lessons on "quantity-per-unit." Questions that lead the analysis include: what kind of notations did they bring to the instruction?; what kind of notations did they generate and modified during instruction?; what had influenced on such notational activities?; and when and where the pupils came to develop significant notations? Preliminary observations show that from the very beginning the pupils select specific notations based on their taste, and that instruction influences the emergence of consciousness of the pupils' uses of notations indirectly. Information on these matters will contribute to revealing how social aspects of learning impact individual's development of proportional reasoning and to designing lessons that take care of each pupil's positive resources to be used in refining their conceptions (Moschkovich, 1998).

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BUILDING NEWTON–RAPHSON CONCEPTS WITH CAS

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In this study the computer algebra system (CAS) Derive in TI-92 calculators was used to encourage an inter-representational approach to the study of the Newton-Raphson method. Students in both New Zealand and Korea were given the opportunity to use CAS to build an understanding of how and why the method works. In addition they were able to construct some of the key concepts of function underpinning the method. The results of the tests and questionnaire suggest that students can integrate CAS into an overall strategy in a way that assists them in their conceptual understanding of this method.

Background

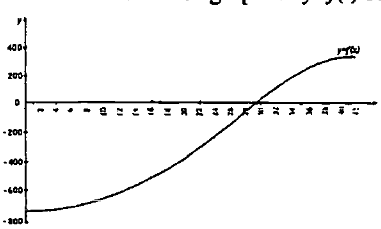
A deep understanding of mathematics involves an appreciation of how different representations contribute in often subtle ways to concept formation. A number of aspects converge in building this type of understanding for students: forming conceptual links between different representations; the ability to translate between representational systems (Lesh, 2000; Hong, Thomas, & Kwon, 2000); and the qualitative nature of the interactions with each representation (Thomas & Hong, 2001). None of these aspects is straightforward. For example, as Greer and Harel (1998) point out, students may have a surface ability to form links between procedures in two representational systems without forming an understanding of the deeper, conceptual links which are imbedded in the transformations between representations. In the case of interactions, Thomas and Hong (2001) have described up to nine qualitatively different types of interaction with representations which students may engage in. These are divided into those which involve procedural interactions, such as calculating the gradient of a function at a point from an algebraic formula, table or a graph, and those which require either a conceptual process (CP) or a conceptual object (CO) perspective of the representation. Examples include investigations of whether a function has points where the gradient is the same (CP), or conservation of properties under transformations of the graph of a function (CO).

CAS calculators provide an opportunity to promote investigation and discovery through a multi–representational approach to problem solving, enabling manipulation of mathematical concepts both within and between different representations. However, it is becoming increasingly apparent that using CAS is not as straightforward matter as it might once have appeared. First there is the question of what prerequisites students need in order to use CAS effectively. Pierce and Stacey (2001) have described a framework to analyse algebraic insight, which they believe is crucial in advance of CAS work. Secondly, others have expressed doubts about the extent to which students generally form inter-representational thinking (e.g. Tall, 1996). Finally, there is the issue of the choices facing teachers and students in the implementation of CAS calculators. Students have to be able to form an overall strategy to solve a given problem and to decide where, and how, to integrate CAS use. They may be faced with a considerable number of choices with regard to such implementation. This involves the issue of instrumentation (Trouche, 2000), the change from calculator as tool to its use as an instrument, with the instrumentation process being separate from the conceptualisation process. This instrumentation process involves the nature of student interaction with CAS, and the integration of pen and paper methods with CAS use (Lagrange, 2000). In the former case the CAS can be used in a procedural way, as a form of 'black box' which simply produces results of calculations on demand, or as a window onto concepts, or a combination of these. In the latter students have to form a partnership with the CAS, making individual decisions within an overall strategy on when thinking with pen and paper is appropriate and when and how to use the CAS.

Method

This study considers the above issues in the context of New Zealand and Korean students using CAS in the Newton–Raphson (NR) method. We investigated whether CAS use improved students' conceptual understanding of this method of calculating approximations to roots of equations; what factors were influencing this; what kind of partnerships with CAS students formed; and whether there were any cultural differences (based on the different languages and diverse cultures).

The diagram below shows the graph of $y=f(t)$ for $0 \le t \le 42$.



3. Use an initial estimate of $x_0=15$. Find the first iterate, x_1, of the Newton-Raphson method to solve the equation $f(t)=0$. Round your final answer to two decimal places.

4. Use an initial estimate of $x_0=15$. Use the above to show geometrically how the first iterate, x_1, and then the second iterate x_2, are found using the Newton-Raphson method. Clearly indicate on the graph x_0, x_1 and x_2.

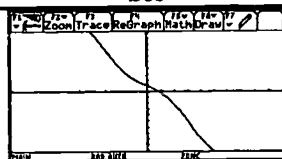
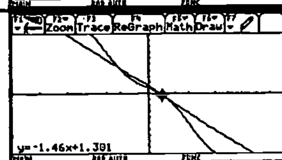
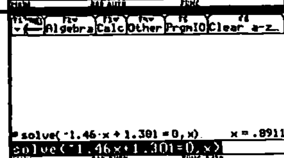
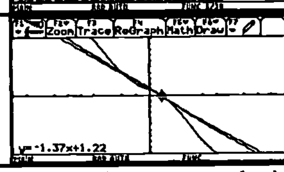
Figure 1. A Newton–Raphson examination question requiring geometric understanding.

The NR method was considered a suitable subject for research because: CAS use is beneficial in calculus (e.g. Heid, 1988; Hong & Thomas, 1997; Trouche, 2000; Drijvers, 2000); it is an area where teaching can concentrate on the procedural symbolic, with little attention to visualisation; and CAS can be used in a 'black box' mode without understanding of the process. For example, for New Zealand university entrance examination questions such as that shown in Figure 1, students often fail to demonstrate any geometrical understanding of the NR method. Typical comments on the geometric aspects of questions like this from New Zealand examiners' reports over a number of years, include: "Very poorly done. It is apparent that many candidates had no geometrical appreciation of the Newton Raphson method. For such students, this topic becomes nothing more than an exercise in evaluating a number of arithmetic expressions." A case study methodology was employed in each country with a partial qualitative and quantitative comparative analysis superimposed.

Subjects: Two high schools were used in this study, one in Auckland, New Zealand (NZ) and one in Seoul, Korea. The NZ school is an independent co-educational school with a relatively high socio–economic intake. The seventeen 16–17 year old students in this advanced Form 6 class had previously covered all the university entrance work, including the calculus and the NR method, but without using the CAS calculators. The twenty 16–17 year old Form 6 Korean students were all volunteers especially interested in mathematics, attending a co-educational school with an average socio-economic intake. The Korean students had no background in calculator use and none had ever used a graphic or a CAS calculator. In contrast, while the NZ school has a positive view of technology use, its students had not previously used CAS.

Since the NR method is not included in the Korean curriculum, after their normal classes on differentiation the procedure was first taught without the calculators. This was to mirror the experience of the NZ students in order to be able better to make comparisons. Also Korean Form 6 students do not learn about exponential function and differentiation of sine and cosine functions, so one short lesson on these was added before the pre-test. The research project was carried out in New Zealand during 16th to 26th July, and in Korea from the 12th to 20th November of the same year.

Example. Solve the equation $\sin x = 2x - 1$ using the Newton-Raphson method. Give the answer accurate to 4 d.p. The first step is to define the graph of $y = \sin x - 2x + 1$ and sketch the graph:

Press	See	Explanation
Method 1. [♦] [Y=] [SIN] x [)] [−] $2x$ [+] 1 [ENTER] [♦] [GRAPH]		The function is defined using $y1 = \sin x - 2x + 1$
F5 A:Tangent [ENTER] Tangent at? 1 [ENTER]		The first tangent line starts at the point $x_1 = 1$. The equation of the tangent is given as $y = -1.46x + 1.301$. This will be used to find x_2 (which should be closer to the root than x_1), by seeing where it crosses the x-axis.
[♦] [HOME] F2 1 [ENTER] [(-)] 1.46 x [+] 1.301 = 0, x [)] [ENTER]		We can see that the next point $x_2 = 0.8911$
Return to [GRAPH] F5 A:Tangent [ENTER] Tangent at? 0.8911. [ENTER]		The equation of the tangent is given as $y = -1.372x + 1.218$ at the point $x_2 = 0.8911$. This will be used to find x_3 (which should be closer to the root than x_2).

This process was then repeated until $x = 0.8882$ was obtained.

Figure 2. A section of the module showing the layout and calculator screens.

Instruments: A module of work using the TI–92 calculator was prepared. It addressed key mathematical concepts and ideas required for understanding NR, including: variable, expression (function), equation, tangent, graph, limit, differentiation, and the NR method, doing so in an inter-representational manner. Each section of the module was set out according to a 'Press', 'See', and 'Explanation' format (see Figure 2 for an example). Two NR methods were presented. The first comprised an intuitive approach seeing visually how we can often get nearer to a zero α of a function by drawing successive tangents and using the CAS to find the equation of the tangent in the graphical mode and its zero on the algebraic home screen. The second method used two derivations of the standard algebraic formula: rearranging the equation of the tangent: $y - f(x_1) = f'(x_1)(x - x_1)$ with $y = 0$ when $x = x_2$,; and a more visual approach, using the triangle created by the tangent and equating the gradient written as the quotient of the two sides and the derivative, i.e. $f'(x) = \frac{f(x_1)}{(x_1 - x_2)}$. A section from the module illustrating one method for solving the equation

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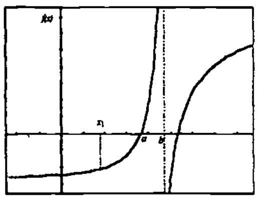
$\sin x = 2x - 1$, is given in Figure 2. Two parallel tests using different numerical values were constructed for the pre-and post-tests. After the post-test the students were also given a questionnaire.

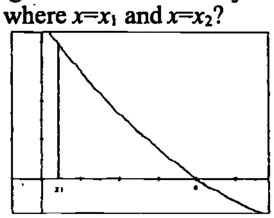
B3. If, in the Newton–Raphson method, for a function $y = f(x)$, $f(x)>0$, $x_1 = 2$ and $f'(x_1)>0$, where x_1 is the first approximation to the root, is $x_2>x_1$ or is $x_2<x_1$, where x_2 is the second approximation to the root? Explain your answer.

B7. How could you use the Newton–Raphson method to find the x-value of the intersection of the graphs of $y=2e^{-x}+\cos x$ and $y=2$? Explain your method clearly.

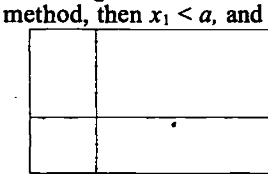
B4. a) Explain why x_1 in the diagram alongside is an unsatisfactory first estimate in the Newton-Raphson method for the root $x=a$ of $y=f(x)$.
b) When would x_1 be a satisfactory first estimate?



B5. For the function $f(x)$ shown below the 2nd approximation x_2 to the root $x=a$ is exactly 0.8 closer than the first approximation x_1. What is: (a) the relationship between $f'(x_1)$ and $f(x_1)$? (b) the gradient of the chord joining the points where $x=x_1$ and $x=x_2$?



B6.b) Draw a continuous function below where, if x_1 and x_2 are the 1st and 2nd approximations to the root $x=a$ using the Newton–Raphson method, then $x_1 < a$, and $x_2 > a$.



c) What determines whether x_2 and x_3 etc. are less than a or greater than a?

Figure 3. Some section B post–test questions on the Newton–Raphson method.

The tests were divided into sections A and B, with the former containing essentially procedural skills but the latter demanding greater conceptual understanding, especially geometrically. We wanted to know if they knew how and why the method worked and could apply this. We were also interested in when and how they would make use of the CAS. In B3, B4 and B5 (see Figure 3) a general function $f(x)$ rather than an explicit function was given, discouraging immediate use of a procedure, instead necessitating understanding and application of relationships.

Procedure: The module, written in English, was given to the New Zealand teacher, and a Korean translation was given to the Korean teacher. Once the teachers were comfortable with the calculator and the material they taught their class for four lessons covering basic facilities of the calculators and describing ways to find a limit, a gradient function, and roots of an equation. Each student had access to their own TI–92 for the whole study, including time at home. In each country the teacher stood at the front of the class, who sat in the traditional rows of desks, and demonstrated each step using a viewscreen while the students followed in the module and copied the working onto their calculator. Afterwards the students worked while the teacher circulated and assisted with any problems.

Results

Comparing the overall results of the Korean and New Zealand students on the pre- and post-tests there was a significant difference on both section A ($m_{\text{Korea}}=3.15$, $m_{\text{NZ}}=5.35$, $t=3.51$, $p<0.005$) and B ($m_{\text{Korea}}=2.7$, $m_{\text{NZ}}=5.6$, $t=3.14$, $p<0.005$) at the pre-test. It was not surprising that the New Zealand students were performing better initially since they had studied the topic previously. However, at the post-test an interesting difference emerged. As expected, for the section A skills questions, the Korean students improved considerably, and although there was no significant difference between the scores on this section ($m_{\text{Korea}}=7.15$, $m_{\text{NZ}}=6.35$, $t=1.17$, n.s.), there was a highly significant difference in the gain scores in favour of the Korean students (see Table 1). In contrast both groups improved about the same on section B (see Table 1), and so the New Zealand students remained significantly better than the Korean students here ($m_{Korea}=5.90$, $m_{NZ}=8.53$, $t=1.97$, $p<0.05$).

Table 1
A Comparison of the Pre– to Post–test Gain Scores

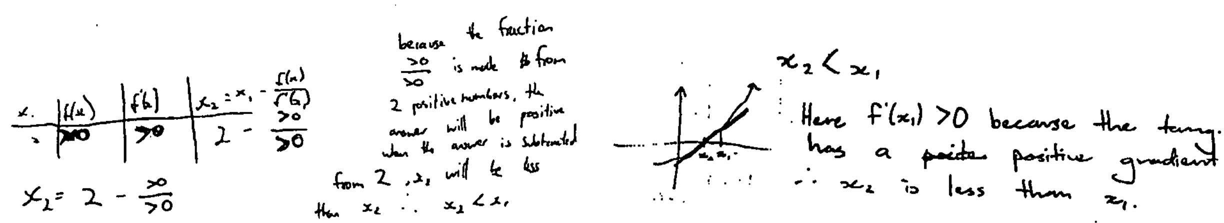
Max score=29 (Sec A=10; Sec B=19)	Mean Korea	NZ	t	p
Total Score (Post–Pre)	7.25	3.94	2.32	<0.05
Section A (Post–Pre)	4	1	5.03	<0.00005
Section B (Post–Pre)	3.25	2.94	0.26	n.s.

Thus it seems that the use of CAS calculators was effective for both procedural and conceptual understanding in both countries, for those familiar and not familiar with the method, but for the former the main benefit was in conceptual understanding.

Understanding concepts: Our analysis of the NR method indicated that students needed to form a partnership with the CAS interacting with its representations (Thomas & Hong, 2001) in a way that would help them to construct an inter-representational understanding of at least four key ideas:

1. The sign of f and $f'(x)$ (gradient of the tangent) affects whether the second estimate is > or < the first estimate.
2. The size of $f'(x)$ (gradient of the tangent) affects the existence and value of the second estimate.
3. The sign of $f''(x)$ (concavity of the curve) at the first estimate affects whether the second estimate is > or < the root a.
4. A geometric understanding of how and why the method works ie tangent intercepts getting progressively closer to a root and a sense of where the algebraic formula comes from.

Solutions to question B3 (post-test mean 0.996 out of 2) gave some insight on student thinking on point one. For example, Figure 4 shows the post-test solutions of two students, NZ1 and NZ2, who were both unable to answer at the pre-test.

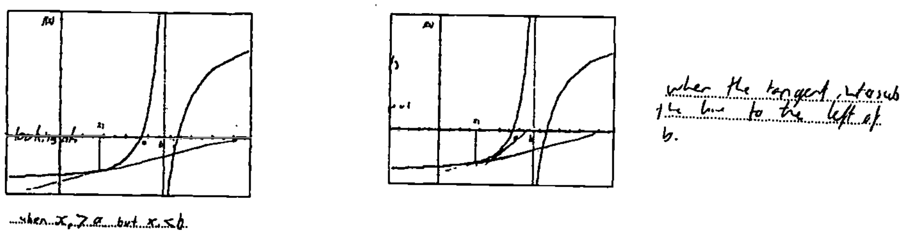


NZ1 NZ2

Figure 4. Student working on question B3 showing appreciation of the sign of f'.

NZ1's solution involved interacting with the algebraic formula representation rather than visualising the graph and the tangent. It was based on general principles of the sign of the two functions $f(x)$ and $f'(x)$, showing that for f and $f' > 0$ $x_2 < x_1$. Evidence that he was also able to think visually about them in the graphical representation was apparent from his comments in the questionnaire. Showing that he had understood the way the tangent influences the solution process he responded to the question 'How does the Newton–Raphson method work?' with "Tangents on the graphs bring you closer to the root of the graph." Thus while he had an inter-representational perspective he chose not to use it on this occasion. This is an important point. Student NZ2, however, connected the symbolic representation of the gradient function, $f'(x)$, with the tangent in the graphical representation and interacted with this representation to demonstrate the influence of the sign of the function (drawn as positive), and the gradient, on the position of the second approximation.

Point two, learning the role of the size of f' can be illustrated by the post-test working of students NZ3 and NZ4 on question B4 (post-test mean 0.947 out of 3, see Figure 5). Interacting with the graph both have appreciated that the first estimate has to be sufficiently close to a for the gradient to be large enough for the tangent to cut between $x=a$ and $x=b$.



NZ3 NZ4

Figure 5. Student NZ3 and NZ4's working showing appreciation of the role of the size of f'.

Further evidence of this understanding was provided in questionnaire comments. When asked 'What is important about the choice of the first approximation to a root when using the Newton–Raphson method?', students showed an understanding that a gradient which was too small or zero would cause problems:

NZ1: You can't choose a max or min point or else you won't cut the x-axis. Also the tangent could go towards the wrong root.

NZ5: It is very important that the approximation is close enough the root and not on a turning point. Otherwise you might be finding the wrong root.

NZ6: It must be close to the root so the tangent gives you the nearest value. Also you can't choose a stationary point as a first value.

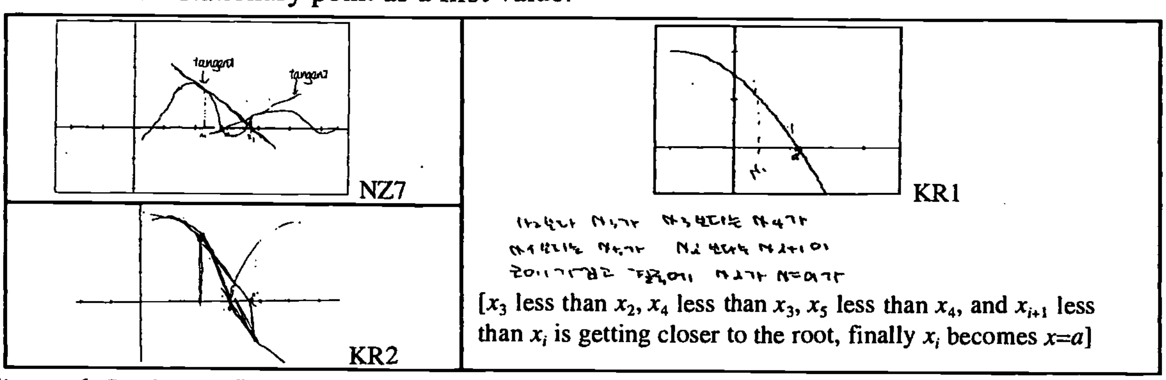


Figure 6. Student NZ7, KR1 and KR2's working showing appreciation of the role of concavity.

Some students gained understanding of the role of concavity in the NR method. Figure 6 shows the post–test working of three students NZ7, KR1 and KR2 on question B6 (post-test mean 1.13 out of 4). In each case they have managed to construct a function whose graph has the required concavity (concave down to the left of a) to put the first and second estimates on opposite sides of the root, as requested (although KR2 has an error - drawing a function which is concave up on the right and mistakenly trying to show the third estimate as greater than a). Interestingly NZ7 has constructed a function with 2 zeros, which was not required, and has arranged the gradient of the tangent at x_2 to be small enough so that it crossed the axis near $x=a$,

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rather than close to the nearer zero. This shows a good grasp of principles in this conceptual interaction with the graphical representation. KR1's written explanation shows that she understands that all further estimates of the root will be greater than *a* for her function. Student understanding of the way in which the NR method works (point 3 above) was provided in answers to the question 'How does the Newton–Raphson method work?' Some of the responses were:

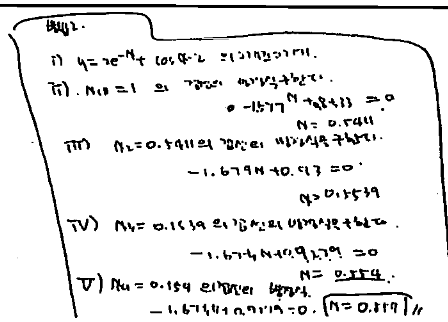
KR3: Take some point x_1, draw the tangent line at the point. Find the other intersection point x_2 on the graph. In the same way, finally find the intersection point x on the x-axis.
NZ2: It finds a root of a graph by finding out where the tangent of a close guess cuts the x-axis.
NZ7: Start with the initial point. Draw the tangent line. Take the approximation closer to the root.

These (one translated from Korean) and others, refer to tangents, and they clearly have the idea of how NR uses these to get closer to the root. They are developing representational fluency (Lesh, 2000) and are no longer limited to working only within an algebraic representation.

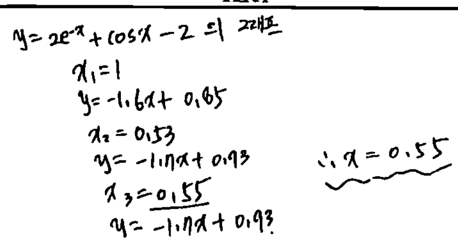
CAS Partnership: One of the methods used in the study for integrating CAS into the method of estimating the roots using NR required the students to have a good overview of the strategy and to see clearly when and how to employ the CAS and when pen and paper techniques (point 4 above).

Translation from Korean:

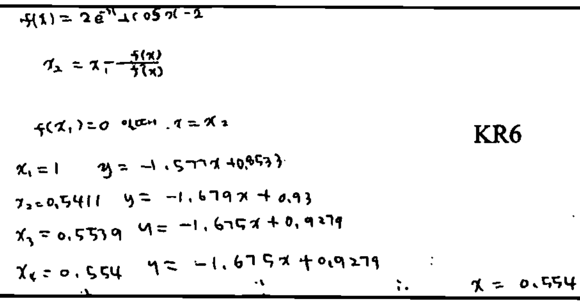
Method 2: i) Draw the graph $y=2e^{x}+\cos x-2$ ii) Find the tangent line on $x_1=1$, $-1.577x+0.8533=0$, $x=0.5411$ iii) Find the tangent line on $x_2=0.5411$, $-1.679x+0.93=0$, $x=0.5539$, iv) Find the tangent line on $x_3=0.5539$, $-1.675x+0.9279=0$, $x=0.554$, Find the tangent line on $x_4=0.554$, $-1.675x+0.9279=0$, $x=0.554$.]



KR1



KR5



KR6

Figure 7. Students KR1, KR5 and KR6's pen and paper working for the CAS method.

Figure 7 shows work from three students on question B7 (post-test mean 0.729 out of 2), which asked them to solve $2e^{-x}+\cos x=2$, without specifying a method. All have correctly identified the function to use and KR6 has also written the standard algebraic formula for the NR method. In each case they have chosen to integrate CAS into an overall strategy, as encouraged in the module. First they select a reasonable initial estimate, $x_1=1$ in all three cases. Moving to the CAS they enter the function using the [Y=] facility and then plot the graph, making sure that the viewing window is satisfactory. Next they use the CAS graphing screen to find the equation of the tangent at 1, and write this down using pen and paper. They have to know what the

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strategy requires them to do with this tangent equation, namely using the CAS algebraic home screen to find its zero. Appreciating the role of this value, 0.5411, as the next estimate in the overall method they can return to the graphical representation and use the CAS to find the equation of the tangent at the point x=0.5411. Finally they have to be able to see that this cycle is repeated until the accuracy required is obtained. This is a fairly sophisticated integration of CAS into a strategy, and shows that these students had become competent users of CAS for the NR method, able to form a good partnership with the technology. Of course, once students have a good overview of the method and its concepts then there is no reason why they should not use the CAS in 'black box' mode and enter *solve*($2e^{-x}+\cos x=2, x$) to obtain the root.

There were some differences between students in the two countries. For example, the Korean students were caused some anxiety by the calculator commands, since these appeared only in English. The teacher translated them into Korean for the students but some confusion ensued until they became more accustomed to the English. The psychological anxiety caused by this could have affected their self-confidence and hence their ability to build mathematical understanding. In spite of this the students from both countries have improved in their understanding of concepts associated with NR method. They have increased the versatility of their representational interactions (Thomas & Hong, 2001) and begun the process of instrumentation of the CAS calculator. This has enabled them to be more comfortable with overall strategies integrating CAS with pen and paper methods.

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HAVE ALL CHILDREN BENEFITED?

A Perspective on Curriculum Initiatives and Low Achievers

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Abstract
This paper considers whether or not curriculum initiatives within England are in fact leading to qualitatively improved levels of thinking amongst children identified as lower achievers in arithmetic. Revisiting data from two earlier studies and introducing current data, the paper draws comparisons in the strategies that 8-year-old children use to solve a range of addition and subtraction combinations on numbers to 20. It considers the outcomes in the context of current theories of concept development and national requirements of expected levels of achievement for such children. The results suggest that these requirements and the way in which they are implemented may seriously impede these children's cognitive development in elementary arithmetic.

Introducing a Perspective

The analysis of the strategies young children use to solve simple arithmetical problems can provide a framework to study the development of their arithmetical skills and the level of sophistication they employ to engage in numerical activity (Gray, Pinto, Pitta, & Tall 1999). It is the purpose of this paper to consider these notions in the light of initiatives implemented within English primary schools over the past decade. In doing so we focus particularly on children who, despite these initiatives, have difficulty recalling number combinations to 20. The evidence suggests that children within the lower quartile of arithmetical achievement are continuing to operate at a procedural level in the sense that they attempt to use a step by step routine such as counting to obtain solutions to elementary number bonds.

In the context of elementary arithmetic the initiatives have placed an emphasis on the ability to recall basic facts and use them to derive others. The latter feature lies at the heart of a strong emphasis on the development of flexible mental approaches to the calculation of two and three digit number combinations. Within this paper we present comparative evidence to illustrate that difficulties remain for some children. Fusing evidence drawn from three distinct studies over the past ten years we consider cognitive and pedagogic reasons as to why the situation has not changed. In doing so we are aware of social and cultural factors that may have a strong influence on the mathematical behaviour of the children we are considering (see for example, Cobb, 1987; Gruszczk-Koczynska & Semadini, 1988) but that is for a bigger story.

Theory and Practice

The construction of mathematical knowledge is a central platform from which the considerations within this paper are developed. It takes the view that hypothesised notions of development may not fit reality, particularly if children at the lower extreme of achievement are considered.

The development of early numerical concepts is heavily associated with physical activity and the pervading belief is that these concepts evolve from an interaction with the environment (Tall, 1995 *after* Piaget, 1973). Actions on physical objects can lead to the development of procedures through which processes are named symbolised and conceptualised. Establishing an appropriate conceptualisation is seen as the product of a suitable form of abstraction (Piaget 1985, Dubinsky, 1991). However, the knowledge and beliefs that learners bring to a given learning situation can influence the meanings that they construct in that situation. This would suggest that learners might select different aspects of an activity to focus upon which in turn leads to different forms of abstraction. When the latter is based upon perception of physical objects it is termed *empirical*. Alternatively such abstraction may not be based on the perception of objects but on the common feature of a series of actions. In such a case it is termed *pseudo-empirical*. The repeated practice of counting leading to the concept of number provides an example of this form of abstraction. The metamorphosis of actions with physical objects through a variety of increasingly abstract representations to form numerical concepts is outlined by Steffe, von Glaserfeld, Richards and Cobb (1983). Through such transformations it becomes possible to act upon the results of carrying out processes without bothering about the processes themselves.

Stages in the process of pseudo-empirical abstraction are frequently discussed in the context of process/object theories (see for example Sfard 1991; Cotterill et al, 1996) and they form the basis for the encapsulation or reification of new objects. There is much to be gained from such a move. Cognitive strain is reduced if it is possible to think of a concept as a single object rather than a lengthy process. However, it is difficult (Sfard, 1991) and its difficulty can lead to qualitatively different outcomes (Gray & Tall, 1994) which can be associated with success or failure.

The observation that some individuals are more successful than others in mathematics has been evident for generations. Piaget provided a novel method of interpreting empirical evidence by hypothesising that all individuals pass through the same cognitive stages but at different paces. Such a foundation underlies the National Curriculum (DfEE, 1999a[1]) with its sequence of levels

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[1] Although only the most recent is referred to within this paper, The National Curriculum has been through three transformations since its implementation in 1991. It outlines both legal requirements (Attainment Targets) and also provides information to help teachers implement these requirements with in schools (Programmes of Study)

through which all children should pass at an appropriate age, some progressing further than others during the period of compulsory education.

Within England and Wales the process of improving standards in numeracy — some aspects of which are seen to improve "confidence and competence with numbers" requiring "an inclination and ability to solve number problems in a variety of contexts" (DfEE, 1999b, p 4) — are specified within two documents. The first, the National Curriculum for Mathematics (DfEE, 1999a), identifies the contents of the mathematics curriculum that should be taught during Key Stage 1 (KS1, age 5 to 7) and Key Stage 2 (KS2, age 8 to 11). The second, the National Numeracy Strategy (NNS) (DfEE, 1999b), identifies, almost through a process of genetic decomposition, how and what should be taught to instill a sense of being numerate. The teaching requires that "teachers will teach the whole class for a high proportion of the time and that oral and mental work will feature strongly in each lesson" (ibid. p.2).

Since the inception of the National Curriculum, formal assessment of the level of achievement of each child against the specified targets has been made through Standard Assessment Tasks, which are taken at the end of KS1 (children 7+, the average to achieve level 2) and KS2 (children 11+, the average to achieve level 4). Since this report is looking at early number development it is appropriate to indicate that for a Level 2 attainment a child would need to demonstrate mental recall of addition and subtraction facts to 10, and recall addition and subtraction facts to 20 for Level 3 (age 8/9). It should be noted that the programme of study for KS1 indicates that 7+ children should be using known facts to 10 to derive facts with totals to 20.

Both the programmes of study and the NNS are fully aligned, the latter providing a detailed basis for implementing the statutory requirements. The development of the early number requirements of the National Curriculum are identified within the NNS by an approach that sees a transition from the use of perceptual items to "more sophisticated mental counting strategies" finally sublimated by the acquisition of basic number facts that are taught. By and large we may see a curriculum that can be compared favourably with the Steffe et al (1983) model

To achieve these ends the NNS strategy suggests that a daily mathematics lesson is appropriate for almost all pupils. Individual needs do not necessarily warrant individual attention although the needs of particular children may be met through differentiated work and other teaching strategies.

Comparing Approaches

The question arises as to whether or not the initiatives within the UK have made substantive differences to the quality of thinking of those who are at the lower end of the spectrum of mathematical achievement. Is it possible to detect qualitative changes in the way in which they interpret mathematical symbolism by comparing the general use of strategies by children who have participated within the NNS with those who have not? If the initiative has been advantageous we would expect to see that there is a greater evidence of the ability to recall basic facts and evidence that the children use these facts to establish the facts not known.

To begin to consider a response we draw upon and compare three pieces of evidence indicating the strategies used by 8/9 year-old children (Y4):

1. The classified strategy responses reported in Gray (1991)
2. Classified strategy responses reported in Pitta (1997)
3. Recent data obtained as part of an investigation into children's participation within the oral phase of mathematics lessons conducted in accordance with the NNS (2001).

Four different schools are represented within the analysis and these range in type from two small town schools (1991), a small suburban school (1997) and large inner city school (2001). Data collection methods for each of the three samples were essentially similar and included semi-structured clinical interviewing, video recording of these interviews and subsequent analysis of the responses.

Children within each sample represented the lower end of the spectrum of achievement. The 1991 sample, in the class teacher's view were representative of the lowest quartile of achievement within the year group. In the second and third cases children were chosen from within the lowest 20% of scores within the KS1 Standard Assessment Tasks. Within the chosen year group (Y4) the average child would be expected to recall all number combinations to 20. Schools from which the two most recent samples had been drawn have subject to recent external inspection by the Office for Standards in Education (Ofsted). Mathematics teaching within both was identified as "mostly satisfactory".

Here we report on strategies used to solve the questions that were common to all children, namely 0+2, 6+3, 3+5, 14+4, 3+16, 4+7, 9+8, 3-3, 6-3, 9-8, 15-9, 13-8, 12-8, 16-10 and 17-13.

	Addition Combinations to 10		
	1991 (n=4)	1997(n=4)	2001 (n=16)
Known Facts	33	42	31
Derived Facts	0	0	2
Count-on	25	50	48
Count-all	42	0	8
Error	0	8	10
	100	100	100

	Addition Combinations to 20		
	1991 (n=4)	1997(n=4)	2001 (n=16)
Known Facts	0	13	5
Derived Facts	0	13	6
Count-on	75	75	53
Count-all	25	0	14
Error	0	0	22
	100	100	100

	Subtraction Combinations to 10		
	1991 (n=4)	1997(n=4)	2001 (n=16)
Known Facts	8	50	29
Derived Facts	0	8	8
Count back	8	17	17
Take-away	67	25	17
Error	17	0	29
	100	100	100

	Subtraction Combinations to 20		
	1991 (n=4)	1997(n=4)	2001 (n=16)
Known Facts	0	19	2
Derived Facts	0	6	3
Count back	6	31	36
Take-away	56	31	0
Error	38	13	59
	100	100	100

Table 1: Comparative responses to elementary addition and subtraction combinations

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Figure 1 indicates the cumulative percentage responses of the approaches used by children within each sample. The numbers of children within each sample are given by "n".

Differences between each year group can clearly be identified, for example:

- Highest use of known facts by the 1997 group, particularly with subtraction combinations to 10
- Evidence of the use of derived facts by the 1997 group
- Highest number of errors by the 2001 group

However, what is more in evidence is the remarkable similarity between the children considered within each year. This can more clearly be seen in Table 2 which identifies distinctions, through the use of percentages outlining the use of processes to do (count-on, count-all, count back, count back to, take away etc) and concepts to know (known fact, derived fact). Essential differences in process driven approaches between the 1991 children, interviewed before the introduction of the initiatives, and the 2001 children are largely accounted for by the errors, mostly procedural, of the 2001 children.

	ADDITION					
	To Ten			To Twenty		
	1991	1997	2001	1991	1997	2001
Concept driven	33	42	33		25	11
Process driven	67	50	56	100	75	67
Errors		8	10			22

	SUBTRACTION					
	To ten			To twenty		
	1991	1997	2001	1991	1997	2001
Concept driven	8	58	38		25	5
Process driven	75	42	33	63	63	36
Errors	17		29	38	13	59

Table 2: Comparative Process/Concept distinctions in elementary arithmetic

The evidence suggests that subtraction combinations to 20 are more difficult for these children than the other combinations. In every case apart from the 1996 group's response to subtraction combinations to 10, the use of processes, always counting, is higher than the use of concept driven approaches. Of course, it is difficult to identify whether or not knowing facts is concept based. A clearer confirmation could be obtained if those facts that are known are used to establish those that are not known. This feature is a specified aspect of the programs of study for KS 1 within the National Curriculum and there is a strong emphasis on this aspect of numeracy within the NNS. The indicators suggest that though we may identify differences between process driven and concept driven strategies, for these children those facts that are known are not used to establish those that are not known. The consequence is that we would see these as isolated pieces of knowledge. Reconsidering Table 1 would suggest that for children at the lower end of the achievement spectrum initiatives developed to improve the sophistication of their thinking have hardly had the desired outcome.

The application of a fixed routine procedure was a feature of the approach used by the children if they failed to recall a fact. In some instances these began to emerge very early within the interviews for example in combinations to ten and in the excessive use of fingers. The application of difficult almost non-generalisable procedures became common when children attempted to apply the same procedure for subtraction combinations to 20. Count-back dominated as a procedure and this caused excessive difficulties with 17-13. Indeed, the greater proportion of errors within all groups is largely accounted for by procedural difficulties associated with counting. Children who successfully used fingers for counting small numbers had difficulty with quantities greater than ten. In some instances this was recognised. For example 17-13 was "*too tricky to count*" (1997 child). In others children attempted to obtain a solution but miscounted.

"9 add 8. .... It's hard. " (1997 child)

"1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..12, 13, 14,15" (2001 child)

Others simply had no effective procedure and tried to keep a mental check (format)

"9 add 8—I ran out of hands"

"1, 2, 3, 4, 5, 6, 7, 8, 9, 10..., 11, 12, 13, 14, 15, 16" (2001 child)

This child's second and third attempts gave the answers 17 and 20.

Thus, though in some instances no suitable counting procedure was identified, in others, particularly when attempting the subtraction combinations, difficulties associated with double counting procedures caused problems:

"I just counted 1,2,3,4... 16, 17 then I went 17,16,15,14,13 and said 5" (1997 child)

Discussion

We are drawing our evidence from those having the greatest difficulty in responding to the new initiatives at an elementary level. It is most unlikely that they will be in a position to respond to the initiatives at subsequent levels of difficulty. However, our evidence should not be used to imply that such children will forever be confined to the use of inefficient procedural approaches. Neither do we wish to place an artificial ceiling on their ultimate performance. We do recognise however that should they persist with what is essentially a fragmented knowledge structure, which in some instances is procedurally inefficient and in others is insufficiently compressed, they may be forced into less powerful strategies through trying to cope with too much information.

In one sense both the requirements of the National Curriculum and the pedagogic approaches associated with the National Numeracy Strategy may be attempting to overcome these problems by firstly giving children too many options to consider and then by focusing on rote-learnt procedures to perform as sequential action schemas. If successful, such knowledge can be used to solve routine problems, as we can see where children are successful in their counting, but it occurs in time and may not be suitable to support thinking about the whole entity, which is formed as a result of encapsulating the process.

By their very nature, the National Curriculum and the National Numeracy Strategy are offering a sequence of activities that teachers have to follow. They are grounded within a stage theory that we would suggest is flawed from the very outset. In addition there is also the sense that the teaching strategy has been identified through a form of genetic decomposition (Cotterill et al, 1996) which relies on the teaching of partial structures that may be qualitatively different from the way the whole is to be understood. Resnick & Ford (1981) questioned the reliability of such an approach, but given the Ofsted measures of mathematics teaching within the schools we would suggest that not only can the children not see this whole structure but it is strongly likely that neither can the teachers. Even if we assume that the NNS contains perfectly constructed, sequenced activities it is evident that the children are not seeing things this way. We have to look at the greater number of errors reported within the 2001 sample. Obviously these children have not interiorised (in the sense of Sfard 1991) an appropriate counting procedure to deal with combinations to 20.

Pitta (1999) suggests that children who have difficulty with elementary arithmetic are either unable to, or simply choose not to, see through actions and objects to embrace more abstract qualities. Their disposition has tended to lead them towards an empirical form of abstraction. Developing suitable stages in the process of pseudo-empirical abstraction in the context of counting actions and the development of number concepts has been one of the functions of teaching. The occurrence of a greater number of errors within the 2001 sample would suggest that the considerable emphasis now given to oral and mental strategies and whole class teaching has limited the time available to these children to interiorise and condense appropriate counting actions in order to make the abstraction. Perhaps teachers do not realise its fundamental importance. In the past we have often heard children told to "count" if they had difficulty.

We would suggest that the emphasis on effective mental approaches and institutional and national demands to achieve them have in fact removed the necessary experiences that these children require in order to make appropriate pseudo-empirical abstractions in the field of elementary number. Their focus is very much on physical objects to support counting. In the absence of aids, such as counters, they search for alternatives. The action is supplementary and we would suggest a secondary focus which has not been effectively interiorised (particularly for those who display errors), nor as our few examples illustrate, condensed. Some children now seem to be seriously disadvantaged. Not only are they demonstrating evidence that they have not encapsulated a counting process but they are also demonstrating that they have not interiorised the procedure.

The cumulative evidence is that if they know facts they fail to use them in the way that the initiatives suggest they should. Some reasons for this have already been discussed elsewhere (Gray & Tall, 1994). Now however we seem to see a higher proportion of children who do not seem to have the procedural competence to consider an alternative approach with confidence. It seems that what may truly be a positive initiative for a great number of children may not be so very positive for some of those it was designed to help.

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A GROUP AS A 'SPECIAL SET'? IMPLICATIONS OF IGNORING THE ROLE OF THE BINARY OPERATION IN THE DEFINITION OF A GROUP

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The study from which this paper originates aims to contribute to the understanding of the difficulties that mathematics undergraduates encounter in a Year 2 course in Abstract Algebra. Analysis of their written responses to a question regarding a generic group on a set of four elements seems to suggest that such a group is not seen as a pair (set plus binary operation) but as a 'special set' where the axioms describe properties of the elements and not of the operation. Here we focus on certain implications of seeing a group as 'a special set': the students' occasional disregard for checking associativity (especially in a case where the group was presented in a table) and their neglect of the inner structure of a group (especially in their claim that two groups can be isomorphic but have different subgroup lattices).

The study of group theory gathered momentum in the mid-nineties with the work of authors like Ed Dubinsky, Uri Leron and Rina Zazkis (see references). The action-process-object-schema (APOS) theoretical framework was used to infer a general pattern for learning concepts like group, subgroup, cosets and normal groups. Also the learning of the concept of isomorphism (see Leron, Hazzan and Zazkis, 1995) was explored, and so was the use of Lagrange's Theorem (see Hazzan and Leron, 1994). The importance of studying this particular subject was suggested in Dubinsky et al. (1994):

> In many colleges, abstract algebra is the first course for students in which they must go beyond learning 'imitative behaviour patterns' for mimicking the solution of a large number of variations on a small number of themes (problems). ...An individual's knowledge of the concept of group should include an understanding of various mathematical properties and constructions independent of particular examples, indeed including groups consisting of undefined elements and a binary operation satisfying the axioms. (p268)

This is still the case in most Universities in the UK, including the university where the present work was based. In this paper we want to show that this first encounter with this 'new' level of abstraction (see Hazzan, 1999) appears to be problematic as the students find it difficult to co-ordinate all the parts needed in a conceptualisation of the notion of group.

Research Issue. The main research issue we want to address in this paper regards the learning of the concept of group. More specifically, our conjecture, in the light of the evidence drawn from our study, is that the most difficult obstacle that the students face while progressing in forming a group schema is understanding that a group is formed by a pair: a set and a binary operation. In a study of the students' written work (see Methodology below) we observed this difficulty in the context of an exercise in which the students had to demonstrate an understanding of the axioms in the definition of a group (closure, associativity, neutral element, inverse element) and appeared to treat these properties, especially that of associativity, as properties of the elements of the group rather than properties of the binary operation. We also note that, while closure, neutral element, inverse element and commutativity are properties that can be checked by applying the binary operation on *two* elements, associativity involves operating on *three* elements and in a particular order – which, in the context of the particular exercise we examine here, was significant.

Methodology. This study is funded by the Nuffield Foundation and will last six months (October 2001 - March 2002). It is a small, exploratory data-grounded theory study (Glaser and Strauss 1967) of the mathematical writing of the students (55 in total) in the Year 2 Group Theory course of the 3-year degree in Mathematics at the University of East Anglia in the UK. It is the third phase in a series of small Nuffield-funded projects run by the two authors during the last two years as an on-going collaboration between the School of Mathematics and the School of Education at UEA (see for example Nardi and Iannone 2000). The aims of the study are: identifying the major problematic aspects of the students' mathematical writing in their drafts submitted to tutors on a fortnightly basis; increasing awareness of the students' difficulties for the tutors at this University's School of Mathematics; providing a set of foci of caution, action and possibly immediate reform of practice; and, setting foundations for a further larger-scale research project. The first two phases focused on Year 1 Calculus, Linear Algebra and Probability courses. The current phase focuses on the Year 2 Abstract Algebra course.

The course ran for 5 weeks at the beginning of the Autumn Semester 2001, with 4 hours of lectures per week. There were 3 seminar sessions for each of three groups of 15-20 students. The lectures were traditional front-teaching sessions. The seminars were run by a seminar leader (the first author) and a seminar assistant. The lectures were observed by the first author in order to become familiar with the examples and the notation the students were exposed to as well as with their general reactions to the new content. At the end of each seminar session the students were asked to submit a selection of the exercises in an exercise sheet. Their responses were then marked by the seminar leader and the seminar assistant. These written responses to the problems sheets administered at each of the three seminar sessions form the bulk of data gathered for this project.

The first level of data analysis, Data Analysis Version 1, involved the production of a student–by–question table that focused on the written responses of 15 out of 55 students registered for the course. This table summarised observations and comments of the first author on the students' responses to the set homework. After this table was completed, informal conversations were held with the seminar assistant to record his observations and comments after having marked part of the homework of the remaining student cohort. Following a detailed discussion of Data Analysis Version 1, the second author produced Data Analysis Version 2, a question by question table where the major issues were summarised, characteristic examples of the students' work were referred to and links with current literature were made. In these analyses it became clear that probing further into the students' thinking would be greatly helped if they could be asked to provide justification for parts of their writing, for example, via interviewing them. An initial step in this direction has been taken with an extensive interview of one of the students (parts of the transcript we use here to support the evidence from the students' written work).

Forming An Image of the Concept of Group. The data we use to raise the issue of the students' understanding of the concept of group originate in the first cycle of data collection and in the students' responses to the following question:

> **Q1.5:** Write down all group tables for a group of four elements. Hence show that there are two essentially different such groups, both commutative. (Consider group tables obtained by merely renaming elements as essentially the same). How are they best distinguished? For each make a list of all the subgroups.

An understanding of group, subgroup and commutativity as well as a "naive" concept of isomorphism (see Leron, Hazzan and Zazkis, 1995: "Two groups are isomorphic if they are the same except for notation") are necessary here. Moreover let us observe that, at this point in the course, students had only just been introduced to the concept of isomorphism, and this is why the question setter suggested a criterion for judging when two groups are "essentially the same".

Fifty-two pieces of homework were handed in that week and all the students but one attempted this question. The most common response consisted of:

- a list of the four tables that can be obtained from a set of four elements;
- a declaration of some of the obtained groups to be isomorphic, hence the existence of "two essentially different groups";
- a list of the subgroups of each of these groups.

Commutativity was dealt with by observing symmetry around one of the diagonals of the tables. In the following we offer certain observations on and exemplify the students' responses.

A major common characteristic of the responses is that none of the students checked whether a group that satisfies these tables actually exists. The students constructed the tables possibly bearing in mind the direction of the lecturer (as recorded in the first author's observation of the lectures) to check that each element of the group appears only once in every column/row. This way of proceeding takes account of the properties of inverse, neutral element and closure but leaves out the checking of associativity. The fact that the table has to be shown to be associative, or else that the students have to produce an example of a group that satisfied the group table, went missed. One interpretation of this fact (see also interview extract later) is that, at this stage, the students deal with the concept of group as a "special set". The schema the students are referring to is the "set" schema and the properties are checked as to be properties of the elements and not of the binary operation defined on the set to form the group. If we agree that, by placing an emphasis on the *order* in which elements are operated on, associativity is the property of a group that refers more to the operation and not so much to the elements, this is not surprising. In short, instead of having a concept of group consisting of three interrelated schemas: set, binary operation and axioms (see Brown et al. 1997), the set schema dominates leaving the binary operation schema, and the checking of certain properties, neglected.

One implication of this domination of the set schema is evident in the response to Question 1.5, offered by Jo:

5 ①

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

②

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

③

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

④

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

① is distinguishable because it is cyclic.
The others can be seen as the same group as they are isomorphic
All tables are symmetrical about the line from top left to bottom right and are therefore commutative.

The subgroups are:

① {e,b} {e} {e,a,b,c}
② {e,a} {e} {e,a,b,c} {e,b} {e,c}
③ {e,a} {e} {e,a,b,c}
④ {e,c} {e} {e,a,b,c}

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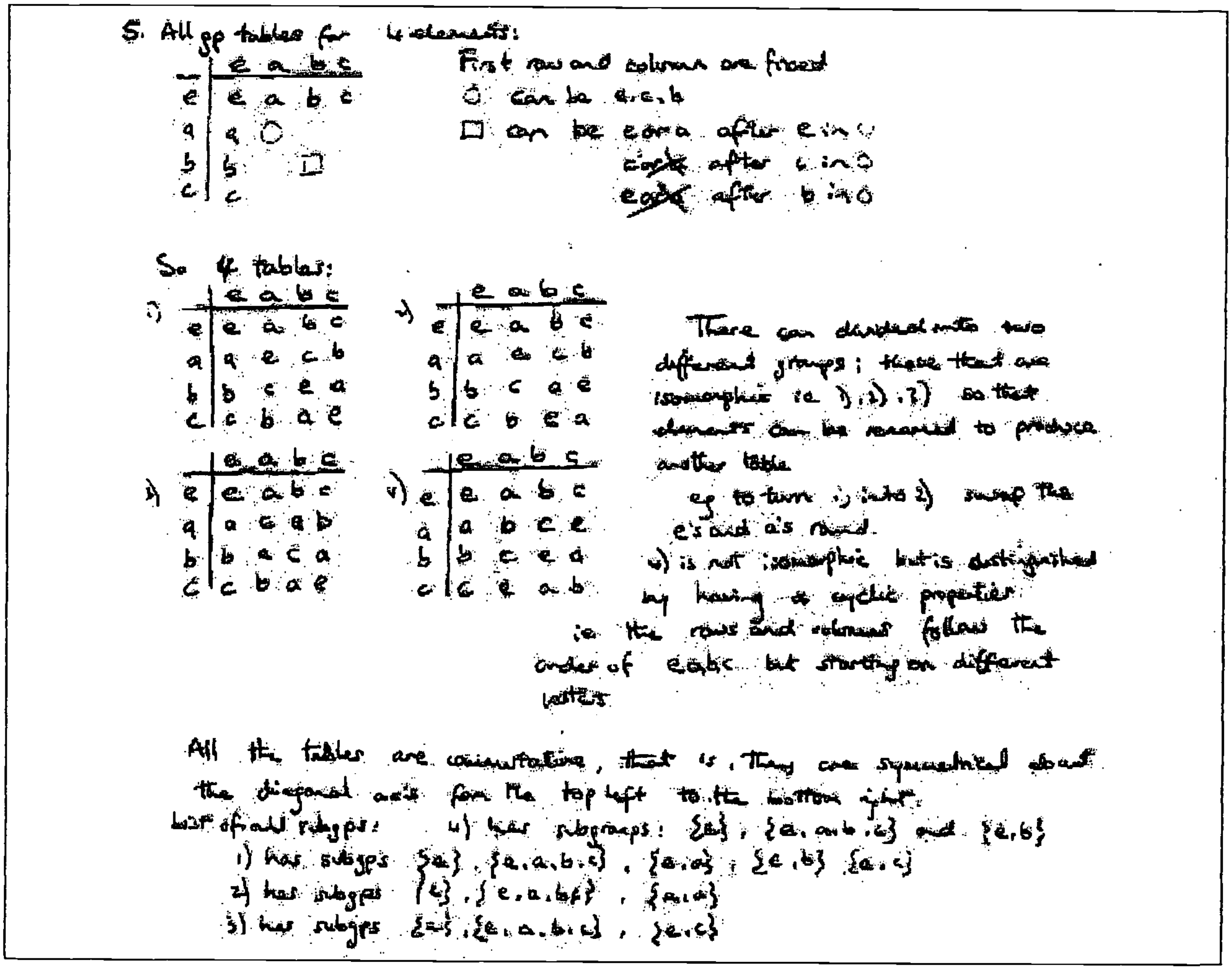
In Jo's script three groups are said to be isomorphic. Yet a few lines later they are shown to have different subgroup lattices. The concept of 'naive isomorphism' seems to be affected by a failure to recognise a group as a pair. The notion that the binary operation defined on the set induces naturally an inner structure of the group (its subgroup lattice) is missed and this inner structure is not regarded as something that characterises a group. Therefore, two groups are seen to be 'essentially the same' even if their inner structure is different. Another student, Hazel, also does not seem to see any contradiction in the claim that two groups are isomorphic but have different subgroups:



5. All gp tables for 4 elements:

	e	a	b	c
e	e	a	b	c
a	a	○		
b	b		□	
c	c			

First row and column are fixed

○ can be e, c, b

□ can be e or a after e in ○

~~c or a~~ after c in ○

~~e or a~~ after b in ○

So 4 tables:

i)

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

ii)

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

iii)

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

iv)

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

There can divided into two different groups; those that are isomorphic ie i), ii), iii) so that elements can be renamed to produce another table

eg to turn i) into ii) swap the e's and a's round.

iv) is not isomorphic but is distinguished by having a cyclic properties ie the rows and columns follow the order of e, a, b, c but starting on different letters

All the tables are commutative, that is, they are symmetrical about the diagonal axis from the top left to the bottom right.

List of all subgps: iv) has subgroups: {e}, {e, a, b, c} and {e, b}

i) has subgps {e}, {e, a, b, c}, {e, a}, {e, b} {e, c}

ii) has subgps {e}, {e, a, b, c}, {e, a}

iii) has subgps {e}, {e, a, b, c}, {e, c}

Hazel's actions, typical in the written responses we have examined, appear to be heavily table-based. Her understanding of the notion of isomorphism seems to involve processes she calls 'swapping' and 'turning into'. Could we suggest that this firm adherence to/dependence on table-based actions, while facilitating the construction of the groups with four elements (and the checking of properties such as commutativity), at the same time places an obstacle in the students' constructing an image of the group's inner structure (as well as perhaps distracting them from checking associativity which requires operating upon three, not two, elements)? That

Hazel's notion of isomorphism is problematic is suggested also by the grammar/syntax she uses when she talks about isomorphic groups: she doesn't talk explicitly about, e.g., 1 as isomorphic to 2 but she writes '4 is isomorphic' (to what?).

This problematic understanding of the 'naive' concept of isomorphism was widespread across the responses we analysed – as was the exclusion from the list of subgroups of the trivial ones ({e} and {e, a, b, c}) such as Steven:

Subgroups

(2): {e,a}, {e,b} and {e,c}
(3): {e,b}

and the inclusion of subsets that were not closed or did not include the identity element. For example Wayne – who also throughout his writing does not adopt a consistent and conventional bracketing system to denote a set:

Can divide into 2 ~~groups~~ different groups
i) Group has 2 self-inverting elements (tables 1, 2, 3)
ii) Group has all self-inverting elements (tables 4)
Both commutative as the tables are symmetric about the left diagonal.
Subgroups for i)
(e), (a), (b), (c) and (e, a*) [and (e, a, b, c)] * a, b, or c depending on which element is self-inverse
Subgroups for ii)
(e)(a), (b), (c), (e, a), (e, b), (e, c) [(e, a, b, c)].

This suppression of the role of the binary operation and the tendency to attribute properties of the binary operation to the elements of the group was also evident in one interview with Wayne held at the end of the Group Theory course (we conducted this interview to test whether interviewing could illuminate us any further about the reasons behind certain parts of students' writing). The student responded to our question about the difference between a set and a group as follows:

> **Wayne:** Hum ... A group has axioms ... certain axioms that have to hold. One is associativity. That is basically if you got elements, I don't know, *a*, *b*, and *c* in the group then ... then, *a* plus *b*, well, say that it is plus the operation [...] Yes, Then *a* plus *b* ... plus *c* is equal the same as *a* plus...*b* plus *c*. That's the first property. You have got some neutral element. Which basically says that, say call it *e*, ... then if you multiply by any of these elements in the group you still get ... you get the same thing out. And then from that ... And also you have got an inverse. You have got *a* and some *d* they come together to come the inverse. *a* plus *d* is the inverse. Basically.

In Wayne's words "a group has axioms" that "have to hold". These axioms include associativity, neutral and inverse element. In this extract the binary operation is mentioned – albeit alternately referred to as 'plus' and 'multiply' (which is an often problematic choice of words - see Nardi 2000). He also seems to confuse the use of the words "inverse" and "neutral" in "*a* and some *d* they come together to come the inverse. *a* plus *d* is the inverse" but still his description of the properties seems to indicate a clear understanding of how they characterise the relationship between the *elements*. Later on he links these properties to his perception of a group:

> **Wayne:** Oh ... I think that the group is a special kind of set, basically, where you got certain properties, basic properties, axioms, basic axioms that have been defined and for groups these hold, no matter what. And so ... that's why I see a group as a special kind of set. That's ... is not a set ... a group is like, ... a group is a set, but is a special kind of set that has properties that can be defined, can be shown to be true. For each group that you have. That's basically...

In Wayne's words a group is a set whose elements happen to have certain properties, it is a "special set". No reference is made to the group's inner structure and to the central role the binary operation plays in the formation of this structure.

Summary and Conclusion. In this paper we discussed Year 2 mathematics undergraduates' developing notions of a group in the context of their first encounter with a generic group on four elements.

Following strictly table-based actions the majority of the students constructed the four possible tables but, by leaving out the checking of associativity, they refrained from checking/showing whether a group that satisfies these tables actually exists. We conjectured that, by strictly remaining within a table-based action schema, the students demonstrated a dominant image of a group, not as a pair (set with a binary operation), but as a "special set". In this schema the properties of associativity, inverse and neutral element are attributed to the group (and in particular to its elements) and not to the binary operation.

There seem to be certain implications of the above conceptualisation of a group as a "special set" in the students' responses. One is that an image of a group that neglects the role of the binary operation – namely, the inner structure that the binary operation yields – makes the emergence of problematic images of isomorphism possible. Such images include: two groups can be isomorphic but have different subgroup lattices.

Another implication of the above involves an image of a subgroup as a subset of a group - evident in the students' responses where subsets that were not closed or did not include the identity element were presented in the list of subgroups of the group.

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GROWTH IN STUDENT MATHEMATICAL UNDERSTANDING THROUGH PRECALCULUS STUDENT AND TEACHER INTERACTIONS

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This paper investigates the role of teacher interaction in the development of mathematical understanding of five students who worked together on a math-modelling task. The dialogue between the teacher/researcher and students is analyzed. Preliminary findings suggest that where the mathematical thinking of the students was understood, interventions helped develop students' thinking. The Pirie-Kiernen model of mathematical understanding guided interpretation of student mathematical thinking and understanding. [1]

INTRODUCTION

The students, engaged in conversation with the teacher, often give explanations for their ideas. A question arises as to what influence, if any, the teacher's response to those explanations have on student progress. This report examines dialogue between teacher and students and seeks to investigate the effect on students' growth in mathematical understanding.

The data come from a two-week summer institute that was a component of a longitudinal study on the development of proof making in students [2]. The students worked in groups on precalculus level mathematics problems. This paper focuses on one group of students and one of the problems they examined.

THEORETICAL FRAMEWORK

Communication is an essential part of the mathematics classroom providing a means for students to express their ideas and explain their thinking (NCTM 2000). Through communication students can share ideas and discoveries about the mathematics on which they are working. Since the communication process helps students create meaning for their ideas, NCTM (2000) includes communication as one of the standards in *Principles and Standards for School Mathematics*. Because of the need for communication in the classroom, an environment can be created where teachers and students engage in very important dialogue. Dialogue is important because it helps teachers assess the student's mathematical understanding and allows the students to clarify and express their ideas. Towers (1998) developed several themes to describe teacher interaction with students and illustrated how these interactions occasion the growth of students' understanding. A teacher should be skilled in interacting with students in order to gain access to students' mathematical understanding. Teacher questioning can help students justify and extend ideas, make connections, and generalize their conjectures (Dann, Pantozzi, Steencken. 1995). The development of these skills is not immediate for the teacher, but once gained the teacher has an effective way to facilitate the growth of a student's understanding (Martino and Maher, 1999).

Being a participant in the classroom discourse, the teacher has an important function. In describing a classroom where students are working in small groups on a task, Maher, Davis, and Alston (1991) indicate that the teacher plays many roles: listening to children, offering suggestions, asking questions, facilitating discussions, drawing out justifications. When students discuss with their teachers the meaning of mathematical notions, students are expected to think about concepts, their meanings and their interrelations (Vinner 1997). If students do think about concepts, they are in a conceptual mode of thinking (Vinner 1997). If students do not think conceptually, but still produce answers which seem to be conceptual, then Vinner (1997) states the students are in a pseudo-conceptual mode of thinking. The teacher must continuously assess whether or not the students have learned the mathematical concept, truly understands the reasoning behind their problem solving approach, and can adequately support and defend their conclusions using their previously learned mathematical knowledge. One way to determine what a student knows is to use a model of understanding. While many models exist, this research analyzes student understanding using the Pirie-Kiernen model of mathematical understanding (Pirie & Kiernen, 1994). This model is "a theory of the growth of understanding which is based on the consideration of understanding as a whole, dynamic, levelled but non-linear process of growth" (Pirie & Kiernen, 1994, p. 83). This theory shows that student understanding is an organization of ones knowledge not an acquiring of categories of knowing.

In a regular classroom, it is not always possible to observe what a student does after an interaction with the teacher. Because this observation is not always possible, it is difficult for a teacher to determine if the interaction was beneficial to the student. Videotape data that follows the student when the teacher leaves make possible gaining a better understanding of a student's actions. Interacting with students is a challenging task for the teacher, who has to make instantaneous decisions. The researcher, who has the benefit of studying and referring to videotape data, however, can learn from the interaction after the fact. What the researcher learns from the interaction can be shared with the teachers, who can reflect on their actions, and help facilitate a growth in students understanding.

METHODOLOGY

Participants. Five students seated at the same table (four males and one female) and one teacher/researcher were subjects in this study. All of the students were entering their fourth year of high school. The teacher/researcher involved in the interaction is an experienced professor of mathematics and mathematics education at the university level.

Task. The students were given a picture of a fossilized shell called Placenticeras. The first part of the task was to draw a ray from the center of the shell in any direction. Then with polar coordinates as a way to describe the spiral of the shell, the students were to make a table of r as a function of theta. After creating the table, the students were asked what they could say about r as a function of theta. The students had graphing calculators, transparencies, rulers, and markers at their disposal for completing this task.

Data Analysis. The data come from a two-hour videotape session during the third day of a two-week Institute. The interactions were coded to consider perspectives of the teacher and the students. For the students, the following codes were developed and used: S(i): Student ignores the suggestion made by the teacher; S(c): Student asks the teacher for clarification of a statement; S(a): Student attempts the teacher's idea or suggestion; S(e): Student engages in conversation for the purpose of explaining their own views. For the teacher: T(r): Teacher restates the problem or returns to an old idea; T(f): Teacher follows the student's idea or suggestion; T(n): Teacher introduces a new idea; T(c): Teacher asks the student to clarify their statements or idea. The codes were used to follow the choices of the teacher and the resulting action by the student. When students became engaged in a conversation, their words were examined for evidence of their understanding.

FINDINGS

The students' own words demonstrate where mathematical understanding occurs, and where their growth about a solution to this problem appears. This dialogue follows a discussion between the teacher and students regarding their solution to the task.

Student 1: S(e) I think it does. I mean if you look, if you look at the regression. It's just like a parabola. And uh your data.

Student 2: It is a parabola.

Student 3: S(e) It is a parabola. A very nice parabola. And like you know. I mean you can't use anything behind past zero on the x obviously because it can't have negative growth. That doesn't make sense. So you can't do that. But I mean the way, the way it goes up and the reason why it goes sharply up is just the fact that. I mean even from here to here like say the distance is 6 then all of a sudden it is 40. It's not going to keep on going little by little. Eventually it's getting wider like this. And that's why it's jumping so high up. It's not the fact that it's off or it's not predicting anything. It's just the numbers are getting larger and larger. It has to go higher and higher. So that's why it goes that steep angle like that.

Teacher: T(f) Ok well. I am still interested in that earlier part because. Are you saying that this animal really started growing where we're saying theta equals zero is.

Student 3: Um hmm.

Teacher: How do you know that?

Student 3: S(e) It's gotta start somewhere. And it doesn't start. You can't start. You can't start.

Student 2: You can't start with anything negative.

Student 3: anything past nothing.

Student 2: Yeah.

Student 3: You know.

Student 2: Cause then it doesn't exist. In which case it's not there.

The students do not look further into the data beyond a visual fit of a scatter plot and their curve. The teacher/researcher returns to the idea about how the model describes the start of the growth of the shell.

Teacher: T(r) See then I am wondering about that fourth power model cause if you go to the left on it. You are sort of going inward on the shell right. You are going backward in time.

Student 3: Yeah.

Teacher: But then suddenly as you keep going left it goes up.

Student 3: S(e) Oh but there is nothing there though. That is the thing. Like you have to set limitations somewhere because some things are just physically impossible you know.

Teacher: I think we're beginning to understand each other.

Student 3: Yeah.

Teacher: Ok, Umm.

Student 3: S(e) I mean its like. I guess its like certain things like if you figure out like differences with like electricity or something or like in physics. Like you can't have things that are. Sometimes you can't have things that are negative. There are things that are just physically impossible to have. And that to have something, to have an animal or a living thing that is a negative distance would mean that it isn't there. So it's not physically possible to have that anything past that zero. You know. It just wouldn't be there. This animal would not be there if there was a negative number. Basically.

The teacher/researcher continues to question the students about why their model does not work for certain values.

Teacher: T(c) Oh, so there's a place. Okay then you are agreeing that there's a place where the regression doesn't model the animal.

Student 2: S(e) You can put it so that the restriction has to be greater than zero.

Student 3: S(e) Yes, but that's necessary for other things too. There's limitations.

Teacher: Okay, Okay.

Student 3: S(e) like like the first graph we did with uh with the running thing, with the uh, with the thing you had to put limitations on it cause there were certain things that went past a certain time.

The teacher/researcher and the students continue the discussion by focusing on the accuracy of the model outside the range of their collected data. The question of what a model would look like if the data were collected again moves the conversation topic to the model's general shape. After this discussion, the teacher/researcher returned to the left side of the student's model. By the left and right side, the teacher/researcher and students are using the origin of the co-ordinate plane as their reference point. Therefore the left side would refer to negative values of time, and the right side would refer to positive values of time. When the teacher/researcher returned to the left side of the student model, the teacher/researcher and students revisited discussing the model's accuracy during negative values of time.

Teacher: T(r) So it looks like we are making sense on the right and then we got questions on the left. Is that fair?

Student 3: Sure, why not.

Teacher: Okay.

Student 2: S(c) What possible questions could you have on the left. It's dead. It doesn't exist.

Teacher: Well I just don't.

Student 3: S(e) Not even that. It's not even born yet.

Teacher: T(r) It's very hard for me, yeah. It's very hard for me to believe that this at some point in the distance past.

Student 2: It doesn't exist.

Teacher: T(r) That that it was very large as the fourth power, as that fourth power curve suggests.

Student 2: S(e) Alright fine, we'll do a third power curve, it'll be very small. It'll be gone.

Student 3: S(e) No, you know what. You know what you got to do. You set a limitation on that graph so there is no left side and then we won't have this problem. Can we do that?

The students have provided a way to adjust the model so that it does not show a large shell when time is negative. Though the two students believe the left side of this model does not accurately portray the growth of the shell, their methods of correcting the inaccuracy are different. Student two wants to change the regression curve to the third power model, which would result in a new equation that models a different rate of change, and continues the inaccuracy of the model before the shell began to grow. Student three explains that he wants to remove the left side and keep the model's representation of the right side.

CONCLUSIONS

The transcript provides evidence of two areas of understanding: the rate of the growth of the shell and the model that represents the growth of the shell. Regarding the first area, the students' understanding of the rate that the shell is growing did not grow during this interaction. For the second area, student three exhibits a growth in understanding from *primitive knowledge* to *property noticing* of the Pirie-Kiernen model of mathematical understanding.

The dialogue showed the students used a fourth power regression to represent the rate of growth of the shell. However, the students used the word parabola to describe the curve. Student one stated, "I mean if you look, if you look at the regression. It's just like a parabola." Student two and three both followed with "It is a parabola". Their early classification of the graph as parabolic demonstrates an *image having* level of understanding. Since parabolic and quartic curves represent different rates of growth, the students are just using the visual image of the function and not the properties of their fourth degree regression equation. Later when student two and three recommended changes for their model, they provided different methods for a correction. Student two suggested changing their regression to a third power, and student three suggested restricting the left side of the model. Since student two's correction used a different regression model, he did not make a connection between the rate of growth and the type of curve needed to model that growth. Neither of the students provided evidence as to why the model is quartic. The students' earlier understanding about the rate of growth did not grow during this interaction because the have not moved beyond a visual inspection of the model's shape.

Despite this misunderstanding, the teacher/researcher did not correct or criticize their comments. Rather the focus of the teacher/researcher was to discuss the students' model for the growth of the shell. By focusing on student three, the transcript shows a growth in his understanding. When questioned by the teacher about the negative values of the fourth power model getting larger, the student responds that there should not be any values because "you have to set limitations somewhere because some things are just physically impossible". Student three's explanation showed a *primitive knowing* level of understanding about their model around zero. The student has demonstrated basic knowledge about when a situation requires limitations. When the teacher/researcher returned to the growth of the shell around zero, the student's level of understanding grew through engagement in the conversation. Without further prompting, the student connected the limitations on the model to other physical situations. This exhibited an *image making* understanding by using previous knowledge about specific situations that need limitations.

Despite a growth in understanding to *image making*, the teacher/researcher asked the student to clarify his ideas about where the regression function does not model the shell. As a response, student three referred to "the running thing", an earlier problem from this workshop. This response moved him to an *image having* level of understanding. He used a mental construct of that activity to further justify his ideas about the need for a limitation. The teacher/researcher returned to the idea of the left side of the model after a discussion of the general shape of the model. During this engagement, student three explained how to change the model so the left side did not exist. He suggested, "You set a limitation on the graph so there is no left side and then we won't have this problem". The student demonstrated another growth in understanding to *property noticing*. The student recognized that the model should be altered to a model that has the same property as other limitation situations. He suggested changing the model so it contains the property of having no values on the graph for negative time. Previously the student explained why the shell could not exist for negative values, but has now moved forward to provide a possible method for representing the limitation on the model.

Through interaction between the teacher and students, the students made public their level of mathematical understanding. By examining the episodes presented, one can see that the teacher/researcher consistently returned to the idea of how the model demonstrated the growth of the shell over the entire domain of the students' model. Additionally, the students' ideas are followed or they are asked to clarify their statements. Using this method of questioning, the students were given the chance to make connections and reorganize their thoughts about their model. By reorganizing his thoughts, student three's understanding grew from *primitive knowledge* to *property noticing*. The opportunity for growth occurred because the teacher continually returned to old ideas. As a consequence, the students had multiple possibilities to become engaged in conversations and articulate their understanding of the mathematics.

This research provides a foundation for continuing a dialogue about the affects of teacher and student interactions in the classroom. These preliminary findings imply that teacher interaction helps the student to express their mathematical understanding. Further research can help to indicate whether the teachers/researchers can learn from their choices during interactions to see if they are constructively contributing to students' progress. More research is needed to provide a better understanding of how teacher interventions, particularly questioning, can contribute to students' mathematical understanding.

NOTES

1. An earlier version of this paper by Ilaria and Maher, Rutgers University, was presented at PME-NA 23, Snowbird, Utah.

2. This work was supported in part by National Science foundation grant #REC-9814846 s(directed by C.A. Maher) to Rutgers, The State University of New Jersey. Any opinion, findings, conclusions, or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science foundation.

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CAN POOR STUDENTS IDENTIFY THE GOOD FEATURES OF A DEMONSTRATED PROBLEM SOLVING METHOD AND USE IT TO SOLVE A GENERALIZATION PROBLEM?

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This study investigated the effect of presentation of a good solution method on poor students' solving generalization problems and the difference between good and poor students' evaluation of a good method. Subjects were fourth, fifth and sixth graders. Three hints were presented :the suggestion of using a "solve a simpler problem" strategy, demonstrating how it is executed in solving a small term problem(n=4,5) and the application of the strategy to the far generalization problem(n=30). The hint was effective for poor students of every grade. Good students evaluated more often the aspect of structure for a good method.

1. Introduction

In mathematics lessons, it is important to learn mathematical thinking, including generalization. Generalization problems are problems that are solvable by finding a pattern of quantitative relationship in a given problem situation. There are three examples in Figure 1. Stacey (1989) pointed out that problems of this type are hard for school students. She contrasted success on and methods used on near generalization problems (where the number of a term is small e.g. finding the number of marbles with 10 marbles on a side in Problem C, Figure 1) and far generalization problems (where the number of a term is large e.g. side of 100 in Problem C). Students even find that near generalization problems are difficult to solve.

Ishida has investigated the processes used by good and poor problem solvers on generalization problems. Good students and poor students showed different approaches. Good students tried to find a mathematical expression when they solve a near generalization problem, and they applied it to solve a far generalization problem. However, poor students tended to use a "draw a figure" strategy on the near generalization and then they tried but failed to find a mathematical expression on the far generalization problem. This observation was consistent across grades 4, 5 and 6 (Ishida,1992; Ishida&Sato,1996).

Sato (1999) showed that it is helpful for some students to be asked to solve a near generalization problem before a far generalization problem. (For example, to be asked to find the number of stars for the 8^{th} figure as a sub question would make Problem A of Figure 1 easier.) Also Sato found that it is helpful for some students to be first asked to draw a small term figure and write a mathematical expression based on it before a far generalization problem. Those hints seem to be useful to remove some of the factors that affect the difficulty of solving a generalization problem. For example, students do not need to realise by themselves that they should use the "solve a simpler problem" strategy or set up a mathematical expression.

Kimura (2001) gave a further hint. In Kimura's study, two hints were presented: the suggestion of using a "solve a simpler problem" strategy (as Sato) and also demonstrating how it is executed in solving a small term problem. These hints were more useful than those of Sato for Grade 4 and equally helpful at Grades 5 and 6. But Kimura did not demonstrate the application of the strategy to a far generalization problem. If such a demonstration is given to poor problem solvers, can they use such a hint to solve a far generalization problem? This is one of the research questions for the present study.

Many studies have demonstrated the need to develop meta-cognition to work in problem solving situations. Evaluation of the mathematical value of methods relates to this meta-cognitive knowledge. Ishida (1998) has studied the choices that students in Grades 4, 5 and 6 make between four methods of solving far generalization problems. One method involved only drawing a figure and counting; the second provided a mathematical expression which did not reflect a relevant pattern; the third provided an expression linked to a simple generalizable structure; the fourth provided an expression in which the way it could be generalized was difficult to see. Most students (about 85 % including both good and poor problem solvers) selected the best method (third in list above). A few students (about 10%) selected the strategy of drawing a figure, which is a poor strategy for a far generalization problem. Their performance on the far generalization problem was lower than that of the others. The study also asked students to write the reason of their choice of method. The reasons were very varied, including references to the generality of the method, its effectiveness in producing an answer, the fact that it displays a simple structure, and the fact that the method is easy to understand. Effectiveness was often written by students in all grades. In the above study (Ishida, 1998), students selected preferred methods from a list of both good and poor problem solving methods. In the present study, only a good method is offered but students' evaluation of the good points of this method are examined. This is of greater relevance to teaching, where good methods are usually presented. Do good and poor students form different evaluations of them?

In summary, the present study asks two questions:

1 Is presentation of a good solution method useful to help poor students solve generalization problems?

2 Are there any differences between good and poor students' evaluation of a good method?

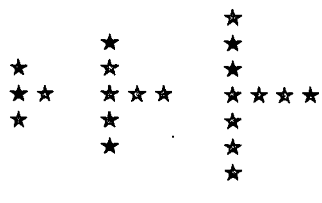
2. Procedure

The subjects of this study are 112 fourth graders, 102 fifth graders and 113 sixth graders, who are students of two elementary schools in Yokohama city, Japan. The three problems in Figure 1 were given to the subjects (each allocated 10 minutes) by a classroom teacher. Problems A and B were used to divide the students into good and poor problem solvers and Problem C was used to investigate the purpose of this study. The 94 students who succeeded on problems A and B were regarded as good problem solvers. The 166 students who failed to solve both problems were regarded as poor problem solvers. The numbers by grade are shown in Table 1. Sixty seven students who succeeded on only one of problems A and B are excluded from further consideration in this study.

Problem C asks students to solve a near generalization (n= 3, Question 2) and a far generalization (n = 77, Question 3), emphasizing the use of the mathematical expression generalized from Akira's solution. It gives the hints used by Kimura (2001) and also demonstrates the application of the strategy to the far generalization problem (n=30). The number 77 was selected to avoid false proportional solutions, which are common when student see simple multiplicative relations between the questions (Stacey, 1989). Question 1 of Problem C asks students to evaluate the benefits of Akira's method. The five choices that are presented are based on reasons offered by

Problem A

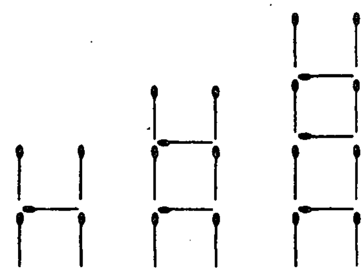
Figures are made as follows. How many stars are needed in the 100^{th} place?



1^{st} 2^{nd} 3^{rd}

Problem B

Using matchsticks ladders are made as following. How many matchsticks are needed to make ladder of 100 rungs ?



1 rung 2 rungs 3 rungs

Problem C

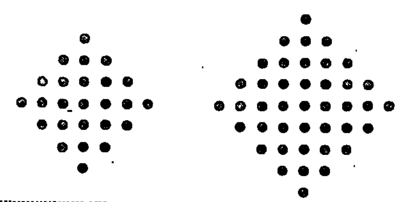
Marbles are arranged like the following figures. Find the number of marbles needed to make a figure of 30 marbles to a side 30.

Akira is thinking of this problem.
He began by thinking it in the case of 4 marbles on a side and 5 marbles on a side.

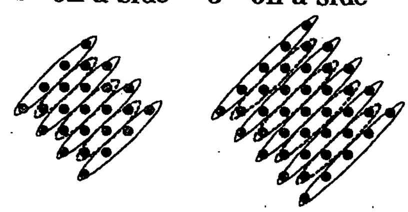
Then, he thought how to find a mathematical expression to get the answer to the problem.

30x30+29x29=1741 Ans. 1741 marbles

4 on a side 5 on a side



4 on a side 5 on a side



4×4+3×3=25 5×5+4×4=41

(1) How do you evaluate Akira's method? Select two viewpoints from the list.

A I can find an answer correctly.

B I can get an answer fast.

C I can use this method when the number of marbles of a side increases.

D He represents a mathematical expression.

E He finds a group of marbles and uses it to make an expression.

(2) How many marbles are needed to make a figure of 3 marbles on a side ?
Using Akira's method, write a mathematical expression to find an answer.

(3) How many marbles are needed to make a figure of 77marbles on a side ?
Using Akira's method, write a mathematical expression to find an answer

Figure 1. Three problems used in this study.

students in Ishida's 1998 study, especially relating to use of mathematical expressions. Items A and B are aspects of effectiveness and item C is generality. Both items D and E refer to mathematical expressions but item E links the expression to the structure of the problem. Item D refers only to Akira's writing an expression.

3. Results

(1) The effectiveness of the hint in problem C.

The effectiveness of the hint given in Problem C is investigated first. In the analysis, we focused on a performance of setting up expression, so the following data was based on a correct answer for setting up a mathematical expression.

Table 1 shows the percentage of good and poor solvers who were correct on question (2) and (3) in problem C. Table 1 shows that the performance of poor students (those who had failed on problems A and B) was good on problem C. In both questions, the percentage correct increases from grade 4 to 6. Therefore the hint used in problem C effectively assists poor students to solve a generalization problem.

How did the hints offered by Problem C assist students? In Problems A and B, in every grade, students frequently gave wrong mathematical expressions by using proportion strategy (for example, 3 times 100 in problem A, 5 times 100 in problem B) and other wrong mathematical expressions which simply combined given numerals. These error types show that they seemed not to try to find a pattern or structure from figures given in problems. The hints in Problem C encouraged them to look at the structure.

Table 1 Percent correct on Problem C for good and poor problem solvers.

	Grade 4	Grade 5	Grade6
Good students	(N =17)	(N=35)	(N=42)
Question (2)	88.2%	100%	95.2%
Question (3)	76.5%	94.3%	95.2%
Poor students	(N=82)	(N= 41)	(N=43)
Question (2)	67.1%	80.5%	90.7%
Question (3)	47.6%	61.0%	81.4%

(2) How common errors reveal understanding of structure

Error analysis for poor students who failed to solve problem C is important to clarify their difficulty. The main classifiable errors were to give no answer, using the form "a x a + b x b" with wrong numbers, or using the incomplete form "a x a +". In total there were 26 students in Grade 4 who failed to solve both question (2) and (3) of problem C. For these, most gave no answer or an unclassified error. The "a x a + b x b" type of error did not appear very often. This trend was the same for Grades 5 and 6 although there were fewer students. On the other hand, poor students who failed only on question (3) in problem C made an error like "a x a + b x b" and "a x a + " . While they set up a correct mathematical expression on question (2), that is 3 x 3 + 2 x 2, they could not generalize it correctly. A high percentage wrote "77 x 77 + " and could not complete this with 76 x 76.

We need to distinguish two types of students who failed on problem C: those who failed on both question (2) and (3) and those who failed only on question (3). In the former case, they did not understand about Akira's method at all. They did not understand that Akira tries to find a pattern in a small term situation to set up a mathematical expression that can be generalized easily. In the latter case, they understand the pattern shown in Akira's method and can easily apply it to the very near generalization problem but cannot apply it to a far generalization problem. This shows that they do not understand the mathematical expression on a small term situation as revealing a mathematical structure.

(3) Comparing good and poor students evaluation of the good method demonstrated in problem C

In question (1) of problem C, students were asked to select two viewpoints to indicate what makes Akira's solution good. The reason why we let them select two items is that this solution method has several good points. Because the hint was not equally helpful for solving problem C for all grades, we analyze the data by grade. Table 2 shows the percentage of students who selected each item. Since the students could select two viewpoints each, the theoretical totals across the rows are 200%. However, not all students selected two, so the totals are less than 200%.

Table 2 Percentage of students selecting items A to E by grade and problem solving group.

		A correctness	B effectiveness	C generality	D expression	E structure
Grade 4	Good	5.9%	11.8%	35.3%	70.6%	70.6 %
	Poor	23.2%	12.2%	42.7%	64.6%	43.9%
Grade 5	Good	17.1%	0%	48.6%	62.9%	68.6%
	Poor	26.8%	9.8%	51.2%	61.0%	43.9%
Grade 6	Good	16.7%	4.8%	52.4%	52.4%	66.7%
	Poor	23.2%	14.0%	58.1%	55.8%	37.2%

The significant difference between good and poor students' selection appears on item E in every grade. This item refers that whether they identify as most important the aspect of structure or pattern to help for generalization. Poor students select this aspect less than good students.

This table also shows that most of good students in each grade commonly identified item D (expression) and item E (structure) as an advantage for Akira's method, while poor students selected commonly item D (expression) more than E (structure). They only paid attention to the surface solution, identifying only that Akira has written a mathematical expression. They did not identify that the expression was linked to the structure.

4. Discussion

(1) The effect of presenting other's method on poor solvers' solving a generalization problem

The hint used in this study assisted poor students to solve a generalization problem. Specially, most of Grade 5 and 6 students succeeded. Many of the poor students who could not solve a far generalization problem without a hint, were able to solve the far generalization problem in Problem C. As they can learn how to solve Akira's problem (n=30) from Akira's method, they understand the way a pattern and its mathematical expression can be used more generally, but they need to learn to find the pattern and expression for themselves. Kimura (2001) pointed out that offering the hint of a figure that presented the structure and a mathematical expression based on this structure in a small term situation assisted poor Grade 5 and 6 students to solve a generalization problem. The result of this study supported Kimura (2001) and extended it.

This result suggested that one difficulty of poor students in solving a far generalization problem was the lack of a way to approach a generalization problem. If they learn Akira's method and have a skill to solve it, their performance will improve. However, the hint was not helpful for some poor students. More research is needed to identify which factor prevented them from solving the far generalization problem.

(2) Poor students could not understand the usefulness of structure in solving generalization problems. Error analysis showed that poor students could not understand the relationships of quantity in small term problems. The result of evaluation of Akira's method also demonstrated that there was a significant difference between good and poor students in selecting a goodness of "structure" from Grade 4 to 6.This may suggest that students who have difficulty for solving generalization problems have difficulty in recognizing the mathematical structure in figures.

It is very interesting that the difference of viewpoint for evaluation between good students and poor students appears to be in noticing a simple structure. Good students learn the importance of finding a pattern through their regular mathematics lesson, but this is not true for all students. Teachers could use this information to be more explicit about developing pattern finding ability through mathematical lessons. Even students from poor group 1 need to develop their ability to notice the structure of a quantitative relationship.

As noted above, the mathematical structure of Problem C (a quadratic relationship) is different from the linear structure of problems A and B. The structure of problem C is more complex than other two problems, so a further study could investigate the effect of the hint on problems similar to problems A and B.

Most of good students from Grades 4 to 6 selected item E (structure). However, the percentage of good students selecting item C (generality) by good students increased from Grades 4 to 6. This result suggests that understanding of structure is different through the grades and that as they grow older, students come to understand the role of a mathematical expression from both aspects of structure and of generality.

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Author's Note

I wish to thank Professor Kaye Stacey of the University of Melbourne for her assistance with English language.

INVESTIGATING COGNITIVE AND COMMUNICATIVE PROCESSES THROUGH CHILDREN'S HANDLING OF SOLIDS

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The article presents the results of an on-going longitudinal research project which was begun in 1993. Initial goals for the research are defined and suitable research tools were determined. Four tasks based on the constructivist approach to teaching were chosen as the tools and these were given to young children, then student-teachers in the Czech Republic and finally to children in the UK. Analysis of pupils solutions of the tasks allowed us to identify and define cognitive processes on a theoretical level and apply these to particular pupils' solutions. The importance of the communicative processes emerged whilst analysing the early experimental work and the goals and tasks modified to take account of this.

INTRODUCTION

This longitudinal research project began in 1993. It was designed to undertaken by a relatively small number of pupils whose work would be subject to a deep qualitative analysis. The first goal was to investigate the contribution of tactile and visual perception on the building of geometrical images. The second goal was to identify cognitive phenomena which occur in school communication when pupils work with solids then characterise and classify them. Early experiments showed that communication was essential in identifying the cognitive processes and a third goal was added namely the investigation of communicative phenomena (Jirotková, Swoboda, 2001).

The goals relate to the writings of J. A. Comenius (Analytical Didactics, Principle 55) 'the greater the number of senses which are involved in putting an image into the mind, the more familiar we are with the image and the more likely we are to retain it.' This means that the pupils who only use visual perception to learn about solids have a lower quality of understanding of the solids than those who employ tactile perception and even manipulate them (building, cutting etc.). Although this is generally known we find that in the classroom insufficient attention is paid to this fact by teachers. The didactics of mathematics should contribute to the changes of this situation by investigating the mechanisms of particular perception processes. Hence the goals which were chosen for this research.

TOOLS AND METHODOLOGY

To help us to determine the tools for our research we set ourselves a series of questions to set the parameters for this work.

· Are there pupils who prefer learning about solids through a tactile perception rather than a visual one and does the reverse apply?

· Is it possible to gain evidence of the extent to which a pupil's memory can store images of solids which were perceived by tactilely or visually?
· Is it possible to set up a situation from which we can gain evidence that pupils perceive certain spatial phenomenon better tactilely rather than visually?
· Can a situation be set up which gives evidence that a pupil has a precise understanding of certain geometrical phenomena but is unable to express this verbally?
· Is it possible to determine the preciseness of the pupil's understanding of certain geometrical phenomena?

From the consideration of these questions and the researchers' experience, the following four tasks were devised which fulfilled the goals which were set. The latest modification of the tasks done in 2001 are presented here.

In the four tasks a total of 16 solids were used in varying combinations. The solids used were: 1 Cube; 2 Square prism; 3 Large rectangular prism; 4 Small rectangular prism; 5 Right-angled triangular prism; 6 Right-angled isosceles triangular prism; 7 Non-convex pentagonal prism; 8 Hexagonal prism; 9 Tetrahedron; 10 Square-based pyramid; 11 Truncated rectangular-based pyramid (x2); 12 Non-convex pentagonal based pyramid; 13 Sphere; 14 Cylinder; 15 Cone; 16 Truncated cone.

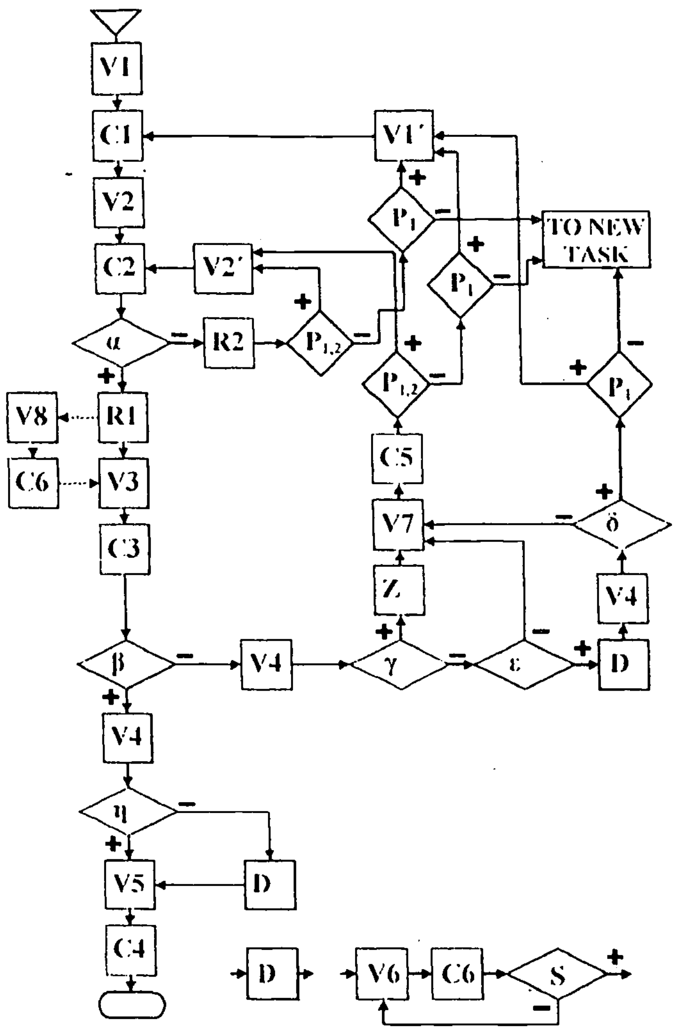
Task 1. Identification

Materials: Two cloth bags A, and B, bag A contains solids 1, 2, 5, 6, 9-16. Bag B contains solid 11.
Instruction: Dip one hand into bag A and feel the solid. Now take your hand out of bag A and dip the same hand into bag B. Try to find the same solid. Tell me if you can find one. Before you take it out give reason(s) why you think it is the same and what is interesting about the solid.
Presentation: Experimenter shows the bags, explains the tasks in language suitable for the age of the pupil, checks to make sure that the pupil understands what s/he has to do. If the pupil does not understand properly, the experimenter will rephrase the task.
Scenario: Individual pupils given task. There was no time limit given for the solution.

It took several experiments before we were able to standardised Task 1 and to represent it in the form of a flowchart. The development of this chart illustrates that although the task appears simple, as soon as you start to analyse the possible scenarios its complex nature is clearly seen. The chart is in an evolutionary state and the latest modification was made in 2001 when the communicative element was added, indicated by the dotted lines.

Legend

Experimenter boxes
a) Instructions
V1 Dip one hand into bag B and feel the solid.
V1' Dip one hand again into bag B and feel the solid.



V2 Dip the same hand into bag A and find the same solid.
V2' Dip the same hand again into bag A and find the same solid.
V3 Take the solid out of bag A and check visually if your choice was right.
V4 Do you think your solution was correct?
V5 Take the solid out of bag B and compare both solids.
V6 Give reasons for your uncertainty about your choice.
V7 Put the solid back into bag A.
V8 Give reasons for your choice and say what is interesting about the solid.
b) *Others*
Z Experimenter doubting asks 'Are you really sure?'
β Experimenter decides if the solution is correct or not.
ε Experimenter decides if discussion needed to make pupil aware of mistake and to learn the reason for the mistake (+) or if a second attempt is necessary (-).
S Experimenter decides if the reasoning was sufficient or not.

<u>*Pupil's boxes*</u>
a) Activities
C1 Pupil perceives solid and makes mental evidence of the solid.
C2 Pupil is searching for the solid and compares the previous and current tactile perception.
C3 Pupil compares current visual perception with the previous tactile one.
C4 Pupil compares current visual perceptions of both solids and confirms the choice was correct.
C5 Pupil replaces the solid in bag A.
C6 Pupil gives reason for the choice of solid.
R1 Pupil stated that the solid was found.
R2 Pupil stated the solid was not in bag A.
b) Decision boxes
α Pupil decides if the solid from B is(+), is not (-) present in bag A.

γ, δ Pupil decides that the choice was correct (+) or incorrect (-).

η Pupil is (+) is not (-) sure about his correct choice.

<u>*Administrative boxes*</u>

P_1 (abacus) After first entry, exit-path is (+), after second entry exit-path is (-).
$P_{1,2}$ (abacus) After first/second entry, exit-path is (+), after third entry exit-path is (-).
The explanation of the transfer to another task will be given orally.

Task 2 - Selection

Materials: One bag containing solids 1, 2, 6, 9, 10, 11, 16.
Instruction: Put one hand in the bag and from the solids which are there choose one which you think is different from the others. Before you take the solid you have chosen from the bag, tell me why you think it is different.
Presentation: Experimenter shows the bag, explains the task in language suitable for the age of the pupil, checks to make sure that the pupil understands what s/he has to do. If the pupil does not understand properly, the experimenter will rephrase the task until understanding is satisfactory.
Scenario: Individual pupils attempting task. No time limit given.

Task 3 - Classification

Materials: One bag containing solids 1, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15.
Instruction: Both hands may be put in the bag. Arrange the solids into two groups, one group having common attribute(s) which you choose, the solids in the other group will not have those attribute(s). Before you take your two groups of solids out of the bag say what are the common attribute(s) or how they differ. Take out your groups and say whether you are happy with your classification. If not change them and say why you have made the changes.
Presentation & Scenario: As for Task 2.

Task 4 - The game OWL

Materials: The following solids were placed on a table : 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.
Rules: One player to choose a solid mentally. The other player is to discover the chosen solid by asking questions which can only be answered by either YES or NO.
Presentation: Experimenter explains the rules of the game; makes sure everything is clearly understood. If necessary the experimenter amends the rules during the course of the game, to allow only questions that refer to geometrical attributes of the solids.
Scenario: The players may both be pupils or a pupil could play with the experimenter. No time limit is set. The experimenter avoids influencing the game.

RESEARCH SAMPLE

In 1993 ten pupils with ages in the range 7 years to 10 years, were observed. They undertook all the tasks including multiple games of OWL. All verbal communications were tape-recorded and then transcribed in the form of protocols. An observer was always present to make notes of non-verbal actions and communications. In some cases photographic evidence was taken. The protocols were analysed. We found the tasks, especially the game OWL appropriate for use with University student-teachers and it is now an integral part of the syllabus. From this work we found that the majority of information regarding cognitive processes can be obtained from the analysis of communication of the pupils/students. Hence the tasks were slightly modified to challenge the pupils to verbalise their perceptions as much as possible. This form of the tasks was given to two nine-year old pupils in England in 2001. The analysis of this latest work and our research experience enabled us to construct the mental mechanisms of some cognitive processes which were involved in solving the tasks. These were the *mechanism of tactile selection* - how the process of choice of a particular solid is made by tactile perception only; the *mechanism of tactile-visual verification of the selection* - how the process of verification their tactile choice is made visually and tactilely; the *mechanism of tactile classification* - how the process of classification of the solids into two separate groups is made without visual perception; and the *mechanism of tactile-visual verification of the classification* - how the process of tactile classification is verified using visual-tactile perceptions. These theoretical descriptions of the mechanisms were applied to the actual solutions of the tasks undertaken by the pupils. In the next paragraph we will describe the mechanism of tactile selection and apply it to the solution of the first task by two pupils. For the description of these mechanisms we used the methodology of analysis elaborated by M. Hejný (Hejný, Michalcová, 2001).

MECHANISM OF TACTILE SELECTION

The process of selection is found in the decision box a in the diagram above. In our research we identified three types of the process and these differ according to the extent the pupil is familiar with the perceived solid.

1. The perceived solid is entirely new for the pupil and s/he does not have any experience of committing tactile images to the short-term spatial memory, that is the pupil has not been challenged to verbalise tactile perceptions. When the pupil is challenged to describe verbally the spatial phenomena perceived tactilely, s/he must perceive the solid analytically that is s/he needs to perceive vertices, edges, shape of face etc. In this case the pupil perceives the solid as a whole, that is s/he perceives its *gestalt* and not its individual analytical characteristics (Van Hiele, 1986). On the pupil's first attempt to find the identical solid in bag A s/he does not know which solids are hidden in the bag, therefore s/he does not know on which attribute to focus his/her attention. The pupil will try to sort the solid from bag A by using his/her global perception first and if not successful attempts to recall and use at least one analytical characteristic of the perceived solid in bag B.

2. The perceived solid is entirely new for the pupil but the pupil has some experience of committing tactile perceptions to the short-term memory. In this case the pupil tries to commit as many dominant characteristics of the solid to his/her short-term spatial memory as possible. When the pupil is then sorting the identical solid from bag A s/he attempts to find the characteristics, which s/he had put to the short-term memory, on each of the shapes in turn.

3. The pupil is familiar with the perceived solid and it is correctly committed to his long-term spatial memory. When the pupil is sorting the identical solid from bag A, the pupil is led by the good image committed to his/her memory.

The above three types of mechanism of tactile selection are described theoretically. The boundaries between them are not well defined and the reality is often on the overlap of them. The images from the long and short-term memory are mutually penetrating. We gain information about how the short-term memory works either from non-verbal expressions (gestures) or from verbal expressions (pupils generate new expressions eg. 'squangle' for square).

APPLICATION OF MECHANISM OF TACTILE SELECTION

The first pupil we wish to discuss is Jan, a Czech boy aged 10 who volunteered to take part in the research. We worked with him in November 1993. Jan had to find the truncated pyramid in bag A but his first choice was solid 2, the square-based prism. He was not certain about this choice and when he took it out he asked 'Is it right?' Then he perceived the solid in bag B again and he realised that his first choice was incorrect. Nevertheless on the second attempt he again made an incorrect choice by choosing the cube. Jan concentrated a lot during his third attempt and his choice was correct. We explain his mistakes by applying the mechanism of tactile selection. We believe that Jan was working according to the first type of mechanism as defined above. When he perceived the solids in bag B only a global perception was committed to the short-term spatial memory. When he tried to find the correct solid he did not find any global characteristic to help him select the solid. He then tried to recall some analytical characteristic of the solid and he found one characteristic which we could call 'four-sidedness'. This idea is supported by two arguments. There are only three solids in bag B which could be characterised by 'four-sidedness'. Jan chose these one-by-one. The second argument is that when he solved the other tasks he expressed that 'being quadrilateral' is, according to him, the dominant characteristic of a solid. For his second choice he was led by the perception of four right angles. This characteristic was perceived when he felt the solid in bag B again. Putting his hand into bag A again the first solid he met which had this characteristic was the cube which he chose without any further reference to any other shapes. Visual perception on taking the cube out of the bag gave him quick information about the metric characteristics of the cube. His third attempt was successful. It should be noted that our second mechanism listed above - mechanism of tactile-visual verification of selection - was also used in this process.

The next example is a good illustration of a pupil perceiving the solid as a gestalt and not its separate properties on meeting it for the first time. Jill was a nine year old English girl when we worked her on the tasks in September 2001. Her teacher chose her for the research. Jill showed unusual mathematical culture and perfect insight into the geometrical world when solving all the tasks. She also had a remarkable mathematical vocabulary, which she used with understanding. When she found the identical solid in bag B, (task 1) before she took it out she was challenged to say why she thought she had made the correct choice. The following is an fragment of the transcript of the subsequent dialogue with her. J4 represents the fourth comment by Jill and E5 the fifth by the experimenter.

J4: Because it feels like the first one.
E5: So what do you think was special about it?
J5: *It was small, it has the same amount of sides.* (She meant faces.)
E6: *How* many?
J6: (A short pause whilst she checked the held solid.) *Six.*
E7: *So you think it has six sides, do you think you have seen a shape like this one before?*
J7: *No.*

Both used words 'feel' and 'small' indicate that Jill perceived the solid as a gestalt. The second part of J5 was surprising because it showed that she tried to give an analytical description of the solid and the use of the incorrect terminology, 'sides' instead of 'faces', which was an unusual occurrence in all her communication, was evidence of the demanding nature of the transfer from global to analytical descriptions (Pegg, Baker, 1999). The drain of energy, which this transfer caused, was the reason she chose the incorrect word for 'face'. J5 shows unusually sophisticated mathematical thinking for a nine-year old girl. She was aware that the global characteristic of a solid was not the perfect answer to the task, hence she felt the need to define the solid by analytical characteristics. We did not meet any similar case of such thinking in our research. This is a good example of the second type of mechanism of tactile selection.

The analysis of our research did not produce any interesting results related to the third type of tactile selection mechanism.

Jill's use of correct mathematical language and her knowledge of the characteristics of both solid and plane shapes can be seen in the following examples. When she was asked by the experimenter at the end of Task 1 what solid did she think would be created if the faces of the truncated pyramid were extended to a point, she immediately replied 'a square-based pyramid'. In the third task she was asked to say what was different about the solids that she had chosen. Her response was 'They all have at least one face which is a quadrilateral in 2D'. The experimenter made sure that Jill fully understood what she was saying by both 'at least' and 'quadrilateral'.

CONCLUSION

We believe that little research has been carried out and written up this area. Our research has produced tools which we consider are beneficial for investigating cognitive and communicative processes. They could be used to advantage in the normal classroom in two ways, the first being that the methodology of the tasks is very much in the format of constructivist teaching strategies (Noddings, 1990) and so would help the children to learn about solids in a concrete, meaningful and understandable way. The second is that all the tasks, develop the pupil's ability to communicate mathematically which we find is lacking in school text-books. These books get the pupils to construct shapes or solve riders with no opportunity to explain or discuss their work and their understanding of geometrical concepts verbally. In particular we found that the game OWL, which can be modified for all age groups, not only helps the communicative ability of the pupils and develops their mathematical vocabulary it can also be used to quickly diagnose deficiencies in these areas (Jirotková, 2001).

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The research was supported by grant GAČR 406/02/0829.

AN INVESTIGATION INTO MY PRACTICE OF TRAINING TEACHERS IN USE OF TEACHING AIDS IN MATHEMATICS TEACHING AND WAYS TO IMPROVE IT

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Abstract

In this paper I report the procedure I followed to investigate my practice of training teachers in use of cultural objects as teaching aids in mathematics teaching and also ways I ought to improve this teaching. Two cohorts of student teachers and two practising teachers were involved. So far two cycles of action research have been completed and the results indicate that my practice is and has been open to improvement.

Introduction

One of my former students told me: "I have tried to do those things (referring to teaching mathematics using aids) but it has proved to be lengthy." "When I finish my present research," I promised, " I am going to come out to schools and work with the teachers. I want to get a class and teach it to show that these ideas can be put into practice and can work".

The issue of teachers' non-use of teaching aids has bothered me over the past ten years. I got more and more concerned each year as I trained teachers in their use and still found no evidence of their use in schools. This was despite the improvements in my training that I tried to make each year.

I have a strong desire to see that all Ugandan children successfully learn mathematics. It is a core subject at primary and lower secondary levels and the grading of certificates at these levels is based upon success in it. I believe that if teaching aids were used adequately by all teachers, students would love and learn mathematics, which would empower them as citizens (D'Ambrosio, 1990). Mwanamoiza (1992) found that students in Ugandan classrooms are treated as passive receivers of knowledge. Many simply cram formulae and methods of solving specific problems in mathematics. The use of aids may encourage other approaches.

The Problem

Despite the training in use of teaching aids in mathematics teaching that Ugandan teachers get in pre-service programs, they hardly use them. This made me question the training they get. Was it adequate? Was it appropriate? If not, how could I improve on the training I was giving them?

How would I train teachers so that they could adequately use teaching aids in mathematics teaching?

Theoretical framework

A major theoretical framework underlying this study is constructivism. The view that knowledge is actively constructed and not passively received (Noddings, 1990). The belief that all knowledge is a product of our own cognitive acts (Confrey, 1990), implying that knowledge is constructed in the human mind and that understanding is constructed through experiences.

The learners have a knowledge base /structure on which they compare incoming knowledge. Then either the new knowledge is modified for taking in, or the knowledge base is modified to allow it to fit in. This is to say that the teachers come to training with some knowledge and beliefs about teaching and use of aids (Ball (1990) to which they compare new knowledge. A series of experiences facilitate this knowledge acquisition and the learner ought to be provided with such an environment. (Piaget, 1969 cited by Von Glasers 1990; Noddings 1990; Confrey, 1990). Freedom to express oneself must be part and parcel of this environment (Vygotsky, 1978). In the training that I give teachers I try to give practical, hands on experiences. The knowledge structure is continuously revised using active knowing (Noddings, 1990). In the intervention phase of this study, students were put in groups, in which they worked and were able to talk freely. Vygotsky (1978) and Bruner (1985) explain that some kind of mulling over takes place in the learner's mind as he/she interacts in the environment. In the intervention phase I attempted to involve students in reflection by requesting them to keep diaries.

Thus the teacher should endeavor to ascertain what knowledge base the learner has and then plan to offer the right environment. A questionnaire and interviews after sessions served this purpose in my study. The teacher's prompting, questioning etc. which Vygotsky calls scaffolding, is integral to this kind of environment. It should be an environment that is social, that is full of sharing ideas, challenging, questioning and explaining and negotiations of meanings (Jaworski, 1994) between teacher and learners and between learners and learners (the peers according to Vygotsky (1978)).

Sometimes such an environment can be created through bringing experts to the classroom (Davidson & Miller (1998), Lave (1988), Lave and Wenger(1991)). Such an approach would be an eclectic one and in my view profitable to learners. This is what justifies my bringing in of an experienced teacher to teach with student teachers acting as students in cycle 2. Jawoski

(1994) refers to this as socio – constructivist nature of learning. Of course, the teacher knows more than the learners do. So by planning in a purposeful way for them the teacher may be able to offer situations that are profitable to learners. The experiences could be through observing and questioning experts who might be other (more capable) students or the teacher.

In my study I used cultural objects as a sample of teaching aids. I claim like D' Ambrasio, (1990) ;Gerdes, (1998) ; Zaslavsky (1994) and others that there exists mathematics in every culture which can be explored through artefacts. While D'Ambrasio refers to it as ethnomathematics or mathematics in specific cultures, I wish to claim that ethnomathematics is mathematics, no different from European mathematics. I agree with Gerdes (1994) that mathematics is preserved in cultural objects. It is ingrained in the cultural activities. Thus aspects of culture can be used to teach mathematics. Using cultural objects in this way allows the learner to work on mathematics as both an unintentional and natural process (Lave 1988) and also through reflective abstraction from purposefully arranged activities (Noddings, 1990).

Vygotsky's theory of social development through the idea of Zone of proximal Development (ZPD) also suggests an area where culture can be used for learning. Teachers or peers tutor the learner until he/she gains mastery of his /her own actions through activities which allow the learner to make a leap. The context could be the use of cultural objects. The use of such objects allows mathematics to take place within. Cultural settings and activities that go on daily and in which learners participate naturally as members of the communities to which they belong (Lave, 1988).

Methodology

The objectives of the study were to:

(i) Document, (ii) observe and critically analyze, (iii) innovatively explore ways of improving my current practice of training teachers in use of teaching aids particularly cultural objects. I chose them because they are in every home and therefore familiar to teachers and pupils. Being in homes they are affordable and they also offer pupils a good opportunity to explore mathematics at home.

Subjects

I used two categories of teachers; (i) 3rd year mathematics' student teachers of academic years 2000/2001 and 2001/2002; (ii) two practising mathematics teachers. The study was organized in two action cycles. Cycle 1 was a pilot phase and involved student teachers of academic year 2000/2001 and two teachers while cycle two was an intervention phase and involved student teachers of academic year 2001/2002 and one experienced teacher from cycle 1.

Instruments

I used video recordings, classroom observations, interviews (audiotaped) and questionnaires. An action research approach was used and it enabled me to (i)document my current practice, (ii) observe it, (iii)discover problem areas through critical analysis alone and with friends, (iv)imagine a solution, (v) try it out and (vi)evaluate my actions. Two action cycles were possible in the given time. Documentation of remembered action in my usual practice enabled me in cycle 1, to act out systematically and purposefully what I usually do as routine. I taught both sets of teachers on the use of cultural objects as teaching aids in mathematics while being recorded on video. After each session with the pre-service teachers I interviewed five of them asking them what new ideas they had got out of the session concerning cultural objects as a resource for teaching mathematics.

With the two practicing teachers I did a two-hour session. Two weeks later I had another two-hour session with them in which we discussed the activities they had prepared exhibiting the knowledge they had obtained in the first session. Two weeks later they used these to teach while I observed the lessons and taped them on video. Throughout, these teachers were free to consult me at any time for clarification on the things we were doing.

Work with these teachers gave me a comparison between in-service and pre-service training.

Eight months later I observed four student teachers teach during school practice. I interviewed each of them concerning what they had learnt from my sessions with them vis a vis their practice in class. I also asked them how other students viewed my sessions and whether they could they suggest some improvements.

Analysis of cycle 1

I watched the videotapes repeatedly noting down my actions and trying to analyze why I did what I did. On separate occasions I watched excerpts of the tapes with five critical others (three individually and two together) and taped their comments which were sometimes prompted by my questions. I transcribed the comments and read through them comparing comments of different viewers with my own observations. I compared the comments from different sources, (the critical others, the student teachers, the teachers, and my own comments). I triangulated these and came up with some discoveries about my practice, which I termed shortcomings.

This work was supported by my reviews of the literature on culture, teachers, teacher educators and learning theories. These reviews also informed my actions in cycle 2.

Thus, from: (i) observations on video tapes and in the classrooms, (ii) interviews with 10 student teachers and the two teachers, (iii) interviews with critical friends, and (iv) critical literature review, I was able to critically analyze my current practice, establish what it is and identify areas of strengths and weaknesses. I worked out a possible course of action that might improve my practice and tried the ideas out in cycle 2.

Findings from cycle 1

The following were the findings:

(i) I talked a lot; i. e. I used the telling mode a lot, although I believe in learners constructing knowledge. The planned activities for the learners to do in the sessions were rather superficial. The time I allowed for each activity was short. (ii) Throughout my presentations I was worried about time and kept mentioning time being short. (iii) Although it was felt that I was clear in my exposition and presentation, the student teachers' hearts did not appear to have been touched. (iv) None of the student teachers were observed using cultural objects as teaching aids in their teaching. v) Although the student teachers would not agree, they did not seem understand what cultural objects are and how to use them. Neither did they appear to conceptualize the possible benefits of using them. (vi) The literature highlights how prospective teachers come into training with their own views about mathematics, its teaching and learning. In the student teachers could be observed (even in their own words) a strong belief that cultural objects are not easy to obtain and that if teaching aids were used the syllabus would not be covered.

Conclusions from cycle 1

Teachers may have knowledge of teaching aids and their use and yet not know how to use them in their classroom teaching. The knowledge they have may only be superficial. That is, it may be of a matter of demonstrating and pointing out to pupils what concept they (the teachers) can see in a teaching aid. They, however, might not know how to engage pupils in related mathematics learning. For example they may not be able to prepare activities that might engage pupils in constructive mathematics learning based on such objects. One possible explanation might be that the teachers are never convinced of the approaches to teaching that they get during training. Since prospective teachers come to training with their own beliefs about mathematics; those that they form over the years of mathematics learning, the limited time of training, within which they get the exposure to new ideas may not be enough to change their views. The reality of conditions in schools does not appear to help either. Teachers' attitudes are that it is impossible to find time to teach children practically using teaching aids and still manage to complete the prescribed syllabus to enable children to pass examinations.

Lesson observations showed that the in-service teacher taught with more expertise and confidence while using teaching aids than those trained in pre-service. This may be due to amount of time on training and/or the experience working in the classroom which might lead to more readiness. It suggests that training teachers at in-service level may be the answer to my question: "How can I train teachers in use of teaching aids?"

Cycle Two –Intervention

Basing on the findings from pilot (cycle 1) and literature review, I carried out the following planned activities for cycle 2 as an intervention measure.

(i) To find out the views that teacher trainees came with to my sessions I administered a questionnaire at the beginning of the first session. In it I sought for their views about mathematics, mathematics learning and teaching. I also asked them to evaluate two of the lessons they had taught during the completed school practice; one they judged as their best and another they judged as their worst. Then, in groups, I asked them to discuss one successful lesson and one failure. I provided them with guidelines to follow as they did this. (ii) To create more time for the course, I gave each student teacher a file with all the materials that we would need during the six weekly one- hour sessions that we were to have. I also gave them notebooks in which to write their thoughts, reflections on the sessions and their own learning which was meant to encourage them to spend more time thinking about the work done during sessions. In order to encourage students to find time outside sessions and to impress upon them the importance of groupwork, I asked them to form groups of five to six in which they would work throughout the course. In these groups they were to discuss and complete unfinished work started during sessions. This arrangement also helped make students talk more while I talked less.

(iii) To practically demonstrate use of teaching aids in a mathematics lesson, I asked an experienced teacher from cycle 1 to come and demonstrate to them. (iv) To monitor the progress of their learning I interviewed three to five students after each session. In each interview I asked each of them what new thing they had learnt concerning mathematics teaching and learning, use of teaching aids, lesson organization, and pupil involvement. I also asked them to say something about teaching aids from culture and those from the environment. (v) To keep record of all information, I taped on video all sessions, and all lessons I observed, I taped on audio tapes all interviews with students and teachers, I kept a diary of all my actions and my thoughts on what I was doing. I requested that students hand in all the assignments and the work done during each session including the diaries and files containing all they had done. After the last session I asked them to fill in a course evaluation questionnaire.

Results from cycle 2

From preliminary analysis of my diary, students' documents (diaries, files, assignments) and interviews, my own session observations I found out that:

(i) The majority of students came to my sessions not convinced that teaching aids are available and can be used. After the sessions they appeared to be convinced that there are many materials that can be used especially from the environment reflecting their culture. Students now said that the teacher only needs to be innovative and willing to look for such objects.

(ii) Students got an idea of what teaching aids are and their possible use. They also understood that cultural objects can be used to teach mathematics and were certain they could use them.

(iii) Teachers could not give a clear distinction between objects from any environment and those, which are cultural. This may be a false distinction or one, which needs more time for assimilation.

Conclusions

The results of cycle 2 still have to be further investigated. However, the changes in my practice appear to have offered the pre-service teachers the opportunity for constructive learning through active group work, and mulling time (Vygotsky (1978), Bruner (1985)). My instructions offered a form of scaffolding. The experienced teacher demonstrated that teaching aids could be used. My expectations of the student teachers' out of session work were this time explicit. All this I would claim has improved students understanding of the ideas I tried to pass on.

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LEARNING OPPORTUNITIES IN A KINDERGARTEN ABOUT THE CONCEPT OF PROBABILITY

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Abstract

In this paper we describe the students' mathematical learning in a kindergarten during a classroom teaching experiment about the concept of probability. We present and analyze the learning opportunities that were created in the classroom as the children tried to resolve their problems, to reason mathematically and to communicate their thinking to others. The results of the research showed that kindergarten children made considerable progress in their probabilistic thinking, when they accepted the process of the experiment to check their different predictions as well as when they arrived at a consensus about the solution of a problem.

As a consequence of the constructivist epistemology, it is nowadays acceptable that the learning of school mathematics is a process in which students reorganize their mathematical activity to resolve situations that they find personally problematic (Cobb et al., 1991). Learning opportunities can arise for students from their personal engagement with the mathematical activities as well as from their interaction with the other members of the classroom. The teaching of school mathematics can be characterized as a process in which the students and the teacher negotiate their mathematical meanings and interactively constitute the truths about a "taken- as- shared mathematical reality" (Cobb et al., 1992).

This acceptable view has oriented research towards the construction of models that specify the development of children's thinking on concrete topics in mathematics as well as to the investigation of the role that interaction among the members of the classroom can play in cognitive development. However, many questions have to be answered for the organization of mathematical education on subject matter specific knowledge and many researchers suggest that empirical data from real classroom settings are necessary.

Towards this effort, the purpose of this paper is to explore and analyze the learning opportunities that occurred in a kindergarten during a classroom teaching experiment concerning the development of probabilistic thinking. The concept of probability is considered as one of the most difficult in mathematics education, as many researches have shown (Shaughnessy, 1992-Kapadia & Borovcnik, 1991). The research reported in this paper is a part of a broader program that try to: a) investigate the range of the abilities that children can develop in the kindergarten on stochastic concepts, b) analyze the learning opportunities that are created in the classroom taking into consideration the cognitive and social processes involved, c) scrutinize the contexts of the activities that can be fruitful for students.

In the research program, we were based on current research literature about the development of student's thinking in probability (Jones et al., 1997- Jones et al., 1999) and these findings influenced the organization of the instructional activities given to the pupils.

Methodology

Fifteen kindergarten children were participated in the classroom teaching experiment in a typical public school of Athens. The research project was realized in November 2001 and it lasted one month. The children were engaged with mathematical activities related to probability three times per week. Every lesson was lasted half an hour. The program was realized in collaboration with an experienced and well-informed kindergarten teacher. In the course of the program we developed a set of instructional activities on the concept of probability concerning the following themes: sample space, probability of an event, probability comparisons and conditional probability. The instructional activities used in the classroom were developed before and during the progress of the program, according to the evolution of the students' ideas, so that they could be problematic to the students. Moreover, all the activities were related with the interests and the experiences that children have at this age. More specifically, the activities were presented through games, small stories in puppet show and dramatic metaphors. A typical lesson could be described in the following way: The teacher introduced the instructional activity to the whole class, the children made their predictions about the concrete problem by explaining their thinking and the teacher recorded the different ideas of the students. Then, the children checked their predictions through the realization of an experiment, they recorded the results and they discussed about the solution of the problem.

The children had not received any previous instruction in probability. Moreover, they had not yet received any instruction in whole numbers. Each child was interviewed for one hour prior to the instructional program, at the end of the program and one month later. The interview included 14 tasks. The children were asked: a) to report the outcomes for a probability situation (for example, what could be happened by throwing a die with different colors), b) to predict the most likely outcome in a random experiment (for example, to predict which color is "the easiest" to be drawn from a box with colored balls), c) to chose the probability situation for the most likely realization of an event and d) to report the outcomes of an experiment, since they had realized one trial. All the lessons were videotaped. The transcripts of the interviews and the lessons provided the data for the analysis of the students' learning. Analytical descriptive narrative was the method used for our analysis (Erickson, 1986).

Results

Prior to the instructional program, all the children participated in the teaching experiment seemed to interpret the tasks in probability in a subjective manner. However, we could identify some qualitative differences in their thinking. More specifically, the students' responses could be classified in two general categories.

On the first category, the children gave answers that they were strongly influenced from their favorite color for all the tasks (4 children). Ada was a representative child for this category. She answered the red color as the only one that could be appeared in all the tasks. So, she reported only one outcome in a one or two stage experiment (e.g. she gave only one ordered pair in a two-stage experiment including the red color) and she insisted that the red color was "the easiest" to come out in all the experiments. Since she had realized one trial in an experiment (with or without replacement) and she was asked to describe the possible outcomes on the next trial, she repeated the color that she has got on the first trial.

On the second category, the children could give all the outcomes for a one - stage experiment. However, they gave only one ordered pair in a two-stage experiment. These children considered that they could not give another combination, because they had only three colors in the bag and so «it is only one that it is left over». That means that it was very difficult for them to understand the context of the task, as they considered that one trial of the experiment was identical with the solution of the problem. They could not imagine that for every trial all the balls could be again inside the bag. Furthermore, these children did not have a consistent way of thinking to answer the tasks for the probability of an event, sometimes they gave a right answer and sometimes they gave a wrong answer. However, in all cases it was very difficult for them to provide an explanation for their answer. In the case that they presented their arguments, they seemed to be influenced by the position of the materials or their favorite color. In the comparison probability tasks, they gave usually the right answer for the probability situation related to spinners and a wrong answer for the one related to boxes with balls. Since they had realized a trial in an experiment, they did not usually mention the color that they picked, but they gave all the other outcomes.

The following episodes were selected for analysis as they represented critical moments in the students' learning. In these incidents that took place in the classroom, learning opportunities were created as the children tried to resolve their personal problems or to communicate their thinking to others and to justify their position/solution to a problem.

First episode

The children were engaged in the following activity (This was the second activity about the probability of an event): The grand mother and the grand father of the squirrel would like to make a scarf for him. The grand mother found a bag with balls of wool with different colors (1 blue and 3 green). As the grand mother and the grand father were in complete disagreement about the color of the scarf, they decided to pick a ball at random. The children were asked to predict the color that was "the easiest" to come out. The following dialogue took place between the children:

Anna: The green, because there are 3 green.
Teacher: Is there another opinion?
Socrates: The blue, because it is higher up in the bag.
Teacher: What do you want to say Paul?
Paul: I say the green ...we close our eyes and we pick a ball.
Teacher: Paul said that we close our eyes when we pick a ball. So, we cannot see which it is up. As we are picking a ball, we can mix the balls. *(He is doing the movement.)*
Socrates: It is the blue one, because the two green are below.
Anna: It is the green, because there are more. Let's do it.
Socrates: Yes!
Teacher: OK! Let's do it.

Anna, Paul and Socrates were three of the children whom their responses at the interview could be classified on the second category. However, in the progress of the program Anna's responses were based on quantitative judgments, she used the words «more» or «less». It was the first time that she used numbers to justify her answer. On the other hand, Socrates was still influenced by subjective judgments, when he tried to justify his answer. The level of Paul' s thinking about the probability of an event could be characterized as transitional (Jones et al., 1997), between subjective and naive quantitative thinking. In this episode, as he tried to justify his answer, he based on the process that they used to make the experiment, and he used an argument which could upset Socrates' argument. This argument seemed to be a fruitful contribution for the development of Socrates' thinking. Although Socrates insisted to his answer, when Anna told him that they could find the solution to their problem by executing the experiment, he willingly accepted her idea. This was the first time that the children legitimated a way to resolve their disagreement by respecting the results of the experiment. Until now, there were children like Socrates that they insisted to their solutions even though they knew the results of the experiment. After the realization of the experiment, Socrates changed his opinion. In this sense, this incident was functioned as a catalyst for the following lessons.

Second episode

In the following episode, the instructional activity was related with the different ordered pairs that the children could make in a two-stage experiment. The children had to construct Christmas cards with two flowers in order to help the Father Christmas to share them to the children together with his presents. The children had to pick one color from a box with three colors to paint the first flower and they had to put it back. Then they had to pick again one color for the second flower. After painting the flowers, they should discuss how many different cards they had made.

It was the turn of Thomas to pick the two colors. After picking the first color (it was the red one), he picked the second one. This was also red. Thomas told to the teacher with surprise that this could not be happened, because «we have to chose two different colors». The teacher tried to pose Thomas' thoughts as a topic of discussion with the rest of the class.

Teacher: What are you saying? Thomas said that this is not right.
Anna: Yes, we cannot have the same color.
Ada: But, this color came out.
Teacher: Ada said that we pick the colors at random. So, can we have the same color twice?

Thomas raised his shoulders puzzled. The teacher did not continue the discussion and the children went on with their work. When they painted the cards, the teacher asked them to find the cards that were the same. The children began to put together the same cards, that is, the cards with the same colors. So, they began to put together the cards with the red and the green color. On the next card, the first flower was green and the second was red.

Teacher: Where will you put this card?
Thomas: Here. *(Together with the other cards)*
Anna: No, this is different. It is firstly the green and then the red.
Socrates: This is wrong. They are not different. They are the same.
Thomas: It' s a little different!
Socrates: Yes, it's a little different.

Teacher: OK. There are not the same, however they are a little different, so can we put this card alone, not with the others?
Thomas: Yes, here. *(He is showing a place near to the other cards).*
Teacher: Can we make other different cards?
Anna: Yellow, red.
Teacher: OK. We have the red-yellow card and we can make the yellow-red.
Marina: Two green.
Teacher: Very nice.
Thomas: Two blue.
Teacher: Bravo, Thomas.

On the above episode, as the children tried to negotiate their different interpretations about the word « different» in the concrete context, they managed to resolve their conflict and to construct an acceptable characterization of the difference of the cards. In this manner they managed to find all the different ordered pairs in subsequent similar activities. Moreover, the argument that Ada presented, showed that she had accepted the notion of random and seemed to influence implicitly Thomas' thinking about the possibility to have the same color twice.

Third Episode

The third episode is presented in brief. It is connected with the probability of an event, when all the events are equally likely. The children had to pose their own problem to be solved about the probability of an event.

Paul: I want to give a notebook to Kostas.
Children: What color does Kostas like?
Paul: The red. (He puts in the box, 2 red, 2 green and 2 blue notebooks).
Teacher: Which color is the easiest to come?
Anna: The red.
Paul: No. It' s easy to come the red and the green and the blue.

Although the teacher had not discussed with the children until this episode probability situations with equally likely events, this problem arose from children's attempts to make their own problems in probability. In this way they provide to us the opportunity to discuss with them similar activities like fair and unfair games.

We should note that at the final interview all the children had done a real progress in their probabilistic thinking. The majority could report all the outcomes in a two-stage experiment and use quantitative arguments to justify their answers.

Conclusions

The results of the research showed that kindergarten children made considerable progress in their probabilistic thinking, when they accepted the process of the experiment to check their different predictions as well as when they arrived at a consensus about the solution of a problem. However, more evidence from different cultural kindergarten classroom settings is needed to investigate the critical incidents that influence children's probabilistic thinking at this age.

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TEACHERS' ROLE IN THE MANAGEMENT OF MATHEMATICAL KNOWLEDGE: AN ANALYSIS OF A PROBLEM SOLVING PROCESS

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The present study deals with a problem solving process in the early grades. Data analysis focuses on pupils' mathematical behaviour and the teachers' management of the mathematical knowledge during the problem solving procedure. The analysis reveals two factors that seem to play an important role in the devolution of the problem and consequently in the construction of the pupils' mathematical knowledge. The former is related with the fact that pupils control the correctness of their outcomes themselves and the second one with the fact that the teacher facilitates the pupils without reducing the mathematical meaning and the cognitive features of the task.

THEORETICAL ISSUES

Current research on teaching and learning mathematics, independently of the perspective taken, accepts the premise that pupils are not passive absorbers of information, but rather have an active part in the acquisition of knowledge. Furthermore, it emphasises the need for mathematics teaching to be much more than a study of ready-made mathematics, which is still so prevalent. Since learners always construct their own knowledge, the critical issue is the nature of the socially and culturally situated constructions (Cobb, 1994). Thus the teachers' task is to challenge pupils by introducing effective mathematical activities, and maintain a classroom culture that encourages and facilitates independent learning.

In this perspective, two are the basic issues, which are of importance for the research in mathematics education: the choice and the formation of the mathematical content and the organization of the mathematical activity within the classroom. In relation to the former, since the common consideration is that mathematical knowledge cannot be reduced to a stock of retrieval facts, but it is constructive and, consequently, is best demonstrated in situations where something new is generated (v. Glasersfeld, 1983), a considerable number of research focuses on the change of the mathematical content in order to teach mathematics as a subject to be created and explored (Cooney, 1988). This framework implies reforms in the classroom organization: a pupil to perform in a correct way on content is not enough but actually contradictory to the consideration of mathematics as a subject to be created and explored. Thus, children should be given opportunities not only to respond to, but also to pose questions, to phrase hypotheses, to explore their ideas, to check and convince others about their correctness and to re-consider their mathematical ideas and constructions (Bowers, Cobb & McClain, 1999, Jaworski, 1994).

Thus, a third issue emerges, which concerns the teacher's role in the management of the classroom as well of the mathematical knowledge. The teacher should adopt such an approach, which will allow children to develop their ideas and actions: to provide them with challenging tasks, to listen to them, to encourage them to express their thoughts, to organize the results of their work (Jaworski, 1994). At the same time, s/he avoids interventions such as providing hints, simplifying the task, giving a sing indicating the expected response. This kind of hints function as external indicators, they are often misinterpreted by the pupils, who adapt them to their existing system of knowledge as this requires less effort (Brousseau, 1977). This is possible to reduce the cognitive value of the task (Henningsen & Stein, 1997) and of the mathematical meaning of concepts and relations (Steinbring, 2001).

For teachers, the shift from familiar instructional practice to a reformed approach is not easily accomplished (Fennema & Nelson, 1997). On the one hand, this is due to the cognitive demands of challenging tasks required for the organization of the new didactical environment. And the difficulties teachers have

> "to determine the preparedness of particular students for investigatory work, and the type of support they may require to successfully engage in investigations" (Diezmann, Watters, English, 2001, p. 258).

On the other hand, the relevant difficulty can be attributed to the teachers' conception of their role with respect to the acquisition of the mathematical knowledge: they attempt to keep control of the pupils' outcomes and their validity, to maintain the students' dependence and to keep for themselves the management of mathematical knowledge (Arsac et al., 1992, Jaworski, 1994, Sakonidis et al, 2001).

In this context one of the important issues concerning the management of the mathematics' classroom is related with the "devolution of mathematical knowledge" (Brousseau, 1997, p.46).

> "the teacher must therefore arrange not the communication of knowledge, but the devolution of a good problem. If this devolution takes place, the students enter into the game and if they win learning occurs" (Brousseau, 1997, p.31)

How pupils interpret the mathematical meaning of a problem, and how the teacher reacts to the different meanings, how s/he helps the pupils to reconsider the problem, its mathematical meaning, without "telling" them the solution and without taking on him/her the mathematical knowledge are central issues for the research.

Based on the above considerations, this paper reports on the analysis of a problem solving process in the early grades (5 and 6 years old). It is a case study and the analysis focuses on the teacher's role in the management of the mathematical knowledge and tries to identify those elements, which led to the successful outcome of the process.

THE STUDY: A PROBLEM SOLVING PROCESS

The data analyzed for this paper come from a problem-solving situation tackled by pupils 5-6 years old. The problem given to the pupils was as follows: The teacher asked the class to divide in groups and each group try to share among its members in equal portions a circular cake. For this purpose, each group was given a cake with a diameter of about 25 cm, made of paper and decorated with non-symmetrical drawings. This problem was chosen because of its certain epistemological characteristics: it includes references to real life situations, it is commensurate with pupils' interests and experiences, it is challenging given that the solution is not obvious, children can apply their basic knowledge, and the material allows children to control in their own the validity of their outcomes

The pupils decided on their own to form two single gender groups, one consisted only of girls and one consisted only of boys. This resulted in two groups of different number of pupils (4 boys and 5 girls) and of different ages [1], which, as it will be seen later, influenced decisively the problem solving. Four sessions were needed all together for the completion of the task, one session per week.

Sessions were video and audio recorded. After each session, the teacher [3] completed a report with her comments. In the end of the whole problem solving procedure, a discussion was contacted with the researcher.

Data presentation

In the following, the development of the whole problem solving process is briefly presented. (numbers between brackets are referred to the transcript, which reports about the group of boys)

First session

In order to tackle the problem, children cut in a random manner the cake in pieces, each of his/her own. These pieces are of different size and a large piece of cake is left over. The children initially are worried about the left over piece and are not concerned with the unequal size of the pieces (1-3). They compare the pieces they had cut only after the teacher's intervention (4). The inequality of the pieces 'disturbs' the children only when their own piece is smaller than of someone's else (5,7). A point worth mentioning here is Costas's pointing out to the difference between senior pupil and junior pupil[2]: it is unacceptable a senior pupil to have a larger piece from a junior pupil! (10). During the process of pieces' comparison, children, while in the beginning they used to compare them by just looking at the pieces, as time went by, they started using another process of comparison, that is, by placing the one piece on the top of the other.

Second session

In the second session, the pupils tried to cut the cake using the process they adopted in the previous sessions: they would cut a piece of cake and they would place it the remaining cake to get the next piece, and so on. They chose this technique, because this way, they know beforehand, the pieces will be equal (15). Thus, the problem that children remained to solve was to cut the whole cake. In order to succeed, they cut larger pieces. However, there is still a remaining piece of cake. It has to be noted the children, in an effort to get an acceptable solution, that is, to be seen as having shared the cake in equal pieces, they suggested to the teacher to take the large piece that is left over (17).

Third session

In the third session, the children tried to solve the problem, to have no left over, using the same strategy as in the second session, but cutting this time larger pieces. However, they still do not manage to get the right size, to solve the problem. Thus, the problem remains. In order to get rid of it, Antonio puts forward the suggestion he had made in the previous session: the teacher to take the piece of cake that is left over (20). The teacher does not reject the participation in the solution of the problem, that is, to also take a piece. However, she insists that she should be treat as an equal member of the group and take a piece of cake of the same size of the others (21). Alexnadros' suggestion that the teacher takes his piece (22) is rejected by the other members of the group: although the older to receive a larger piece is acceptable by the pupils, a partner to receive a larger piece is not; the equilibrium of the system is destroyed. Thus, the pupils, although they are disappointed by the difficulty of the task (25-27) and ready to give up, they are finally forced to carry on with it and the problem is left to be tackled in the next session.

Fourth session

Accordingly to the teacher's comments, children expressed great anxiety in relation to whether they would be able to finally solve the problem.

The girls decided to drop Vaso, who is junior, in order to be as many as the boys, that is 4, and solve the problem. This action led to the formation of three groups and to the change of the situation, as one of the three groups had only two members. In this latter case, the problem was similar to the original, but much easier for the pupils, who were familiar with the concept of 'half' from their everyday life experiences (34-35). Thus, the change of the situation final led to the solution of the problem, passing through the solution of a simpler problem.

ANALYSIS OF DATA - DISCUSSION

The analysis of the data will focus on two issues: the mathematical behavior of the pupils and the management of the mathematical knowledge by the teacher.

With respect to the first, the following points are of importance: the children adopted the problem, but the attached to it a different meaning from that intended by the teacher. Thus, for example, in the first session, the pupils tried to solve the problem in such a way as to each of them take a piece, in the second session they focused on sharing in equal pieces and from the third session onwards they attempted to consider the problem as a whole. We could say that the problem the children effectively solve during each session is the following: in the 1st session the try to share the cake, in the 2nd to take an equal piece of cake, in the 3rd to solve the problem and to 'get rid of it' and in the 4th to solve finally the problem: the problem's devolution takes place during all 4 sessions. The pupils' solutions satisfied each time their own interpretation of the problem and it was only through the evaluation of the result, performed after the teacher's intervention, that the children adopted a new interpretation and attempted to approach differently the problem. The difficulties encountered by the pupils were of epistemological nature and concerned the equality of the pieces and the properties of the circle. With respect to the former, it is possible that the relation of the problem to the everyday situations functions as an obstacle. This is because, as the pupils tried to overcome the difficulty of finding a satisfactory solution, by appealing to real life rules (the youngest takes a smaller piece, the older a larger), which could make their solution acceptable (thus helping them to get rid of the problem). The final agreement that all (junior pupils, senior pupils, teacher) should receive equal pieces constitutes an epistemological advancement towards the mathematical concept of equality. As far as the properties of the circle are concerned, it could be argued that it is a lack of knowledge (Diezmann et al., 2001) and that the pupils learn about these properties (folding like a cross-symmetry) through this problematic situation.

The above analysis of the problem's devolution is related to the manner in which the teacher manages the mathematical knowledge and the mathematical learning. She intervenes by posing questions concerning whether the cake was cut in equal pieces or whether the piece that was left over is equal to the others (1,4,8,16,19), avoiding to say whether pupils' outcomes are right or wrong. Thus, her questions focus on the need for the pupils to check their results. This checking is the crucial element, which will allow them to proceed to the solution of the problem, by re-considering their ideas. This way, she facilitates the children's approach, without indicating what is right or providing hints, or simplifying the problem, but by helping them to stay focused on the problem (to not give up) and to cope with it, its cognitive demands and the mathematical knowledge it involves. In other words, she allows the devolution (Brousseau, 1997) of the mathematical knowledge. Another important point is when the pupils 'desperate' suggest that she takes the larger piece (22). She refuses to do so (if she had accepted, she would have reinforced their conceptions), but she does not refrain from participating in the solution of the problem with the condition of being treated as an equal member of the group. Thus, by 'rejecting' the role of the adult who deserves a larger piece and as teacher who does not solve the problems (they are pupils' tasks), she reinforces her facilitating role, as described earlier.

The simplification of the task (the creation of two groups of 4 pupils and one group of 2 pupils) was initiated by the pupils themselves. This seems to be a special feature of this particular case. However, if this simplification did not occur in the fourth session, the teacher could have intervened by suggesting the formation of groups of 2 children, without reducing the cognitive and mathematical benefit of the task (it was the right moment).

CONCLUDING REMARKS

Summarizing, it could be argued that there are two factors which seem to play a decisive role in the successful solution of challenging tasks: the first is related to the task itself and it is the possibility for control of the outcomes by the pupils' themselves. The second one is related to the management of the knowledge by the teacher: facilitating and supporting the pupils without reducing the cognitive value and the mathematical meaning of the problem. In this way the problem's devolution takes place and thus the related mathematical knowledge is constructed.

Transcript

First session [...]

01 Teacher: Has the cake been cut into equal pieces?

02 Dinos: There is some leftover, because there are not many of us.

03 Alex: I should have cut bigger pieces...

04 Teacher: Are all the pieces the same?

05 *Both children compare the pieces they have cut visually. Dinos is taking Alex's piece and as he puts it on the other piece of cake, he finds out that his piece is smaller and he gets angry.*

06 Dinos: I will take the big one (*he means the piece of cake that is left over*)

07 *All the others strongly disagree with his action*

08 Teacher: Has the cake been cut into equal pieces?

09 Costas: I have a smaller piece than Antonio does (Antonio is junior-pupil)

10 *Costas reaches this conclusion by comparing the two pieces via putting his piece on top of the other piece of cake.* [...]

Second session

11 Alex: We are going to cut the cake two by two pieces.
12 Teacher: Hmm, how are you going to do that?
13 Alex: I will cut one piece and I will put it on the top of the first.
14 Teacher: So, then?
15 Alex: These two pieces will be the same. [...]
16 Teacher: Is the leftover piece of cake the same as yours?
17 Antonio: You are going to eat it Miss, because it is big and you are the older.
18 Costas: Alex should have cut the cake into bigger piece [...]

Third session [...]

19 Teacher: Is this piece of cake in the middle the same size with all the others?
20 Antonio: It is big. You should eat it!
21 Teacher: I would like to have a piece of cake the same size as yours.
22 Alex: Take my piece, Miss.
23 Dinos: So, are you going to be the one who will eat the bigger? Oh, no. I don't agree with this.
24 Teacher: So, the cake has not been cut into equal pieces!
25 Pupils: It is difficult, Miss!
26 Teacher: We are going to try again!
27 Pupils: Yes, but not today! [...]

Fourth session

28 Dimitra: We would like to be four of us Miss, like the boys did!
29 Alex: Girls are jealous of us.
30 Dinos: Vaso you better cut an other cake with Mrs Evi. (*Vaso is a junior-pupil and Evi is the other classroom teacher*)
31 Teacher: I agree, if Vaso agrees too.
32 Vaso: I agree.
33 Teacher: Well, today we are three groups and we are going to cut three cakes. I am wondering which group will be the one, which will successfully cut the cake into equal pieces. I would love to hear what you are thinking about [...]
34 Dimitra: You are only two. So you are going to have half cake. This is how my mother does when the chocolate bar is too big.
35 Vicky: It is so easy! You will cut it in the middle.[...]
36 Dinos: I will fold it, as my mother does when I need a paper to draw.
37 *Dinos is folding the paper and cuts it* [...]

NOTES

1. In the early grade, which usually lasts 2 years, enroll pupils of 5 to 6 years old. Children of the first year are called junior pupils, whereas children of the second year senior pupils.
2. After this first problem was solved, four more sessions followed, where pupils engaged in sharing cakes of other shapes: square, rectangle, triangle (isosceles and equilateral).

3. There were two teachers in the classroom, exchanging roles, according to the content of the teaching.

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EXPLOITING CLASSROOM CONNECTIVITY BY AGGREGATING STUDENT CONSTRUCTIONS TO CREATE NEW LEARNING OPPORTUNITIES[1]

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Abstract: We examine new activity structures that involve aggregating personal mathematical constructions built on hand-held devices such as graphing calculators on larger computers in publicly shared displays. Among the many possible applications of classroom connectivity, we focus on situations engineered to require students to coordinate mathematical ideas and representations within systematically varying families of functions.

INTRODUCTION: THE CONTEXT AND GOALS FOR THE STUDY

Data, Subjects, Course-Context, and Technology

The context for the study is the same as reported in (Hegedus & Kaput, this PME). In particular, video, field notes and student work in a course for academically weak university freshmen were analyzed. The 12 students reported on here, averaging 19 years of age, were mathematically similar to 14-16 year olds. The three 1-hour sessions from which the observations were made followed those reported on in the accompanying paper, and occurred slightly past the midway point of a 15 class SimCalc exploratory teaching experiment. Students sat at tables in groups of 3-5 in a crowded space, where each used a TI-83Plus graphing calculator running MathWorlds software. Calculators could be connected, by 2m wire, to a hub that could serve up to 4 wires "quadrapus"-style, where different calculators might be interchanged to use a single wire if needed. The hub, in turn, could wirelessly send to and receive data from an external server. This was a prototype version of the TI-Navigator™ system. At the front of the class, in control of the teacher and his assistant, was a computer with display that could upload and aggregate (as described below) student work from the server while running a computer version of MathWorlds. The actual mechanics of the prototype system are ignored for the purposes of this description since they are subject to change as the system evolves.

Goals of the Study

Our aim was to understand the affordances of this level of classroom connectivity, both in terms of the new kinds of activity structures it could support as well as the teaching and learning opportunities it might support, both planned and spontaneous.

COORDINATING MULTIPLE DESCRIPTIONS OF SITUATIONS IN SIMCALC CLASSROOMS

Essential Goals of the SimCalc Project

The SimCalc Project, underway for almost a decade as of this writing, seeks to democratize access, beginning at as early an age as possible, to the core ideas of the mathematics of change and variation, especially the ideas underlying calculus (Kaput, 1994; Kaput & Roschelle, 1997; Kaput, Roschelle & Stroup, 2000). The essential means for the effort are a combination of representational and curricular strategies beginning with two foundational representational strategies that change the representational infrastructure embodying the ideas, from algebraic to visual:

(1) To use interactive simulations hot-linked to new forms of visually editable graphs and visualization tools, and

(2) To build the fundamental relationships between rates and accumulations (what is normally referred to as the "Fundamental Theorem of Calculus") into both the software and associated curriculum.

Much of the student work using these representational strategies takes the form of coordinating multiple descriptions of the physical or simulation phenomena involved. For example, relating to #1, students must connect time-based coordinate graphs of position or velocity functions to screen animations driven by those functions, e.g., understanding that where a velocity graph is flat, the object must be moving at constant velocity, and when a position graph is flat, the object must be stationary. Relating to #2, they must learn to coordinate velocity and position descriptions of the same motions (or rate-accumulation relationships for other change-phenomena), e.g., understanding that a horizontal velocity graph corresponds to a straight-line position function, and vice-versa.

Further, students must also coordinate multiple representations of the same functions, e.g., algebraic formulas and coordinate graphs. Not only is the coordination among different descriptions and representations an essential part of the SimCalc strategy, it is also a more general goal of mathematics education extending well beyond the mathematics of change and variation.

Exploiting Diverse Devices and Classroom Connectivity

Technologically, the SimCalc Project has built software to support the above representational strategies, first for computers, and more recently for hand-helds, particularly the popular TI-83Plus graphing calculator (see www.simcalc.umassd.edu for downloadable software and curriculum materials for each). Most recently, we have begun development and extension of technologies that attempt to exploit the strengths of these different kinds of hardware platforms by using them in combination, particularly where each student has the personal at-handedness of a graphing calculator, and where the teacher has available the processing power and high resolution/color display of a larger computer. Further, we attempt to exploit the newly available classroom connectivity as sketched above. Other work exploiting classroom connectivity and diverse hardware platforms is underway by Stroup and Wilensky (Wilensky & Stroup, 2000) and Resnick and Collela (Collela, 1998), where the focus is also on integrating individual student constructions into larger classroom structures, especially participatory simulations, where each student plays the role of an agent in a larger system with emergent behavior.

The Set-Up: Unique Two-Digit Student Identifiers to Support Aggregation

We used the same three levels of activity-organization as described in Hegedus and Kaput (this PME). Individual, small-group (students sharing a hub, so we refer to these as "hub-groups"), and whole-class. Each student in the class has a unique 2-digit identifier rather arbitrarily defined as follows. The first digit is determined by their hub-group number (which ranged from 1 to 3 for these sessions), and the second digit is determined by simply counting-off in each hub-group beginning with 1. The 3 respective hub-groups had 5, 3 and 4 members for these sessions, so the identifiers ranged from 11 to 34 and were fixed across the three sessions because the students stayed in the same groups, with the same count-off numbers.

The Two-Prior Aggregation Activities

In Hegedus and Kaput (this PME), we described two initial aggregation activities using the 2-digit identifiers to create parametrically varying families of linear position functions where the variation depends directly on the students' identifiers, which in turn means that the variation depends on the students themselves. The *Staggered Start, Staggered Finish* activity involved a simulated "race" with a Y=2X target position function controlling the motion of one screen object ("*A*"), where each students created, in Y=mX+b form, a linear function controlling a second screen object ("***B***") where their second object ***B*** started at their count-off number but traveled at the same rate as ***A***. Thus they finish at the same distance apart as they were when they started. Within a hub-group each student's function was different from each other student's function, varying in the Y-intercept, so each was parallel to *A*'s Y=2X function. Each student then sent their position function for ***B*** to the teacher where it was aggregated with all the other students' position functions. While each student's object ***B*** was represented as a dot in the aggregated collection of objects, there were only 5 different (parallel) position functions because there were only 5 count-off numbers. Part of the whole-class discussion involved identifying each student's dot and function-graph, where as many as three students' position graphs overlapped.

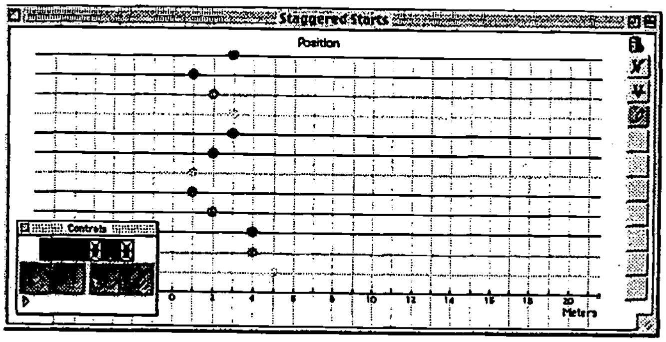
The next aggregation activity was a more challenging variation, the *Staggered Start, <u>Simultaneous</u> Finish* activity, where each student started at 3 times their count-off number and was to finish in a tie with *A*. After aggregation in this case, the motion showed a series of dots starting in staggered positions and traveling at constant velocities which depended on how far they needed to travel in the given 6 seconds. And the slopes of their respective Position functions likewise varied depending on the starting positions (Y-intercepts) since all the graphs "focus" in to intersect at the common (6, 12) endpoint, reflecting that the race must end in a tie.

Of special interest in this paper is how we are able to exploit further the students' personal connection with their constructions in the aggregated publicly displayed set of student constructions in order to serve developing the two kinds of critically important coordination skills described above.

AGGREGATION-BASED ACTIVITY STRUCTURES: WHERE ARE YOU?

New Student Coordination Activities: Where Are You?—Example 1

These activities move to a higher level of complexity by involving both the student's count-off number and their hub-group number in their individual mathematical constructions. Each activity was designed to put the student in the position of needing to coordinate important information about their mathematical construction in order to identify either their motion-object, their function, or a closely related function in the aggregated set in the classroom display – to "find yourself." The first case involves coordinating position graph information with motion information.



Example 1: Make a Y=mX+b Position function formula for a 5-second motion for **B**, where your starting point is your count-off number and your slope is your group number. Then send it up and we will examine where you are.

Figure 1: Initial Positions Equal to Count-off Numbers at Time = 0 Sec

In *Figure 1*, is what the students saw when the functions were first aggregated. In particular, note that we deliberately did not show their function graphs, which they had already seen on their calculators, and, in fact could—and did—refer to as the discussion proceeded. In response to the teacher's question "Where are you," there followed an animated 30 minute discussion that occurred in 3 stages, the latter two of which were based on revealing, respectively, additional information when a consensus was formed that the information was needed: (a) initial position as shown in *Figure 1*, (b) approximate velocities and ending points, and (c) exact velocities as shown by the "Marks" in *Figure 2*. Also revealed in stage (b) was the fact that one student had created a 6-second motion and that one student had entered an incorrect slope (2, which should have been 3). Space limitations prevent inclusion of the transcript of the extremely rich classroom discussion (a full account is in progress), although a few observations are central to understanding the activity design and how the students responded.

In stage (a) students recognized those dots that might represent them but decided that they could not exactly identify themselves, except to know that their companions on the "starting-line" all shared their count-off number. The one exception was Clive, who was the lone member of the class with count-off number equal to 5. He appears in the bottom position, and, when he identified himself, seemed to stimulate and consolidate the consensus that more information was needed by the remainder of the class. Interestingly, Clive had been among the quietest students in the class and barely spoke publicly during the first 6 weeks of the course. By the end of the 3rd class where he was the only student with 5 as a count-off number, he was acknowledged as "famous" by another student and frequently contributed to classroom discussions.

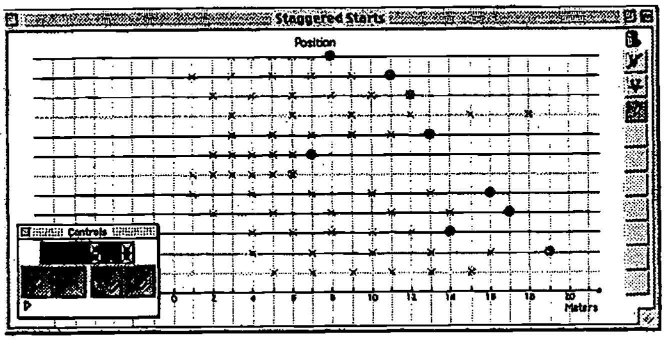


Figure 2: Ending Positions with Marks Shown

In stage (b), as the class came to recognize that some groups would be faster than others and hence would yield greater total distances traveled, they were able to make progress in identifying themselves, although the errors mentioned above added sufficient uncertainty, as did the fact that the starting positions disappear when the animation is run. The need for specific identification forced a move from qualitative analyses (e.g., "He went the farthest so is probably in Group 3") to quantitative analysis.

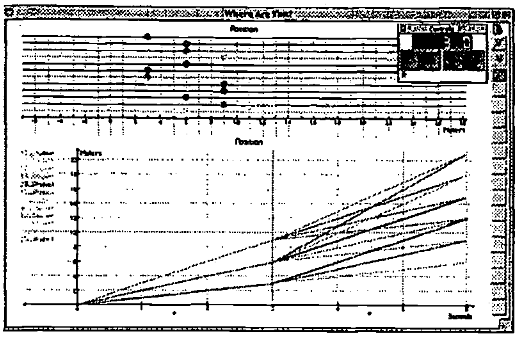
An important visualization feature built into MathWorlds is to have the moving objects "drop marks" (location-traces) for specified time intervals as they move, with the default being 1 second. The students had come to use this feature regularly when the velocity of an object was in question. Hence they called for marks to be dropped. This resulted in *Figure 2*, which shows the positions of each object at 1-second intervals from the initial position to the ending position.

Since the velocities are constant in this case, we can now read off the respective velocities of each object and coordinate it with the starting position to uniquely identify the identity of each dot.

Reflections on Example 1: Using Personal Identity to Coordinate Linear Function & Motion Information, Especially X-Coefficient & Rate-of-Change

Example 1 indicates how, by selectively hiding certain information, in this case the coordinate graphs for the algebraically defined linear functions, and revealing additional information about the associated motion in a pedagogically functional way, students' personal identities act to motivate and focus attention on key features of a display. Exactly the kinds of cognitive things we want to happen in traditional motion-representation coordination activities at the heart of SimCalc representation strategy #1 identified above (e.g., identify the motion that goes with this formula, or make a formula that matches this motion) can occur very naturally when the activity is of the *Where Are You?* type. Far less obvious in this sketchy, abbreviated and static-medium account is the high level of personal engagement of the students, and, more importantly, how that engagement helped structure the coordination process.

Example 2: Coordinating Velocity & Position Descriptions of Motion—Part 1



Example 2: Make a 2-step velocity function, each step 3 seconds long, where your velocity for the 1st segment is your Group # and your velocity for the 2nd segment is your Count-off # in the group. Everyone starts at 0. Send it up and we will look at the position functions and examine where you are.

Figure 3: Branching Position Graphs – Group # = First Velocity

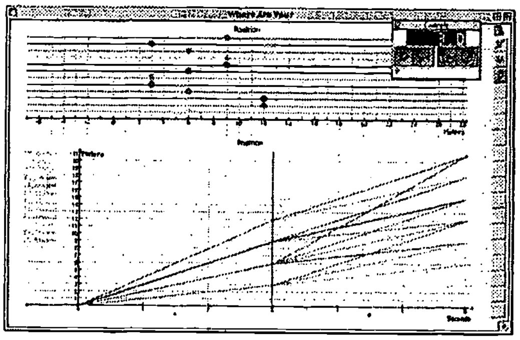
MathWorlds on the calculator (as on the computer) enables students to make step-wise varying graphs directly by systematically adding and manipulating segments. We displayed the dots and position graphs for the whole class and then asked "Where Are You?" The goal here was to coordinate velocity and position descriptions of a motion—using the ideas that velocity is slope of position graphs and that area under velocity graph segments gives the position-change during those segments. In this case the geometry of the configuration comes to play an interesting role relative to the motion.

Note that *Figure 3* is a cleaned-up version of the initial one that appeared—one student sent up a constant velocity graph steeper than the rest and was discovered to be in error. The set of position graphs consist of a 3-branched tree from the origin, one branch for each of the 3 groups, each of which then branches for each of the counting numbers in the group, which varied from 3 to 5. The discussion leading to the identification of individuals was extraordinarily rich, and filled with excitement as the students gradually recognized the two separate roles for their two numbers in the shape of the graphs and the resulting motions. Interestingly, as can be seen from *Figure 3*, all members of a group travel together, side-by-side, for the first 3 seconds, at which time they diverge to travel at their different count-off number velocities, a fact noted by the students.

Note that Clive, the only "5" was an outlier, and went the fastest at the end of the trip, tied for the longest distance, etc. One student observed that those people with straight graphs and constant motions were those whose count-off number equalled their group number. They also kept their velocity graphs available for reference in the coordination and used the motion information to anchor the two descriptions, exactly as we use it in our traditional SimCalc activities, most of which take the form of matching activities (e.g., make a position function for B to match A's motion exactly, where A's motion is given by the given velocity graph).

Example 3: Coordinating Velocity & Position Descriptions of Motion—Part 2.

Here, members of a group split apart at the outset, where student with the same count-off number were side-by side. This version involved a very different configuration of graphs and motions appearing—see *Figure 4*—except, of course, for those students whose Group # = their Count-off #. Of interest is how group identity was vocalized in this case, as each group "broke apart" in the 1st 3 seconds and never came together, despite the fact that they all traveled at the same velocity for the last 3 seconds. It appears that these kinds of activities also provide a prime context for purposeful and logical problem solving, because both their and their classmates' identities are involved. In this case, there was someone unaccounted for and that person was identified by elimination that involved coordination between the slope-information on the position graphs and the velocity information.

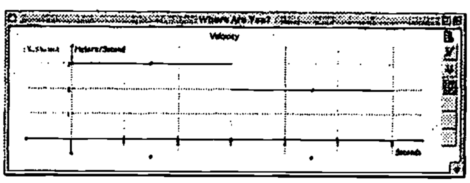


Example 3: Make a 2-step velocity function, each step 3 seconds long, where your velocity for the 1st segment is your Count-off # and your velocity for the 2nd segment is your Group #. Everyone starts at 0. Send it up and we will look at the position functions and examine where you are.

Figure 4: Position Graphs With Count-off #'s as 1st Slope.

Example 4: Velocity Graphs Provide No Position Information.

In this case, we included the example to drive home the point, which had arisen previously, that velocity information tells us "how fast" but not "where." This shows up quite dramatically where we showed what appears to be a single velocity graph (they all overlapped—See *Figure 5*) and asked "Where are you?" We temporarily hid the dots, which would have given away the position information. Of course, it is still inconclusive in this case even when coupled with the motion shown—where all the dots move at the same velocity, and end in the same configuration in which they started.



Example 4: Make a 2-part position function, where you start at the sum of your count-off and group numbers, and each part is 3 seconds long. The slope of the first segment should be 1 and the slope of the second should be 2. Send it up and we will look at the velocity functions and examine where you are.

Figure 5: Velocity Functions for 12 Students—all overlapping

REFLECTIONS ON THE EXAMPLES: USING TECHNOLOGICAL CONNECTIVITY TO GENERATE PERSONAL CONNECTIVITY

These early examples only scratch the surface of what we believe to be possible in exploiting new classroom connectivity for educational purposes by tapping into personal identity as a resource for focusing attention and generating engagement in complex mathematical activity. Many more examples are being examined, and the reader could surely generate more. For example, we are currently studying the use of CBR-based data, where groups separately create motions based on their own physical motion, which are then aggregated in a class to form dances and marches and then re-animated on the computer screen. Again, the kind of thinking needed to plan out a dance that, of course, is executed simultaneously, but produced serially, is exactly the kind that we would want to produce by traditional means.

As pointed out by Donald (2001), humans accomplish extraordinarily complex tasks of management of their mental resources and communication in everyday social contexts using the cultural tools of language and, as needed, other representational tools. Most mathematical activity ignores these resources despite the unanimity of recognition of the power of classroom talk and norms that support inquiry and purposeful discussion. The kinds of aggregation activity structures described above deliberately build systematic mathematical variation, personal identity and ownership into a functional classroom role by making them an intrinsic part of the activity structure itself at one or another levels of group organization. Future work will continue to explore and map out this extremely rich opportunity space.

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1. This work was funded by National Science Foundation Grant # REC-0087771, Understanding Math Classroom Affordances of Networked, Hand-Held Devices. Assertions and conclusions are those of the authors and not necessarily those of the Foundation.

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FUNCTIONS, MANY YEARS AFTER SCHOOL: WHAT DO ADULTS REMEMBER?

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This paper presents findings from a qualitative study regarding adults' recalled definitions of function, and their attempts to draw simple linear functions. The subjects in this research were 24 men and women between the ages of 30 to 45, all engaged in successful careers. Findings support the principal conclusion of Bahrick and Hall (1991), who claimed that retaining of high school mathematical material strongly depends on the length of acquisition period and the amount of practice. Also, some of the findings exemplify Bartlett's theory (1932) concerning reconstruction vs. reproduction in the mechanism of recalling.

Introduction

Consider the phrase "curricular consensus"; sounds like an oxymoron, does it not? So many different views exist about what should be taught in schools, that it seems a great achievement if an agreement can be reached within a single school. Nevertheless, we might find a consensus in the shared implicit expectation of all curricula, which is – as we dare to assume – that graduates should be knowledgeable to a certain degree about the contents they have learned. Such an expectation evokes a cardinal question: What do adults remember from their past studies? This intriguing question was scarcely addressed by research until two decades ago, much to the explicit disappointment of cognitive psychologists specializing in memory research (Neisser, 1978, Bahrick, 1979). Since 1980, some progress has been made in this area (see a review by Semb et al., 1993), but the number of relevant researches is still relatively low, probably due to unresolved methodological problems. In regard to mathematics, a unique work is that of Bahrick and Hall (1991), a large-scale quantitative study designed to identify variables that affect losses in recall of high school algebra and geometry contents. A major finding of this work was:

> "When the acquisition period extends over several years, during which the original content is relearned and used in additional mathematics courses, the performance level at the end of the training is retained for more than 50 years, even for participants who report no significant additional rehearsal during this long period. In contrast, those whose acquisition period is limited to a single year perform at near-chance levels…" (p.30).

The performance of Bahrick and Hall's research participants (about 1700 in number) was measured by psychometric means (that is, correct/incorrect answers); indeed, a statistical study of such a large scale does not usually involve cognitive analysis of answers to open-ended questions. Since cognitive analysis could potentially reveal phenomena that remain unnoticed by psychometric analysis (Karsenty & Vinner, 1996), it seems that a qualitative study, scrutinizing adults' attempts to recall mathematical material, could add to the general picture in the issue of maintenance of high school mathematical contents. In this article we chose to present attempts to recall the definition of function and graphical representations of linear functions.

Theoretical background

Mechanisms of recalling. The cognitive aspect of recalling contents has long been an attractive subject for researchers (mostly referring to contents acquired in research laboratories). The well-acknowledged classical work of Bartlett (1932) laid the foundations of this field. Bartlett suggested that recalling is a mechanism of *reconstruction* rather than *reproduction*. In other words, when requested to repeat a story heard or an event experienced, people are likely to produce an interpreted version, though they may be unaware of doing so. Thus, some details might be omitted, others emphasized or even added. In this article we present examples of attempts to recall the graphical representation of a linear function, in a way that suggests that a similar mechanism of reconstruction takes place. Moreover, we intend to show that such reconstructions, idiosyncratic as they might be, sometimes follow a certain inner logic that is recruited in the absence of accessible relevant details.

The concept to be recalled: What is a function? Math educators and researchers have debated on proper ways of introducing the concept of function, one of the most difficult and complicated concepts that secondary school mathematics students encounter (Buck, 1970). One of the core questions in that matter concerns the kind of definition that should be used. Roughly speaking, we can divide definitions into two groups, based on the historical development of the notion of function (see Kleiner, 1989). The first group is what we might call "old definitions" – definitions in the spirit of 18th century mathematicians Euler and Lagrange. For them, a function was "an analytic expression representing the relation between two variables with its graph having no corners" (Malik, 1980, p. 490). The second group of definitions can be referred to as "modern". In these definitions the key word is correspondence, as in the Dirichlet-Bourbaki definition. This major change in the definition of function was reflected in most of the mathematical secondary school curricula all over the world. Examination of ten 20th century Israeli textbooks, dating from 1923 to 1995, reveal a shift from "old" definitions to "modern" ones, around the end of the sixties and the beginning of the seventies (Karsenty, 2001). The question of whether this curricular shift contributes to a better understanding of functions, stands at the heart of the debate mentioned above. Malik (1980) criticizes the use of modern definition of function in courses intended for average students:

> "The necessity of teaching the modern definition of function at school level is not at all obvious and most of the instructors feel that pedagogical considerations were ignored while designing the course content and the mode of presentation" (p.490)

Vinner and Dreyfus (1989) doubt the appropriateness of a modern definition for certain courses, and suggest that an intensive example-based introduction should precede and accompany the formal definition. Markovits et al. (1986) suggest ways to overcome some obstacles and enable a successful use of the modern definition.

In light of these opinions, it was interesting to look into the following question in regard to our research subjects. Considering the fact that they had attended high schools around the decade of change (see methodology section below), what kind of definitions will they use when requested to define a function in an open-ended manner? Some results follow the methodology description.

Methodology

The data and analyses reported here are part of a study, whose aim was to investigate cognitive and affective characteristics of adults' recollections in regard to their mathematical experience and knowledge (Karsenty, 2001). The study was defined as a "collective case study" (Stake, 1994), and focused on adults with considerably high level of education. The subjects' selection procedure was described in detail in a previous article (Karsenty & Vinner, 2000), and will be repeated here briefly.

12 men and 12 women were selected for the study from an upper-middle class community whose residents came from all over the country. At the time of the interviewing process the subjects were between the ages of 30 to 45, and their high school graduation year ranged from 1968 to 1984. All of them were graduates of college or university, but they varied in the level of math they had taken in high school[1] and the type of their current profession[2]. Each participant attended an individual session, devoted mainly to an interview, which was semi-structured and lasted about 2 hours[3]. The first part of it intended to explore personal affective components of learning mathematics (see Karsenty & Vinner, 2000, for some findings). In the second part of the interview, subjects were asked to "think aloud" about fourteen mathematical tasks involving basic concepts and procedures. The interview was intentionally held in a moderately-intervening manner. Thus, when a subject persisted that he or she "doesn't remember anything", small hints were suggested as triggers for eliciting any kind of respond associated with the question at hand. All interviews were recorded and transcribed.

In the next section we present results in regard to two interview tasks, introduced to the research participants as follows:

What is a function?
Can you sketch a graphical description of the function y=2x?[4]

[1] In Israel mathematics is a compulsory subject throughout high school, and can be studied in three levels, herein referred to as high level, medium level and low level.

[2] Professions were categorized by The Standard Classification of Occupations, a scale of 10 categories published by the Israeli Central Bureau of Statistics.

[3] Subjects also filled two questionnaires, findings from which will be published elsewhere.

[4] In some cases the given function was y=x or y=x+1 or y=2x+1.

Results and Discussion

I. Responses to the interview question "what is a function" were grouped into six categories, which will be described and illustrated here in short. The categories are partly adapted from Vinner and Dreyfus (1989).

Category A. The subject's response includes, as a dominant component, one of the following expressions: "relation", "dependence", "influence", in regard to variables or number values.
Example:

"A function is, I think, a relation or a connection between two variables"

(Nurit (female), 38, psychologist, studied math in a high level track)

Category B. The subject's response includes, as a dominant component, one of the following words: "equation", "formula", "operation".
Examples:

"A function is an equation. An equation that describes some kind of a line"

(Gadi (male), 41, architect, studied math in a medium level track)

"A function is... using a... a general formula, and then you can plug in different numbers to reach certain goals. That is, it's some kind of a general formula that afterwards you can use each time with variables that are appropriate in a certain situation"

(Irit (female), 33, senior banker, studied math in a medium level track)

Category C. The dominant aspect in the subject's response is visual; the notion of function is mainly identified with a graphical representation.
Example:

"Ah... What is a function? A function is that thing with the... inside the graph, which has all kinds of shapes, and it looks differently according to its elements. [...] I remember that there are all kinds of functions [draws 3 different sketches of graphs]. Some are like this, some are like this, it depends on how the x relates to the y."

(Amira (female), 31, museum director, studied math in a medium level track)

Category D. The subject recalls the concept of function mainly in a context of exercises that he or she used to work on.
Example:

"I recall some story of substituting points, of a formula that you check. There's a question or instruction to plug in points and then every point to put on a graph [...] I remember the requirement, say in an exercise, of the peak point, something like that, and the lowest point, such things I remember. Or questions that dealt with the relation between the y-axis and the x-axis. Below zero, above zero, such things I recall, very vaguely. It's been some 20 years ago, you know. If you ask me what happens with this and what is it used for, I won't be able to tell you"

Eli (male), 40, principal of a post-secondary institution, studied math in a low track)

Category E. The subject can only elicit blurry associations related to the process of sketching graphs.
Example:

> "I remember that function was... we used to set up the... we used to set up on the graph there, on the x-axis and the y-axis, points... what was it? x-axis, y-axis, the points, and construct the function accordingly"
>
> (Arie (male), 38, insurance representative, studied math in a low level track)

Category F. The subject can neither define, nor exemplify, or even elicit associations to the word 'function'. It seems like the concept has been "deleted".

The distribution of subjects within the six categories is shown in table 1. Several observations can be made in regard to the data presented. First, it should be noted that none of the 24 subjects has defined a function in a way that could be regarded as modern. Generally it can be said that half of the subjects remembered the concept of function and defined it mostly in the spirit of old definitions (relation, dependence, equation, formula, operation, or via smooth graphs). The fact that the modern definition was not preserved in subjects' memory might suggest that it was not well assimilated, in spite of the favorable learning circumstances mentioned before (i.e., when the modern definition became popular in textbooks). It could be claimed, however, that this state of affairs is due to teachers' dissatisfaction with the use of the modern definition, as noted by Malik (1980) (see also Cha & Wilson's report (1999) concerning the inclination of prospective math teachers to define functions as equations or machines rather than by way of sets). One might suspect that, as a result of such dissatisfaction, the modern definition of function has played a lesser role in the implemented curriculum than it did in the intended curriculum.
Second, a connection can be noticed between high school math level tracks and the category distribution: Most of the subjects who participated in a high level track were classified to categories A-B, expressing familiarity with the concept of function. All

Category	No. of subjects assigned to this category, distributed by level of math taken in high school			
	High level:	Medium level:	Low level:	Total in this category:
A: Relation, dependence, influence	3	-	2	5
B: Equation, formula, operation	2	2	1	5
C: Visual response	-	1	1	2
D: Context of exercises to solve	1	2	2	5
E: Blurry associations	-	-	3	3
F: The concept has been "deleted"	1	-	3	4

Table 1. Distribution of subjects within the six categories of function definition (N=24).

medium-level graduates were classified to categories B-D. Most of the low-level graduates (8 of 12) were classified to categories D-F, which can be characterized by vague recollections, if any at all, about the definition of function. This finding is in accordance with the general conclusion of Bahrick and Hall (1991) in regard to lifetime maintenance of mathematical material, which is basically that the rate of loss is most affected by variables concerned with the amount and distribution of practice. Finally, attention should be drawn to the other half of the subjects, those assigned to categories D-F. Our general impression was that for them, the concept of function was absent from what might be referred to as an "available concept pool". Using Vinner's (1983) terms, it can be said that the cell of "concept definition" became devoid; any associations, if elicited at all, were based on random pieces from the "concept image" cell. This finding is noteworthy, considering the fact that these people are educated adults who are well positioned in modern society. One can only conjecture on the extent of loss, in regard to this important concept, among the wider population of high school graduates.

II. Subjects' attempts to draw graphs for simple linear functions were classified into seven categories. We will not describe these categories here, due to space limitations. Instead, we present three examples of drawings made by subjects. However, we would like to point out that of the 24 adults participating in the study, only 7 could sketch, without intervening clues, a graph that was correct in its general idea.

Example 1. Figure 1 presents a drawing of the function y=x, made by Dov (male, 37, government official, studied math in a low-level track).

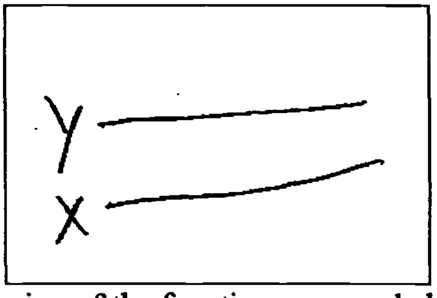
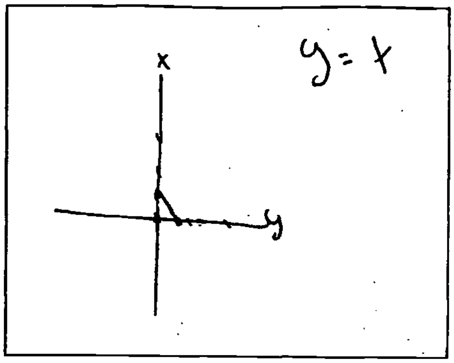


Figure 1. Drawing of the function y=x, made by Dov.

As can be seen, Dov does not recall the Cartesian system. In the absence of this frame of reference, he tackles the task in an idiosyncratic way. In other words, since Dov's knowledge of linear graphs is inaccessible, and thus can not be reproduced, Dov reconstructs the idea of y=x based on his present interpretation. Apparently, y=x is seen as an equality between quantities, and the two identical line segments illustrate this understanding. Two other subjects gave quantity illustrations; one described y=x as two identical circles, the other described y=2x as two small circles and a bigger circle.

Example 2. Figure 2 presents drawings of the functions y=x and y=2x, made by Tamar (female, 42, high school humanities teacher, studied math in a low-level track).



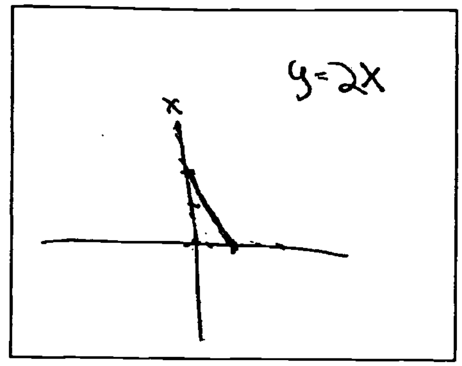


Figure 2. Drawing of the functions y=x and y=2x made by Tamar.

Tamar recalls the axis system, but not the Cartesian representation of points. Like Dov, she reconstructs the idea of linear graphs on the basis of her current common sense. While drawing y=x she says: "Well, anyway, there's got to be something equal here". Thus, she allocates equal segments of one unit on both axis. She then joins the two endpoints by a line. This last action might be ascribed to vague residues of the notion of function, elicited earlier by Tamar: When requested to define a function, she said - "There is a horizontal axis and a vertical axis, and there are points. You join the points and you get a function". Note that Tamar persists with the same logic when drawing y=2x. This time the segment allocated on the x-axis is twice as long as the one allocated on the y-axis, thus reflecting a common proportional misconception. Similar descriptions of linear functions were given by two other subjects.

Example 3. Figure 3 presents a drawing of the function y=3x+2 made by Judy (female, 35, lawyer, studied math in a medium-level track).

It is the second graph drawn by Judy. She first drew a correct graph of the function y=x+1, by calculating values of several points and joining them with a straight line. However, while doing so, she remarked: "It's a straight graph because it's very simple, it only has 1 added, and no multiplication". This remark led the interviewer to ask her to draw y=3x+2. This time, although Judy correctly calculates five point values, she joins the points with a curved line, as can be seen in Figure 3. She claims that due to the multiplication of x by 3, "what happens is that it's going to be higher and higher".

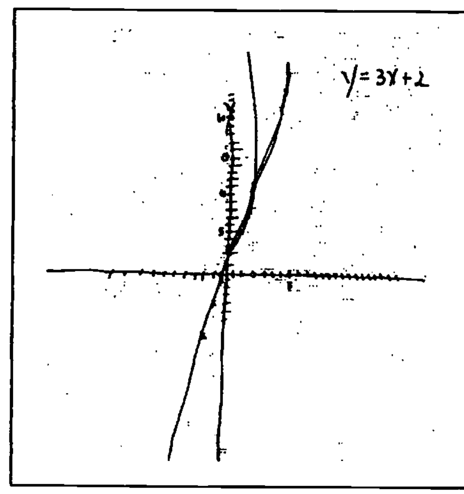


Figure 3. Drawing of the function y=3x+2, made by Judy

Again, we can see that a mechanism of reconstruction of knowledge takes place, even when the basic ideas of function graphical representation can be recalled. As a final comment we would like to emphasize, that the examples given above are merely illustrations of interesting phenomena of recalling mathematical material, yet to be explored.

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FROM CONTROLLING TO COMMUNICATING: INNOVATION IN BOTSWANA'S TEACHER PREPARATION PROGRAMME ON INTEGRATION OF ASSESSMENT AND INSTRUCTION

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This paper reports on an analysis of part of a wider study to investigate innovative approaches in the preparation of mathematics preservice teachers in the integration of assessment and instruction (IAI). Student-teachers' teaching practice was analysed to reveal the effects, if any, of IAI training that emphasised communication over control. The reported case studies show that training student-teachers in this way facilitated their application of IAI, and led to increased reflexive use of information from pupil responses in the their teaching, compared to that observed with previous cohorts of students.

Introduction

The integration of assessment and instruction (IAI) refers to teaching and assessment being intertwined, to assessment being used formatively. Torrance and Pryor (1998) suggest that IAI occurs when assessment is intended to generate information about children's knowledge and understanding while at the same time contributing to the process of creating that understanding. In mathematics teaching, the main aim of IAI, as Chambers (1993) explains, is to access students' mathematical thinking, and reflect on how to use that knowledge as the basis for instructional decisions for both individual students and the class as a whole. Feedback from such assessments should inform the student of what he or she can do to improve or move further in his or her learning, as suggested by Black and Wiliam (1998). IAI in this study made limited use of techniques of assessment that rely on written work such as tests. More focus was placed on assessment techniques that may be used hand-in-hand with on-going teaching, such as oral questioning and observations of students on task.

The wider study

The wider study from which this report is taken is an action-research project on my own practice as a mathematics teacher educator. It was formulated in the context of the purpose of equipping student-teachers with the necessary tools for their teaching career. It was conducted at Tonota College of Education (TCE) in Botswana, which trains teachers for a Diploma in Secondary Education qualification (DSE). A class of 21 student-teachers training to teach mathematics was involved in the study. The practical element of the study was undertaken over a period of six months covering college-taught sessions and Teaching Practice in placement schools.

In my previous work with student-teachers, I had observed a general failure in their application of IAI during Teaching Practice, despite them receiving several lectures about its techniques and methods, and its importance in the teaching and learning of mathematics. The students of previous cohorts could not assess students' mathematical thinking effectively or use the knowledge to make appropriate instructional decisions. Invariably, their teaching was characterised by "telling" students rather than engaging them in the negotiation of meanings or construction of knowledge. This observation was also made by educators from the University of Botswana (UB), which validates the DSE programme, and questions were raised about the nature of teaching at the college, which is dominated by the lecture method. The UB, through Hopkin (1999) recommended that:

> The College (TCE) should explore the use and development of co-operative teaching in their training programme.

In recognition of this problem, I focused my investigation on how my methods of training affected student-teachers' learning. In other words, the study addresses what Russell (1997) calls the 'pedagogical turn'. I also investigated the 'content turn' by training student-teachers in IAI. However, as indicated by Webb (1993), IAI is not easy and requires teachers to be prepared for its application. As suggested by Heid et al (1999), mathematics teachers do not automatically attempt to determine students' understandings. They need to be empowered to act in that sense, for example by learning to listen with student understanding in mind, rather than as part of a method to control student responses.

In order to make a deliberate difference I designed a new teaching programme to illuminate aspects of IAI that I wanted student-teachers to use. It involved (i) oral questioning (ii) responding to students' responses and (iii) practical ideas that a teacher can draw from to facilitate his or her use of the proposed approach. This was designed to enable student-teachers to reduce control over the students' responses and allow the students to communicate their mathematical thinking.

As this study was a practitioner-based research seeking to improve practice, I employed action research methods that allowed me to monitor and modify the training as it unfolded. Therefore, it is likely that my influence on student-teachers' applications of IAI was not constrained by the research process, in that I had the freedom and experience to assist professionally. However, the fact that I was learning at the same as the students may have influenced them.

Data collection and analysis procedures

A lesson observation instrument was used to collect data on student teachers' implementation of IAI, focusing on their planning, on their questioning and other strategies for identifying and eliciting students' mathematical thinking, on their feedback to students, and on their self-evaluation. The lesson observation instrument was supplemented by other data sources such as detailed notes of class incidents, particularly the exact questions and responses exchanged by the student-teacher and their students, and student-teachers' self-reports. The analysis of data focused on the characteristics of student-teachers' application in comparison to each other and in reference to the aspects of the IAI approach covered during training. The overall objective of the analysis was to establish whether training student-teachers in IAI in the way I adopted enabled them to apply it during Teaching Practice.

Results

Three case studies are reported in this paper. They have been chosen to illustrate the range of degree of application of IAI in the cohort studied.

Elvis

Elvis's lesson planning explicitly made use of information obtained from previous classroom assessment, a point repeatedly emphasised during training. For example, in one of his lesson plans, under the introduction section he wrote:

Time	Content
10 min	<u>Introduction</u> General comments on marked exercise from last lesson (errors detected) $5\frac{1}{4}$ not $5\,\frac{1}{\ }$ 4 make reasonable statements after solving word problems ie (encoding) add: $2\frac{3}{5} + 4\frac{1}{2}$ If 6 (2 + 4) is done Therefore $6 + (\frac{3}{5} + \frac{1}{2})$

In his lessons, Elvis evidently elicited alternative problem solving strategies by advising students to make realistic decisions on the strategies they preferred to use. He commented that the strategy chosen should be what the individual student understood rather than that it was used by the teacher or other students. He posed questions that informed students clearly that there were alternative strategies that may be used. This is illustrated in the following dialogue:

T: How do we go about simplifying this? (Pointing to 2.4 : 2.7 on the board)
(Students suggest changing the numbers into fractions and the problem is worked through on the board.)
T: Is there another way of doing it besides changing it to fractions?
S: Multiply by ten and simplify
T: Why did you decide to multiply by ten?

This line of questioning required students to feel free to express themselves. Nevertheless, Elvis found it hard to promote the student-to-student interaction that he wanted so that students would communicate their thinking and share their strategies. His recognition of the importance of this revealed his understanding of students as well as that of the demands of IAI. Indeed the issue of giving students opportunities to express their understanding is recognised by Black and Wiliam (1998, p11) to be essential in the design of teaching for formative assessment to aid learning. I assisted him in planning and implementing activities that allowed students to interact with each other in one of the lessons. We agreed that he could ask me to intervene during the lesson and I could participate in ways demanded by the situation. After receiving this moral and practical support he regained his confidence in posing questions that required students to communicate their thinking. Without this extension of the training into the Teaching Practice period, Elvis may not have been able to sustain his initial application of IAI due to a failure to get students to express their understanding.

As a result of it he became increasingly able to ask questions that illustrated or revealed certain aspects of the mathematical concepts, and this provided students with an opportunity to learn in the process. In one lesson, students were asked what they would do to simplify a ratio. The response was that they would divide by the LCF and the teacher decided to illustrate that the LCF of any two numbers is 1 by using factors of 10 and 20. He then asked them what they could say about 1 and 10 from the lists of factors. The fact that 1 is the LCF of the two numbers was illustrated and they were asked which of the two would they use to simplify a given ratio. This exchange was giving students an opportunity to learn about the difference between LCF and HCF as well as for them to realise that 1 would be the LCF of any two or more whole numbers, which is usually not considered when teaching the topic.

Elvis often provided students with feedback that supported their knowledge and also extended it in a challenging manner. This was done by asking students to account for their solutions, for example by questions such as:

Why do you multiply 4 by 500? Say it out, I'm not saying you are wrong.

The student is supported by being told that questioning them does not mean that they are wrong, but is intended to extend their knowledge.

What about if this was 10m and 2kg? Can we compare them?

Here, Elvis wanted students to realise that in ratios the quantities must have the same units. He then moved on to m and km and to conversion of units to express the quantities in ratio form. This type of questioning, to provide challenges to students' knowledge in the process of giving them feedback on what they already knew, was explicitly encouraged during training.

Elvis also reflected on how to improve his practice. Motivating students to feel free to communicate their thinking was identified as an area for improvement in his teaching. He pointed out that making connections between topics explicitly was also found to enhance such communication for his students. He suggested that some topics might be easy to understand, thus making it possible for students to say something on them spontaneously. By the end of the last lesson that I observed he engaged comfortably in an extended exchange with one student. However, he continued to experience problems with making decisions about when to persist in the extended exchanges with individual students. This was a reminder to me that there can be no recipe that would work for everyone, as observed by Wiliam (2000, p22).

Tina

In Tina's case, lesson planning was significantly different from other student-teachers' planning and this seemed to have an effect on how the lessons progressed. Her initial lesson plans reflected a shallow consideration of content coverage and insufficient student involvement in the development of the concepts. This was evidenced by her failure to recognise alternative solutions raised by students. In one of her self-evaluation statements, she wrote:

> *"Most of the students were responding well except in the last part where I was unable to use some of their wrong answers well as they gave unexpected answer but some were correct e.g. They gave 18 x 2 while I expected 9 x 4. I could have asked them to expand 18 to get 9 x 2 then have 3 x 3 x 2 x 2."*

Either Tina had not considered various ways of expressing 36 as a product of its factors in her preparation, or she had a shallow understanding of the subject-matter. I had no reason to believe that the latter was the more likely explanation since she was one of the two student-teachers in the class with the highest 'O' Level grade. Whatever the reason, it resulted in a restriction of the use of students' responses in developing the mathematical concepts and thus assessment was not effectively integrated with instruction according to its interpretation in this study.

However, through processes infused in the training program to assist student-teachers to apply and sustain initial application of IAI, like self-evaluations, class observations, practical assistance and conversations after lessons, Tina managed to make progress towards flexibility in both content and use of students' mathematical thinking. In particular, discussions of students' responses that focused on why they may have responded that way made her view ideas from different perspectives. These reflections enabled her to learn to view the situation from the students' perspective; the kind of reflection that Wood (2001) suggested to be essential for pedagogical reasoning. Her line of questioning gradually changed from the more controlling, 'what' type of questions to include more 'why' and reasoning type questions. In the last lesson that I observed, I noted an exchange between Tina and a student that reflected this change of questioning, as follows:

T: What do you think you are going to get?
S: Negative two.
T: How did you get that? You have to try a step-by-step approach.

I viewed this type of questioning as encouraging students to think and communicate their thinking rather than controlling their responses.

Tina also observed changes in students' motivation as a result of allowing them to express their thinking orally and this encouraged her to pose questions that enabled everyone to say something. In one of the lessons, she based the development of the concepts on students' responses to the question "What have you observed?" Although she struggled to provide useful feedback to the students, it was clear that she wanted to make use of their responses to enhance their understanding. She viewed this as important, as reflected in her Teaching Practice report where she stated:

> *"As the lessons went on some lessons were conducted by questioning asking questions and the lessons are now interesting as they are well motivated. Also pupils are able to answer orally asked questions than written ones."*

The fact that Tina recognised how such assessments affected students' motivation and that she could use the situation to benefit the students made her pay more attention to how to sustain the situation.

It seemed that Tina's difficulties in applying IAI were rooted in inadequate lesson preparation during the initial stages. It appeared as though the emphasis on lesson planning during training was not done effectively for her to understand its importance. Though she persistently experienced difficulties with providing appropriate feedback, there were signs of improvement arising from the continuation of training through my active support during Teaching Practice.

Bowie

The most developed application of IAI in terms of lesson planning and the level of performance during the lesson was shown by Bowie. Lesson plans were detailed in terms of content, students' activities and the teacher's activities. It was evident that questioning was given consideration during planning just as had been recommended during training. The questions that he was to draw from were often stated in his lesson plans. For example:

> *"Some questions to be asked by the teacher: 1) What can you see in your reading that is common? 2) What do you think is the use of the decimal point?"*

From the first meeting Bowie exposed students to group work and to communicating their thinking to the rest of the class. He seemed to have internalised the assumption of active involvement by the students that IAI is based upon. The deliberate emphasis on the notion of learning from each other, including himself in that he was ready to learn from the students, can be associated with socio-cognitive theories of learning, that include ideas of 'learning from learners'. He achieved this purpose through posing questions such as "Can somebody explain what he said to me?" A working atmosphere in which students became free and keen to express themselves was established. In Bowie's lessons, students were observed posing questions and volunteering explanations to the class, unlike in other classes. This did not happen by chance, but because he created room for it through deliberate tactics to involve students, such as posing questions back to students, appropriate waiting time, and being honest about things he was not sure about, thus giving students confidence that he could actually learn from them. Throughout his trials of these tactics, his reflections drew heavily from the training. In our conversations after his lessons, he often related students' experiences to his college experiences, particularly the struggles encountered in group work and benefits gained from discussions with other student-teachers. It is presumably this experience or the practical exposure provided by the training program that enabled him to try out ideas to motivate students to reveal their mathematical thinking freely. How he was taught during training seems to have had an influence on his teaching.

Good questioning skills was one of Bowie's strengths in the application of IAI. He used the type of questions recommended during training that were perceived to enable students of different abilities to come up with something to say, such as:

> *"What similarities can you observe? What is common among these readings?"*

Students' responses were commonly used to ask further questions that in some cases supported the acquired knowledge and sometimes led students towards the mathematical concept being developed. This can be seen in the question:

> *"The three is a fraction of a centimetre. Should we write mm or cm (written on the board) now that we have three over ten* $(\frac{3}{10})$*?"*

Bowie's questions and the dialogue that followed bear similarities to that which Black and Wiliam (1998, p12) recommended, in stating that:

> The dialogue between pupils and a teacher should be thoughtful, reflective, focused to evoke and explore understanding, and conducted so that all pupils have an opportunity to think and to express their ideas.

Bowie employed a variety of assessment techniques such as observations to identify students' strategies that he then used in further instructional decisions. The observations of students' strategies served a meaningful purpose of exposing them to alternative strategies. They were not merely to check who was correct. In one lesson, the following strategies were identified through observations of students' work and were put up on the board for a class discussion:

(a)	12 x 10	(b)	12 x 10	(c)	12 x 10	
	H T U		12		12. 0	
	1 2		x10			1 place to R
			00		= 120	
	1 2 0		12			
			120			

This exposure to alternative strategies was followed by questions that reinforced knowledge already acquired by the majority of the students, if not all;

> *"What observation have you made when you multiply by 10?"*

Another demanded thinking from a different perspective;

"What happens to the digits when the decimal point moves to the left?

The latter demonstrates how students' knowledge can be challenged by way of providing feedback in the form of further questioning without necessarily posing difficult questions. Bowie also used questions such as "What else is missing?" to stimulate students to think of more, and from different perspectives. To me this was a sign of his perceptions of students: that he saw them as capable of thinking beyond their current thinking, as people who are capable of thinking for themselves instead of being there to be 'filled' with knowledge.

Conclusion

The case studies reported in this paper show how moving from controlling to communicating was achieved by empowering student-teachers during a preservice training programme. Previous cohorts did not progress to communicating because they did not have the advantage of the training methods I devised for this programme. The case studies also emphasise how individual student-teachers' learning differences play a significant part in their progress, and illustrate the value of training programmes having a means of incorporating further training for some student-teachers within Teaching Practice. In the study being reported, this was achieved by making supervision of student-teachers' first Teaching Practice formative, rather than to serve the purpose of grading currently practised at TCE.

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THE ACCURACY OF MATHEMATICAL DIAGRAMS IN CURRICULUM MATERIALS

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In this paper, two diagrams are analysed, for accuracy, using a set of principles generated from a review of the research literature from the fields of mathematics education, cognitive science, computer-aided learning, computer graphic design and semiotics. The diagrams are typical of those found in an Integrated Learning Systems (ILSs) evaluated in Queensland schools and a student workbook. Using this set of principles, it is explained why many of the diagrams contained within these curriculum materials do not facilitate the construction of mathematical knowledge.

BACKGROUND

The general aim of this pilot study is to investigate the accuracy of mathematical and scientific diagrams available in curriculum materials. The focus of this paper is on the mathematical components of the study. The study is timely in four ways. First, it focuses on mathematics and science education that are disciplines crucial for economic developments for Australia. Secondly, it integrates science and mathematics. Making connections across the curriculum has been recognised as an important learning outcome (International Society for Technology in Education, 1999; National Council of Teachers of Mathematics [NCTM], 1989; National Research Council, 1996). Third, it concentrates on diagrams, important in our iconic world. Finally, it considers an area with scant literature concerning diagrams in curriculum materials. No other studies have investigated science and mathematics coupled with an accuracy analysis of diagrams contained in curriculum materials. Mukherjee and Edmonds (1994) made the observation that diagrams in many Integrated Learning Systems (ILS's) often seem to have been developed in a vacuum by individuals or teams that have no background in graphic design or visual literacy. Diagrams therefore have the potential to be erroneous and misleading.

Curriculum materials. Most curriculum materials suffer from a lack of coherence and focus (Schmidt, McKnight & Raizen, 1997). They do little to promote critical thinking about mathematics (Risner, Skeel & Nicholson, 1992). Many teachers' lack of confidence and knowledge in relation to mathematics teaching is a world-wide problem with most attempts to remedy the problem achieving limited success (Peacock, 2001). As a consequence, teachers throughout the world rely on diagrams in both print and electronic based curriculum materials to provide the knowledge, and techniques of mathematical ideas to their students, and so such diagrams greatly influence the content of lessons. The only way to gain information relating to the suitability of instructional material is through an evaluation of the instructional materials available. However, teachers have neither the confidence nor competence to make these decisions in mathematics (Peacock, 2001).

Teachers may not realise that diagrammatic errors are present in some curriculum materials (Kidman, 2000). They reason that if several materials use a similar diagram to teach the same concept in exactly the same way, how could all those materials and diagrams be wrong? (Beaty, 1996). While publishers and other curriculum material developers are eager to claim that their materials have content accuracy, few schools and individual teachers are able to devote the time and resources necessary to judge the accuracy of scientific diagrammatic content themselves.

Relations between diagram and text. Diagrams are frequently used in mathematics curriculum materials at every academic level. Most texts have several diagrams on each page (Iding, 2000). Diagrams are included for two purposes: that of instruction and that of decorative purposes. High quality diagrams enhance instruction by encoding new information not referred to in the text, thus compensating for text deficiencies, and also by verifying each clause in the text, to develop an initial representation (Hegarty & Just, 1989). Hunter, Crismore and Pearson (1987, p. 122) identified five possible relations a diagram has in relation to the text it accompanies:

1. *Embellish* - provides completely new information not discussed in the text.
2. *Reinforce* - repeats all information presented in the text.
3. *Elaborate* - partly repeats and partly adds to information presented in text.
4. *Summarize* - provides a broad overview of text, much like an advance organizer.
5. *Compare* - provides information intended to be compared with a previous graphic.

This categorization, and the one that follows, are particularly appropriate for investigating the accuracy of mathematical diagrams because they focus on the relationship between textual and diagrammatic information and on the typical diagram types that are reasonably representative in mathematics curriculum.

Role of diagrams. A review of the research literature indicates that diagrams can play at least 4 different but interrelated roles in learning and instruction:

1. *Identification* - Diagrams that point out or identify parts of things (Cook & Mayer, 1988, Charles & Nason, 2001). For example, most students are familiar with geometry diagrams of the circle upon which the names of the diameter, radius and circumference are labelled (Iding, 2000; Kidman, 2000; Lowe, 1993).
2. *Comparison* - Diagrams that compare one kind of thing to another (Lemke, 1999; Hunter, Crismore & Pearson, 1987; Winn, 1989). For example, two different kinds of time-pieces, analogue and digital, might appear next to each other.
3. *Sequence* - Diagrams that point out stages in a chain of events (Cook & Mayer, 1988; Hunter, Crismore & Pearson, 1987). Typical examples would be diagrams of a sporting race showing the ordinal finish.
4. *Combination* - Possibly the most frequently occurring type of diagram, particularly in scientific texts, is one that combines two or more of the above

functions (Cook & Mayer, 1988; Hunter, Crismore & Pearson, 1987; Iding, 2000; Lowe, 1993; Winn, 1989). For example, a diagram of a scaled map can provide more than one view, for example, an additional cross-section or enlarged view (i.e., comparison). Aspects of the map can be labelled (i.e., identification) and the pathway to follow can be indicated via the use of arrows (i.e., sequence).

Accuracy of diagrams. Diagrams contained in curriculum material, with the purpose of instruction rather than decoration, need to be evaluated as to how well they address instructional criteria. Instructional criteria can be arranged in seven broad principles (see below), and are consistent with effective mathematics learning and teaching found in the National Council of Teachers of Mathematics current standards (NCTM, 2000).

1. Diagrams should help students to *recall knowledge* and *skills*, and *make connections* between prior knowledge and new situations (Charles & Nason, 2001; Derrori & Lemut, 1995; Gentner, 1982; Janvier, Girardon, & Morand, 1992; Kidman, 2000, 2001; Kidman & Nason, 2000; Lowe, 1993).
2. Diagrams should *introduce terms* and *procedures*; *represent ideas accurately*, and *demonstrate/model procedures* (Gentner, 1982; Kidman & Nason, 2000).
3. Diagrams should provide *conceptual links* within the representation, and allow the learner to *abstract and understand the important notion* underlying the diagram (Fish & Scrivener, 1990; Kidman & Nason, 2000).
4. Diagrams should *reduce the working memory demands* of the problem solving process (Fish & Scrivener, 1990; Kidman & Nason, 2000).
5. Diagrams should *allow for exploration* of ideas and understandings not possible *from natural language* (Lemke, 1999; Kidman, 2001; Kidman & Nason, 2000).
6. Diagrams should contain elements of natural language *facilitating links between* scientific and mathematical *expressions and natural language* (Lemke, 1999).
7. Diagrams should allow the learner to *interpret underlying scientific and mathematical notions*, and allow the learner to participate in an *expressive learning* activity (Gordin, Edelson, & Pea, 1996; Kidman & Nason, 2000).

DATA ANALYSIS

Data comprised the diagrams. The relations between the diagram and the text, and role of the diagram were considered. The analysis of accuracy of a diagram was informed by a set of seven principles (above) for analysing diagrams within mathematical curriculum materials (Kidman & Nason, 2000).

RESULTS AND DISCUSSION

Two examples are presented for discussion. One example is drawn from an Integrated Learning System (ILS) (Computer Curriculum Corporation, 1996) (a collection of electronic worksheets, divided into a range of strands (e.g., fractions)). 550 instructional diagrams were sampled from the ILS and evaluated. 220 of these diagrams were found to be deficient in at least five of the seven principles. A second example is drawn from a Year 5 student workbook (Boswell, 1998). Of the 501 instructional diagrams found in the workbook, 38 were found to be deficient in at least five of the seven principles. Both the ILS and workbook series are found in many classrooms throughout Queensland, Australia.

In order to illustrate the nature of the diagrams found in the ILS and workbook, a detailed analysis of the two diagrams is now provided. The deficiencies identified in the ILS example are typical of those found throughout the 220 diagrams (e.g., font problems, unclear purposes and layouts, forced formatting and poor links between natural language and visual representations) as is the case for the workbook.

The diagram presented in Figure 1 comes from the repertoire of exercises contained in the 'Fraction' strand of the ILS. The relation between the diagram and text in Figure 1 is that of elaboration. The diagram partially repeats text information in the form of the triangle having a base (b) and height (h), and partly adds the information that height is perpendicular height. The role of Diagram 1 is one of identification as it identifies where the base and height are to be found on the triangle. The intended skill of this exercise is the multiplication of fractions (see upper left hand corner of Fig. 1: fr840 = Fractions, Year 8, 40 percent of the way through Year 8).

The diagram presented in Figure 2 comes from the exercises contained in the 'Number' strand of the workbook. The relation between the diagram and text in Figure 2 is that of embellishment. The diagram provides completely new information not covered in the text. The role of Diagram 2 is one of identification as it identifies the spatial relationships between five cities. The intended skill of the exercise associated with Figure 2 is subtraction "from four-digit numbers, with regrouping" (Boswell, 1998, p. 93). The text accompanying Figure 2 is "Here is a chart showing some airports and their distances from one another. Use this information to write three subtraction problems" (Boswell, 1998, p. 93). To the right of the diagram is a highly structured area for the student to write the three problems.

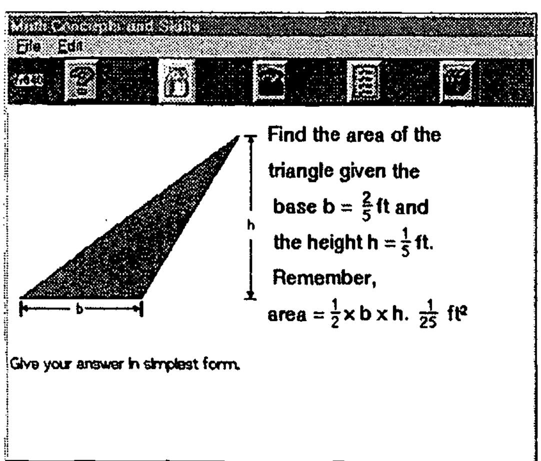

Figure 1. ILS Example

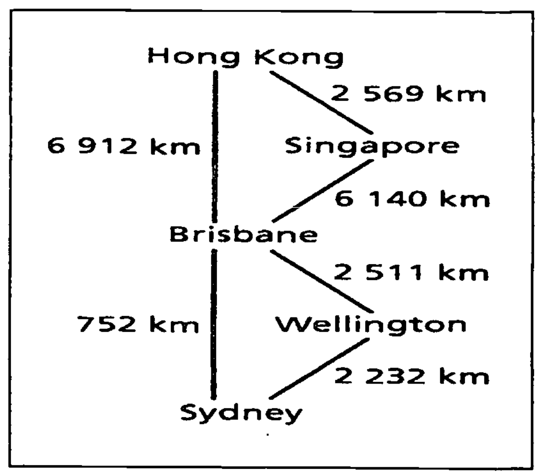

Figure 2. Workbook Example

The analysis of the diagrams in Figures 1 and 2 are presented in Table 1. Both of the diagrams analysed in this paper are clear and relatively uncluttered. However, neither of the two diagrams facilitates meaning-making.

Table 1 Analysis of Figures 1 & 2

Principle	Compliance	Commentary
1	Low	The font and the layout of both diagrams are clear to read. A major problem found in both diagrams making understanding difficult is that neither diagram is drawn to scale. Understanding of the diagram in Figure 1 is hindered by the text leading the reader to believe the exercise is on area measurement and not on fraction concepts. Understanding of the diagram in Figure 2 is hindered as the reader may be led to believe that a distance of 6912 km is the same as 752 km on the chart.
2	Low	The diagram In Figure 1 does not relate the area of the triangle to the notion of the unit (i.e., a 1x1 square), so it does not enable students to focus on deep structural knowledge. It does not encourage the use of intuitive knowledge about fractions. The diagram in Figure 2 does not relate the notion of the different distances between cities and geographical locations, so it does not enable students to focus on deep structural knowledge. It also does not encourage the use of intuitive knowledge about subtraction with regrouping. Neither diagram encourages spatial awareness.
3	Low	An important notion in Figure 1 that needs to be abstracted from the diagram is that 1/2 x 2/5 x 1/5 is equivalent to 1/25 because if two fifths is halved, then a fifth is generated; and if a fifth of a fifth is then found, then a twenty fifth is generated. Because the diagram does not relate the area of the triangle to this, the "Correct" answer is not conceptually linked back to diagram. An important notion in Figure 2 that needs to be abstracted from the diagram is that in subtraction algorithms, the bottom number needs to be subtracted from the top number, even when at first sight it cannot be done. Because the diagram does not relate this notion, the "Correct" answers written by the child are not conceptually linked back to diagram. An adequate environment for learners to abstract and understand this notion is not provided in either diagram.
4	Low (Fig 1) High (Fig 2)	Students are unable to add notes to the diagram in Figure 1. Unless they are instructed to make notes on paper much information has to be memorised potentially overloading the working memory capacity. Students are able to add notes to the diagram in Figure 2, and a highly structured area is supplied for the student's working. Very little information has to be memorised, freeing up the working memory capacity.
5	Low	It is very difficult to adequately represent, in natural language, the important notions noted in Principle 3 above for both diagrams. Neither diagram enables students to explore these notions. Therefore, students probably will not construct the iconic

		understandings of the relationship between 1 whole, 1/5, and 1/25 from Diagram 1, nor understandings of subtraction with regrouping from Diagram 2.
6	Low	The diagram in Figure 1 contains no natural language, and because the diagram in Figure 2 contains only a little natural language (but as it is not to scale - even remotely), neither facilitate links to natural language.
7	Low	The diagrams allow for some interpretation but because of the lack of compliance with the majority of the above Principles, only low levels of interpretation can occur. The diagrams cannot be used for expressive learning activity.

For example, neither diagram:

1. highlights the relationships between the problem information or prior knowledge and skills;
2. enables the reader to focus beyond the surface level aspects of the task., and
3. provide students with the means to construct a deeper appreciation of the concepts beyond that which can be achieved through the semantics of natural language.

Because of these limitations, it is highly unlikely that either of the visual representations would do much to facilitate the construction of deep-level, principled knowledge about fractions or numeration.

However, the effects of these limitations may be more serious than this. The covert geometry curriculum presented in both these diagrams is not given the attention it deserves. The barriers to the development of spatial intuition created by these two diagrams are of concern due to the lack of attention to scale. The development of spatial awareness is informal. “It is the use of space, shape and form at an intuitive, personal and unstructured level such as interpreting a map” (Booker, Bond, Briggs, & Davey, 1997, p. 270). Geometry allows many ideas to be pictured and thus, facilitates problem-solving, therefore placing geometry in a unique position in relation to other branches of mathematics.

Both diagrams unnecessarily lack an attention to scale. In Figure 1, the base of the triangle ought to be twice that of the height. This is not even approximated on the diagram. Figure 2 is a map supposedly showing airports. It actually shows only two airports (Brisbane and Wellington) and three cities (the names of these three cities airports differ to their city names). The map indicates that Wellington is somewhere between Brisbane and Sydney. This is not correct, it is geographically south east of Sydney, in a different country to Sydney! The lack of attention to these details, and many others in Figure 2 are inexcusable. Present curriculum documents like those produced by the National Council of Teachers of Mathematics (2000) call for integrated links between mathematical topics and concepts. This is clearly not being done in the diagrams assessed in this and other studies (Kidman, 2000; Kidman & Nason, 2000).

CONCLUSION

The findings from this study and the previous studies by Kidman and Nason indicate that instructional diagrams are not facilitating mean-making. The studies have shown that a significant proportion of diagrams are not facilitating the construction of mathematical knowledge. This study has shown that while the diagrams may be attractive, and possibly attract and maintain a student's attention it is highly unlikely that either of the diagrams would do much to facilitate the construction of deep-level, principled knowledge about fractions or numeration. The lack of attention to scale in both diagrams is a problem for the development of spatial awareness.

The National Council of Teachers of Mathematics (2000) has called for a coherent curriculum where mathematical ideas are linked and built on one another. It argued that this would facilitate understanding, deepen knowledge and expand application. It is evident that this coherence is not present in these diagrams.

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CONCEPT DEFINITION, CONCEPT IMAGE, AND THE NOTION OF INFINITE SUM IN OLD AND NEW ENVIRONMENTS

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The literature is rich in studies on the conceptual difficulties with the notion of limit of a sequence of numbers and infinite sum of numbers. In this study we analyzed the students' perceptions of the infinite sum of functions. Two different approaches were used: analytical and algebraic. In the first, the infinite sum is represented as a process and in the second, as an object. Mathematica software permits multiple representations of the same concept. The students used animation to focus on the "process itself" and not on the divergent process of adding terms to the sequence. Symbolic manipulations were utilized to create the illusion of an infinite number of calculations performed at once. The infinite sum as an object appeared clearer. However, not all the conceptual difficulties were resolved.

Students' conceptual difficulties in learning the notion of infinite sum are well documented. A number of double strands are observed when students try to understand the concept of infinite sum: the infinite sum as a *process* or as an *object*, the intuition of the infinite process as a *potentially infinite* process or as an *actual infinite* sum, and the reading of the equality $S = a_0 + a_1 + ... + a_n + ...$ from left to right or from right to left, which is cognitively different.

Herein, we describe a study that examined the effect of introducing the concept of infinite sum of functions using two different approaches in which those double strands are present. The research was conducted in the context of an experimental course on Approximation theory that was given to eleventh grade high school students. The concept of infinite sum was introduced by developing a function in infinite series. In this way an infinite decimal is obtained by substituting a particular value of x. Mathematica software was used. In particular, we utilized the software's ability to support multiple representations of the same mathematical concept.

Multiple representations were applied in two interpretations:

In the first, two different mathematical approaches were used, analytical and algebraic, the first representing the infinite sum as a *process,* a *potentially infinite* process, and the second as an *object,* an *actual infinite* sum. In the second interpretation, each of these two approaches to the infinite sum was investigated by multiple attributes of the Computer Algebra System: symbolic computation, graphics, and animation.

The use of animation in teaching the limit concept is discussed in Kidron et al. (2001). Animation gives the illusion of completing an *ongoing* process.

This article describes some efforts to partially overcome the difficulties that were pointed out by Núñez (1994). Núñez explained some difficulties concerning the concept of infinite sum by the fact that two types of iterations, of perhaps a different nature, are involved simultaneously: *the process itself* and *the divergent process of adding terms to the sequence* (an increasing number of steps).

We utilized Mathematica's capabilities in a way that reduced the two-dimensional analysis, which stems from the two types of iterations, to a one-dimensional analysis.

In the analytical approach, in which the infinite sum is represented as a *potentially infinite* process, the use of animation might help in focusing on *the process itself* and in differentiating it from *the divergent process of adding terms to the sequence.* In the algebraic approach, in which the actual infinite sum is represented as an object, the software creates the illusion of symbolic manipulation taking place on the actual infinity of terms. The illusion that the infinite calculations are performed at once might facilitate the transition from symbolic manipulation to a symbolic object.

Our study examined to what extent the interrelationship between the two approaches and the different attributes of the Computer Algebra System helped the students to perceive the infinite sum as a limit of the infinite process. We also investigated the readiness of the students to grasp the formal definition of infinite sum.

CONCEPTUAL DIFFICULTIES WITH INFINITE SUMS

The "finitist" character of our intellectual schemes might cause difficulties when we deal with the notion of Infinite Sum. Fischbein et al. (1979) observed that the natural concept of Infinity is in fact the concept of "Potential Infinity", which is simply a process that goes on without end, like counting without stopping. Lakoff & Núñez (2000) suggested that metaphorical thought might be necessary to conceptualize another infinity concept: the "Actual Infinity".

The intuition of infinity might become an obstacle to learning the formal definition of the concepts related to infinity (Cornu, 1991). Vinner & Hershkowitz (1980) introduced the terms "concept image" and "concept definition". The term "concept image" describes the *total cognitive structure that is associated with the concept which includes all the mental pictures and associated properties and processes.* The term " concept definition" is defined *as a form of words used to specify that concept* (Tall & Vinner, 1981). The concept definition of Infinite Sum is not necessarily linked to the concept image. There may be a gap between the mathematical definition of the concept and the way one perceives it.

Symbolic notation is another source of conceptual difficulty. The symbol $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ represents both the process of n tending to infinity and the concept of limit. It is a *procept* (Gray & Tall, 1994). Another source of cognitive complexity is embedded in the symbolic notation "=" in $S = a_0 + a_1 + \ldots + a_n + \ldots$. An understanding of what "=" means requires a cognitive analysis of the mathematical ideas involved (Tall, 2000).

STUDENTS' PERCEPTIONS OF INFINITE DECIMALS IN PREVIOUS STUDIES

The following perceptions of infinite decimals were observed:

Infinite sums exist only in theory (Kidron, 1984). *The infinite decimal is perceived as one of its finite approximations* (Kidron & Vinner, 1983). *The infinite decimal is perceived as a process not as a product* (Kidron & Vinner, 1983; Monaghan, 1986). The students also expressed a generic concept of *measuring infinity:* 1+1/2+1/4+1/8+...is stated to be $s = 2 - 1/\infty$ because "there is no end to the sum of segments" (Fischbein et al.,1979). In Kidron (1984), the students in comparing $0.\dot{9}$ and 1, claimed that $0.\dot{0}1$ is a legitimate number that is equal to $1-0.\dot{9}$. The existence of a conflict in comparing $0.\dot{9}$ and 1 and the path dependence of the decisions are mentioned by Tall (1976).

The following perceptions of infinite decimals were observed in some computer environments: Monaghan et al. (1994) pointed out difficulties that result from attempts to use the computer for learning the concept of limits. Nevertheless, the limit sum as an object appeared clearer. Sacristan (2001) presented some results demonstrating that in exploring different types of representations, including visual ones, the students gradually understand how a process can continue infinitely and not grow to be infinite. Tall (2000) indicated that reading the equality $a_0 + a_1 + ... + a_n + ... = S$ from left to right or from right to left can relate to ideas that are cognitively different. For example, most of the students regarded 0.1+0.01+0.001+...= 1/9 to be false but 1/9 = 0.1+0.01+0.001+... to be true. The students were interviewed and as a possible explanation, Tall suggested that when reading from left to right, the first statement seems to represent a potentially infinite process that can never be completed, whereas the second shows how 1/9 can be divided out to get as many terms as are required. Tall also noted that the students were reluctant to accept a formal definition of infinite sum that does not agree with their personal experience.

THE TEACHING EXPERIMENT

High school students learning at the highest level (grade 11, N = 63) participated in the research. The laboratory consisted of 20 PCs, each equipped with Mathematica software. The author taught the students six hours a week; two of the six hours in the PC lab were devoted to topics in Approximation theory.

The concept of infinite sum was introduced from two different points of view. The algebraic approach represents the polynomials with "an infinite number of terms" as an *object*. The analytical approach describes the *process* of the different polynomials approaching a given function. We pointed out the importance of interacting with each of the different representations. Moreover, we emphasized the importance of establishing links between the different types of representations. The **algebraic approach** is Euler's intuitive idea of expressing non polynomial functions as "infinite polynomials"; it relates to the concept of Infinity as Actual infinity. Euler's approach is from left to right: you start from a given function f(x) = P(x)/Q(x) and you seek a "polynomial with an infinite number of terms" such that $f(x) = P(x)/Q(x) = a_0 + a_1x + a_2x^2 + ...$ The students used Mathematica to *follow the original text of Euler* (translated into English in Euler, 1988). Following Euler's instructions, the students carried out the "continued division procedure": they assumed that there is an infinite series such that $P(x)/Q(x) = a_0 + a_1x + a_2x^2 + ...$ and then calculated the coefficients $a_0, a_1, a_2, ...$ that satisfy the equality by multiplying both sides of the equality by Q(x) and comparing the coefficients as Euler did.

Using the **analytical approach** the students plotted two functions, f(x) and g(x), whose formulas were different but whose plots were similar, near x = 0. The notion of the order of contact was introduced. The two curves, y = f(x) and y = g(x), have an order of contact n at x = 0 if: f(0) = g(0) , f '(0) = g'(0), f''(0) = g''(0), $\ldots, f^{(n)}(0) = g^{(n)}(0)$. As an application, the students were asked to find the polynomials of degree 2, 3, 4, ... that have the highest possible order of contact with a given function at x = 0. Mathematica helped the students solve the relevant system of equations. Using the analytical approach, the students calculated polynomials with a *finite number of terms,* the Taylor polynomials, which approach a given function. Acting with this representation, they observed that when the degree n of the approximating polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ increases, the approximation is better. The students made graphical representations of their results. By means of animation (Kidron, 2000, example 1) they were asked to "encapsulate" the *process* into an *object* (Dubinsky, 1991). The animation permitted them to see the dynamic process in one picture. Animating the approximating polynomials $a_0 + a_1x,\ a_0 + a_1x + a_2x^2, .., a_0 + a_1x + a_2x^2 + ... + a_nx^n, ..$ approaching the given function f(x), may help in identifying the equality $a_0 + a_1x + a_2x^2 + ... + a_nx^n + ... = f(x)$. In both representations the students performed algebraic manipulations of the two sides of the equality in order to find the unknowns $a_0, a_1, a_2, ...$ which are the coefficients of the Taylor series.

DATA ANALYSIS AND RESEARCH METHODOLOGY

Each year, one class (grade 11) participated in the experimental course throughout the academic year. The course was given three times. All together, 3 grade 11 classes participated in the experiment. We noted the students' questions and remarks during the sessions, and collected the Mathematica files of their examples. We gave the students a questionnaire that was designed to elicit their conceptions of infinite sum. The same questionnaire was given each year. In the second and third years, a full session was dedicated to the formal definition of the infinite sum of functions as the limit of an infinite sequence of partial sums. We present here a class discussion and some findings from the questionnaire in different years.

The class discussion When we replace a given function f(x) by its Taylor polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, we just leave off the "infinite tail". Some students had problems with this "infinite tail".

Dina: How could we speak about a graph that describes the error $|f(x) - P_n(x)|$? This difference is the "infinite tail": $a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + ...$ How could this difference be well defined?

A similar reaction is shown in the following class discussion. The teacher proved Taylor's theorem at x = 0. In one step of the proof, in order to calculate the derivative of a certain term, she mentioned that the error term $d = f(x_0) - (a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_n x_0^n)$ is a constant.

Julia: Why is d a constant?

Teacher: d is the difference between two constants when we approximate f(x) with a polynomial with a given exponent n at a given point.

Julia: But d is the "infinite tail"... how could it be a constant?

Tomer: We compute the error for a given n.

Julia: But $d = a_{n+1} x_0^{n+1} + a_{n+2} x_0^{n+2} + ...$ so how could it be that d is a constant?

Tomer: $d = a_{n+1} x_0^{n+1} + a_{n+2} x_0^{n+2} + ...$ is an infinite sum that is equal to a given number.

Ron: Yes! For example, you have 1/2+1/4+1/8+1/16+1/32+...which equals 1.

Julia: Will it not be bigger than 1 when we continue to add terms?

Dan: In the example 1/2+1/4+1/8+1/16+1/32+...the infinite sum is a defined number but there are other examples in which the infinite sum is not a given number. It tends to ∞!

Adi: So, how could we know if $d = a_{n+1} x_0^{n+1} + a_{n+2} x_0^{n+2} + ...$ is an infinite sum which is equal to a given number?

Yifat: In the last lab we have seen animations showing that when $n \to \infty$, the expression $f(x) - (a_0 + a_1 x + a_2 x^2 + ... + a_n x^n)$ tends to 0. Therefore the expression $d = a_{n+1} x_0^{n+1} + a_{n+2} x_0^{n+2} + ...$ is a given number and not an expression which tends to ∞.

The questionnaire We report on responses to three of the questions.

Question 1 deals with the meaning of the statement:

$$1/(1-x) = 1 + x + x^2 + x^3 + ... + x^k + ... \quad \text{for } -1 < x < 1.$$

We observed three categories of perceptions of the infinite sum of functions:

I The infinite number of terms is perceived as a very large but finite number of terms: **the infinite sum is perceived as a finite approximation.**

II **The infinite sum is perceived as a process and not as a product**; it is perceived as an ongoing process that passes through an infinite number of terms: $1,\ 1 + x,\ 1 + x + x^2,\ 1 + x + x^2 + x^3, ...$

III **The infinite sum is perceived as a limit**, as the product of the infinite process.

	First year N=22	Second year N=22	Third year N=19
Category I	14%	9%	10.5%
Category II	46%	45.5%	26.5%
Category III	36%	45.5%	47.5%

Table 1: Distribution of the categories of perceptions in question 1

Question 2 Complete: $1+1/2+1/4+1/8+.....+1/2^K+...=$

Our aim was to check the students' perception of the infinite sum of numbers as a limit.

	First year	Second year	Third year
$1+1/2+1/4+1/8+.....+1/2^K+...=2$	50%	57%	61%

Table 2: Distribution of answers to question 2

In the first year, no one linked question 2 to the expansion of 1/(1-x) in the power series in question 1. In the second year, 33% of the 57% linked question 2 to the expansion $1/(1-x) = 1+x+x^2+x^3+.....$ (in the third year, 36% of the 61%). These students indicated that they could look at the process from the opposite side, from right to left, $1+x+x^2+x^3+.....= 1/(1-x)$ and then substitute x= ½. This ability did not eliminate the students' difficulties in perceiving the infinite process as an object:

Daniel: As a consequence of the expression $1/(1-x) = 1+x+x^2+x^3+.....$, $1+1/2+1/4+1/8+.....+1/2^K+...$ is supposed to be equal to 2 if $K = \infty$, but since there is no such thing 'infinite' the expansion tends to 2.

Question 3 Find: $\lim_{n\to\infty}(9/10+9/10^2+...+9/10^n)$. Is $0.\dot{9}$ equal to 1 or less than 1? Justify your answer.

	First year	Second year	Third year
$\lim_{n\to\infty}(9/10+9/10^2+...+9/10^n)=1$ and $0.\dot{9}=1$	14%	27%	32%
$\lim_{n\to\infty}(9/10+9/10^2+...+9/10^n)=1$ and $0.\dot{9}<1$	57%	41%	21%
$\lim_{n\to\infty}(9/10+9/10^2+...+9/10^n)\to 1$ and $0.\dot{9}<1$	29%	32%	47%

Table 3: Distribution of answers to question 3

Some perceptions already seen in previous studies reappear:

Orit: The lim tends to 1, but will not reach it since infinite exists only in theory

Aviv: 0.999…is less than 1 because it always lacks 0.000….01 (an infinite number of 0) in order to be 1.

The legitimate $0.\dot{0}1$ re-emerges!

DISCUSSION AND CONCLUSIONS

The class discussion indicates that a conflict exists. The teacher presented the error term as the difference of two constants $d = f(x_0) - (a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_n x_0^n)$. Julia looked at the representation of d as an infinite sum $d = a_{n+1} x_0^{n+1} + ...$ She realized that the difference between two constants is a constant and yet she was not ready to accept the infinite tail as a constant. Julia looked at the infinite sum as an unending process. In the class discussion, the dynamic images produced by animation were present in the students' minds even when the computer was turned off. The animation enabled them to concentrate on *the process itself* (the error term tends to zero) and to differentiate it from *the divergent process of adding terms to the sequence*. It helped the students understand that an infinite sum is not necessarily an expression that tends to ∞, and that it could be equal to a given number. One of the purposes of the study was to investigate the students' ability to perceive the infinite sum as a limit, as an object. In our research we encountered some perceptions of the infinite sum that were observed in previous studies. However, there was also an indication that some students were beginning to see the infinite sum as an object (43% of all the students expressed their perception of the infinite sum of functions as a limit; 56% of all the students wrote $1 + 1/2 + 1/4 + 1/8 + ... + 1/2^K + ... = 2$). A possible explanation for this is that the students, looking at the same concept from different approaches, developed a more balanced view of the infinite sum as process and concept. We quote also Euler's reason for applying an algebraic approach to subjects that are usually discussed in analysis: *in order that the transition from finite analysis to analysis of the infinite might be rendered easier.* We also wanted to investigate the students' ability to grasp the formal definition of an infinite sum. The percentages of answers where students perceived the notion of the infinite sum as a limit, as an object, were slightly higher in the second and third years (these students were given the formal definition of an infinite sum). It seemed that the influence of providing the formal definition was minor. However, this impression changed examining the students' explanations to their answer $1 + 1/2 + 1/4 + ... + 1/2^K + ... = 2$. Some of the students who received the formal definition were able to link between this question and the first question, where $1/(1-x) = 1 + x + x^2 + x^3 + ... + x^k + ...$ for $-1 < x < 1$. In order to do so, they had to look at the equality from the other side (from right to left) and to replace ½. Yet, Daniel's answer to question 2 illustrates the fact that the flexibility to read the equality from both sides does not eliminate the conceptual difficulties in *the transition from finite analysis to analysis of the Infinite*.

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FACILITATING STUDENT LEARNING THROUGH MATH JOURNALS

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Abstract. *In the last two decades, mathematics teachers have shown a great deal of interest in how students learn mathematics through journal writing. I have also used journals in my mathematics classes for the last five years. Based on the analysis of more than 1800 journal entries written by approximately 200 students in the last five years, I conclude that journal writing have potential to aid in student mathematical learning even though teachers need a large amount of time to examine student journals and provide feedback. Math journals not only help instructors in understanding students' feelings, likes, and dislikes about classes but also help students to demonstrate their mathematical thinking processes and understanding.*

Introduction and Theoretical Framework

The National Council of Teachers of Mathematics (NCTM, 1989, 2000) and the American Mathematical Association of Two Year Colleges (AMATYC, 1995) emphasize that students should be able to communicate mathematically, both in written and oral forms, using mathematical vocabulary and notations. NCTM (2000) argues that "reflection and communication are intertwined processes in mathematics learning" (p. 61) and writing in mathematics provides opportunity for students to express their thinking. Teachers also benefit from student writing by getting access to student thinking, which can be used to improve instruction. Following these recommendations, many teachers have used journals as a regular feature of their school and college mathematics classrooms and report that journal writing helps students reflect and learn math concepts (Burns & Silbey, 2001; Chapman, 1996; Dougherty, 1996; McIntosh & Draper, 2001; Pugalee, DiBiase, & Wood, 1999).

The use of journal writing in a mathematics classroom is not only being supported from the experience of classroom teachers but also from research. Qualitative researchers working from a constructivist framework often use journals to inquire and understand participants' mode of thinking (Kroll & LaBoskey, 1996; Mewborn, 1999; Miller, 1992; Pugalee, 2001). Because the construction of knowledge is a continuous process, it must be regularly communicated and reflected. Mewborn uses journals as tools for reflection. Miller points out to the benefits that teachers get from reading students' writing. Pugalee claims that writing is a necessary part of metacognitive thinking, which helps in constructing mathematical knowledge. The present study focused on the role of journals in mathematics content course designed for prospective elementary school teachers. The main purpose of the study was to explore the ways in which math journals influence the teaching of mathematics to prospective teachers. Consistent with previous studies, this study claims that math journals help preservice teachers in their learning of mathematics as they express their thinking and get feedback from the instructor. Furthermore the instructor of the course consistently valued student math journals as they provided helpful hints on how to modify and improve classroom instruction.

Research Methods

The setting for this study was the course "Number Systems," which is a required mathematics content course for prospective elementary school teachers at a North Eastern university in the United States. Most of the students in this course intend to become future elementary school teachers. The main goal of the course has been to encourage students to embrace the challenge of learning mathematics through inquiry and exploration. In the past five years of my teaching, the course has taken a problem-solving approach in which students were required to monitor their thinking process while solving mathematical problems. Students monitor their thinking through math journals, which is a necessary component of this course. So far more than 200 students have participated in journal writing.

In the first two years of teaching, the course required all students (approximately 100 students) to write weekly journals, 1 to 3 pages long. Each student wrote a total of twelve entries for the course during the semester, totaling approximately 1200 journal entries. Because of the large volume of journals in those two years, the frequency of journal writing was reduced to bi-weekly in the last three years, which generated approximately 600 journal entries. All 1800 entries were carefully analyzed in this study.

The students wrote journals in a variety of forms. In some entries they simply reflected on their excitement or frustration regarding their learning of a particular mathematics concept that was discussed in the classroom. For example, they might describe a new mathematics concept they learned that week or reflect on the confusion caused by a new concept. For other entries, the instructor provided them with specific questions or scenarios for response. For example, in one of the entries students were asked to respond to the following scenario developed by this researcher:

> Senator *X*, an expert mathematician, proposes a new number system to be adopted in the United States. She said that the new system will have only two numerals 0 and 1. According to her these two numerals can be used to represent any numbers of the present Hindu-Arabic numeration system. "If we can do all the work with two numerals, why bother using the Hindu-Arabic system with ten numerals", argued senator *X*. Other senators, however, are totally confused with the senator *X*'s proposal. Senator *X* further argued that adults are confused with the new system

because they have used the Hindu-Arabic system for a lifetime. "But look at the future of the millions of children who will learn only two numerals to represent all numbers," she contended.

The Senate is now interested in getting responses from public. Write a letter to the Senate describing your position on whether or not the new system should be adopted in the US. Remember that most of the senators have difficulty understanding this system and you should clearly describe what the system is and the advantages and disadvantages of the new system over the Hindu-Arabic system. Please provide examples to demonstrate your understanding of the new system in your letter.

Additionally some selected students were interviewed about their experience in journal writing. All the students interviewed valued the role of journals in their mathematical learning. Because of the space limitation, only the analysis of journals is provided in this paper.

Data Analysis

The researcher/instructor read all the responses from student journals and responded to them during teaching. In addition to reading and providing feedback to students, the researcher also coded, analyzed, and categorized each journal using qualitative data analysis methods, in particular, the constant comparative method (Guba & Lincoln, 1989) and the interactive model (Huberman & Miles, 1994). The researcher also made an overall reflection about student journals at the end of each semester.

Two kinds of questions were considered in the analysis. First, what were students' responses in a particular week's journal? For example, what were various responses provided by students to the entry related to Senator *X*'s proposal of the binary system of numeration? Second, how did journals helped students for their growth of mathematical understanding? These questions led the analysis mainly in two categories: Journals that only described students' attitudes and feelings and the journals that demonstrated student mathematical thinking. Because the constant comparative method of data analysis was used students' responses were checked for consistency and patterns.

Results and Discussion

In response to open topics related to course materials, more than 80 percent of the students simply chose to express their beliefs about mathematics. Although they were asked to demonstrate their mathematical understanding in their journals no such understanding was noted. They made statements like "I liked the last week's class because it was fun" or "I did not understand what you were trying to do in the last class." Moreover, the journals were rather brief and did not demonstrate any in-depth reflection from students. I continuously encouraged them to express their thinking and problem-solving methods in their journals. Sometimes, I asked them to read specific materials and at other times I asked them to reflect on particular concepts that we discussed in class. These responses encouraged them to think at a deeper level. As the course progressed, the students developed sufficient confidence to express their concerns, difficulties, frustrations, and excitement in their journals. They also began to identify mathematics concepts in their journals. Some of the examples from students are provided below.

> I wish to begin this journal by reflecting on my frustration that I have experienced during this first week of class. The two words "Problem Solving" frighten me, as I have never been successful at solving any math problems (word problems) ever.
>
> I was frustrated this week when we started discussing the binary system of numeration. I liked the Dienes blocks and Unifix blocks [cubes] that we were using and I understood the Base-10 number system (probably because I've been using it for years!), but the binary system was confusing to me when we were using the Unifix blocks. I wasn't sure what you meant by "trade" this one for that one.
>
> This week in math we did more fun things. The first thing we did was the chart to find your age. ... When we first did this I didn't understand how you were getting the answer. I thought that you were looking at every number in each of the columns but then I found out that you add all the numbers together from the top row to get the answer. ... I tried it on some of my friends and they enjoyed the game also.

While the information obtained from students' general entries were helpful to address their problems and concerns, it was difficult to explore their mathematical thinking. Students' mathematical thinking was more evident on specific journal entries, such as those in the senator *X*'s proposal even though there was still a significant proportion of students (about 25%) who provided very general response to this scenario. A student who was struggling to understand the materials in the course responded to the journal as follows:

> Having only two numerals, 0 and 1 would lead to much confusion. As an example telephone numbers, bank accounts, social security numbers etc., would be impossible to distinguish for all citizens in the United States.

This response does not provide good evidence that the student understood the binary system of numeration. It is so general that anyone can state this without actually understanding the system. Some other students, however, wrote what they understood. For example, one student responded like this.

> This is how the senator feels this new system should work. .000000001 would represent 1. After all the zero's are represented by 1, the next 1 will be represented on the other side of the decimal point. This is what the system would like:

.000000000=0
.000000001=1
.000000011=2
.000000111=3
... ...
.111111111=9
1.000000000=10
11.000000000=20 and so on.

The student knew that the binary system of numeration has only two numerals 0 and 1, but did not know how these numerals can be combined to write numbers. The student was simply adding the digits to represent the base-10 numbers. But what was important in the journal was that the student clearly communicated what she understood about the system.

There were some students who clearly demonstrated their understanding. For example, one student responded to the proposal as follows:

> Ladies and gentleman of the Senate, this new system of numbers is very disturbing to me. Let us say we want to call someone. Their phone number is currently 555-2233. To fit this new system we will have to change the number to 101 101 101-10 10 11 11. What kind of phone number is that? Are you really going to remember that kind of a number? It's hard enough to remember a 7-digit code not including the area code. It would be impossible to remember that number.... How old are you? Not 33 anymore. Now the person would be 100001 years old. I am sorry but there is no need to have an age that looks like that. Imagine trying to write that on job applications. If you live at 5 Main street, you now live at 101 Main street. Whoever lives at 101 Main street now lives at 1100101 Main street. Does anyone want to write that for an address? ... Ever buy a computer? The one I am using costs about 1899.00 dollars. In the new system it would be approximately $11101101011. How can we change all of our monetary units to match this system?

This student clearly understood how to convert the base-10 numbers to the base-2 and wrote the journal in a powerful way. The journal was particularly powerful because all the examples chosen are from real life contexts and there are no mathematical errors in the journal. There were other journals in which students demonstrated an excellent understanding of the binary system. For example, let us look at an excerpt from one student below.

> Senator *X* proposes changing our current numeration system (Hindu-Arabic), which is base 10 to a Base 2 system, in which the only numerals used would be 0 and 1. Let me first identify some properties of this Binary System.

1. Remember that all values can be represented, but numbers other than 0 and 1 do not exist as symbols.
2. Hence, the first 10 numbers in the Binary System that Senator X proposes would be represented as: 0, 1, 10, 11, 100, 101, 110, 111, 1000, 1101.
3. For your understanding let's use our (Base 10) number 27 as an example. Instead of a two digit number to represent the value of 27, the binary representation would be 11011! Imagine the length of the number used to represent larger values!

This student demonstrated that she understood not only how to convert the base-10 numbers to the base-2 but also how to perform operations such as addition and subtraction in the new system. The journals that were more specific like the above ones allowed the instructor to explore students' understanding of mathematical concepts required for the course. Moreover, students themselves found that journals were powerful tools for them to express their concerns. As one student commented, "I like the journals because they give me time to think and reflect about what I learned and ask questions as they come up, and not have to wait until class-time." Another student noted:

> I've never really liked math before, but I did in this class. I feel like I had a chance to go back to basics, and find out why everything works the way it does. All through my schooling in mathematics, teachers have said, do this, do this, do that, and never really given a reason, how we know why things really work. Now we know why things really work. I really loved all of the specific journals we did. They really made us think.

The case was similar when students had to reflect on why minus times minus is plus. Many students really had a hard time providing a logical explanation. They knew the rules but they did not know how the rules worked. Following are some typical responses:

> I must be honest about this journal entry. I've never encountered this question. My teachers always told me the facts, such as positive plus positive equals positive or negative times negative equals positive, but never why. I was told to memorize the equations but never understood why. "That's just the way it is;" that's what I was told.

> For this week's journal our assignment is to explain why a negative multiplied by a negative equals a positive number. I had a very hard time finding a way to explain why two negatives multiplied together equal a positive because I never questioned why this was true.

> It is very strange how we are taught things in school and we just think that this is how something is done without even asking why. I'm sure this occurs in many other

subjects besides math and I think maybe we should ask more questions, and be more critical and make people explain what they mean.

The above responses made it clear to me that many of our students who intend to be elementary school teachers in future never had the opportunity to learn mathematics in a meaningful way. They were taught to accept rules without asking why those rules worked. The responses from students provided me with an opportunity to discuss the importance of understanding mathematics conceptually.

Conclusions and Implications

The use of journals in this course helped the instructor improve his teaching of mathematics in two ways. First math journals were effective in soliciting students' mathematical thinking, both cognitive and affective. In this case, math journals were serving as tools for thinking. Second, the instructor was able to understand students' mathematical thinking and was able to respond to their concerns. As a result of this interaction, students' mathematical understanding was improved.

Despite the roles journals played in students' mathematical growth of understanding, it was clear that only journals entries such as senator *X*'s proposal solicited student's specific mathematical thinking. The non-specific entries provided information about students' beliefs and attitudes, which were also helpful for the course. Because of these positive atmosphere created by journals, the author of this paper believes that journals are important tools for student growth of mathematical understanding despite the fact that the use of journals demands substantial amount of time on the part of the instructor.

The findings of this study are similar to those found by Miller (1992) and Pugalle (2001) that journal writing is useful in mathematics class to improve instruction and student learning. In spite of substantial amount of time I spent in examining student journals, I find that they are powerful means of communication and reflection. It is refreshing that many students who hesitate to speak in the class express themselves well in their journals. This study implies that writing about mathematics improves student mathematical reasoning. This is an important area that needs further exploration by researchers who are interested in student thinking.

Another implication of this study is that student responses in journals depend mostly on the tasks assigned. It the tasks are general such as "write what you think" the students' responses are also bound to be general, in which it is hard to find evidence of their mathematical thinking. These general responses are usually affective, which are related to student feelings and attitudes. Although these affective domains contribute to teaching, there needs to be some specific mathematics tasks such as Senator *X*'s proposal to generate specific mathematical thinking from students.

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MISUNDERSTANDING OF VARIABLES BY STUDENTS IN AN ADVANCED COURSE

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The paper proposes a theoretical framework to analyse the understanding of the roles of literal symbols in algebraic tasks basing it on the distinction between free and bound variables. This approach is applied to study the difficulties of advanced high school students with the notion of the parametric representation of a plane. It is shown that algebra courses tend to form a limited understanding of the notion of variable which creates an obstacle for a learner in advanced courses.

INTRODUCTION

The students' difficulties with the mathematical language has been the traditional concern in the introductory algebra courses. Some time ago the issue has also got an attention in the field of advanced mathematical thinking: for example in relation to the teaching of linear algebra. It was shown, for instance, that students do not interpret rightly the formulas of linear algebra and do not have for the latter appropriate set-theoretical meanings (Dorier, Robert, Robinet & Rogalski, 2000). Other researchers had also suggested that studying linear algebra requires from a student a good understanding of the mathematical syntax and use of variables (Sierpinska & Nnadozie, 2001).

There are enough signs, however, that many of the difficulties with understanding the mathematical language originate as early as at school and are not overcome by the students in their passage from school to college (Ursini & Trigueros, 1997). It was repeatedly shown, for example, that an average high school student doesn't realise the difference between unknown, parameter and functional variable. Some studies had related this obstacle to the historical development of algebra and deficiencies of the dominating structural approach in teaching of elementary algebra (Sfard & Linchevski, 1994), whereas others, to inherently complex propositional nature of algebraic tasks, especially when they contain parameters (Bloedy-Vinner, 1994). At this moment there is a growing agreement that the 'algebraic sense' developed by a learner may be to a large extent equated with his or her mastery of the concept of variable. Ursini and Trigueros suggested that a learner must understand the three main uses of variable: as unknown, as a general number, and in functional relationships (Ursini & Trigueros, 1997). Bills showed in a recent study that there are many problem solving situations which require from a student not only to recognise different uses of literal symbols, but to shift attention from one to another meaning of the same symbol in order to succeed with a task (Bills, 2001). The latter, like other studies, used its own, though more extensive than others, list of the roles of literal symbols in the algebraic tasks. This leads to the following methodologically important questions: To what extent do the lists proposed by different studies correspond to each other? What is their relation to the mathematical status of literal symbols as defined in mathematical logic? And can the research on algebraic thinking draw on it to describe the roles of literal symbols a more unified and simple way?

The author believes in a positive answer to the last question and suggests as a guiding principle the distinction between free and bound variables used in mathematical logic. This theoretical framework is then used to describe the difficulty of advanced high school students with such notion of linear algebra as the parametric representation of a plane. It is shown that their difficulty follows from the limited understanding of the notion of variable. The paper ends with the discussion of the historical and didactical aspects related to the teaching the notion of variable in introductory and advanced algebra courses.

THEORETICAL FRAMEWORK

Whereas there are numerous terms used by researchers for the roles of letters in algebraic expressions, in mathematical logic a literal symbol which doesn't denote a mathematical constant is either free or bound variable. This useful distinction was introduced in mathematics by Peano and has become the standard approach in any formal analysis of the mathematical language (Taylor, 1999). According to this distinction, the free variable in an expression stands for any one unspecified element of a set, while the bound variable denotes collectively all the elements of a set. The bound variables are used in such mathematical expressions as sums, integrals, limits, equations of curves, etc. An assignment of a certain value to the bound variable is irrelevant procedure, as far as the corresponding mathematical entity is defined by all the possible values of that variable. When variable in an expression is treated as free, it may, on the opposite, be assigned any value, and the resulting expression is treated as a particular case of the initial generic form. The main experiences with free variables in algebra courses are related to their use as generalised numbers and to equations. Finding among the values of a free variable those which satisfy an equation, corresponds to solving the equation in relation to this variable, which is then called unknown. If the equation contains besides the unknown an additional free variable, then the latter is called the parameter for this equation. The parameters may also appear in the formula of a function in addition to the independent variable, which has the status of the bound variable. The status of variable in a certain expression may depend on the goal of a task and may be changed in the course of the task. For example, a parameters of an equation or function may be 'binded' when one is interested to consider instead of one equation or function corresponding to an arbitrary value of a parameter, the properties of the entire family of equations or functions corresponding to all the possible values of the parameters. Another important example is the introductory course of linear algebra: binding of variables in an arbitrary linear combination produces a subspace. Many additional examples may be found in the theory of groups and functional analysis. On the school level more often happens 'unbinding' of variables: for example, in evaluating functions for certain values of independent variable or in finding certain points on a curve.

METHOD

The reported here study is a part of a larger project evaluating the difficulties of the Israeli 12th grade students with their regular vector geometry course. The sample consisted of 10 classes (N=214) of most advanced students (top 10% of the student population which pass the mathematics matriculation exam on the highest, '5 units', level) chosen from 5 academically high standing schools. The teachers, who volunteered to participate in the study, all had at least the second academic degree and included also the Heads of the mathematics departments of these schools. The purpose of the described below questionnaire was to test the students' general understanding of the parametric representation of a plane which had been one of the core notions of the course. The teachers were acquainted with the questionnaire in advance and recommended to deliver it in their classes in the most appropriate moment after the notion of the parametric representation would be well mastered by the students. Some of the students were interviewed after they had completed the questionnaires, and it allowed to clarify their written answers and the meanings they hold about the parametric representation. Another way to get a better insight into the students understanding of this notion was to suggest the same questionnaire to the academics from the mathematics and theoretical physics departments and analyse the differences between their and the students' approaches. Due to the space limitations, the results for only two of the items of the following questionnaire will be presented.

> Let $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$ be the parametric representation of a plane. Find out for each of the following two representations whether they may represent the plane identical to that of the given representation. Explain your answers!
>
> 1. $\underline{x} = \underline{a} + t\underline{u} + t\underline{v}$ 2. $\underline{x} = \underline{a} - t\underline{u} + s\underline{v}$

RESULTS

Though the first two items were not expected to present any difficulty, they revealed a serious misunderstanding by the students of the role of the parameters in the parametric representation of a plane. Only 32% of the students in the sample had answered correctly that the first item didn't represent the same plane, whereas the rest wrongly claimed that the item could be a representation of the plane identical to that of the given representation. These students treated the expression $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$ as a symbolic name of a plane which is completely specified, if each of the lettered symbols were assigned a certain value. It included the parameters t and s which were understood by the students as two certain numbers. The students' explanations could be be divided into three described below categories.

<u>Category 'Formal'</u>. The students in this category had considered the parametric representation as a formal expression. In order to find out whether two representations correspond to one and the same plane, the students equated them and

solved corresponding equations, treating the letters as unknowns. Thus the common conclusion in this category was that the item may represent the same plane as the given representation, if t equals to s :

'The planes do not coincide, since one of the direction vectors is not identical to that in other representation The planes will coincide only if t=s'.

'Yes, t and s are given numbers. It may happen to be the case that these s and t will be equal'.

'No, it may represent only the planes for which t=s, but all other planes are not possible to represent by this equation'.

<u>Category 'Geometric'</u>. These students assumed that the parametric representation is a way to express the geometric fact that two intersecting lines, or correspondingly, two vectors applied to a certain point, determine one and only one passing through them plane. For these students the main elements of the parametric representation are vectors $\underline{u}$, $\underline{v}$ and $\underline{a}$, since geometrically they are enough to determine a plane. The values of the parameters t and s are not important, because a pair of vectors $t\underline{u}$ and $s\underline{v}$ still determine the same plane as vectors $\underline{u}$ and $\underline{v}$. Therefore, the parameter was considered to be a free variable which may be substituted by any letter or number without changing the corresponding plane. The following are the examples of the explanations in this category:

'Yes. It doesn't matter whether s=t or s ≠ t , what matters is that the vectors are the same'.

'Yes, since s and t in the given representation are variables, so that even if we substitute s=t, like it is in this item, the representation will represent the original plane'.

'Yes, it may be, because s is a variable, so it is possible to write instead of s anything, including t'

Interviews with the students had confirmed that for many of them any two given values of the parameters were considered to be enough to determine a plane. The following is an excerpt from an interview with one of the students Ron after he completed the questionnaire:

I: So you say, that it doesn't matter what the parameters are. Can you describe me the case, of say, t=s=2?.

Ron: OK, we have a direction vector which is in the same direction but twice as long as vector u and another one which is also twice longer than v. But still it is the representation of the same plane.

I: ...How?

Ron: Well, imagine two vectors $\underline{u}$ and $\underline{v}$ in space. I can put a plate on them in only one way, so that its position will be quite determined. Now you have vectors $2\underline{u}$ and $2\underline{v}$. They still define that same plate: it doesn't matter infinite or not. Do you see?

I: Yes, I see...

To provide more evidence that the students saw nothing wrong in the substitution of a parameter by a specific number, an additional item $\underline{x} = \underline{a} + 100\underline{u} + s\underline{v}$ was included in the questionnaire delivered in four classes of the sample (N=74). The success with the first item in this sub-sample was 32%: the same as in the whole sample. The success rate with the additional item turned out to be only slightly higher: 42%. Most of the students, who erroneously claimed that the first item could represent the plane identical to that of given representation, answered the same to the additional item: 35 students out of 41. In the category 'Formal' this proportion was the highest: 26 out of 28; that is, 93% of the students in this category gave the same answer to the both items.

Category 'Analgebraic'. The term 'analgebraic' was first introduced by Bloedy-Vinner (Bloedy-Vinner, 1994) to describe the students' behaviours which go against the basic algebraic conventions. The students in this category not only assumed that the parameters t and s denote two certain numbers, but that these numbers must be necessarily different, since they are denoted by different letters:

'No *t* is not equal to *s*'.

'No, the direction vectors are not the same'.

'No, the lengths of the direction vectors have become equal'.

.The distribution of the students' answers to this item according to the described categories is presented in Table 1:

Table 1: Distribution of the students' wrong answers according to the categories.

'Formal'	'Geometric'	'Analgebraic'	no explanation
55	40	18	28

Considering the item $\underline{x} = \underline{a} - t\underline{u} + s\underline{v}$, most of the students answered, like they did for the previous item, that it represents the same plane as the given representation $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$, which in this case was the right answer. The close analysis of the students' explanations had shown, however, that many of them were based on the same erroneous assumption that parameters in the parametric representation of a plane are certain fixed numbers. For example, one of the very common answers was: *'the same plane, because the direction vector -t$\underline{u}$ is just opposite to t$\underline{u}$'*. This utterance means that for a student *-t$\underline{u}$* denotes only one, rather than a set of vectors corresponding to all real values of *t* and that the letter *t* is assumed to denote in different representations one and the same number. Another type of erroneous explanations which accompanied the correct answer to this item was: *'yes, it is a linear combination of the same direction vectors'*. This answer indicates that a student doesn't understand the difference between a generic linear combination and the parametric representation of a plane which is the set of all linear combinations described by this generic form. .

Among those who didn't answer correctly to this item, most had formally equated the item and the given parametric representation of a plane and, after subtracting the terms $s\underline{v}$ in the resulting equation, concluded that *'the item represents the same plane only if t=0'*. Only few among these students had realised the contradictory nature of this conclusion: *'Not the same plane. If t= -t, then t=0, rather than being any real number'*. In fact, by equating the representations in order to get the conditions of their equivalence, the students were employing a sound mathematical procedure. However, the performance even of those who knew that the parameter should be any real number, was lacking the understanding that the parameters in different representations vary independently one from another and for that reason can't be denoted in the resulting equation by the same letters.

DISCUSSION

The study has revealed the students' serious misunderstanding of the role of the parameters in the parametric representation of a plane. That this misunderstanding is rooted in the students' one-sided experience with variables, becomes evident if one compares the answers of the students and the experts to the same questionnaire. Most of the students saw the item $\underline{x} = \underline{a} + t\underline{u} + t\underline{v}$ as a result of the assignment of a particular value to one of the variables of the representation $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$. For all the academics, on the other hand, it was obvious that the number of the parameters in the two representations was different. Unlike the students, the academics knew that the parameter in the parametric representation of a plane is a variable which is not characterised by its particular values. The students' unawareness that there are different types of variables and that the parametric representation requires the bound ones, made them readily accept that $\underline{x} = \underline{a} + 100\underline{u} + s\underline{v}$ is also the parametric representation of a plane.

The fact that the introductory algebra courses still do not include the issue of different types of variables, has several historical and didactical reasons. First, it should be noted that the distinction between free and bound variables is of relatively recent origin. Almost up to the beginning of the 20th century, the mathematics were quite satisfied with the Euler's simple distinction between variable and constant quantities and meeting, for example, something of the type $ax+by+c=0$, one knew exactly which is which. It was not until Russell had challenged this tradition and proclaimed on the first pages of his 'Principles of mathematics' a new, 'only variables' age (Russell, 1903/1964):

> '...But unless we are dealing with one absolutely particular line, say the line from a particular point in London to a particular point in Cambridge, our *a, b, c* are not definite numbers, but stand for *any* numbers , and are thus also variables'.

Russell had, of course, good reasons for this claim. He developed the notion of variable as a set-building device which allowed to deal with a family of equations, rather than with its arbitrary representative; to form functions, functions of functions -

all this by just prescribing for a free variable in a mathematical expression to become a bound one. He had also suggested special symbols to mark bound variables, but this and other similar suggestions had never been universally accepted, so the type of variable is still inferred only from the context. On the other hand, the mathematicians find even some advantage in this notational ambiguity, since it allows to deal with different meanings of the same expression by just imagining that corresponding variable had changed its type. Furthermore, this ability for mental 'binding' of variable has become a prerequisite for learning advanced mathematics in which this feature has become widely used.

One of the conspicuous features of the school mathematics curriculum is that though it is permeated with the distinction between bound and free variables and is concerned that the students master it, there is no any attempt to teach it explicitly. Probably the first occasion when the students have to deal with the notion of a bound variable is related to functions. However, as this study shows, this doesn't necessarily led even the advanced students, who had a year-long introductory calculus course, to master this notion in a way that allowed to them to recognise and apply it in linear algebra. Avoiding the issue of variables also in this latter context, may only multiply the students' misunderstanding, because the dual use of variables there is quite common. For example, in the expression $t\underline{u} + s\underline{v}$ the letters t and s are bound variables, if it is meant as the parametric representation of a plane, but are free variables, if it is intended as a general linear combination of vectors $\underline{u}$ and $\underline{v}$. The difference is not well expressed in the ordinary language, as in the both cases one would likely to say that the lettered symbol stands for any number. Indeed, in their explanations that the item $\underline{x} = \underline{a} + 100\underline{u} + s\underline{v}$ is the parametric representation of the same plane as $\underline{x} = \underline{a} + t\underline{u} + s\underline{v}$, some students wrote that the parameter may be 'any' number, meaning by it **any some** number; what a teacher or a textbook meant by using the word 'any' in relation to the parameteric representation, was that the parameter should take on **all** the real numbers, that is to be the bound, rather than free variable.

The 'silencing' of this distinction is the worst possible choice in the case of the parametric representation of a plane, because the textbooks introduce it starting with an arbitrary point on a plane and then describe its possible positions on a plane as all the linear combinations of the basis vectors. This passage remains an impossible mental exercise for a learner who is neither prepared for set-theoretical arguments, nor is proficient in bound variables. This shows the dilemma of the current algebra courses: on the one hand, due to certain didactical tendencies of the last decades the textbooks try not to use the set-theoretical notations, in order not to embarrass the students; on the other hand, they by all means avoid any informal or kinematically coloured expressions, like 'parameter ranging over all numbers' or 'a point moving along a plane'. It is worth to note that the university textbooks seem to have much less constraints in this sense and do not hesitate to explain to a reader that 'the parameter t runs through all numbers' (Lang, 1966, p.13).

CONCLUSIONS

The study has described the difficulties experienced by advanced high school students with the notion of the parametric representation of a plane. It is shown that this difficulty followed from their lack of understanding of the difference between free and bound variables in mathematical expressions. This distinction, though basic in mathematics, has not received any attention in the current algebra courses. It is suggested to direct more research and didactical efforts to this issue which may significantly improve the situation.

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Promoting students' meaningful learning in two different classroom environments

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Abstract: The paper continues the research presented in (Kubínová, Mareš & Novotná, 2000) and (Kubínová & Novotná, 2001). It reports on an observational study focused on the analysis of concrete situations in two classes taught in different ways by different teachers in the past but taught by one teacher at present. The differences in students' behaviour, teacher's approaches and results achieved by students are diagnosed and illustrated by the topic Functions.

1. Theoretical framework and related research

The teaching experiment dealt with in this paper is a part of a longitudinal research focused on the transition from the traditional transmissive, instructive way of teaching to the constructive one.

In (Kubínová, Mareš & Novotná, 2000), four schemes *subject matter – teacher – students* were analysed and characterised: **IR** (instructive teaching method, direct teaching of ready information or learning from text), **ID** (instructive teaching method, attempt for students' independent transfer of acquired knowledge), **CR** (constructive teaching method, learning from text), **SC** (social constructive teaching method). The main consequences of the use of constructive teaching methods are: The subject matter becomes an intermediary which enables the development or modification of students' existing concepts and the creation of new ones; social relations among individual cognising subjects are accepted; the role of the social relationship between students and teacher are accentuated, and the social relations among students are taken into account.

In (Kubínová & Novotná, 2001), the differences in students' behaviour, teacher's approaches and results achieved by students were diagnosed by analysing concrete situations in two classes taught in different ways by different teachers in the past but taught by one teacher at present. It is shown that even if the teacher who wants to implement the change from instructive to constructive teaching, is sufficiently qualified, has long-term experiences with constructive teaching strategies and has no obvious external obstacles for implementing their plans, has to be open-minded and respect students and their prior experience.

The present paper focuses on the development of students' understanding of mathematical concepts when the constructive teaching method **SC** for 4 years had been applied. The findings are illustrated by the topic *Functions* which is commonly agreed to be one of the difficult topics of school mathematics.

Function concept

(DeMarois & Tall, 1999) undertakes the complexity of the function concept. The function concept is studied as an organising principle for algebra and beyond. The aspects studied include the function notation and the symbolic, numeric and graphic representations. The pre-procedural, procedural, process, object, and proceptual levels are studied. It is documented that for many students, the complexity of the function concept is such that the making of direct links between all the different representations is a difficult long-term task.

In (Even, 1990), ideas giving basis for the difficulty that functions represent to students are summarised: the arbitrary nature of functions; the univalence requirement; the function as a unifying, complex concept at one side and their different behavior, representations, notations on the other; alternative ways of approaching functions - point-wise, interval-wise and objects-wise; the richness and accessibility for students of the basic repertoire of functions; the importance of both procedural and conceptual knowledge and the relationship between them.

In the Czech Republic, the function is usually the first concept that students meet at the Basic school and that contains a certain dynamics, movement. This fact markedly influences the process of constructing a notion. A significant progress was reached mainly in that the curriculum pf all three valid educational programmes for the Basic school suppose that the stage of preparing the function begins in an implicit form already at the first years of schooling with a more significant use of a variety of inter-disciplinary links and with a systematic stabilisation and refreshing of the mutual connection *function – (real) story, notion*. The programme authors appreciate that in the didactic interpretation in the concrete schooling, the dynamic nature of the function enables very well among other the use of experimenting, solving problems intuitively, modelling, but also a timely preparation of further concepts.

Social interactions

Social situations exist in the school regardless of the significance we give them (Kubínová, 1999). If an opening of the space for effective teaching is to occur, it is necessary to create more natural conditions for teaching, i.e. a situation which enables this to happen. The teachers should

- admit that they are not the only source of information for students, that discussions with other people, TV programmes and Internet access significantly influence students' knowledge and way of learning,
- understand that each of their students creates their own concepts and these concepts are multileveled with respect to the student's own concepts as well as to their peer group, and that many of these concepts are not complete, sometimes even not correct but used by them for experiencing the world,
- suppose that students already created a certain concept through various resources, not only during the work in class.[1]

Steinbring (2000) has studied social discourse and a reflexive discussion, and shows by analysis of teaching episodes how individual learning strategies and social-interactive constructions of knowledge favour different forms of an epistemological

[1] Similar ideas can be found already in the work of J.A. Komenský (Comenius).

development of new mathematical knowledge. Construction of a new knowledge is not only an individual process, but collective processes make potential development of new knowledge possible.

2. Our research

When studying questions related to using constructive approach to teaching (mathematics) we use variety of methods: longitudinal evaluation of teaching effectiveness by comparison of periodic testing of parallel classes, direct observation of the milieu of the classroom and analysis of teaching strategies, the teachers accounts of their own classroom experience, analysis of audio/video recordings of lessons and of students' products, as well as the direct teaching at one Basic school. In the school year 2000-2001 we faced a singular opportunity. One of the authors of this paper taught mathematics in two parallel classes of the ninth grade (9.A and 9.E referred to below where students are age 14). In 9.A she taught the class for five years and using constructive teaching strategies (Kubínová & Novotná, 2001). The class as a whole showed very good results in mathematics, in the inquiry most students put mathematics among their favourite subjects. She has never taught the other class (9.E) before. In this class, there was never one teacher teaching longer than during one year. The class was commonly considered as average, two students showed excellent results. Only two students put mathematics on the list of their favourite subjects.

Following the direct observation, interviews with students and teachers who had taught in both classes in the previous school year and the analysis of students' written products, the input diagnosis of both classes was created. For the purposes of our paper we will focus in the further text on the phenomena related to the creation and fixing of the function concept.

INPUT DIAGNOSIS (15.9.2000)

In the previous period in the mathematics teaching

- ***the emphasis was put on***:

9.A	**9.E**
Long-term preparation of the function concept and of accompanying phenomena	Transmission of ready-made knowledge about the function – definition, graph
Constructing of the function concept	Instructive teaching strategies
Cultivation of communicative abilities including mother tongue development and the work with different function representations	Assigning and elaborating standard tasks mainly in the written form
Work with diverse information sources in and out of school	Work with textbooks and mathematical tables as the only "legal" information sources
Long-term building of the function concept in the student's cognitive structure, creation of separated and universal models	Immediate student's performance based on memorising

Work with an error as a source of cognition	Error as an indicator of the immediate student's performance
Evaluation of qualitative changes in the student's work during a certain period	Evaluation of the immediate student's performance when solving standard problems
Use of the function concept as an intermediator when using inter-disciplinary links and solving real life problems	Use of the function concept when solving standard mathematical problems
Cooperation, team work and support of social links in and out of the class	Individual students' work

- ***the teacher was apperceived by students as***:

9.A	**9.E**
A person guiding the teaching/learning process	The only person who has the right to decide about the teaching/learning process
An authority providing enough space also for the ideas of students	An authority who does not need to provide students with space for their ideas
A facilitator and advisor (also in matters not directly connected with mathematics teaching)	A person whose duty is to transmit ready-made knowledge and instruct students what they are to do

The diagnosing tool evaluating the input analysis was the student project *What do the graphs say* (see Table 1).

What do the graphs say
Describe by another way the dependences those graph is on the figure.

Table 1

At the end of September 2000, pairs of students in both classes elaborated the project individually outside of school (in order to be able to use various sources of information). After that period they showed written materials and presented their results in a mathematics lesson.

Working pairs in 9.A presented divers solutions (often unexpected for us) and were able to defend their results also during the following discussion about the project outputs. In Table 2, all types of 9.A students' solutions (including the incorrect ones) and some of their statements are summarised. Students' solutions (*I*), (*V*), (*T*), (*S*), (*U*) and (*R*) represent separated models of dependences, (*F*) an universal model.

What do the graphs say		
Identified dependence/ number of occurrences	***Quantities monitored***	***What happens (in the student's statements)***
Trip graphs (*I*)/7	Time, distance	• Train (car, student, dog, ...) moves between two places with different velocity or is staying • It is possible to continue the graph (time is running always).
Water in tank (*V*)/8	Time, water cubage	• In the tank there is a certain amount of water, the bleeder is opened, amount of water decreases, after some time it is closed again and then the feed-pipe with another velocity is opened and the amount of water is increasing. • Instead of filling in and out we can take out water from the tank by cans and add some using other cans. • It is possible to do it as along as the tank is empty (e.g. they forgot to close the bleeder) • Water may also flood if we let the feed-pipe opened too long.
Changes of temperature (*I*)/9	Time, temperature	• We follow the outside temperature regularly. It stabilise after a while, then descends, then is again stable and then increases. • From a digital thermometer we would not have such a nice line.
Changes of state (*S*)/3	Time, amount of heat	• We are cooling boiling water until its temperature starts to descend. When it is at zero, we still cool it until we get ice. After a while we will heat the ice and we get water that will start to heat. • Time when nothing happens cannot be extended as we want. After a certain time the state changes and the temperature starts to change as well.
Savings (*U*)/2	Time, amount of savings	• We have some savings and after a certain time we begin to spend it regularly. Then we stop it for a while and then we start to save again. • Or we have also some money and have to pay interests. After paying them, we do not do anything with our money and then we put them on an account and they give us interests, so out money begins to increase.
Changes of velocity (*R*)/1	Time, velocity	• We are at the motorway in a super-fast car with the same velocity, then we have to brake in order to move with a smaller velocity and at a certain moment we can increase the velocity again.

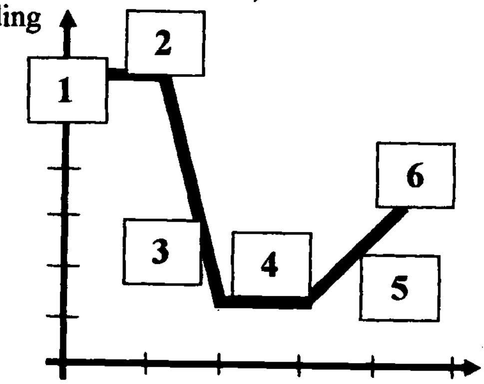
Universal description (*F*)/4	*x*, *y* not stated	1. beginning 2. nothing is happening (bigger value) 3. graph is descending 4. nothing is happening (smaller value) 5. graph is ascending 6. end 

Table 2

For our work with 9.A the following findings were most important:

- Students interpreted the open assignment of the problem in various contexts, even from non-mathematical once. Here we were mainly surprised by the physical interpretations (changes of temperature and above all changes of state depending on time).
- Students were aware of limiting conditions for individual dependence descriptions (e.g. limitations for the range given by the volume of the tank or by the size of the latent heat of the dissolution of ice).
- In the cognitive structure of our students the universal model of a continuous (discrete) function was not created yet. They used a "continuous" line for interpreting discrete actions (measurement of temperature in given intervals, saving money, supplying water in a tank using a can).
- Students reflected the existence of more precise tools for the graphic record of dependences (e.g. digital thermometer).
- Four of eleven working pairs offered, besides the separated models, also the universal one (*F*), see Table 2.

The situation in 9.E was different. Only three from twelve working pairs accepted the openness of the project assignment and presented separated models (*V*) (2 pairs) and (*T*) (3 pairs). One of these three pairs labelled the coordinate axes and presented the universal model (*F*). Four pairs did not solve the problem anyhow while they did not considered the curve to be a graph of a dependence. (it did not correspond with any of graphs they had in their register, i.e. the graph of linear or quadratic dependence or indirect proportion). The last five pairs modified the task first to a closed one (by a concrete labelling of axes) or simplified it (omitting parts of the graph) and presented the unique solution. In four cases, the dependence description was the dependence of distance on time, in one case of temperature on time.

Using the input diagnoses, teaching strategies based on constructive approaches in both classes were stated for the period of one school year. In 9.E they could not be developed to the whole versatility because students grasped only step-by-step the cooperative ways of working, learned to work with open problems, solve non-standard problems, trust their own decisions etc.

The **output diagnoses** based on participating observation and analysis of written products showed that having worked with the function concept during one school year 9.A students précised their knowledge to such extent that they

- did not failed when solving standard school problems,
- were able to work with the universal model of a dependence (function), solve non-standard problems, use various representations of the concept when solving school and practical problems,
- used systematically various information sources (also out of school),
- identified at the intuitive level important properties of functions (continuity, discreetness, monotony, extremes, ...)
- classified some classes of functions (linear, quadratic, ...).

In 9.E, the class climate changed significantly towards the cooperative one. 9.E students did not show a significant shift in grasping the function concept during the school year. We identified only the enlargement of the group of separated models and more frequent attempts to solve non-standard problems. There survived the link to the function assignment and creation of graphs of "known" functions following the rules given beforehand. The influence of the preparatory period neglect was significant.

3. Concluding remarks

In (De Corte, 2000) it is stated that: "... we should realise that powerful learning environments ... require drastic changes in the role of the teacher. Instead of being the main, if not the only source of information – as is often still the case in average educational practice – the teacher becomes a 'privileged' member of the knowledge-building community, who creates an intellectually stimulating climate, models learning and problem-solving activities, asks provoking questions, provides support to learners through coaching and guidance, and fosters students' agency over and responsibility for their own learning." From the evidence above it is clear that the teacher's role is crucial, he/she has to understand and respect the situation in each individual group of students. It is not possible to transmit the methods and forms of work which were successful with one group of students to another unmodified, however it is possible to use experiences gained with one group of students to organise work in another group.

In (Edwards & Jones, 1999), the grouped categories of students' views of learning mathematics in collaborative groups were classified. The following were clearly identifiable in our analysis of 9.A and 9.E performances in school mathematics:

benefits of working together, respecting others in the group/sharing knowledge, confidence building and speed/volume of learning. In the class where the interactive teaching strategies were newly introduced (9.E) it influenced the climate in the class first, the influence on the mathematics behaviour and knowledge was significantly milder and occurred later. To change students' gained norms of acting to a greater extent demands a long period of phased transition from transmissive teaching strategies to constructive ones (in our case, after 8 years of schooling, one school year was not sufficient).

In our experiment the role of peer interactions in the process of cognitive development was important. To profit from them needs a long experience of students in the similar activities, the enlargement of their self-confidence as well as the changes in their attitudes towards the subject. Students urgently need to see clearly mathematics as the subjects having narrow links to other subjects and mainly to the life situations.

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Acknowledgement: The research was supported by the Research Project MSM 114100004 Cultivation of mathematical thinking and education in European culture.

STUDENTS' UNDERSTANDING OF A LOGICAL IMPLICATION AND ITS CONVERSE

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In this paper we present an analysis of student responses to a written question involving logical implication and its converse in the context of simple number theory. Our sample consisted of relatively high attaining 13 year old students in randomly selected English schools. Most of the students regarded the statements representing the implication and its converse as essentially the same, and though nearly half were able to make a correct deduction on the basis of one of the statements being true, very few students used deductive reasoning as warrants for their conclusions about the truth of the statements. Students preferred to argue on the basis of empirical evidence, though many were not able to use the evidence in a mathematically appropriate way.

There is a considerable body of research over many years into argumentation in the mathematics classroom and how the processes of explanation, justification and proving can be fostered by teachers in order to develop a culture of mathematical interchange. Yackel (2001) for example took Toulmin's scheme (Toulmin, 1958) comprising conclusion, data, warrant and backing, as elaborated for mathematics education by Krummheuer (1995), to analyse interactive argumentation. In this paper, we report on student responses to a written question in which students are asked to evaluate the truth of a logical implication and its converse, and to justify their conclusions. Though a written question is far removed from the kind of interaction reported by Yackel, Toulmin's scheme provides a useful framework for discussing our results.

School students have a propensity to use inductive reasoning to validate conjectures in mathematics (e.g. Bell, 1976; Van Dormolen, 1977; Balacheff, 1988) rather than to prove them deductively. Even when students seem to understand the function of proof in the mathematics classroom (e.g. Hanna, 1989; De Villiers, 1990; Godino & Recio, 1997) and to recognise that proofs must be general, they still frequently fail to employ proof to secure beliefs in the truth of their warrants, preferring instead to rely on more data (e.g Fischbein, 1982; Vinner, 1983; Coe & Ruthven, 1994; Healy & Hoyles, 2000; Rodd, 2000; Simon, 2000). At the same time, there is some evidence from cross-sectional studies of logical thinking, that the use of 'child logic', whereby an implication and its converse are deemed as equivalent, decreases with age (O'Brien et al, 1971). In the 1970s there was considerable research on students' understanding of logical connectives, and recognition of the importance of domain knowledge (eg, Wason & Shapiro, 1971), but no systematic study of the development of logical reasoning over time in mathematical contexts, perhaps surprisingly given its importance for success in the subject.

How do students who are not taught explicitly about logical implication come to appreciate its structure from a basis (we surmise) of everyday or inductive reasoning? How do students exploit inductive reasoning (that provides a way of forming and testing conjectures, but not of proving them) to support their claims about the meaning, outcome and truth of "$p \rightarrow q$", but also, in Toulmin's language, begin to develop more general and explanatory warrants for their conclusions? In this paper, we begin to address these questions through an analysis of students' responses to a written question, L1, which we describe below.

Question L1 involved logical implication in the context of simple number theory. It involved the concepts odd, even, sum, and product, which would be familiar and meaningful to most of the relatively high attaining students that formed our sample. It was one of nine in a 50-minute written proof survey that was given to about 3000 students in June 2000 when they were approaching the end of Year 8 (age 13 years). A detailed scheme was devised to code the responses to all the questions, based partly on theoretical distinctions and partly on samples of pilot data. The students came from the highest attaining class (or classes) in 63 state funded secondary schools randomly selected within nine geographically diverse English regions. The same classes were also given a baseline mathematics test (using a broad selection of TIMSS items) about a month before taking the proof survey. We report here only on the students who were present on both occasions (N = 2663)[a]. All the students had followed the statutory National Curriculum in which they are encouraged to engage in informal argumentation about mathematical problems but where the nature of logical implication is not explicitly taught. Our assumption is therefore that analyses of cross-sectional responses will provide a picture of students' understandings of the deductive process in a mathematical context, after having engaged in classroom experiences of justifying and explaining, either verbally or in writing.

RESULTS AND DISCUSSION

Question L1 was devised to investigate whether students could distinguish between a logical implication (Joe's statement) and its converse (Fred's statement). It was based on a classroom activity suggested by Watson (1995) for highlighting the directional nature of some proofs and the role of counter example. The introduction to the question is shown in Figure 1, below.

The question had four parts. In part a) students were asked "Are Joe's and Fred's statements saying the same thing?". In b) students were told that the product of two whole numbers is 1271 and to suppose that Fred is right. They were then asked to select one of three conclusions, namely that you can be sure that the sum of the two numbers is i. even or ii. odd or that iii. you can't be sure whether the sum is odd or even until you know what the two numbers are. Students were then asked to state whether c) Joe's and d) Fred's statements were true and, in each case, to explain their answer.

L1	Joe and Fred are thinking about the pair of numbers 3 and 11.
	They notice that the SUM (3 + 11) is EVEN.
	They notice that the PRODUCT (3×11) is ODD.
	Joe says: If the SUM of two whole numbers is EVEN, their PRODUCT is ODD.
	Fred says: If the PRODUCT of two whole numbers is ODD, their SUM is EVEN.

Figure 1: Introduction to Question L1

In part a) 13 percent of students correctly stated from the outset that Joe's and Fred's statements are not saying the same thing, with a further 15 percent changing their answer from Yes to No at some stage. The vast majority of students, 71 percent, stated that they were saying the same thing.

In b), the large and rather obscure number 1271 was chosen to discourage students from trying to find the actual numbers (31 and 41) whose product this is. The aim of the item was to see whether students could play the mathematical game of supposing a statement is true, whether it is or not, and then making a correct deduction on this basis. Nearly half the students in the sample (47 percent) chose the correct option (that the sum must be even) with another 47 percent choosing the empirical option (you can't be sure until you know what the numbers are).

In c), 36 percent of students correctly stated that Joe's statement (Sum-even $\rightarrow$ Product-odd) is false *and* supported this with one or more appropriate counter example, as in this response: "No. Because if you do 2+4=6 (even) then $2\times4=8$ (even) so they are both even". Of this 36 percent, most (28 percent of the total sample) gave specific counter examples (usually just one, as in the above response), but some students (8 percent of the total sample) described the counter examples in general terms (namely, that both the initial numbers are even). A general explanation of this sort represents a shift from just looking at data to looking at structure: it goes beyond showing *that* the statement is false, though this is all that the item requires, by giving some insight into *why* it is false, and so provides some insight into the warrants students held to support their conclusion.

In d), 24 percent of students correctly stated that Fred's statement (Product-odd $\rightarrow$ Sum-even) is true *and* supported this with appropriate examples, as in this response: "Yes. $3\times3=9$. Product = odd. 3+3=6. Sum = even". Another 9 percent of students supported their correct evaluation of Fred's statement with a general explanation, of the sort "If the product is odd, then the numbers must be odd, so the sum is even".

In contrast to c), a general explanation is needed to answer d) satisfactorily as empirical examples can not show conclusively that Fred's statement is true.

The response frequencies for the four parts of question L1 are summarised in Tables 1 - 4, below.

L1a Response	YES (incorrect)	YES changed to NO (correct)	NO (correct)	Other
Frequency (%)	71	15	13	1

Table 1: Distribution of responses to L1a (N = 2663)

L1b Response	Sum is EVEN (correct, deduction)	Can't be sure (incorrect, empirical)	Sum is ODD and other incorrect
Frequency (%)	47	47	6

Table 2: Distribution of responses to L1b (N = 2663)

L1c Response	Correct, incorrect or no decision; no valid justification or incomplete	Correct decision; valid justification, specific	Correct decision; valid justification, general
Frequency (%)	64	28	8

Table 3: Distribution of responses to L1c (Joe: Sum-even → Product-odd) (N = 2663)

L1d Response	Correct, incorrect or no decision; no valid justification	Correct decision; incomplete justification, empirical	Correct decision; valid justification, general
Frequency (%)	67	24	9

Table 4: Distribution of responses to L1d (Fred: Product-odd → Sum-even) (N = 2663)

One striking feature of the frequencies in Tables 1 - 4 is that the proportion of students who seem to use general, deductive reasoning is far higher in part b) (47 percent) than in parts c) and d) (8 percent and 9 percent respectively). This difference might be partly due to the greater complexity of items c) and d) compared to b), although the fact that our multilevel statistical analysis showed a correct response to b) to be a significant predictor of overall proof score suggests that some appreciation of the structural nature of logical implication influences approach and success to questions involving deduction. However, it also suggests that many students who use an empirical rather than a more general, deductive approach in c) and d), do so out of choice or habit rather than necessity. Of course, the point has been made that a satisfactory mathematical explanation in c) does not require a deductive approach, though it does in d).

It is also striking that the corresponding frequencies in c) and d) are very similar. Care must be taken in interpreting this, however. It does not necessarily mean that the two items are similarly demanding. Indeed a good case can be made for expecting the frequencies in the last two columns of Table 4 to be substantially higher than the corresponding frequencies in Table 3[b]. That this is not the case could be due to the order of presentation of the two items, in that many students when they reach d) will be influenced by their response to c), given the widely held belief that Joe's and Fred's statements are essentially the same. In particular, this is likely to increase the number of students who say No in d), as roughly $1^1/_2$ times as many students say No in c) as say Yes.

If the raw frequencies in c) and d) can not easily be compared, the relative frequencies are still of interest. In particular, it is notable that the students who give a correct deductive reason in d) are greatly outnumbered (by nearly 3 : 1) by those who give a correct (as far as it goes) empirical reason. This fits with the findings of other studies (e.g. Bell, 1976; Van Dormolen, 1977; Balacheff, 1988) and supports the observation made earlier that students in our sample are for more likely to favour an empirical, data-driven approach to evaluating a mathematical statement than to consider mathematical structure and to argue deductively. However, it is of interest too that the corresponding ratio in c) is similar (just over 3 : 1), bearing in mind that a general, deductive reason is not necessary here.

An examination of individual students' scripts indicates that most students' explanations in parts c) and d) start with one or more empirical example. Usually, the explanation stays at this level, though occasionally, as the frequencies discussed earlier indicate, the empirical examples are followed by a more general statement which is an empirical generalisation of the examples or, as seems to be the case below, for which the examples serve as illustrations:

d) (Fred: Product-odd → Sum-even)

9=3×3 3+3=6 21=3×7 3+7=10

I believe that it is correct as only an odd times an odd can equal an odd and if you add together the two odds you always get an even.

Students who provide appropriate empirical examples (in support of a correct conclusion) commonly only provide one empirical example, or, where they provide several, the examples are often quite similar. This suggests that rather than starting with fairly random examples with which to explore Joe's or Fred's statement, the example is carefully chosen after 'inspecting' the statement in some way. Interestingly, too, the example chosen is often quite special, for example consisting of a pair of small numbers, often identical, such as 4 and 4 in c) or 5 and 5 in d). Such numbers would seem to be the antithesis of numbers that would provide a 'crucial experiment' (Balacheff, 1988), and nor would they seem well suited for a generic example; however, one should not discount the possibility that some students thought of the numbers in one of these two ways.

There seemed to be a tendency for students who generated a lot of data to go wrong, by making incorrect conclusions from the data, perhaps because they were overwhelmed by the data or perhaps because they lacked a clear idea of the nature of the data and how it could be used to validate a claim. We have tried to describe some of the warrants students seemed to use to explain why the data they presented supported (or not) their conclusions. This analysis suggests that students may not only fail to distinguish logical implication from its converse, but also fail to appreciate how data can properly be used to support a conclusion as to whether "$p \to q$" is true or not. For example, some students would take "$p \to q$" to be true if they found data to support it, even if they had other data that would serve as a counter example, as in the response below. This could arise because the counter example was ignored or in the belief that 'a statement is true if it is sometimes true' (perhaps by analogy with 'a statement is false if it is sometimes false').

c) (Joe: Sum-even → Product-odd)

Yes.	4+8=12.	2+4=6	1+15=16	3+15=18
	4x8=32.	2x4=8	1x15=15	3x15=45

Other students deemed "$p \to q$" to be false if data could be found that falsified *both* propositions in the given statement, as illustrated below, suggesting that the statement was being seen as a conjunction rather than an implication.

c) (Joe: Sum-even → Product-odd)

No. Certain numbers added together will become ODD for example: 7+2=9. The product of those two numbers is also now even: 7x2=14.

Interestingly, the student who gave the above response switched, in answer to part d), to seeing Fred's statement in the conventional way, as is shown below. Such inconsistency was not uncommon, which has led us to wonder how far a student's expression of a warrant for their conclusions (either in writing or verbally during an interactive episode) can be assumed to be based on the appreciation that if a warrant is to be mathematical, it must be applied consistently (even if it might not be correct from the teacher's point of view!).

d) (Fred: Product-odd → Sum-even)

Yes. If the product of the numbers is known to be odd the sum will be even: 7x3=21 7+3=10 5x5=25 5+5=10 9x5=45 9+5=14

Some students were at what we tentatively suggest might be a transitional level[c]. Typically, they initially answered Yes to a) but changed this to No, presumably after reflecting on their answers to c) and d). Interestingly, those who changed their answer to No had a somewhat higher mean score on the other numerical/algebraic items in the survey and a higher baseline test score than those who stuck with No from the outset and than those who answered Yes (mean score of 6.7, 6.0 and 5.8 respectively for algebra and 16.6, 15.7 and 15.0 for baseline, compared with means of 6.0, out of a possible 13, and 15.3, out of a possible 22, in algebra and baseline for the sample as a whole). These students could cope with the suspension of disbelief and the deduction required in b) and were able to answer c) and d) competently using empirical data, and perhaps using large numbers as a crucial experiment to test their conclusion in d) more thoroughly, as in the response below:

d) (Fred: Product-odd → Sum-even)

Yes. Fred said the product of two whole numbers is odd, their sum is even. So:-
3×5=15 (odd) ✔ 3+5=8 (even) ✔
I tried 5 and 7, 9 and 17, and lots more, they all worked. Fred is right!

Students are more likely to adopt a general, deductive approach if they reflect carefully on the data they have generated or, indeed, if they reflect on the structure of a mathematical task before generating data. This is not something that is particularly emphasised in the English National Curriculum – perhaps even the opposite is the case, with students commonly being encouraged to formulate and test conjectures by first generating lots of data. It will be interesting to see, therefore, how the different types of warrant that students call upon to justify their conclusions, together with the consistency of their use, change over time, and to what extent the use of deductive reasoning to justify conclusions increases in our sample as the students get older.

ACKNOWLEDGEMENT

We gratefully acknowledge the support of the Economic and Social Research Council (ESRC), Project number R000237777.

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[a] Virtually the same question was given to the same sample of students one year later in 2001.

[b] Consider, for example, someone who works on an empirical basis and whose first step is to find a pair of numbers that fits the first proposition in the statement under consideration. In d) (Fred's statement), the numbers will both be odd and so will of necessity fit the second proposition, thereby supporting the correct conclusion that Fred's statement is true. On the other hand, in c) (Joe's statement) the numbers that fit the first proposition will both be even or both be odd, so they might (both even) or might not (both odd) support the correct conclusion that Joe's statement is false.

[c] Our analysis of responses to the survey administered in 2001 will assist in this interpretation as we will be able to describe the trajectory of student responses over time.

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From the Lotto Game to Subtracting Two-Digit Numbers in First-Graders.

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The purpose of this study was to design a context for the subtraction of two-digit numbers that would survive the transfer into the classroom situation, make sense to the children and thus have the power to elicit their outside-school intuition and would lead to the invention of an efficient computational method. A modified version of the Lotto Game was found to be such a context. All the first-graders who participated in this study invented and used the "Overshoot-and-Come-Back" method in the context of the game. Moreover, this method was used by the children also when later presented with abstract subtraction expressions where the unit digits of the subtrahend were larger than the unit digits of the minuend.

Background

It has been found that grade 2 and 3 students who use their own invented-strategies to solve multidigit subtraction make significantly fewer errors than students who were taught to use the standard subtraction algorithm (Carpenter et al, 1998). Moreover children who use invented-strategies develop knowledge of base-ten number concepts earlier (Hiebert & Carpenter, 1992) and gain higher levels of understanding (Hiebert & Wearne, 1996) than students who were taught from the beginning to use the standard algorithm. First grade students, however, usually do not demonstrate ability to invent algorithms for subtraction of two-digit numbers, although they are able to for addition (Carpenter et al, 1998) even though they had been previously provided with opportunities to solve both addition and subtraction word problems that involved two-digit numbers and solved them meaningfully.

Typically, pupils who are encouraged to develop computational strategies learn in mathematics classrooms that are characterized by instructional designs that afford the construction of mathematical knowledge via solution of context problems (word-problem situations). Well-chosen context problems provide the students with opportunities to develop specific-situation solution strategies. When these informal strategies do develop, they can then be modeled. These models can then be generalized to develop into entities that may become models for mathematical reasoning (Gravemeijer & Doorman, 1999).

Study

Following the above-reported results that indicate that invented strategies develop students' understanding and result in more correct solutions, it would seem disadvantageous to teach first-grade students algorithms for two-digit subtraction expressions. However, since situations that involve subtraction of two-digit numbers are meaningful to them (and they are able to solve them albeit in primitive basic methods like one-digit back counting or modeling with unit counters) it seems challenging and advantageous to look for situations that are likely to provide new affordances for fostering higher invented strategies (Schwarz & Linchevski, 2002). For instance, children have been party to situations where they paid for an item out of a given amount of money they had in their possession. For example, a child might have 64 cents in his or her pocket – 6 dimes and 4 one-cent coins – and buy candy for 28 cents. Thus, if it were possible to "import" such a situation into the classroom in a context that would evoke in the children their outside-school intuition (situated intuition), an invented computational strategy might emerge and then be developed into more formal mathematical thinking.

Two questions arise:

1) Is it possible to design a context also in the classroom that would afford the development of an invented computational strategy for the subtraction of two-digit numbers in first graders?
2) If so: a) What strategies would be invented for this context?
   b) Could the model of this situation become a model for abstract subtraction problems?

Our hypothesis was that

a) A " buying-selling" situation could be a context for the developing of invented strategies for the subtraction of two-digit numbers.

b) Since invented subtraction strategies generally start with the larger unit (Hiebert & Wearne, 1996; Fuson et al. 1997), we expected two possible invented strategies: i) Decompose-Tens-and-Ones; ii) Overshoot-and-Come-Back. For example, in strategy i) the children would first pay 2 tens out of the 6 tens, change another ten to 10 ones and then pay 8 ones out of the 14 ones leaving 6 ones. Answer: 36. In strategy ii) the children would pay the 28 with 3 tens, get 2 ones back, so they are left with 34 + 2. Answer: 36. This strategy we coined the "Change" method (sometimes referred to as "Compensation" or "Overshoot-and-Come-Back").

Students who would develop strategy (i) could be guided to model it mathematically as, for example: 60-20=40, 10(out of the 40)+4=14, 14-8=6, 30+6=36. Students who would invent strategy (ii) could be guided to model it mathematically as, for example: 64-30=34, 30-28=2, 34+2=36.

Study Design: The study involved two sets of teaching episodes in heterogeneous groups in a cooperative learning environment (Linchevski & Kutscher, 1998). The children were first-graders who had not yet received any instruction in subtracting two-digit numbers, although they were acquainted with addition and subtraction of one-digit numbers. One of the researchers was both teacher and interviewer. All the meetings were videotaped to allow further analysis. The sequence of episodes was designed a) to elicit out-of-school spontaneous strategies for the subtraction of two-digit numbers, b) to allow the shift of the spontaneous strategies from intuitive meanings to mathematical meanings expressed in mathematical sentences (Linchevski & Williams, 1999).

Research population: An average, suburban first-grade class was chosen for our research population. Students who participated in the study were those who on a pretest were able to a) read/identify two-digit numbers b) decide which of two given two-digit numbers is the larger c) understand that a two-digit number like 27 is 20 plus 7. The two top students of the class were excluded. 20 students participated in our research.

Teaching episodes: During the first set of teaching episodes a "buying-selling" game – a modification of the well-known children Lotto Game - was played by the groups. Each of the participants was given a different lotto-board that had 9 picture squares drawn on it. For each lotto-board there were 9 square picture-cards identical to the ones drawn on the board that the players had to accumulate in order to cover the pictures on their lotto-board and thus win. We modified the game by writing prices of two-digit numbers on the pictures; their values were between 11 to 45. Each child started off with the same amount of money – 77 IS - 7 strips with division lines, each strip representing a 10 IS note and 7 squares representing the ones. Each child had an empty 200-number-board on which he or she could store and arrange their money. There was also a communal bank of money where there were ample tens and units that the students could change from tens to units and vice versa, so that they could apply any strategy they wished. The cards were distributed equally among the children. If a player did not have a picture-card for his or her lotto-board they had to buy it from another player in the group according to the "price" of the picture-card. This "buying" process laid the base for the subtraction.

During the second set of episodes the children reflected on concrete cases that they experienced during their game and were offered ways of translating the intuitive strategies they used during the game to mathematical sentences. The translation was done in four stages:

a) A verbal formulation of the solving strategy in shared spoken-language terms for concrete cases from their game. The verbal formulation was based on the shared verbal communication they had developed during the game.

b) (i) A written account of the solving process for concrete cases using the verbal formulation.

(ii) Solving expressions written in verbal formulation of hypothetical cases that could occur in a game.

c) (i) A written representation in mathematical symbols for concrete cases.

(ii) Solving expressions written in mathematical symbols of hypothetical cases that could occur in a game

d) Solving written abstract subtraction expressions.

Results

First set of teaching episodes: The lotto game.

When the children were faced with the problem of having to pay money for a picture-card, having enough money on the board, but not enough ones, **all** the children who participated in this study used the "change" method to solve their problem.

The process of the invention of the "change" strategy: 1) Identification of the problem 2) Solution of the problem.

1) Two patterns of identification were recognized:

a) Identification of the problem after trying to pay the exact amount:

Y wishes to buy a flag. Its price is 17. She takes off 1 ten-strip while she counts the units. She takes off money of value 16. There are no more units on the board. She immediately takes off another ten-strip – the buds of the "change" strategy.

b) Identification of the problem before paying by comparing the number of units on the board to the number of units that need to be paid.

K wants to buy a car. Its price is 29. *"I don't have"*.
Interviewer: *What do you mean when you say 'I don't have'?*
T (another player in this group): *That he doesn't have the number.*

Interviewer: But you have even more (than you need).
T: *That we don't have like these;* and he points at the units.

2) The solution - a joint learning effort: Report and Analysis.

The following is representative of how the groups constructed cooperatively their strategy:

B is interested in buying a sofa. Its price is 47. He removes 4 ten-strips and says *"I don't have"*.
Interviewer: *Does anyone have an idea?*
A: *To give one like this.* And she points at one of B's ten-strips.
B removes 1 ten-strip from his 200-board.
Interviewer (to V, the owner of the sofa-card): *What will you do?*
V: *I'll give* change.
Interviewer: *How much change? You were supposed to get 47 and you got 50.*
V and A announce a few numbers: 2, 5, 7. And A shouts: *"Three!"* This answer is approved by all.

Every member of this group was involved in the construction of the solution. B identified the problem, A suggested the need to "overshoot", V the need to "come back" and A found the amount you "come back" with.

As reported earlier, **all** children adopted the "change" method in the game. But none found it necessary to find, after a transaction, the amount of money left on his or her 200-board: the answer to the multidigit subtraction problem. Their behavior corresponded to the situation that elicited their spontaneous strategy: in a buying-selling situation one does not usually calculate the amount left in the wallet – unless there is some concern that the amount will not suffice. Thus the children's disregard of the amount left on their 200-board is not surprising. The teacher would call their attention both to the initial amount on the 200-board and to the amount left after each transaction in the second set of teaching episodes, when the transactions would be translated to mathematical subtraction- expressions.

When encountering the problem of not having enough units to pay, in no instance did the pupils suggest that the buyer exchange a ten-strip for ten ones albeit the 'bank' was on the table. The decomposition of tens strategy did surface once, when the seller did not have enough change:

V is interested in buying a flute. Its price is 16. She removes 2 ten-strips and gives B. B doesn't have change of 4 ones on her board .
Interviewer: *Does anyone have an idea? V doesn't have 6 units and B doesn't have 4 units to give her change?*

A: *We'll change this* (points to a ten-strip on B's board) *to small ones.* B removes a ten-strip, puts it in the 'bank' and takes 10 unit squares and sticks them on his board on the row that was vacated when he removed the ten-strip. Only then does he give V 4 square-units.

The preceding vignette allows us to observe the powerful effect of situated-intuition. Why is it that here "decomposition-of tens-and-units" was the spontaneous reaction and that in all the other instances only the "change" strategy was evoked? It might be that in 'real' life experiences, it is usually the responsibility of the cashier to change the "big" money into smaller denominations in order to be able to give change to the buyer. Furthermore, just as was observed in B's behavior, in real life when the seller changes money into smaller denominations he places it in the till and only then gives the buyer his or her change. Thus, when both the "buyer" and the "seller" did not have the necessary units to complete the transaction, the solution that was offered in the game, mirrored a real-life problem-solving strategy. However, this decompose-tens strategy although understood and approved by the group was not used by the students later on.

<u>Second set of teaching episodes: Translation of spontaneous strategies from intuitive meanings to mathematical meanings expressed in mathematical sentences:</u>

This teaching episode had 3 parts:

Part 1) The children played a shortened version of lotto in the presence of the interviewer who documented their transactions using their shared language. This time the game ran very smoothly: all children applied the change method to complete their transactions. At the game's completion the interviewer reconstructed with the children the game's transactions and translated them first into (i) shared verbal formulation they developed and then (ii) into written spoken-language. The translation was from the viewpoint of the buyer as this was the language the children had used during the game. The transaction of money from the buyer was coined "I gave". The transaction of money to the buyer was coined "I received". Reference to the amount of money left on the 200-board was coined "I am left with". The reconstruction was done in 3 stages: a) The interviewer recalled a transaction from the game: "*B had 68 and she had to give 38.*" b) Each child reconstructed this by first sticking 68-worth of strips and/or squares on the board, then "gave" 38 and reported the money that "I am left with". c) The children documented the transaction using a combination of numbers and spoken-language stickers. This was done first with "simple" transactions such as 56-32 that was translated to:

56 $\boxed{\text{I gave}}$ 32 $\boxed{\text{I am left with}}$ 24

A "complex" transaction (needing change) such as "There were 74 and one had to give 38" was reconstructed and executed via the 200-board and translated in 2 stages:

a) verbal spoken language: "I had 74, I gave 40, I received 2, I am left with 36",

b) written spoken-language (the expressions "I gave", "I received", "I am left with" were written on stickers to facilitate the first-graders' 'writing'): 74-38= ;

74 $\boxed{\text{I gave}}$ 40 $\boxed{\text{I am left with}}$ 34; 34 $\boxed{\text{I received}}$ 2 $\boxed{\text{I am left with}}$ 36

The children seemed to be able to reconstruct the transaction and follow the corresponding steps in writing quite easily. But despite the fact that they counted the strips and/or squares left on the board, when the interviewer summarized: *"There were 74, I gave 38, how many are left?"* the children could not answer the question immediately. This difficulty repeated itself a few times in each group. Some children re - counted the strips and/or squares on the 200-board, some recalled the end result on their board and some re-looked at what was written on the paper.

Part 2: Each child was given "Work-Sheet1" with written abstract subtraction problems such as "63 $\boxed{\text{I gave}}$ 26= " Some children disregarded the equal sign completely, verbalizing the expressions in shared spoken-language as "I had 63, I gave 26" and put a sticker " I am left with" next to the "= " sign: 63 $\boxed{\text{I gave}}$ 26 $\boxed{\text{I am left with}}$ 27. It seems that they perceived the equal sign as a "start-working" instruction. For these children the mathematical expression seemed to be part and parcel of the game situation. The children learned to document each part of the process in written spoken-language in the same steps as is reported in *Part 1.*

There were some children who developed dependence on "reconstruction" as a model for documenting the solution. They would solve the entire problem with strips and /or squares, then verbalize these steps and only then use stickers to write all the steps of the solution.

Part 3: The spoken-language expressions "I gave", "I received", "I am left with" were replaced by corresponding mathematical symbols -, +, =. Thereafter the interviewer reconstructed with the children two transactions from the lotto game they had previously played, using the strips and/or squares to calculate the result. Following each game move, she translated it with the children into mathematical sentences using only mathematical symbols, in the same steps as had been previously made with the spoken-language stickers. Each child was then given Work-Sheet2; they were asked to solve written abstract subtraction expressions such as "57-13=" and "42-36=". All the children solved these problems correctly using their strips and/or squares. However, they found the documenting process to be a burden as it broke their line of thought. Thus most children preferred to write only the numerical answer to the problem. Two children did document, but with the aid of stickers. One child documented using only mathematical symbols.

Discussion

The results of this study suggest that first grade students are able to solve two-digit subtraction problems. We infer that the reason our first-graders were able to invent quite a sophisticated computational method for the subtraction of two-digit numbers, despite the fact that children of this age and mathematical experience, are generally not at the appropriate stage of number-concept development, is because of the specific context-design and documenting device. The buying-selling situation - embodied in a modified game of lotto - survived the transfer into the classroom context, made sense to the children and thus had the power to elicit the outside-school intuition. This led to the invention of a computational method - "overshoot-and-come-back" - for subtraction problems where the unit digits of the subtrahend were larger than the unit digits of the minuend. Moreover this computational method was used by the children also when later presented with abstract subtraction expressions.

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TESTING A DEVELOPMENTAL MODEL OF MEASURING PROBLEM SOLVING SKILLS BASED ON SCHEMA THEORY

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The aim of this study was to provide construct related evidence of a model measuring problem solving (PS) skills based on Marshall's schema theory (ST) using responses from 712, 4th and 5th grade Cypriot students to a battery of tests on PS. The Extended Logistic Model of Rasch was used and a scale was created for the battery of tests and analysed for reliability, fit to the model, meaning and validity. It was also analysed separately for each of the two types of knowledge proposed by Marshall in order to examine the appropriateness of ST in building a model of measuring PS skills. The analysis reveals that the battery of tests has satisfactory psychometric properties and supports the conceptual design of the proposed model. The findings are discussed with reference to intended uses of the assessment of PS skills and suggestions for further research are drawn.

INTRODUCTION

Problem solving (PS) is a central issue in mathematics education theory, as it can be documented by recently published literature on curriculum and assessment in Mathematics (see e.g., NCTM, 2000). In this context, Marshall (1995) developed a comprehensive proposal for teaching and assessing PS, which is grounded on schema theory (ST) and the acquisition of basic schemas by the learners. Mayer (1992) points out that "a schema is an organised structure consisting of certain elements and relations which are related to a situation and it can be used for understanding incoming information" (p. 228). ST was based on the assumption that the external representations used to describe the structure of a problem (i.e. diagrams) can serve in constructing a mental model which can be retrieved and used in solving analogous problems of the same structure (Goldin, 1998). Thus, ST aims to help students systematise PS experiences by providing them with simple diagrams for solving problems of additive and multiplicative structure.

Previous research on PS (Schmidt & Weiser, 1995) revealed that the crucial element in solving a problem lies in its structure and not on characteristics such as its context, content, or the number or type of the operations needed for solving the problem. Consequently, attention should be paid to the construction of the mental schema, which mirrors the structure of the problem. Diagrams or appropriate physical schemata can serve as vehicles for developing a solution plan (Marshall, 1995), since they provide access to similar structure problems that have been encountered in the past, and the means for reformulating or simplifying a problem. It is also recognised that retrieving the appropriate schema facilitates the design of a solution strategy, which is the most important part in the whole endeavour. Nesher and Hershkovitz

(1994) argue that schemas constitute the bridge between the verbal formulation of a problem and its mathematical structure. Schemas are, therefore, of primary importance with respect to cognitive processes involved in PS, since they facilitate the comprehension of the semantic relations in given text and serve as generalised frames for action in given situations (Philippou & Christou, 1999). In brief, schemas help students construct deeper understanding of problems, clarify their thinking and justify their ideas.

Marshall (1995) proposed five distinct problem situations: change, group, compare, restate and vary. The former three situations can be used to solve additive structure problems, while the last two situations are mainly used for solving multiplicative structure problems. For each situation, Marshall (1995) proposed an appropriate diagram, which is expected to help students recognise the problem situation and solve the problem. In this process, students are expected to proceed through four stages, each representing a different type of knowledge: identification knowledge, elaboration knowledge, planning knowledge and execution knowledge. The first stage refers to specific characteristics, features, and facts that help students recognise the schema; identifying the main elements of a problem can be considered as the most important part for schema activation, since it is this understanding that contributes to the initial recognition of a situation. The second stage refers to working out rules and limitations having to do with the use of the schema; at this stage the students recognise the details that are distinct to each schema, and consequently choose or construct the correct schema to solve the problem. The planning knowledge refers to ways through which students make decisions about the schema that is appropriate for the solution of a problem. Finally, the last type of knowledge includes a set of procedures, rules, or algorithms that can be applied to reach sub-goals and the final goal.

To the best of our knowledge and belief, no systematic attempt has so far been made to verify the sequence of these four stages, or to investigate whether these stages, representing different types of knowledge, may help us form a developmental model measuring PS skills based on ST. We consider the development of a model as highly important, since it may help teachers design activities that will enable students to progressively develop the required abilities to solve routine problems.

In this context, the main purpose of this study was to collect empirical data in order to examine the validity of a model measuring pupils' skills in PS. The developed model refers to additive structure problems of three situations, namely: change, group and compare, with regard to the former two types of knowledge that a student is expected to acquire (identification and elaboration knowledge) in order to be able to use appropriate schema(s) to solve a problem. Hence, another purpose of this study was to reconfirm the two aforementioned types of knowledge proposed by Marshall which could be helpful for using ST in teaching and assessing PS.

THE DEVELOPMENT OF THE BATTERY OF TESTS ON PS

To answer our research questions, a battery of tests on PS was constructed guided by existing research and theory on assessment of PS skills in Mathematics and by taking into account ST. Further, we sought a developmental ordering of tasks on a continuum of difficulty, which is an essential concept derived from research on developmental assessments for measuring proficiency in cognitive abilities and content areas (Hambleton, 1995). A final key requirement in designing the tests was its alignment with the mathematics curriculum that was operative in the country where the study was conducted.

The specification table of the tests (Table 1) included eight levels of PS skills which belong to the first two types of knowledge proposed by Marshall (1995); the former four levels referred to the identification knowledge of schemas and the remaining four to the elaboration knowledge.

Types of knowledge	Levels	Items of the battery of tests
Identification knowledge	1. Verbal recognition of problems	1-26
	2. Diagrammatical recognition of problems	27-52
	3. Selection of a problem reflecting the structure of a given diagram	53-78
	4. Posing questions from a mathematical situation	131-139
Elaboration knowledge	5. Filling in the boxes of given diagrams	105-130
	6. Correct placement of the unknown quantity	79-104
	7. Problem Posing based on given contexts	140-159
	8. Problem Posing based on given numbers	160-178

Table 1: Specification table of the tests on PS based on ST

The eight levels were mainly based on tasks proposed by Marshall (1995) for assessing the acquisition of each type of schema-knowledge. Specifically, the first two levels included tasks examining the verbal and diagrammatical identification of the schema needed for solving a problem (i.e. students were requested to identify the structure or the appropriate diagram which could be used to solve a given problem). The third level included tasks examining students' ability to select problems that could be solved using a given diagram, while the fourth level referred to their ability in posing a question to produce a problem of a certain structure. The four remaining levels included tasks such as filling the data in a given diagram to represent the structure of the problem (5th level), placing the unknown quantity in the correct position of a diagram (6th level), and posing a problem given either the context of the problem (i.e., a given diagram including words and numbers) or solely numbers (7th and 8th level, respectively). The specification table guided the construction of a battery of tests with 178 items, representing the eight aforementioned levels. Each level included tasks of all three problem-situations (change, group and compare situation).

METHODS

The items in the final version of the battery of tests were content validated by three experienced primary teachers and two university tutors of Mathematics Education. The "judges" of the tests were asked to mark-up, make marginal notes or comments on or even rewrite the items. Based on their comments, minor amendments were made particularly where some terms used were considered as unfamiliar to primary pupils. The final version of the tests (available on request) was administered to all 4th grade (314) and 5th grade (398) pupils from 21 primary schools selected by stratified sampling (336 of the subjects were boys and 376 were girls).

The Extended Logistic Model of Rasch (Rasch, 1980) was used and the data were analysed by using the computer program Quest (Adams & Khoo, 1996). The data were initially analysed with the whole sample (n=712) for all items together; it was found that all items fit the model. The analysis was repeated with each of the four groups (grade 4, grade 5, boys and girls) of the sample, to investigate whether the battery of tests is used consistently by each group of the sample. By taking into account the item difficulties derived from the analysis of the whole sample, we used the procedure for detecting pattern clustering in measurement designs developed by Marcoulides & Drezner (1999) in order to examine whether levels of PS skills similar to those described in the specification table could be identified. Moreover, separate analysis of the two sub-scales, which refer to the first two types of knowledge mentioned by Marshall, was conducted to analyse the meaning of the general scale and the trait it measures.

FINDINGS

Figure 1 illustrates the scale for the 178 test items with item difficulties and the whole group of pupils' measures calibrated on the same scale. Both figure 1 and the item fit map for the 178 items fitting the model reveal that all the items of the tests have a good fit to the measurement model. Moreover, pupils scores range from –3.44 to 3.58 logits and the item difficulties range from -3.66 to 3.62 logits. This implies that the 178 items of the test are well targeted against the pupils' measures.

Table 2 provides a summary of the scale statistics for the whole sample and for each of the four groups of the sample. We can observe that for the whole sample and for each group the indices of cases and item separation are higher than 0.85 indicating that the separability of the scale is satisfactory (Wright, 1985). We can also see that the infit mean squares and the outfit mean squares are 1 and that the values of the infit t-scores and the outfit t-scores are approximately zero. And since the mean squares are within 30% of the expected values, calculated according to the model, it can be claimed that there is a good fit to the model. Moreover, the analyses of each of the four groups separately revealed that almost all items (176 out of 178) have difficulties, which could be considered invariant among boys and girls, within the measurement error (0.15 logits). The difficulties of 171 out of 178 items are invariant between the two age groups but 7 items vary markedly across the two age groups. Thus, an important aspect of creating a scale (sample-free item difficulties) has not been completely achieved.

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High Achievement in Problem Solving      Difficult items
 Thresholds                             |
                               X        | 172 174 177 178 175 176 166
                              XX        | 155 157 159 162 170 171 165 167
                              XX        | 146 156 158 160 152 153 164 173
 3.0                         XXX        | 144 145 161 140 147 151 163 168
                            XXXX        | 142 143 150 141 148 149 154 169
                              XX        |
                              XX        |
                             XXX        |   80  81  83  93  94 95 96
 2.0                          XX        |   87  89  91  100 97 98 99
                            XXXX        |   79  85  86  92  103 101
                            XXXX        |   82  84  88  90  102 104
                           XXXXX        |
                       XXXXXXXXX        |  126 129 130 123 125
 1.0                  XXXXXXXXXX        |  120 124 128 110 121 115 122
                    XXXXXXXXXXXX        |  107 111 114 112 116 118 119
                  XXXXXXXXXXXXXX        |  105 109 106 108 113 117 127
                 XXXXXXXXXXXXXXX        |
              XXXXXXXXXXXXXXXXXX        |  137   135   139
  .0         XXXXXXXXXXXXXXXXXXX        |  132   136   134
              XXXXXXXXXXXXXXXXXX        |  131   133   138
               XXXXXXXXXXXXXXXXX        |
                  XXXXXXXXXXXXXX        |
                   XXXXXXXXXXXXX        |   74  78  70  72  75  76
-1.0                XXXXXXXXXXXX        |   58  68  60  56  65  66  77
                      XXXXXXXXXX        |   71  55  53  59  64  67  73
                        XXXXXXXX        |   62  54  57  61  63  69
                         XXXXXXX        |
                           XXXXX        |  51  52  50  49  48
-2.0                         XXX        |  28  30  31  32  41  45  46
                            XXXX        |  29  34  38  33  40  44  47
                             XXX        |  27  35  36  37  39  42  43
                              XX        |
                               X        |  6  21  22
-3.0                          XX        |  1  3   7   13   16   17  19
                              XX        |  5  11  9   14   18   20  23
                               X        |  2  4   8   10   12   15  24
                                        |  26  25
                                        |
Weak achievement in Problem solving      Easy items
----------------------------------------------------------------------
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Note: Each X represents 3 pupils

Figure 1: Scale for the battery of tests on PS (N=712, L=178)

The Rasch model was also helpful in analysing the conceptual design of the battery of tests. The indices of cases and item separation of each sub-scale (representing the two types of schema-knowledge) for the whole group and for each of the four groups of the sample are higher than 0.85 indicating that the separability of each sub-scale is satisfactory. Moreover, the infit mean squares and the outfit mean squares for the whole sample and for the four groups are one and the relevant values of the infit t-scores and the outfit t-scores are approximately zero. Thus, both sub-scales have satisfactory psychometric properties.

Statistics	Whole (n=712)	Boys (n=336)	Girls (n=376)	Grade 4 (n=314)	Grade 5 (n=398)
Mean (items)	0.00	0.00	0.00	0.00	0.00
(persons)	-0.18	-0.21	-0.16	-0.48	0.39
Standard deviation (items)	1.67	1.84	1.61	1.84	1.32
(persons)	1.19	1.45	1.06	0.96	1.22
Separability* (items)	0.99	0.99	0.99	0.96	0.99
(persons)	0.89	0.88	0.89	0.86	0.91
Mean Infit mean square (items)	1.00	1.00	0.99	1.00	1.00
(persons)	1.00	1.00	1.00	1.00	0.99
Mean Outfit mean square (items)	1.02	1.00	1.03	1.03	1.00
(persons)	1.02	1.01	1.04	1.02	1.01
Infit t (items)	-0.04	-0.05	-0.03	0.02	-0.02
(persons)	-0.01	-0.01	-0.01	-0.03	-0.01
Outfit t (items)	-0.05	-0.06	0.01	0.02	0.07
(persons)	0.04	0.08	0.05	0.05	0.02

Separability* (reliability) represents the proportion of observed variance considered to be true.

Table 2: Statistics relating to the scale for the whole sample and the four groups

Comparing the difficulties of the items of the two sub-scales, we can observe that the measurement model places a significant number of the items concerning identification knowledge at the easiest part of the scale and a significant number of items on elaboration knowledge at the harder part of the scale. In order to examine further this finding, the procedure for detecting pattern clustering in measurement designs developed by Marcoulides & Drezner (1999) was used. The cumulative D for the seventh cluster solution is 86% and the eighth gap adds only 2%. Moreover, all the gaps after the seventh gap are very small and this indicates that the 178 items are separable into seventh clusters. Thus, seven levels of PS skills based on ST can be identified. These levels are similar to the levels mentioned at the specification table of the test. More specifically, pupils who are at the first level (i.e. below –2.60 logits) are able to recognise verbally problems. The second level (-2.47 up to –1.78) refers to the diagrammatical recognition of problems. Pupils who are at the third level (-1.54 up to –0.85) are able to select a problem which reflects a given diagram. After the third level, there is a relatively big area where none item is included. This implies that there is a gap between the third and the fourth level (-0.24 up to 0.36) which refers to pupils' ability to pose questions from a mathematical situation. This gap can be attributed to the fact that although the skills of the first four levels refer to the first type of knowledge mentioned by Marshall (1995) pupils have to make an important progress in order to be able to pose questions. At the fifth level (0.45 up to 1.20), pupils are able to fill the data in a given diagram to represent the structure of a relevant problem and at the sixth level (1.50 up to 2.16) they are able to place the unknown quantity in the correct position of a diagram. Then there is a relatively big gap between the sixth and the seventh level (2.80 up to 3.62), which refers to pupils' ability to pose a problem based on a given diagram including either words and/or numbers.

DISCUSSION

The findings of this study provide support to the conceptual design of the proposed model of measuring PS skills. The underlying trait, that is pupils' abilities to use appropriate schemas to solve routine problems, seems to be an overarching concept comprised of the two types of knowledge mentioned by Marshalll and upon which the specification of the test was based. It would theoretically be expected that primary pupils would find it easier to develop skills concerning the identification knowledge rather than the elaboration knowledge. The findings of this study provide further support to this argument. Moreover, the procedure for detecting pattern clustering in measurement designs developed by Marcoulides & Drezner (1999) was found useful in supporting the theoretical background upon which the construction of the battery of tests was based. The seven levels of PS skills, which were identified, were similar to those described in the specification table of the test. However, this technique did not identify two different levels in relation to the problem posing skills concerning the elaboration knowledge, as it was expected. Moreover, a gap among the levels of each sub-scale was identified which revealed that problem posing skills are more difficult to be achieved than any other skill concerning each type of knowledge. This finding provides support to the assertion that problem-posing tasks are more complex and difficult for students than PS tasks (Silver, 1994; English, 1997).

The battery of tests on PS skills based on ST and its Rasch scale may help teachers decide how to identify and meet pupils' learning needs in relation to the seven levels of PS thinking and how to use their teaching time and their resources. An important implication of the identification stage is that it works as the first in sequence, facilitating pupils to make decisions, improve their abilities and move to the next stage (i.e. elaboration knowledge). Teachers should also be aware of the fact that the aforementioned two stages consisting of seven levels of thinking follow a linear sequential hierarchy. However, some pupils could be at the same level even though their abilities may differ. There is no clear distinction between consecutive levels, except between the levels concerning problem posing skills and those which are lower than them.

It goes without saying that further research is needed regarding the levels of PS skills based on ST. Specifically, further studies could explore whether the developmental model which emerged from this study and the seven levels, which were identified, may also derive from a study measuring pupils skills in solving problems of multiplicative structure. Furthermore, pupils' skills in solving two or three step-problems should be examined. Finally, there is a need to expand the model of measuring PS skills in order to refer to the next two types of knowledge (i.e. planning knowledge and execution knowledge). The findings of these studies may contribute in building a comprehensive model for PS skills based on ST.

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THE ROLE OF INSCRIPTIONAL PRACTICES IN THE DEVELOPMENT OF MATHEMATICAL IDEAS IN A FIFTH GRADE CLASSROOM

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The purpose of this report is to examine the role of classroom practices that emerged within a fifth grade classroom teaching experiment that helped coordinate and support student understanding of the Quotient construct through the invention, modification, and interpretation of inscriptions. Two types of inscriptions supported classroom discourse about quotients: Introduced inscriptions (those drawings and written marks offered to the class by the teacher or text materials as culturally appropriate forms of representing and communicating); and Ad-hoc inscriptions (those marks developed *in situ* by the students to convey their thinking). Through the use of public depictions of thinking, the classroom coordinated its practices by modifying Introduced forms to fit immediate demands of problem solving, and adopted the ad-hoc contributions of students to form a collective inscription retaining surface features and underlying meanings of both.

Theoretical Framework

The acts of doing mathematics and making sense of mathematical ideas takes place within the microecology (Lemke, 1997) of the classroom. They are reflected in the products created and used within the classroom microculture (Cobb et, al. 1997).

Most research in the domain of rational numbers has been conducted from an individual psychological perspective, whereas most classroom instruction is done in a social system. The analysis conducted in this paper attempts to account for the developments that occurred in a fifth grade classroom during mathematics instruction over a five week period. Our reasons for conducting the analysis was motivated by understanding how students co-constructed a unified understanding of the quotient construct and also the mediational means through which knowledge was constructed within the microculture of the classroom. Therefore, the practices and the products that were created and used as representations of mathematical ideas within the context of the classroom were examined. The emergent perspective takes into account that mathematical learning is a process of active social construction and a process of enculturation (Cobb et al, 1997).

A representation is an idea, experience, or object symbolized in another form. Because mental operations take place inside the mind, it is difficult to see or understand another individual's thoughts without some sort of mediational process (Davis, Hunting, & Pearn, 1993). Therefore, individuals need to explicate their thinking using external forms of representations such as through *inscriptions* (e.g., concrete objects, pictures, and symbols (Roth & McGinn, 1998). These external signifiers (Whitson, 1997) serve as cultural models through which communities of individuals can coordinate thinking (Ball, 1993; Brown, Collins & Duguid, 1989, Greeno, 1991, Kaput 1994).

Representations not only serve as tools for thinking but also for communicating thinking. When a base meaning has been negotiated for a particular inscription, then this taken as shared meaning serves as a "boundary object" to coordinate activities and divergent view points of several individuals, thereby facilitating the flow of resources (such as information, practices and materials) among the variety of individuals in a social setting (Roth & McGinn, 1998).

This means that as students think about a particular concept, they use various forms of inscriptions to record pertinent aspects of their thinking, and then use the set of inscriptions themselves as proxies for operating on the concept. Much of modern mathematics is dependent upon the systematization of inscriptions. In particular, when solving algebraic equations it is necessary to manipulate inscriptions and then retranslate the inscriptions back into conceptual forms is necessary (Kaput, 1999). New ideas are generated when new patterns of inscriptions signify different features of a concept than previously encountered (Ball, 1993).

Method

A five-week whole class teaching experiment was conducted in a fifth grade classroom in an urban school district in the Southwestern United States. Nineteen students and the classroom teacher participated in the study. An instructional unit based on Toluk's (1999) model of students' development of Quotient understanding was developed by the researchers and utilized as part of the instructional sequences. An Anchored Instruction (CTGV reference here) video was also developed as part of the instructional unit. The teaching experiment consisted of the following phases: classroom observation, clinical interview, teaching, analysis and hypothesis testing.

During the *classroom observation* process, whole group development and individual development were examined. The purpose of the classroom observation phase was to establish the physical, social, and cultural context of the classroom as a contextual referent within which lesson activities and thinking took place. The *clinical interviews* of the teacher and selected students provided additional information on the cultural context of the classroom. The *teaching phase* consisted of the actual implementation of the instructional unit in the classroom over a period of four weeks. These lessons were videotaped. The researcher collaborated with the participating teacher informally to modify and discuss the progression of lessons immediately preceding and following each teaching episode. The researcher also served as a recorder of information by making field notes during the teaching phase.

During the *analysis* and *hypothesis testing phases,* the researcher analyzed the data collected from the classroom observations and student work. Student work was analyzed and coded. The videotapes were transcribed. The classroom discourse was analyzed in relation to the activity of creating and translating inscriptions to make meaning of the quotient construct. The hypothetical learning trajectory of the whole class was examined. The teacher and the researcher formally met once a week to discuss the findings and make modifications to the instructional unit based on the findings. The researcher identified the "essential mistakes" students made during the lessons from the analyzed data and used it as a basis for discussion with the teacher. The teacher thought about the essential mistakes students made, and reflected on how she could help her students to overcome them. These reflective sessions resulted in the teacher modifying her teaching practices and the researcher modifying the instructional unit.

Both whole group interactions and a focus group of 4 individual students were videotaped. Digital images of all marks made in students' notebooks were recorded. In the analysis, the creation and modification of inscriptions were used as a focus for examining the connections between students' individual knowledge construction, their contribution to the knowledge of the collective, and the ways in which the practices of the collective constrained those of the individual students. For example, students' written marks in their notebooks often were different than those depicted during whole group discussion. When a student chose to annotate or alter a mark at the board, we cited this as evidence of individual contributions to the collective inscription. The relationship between individual conceptual development and collective development was made by comparing these marks and how they changed over time. For example, when students' annotations in their individual notebooks changed following a whole group discussion, we cited this as evidence of the collective practices constraining those of the individual. In recording the changes in inscriptions and the discourse that accompanied these changes, we were able to abstract a learning trajectory for the whole class, and note the individual inscriptions and thinking in the hypothetical learning trajectories of the individuals within the class. Normative inscriptional practices for creating and translating inscriptions within were noted during the teaching episodes as students engaged in whole-group discussion on a flip chart (all pages of the flip chart were displayed on the board throughout the 5-week teaching experiment).

Results

Students engaged in the practices of creating and translating inscriptions to make mathematical sense of the relationship between fractions and division with regard to the quotient construct. The whole-class learning trajectory emerged out of these practices. First the emergent collective learning trajectory will be briefly described, and then the inscriptional practices through which this learning trajectory emerged will be explained. Discussion will center on how these inscriptional practices became part of the classroom norms.

The learning trajectory established in the classroom corresponded to key uses of drawings and written marks. These changes roughly followed three Phases: In Phase I, students pictorially represented and solved fair sharing problems. They partitioned the unit into equal sized pieces and distributed them into groups to make equal shares. Initially students represented the answer as a whole number indicating the "number of pieces that made up each share. Then when asked "how much" the number of pieces represented in relation to the whole, students were able to symbolize the answer as a fraction. In Phase II, students focused on understanding that a fair-sharing problem represented a division relationship between two extensive quantities. They began to realize that this division relationship could be symbolized using the conventional notation of the fraction bar or a division symbol (÷, /).

In Phase III, students began to symbolically represent the fair-sharing problem and solution and started to think about the relationship between the division problem and solution in terms of quantities. Prior to this phase, students had been solving the fair-sharing problems using pictorial representations and then symbolizing the answers. Students first represented the answer of the fair-sharing problem as a fraction or a whole number and fraction on the right side of the number sentence. Then they proceeded to figure out how to symbolically represent the left side of the number sentence containing the division relationship between the quantities of the fair-sharing problem. Therefore, students treated symbolizing the number sentence as a two-step process where they first symbolized the answer and then the division relationship. They thought about each side of the equation as quantities, but they did not focus on the relationship between the fair-sharing problem and answer as a number sentence.

Focusing Attention to the Immediate Problem Context

Students and teacher generated inscriptions on chart paper during whole group discussion to make thoughts and ideas explicit. The whole class focused their attention on the representational context of the person creating an inscription as he or she drew pictures, symbols, or wrote words. The person usually explained what the inscription represented and verbally recounted a running narrative of what he or she was producing. Simultaneously that person used gestures such as pointing and engaged in the act of writing or drawing on the chart paper to draw the whole class' attention to the particular thought or idea being communicated.

The act of creating an inscription and providing an explanation coordinated the group's attention to the immediate problem context. Therefore, the act of creating the inscription and the whole class's focus on this action was situated. Even though there were other inscriptions (other parts of the same inscription) that had already been created and were visibly displayed, the whole class focused their attention on the part of the inscription that was in the process of being created.

Record of Distributed Thinking

Once an inscription was created during whole group discussions, then it became a historical record of the dialog that took place and hence, could be recalled as a common exemplar to which all participants could attach meaning. Furthermore, the inscription also represented multiple viewpoints of several different students within the group—i.e., such an inscription represented the distributed expertise of the group.

The types of socio-mathematical norms that emerged included: 1) proving or defending one's idea in relation to another student's verification; 2) model competition, and 3) explaining and clarifying another student's thinking (e.g., perspective taking).

Inscription Became a Tool to Coordinate Student Attention to the Problem Context

Inscriptions were used as tools to focus student attention to the problem context. The inscription was used as a reference point to re-create previous conversations and actions. Most whole group mathematical discussions that took place focused on uncovering "essential mistakes" or misconceptions that student were experiencing during that day's lesson or the previous day's activities.

The teacher influenced the focus and direction of the whole group conversation, which became part of the inscriptional practices that took place. In other words, the topic of conversation and the nature of the inscriptions were influenced by the actions of the teacher. The teacher also played an instrumental role through her questioning and statements to facilitate the coordination of inscriptions and dialog to make mathematical meaning.

<u>Analyzing Another Student's Thinking About a "Taken-as-Shared" Inscription</u>

Analyzing another student's thinking during the whole group discussions became another mathematical norm. In doing so, students were forced to actively think about another student's strategy and reasoning and relate it to how they thought about the problem.

<u>Continuous Reconstruction of Meaning</u>

Ideas recorded in inscriptions were continually re-examined throughout the teaching episodes and reconstructed in the process of developing meaning. Inscriptions were initially interpreted on a surface level. This meant that the students were focusing on examining what the immediate problem represented and what was required to solve the problem. A deeper level of understanding was reached when the students were able to connect their understanding of the immediate problem to the mathematical concepts that could be abstracted to another situation.

Discussion

In this paper, we have documented the role of inscriptional practices in the development of student understanding of the quotient construct over a five week period. In doing so, we have pointed out that the inscriptional practices that took place mediated the creation and translation of inscriptions which ultimately influenced the type of knowledge constructed as reflected in the learning trajectory that emerged. The inscriptional practices that took place were part of the classroom microculture and became part of the classroom norms. Therefore these practices must be taken into consideration when examining how knowledge is constructed in the classroom.

Agents and cultural tools mediate human action Wertsch (1998). The agents were the students and the teacher and the inscriptions represented the cultural tools produced through interactions between the agents and the tools within the classroom. These interactions of creating and translating inscriptions to make sense of the quotient construct became the mediational means through which the learning trajectory emerged.

The inscriptional practices that are described in this paper also illustrate how ideas that were recorded as inscriptions were adapted and modified through time. Inscriptions became used as tools. The nature of a tool has meaning and significance based on the context in which it is used. A tool by itself does not have any significance. The use of the tool changes the situation and perception of the user (Wertsch, 1998). In this whole class teaching experiment, students created inscriptions as tools to make their thinking explicit and communicate their thinking to others. They translated inscriptions by examining them to make meaning. Inscriptional practices that took place represented individual contributions as well as the distributed thinking of the group. The usefulness and the meaning of these inscriptional practices must be understood with regard to the learning trajectory that emerged.

Lastly, the inscriptional practices described in this paper, because they packaged a complex discourse into relatively few retrievable exemplars whose structure embodied key features of quotients, afforded students the opportunity to revisit and build upon ideas efficiently to make sense of the quotient construct. The purpose of making sense of the division and fraction relationship to understand the concepts guided the meaning making that took place within the inscriptional practices. There was a continuity of ideas that were revisited and build over time through these inscriptional practices, as opposed to the inscriptions being generated as a series of disconnected activities (e.g., as "representations"—temporally static pictures with a fixed meaning). Therefore the nature of the inscriptional practices in a classroom can either afford or constrain concept building that takes place in the classroom.

Further investigation is needed to fully understand the role of inscriptional practices within the classroom. How do inscriptional practices get established as classroom norms in the classroom? What elements of the inscriptional practices can afford or constrain thinking?

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THE NATURE OF PUPILS' PROBABILISTIC THINKING IN PRIMARY SCHOOLS IN CYPRUS

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In this research work we explored the nature of 9-12 year old pupils' responses to probability problems. Analysis of pupils' arguments in 'Explain why' questions uncovered their thinking strategies, which we compared for pupils of different age and gender. The results revealed the existence of subjective elements and other errors in pupils' probabilistic thinking. The data were generated in year 2000 when the new mathematics books had just introduced probability extensively in the primary curriculum. Since the relevant literature in Cyprus is sparse, the results of the study form a general overview of the pupils' errors and build the basis for further in-depth and more focused research.

INTRODUCTION

Probabilistic thinking is a mode of reasoning attempting to quantify uncertainty, as a tool for decision making. A worldwide increased attention to probability and statistics is given and in Cyprus new mathematics books were written that introduced probabilities extensively in the primary curriculum. Nevertheless, pupils' probabilistic thinking is influenced by culture and is sensitive to cultural experience (Amir & Williams, 1994, 1999). There is, however, very little research in Cyprus focused on pupils' errors and probabilistic thinking (Gagatsis et al, 2001).

Research into pupils' capacity to compare two probabilities started with Piaget and Inhelder (1951). Following Piaget and Inhelder, other researchers have undertaken the study of pupils' abilities to compare probabilities (Green, 1983; Fischbein & Gazit, 1984; Canizares et al, 1997). Since comparing probabilities entails the comparison of two fractions, proportional reasoning is considered to be a basic tool of probabilistic reasoning. An important difference between comparing fractions and comparing probabilities is that the result of a proportional problem refers to a certain event, while the result of a probability problem implies a degree of uncertainty. Therefore, pupils' answers might be influenced by intuitive judgments. These intuitions are somehow some cognitive beliefs that sometimes may coincide with scientific accepted statements but some other times they may not or may contradict them. Furthermore, pupils consider subjective elements to assign probabilities.

In a previous study (Canizares et al, 1997) pupils' strategies, when comparing probabilities in tasks, were analysed and pupils' arguments were classified according to the following strategies: (a) Single variable strategies (comparing the number of possible cases; comparing the number of favourable cases and comparing the number of unfavourable cases), (b) two variables strategies (additive strategies, correspondence and multiplicative strategies) and (c) other types (equiprobability bias, outcome approach, taking decision depending on other irrelevant aspects in the task).

It has long been known that pupils' errors and misconceptions can be a starting point for effective diagnostically designed mathematics teaching (Williams & Ryan, 2000). If the teachers are aware of the most common errors and misconceptions that pupils have in probabilities, they will try to develop classroom strategies for helping students to confront them (Fischbein & Gazit, 1984; O'Connell, 1999).

This study aims to explore the probabilistic thinking of primary school pupils in Cyprus aged between 9-12 year old and to study the effect of age and gender.

METHOD AND INSTRUMENT

The research instrument was developed through several pilot steps. Informal unstructured interviews with pupils and teachers helped us to evaluate whether they interpreted the questions correctly. The pilot test was administered to pupils in Year 4 (90), Year 5 (51) and Year 6 (47).

The final version of the test consisted of nine questions clustered into three subtests. The first subtest consisted of Questions 1 and 2. Both tested the ability of the pupils to identify the most likely event from a single sample space (Single Variable questions). Question 1 was:

- Q1: There are 5 blue, 4 yellow and 3 green rubbers in a drawer. I pick a rubber randomly, without looking into the drawer. (a)What colour of rubber is more likely to pick? (b)Why?

The second subtest is cognitively more demanding than the first one. Questions 3, 4, 5 and 7 tested the ability of the pupils to define the probability of two independent events and to compare the two probabilities to decide which of the two events was the most likely.

- Q3: I have two bags with marbles. Bag A has 4 marbles, 2 blue and 2 green. Bag B has 6 marbles, 3 blue and 3 green. (a)From which bag do I have the largest probability of picking randomly, without looking in the bag, a blue marble? (b)Why?

- Q4: The yellow box contains 3 chocolate biscuits and 6 strawberry biscuits. The green box contains 1 chocolate biscuit and 2 strawberry biscuits. (a)From which box would you prefer picking randomly a biscuit in order to have a larger probability to get a chocolate biscuit? (b)Why?

Questions 6, 8 and 9 tested the reactions of the pupils to additional but irrelevant information gathered from the every day life. It was noticed from the pilot test that some pupils omitted the quantitative information they were given (numbers) and focused on irrelevant information like the size of an animal, the size of a marble and the number written on a card.

• Q6: Andrew wants to decorate his Christmas tree. On the carpet, there are 4 big and 6 small golden balls. Since he does not care if he starts the decoration with a small or with a big ball, he selects randomly a ball from the carpet. (a)What is it more likely to select firstly, a small or a big ball? (b)Why?

• Q8: In a zoo there are 2 elephants and 4 monkeys. Today the staff of the zoo wants to choose randomly an animal to wash. (a)What is it more likely to choose, an elephant or a monkey? (b)Why?

THE SAMPLE

The final instrument was administered to 426 pupils in four different district schools in Cyprus, 222 boys and 204 girls. The sample consisted of 169 pupils in Year 4, 132 pupils in Year 5 and 125 pupils in Year 6.

RESULTS & DISCUSSION

Single Variable Strategies: Pupils' correct responses can be based on intuition

In Question 1a, 381 pupils (90%) answered correctly but only 32 pupils (7.5%) explained their thinking strategy mentioning the concept of probability. Sub-questions Q1c, Q1d and Q1e asked pupils to identify the probability of drawing randomly a blue, a yellow and a green rubber. 44.8% of the sample gave a correct response to all three questions but 106 pupils (25%) failed to identify any of the probabilities correctly. 90 of them (21.2%) answered correctly that a blue rubber was more likely to be picked than a yellow or a green rubber. This result indicated that one to five pupils answered the question intuitively without being able to express in written any formal probabilistic thinking. Indeed, 87 of those 90 pupils explained that a blue rubber was more likely to be picked because "there are more blue rubbers in the drawer than rubbers of the other colours" but they failed to identify any of the probabilities of the three alternative events.

Single Variable Strategies: The effect of subjective elements

In Question 6, 69.7% of the sample answered correctly that it was more likely to pick randomly a small than a big ball. However, 72 pupils, approximately 17% of the sample, responded that it was more likely to select a big ball. Approximately 11% supported their response by saying that "it is easier to pick a large ball than a small one" or "because the small balls are harder to pick" and other similar statements.

In Question 8, 76.8% of the sample answered correctly that it was more likely for the staff to randomly select a monkey than an elephant. In total, 19% explained their thought mentioning physical and other characteristics of the two animals. For example, 15% of the pupils based their answer on the physical size of the elephant. Other wrong explanations included information about the elephants being quieter or dirtier than monkeys. Four pupils from different classes mentioned that one elephant equaled two monkeys (probably in size), therefore, two elephants equaled 4 monkeys and, consequently, the probability to select a monkey or an elephant was the same.

In Question 9 we had comparable results. The results showed that the probabilistic thinking of some pupils might be influenced by subjective elements. The percentages of those pupils according to the three questions above might lie in the area of 11% (Q6), 19% (Q8) and 10% (Q9). These percentages may be significantly underestimated because sometimes it was difficult to distinguish from the written explanation whether pupils' thought was influenced by subjective elements or not.

Two Variables Strategies: Comparisons between favourable or unfavourable events

In Question 3, only 21.1% of the pupils realized that the probability of getting a blue marble was the same in both bags. 257 (60%) of the pupils declared that they would have a larger probability of picking a blue marble from bag B. 246 of them (57.8%) explained their response in Question 3b by saying "bag B contains more blue marbles than bag A". From the responses of the pupils, it seems that they mainly compared the number of blue marbles in the two bags instead of the probability of getting a blue marble from each bag.

106 pupils (24.9%) gave correct responses to Q3c and Q3d that asked for the probabilities of getting a blue marble out of each of the two bags (they were ½ in both cases) but also claimed that it was more likely to get a blue marble out of bag B. Most of them explained that there were more blue marbles in bag B indicating that although these pupils knew how to find the relevant probabilities of the two events, they still used their own intuitive rules to take the decision of the most likely event.

In Question 4 we had similar results. Only 18.1% of the pupils realized that both boxes contained the same proportion of chocolate biscuits. 232

(54.5% of the sample) responded that the yellow box would give a larger probability to pick a chocolate biscuit. 218 of them (94%) explained their response by saying "there are more chocolate biscuits in the yellow rather than in the green box". Again, it seems that they focused on the number of chocolate biscuits in the two boxes instead of the proportion of chocolate biscuits in each box. 69 pupils (16.2% of the sample) gave correct responses to Q4c and Q4d that asked for the probabilities to get a chocolate biscuit out of each of the two boxes (1/3 in both cases) but used their own intuitive rule to answer incorrectly Question 4a. Questions 5 and 7 gave similar results with Questions 3 and 4.

Construction of an 'ability' measure

Eight of the nine questions of the test (excluding question 2) ask pupils to indicate the most probable between two events (e.g. to draw a blue marble from bag A or bag B) and to explain verbally their thinking strategy. A correct mark was awarded for a correct response to the question and one additional mark was awarded to the pupils whose explanation indicated a 'correct' probabilistic thinking strategy.

A Partial Credit Rasch model (Wright & Masters, 1982) was fitted on the data, using Quest. The mean difficulty for the questions was constrained to zero (SD=1.18 logits) and the reliability of question difficulty estimates was 0.96. The mean of the pupils' ability was -0.18 logits (SD=0.98 logits) and the reliability of pupil estimates was 0.66. All questions had a satisfactory fit. The test and the sample may be interpreted as falling into a hierarchy of three levels (Figure 1).

At level A (-2.5 to –1.5 logits) pupils can succeed on answering correctly questions that tested for the identification of the most likely outcome from a single sample space (Single Variable problem). At level B (-1.5 to 0 logits) pupils can succeed on answering correctly questions that tested for the identification of the most likely outcome from a single sample space (Single Variable problem) without being carried away from subjective elements (e.g. elephants are quieter than monkeys). At level C (0 to 2.5 logits) the pupils can succeed on answering correctly even harder questions, which ask them to compare the probabilities of two independent events and decide which of the two is the most likely to happen (Two Variables problems). Pupils at higher level can also answer correctly the questions of the previous levels.

According to Figure 1, a percentage of pupils was influenced by the 'Subjective Elements' effect (from 10% to 19%). In general, this type of error is more representative of pupils of low ability. The mean ability of the pupils who made the errors associated with the 'two variables' is near to the mean ability of the sample. The percentage of pupils making this error is relatively large (47% to 66%).

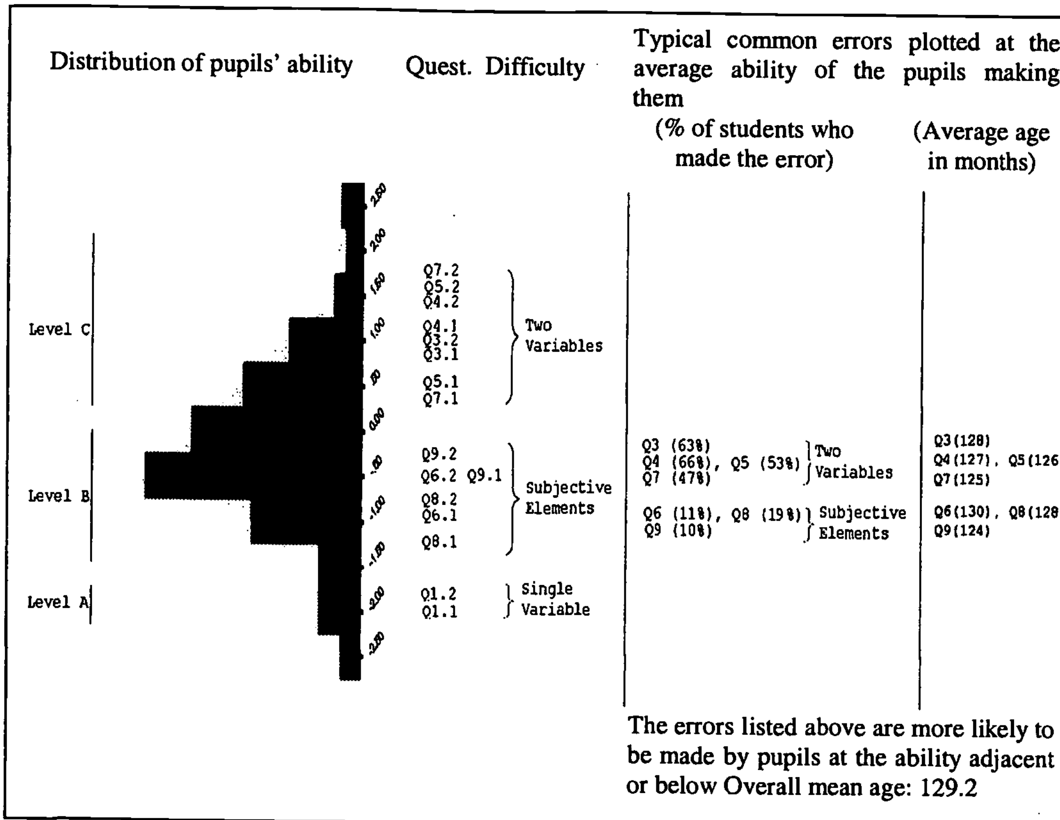

Figure 1: A 'probability performance' measure

The pupils in Year 4 are significantly less able than pupils in Year 5 and Year 6 ($F=17.35$, $df=2$, $p<0.01$). The difference in the average ability of the three year groups (average ability: Year 4= -0.52; Year 5= 0.03; Year 6= 0.12) was significant only between pupils in Year 4 and in Year 6.

Construction of the 'likelihood ability' measure

Some of the questions of the test were followed by sub-questions, which asked the pupils to identify the likelihood of certain events. A mark was awarded for each correct response and the simple Rasch model was fitted on the data, using Quest. The mean difficulty for the 13 questions was constrained to zero (SD=1.2) and the reliability of question estimates was 0.98. The mean ability for the pupils was –0.05 (SD=2.03) and the reliability of ability estimates was 0.83. The fit statistics were satisfactory except for one question, which was only marginally satisfactory.

The correlation between the 'likelihood ability' (ability to quantify the likelihood of events) and the 'ability' (performance to the test) of the pupils is 0.58, which is not as high as expected. Indeed, this showed that it was not necessary for the pupils to be able to quantify the probabilities of the events in order to give correct responses to the questions of the test.

It is possible that the pupils were able to answer the questions intuitively and the ability to numerically identify the likelihood of the events was not a prerequisite.

Predictors of pupils' 'ability'

The test included 8 questions, which asked the pupils to compare two fractions, similar to the ones used in the probability questions. This subtest was administered only to 223 pupils. A mark was awarded for each correct response and the distribution or pupils' scores had a mean of 4.7 and SD=2.5. The correlation between the 'ability' measure and the raw score in the fractions sub-test is 0.37; not as high as might be expected.

A multiple regression model was attempted using pupils' 'ability' as a depended variable and the 'fractions raw score', the 'likelihood ability', the gender and the age as predictors. The stepwise procedure accepted in the model only two terms: the 'fractions raw score' and the 'likelihood ability' (R2=0.38). When the 'fractions raw score' was not included in the predictors, the stepwise procedure kept only the predictor 'likelihood ability' in the model with almost no loss of information (R2=0.33). Gender and age were not good predictors of 'ability'.

CONCLUSION

We have managed to develop two scales describing pupils' responses to the instrument, which is revealing about their probabilistic thinking, especially as regards their inappropriate use of intuitions. We have further identified from pupils' responses that some pupils were influenced by other irrelevant aspects in the tasks (subjective elements). The knowledge that teachers would collect from these scales would enrich teachers' mental models of their learners and would help them improve their classroom practice.

Having collected these data, however, we should further continue interviewing pupils in order to shed more light into pupils' probabilistic thinking. Further research into pupils' probabilistic reasoning would be an essential step for selecting teaching and assessment situations. We will be studying this aspect in the next stage of the work.

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"What if not?" Problem Posing and Spatial Geometry - A Case Study

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Abstract

The aim of the study was to try to find out what kinds of problems are posed by prospective teachers on the basis of complex spatial geometry tasks using the "what if not?" strategy, and what is the educational value of such an activity. Analyzing the posed problems revealed a wide range of problems ranging from problems including a change of one of the numerical data to another specific one, to a proof problem. We also discuss the educational aspects of problem posing in spatial geometry using the "what if not?" strategy, which could encourage the learner to rethink spatial geometry concepts and make connections between them.

Background

Generally the term "problem posing " refers to three kinds of cognitive activities: (a) posing sub-problems within the process of solving a complex problem; (b) posing new problems on the basis of a solved problem; (c) posing problems regardless of the referred problem solution (Technai, 2001). Three types of problem posing can be distinguished in regard to posing new problems on the basis of a given problem: problem posing on basis of free, semi-structured and structured problems (Southwell, 1998). The term "problem posing" in the literature usually refers to an activity in which the problem posing itself is the focus of attention and not as a problem-solving tool.

This study focuses on problem posing on the basis of a given structured problem using the "what if not?" strategy (Brown & Walter, 1993). According to "what if not?" strategy, we examine each component of the problem and manipulate it through the process of asking "what if not?"

The problem posing role in students' mathematics learning

During the study of school mathematics, students experience problem solving. Usually they get the problems from the math teacher or from text books and only rarely are they asked to pose problems of their own. Mathematics educational researchers emphasizied the important educational value of problem posing by students and suggested incorporation of activities of problem posing within the mathematics sessions in school (Goldenberg, 1993; Leung & Silver, 1997; Mason, 2000; NCTM, 2000; Silver, Mamona-Downs, Leung & Kenney, 1996). The importance of an ability to pose significant problems was recognized by Einstein and Infeld (1938), who wrote:

"The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old questions from a new angle, require creative imagination and marks real advance in science"(p.92) (in Ellerton and Clarkson (1996).

Students who engage in problem posing activities become enterprising, creative and active learners. They have the opportunity to navigate the problems they pose to their domains of interest according to their cognitive abilities (Goldenberg, 1993; Mason, 2000). Studies show that problem posing might reduce mathematics anxiety (Brown & Walter, 1993; Moses, Bjork & Goldenberg, 1990). Including activities of problem posing might improve the students' attitude towards mathematics and make them more responsible for their learning (Brown & Walter, 1993; Silver, Mamona-Downs, Leung & Kenney, 1996). Researchers emphasized the inverse process in which developing of problem solving skills can be helpful to the developing of problem posing skills (Brown & Walter, 1993).

The problem posing role in teachers' mathematics learning

Developing skills of mathematical problem posing is important for mathematics teachers as well (Silver, Mamona-Downs, Leung & Kenney, 1996; Southwell, 1998). Southwell (1998) found that posing problems based on given problems could be a useful strategy for developing the problem solving ability of prospective mathematics teachers. Integrating problem posing activities in their mathematics lessons enabled them to get to know their students' mathematical knowledge and understanding better. Since teachers have an essential role in the use of problem posing activities in their mathematics classrooms, they should develop their own problem posing skills (Leung & Silver, 1997).

The study

In the present study we explored two questions:

1. What are the kinds of problems posed by prospective teachers on the basis of complex spatial geometry tasks using the "what if not?" strategy?
2. What is the educational value of posing problems using the "what if not?" strategy referring to spatial geometry?

Eighteen prospective teachers participated in the present research.The participants were participating in the "teaching methods for secondary school mathematics" course in which they were introduced to methods and activities referring to central topics studied in school.

The participants were asked to pose as many problems as possible on the base of the following problem:

> Find the distance from the center of the base of a regular triangular pyramid to the pyramid's lateral face given that the pyramid's height is 10 cm and the dihedral angle (the angle between the two faces) is 67^0.

Our data comes from two sources:
(1) written protocols submitted by the participants.
(2) clinical interviews with several participants.

Results and discussion

The problems posed by the participants can be divided into two main categories: changing one of the given problem components; changing the problem question. Each one of the above categories was refined into subcategories, which we describe below. The changing of one of the components given in the problem can be divided into two subcategories: 1.Changing of the numerical value of the data; 2. Changing of the kind of data. The subcategory "Changing of the numerical value of data" refers to:
1.1. A change of a specific numerical value, given in the problem, to another one. For example, the pyramid height was changed from 10 cm into 12 cm.
1.2. A change of a specific numerical value, given in the problem, into a range of values. For example, the angle between the lateral faces was changed from 67^0 into an angle between 67^0 and 90^0.
1.3. Negation of the specific numerical value given in the problem. For example, what if the hieght is not 10?
1.4. Implied generalization of the given numerical value of data. For example, the height of the pyramid is not 10 cm (for example, 12, 20 and so on).
The subcategory "Changing of the kind of data" refers to:
2.1. A change of a specific kind of data, given in the problem, into another kind. For example, from a triangular base pyramid to a square base pyramid.
2.2. Negation of the given data kind. For example, what if not a pyramid?
2.3 Generalization of the given data kind. This sub-category can be divided again into implied and formal generalization. The term "implied generalization" refers to the participants' answers in which they offered a number of alternative examples to the negated component of the problem finishing their answer by the words: "and so on". This term is taken from a similar classification of students' utterances in Lavy (1999).
2.3.1. Implied generalization of the given data kind. For example, a pyramid does not have a triangular base (for example, quadrangular, pentagonal and so on).
2.3.2 Formal generalization of the given data kind. For example, the pyramid has an n regular polygon base.

The second category " Changing of the problem question" refers to another specific question. For example find the pyramid base area. It can be referred also to the changing of the given problem into a proof problem, for example, proof that sin $\alpha/2 = 5/8$ while the relation between a lateral edge of the regular triangle pyramid to the base edge is 5/9.

The sub-categories related to the negation of one of the problem data components could be considered as an uncompleted new posed problem, since the participants did not offer an alternative to the negated data component.
To visualize our findings, we draw the following diagram:

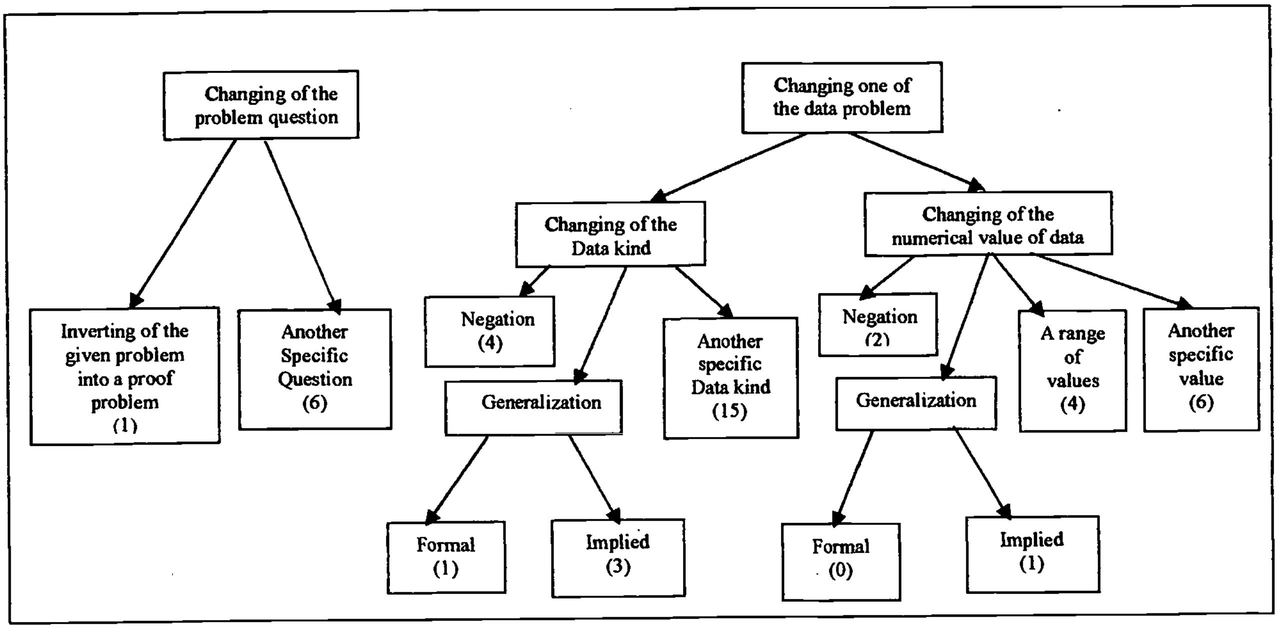

Diagram 1:
Classification according to data and question change

In cases where we had difficulty deciding to which category a suggested posed problem belongs, we were aided by the participant's interview. For example, the participant's utterance "what if not a triangle?" was interpreted as an implied generalization. Since in other posed questions of the same participant when he or she indicated another specific shape, he or she wrote so explicitly.

Classification of the posed problems reveals a wide range of problems ranging from easily phrased problems to the inverting of the original problem to a proof problem. The term "easily phrased" problems, refers to the change of one (or more) components in the original numerical data to another specific one.
The study revealed that the number of posed problems with another numerical value had a greater frequency than generalized posed problems (9 versus 3). It could be explained that posing another numerical example is a common activity in mathematics lessons in school and students are more familiar with this kind of activity than with posing generalized problem on the base of the numerical data. In other words, it could be said that when students are asked to pose a new problem on a basis of a given numerical data problem, the technical and the immediate widespread strategy is to pose a similar problem with another numerical value. Analyzing the solution of numerical data problems during school mathematics, is not a common activity (Polya, 1981; Schoenfeld, 1985). The solution analysis

including the testing of different numerical values could result in problems with another type of solution (if any) and might lead eventually to problem generalization. In addition, posing a generalized problem on a basis of a given problem is a complex cognitive activity rather than a technical change .

Another issue refering to posing problems with different numerical data emerged from Maria's interview. Observing her posed problems revealed that none of them included similar problems with new numerical values.

> Maria:"What if the height is not 10 cm? If I had the problem solution I would have seen how the change of height influences it. Without the solution it is difficult for me to decide what to take instead of 10. Maybe there is a number that doesn't influence the solution, but may be there exists a value that will change the problem to another one with a different way of solution".

From the above excerpt it follows that Maria added another constraint to the given task. She strove to pose problems that will have another mathematical solution, different from the original one and not just a cosmetic solution change. If she had solved the problem first, it could have helped her to estimate new numerical values that could lead to different ways of solution.

> Maria: "It was difficult to me at the beginning. I did not know where to start. When I solve the problem I know what I am using, and what can influence the solution. Without the solution it is difficult for me to visualize influential changes, that will lead eventually to a reasonable solution. I felt I needed to solve the problem first. I tried to pose a problem, but I could not come up with anything".

According to Maria's interview the fact that the participants were not asked to solve the problem first prevented her from posing meaningful new problems.

Julia, on the other hand, said that it was easy for her to pose many problems on the basis of the given one, emphasizing that if she had to solve the given problem first, she would have difficulties in posing many new problems.

> Julia: "Because you do not see the spatial problem, you are in the computing process. You become technical and after that you cannot disconnect yourself".

Julia needed the overview of the problem to be able to pose many new problems on a basis of the given one. For her, getting into the solution process diverted her attention from having the global viewpoint that enabled her to focus on the activity of posing problems.

Students differ in their view as to the assistance that a familiarity with a solution provides in problem posing.

One of the interesting phenomena is the lack of posed problems including formal generalization and the relatively small number of posed problems including implied generalization. As to posing problems including implied generalization, there is a difference between the number of posed problems including changing of numerical data versus posed problems including generalization of the given geometrical shape. The process of generalizing the numerical value of data in geometric problems leads to the analysis of extreme cases such as zero. Most of other numerical data changes cause mainly a cosmetic change of the problem, while the change of the data kind usually offers a new problem with a totally different solution.

Observing carefully the data reveals also that in "data kind change" posed problems there were more changes of plane data kind than spatial data kind (9 versus 2). There was a small scope of spatial shapes than of plane shapes in the posed problems. The most common plane shape was a square and it can be explained that the common students' example for a quadrangle is a square (Hershkowitz et al., 1990).

Considering the educational aspect of problem posing in spatial geometry using the "what if not?" strategy, such an activity makes the learner rethink different spatial geometry concepts. For example, questions such as "what if not a pyramid?" or "what if not a regular pyramid?" require the learner to know and understand the relevant geometrical concepts appearing in the problem and at the same time to think of the meanings and outcomes emerging from the negation of them. For example, if the concept in the given problem is a pyramid, the learner should know and understand the pyramid definition and think of the potential alternative concepts that might replace this concept, such as a cube, in the process of posing problem using the "what if not?" strategy. Concerning the question: "what if not a regular pyramid?" we will refer to the following paragraph from Maria's interview:

Interviewer: "Let's have a look at your posed questions. Could you clarify some of them? You wrote, what if not a pyramid? What did you mean?"

Maria: "Another shape, cone or rectangular parallelepiped. What if not a triangular pyramid: quadrilateral, pentagonal or any other polygon. What if not a regular pyramid: regular means that all the triangles are equilateral triangles...no ...[mumbling] only the base, the others are isosceles. What if not equilateral triangular base, but isosceles triangle or anything."

Observing Maria's posed problems during the workshop made us decide to interview her, since Maria posed, in comparison to others, many problems which had a foggy formulation and we thought it would be helpful for our analysis to get some clarifications. We asked Maria to tell us what she meant by her posed problem: "what if not a pyramid?" and her answer appears in the first part of the quoted paragraph. As to the question "what if not a regular pyramid?", Maria was thinking aloud about the meaning of a regular pyramid: a pyramid composed of equilateral triangles which later on she corrects herself to the definition that only the pyramid base has to be equilateral and the pyramid faces should be isosceles triangles. Maria's rethinking creates a new problem, which negates the regularity property of the pyramid. ... "What if not equilateral triangular base, but isosceles triangle or anything else."

Maria posed a new problem using the negation of the pyramid base regularity property. By doing so, she created a non-regular pyramid since its base is not an

equilateral triangle anymore. As an alternative to the given pyramid base, she offered an isosceles triangular base or non particular triangular base.
In order to be able to raise problems using the "what if not?" strategy, Maria deconstructs the term "regular pyramid" into a list of properties which define this term, negates one of them and builds an alternative geometric entity. The act of thinking reflectively about the properties of a certain geometric shape and the geometric implications emanating from the negation of one of the properties is a cognitive activity leading to deeper understanding of the relevant geometric terms. If we continue the same line of thinking, several other problems could be raised such as: what if the pyramid is not right? What if the pyramid faces are also equilateral triangles? What if the pyramid consists of scalene triangles? etc. Such a discussion could contribute to strengthening the connections between the various geometric terms and as a result to deepen the understanding of them.

An additional educational benefit of problem posing using the "what if not?" strategy is connected with the following issues: Whether the data change of one (or more) of the given problems leads to a solvable or unsolvable problem? Whether the created problem solution is totally different from the original one or are there only cosmetic changes in it? These issues appear in diagram 2.

The second classification we made was the influence of data change on the solvability of the problem and its solution.

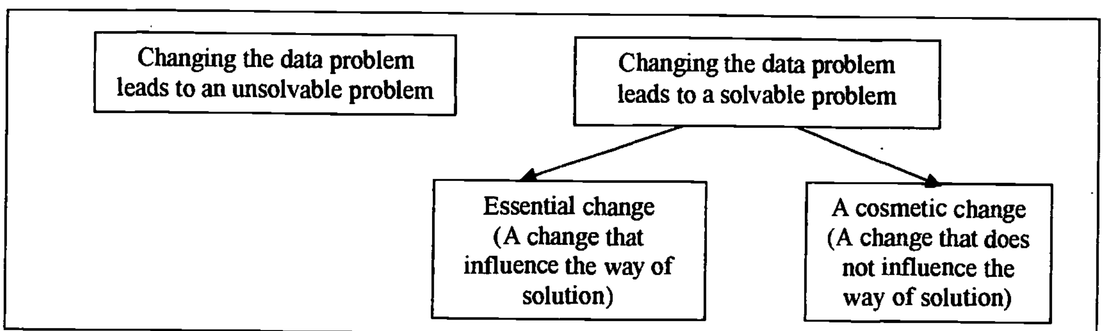

Diagram 2:
Classification according to data influence on the problem solution (if any)

Although our study is a case study, we found interesting phenomena with educational and learning benefits. The present case study raises some questions, which could be the basis to further studies, such as:

1. How can we cause learners to "produce" new correct mathematical problems?
2. How can we explain the phenomena of raising many problems (or few) and the reasons for it?
3. Does solving the given problem first influence the posing of problems based on it?

The activity of posing new problems on the basis of a given spatial problem made the students interact with their geometrical knowledge and use it as a basis for both a widening and deepening of their understanding.

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UNPACKING STUDENT MEANING OF CROSS-SECTIONS: A FRAME FOR CURRICULUM DEVELOPMENT

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Recent literature indicates that there has been some change since Freudenthal (1973) expressed concern for the future of geometry instruction. This has mainly been due to research based around the van Hiele Theory. However, in the field of 3-dimensional geometry there is still room for great concern. In analysing the understanding expressed by students in their descriptions of a cross-section of a solid, the authors present a frame that has the potential to be used for curriculum development in 3-dimensional geometry.

While researchers lament the lack of importance attached to geometry in school curricula (Freudenthal 1973; Meissner 2001; van der Sandt 2001), they are unanimous in their espousal of its importance. Geometry offers a way to interpret and reflect on our physical environment, while spatial thinking has been suggested to be essential to creative thought. Recent literature (Meissner 2001; van der Sandt 2001) suggests that little has changed since Freudenthal (1973, p.402) expressed fears that "the days of traditional geometry are counted, ... there is cause to be concerned about the future of geometry instruction. The lack of attention to solid geometry is matched by the lack of published research on the topic. The only research reports on 3-dimensional geometry published in recent years appear to be on nets of solids (Despina, Leikin & Silver 1999; Lawrie, Pegg & Gutiérrez 2000; Mariotti 1989). Mariotti (p.260), in attempting to identify specific didactic variables related to the utilisation of mental images, hypothesised that there are two levels of complexity; the first level is when the image is global, and the second occurs when an operative organisation of images is required for coordination. Lawrie, Pegg & Gutiérrez (2000), in their analysis of students' responses expressing understanding of a net, demonstrated that there is a hierarchy of difficulties within Mariotti's second level.

If one considers geometry to be the poor relation of mathematics, 3-dimensional geometry must surely be the Cinderella. If the above situation is to be addressed, there is need first to address development of school curricula. A framework demonstrating the didactic hierarchy in 3-dimensional geometry needs to be determined.

RESEARCH DESIGN

The authors developed a test to investigate secondary-school students' understanding of various aspects of 3-dimensional figures, i.e., the solid form and its cross-sections, and nets of 3-dimensional figures. The analysis of students' perceptions of nets of solids has already been presented by Lawrie, Pegg & Gutiérrez ((2000). This report considers the coding of the students' responses to a second aspect, one in which students express their understanding of cross-sections of 3-dimensional figures. A cross-section is defined as:

> A section is a plane geometric configuration formed by cutting a given figure with a plane. A *cross-section* is a section in which the plane is at right angles to an axis of the figure. Daintith & Nelson 1989, p.290

Two frameworks, the SOLO Taxonomy (Biggs & Collis 1982) and the van Hiele levels of understanding (van Hiele 1986) are used in coding the nature of thinking displayed by the students in their responses.

BACKGROUND

The SOLO taxonomy

The SOLO Taxonomy (Biggs & Collis 1982) has been identified by several writers (e.g., Pegg & Davey 1998) as having strong similarities with the van Hiele Theory, despite some philosophical differences. It is concerned with evaluating the quality of students' responses to various items. A SOLO classification involves two aspects. The first of these is a mode of functioning, and, the second, a level of quality of response within the targeted mode. Of relevance to this study are the three modes, ikonic, concrete symbolic, and formal. Within each of these modes, students may demonstrate a unistructural, multistructural or relational level of response.

In the *ikonic mode* students internalise outcomes in the form of images and can be said to have intuitive knowledge. By contrast, in the *concrete symbolic mode,* students are able to use, or learn to use, a symbol system such as a written language and number notation. This is the most common mode addressed in learning in the upper primary and secondary schools. When operating in the *formal mode*, the student is able to consider more abstract concepts, and to work in terms of general principles.

A description of the levels that occur within the modes is given:

The *unistructural level* of response draws on only one relevant concept or aspect from all those available.

The *multistructural level* of response is one that contains several relevant but independent concepts or aspects.

The *relational level* of response is one that relates concepts or aspects. These relevant concepts are woven together to form a coherent structure.

The targeting of the concrete symbolic mode for instruction in primary and secondary schools, and the implication that most students are capable of operating within the concrete symbolic mode (Collis 1988) has resulted in the exploration of the nature of student responses within that mode. This has led to the identification of at least two unistructural (U) - multistructural (M) - relational (R) cycles within the concrete symbolic mode (Pegg & Davey 1998). One noticeable characteristic of the cyclic form of the levels is that the relational response (R_1) in the first cycle becomes a single entity as the unistructural element (U_2) in the second cycle. This cyclical nomenclature has been used in the coding of concrete symbolic responses for this study.

The van Hiele Theory

Pierre van Hiele's (1986) work developed the theory involving five levels of insight. A brief description of the first four van Hiele levels, the ones commonly displayed by secondary students and most relevant to this study, is given for 3-dimensional geometry (Pegg's (1997) differentiation between Levels 2A and 2B is used):

Level 1 Perception is visual only. A solid is seen as a total entity and as a specific shape. Students are able to recognise solids and to distinguish between different solids. Properties play no explicit part in the recognition of the shape, even when referring to faces, edges or vertices.

Level 2A A solid is identified now by a single geometric property rather than by its overall shape. For example, a cube may be recognised by either its twelve equal edges or its six square faces.

Level 2B A solid is identified in terms of its individual elements or properties. These are seen as independent of one another. They include side length, angle size, and parallelism of faces.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised. Symmetry follows as a consequence. Simple proofs and informal deductions are justified. Families of solids can be classified.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions in definitions are used with understanding, as are equivalent definitions for the same concept.

Van Hiele saw his levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level(s). Van Hiele also saw (i) the levels as discontinuous, i.e., students do not move through the levels smoothly; (ii) the need for a student to reach a 'crisis of thinking' before proceeding to a new level; and (iii) students at different levels speaking a 'different language' and having a different mental organisation.

ANALYSIS OF RESPONSES

The 3-dimensional geometry test designed by the authors was given to students from all except the final year (i.e., to Years 7 to 11, ages 12 to 17 years) of four secondary schools in a rural city in New South Wales, Australia. The students were of mixed ability, and were not drawn equally from across the years; rather, the researchers were dependent on the availability of classes and the goodwill of the schools and their teachers. This resulted in there being many more students from Years 7 and 11 than from Years 8, 9, and 10.

Because of page limitations, this paper considers in depth the responses to one of the questions on cross-section of 3-dimensional figures, namely:

"Describe in as much detail as possible what is a cross-section of a solid."

An indication of the difficulty students experienced with the notion of cross-sections is that only 181 students attempted the question. The response rate differed across the various years, with 18, 12, 31, 51 and 69 from Years 7, 8, 9, 10 and 11 respectively. It should be noted that in Australia cross-sections are considered in a global sense in primary schools and are not considered in any greater depth until late in Year 8, with further treatment in Years 9 and 10.

Analysis of responses (SOLO)

The responses were coded first with relation to the SOLO Taxonomy. Of the 181 responses, it was possible to code 177 (98%); only 4 responses were considered uncodable. No responses were identified as belonging to either the ikonic or formal modes, i.e., it was considered that all responses were concrete symbolic.

Initially, the responses were classified according to their cycle within the mode. This was done depending on whether a response expressed the notion of a cross-section being a single cut or halving of the solid (first cycle), or whether the response inferred multiple cuts or slices (second cycle). After considering several features in responses, e.g., the cut, direction, result of the cut, together with various combinations of these, it was felt that the significant factors for indicating levels in the responses were:

- single versus repeated cut(s) or slice(s);
- whether the result of the cut(s)/slice(s) was acknowledged;
- qualification of the result, i.e., the linkage between the cut(s)/slice(s) and the result(s).

The term 'slice' rather than 'cut' was used frequently by students giving second-cycle responses, yet did not appear in responses of the first-cycle. In the first-cycle responses, students frequently used the terms 'cutting in half", 'cutting down the middle', 'bisecting'. Almost all students applied their notion of a cross-section to figures with a constant cross-section (prisms and cylinders), not mentioning cross-sections of pyramids nor 3-dimensional figures in general.

This initial classification of responses led to the following description of each SOLO level, U, M, R, for the two cycles in the concrete symbolic mode (Table 1).

Table 1: Response descriptors for SOLO levels with examples of student responses

Level	Description
Single cut	
U_1	One cut e.g. *cut a solid in half; a solid is divided through the middle.*
M_1	One cut and look at the result e.g., *what you see when you cut a solid into two pieces; cut down the middle and the cross-section is the result.*
R_1	The effect of the cut; qualifying the 'look' e.g., *It is when you cut it in half and both sides match.*
Multiple cuts/slices	
U_2	Multiple cuts/slices e.g., *when the solid is cut straight anywhere; ... is cut into slices.*
M_2	Multiple cuts/slices and the resulting shape stays the same e.g., *A cross-section of a solid is an imaginary line that can be cut anywhere so that the same shape remains throughout.*
R_2	Direction of the cut is the integrating factor that links multiple cuts and same shape e.g., *The section where the solid is cut down, parallel to the front face. The cross-section is the same shape as this face.*

Of the four responses considered not able to be classified in accordance with the descriptors listed in Table 1, two very similar responses appeared to reflect the appearance of a pyramid if viewed from above the vertex;

> *The cross-section of a solid is the section where, when drawn, all the vertices intercept each other; straight down the middle.*

In the third response the student, while demonstrating understanding of the general concept of a cross-section, gave a response for topography;

The shape meaning the ascension and descensions of the shape – just like the cross-section from one point to another on a topographic map.

The fourth student's response seemed to reflect general geometric awareness, but no specific knowledge of a cross-section:

A particular part to the solid, looked at thoroughly unto detail. Things which might be taken into account are length of sides, edges.

The results of the analysis of the 177 responses are shown in Table 2. As the number of students giving a response to the question varied, results of the analysis will be given as percentages in parenthesis after the number of students.

Table 2: Results of analysis for SOLO levels of student responses (with percentages)

Level	Total	Year 7	Year 8	Year 9	Year 10	Year 11
U_1	43(24)	9(5)	6(3)	7(4)	11(6)	10(6)
M_1	63(36)	7(4)	2(1)	16(9)	18(10)	20(11)
R_1	37(21)	0	4(2)	7(4)	10(6)	16(9)
Subtotal	143(81)	16(9)	12(7)	30(17)	39(22)	46(26)
U_2	5(3)	1(1/2)	0	0	1(1/2)	3(2)
M2	18(10*)	1(1/2)	0	0	4(2)	13(7)
R_2	11(6)	0	0	0	4(2)	7(4)
Subtotal	34(19)	2(1)	0	0	9(5*)	23(13)
TOTAL	177(100)	18(10)	12(7)	30(17)	48(27)	69(39)

* percentages may not tally because of rounding

Analysis of responses (van Hiele)

The students' responses were also coded according to the van Hiele levels of understanding. It was considered that there were no students displaying understanding at either Level 1, the visual level, or at Level 4, the formal level in which logical reasoning can be demonstrated. All students were considered to be displaying reasoning at Level 2A, i.e., identifying a single geometric element, Level 2B, i.e., identifying individual elements, or at Level 3 in which students display a degree of analysis or logical ordering. Table 3 gives descriptors of the van Hiele levels and the correspondence with the SOLO levels, and Table 4 gives the results of the analysis of student understanding as van Hiele levels. Again, percentages are given in parenthesis.

Table 3: Response descriptors for van Hiele levels

van Hiele Level	Description	Corresponding SOLO level
2A	Perceiving cut(s)/slice(s)	U_1; M_1; R_1; U_2
2B	Acknowledging that there are multiple cross-sections which are the same shape	M_2
3	Recognising that the multiple cuts/slices are in a certain direction – the integrating factor	R_2

Table 4: Results of analysis for van Hiele levels of understanding (with percentages)

Van Hiele Level	Total	Year 7	Year 8	Year 9	Year 10	Year 11
2A	148(84*)	17(10)	12(7)	30(17)	40(23)	49(28)
2B	18(10*)	1(1/2)	0	0	4(2)	13(7)
3	11(6)	0	0	0	4(2)	7(4)
Total	177(100)	18(10)	12(7)	30(17)	48(27)	69(39)

*percentages may not tally because of rounding

CONCLUSION

The analysis given in this report demonstrates a hierarchy of difficulties in the understanding of cross-sections of solids. Not only does the hierarchy fit into Mariotti's (1989) second level of complexity in the manipulation of mental images in which an operative organisation of images is required for coordination, it also provides a frame for the development of a curriculum for instruction in cross-sections of 3-dimensional figures. This frame could be expanded to encompass the study of the whole of 3-dimensional geometry and spatial reasoning. However, care should be taken that within a curriculum, adequate time is allotted to the various hierarchical strata demonstrated above as levels in the two SOLO cycles. In particular, there should be adequate time allowed for students to develop through the first SOLO cycle, thus allowing students to come to an understanding of Level 2A in van Hiele. Instructors, in observing students from their higher level of understanding, do not always consider the importance of establishing sound understanding at the lower levels. To move students from one level to the next requires interaction between the teacher, students and subject matter and interaction requires time.

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Mathematics Pedagogical Value System Oriented Towards the Acquisition of Knowledge in Elementary School

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The purpose of this study is to investigate the mathematics pedagogical values of a fifth grade elementary school teacher. The main methodology encompasses classroom observations and interviews. The result of the research reveals that the teacher has three mathematics pedagogical values, which are emphasized differently. However, the statement "the goal of mathematics teaching is to induce students to acquire knowledge from the textbook" serves as the main core of these three values.

Introduction

After six years of development and pilot tests, a new national elementary mathematics curriculum in Taiwan was launched in 1996. The new curriculum emphasizes on problem solving, communication, reasoning and mathematical connections, as the Standard in the United States stressed (NCTM, 1989). The reform-oriented modes of teaching tend to transfer from teacher-centered lecturing into pupil-centered learning, and put more emphasis on group discussions than classroom lecturing. Taiwan's elementary schools' mathematics curriculum and teaching is deeply influenced by that of the West. Do we have to cogitate if this approach is suitable in Taiwan's education environment? Is this approach worthy of advocating? When we are considering if it is worthwhile, we shall discuss about the mathematics pedagogical values we hold.

Mathematics pedagogical value is a new research domain (Chin, Leu, & Lin, 2001; Bishop, FitzSimous, Seah, & Clarkson, 2001). According to the view of Bishop et al, there is widespread misunderstanding that mathematics is the most value-free of all academic subjects. Since mathematics is human and cultural knowledge, teachers inevitably teach values at mathematics class. It is generally agreed that the quality of mathematics teaching would be improved if there were more understanding about teachers' values and influences. But what are those salient values regarding mathematics, mathematics education and education in general, that Taiwan's experienced elementary school mathematics teachers have?

Do those values accord with current education policies?

In this paper, the values regarding mathematics education are investigated. We identified an elementary school teacher's three mathematics pedagogical values, disclosing the relationship among these values.

Sample and Methodology

The research sample of this study was Ms. Lin, who has had nine years of teaching experience in elementary school. She is currently teaching one class of twenty-seven fifth graders. In the first interview, she mentioned that "I would give extra exercises to my class in addition to the official workbook", "I felt a lot of pressure following the syllabus since there was too much material in the textbook for fifth or sixth graders", and "I struggled a lot because I felt that I was unable to help students obtain good scores". These thoughts were typical of the views expressed by most elementary school teachers when asked about their mathematics teaching experience in Taiwan. They emphasized good scores, asked students to do more exercises, and felt lots of pressure to follow the syllabus. Hence, we chose her as one of our research subjects.

According to the theory of the process of valuing by Raths, Harimn & Simon(1987), values are defined as any beliefs, attitudes, activities or feelings that satisfy the following three criteria: choosing, prizing, and acting. What satisfies the criterion of choosing is the belief or attitude chosen under free will, among several differing options or after thorough consideration. What satisfies the criterion of prizing is the belief or attitude of cherishing showing pride of or willingly made known publicly. The belief or attitude that satisfies the criterion of acting is a performance acted out repeatedly.

Based on Raths et al's theory, we used classroom observations and interviews to collect data. The purpose of classroom observation is to notice repeated behavioral patterns during the mathematics lessons. The purpose of the interviews is to recognize the reasons why Ms. Lin developed these behavioral patterns in order to form some value indicators, as well as to examine if the value indicators met the criteria of "choosing" and "prizing". The time for research is one year. Ten lessons are observed and 18 interviews are conducted.

Results

Ms. Lin's three mathematics pedagogical values are of the following goals: to teach students to acquire mathematics textbook knowledge, to have students enjoy learning mathematics, and to use mathematics in solving their daily problems. According to Raths et al's theory (1987), we present our data in attempt to explain Ms. Lin's first acting value under the criteria of acting, choosing, and prizing. The latter two values are only outlined briefly.

In the following excerpts, "I" stands for the interviewer and "S" for Ms. Lin's students.

1. Acquiring knowledge from the mathematics textbook

Acting Ms. Lin instructed her students to preview the mathematics material before the respective class because she believed that preparation would help her students to understand the lectures more easily. Besides, preparation also allowed the more capable students to help explain the contents in the mathematics textbook to the class as teaching assistants.

Ms. Lin's teaching method consisted in teacher-centered lectures. She admitted that she liked being given the leading role to lecture. She usually requested students to read out loudly the textbook problems and then they were explained by the teacher. Then, she guided students to solve problems by additional practice testing questions (Ainley, 1988), as demonstrated in case 1.

Case 1

(A problem on page 93 of the textbook: Draw a circle with a diameter of 5 cm, and then compute the circumference of this circle.)

Lin: What isπ? 3.14. In other words, how many times of diameter is the circumference? 3.14. Since we know the diameter, how do you solve this problem then? How many times should the diameter be multiplied by?

S: 3.14.

Lin: The answer is the diameter times by 3.14, isn't it? Thus, the answer to this problem is $5 \times 3.14 = 15.7$.

A self-answering questions style by the teacher is often adopted in the classroom, such as "What isπ? 3.14." in the above case. There are also some testing questions such as, "How many times should the diameter be multiplied by?". Taking a forty-minute classroom observation on June 8th, 1999 for example, there were 24 questions which she raised and answered on her own, as well as providing 69 practice testing questions in that class.

Sometimes, Ms. Lin would have students solve problems on the podium and explain the solutions to all students. She said, "There are about two problems on a page in the textbook. I do one first, and then have some students to do the other similar one in front of the whole class." When asked would she have students try the first problem? She replied, "Yes, but not often. If the students do not give good explanations, it will take me more time to clarify the confusions".

From the interview described above, we realized that Ms. Lin emphasized more on whether students learned how to solve problems in the mathematics textbook.

And that was the reason why she would always solve and explain those problems one by one in detail. Meanwhile, she does not stress the cultivation of students' active attitude in solving problems on their own. Therefore, she did not let students first try to solve problem on their own, in order to avoid spending more time to correct students' misconceptions. Nonetheless, we also found that in her class, she seldom allowed the students to express different solving strategies from those written in the mathematics textbook. Neither did she discuss the incorrect problem solving strategies of some students.

Why didn't she have students express their different strategies in solving mathematics problems or discuss those incorrect problem solving strategies that students used?

According to interview data, there are two reasons to explain why Ms. Lin did not encourage students use their own strategies different from those in the mathematics textbook. One is the obedience of authority. She believed strategies developed by the experts were more suitable for students. The other is the examination score issue. She was afraid that the slow speed of students' own strategies would affect their scores on paper-and-pencil tests.

The reason why Ms. Lin would not discuss the incorrect problem solving strategies is that she found more errors raised in Chinese language workbooks after showing students those errors during Chinese language class. She extended her disastrous experience from the Chinese language class to the mathematics class and thus would try not to discuss mathematics misconceptions in class.

In addition to the teaching behaviors described above, we also discovered that Ms. Lin would ask her class to review what they had learned before the end of each lecture. "The object of review on the same day is to seize the golden time for memorization when students still can remember clearly what had just been taught", she explained. From the word, "memorization", it is obvious that she emphasized not the 'process' of learning but rather the actual 'results' of learning mathematics from the textbooks.

Judging from the request for preview and review, the development of testing questions, and the avoidance of explaining incorrect problem solving strategies, we concluded that Ms. Lin's value toward goal in mathematics teaching was to have students only acquire knowledge from the textbook.

Choosing We raised two different ways of teaching and Ms. Lin commented on them. The first one was for teachers to take a leading role and for students to follow the textbook. The other was for students to initiate learning and for them to discuss and solve problems in a group. Ms. Lin favored the first approach for the speed and effectiveness of the whole learning process. But the drawback of this approach was that some students may become absent-minded. She said, "The advantage of the latter approach would be to develop an awareness of the possible incorrect problem solving strategies and mistakes and to avoid them. However, if students are too lazy to participate in discussion, the group leader will let me know. I will then instruct the distracted students to be attentive." She said at the interview, "The drawback is schedule delay. Sometimes, I felt that even if I spent twice as much time, I still could not finish all the materials." "Some students might even be misled with the incorrect problem solving strategies discussion and become confused, consequently."

Ms. Lin reviewed all the pros and cons for these two approaches of mathematics teaching, considering also the time constraints involved and students' misunderstandings over incorrect problem solving strategies. She still chose the first approach where teachers took the leading role in order to instruct students to learn knowledge from the textbooks.

Prizing Ms. Lin emphasized previews and reviews on mathematics textbooks because she said, "Going over them three times is better than going over them twice; and twice is better than once." These words revealed that she stressed that students must acquire knowledge from the textbooks well.

According with the three criteria of acting, choosing and prizing, we are able to identify that the mathematics pedagogical value of Ms. Lin is to induce students to obtain mathematics knowledge from the textbook.

2. Learning mathematics happily and using it to solve daily problems

In her mathematics class, we sometimes saw Ms. Lin use daily life examples to demonstrate mathematical concepts. For example, when she was teaching how to compute the area of circles, she mentioned cakes, and cake diameters. Why would she mention things such as cakes? She said, "To mention something edible would revive tired students, draw attention, and regain classroom focus. In the mean time, I hope students would apply what they learned in class to aspects of daily life. Mathematics is not as hard as someone would imagine. It relates with one's daily life."

In one interview, she also said, "I asked students to do an assignment about Kao's lottery last year. I gave the data from the lottery enrollment showing the winners from the newspaper, and then asked them to figure out the probability of winning a car based on the data given. It turned out that the students were very interested in this assignment, since figuring lottery probability is applicable to daily life. Through this assignment, they realized and accepted that is one of the reasons for learning mathematics in school."

From the above quoted, we know that Ms. Lin preferred students to relate mathematics with aspects of their daily life, in hope that this would revive an interest for mathematics. She added, "I tried to use examples from daily life, making those examples as vivid as possible with jokes and humor during mathematics class, thereby transferring more joy in the learning process. Besides, operating with tools for learning on their own, students justify more joyful learning. Students like using their own instruments for learning mathematics because they could relax with their group members when solving a problem. Of course, they may converse among the group members, however, it provides for a better learning atmosphere, since students feel comfortable talking and focusing in class".

Although she had the idea of grouping students together to operate with their own learning tools, she seldom conducted this because she had a tight schedule to keep up with. She would instruct the class to be absolutely quiet, or else she would scold the students, timing them to stand when they were noisy. Therefore, Ms. Lin acknowledged the learning tools to allow for more joyful learning, but she was unable to use this systematic learning process and to have the students to learn actively the mathematics knowledge.

According with the above situation, note that two of Ms. Lin's value indicators are that, "Mathematics teaching's goal is having students learn mathematics happily" and "Mathematics teaching's goal is for students to use mathematics in solving their daily problems".

However, Ms. Lin chose to help students learn mathematics happily and relate mathematics problems to things in daily life only when she could keep up with the syllabus and students' examination scores. In addition, she believed, "It was most important to have students learn mathematics without pressure and happily." She also said, "Through solving specific problems in daily life, students became more aware the reasons to learn mathematics." Those words showed that Ms. Lin emphasized the importance of these two value indicators. Based on the criteria of acting, choosing, and prizing, we concluded that the above two value indicators enlist as two of Ms. Lin's mathematics pedagogical values.

3. Relationships among mathematics pedagogical values

Ms. Lin mentioned that one of the most important things about teaching mathematics is allowing students to happily learn mathematics. However, learning happily became just a means for learning mathematics well in the interview Case 2. Case 3, showed that teaching mathematics contents of the tests took priority over solving daily mathematical problems and learning happily.

Case 2

I: Ms. Lin, You said that when students learn with joy, it was to induce their motivation for learning even if they could not learn well. Hence, it seems that the ultimate goal of mathematics teaching is to motivate students to learn mathematics and learn well. Thus, learning with joy is not the ultimate goal, isn't it?

Lin: No, learning happily would not be the ultimate goal.

Case 3

I: Earlier in the interview, you mentioned that some abstract and mechanical topics, would later become the base for future learning, like the common divisor and common multiple. If there were some topics you had to teach within a given period before an examination, including such abstract and mechanical topics mentioned, along with those related with daily life and enjoyment, how would you schedule them under time limitations before the examinations?

Lin: I probably would compromise with the examination, giving priority in teaching topics about the common divisor and common multiple.

To synthesize teaching behaviors of mathematics at interviews and analyze mathematics pedagogical values of Ms. Lin, we maintain that relating mathematics to daily life was a means only for Ms. Lin to encourage students to joyfully learn mathematics. Correspondingly, learning with joy also meant to facilitate learning mathematics. That is, the value that mathematics teaching aims to help students use mathematics to solve their daily problems was only the instrumental value (Chen & Chen, 1990) which is a means to get the other value that teaching mathematics is to have students learn mathematics happily. While the value that teaching mathematics is to have students learn mathematics happily was also only an instrumental value which is a means to get further value that the goal of mathematics teaching is to teach students to learn the knowledge in the textbook. Therefore, "the goal of mathematics teaching is to induce students to acquire textbook knowledge." became the core of the above three values.

Implication

In exploring the mathematics pedagogical values of Ms. Lin, we discussed the pros and cons of teacher-centered teaching and group discussion. Ms. Lin only accepted them as teaching strategies, without realizing their hidden values and implications. For example, allowing students to solve problems in discussion groups would cultivate students' to initiate problem solving, thinking, and reasoning abilities. The findings revealed that during teacher education of mathematics teachers, educators must not only discuss effective teaching strategies, but also those values hidden behind the strategies. Accordingly, teachers will know which strategies to choose and why those strategies are selected for teaching mathematics.

We identified, "The goal of mathematics teaching is to teach students to acquire the knowledge from the textbook." "Teaching mathematics is to have students joyfully learn mathematics." "Part of mathematics teaching is to help students use mathematics to apply to daily life problems." These were the three mathematics pedagogical values of Ms. Lin. The relationship among the values are illustrated in Figure 1.

toward toward

applying to daily life problems | joyfully learning math | learning math knowledge

instrumental value | instrumental value | core value

Figure1: Mathematics Pedagogical Value System, centering on mathematics knowledge acquisition in elementary school

The mathematics pedagogical value system in Figure 1 provides mathematics teacher educators the implications. Though teachers in elementary schools have many mathematics pedagogical values, their values are emphasized differently. The relationship among the values must be clarified and the core values must be identified first. Then, we may examine further whether the core values accord with current education policies. If not, then in modifying teachers' mathematics pedagogical values, changing of those core values becomes an underlying solution. Of course, how to change teachers' mathematics pedagogical values is a new research subject.

Acknowledgements: Thanks is given to the teacher Ms. Lin, who was kind and open in sharing her ideas with us. We thank our assistant Ms. I.H. Hong, who did a great job on the audio-video tape transcriptions. We are grateful to NSC for funding the project. The opinions expressed do not necessarily reflect the views of NSC.

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EXPLORING THE RELATIONSHIP BETWEEN SIMILAR SOLUTION STRATEGIES AND ANALOGICAL REASONING

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Analogical reasoning is a powerful problem solving strategy that exploits the isomorphic relationship between two problems. The result of this exploitation is the production of similar solution strategies for the two problems. However, the presence of similar solution strategies is not a result exclusive to analogical reasoning. This study examines the use of similar solution strategies as an indicator of analogical reasoning in pre-service elementary school teachers' attempts to solve repeating pattern problems. Findings show that an awareness of problem similarity is not a necessary requirement for the production of similar solution strategies. As a result, the use of similar solution strategies as a measure of analogical reasoning needs to be modified.

INTRODUCTION

Making connections and utilizing similarities between problems is at the core of mathematical reasoning (English, 1998; National Council of Teachers of Mathematics, 2000). There are several proposed mechanisms by which learners weave a thread between what is known and what is new. One such mechanism, referred to as analogical reasoning, builds this connection by exploiting the tension between the similarities and the differences of two situations (Jardine & Morgan, 1987). However, a "subjects' conceptions of similarities between situations are analyzed relative to normative criteria with respect to predetermined mapping between the situations, built into the experimental set-up by the researcher." (Greer & Harel, 1998, p. 11)

This article examines the use of similar problem solving strategies as an indicator of analogical reasoning. I make the argument that strict adherence to similar solution strategies as an indicator of analogical reasoning causes improbable results to emerge from the data. An alternate form of reasoning is proposed that can be used to explain situations in which similar solution strategies are utilized in the absence of analogical reasoning.

ANALOGICAL REASONING

Mathematical reasoning in general, and analogical reasoning in particular are most closely associated with how students solve mathematical problems. Polya (1957) acknowledged reasoning by analogy as an explicit part of problem solving with such strategies as "think of a related problem", and "use a simpler (but similar) problem". These are very general recommendations to problem solvers as to how to proceed when faced with a challenging problem. Reasoning by analogy is a more specific description of the mechanism that forms the underpinnings of these strategies. As it applies to problem solving, analogical reasoning involves mapping the relational structure of a known problem (that has been solved previously, referred to as the *source*) onto a similar problem (referred to as the *target*) and using this known structure to help solve the similar problem (English, 1997, 1998; Novick, 1990, 1995; Novick & Holyoak, 1991). If the source problem and the target problem are almost isomorphic (that is, the source problem is isomorphic to part of the target problem or vice versa) then some adaptation or extension of the solution strategy may be required (English, 1998; Novick & Holyoak, 1991)

ANALOGICAL REASONING: A FRAMEWORK FOR ANALYSIS

Studies examining students' abilities to utilize analogical reasoning can be classified by what I refer to as their degree of openness (ranging from closed to open). Closed studies are ones that direct students, through the experimental set up, to use analogical reasoning skills. Experiments involving problem sorting, grouping tasks, or possible source problem identification from a list of problems are examples of such studies (e.g. English, 1998). These studies do not determine whether or not a student uses analogical reasoning skills but how well they use it as indicated by correct selection of source problems or normative sorting of problems.

Open studies on analogical reasoning abilities are ones in which the experimental design allows the students the freedom to invoke whatever problem solving strategies they see fit and then to analyze the results for evidence of analogical reasoning skills as indicated by similar problem solving strategies. In order to assure that participants are utilizing analogical reasoning it is necessary that the problems presented to the participant have many possible solution strategies (Bassok & Holyoak, 1989; Novick & Holyoak, 1991).

A particular study's openness is dependent on the impact that the experimental set-up has on the participants' choice of problem solving strategy. A partially open study would be one in which the participants are operating in a problem solving situation but the experimental structure may guide them into using analogical reasoning. Studies using a hint/no hint paradigm (e.g. Novick & Holyoak, 1991) to measure the impact of retrieval on successful solution strategy transfer are an example of a partially open study. Successful use of analogical reasoning is again indicated through the use of similar problem solving strategies.

With open, or partially open studies, however, analysis of data operates on two assumptions. The first is that the use of different problem solving strategies on similar problems indicates a lack of analogical reasoning, and the second is that the use of similar problem solving strategy indicates analogical reasoning. Analogical reasoning hinges on a solver's explicit identification of the similarity between two problems. This is a description (or a definition) of the mechanism of analogical reasoning. Similar solution strategies are a product of this mechanism. The question is, is the product an indicator of the process? Is the presence of similar solution strategies an accurate indicator of analogical reasoning?

METHODOLOGY

This study was designed to probe the relationship between the students' explicit identification of problem pair isomorphism and the use of similar solution strategies. Participants in this study were preservice elementary school teachers enrolled in a "Foundations of Mathematics for Teachers" course. Twelve of the students enrolled in the course volunteered to participate in clinical interviews. The tasks, along with some of the questions they were asked to solve were:

The Calendar Problem.

> I've chosen a calendar page, October 2000, and I'm going to place a red marker on the 1, a blue on the 2, a green on the 3, and a yellow on the 4. Now, I'm going to repeat this pattern; red on the 5, blue on the 6, green on the 7, and yellow on the 8.

a. What colour will number 13 be? What colour will 28 be?
b. If the calendar continued on forever, what colour would 61 be? 178? 799?
c. If there were five colours (red, blue, green, yellow, and black), what colour would 799 be? If there were 6 colours, what colour would 799 be?

The Sequence Problem

a. Consider the sequence 1, 5, 9, ...What will the next few numbers in the sequence be?
b. Will the number 48 be in this sequence? Will 63?
d. Can you give me a big number that you know for sure will be in the sequence?
e. Consider the sequence 5, 12, 19, ... Is 96 going to be in this sequence?
f. Can you give me a big number that you know for sure will be in the sequence?
g. Consider the sequence 8, 15, 22, ... Can you give me a big number that you know for sure will be in the sequence?
h. Consider the sequence 15, 28, 41, ... Is 1302 going to be in this sequence?

Comparison of the Two Problems

a. Consider the two problems you have worked on here: the calendar problem and the sequence problem. Is there any similarity between the two problems?
b. Is there a similarity between the strategies you used?

The interviews were conducted in the later part of the course. Students had studied the topic of 'arithmetic sequences' in the earlier part of the course.

The calendar problem and the sequence problem were chosen in order to present students with two situations that were almost isomorphic. The questions in the sequence problem were chosen to be 'twist' or 'inverse' problems (Zazkis & Hazzan, 1998). Rather than ask standard questions of finding the n^{th} element or which term a given element is, these questions 'twist' what is sought and what is given. Similar problems were used in Zazkis and Liljedahl (2002) and have been shown to be useful in getting students to examine the situation rather than simply use established algorithms.

Specific questions within each of the two tasks were removed or added (at the discretion of the interviewer) in order to help establish solid understanding of the problems before proceeding with the interview.

The two main tasks, the calendar problem and the sequence problem, were chosen for their implicit isomorphic relationship. Comparing the two it becomes clear that arithmetic sequences can be viewed as a special subset of the calendar problem. That is, the sequence 1, 5, 9, ... represents the sequence of red markers in the calendar problem.

RESULTS AND ANALYSIS

The transcripts of the clinical interviews, written work, and the audiotapes themselves, were used to analyze the problem solving strategies utilized by the students. A simple partitioning of the data using solution strategies as a gauge produced two groups: those who did not use similar solution strategies, and those who did.

Initial analysis revealed that five of the participants utilized different strategies, six utilized the same strategy, and one used multiple strategies for both questions. Of specific interest to this study were the six students who used similar solution strategies to solve both the calendar problem and the sequence problem.

Although the use of similar solution strategies for the almost isomorphic problems could be taken as indication of the use of analogical reasoning in solving the sequence problem, the responses to the probing questions indicated otherwise.

When asked to comment on the similarity between the problems both Deanna and Helen have difficulties identifying any.

> Deanna: *They're kind of hard to connect right away, or at all. [...] But it's hard to see that they're similar, for me at least. It's hard to recognize that and see that, it's very hard to play around with numbers and to kind of see the relationships between the different numbers.*

> Helen: *I used multiples of 4 for all of them, . . .*
> Interviewer: *Okay, anything else?*
> Helen: *Um, they all continue in a pattern. . .*
> Interviewer: *Okay, anything else?*
> Helen: *No. . .*

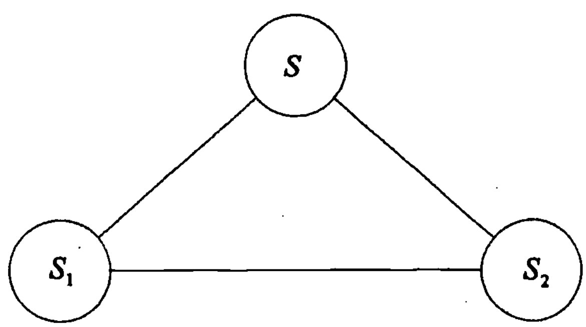
John, on the other hand, has no difficulty identifying the similarity between the problems when prompted to do so. However, probing for his views on the similarity of his strategies reveals that he cannot recall the strategy he used in the calendar problem.

> Interviewer: *Um, the strategy you used here for solving those types of questions, would you say it's the same as the strategy you used here?*
> John: *(pause) You would ask me, so if I was going to say. . .*
> Interviewer: *Do you remember the strategy you used here?*
> John: *Uh, I would like to go over one example just so [...] So yeah they are the same.*

After some time to refresh his memory, John indicates that he sees the strategies as being the same. This delay, however, shows that he was not exploiting the similarity of the problems and utilizing his strategy for the calendar problem while he was working on the sequence problem.

Each of the six participants who produced similar solution strategies either failed to explicitly identify the similarities between the problems or gave indication that, although they now saw the similarity, they failed to capitalize on it during their efforts to solve the sequence problem.

Greer and Harel (1998) take a broader view of the analysis of solution strategies of isomorphic problems. They classify the isomorphic relationship into three cases. The first two are identical to the processes of analogical reasoning (see English, 1998 for more details). The third is a case which Greer and Harel refer to as 'mediated isomorphism'.

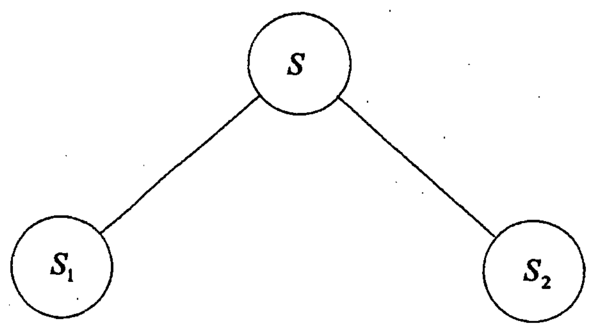


Mediated Isomorphism (Greer & Harel, 1998, p. 12)

Figure 1

Mediated isomorphism relies on the solver first identifying the similarities of the two problems S_1 and S_2 to the more general problem S, and then using these similarities to help establish an isomorphic relationship between the two problems S_1 and S_2. This mechanism has many of the same markers as analogical reasoning: two similar problems, similar solution strategies, an awareness of problem similarity. However, it is a mechanism different from analogical reasoning in that it lacks the willful exploitation of problem similarity for the purpose of solving the second problem. In fact, problem similarity is more a byproduct of the solutions than a mechanism towards the solution. Two of the participants showed evidence of this mediating effect as they became aware of the problem similarity while answering the comparison questions.

However, this mechanism fails to explain the results displayed by Deanna. Deanna's solution to both the calendar problem and the sequence problem indicated that she treated both problems as special cases of repeating pattern problems. Yet, she was not aware of the inherent relationship that existed between them. As an isolated case it could be classified as an incomplete execution of mediated isomorphism. However, the prevalence of this result in this study necessitates the recognition of this process as a different form of reasoning. I call this mechanism *non-mediated generalization*, where, as shown in *figure 2*, there is no direct connection between S_1 and S_2.



Non-mediated Generalization

Figure 2

All six of the participants who produced similar solution strategies for the calendar problem and the sequence problem indicated (through their responses to the comparison questions) that they were unaware of the similarity between the problems while they were actually engaged in the solving of the problems.

CONCLUSION

There is no denying that reasoning by analogy is a powerful problem solving strategy. As mathematicians we have come to rely on its strength. As teachers we make it our goal to give students the experiences and guidance necessary to improve this skill. As researchers we try to measure its use and effectiveness and try to identify the factors that aid or impede its success. However, there are other forms of reasoning, valid and invalid, available to the student in an analogical problem-solving situation. Carefully constructed instruments will be able to guarantee the willful use of analogical reasoning, but only in very controlled environments. Tasks such as problem sorting and source/target pairing direct participants to utilize reasoning by analogy through the experimental set-up. Studies using such tasks will, therefore, be able to discern the sorts of features that are attended to and allow analysis as to how well analogical reasoning is invoked. But if we utilize the same sort of analysis in less controlled environments we are making the assumption that solvers are, indeed, intending to utilize analogical reasoning. If true problem solving efforts are to be evaluated for the application of analogical reasoning then there must be a contingency for alternate forms of reasoning in any subsequent analysis.

This study has shown that even seemingly successful cases of analogical reasoning as indicated by similar solution strategies must be treated carefully. The theory of analogical reasoning is constructed on the explicit awareness of the similarity between the source problem and the target problem and the willful exploitation of this similarity in solving the target problem. Although a similar solution strategy is a good indicator of this form of problem solving it is not an exclusive outcome of analogical reasoning. Mediated isomorphism also relies on the explicit awareness of the similarity. However, because the similarity between the problems is not identified through direct comparison of the two problems as suggested by the theory of analogical reasoning, the purposeful use of the similarity is absent. Non-mediated generalization is a theoretical refinement of mediated isomorphism in that it lacks not only the willful use of the problem similarity, but more fundamentally, it lacks the very awareness of the similarity. However, both mediated isomorphism and non-mediated generalization have been shown to be present in cases in which similar solution strategies were used. Researchers need to be conscious of these subtleties both when considering experimental design and when analyzing data in order to avoid misattributing student's efforts to analogical reasoning.

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SUPPORTING TEACHERS IN BUILDING MATHEMATICAL DISCOURSE COMMUNITIES BY USING RESEARCH-BASED CASES

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Abstract

This study was designed to examine the effect of using research-based written cases on supporting teachers in building mathematical discourse communities. Four first-grade teachers enrolled in five workshops in which they studied the cases about first-graders' learning and teacher's roles in discourse. Through case discussion, the participants were supplied with needed experience and support for evolving their pedagogy. Asking students to explain, to clarify and justify their ideas orally, and criticizing for challenging their thinking were the three ways the participants used most frequently to encourage students to engage in discourse. The set of discussion questions in each case appeared to be an essential contributor to have a salient focus for the case discussion.

Key words: cases, narratives, mathematical communication, discourse.

Introduction

Communication is central to the current vision of desirable mathematics teaching (NCTM, 2000; MET, 1993). The process of creating mathematical discourse communities dealing with complex and multifaceted undertaking is a challenge for teachers (Lin, 2000; Silver, 1996). Teachers are challenged by the interplay between the reform vision of instruction and their own experience with more traditional tasks and pedagogy. Helping teachers toward an instruction rich in communication is likely to require new experience of learning mathematics in a manner that emphasized discourse and require needed support from collaborative communities of practice in which mathematical discourse occurs. One of the ways to meet the needed support and acquire new experience for teachers is through the use of cases that reflect others' experience on encouraging students to participate in discourse and centering discourse on mathematical ideas (Harrington, 1995). Therefore, case discussion is considered to be the kernel part of the study for helping teachers creating mathematics discourse communities.

The research-based cases involved in the study refer that the cases were constructed collaboratively by the researcher and teachers participating in a previous school-based teachers' professional program. The program is designed to help teachers keep with the tenet of 1993 vision of curriculum reform that emphasized engaging students with challenging mathematical tasks and enhancing students' levels of discourse about mathematical ideas. The effect of cases on supporting teachers' professional development has been examined (Lin, 2000). It indicates that both case discussion and case writing are two critical aspects of cases on developing teachers' thinking (Lin, 2000;

Richardson, 1993). Since participating in the research context in which cases were constructed, teachers are more likely to empower the effect of case writing and case discussion. However, it is still considerable whether the research-based cases in a written form would contribute to building discourse communities in mathematics classrooms of those who did not participate in the research context in which the cases were constructed.

Cases referred to in part of previous studies are conducted by personal practical experience (Barnett, 1998; Shulman, 1992). Thus, such kind of cases is intimately tied to personal practice and lacking of multiple levels of interpretation and analysis. As suggested by Merseth (1996), the research-based written cases referred to in the study are characterized in four essential ways. (1) Cases are based on careful research. The first-grade case-teachers participating in the previous professional program have been collaborated with the researcher to enhance their mathematical instruction through an emphasis on mathematical communication within the context of innovative curriculum. (2) Cases present reality. (3) Cases are developed to stimulate thought and debate for expanding the perspectives of users, but not to provide best practices. (4) Cases are potential to help users to recognize salient aspects launched from a set of discussion questions.

Researchers have examined how teachers interact each other through case discussion (Levin,1995; Richardson, 1993), but little empirical work describes how case discussion influences teachers creating mathematical discourse communities. Thus, this study was designed to examine what teachers learn and investigate how teachers establish mathematical discourse communities in their classrooms through the discussion of research-based written cases.

Theoretical Perspectives

The theoretical rationale for this study stems from the constructivist's perspectives. Piaget (1932) suggests that children's conflicting ideas are resulted from peer interactions. Cognitive conflicts result in an imbalance providing the internal motivation for an accommodation. Similarly, conflicts can occur for teachers, when they learn to teach individually or socially. The social interaction in a group discussion has the potential for initiating cognitive conflicts, hence to result in teachers' change. Vygotsky (1978) asserts that what is learned in the social interaction of the group is prerequisite to cognitive development. From this perspective, the interaction and the content of the group discussion are crucial to what is learned from cases. This would manifest itself in differences in understanding the issues in cases between those who do and do not participate in the process of constructing cases. In addition, his notion about "the Zone of Proximal Development" can be interpreted as suggesting that cognitive conflicts, caused by discussing, debating and negotiating in interactions between learners and more capable peers, act as a catalyst for reaching a higher development level. This indicates that case-teachers with more experience should influence the thinking of other teachers with less experience who interact with them in the case discussion. The two theoretical perspectives from development and social psychology provide a basis for helping us think about how knowledge is constructed both individually and socially. Case discussion fosters personal reflection through an external process (Shulman and Colbert, 1989). This notion suggests that social interaction during case discussion among the case-teachers and the teachers in a group seems to be likely to initiate cognitive conflicts and then contribute to reaching a higher level of psychological development.

Method

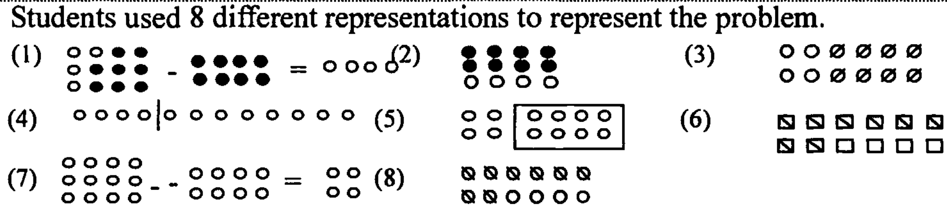
The participants in the study were four first grade teachers (Lao, Pan, Hu, and Wu) at two schools, located in a suburban area. They were experienced teachers with at least 10 years of teaching experience, but they were beginning teachers in their second year of teaching the 1993 vision of innovative curriculum. The participants were volunteers to participate in five workshops as part of a teacher education program that was designed to help teachers understand how first-graders develop mathematics concepts and identify various aspects of a teacher's role in mathematical discourse.

The research-based cases have the following major issues: (1) a mismatch between teacher's goal and objectives of a lesson, (2) students' various solutions of resolving a problem, (3) inappropriate tasks, and (4) framework of underpinning the curriculum. Through the cases, teachers are expected to deepen their understanding and expand their views of students' ways of thinking, hence to identify various situations in deciding what to pursue in depth, when to model, and how to encourage student to engage in tasks.

A set of discussion questions is one of the six components included in a case (Lin, 2000). The discussion questions", as kernel part of each case, are incorporated the reflections of case-teachers who involved in constructing the case into the major concepts to be learned of the lesson. The questions listed in cases are to stimulate teachers' rethinking about mathematical teaching and reflect to their practices. A set of focus questions of the case entitled "How do you categorize them?" serves as an example (see Figure 1).

There are 12 toys. Kei-Hua took 8 of them away. How many toys are left?

Students used 8 different representations to represent the problem.



Discussion Questions

1. How would you categorize the various representations? Why did you do so?
2. Which of the representations would be presented in your teaching?
3. Do you consider the factors including the forms, arrangements, and the order of crossing out the pictures when you categorize them?

Figure 1 Discussion Questions in a Research-Based Written Cases

As part of the workshops, the discussion of each case is used to probe for responses to the questions in the written cases. To structure case discussion, I, as the leader of the discussion, played various roles in order to guide, probe, give feedback, and observe each participant's thinking about cases. There were two common questions to be addressed across cases in the case discussion session. (1) How would you respond to each question of the "Discussion questions"? (2) What are some of the issues in this case coming up in your mind in this case?

The participants read and discussed eleven cases in the five workshops. They were observed during their mathematical instructions in the entire year. To document the efficiency of cases resulted in teachers' change, each workshop determined the time of each observation. There was around one-month difference between two workshops. The time of the four classroom observations and five workshops was depicted as O_1-W_1-W_2-O_2-W_3-W_4-O_3-W_5-O_4, O_i: the i^{th} observation, W_j: the j^{th} workshop. Available data for each participant's four curriculum units include 12 classroom observations. O_{pre} including the O_1 and O_2 observations occurred in the first school term and O_{pos} including the O_3 and O_4 observations in the second term.

The data were collected from the participants' responses to questions of each case in the five workshops and their instructions to be observed. The videos and audiotapes were recorded and transcribed verbatim. The data was analyzed by using Strauss and Corbin's grounded theory (1994). Through recursive reading the transcriptions of each video teaching, a coding scheme including ways of communication and contents of teaching was developed. There were nine ways of commutation in mathematics emerged: restating students' statements (R), asking students to explain what they understood (E), asking students to clarify and justify their ideas orally (C), criticizing for challenging students' thinking (J), giving students hints when students struggle with a difficulty (H), encouraging students to share their answers (S), asking students to model or demonstrate their understanding (D), motivating students to provide multiple ways of thinking (M), and inviting other students' follow-up ideas for making up an incomplete interpretation (I). Each lesson was categorized into various phases of teaching across mathematics contents when discussing the problem to be solved, the use of hands-on, symbols and number sentences given by students. Each sentence in the transcription under a unit of analysis was encoded and the frequencies were counted for each lesson. The constant comparative analysis was used to compare data segment, determine similarities.

Limited space prevents to report what the teachers learned from each research-based written case and how they learned about building mathematical discourse communities. An illustration of the case influencing on teachers' thinking about cases is taken from one case that deals with three levels of representation. To illustrate in-depth the effect of case on teachers' ways of communicating mathematics with students, a participant's lessons as an example will be the focus of analysis reported in this paper.

Case discussion on teachers' awareness of students' learning mathematics

Initially, the four teachers were puzzled with their students' early use of number sentences prior to teaching it. They were not aware of the importance of the development from concrete, pictorial and to symbolic representations to first-graders. The issue of representations listed in the discussion questions of a case was brought up in the second workshop to discuss as follows.

> As observed, a student wrote a number sentence 8-5=3 first, and followed by drawing the circles oooooooo - ooooo = ooo representing the given problem: *"There are 8 children and 5 presents. Each child gets a present. Are the presents enough?"* Do you accept the student's representation, oooooooo - ooooo = ooo?Why? What instructional strategy would you take to reduce the use of oooooooo - ooooo = ooo? (2nd workshop, 11/08/2000).

In the discussion, Ms. Lao and Ms. Wu reflected their students' high frequencies of using number sentence to their first year of teaching using the innovative curriculum. They faced the challenge of students' disordering learning of pictorial with symbolic representation, but they did not realize its hindrance of learning in multiplication. They perceived that their students did not make sense to the meaning of the number sentence at this time, so that the circles most first-graders drew merely meet teacher's needs rather than represent a process of student's thinking. Understanding a mathematics concept meant by them was that students are capable to use it. Thus, prior to teaching it, they accepted students' use of the number sentence 8-5=3 followed by oooooooo - ooooo = ooo no matter who they learned from. Lao suspected why students must go through the use of hands-on and drawings, since they had used 8-5=3 to solve the problem. According to Lao's response to the issue, she did not appreciate the importance of the developmental processes among the three levels of representation to the future learning.

To resolve the puzzle, Ms. Hu offered an instructional strategy in the group discussion. "Decomposing a step into sub-step" means that "a word problem is divided into several parts, one part is presented at a time and students' follows". A problem described by *"There are 8 children and 5 presents. Each child gets a present. Are the presents enough?"* is a typically complete statement. Alternatively, the problem is posed by decomposing it into three sub-steps as "*There are 8 children, representing the 8 children by chips; 5 presents, showing the presents by chips. Are they enough? (counting the amount of chips indicating the answer)*". In the discussion of the case, Hu not only introduced its meaning but also explained the merits of the alternative approach to other three teachers who have never heard this term before.

They supported mutually. Lao, Pan, and Wu learned the meaning and the strategy of "Decomposing a step into sub-step" from Hu and put it into their following lessons. Moreover, they became an opponent of the approach, since the strategy indeed resulted in the low frequencies of their students' early use of number sentence. Lao stated the efficiency of cases in the third workshop.

> The previous case discussion helped me perceive my blind spots in instruction. Reading and discussing it ahead of my teaching make significantly effectiveness of teaching the innovative curriculum in this year. I re-taught previously unsuccessful lessons by taking "Decomposing a step into sub-step" approach aiming at the use of hands-on or pictorial representation of a word problem. Through the approach, I found that first-graders' at this stage did not use the symbolic representation to solve the word problem any more. Thus,

if I can understand the main ideas of a lesson in which they are focus questions of a written case, then I would not be frustrated with students' difficulty with learning. Moreover, I could predict students' various solutions in advance and then help them have better understand (Lao, 3rd workshop, 12/06/2000).

Case discussion on supporting teacher in building mathematical discourse

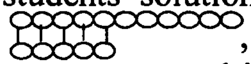
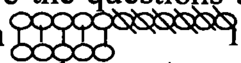
The topic of the lessons determines what and how teachers communicate mathematics with students. To be coherent the effect of cases on teachers' thinking about cases with supporting them in building mathematical discourse communities, the analysis of this section will be only focused on the lessons relating to number area. The lessons in a curricular unit is determined by each teacher's instructional activities. The textbook Lao and Pan used is different that of Hu and Wu used. The lessons of number topic included in the textbooks scheduled on the list to be observed in the study were included only in Lao's and Pan's six lessons. Lao's six lessons serve as an example shown in Table 1.

Table 1: Frequencies of Ways and Contents of teaching in Lao's Mathematical Discourses

Ways / Frequencies. / Content		Restatements (R)	Give students hints (H)	Explain (E)	Model (D)	Share answers (S)	Clarify justify (C)	Criticize (J)	Multiple thinking (M)	Compensate incomplete answer (I)	**Totally**
problem posing	O_{pre}	18	4	16	2	3	20	6	5	1	**75**
	O_{pos}	0	2	31	10	13	13	13	10	8	**110**
hands-on	O_{pre}	12	1	11	11	1	19	12	0	11	**78**
	O_{pos}	7	0	23	18	3	29	33	3	18	**134**
drawings	O_{pre}	1	1	13	1	8	3	5	0	1	**33**
	O_{pos}	1	2	41	2	3	46	7	39	0	**141**
symbols	O_{pre}	3	1	19	17	9	20	26	3	4	**102**
	O_{pos}	8	0	26	25	3	14	24	13	3	**116**
Totally		**60**	**11**	**180**	**86**	**43**	**164**	**126**	**73**	**46**	**789**

O_{pre}: observations occurred in the first school term. O_{pos}: observations occurred in the second school term.

The data of the Table 1 shows that the most three frequencies of ways Ms. Lao used to encourage students to participate in classroom discourse were asking students to explain what they understood (E), asking students to clarify and justify their ideas orally (C), criticizing for challenging students' thinking (J). The frequencies of the three ways used by Lao were 180, 164, 126, respectively. Lao's ignorance of the importance of the pictorial representation described earlier was also evidenced by the data shown in her communication in classrooms. The data suggest that Lao did not frequently encourage students to draw circles representing the process of their thinking in the first school term (33 frequencies) as compared to the second school term (141 frequencies). The result indicates that the cases influencing on Lao's more attentions to the discussion of drawings with students. Lao improved her skills in asking students to explain (41 frequencies) and to clarify what they understood (46 frequencies) as she sought to engaging students in the meaning of drawings they drew.

As observed, Lao created a mathematical discourse community in discussing various strategies of a compare word problem. After posing the problem "*Shiao has 11 marbles. Mei has 5 marbles. How many more marbles does Shiao have than Mei? Drawing pictures represents your thinking and writing its number senescence with subtraction.*", she furnished opportunity for students to share their solutions and justify their thinking publicly. In the class discussion, Lao frequently asked students the question "Does anyone of you have different solutions?" to encourage students showing their different solutions from others'. To provoke thoughtfulness in the discourse, she asked students various questions about their answers. The questions were varied with students' solutions. For instance, after Kao-Bin offered his solution as 11-5=6 , Lao asked the questions to Kao-Bin "what do the circles mean presented in the above row?" "why did you use the way?" "What does 11 represent in 11-5=6? How about 5?How about 6?". Likewise, Lao asked Uein-Jye the questions to clarify and justify his answer, after he provided the solution  11-6=5. "Why did you write 11-6=5?" "What did the part crossed out represent?" Could you tell us which part represents Shiao's more marbles than Mei's?" "What does the difference between your drawings and Kao-Bin's?". Lao did not stop the discourse on students' various solutions until Shin-Jane presented the solution  6=11. Because Lao learned first-graders' possible solutions coming up in the problem from previous workshops, she expected the third solution showing up in the class discussion. Meanwhile, she asked a question to criticize for challenging students' thinking, "Did 5+6=11 given by Shin-Jane meet the demand of the problem in which number sentence is expressed by the subtraction?" From the above scenario, Lao clearly had progressed in satisfying the demands of listening carefully to students ideas, encouraging students' engaging in thinking about mathematical ideas, and asking students thoughtful questions to clarify and justifying their ideas orally.

Discussion

It is found that the use of research-based written cases enhanced the teachers' ability in creating mathematical discourse communities. In the discussion of cases, the teachers were supplied with new experience and needed support of mathematical discourse community from the member of the discussion group. They learned about the role of the researcher in creating mathematical discourse communities from the case discussion in which the manner is similar to that of mathematical discourse in classroom. Likewise, the researcher learned from the teachers about what they responded to the discussion questions described in each written case and how they learned to teach. The teachers supported mutually in understanding how first-graders learned mathematics by the discussion of the case. Lao's acquisition of students' various solutions from case discussion was an important contributor to her success in moving toward the creation of authentic mathematical discourse communities in her classroom. The new experience and needed support indicate that the teachers appeared to evolve their thinking and pedagogy from a traditional form toward a form of instruction centered with mathematical discourse. In Piaget's (1932) and Vygotsky's (1978) notions, the case discussion in a group created the opportunity of social interaction for the teachers. The set of discussion questions integrated with case-teachers' various perspectives are readily to initiate the teachers' reflection to their practices and cause their cognitive conflicts of mathematical teaching, hence to trigger change. The set of discussion questions are integrated with multiple perspectives of those who participated in practice-oriented research. The case-teachers embedded in research-based written cases play the significant role of more capable peers.

The research-based cases were not expected to provide best practices but to initiate the users' cognitive dissonance. Instead, the cases referred to the merits and breakdowns in classroom practices that call for solutions within the context of that practice. Thus, the use of research-based cases was more likely to support reflection-on-action and then oriented toward reflection-in-action (Schon, 1987). The discussion of written cases seemed to be a catalyst for the teachers creating mathematical discourses in classrooms. The set of discussion questions described in each written case appeared to be an essential contributor to have a salient focus for the case discussion. The set of discussion questions as a kernel part of the written cases become distinguished characteristics that are not possessed in the videotaped cases. The comparisons between the effect of cases presented in a written form and in a video form on either in-service or pre-service teachers' thinking about cases and decision-making of classroom are valuable for further investigations.

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MATHEMATICAL REASONING IN CONTEXT: WHAT IS THE ROLE OF SCHOOL MATHEMATICS?

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This report considers the appropriateness of modelling and of a radical situated paradigm for working out advanced (non-intuitively obvious) mathematical tasks in work contexts. A case study of vocational school students and expert practitioners solving geometry problems in a computer-drafting context indicates that neither of these paradigms is appropriate. For the students school mathematics and drafting context are, above all, important as frames of reference. The practitioners do employ school mathematics but their way of mathematising undergoes a complex transformation that cannot be simply explained by either paradigm and which requires further conceptualisation and research.

INTRODUCTION

The relation between school and out-of-school mathematics is an important issue which has been the object of many research studies. The apparently self-evident idea that mathematical knowledge, learnt at school, is simply applied in out-of-school practices via some sort of mathematical modelling is more of a myth than a matter of empirical observation (Bishop, 1988, p. 8). Situated cognition and related theories have pointed to a marked discontinuity between school and out-of-school practices (Lave, 1988). In short, mathematical practices in specific activities, e.g. in work or buying context, were found to have 'little' in common with school-learnt mathematics. Many studies (e.g. Masingila, 1993; Millroy, 1992; Scribner, 1984) have confirmed that contextual activities impose many conventions, social and activity-related constraints, indeed, different conceptualisations of apparently purely mathematical problems, and that practitioners, as they solve such problems, do not base their mathematical reasoning on school-learnt knowledge.

In this report I present a self-contained part of a recently completed broader study about geometric thinking in an out-of-school context. The research was motivated by the observation that studies about the discontinuity between school and out-of-school mathematics considered mathematical practices where it would hardly make sense to use school-learnt procedures, for they are either ineffective or the problems are too complicated to be solved by analytical means or they are so simple that the solutions can be easily learnt from peers or can be even self-invented. The research considers the situated vs. modelling dichotomy in activities and situations where advanced (i.e. non-intuitively obvious) school-like mathematical knowledge could be profitably applied. Several aspects of the situated vs. modelling paradigm of mathematisation in out-of-school contexts are examined: the mathematical actions in activities (Magajna & Monaghan, 1998), learning mathematical concepts in activities (Magajna, 1999), and mathematical reasoning in activities. In this report I focus on the last issue: how do people reason when they solve non-trivial geometric problems which arise in the case of designing geometric shapes on computers using professional software for computer aided design (CAD). In particular I consider the following questions:

1. Do practitioners in working out geometric problems (where school mathematical knowledge can be profitably used) switch to school mathematics by using some sort of modelling or do they stick to activity practices?
2. Do practitioners, as they solve geometric problems in context, take into consideration, in any way, the context (e.g. the available tools, allowed ways of working, the required precision)?
3. How does personal mathematical knowledge and work expertise reflect in the way practitioners work out geometric problems in activity?

METHODOLOGY

The research questions were explored in two activities closely related to computer technical drafting: designing moulds for glass containers (a work activity) and learning computer-drafting in a school for machine technicians (a school-learning activity). Since the aim of the study was to find cases of qualitatively different ways of mathematical reasoning in context the participants were purposefully selected from those that took part in the broader research. In this part of the research eight participants were studied as they solved problems given in a computer-drafting context. Two of the participants were machine technicians with six and 15 years of experience, selected from a group of six designers and technologists in a small mould-making factory in Slovenia. As part of the broader research the group was studied using several techniques (interviews, ethnographic observations, scheduled observations, document analysis). The other six participants were selected from a class of 22 students, aged 18, from a vocational school of machine technicians in Slovenia. The students attended a computer-drafting course, which lasted several months (two hours per week) and were observed as part of the broader research. The six participants were selected on the basis of a special filtering procedure aimed at selecting students that avoid and students that are inclined to use non-intuitively obvious ideas in computer-drafting.

The participants' reasoning in solving geometric problems was studied by analysing verbal protocols of participants' speech as they solved geometric tasks in a drafting context under think-aloud instructions. In simple terms, the method consists in recording and analysing the vocalisations of subjects who work out a task and simultaneously talk, i.e. they are verbalising what is 'going through their mind' as they work on a task. More details about the method are in (Ericsson & Simon, 1985).

Each of the eight participants was asked to work out, under think-aloud instruction, 3-6 drafting tasks which contained non-trivial geometric problems. Figure 1 shows two examples of such tasks. The participants worked in front of a computer using the drafting program they commonly used. Their utterances (while thinking aloud) were tape-recorded, and their hand-sketched and computer drafts were also recorded. The protocols were literarily transcribed and segmented using easily applicable criteria of pauses between thoughts and the timing of each segment was measured.

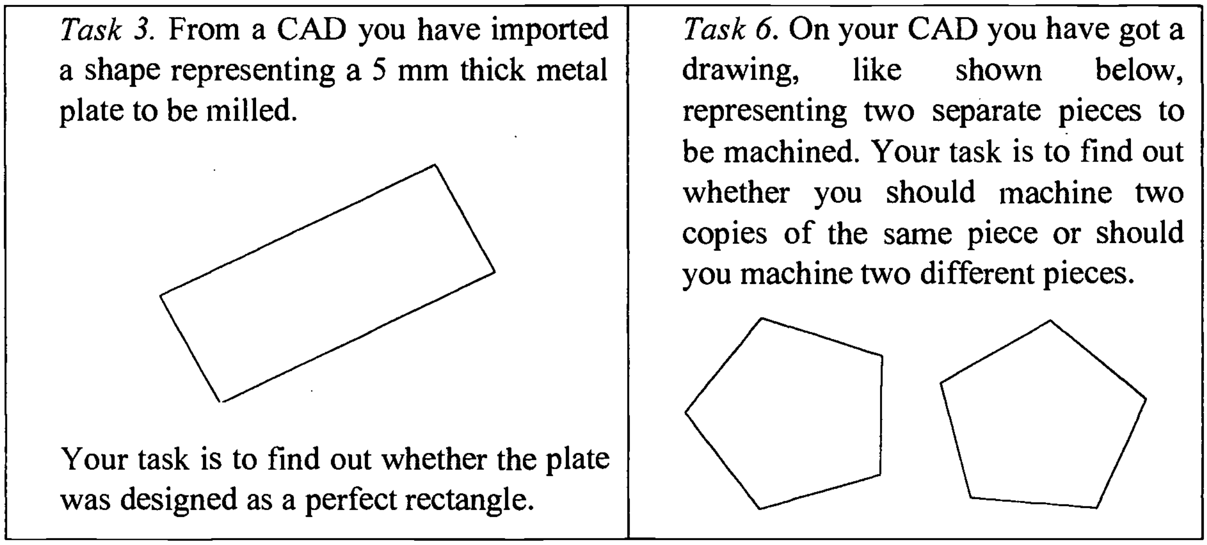

Figure 1. Two examples of tasks used for verbal protocols.

ANALYSING THE PROTOCOLS AND RESULTS

Of the eight selected participants one was not able to think aloud. The other seven participants solved altogether 32 geometric problems. The recorded protocols gave rise to 2843 segments, 773 of which were excluded from further consideration because were not directly related to the solved problems (e.g. comments on saving files to the computer disk). The analysis proceeded along two directions: a qualitative analysis of the solutions of the tasks, based on clustering techniques, and a quantitative analysis of the protocols, which is described below.

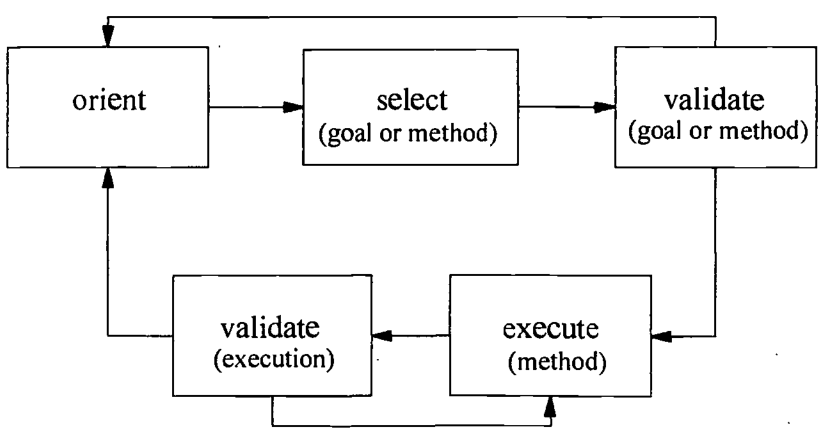

Figure 2. The assumed processes in reasoning.

The quantitative analysis of the protocols has to be considered to be exploratory because of the small number of observed tasks and because of the fact that the participants were purposefully selected. In the analysis two elements were considered: 1. the cognitive processes during the problem solving, and 2. the context (school mathematics or computer-drafting) to which the participants were referring while solving the tasks. The cognitive processes were analysed using a cognitive model, based on Saxe's (1991, p. 16-23) description of cognition. Figure 2 shows a simplified schema, derived from the model. The related categories used for coding the protocol segments are listed in Table 1.

Category	Description of segments
Context	Referring to the problem at a meta-level, e.g. any reference to drafting or school-learning activity, tools, artefacts, conventions, or social relations.
Ignored	Making statements related to the problem but not to the solving process, e.g. students' utterances solicited by experimenter for clarification.
Orient	Looking for a goal or method; stating given, observed or inferred facts; quoting facts or methods without relating them to the worked task.
Select	Selecting, declaring, naming a goal or method with the expressed intention to work on it or to consider it as a possible direction of work.
Execute	Executing a method either mentally, on paper or on computer.
Validate	Checking whether the selected goal or method would lead to the desired result, e.g. expressing arguments, questions or confirmations about the correctness or appropriateness of a goal or method.

Table 1. Categories related to the assumed cognitive model.

The assumed cognitive model and the categorisation were satisfactory in the sense that the segments could be reliably coded and reasonably followed the flow from Figure 2. For example, the segments from the Orient category were followed by another Orient segment in 61% of cases, by a Select segment in 15% of cases, by an Execute segment in only 6% of cases and by a Validate segment in 3% of cases.

Another categorisation of the segments of the verbal protocols, shown in Table 2, was related to the important issue of whether participants, in solving a geometric problem in a drafting context, were 'reasoning' in terms of school mathematics or in terms of contextual (drafting) activity. It was initially envisaged that it would be extremely difficult, if not impossible, to determine whether someone is referring to one or another context but this was not the case because there are concepts and operations in computer-drafting that are very unusual or not even meaningful in school-geometry context (e.g. setting the snap interval of the computer mouse – an important issue in defining geometric entities in computer-drafting) and vice versa (e.g. quoting a theorem or writing down an equation). However, in most cases the segments are meaningful in both contexts (e.g. measuring a distance or drawing a line segment). It was fortunate that the participants were thinking aloud in Slovenian while the software they were using was in English, so that it was easy to identify segments that certainly referred to drafting commands on computer.

Category	Description of segments
Context	Referring to the problem at a meta-level, e.g. any reference to drafting or school-learning activity, tools, artefacts, conventions, or social relations.
Ignored	Making statements related to the problem but not to the solving process, e.g. students' utterances solicited by experimenter for clarification.
School geometry	Referring to geometry related ideas which are clearly not meaningful or common in computer-drafting activity and which are clearly not related to any computer (drafting) command.
Unclear	Referring to geometry related ideas that are meaningful in the context of school geometry as well as in relation to computer-drafting.
Drafting	Referring to geometry related ideas that are clearly not meaningful in the context of school geometry.

Table 2. Categories related to the context of thinking.

Here is a brief summary of the results of the (exploratory) quantitative analysis of the 32 geometric tasks. Recall that the tasks were solved by eight participants (one of which was not able to think aloud). The quantitative and qualitative analysis of the protocols (as well as other performances not considered in this report) indicated that the participants clearly split into three groups: the two expert practitioners (the WORK group), three students that avoided using advanced mathematics (the CAD group), and two students that showed a preference for applying advanced mathematics in drafting context (the GEO group). The unit of analysis used in comparing the three groups was the solved task. For each solved task the fractions of time spent in each category related to cognitive processes was computed (and similarly for categories related to contexts). To determine significant differences in the distributions of codes between tasks worked out in the three groups a one way ANOVA was performed for each code. The results are presented in Table 3. From the table it can be read, for example, that validation took on average 16% of time for the students from the CAD group, on average 13% of time for the students of the GEO group, and on average 21% of time for the participants of the WORK group. The value p ($p=0.297>0.05$) indicates that in this respect the groups do not differ significantly. From Table 3 it is evident that the three groups do not significantly differ with respect to the assumed cognitive model. However, the participants from the three groups behaved quite differently regarding the context they referred to as they were solving the tasks.

	Common		Cognitive model				Context of thinking		
Group (Tasks)	Context	Ignore	Orient	Select	Exe cute	Vali date	Draft	Unc lear	Sch ool
All (32)	1%	13%	23%	10%	35%	17%	27%	50%	8%
CAD (13)	2%	16%	20%	9%	37%	16%	40%	40%	2%
GEO (8)	0%	15%	26%	12%	34%	13%	25%	50%	10%
WORK (11)	2%	9%	24%	10%	34%	21%	13%	63%	13%
Significance p	0.475	0.387	0.687	0.609	0.949	0.297	0.004	0.031	0.119

Table 3. Average fractions of time per task spent by participants of different groups in categories related to cognitive model and the categories related to context of thinking.

DISCUSSION OF THE RESULTS

School mathematics in drafting context. In spite of the fact that in drafting activity procedures tend to be routinised and, for the sake of productivity, reduced to push-button operations, some students and practitioners use methods and reasoning patterns similar to those commonly found in school mathematics. A practitioner, for example, solved Task 3 (see Figure 1) by measuring the lengths of both diagonals: since they were not congruent, he deduced that the quadrangle is not a rectangle. Practitioners and some students also wrote down equations containing trigonometry functions or ratios related to similar triangles and tried to solve them. In general, the students from the GEO group and the practitioners from the WORK group did not hesitate to draw attention away from the computer drawing and to study the geometric properties of the figures. In contrast, the students from the CAD group, in general, tried to solve the problems by looking for appropriate computer commands. Though the tasks were set in computer-drafting context, some participants did not automatically stick to the computer-drafting practices.

The role of the context. The practitioners very sporadically mentioned anything about context. Altogether they spent only 1% of time on segments anyhow related to the contextual activity, this occurred mostly when they were stuck. The fact that the participants did not reason about the context at the beginning of the task (and similarly not at the end of the tasks) by itself indicates that they their way of working out geometric tasks was not based on mathematical modelling.

Though the participants spent very little time considering anything related to the context, the context played an essential role in solving geometric problems. The students from the GEO group evidently placed the task immediately (without even thinking about it) in school-geometry context, and similarly the students form the CAD group placed the tasks in drafting context. The students from the GEO group spent significantly less time in drafting considerations (e.g. looking for an appropriate command, working on a command) and used more advanced mathematical ideas than the students from the CAD group. This indicates that, for students (novices), the context serves as a frame of reference in making sense of a task and in indicating the way of working out the task. Though the tasks were given in a drafting context, some students perceived them as 'purely' mathematical tasks to be solved with mathematical means, and some perceived them as drafting tasks to be solved with techniques specific to the used software or activity. Typically, the students from CAD group solved Task 3 by interactively rotating the figure to an almost horizontal position and reasoning about whether the line segments are parallel on the basis of the discontinuities of lines due to the resolution of computer display.

Expert practitioners regarded the context quite differently than the students. On one hand they had considered the broad context of the task – in solving Task 6 (see Figure 1) a practitioner, for example, asked whether the shapes have to be milled from inside or from outside and he mentioned what one should take care of when positioning the workpieces on the milling machine. On the other hand, though the practitioners claimed they had forgotten all mathematics they learnt in school, in solving the tasks the mathematics they used was much richer than even those of the students from the GEO group. They used trigonometry, wrote equations and made non-trivial deductions. This apparently paradoxical behaviour is perhaps due to two causes. First, the practitioners spent very little time on drafting considerations (see Table 3), for they used the software in a very fluent, almost automatised way. The fluency in managing the activity-related apparatus seems to be a necessary condition for 'advanced mathematical reasoning' in other activities. Second, compared to the students, the two participants used many mathematical ideas and procedures similar to those found in school classes even though they did not perceive these to be mathematical. I hypothesise that, as participants join an activity, their mathematical knowledge undergoes a transformation. Apparently, on one hand their visible mathematics reduces to some more or less routinised practices, but on the other they learn to relate school mathematics to the activity environment, how to 'reason mathematically' using the structural resources found in their activity and how to govern their mathematical thinking by the contextual activity.

Individual differences in solving mathematical tasks in context. One of the aims of the research was to point to different ways of using school mathematics and activity-related knowledge in solving mathematical tasks that occur in other activities. The most important differences in this respect have been considered above, but other differences between the participants from the three groups also emerged. Expert practitioners, for example, in solving harder tasks worked in parallel on two or more different directions, while the students commonly stick just to one solution. Another interesting trait of expert practitioners was to work on solutions for which they knew in advance to be only apparently correct. Usually they were able to use such solutions as a step towards the final solution. Such differences are probably due to the expertness of the practitioners.

FINAL REMARKS

This study indicates that when the use of advanced mathematical ideas in out-of-school practices is considered neither the radical situated view (according to which people work out tasks by relying on activity-related practices) nor the modelling paradigm (according to which tasks in activities are worked out by applying school-learnt knowledge) seem to be valid. Inexperienced students solve geometric problems given in an out-of-school activity by placing them in a context - which can be school mathematics as well as the contextual activity - and solve them according to conventions common to the considered context. For them, the context is, above all, a frame of reference for setting the meaning of the task and for setting the way of working out the task. As practitioners acquire expertise their school mathematical knowledge and mathematical reasoning appear to undergo a complex transformation which is not explained by either paradigm. Further research and different conceptualisations are required to describe mathematical thinking in (work) context (Noss, Hoyles & Pozzi, 1998).

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'WHY ARE WE DOING THIS?': A CASE STUDY OF MOTIVATIONAL PRACTICES IN MATHEMATICS CLASSES

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In this paper I argue that looking at the meanings that mathematics classroom practices have for learners and the subject positions that these make available are central to understanding who succeeds, who fails, who enjoys and who detests maths, and so who decides to continue studying it. I explore this using a case study of the, often overlooked, motivational strategies used in an AS maths class arguing that the stress on the exam, competitiveness and procedural work engaged in make it difficult for students to come to think of themselves as mathematicians and so unlikely that they will study the subject further.

> Veronica Sawyer [1] has just marked the second homework of her new first year AS-level group [2]. The topic of the homework was completing the square although a lot of the errors occurred in the students' manipulation of fractions. As a result she decides to suspend the normal curriculum and spend an hour looking at prime factors, highest common factors, lowest common multiples and then the four operations on fractions. She describes this as a "very, very simple" topic "dealing with very, very basic things" and further as "babyish". The students progress through the work largely in silence and with few problems. Their teacher interrupts them at regular intervals to offer advice, call out answers, deal with queries and remind them how to deal with the operation rehearsed in the next set of questions. One particular student, Vicky is one of the first to complete the fractions exercises and has had no problems with it. This surprises her teacher who had commented at the beginning of the lesson that, given the mistakes Vicky made in her work, this lesson will be particularly useful for her. Mrs Sawyer asks her, "so it wasn't that that was the problem? What was it?" Vicky pleads ignorance and suggests that, "perhaps it was just a bad day". Mrs. Sawyer simply adds that it cannot be that because she has made these same errors twice, and the discussion ends.

This occurred during my observation of classes at a London college and struck me as typical of how, often in maths lessons, the assumptions of teachers about how maths is learnt cannot explain learners' behaviours. The teacher's conversation with Vicky indicates that she has assumed that the latter's inability to apply a skill (in this case, the manipulation of fractions) in one context (in this case, completing the square) indicates a general inability to apply that skill in all contexts, and one that could be remedied by decontextualised drill and practice of the given skill. In spite of a mounting body of evidence to the contrary, maths teachers continue to act as though they are 'delivering' transferable skills to students and, as in this case, disappear the parts of their lived experience that cannot be understood within such a model.

In this paper I highlight some of those disappeared features. I argue, drawing on insights from situated cognition, post-structuralism and psychoanalysis, that it is the practices in which maths students and teachers engage, the meanings they give to them and the possibilities these make available for their own identity work [3] which provide a better model within which to understand who comes to succeed and who to fail in the subject and crucially why some people choose to continue with the subject and some reject it. I focus on the practices through which teachers, explicitly and implicitly, answer the frequently asked student question "why are we doing this?" since constructing a rationale for one's actions is a central part of identity work.

SITUATING THE RESEARCH: WHERE, HOW AND WHY?

The case study data used here are taken from an on-going project on gender and AS-level maths that combines classroom observations with student interviews at three sites: Grafton School, Sunnydale Further Education College and Westerburg Sixth Form College [4]. They focus on subject choice, since it is in participation, rather than in achievement, that significant gender differences remain in the United Kingdom. The starting point is the idea that subject choices are a public disclosure of identity, a way of telling people who we are and who we are not. As such, they are intimately connected to other aspects of our identity work: importantly gender, class and 'race'. This orientation to the project, together with my overtly feminist stance, guided the methodological and theoretical choices that I have made. In particular my methods involve the detailed discursive exploration of qualitative data and my theories aim to challenge the authority of mathematical knowledge and to politicise maths classes.

I interviewed 12, 13 and 18 students at Grafton, Sunnydale and Westerburg respectively during their AS course. Also, during Autumn 2001, I have observed two weeks of lessons in each of the classes attended by my participants. In this report I concentrate on some of the findings from my observations, looking in detail at Mrs. Sawyer's AS class at Westerburg. Westerburg is a college with about 800 ethnically diverse 16-19 year-old students and a mostly academic curriculum; it is always oversubscribed and so has a highly competitive entry policy. It has a large maths department, with over 300 students being taught advanced maths courses by six full-time and one part-time members of staff. In order to qualify for entry to an AS maths course students must have at least a grade B pass from the higher level GCSE examination, although occasionally exceptions are made [5]. I have drawn on my observations in Westerburg here, rather than those in Grafton or Sunnydale, because sixth form colleges are now the most common route through which young people gain AS and A-level maths qualifications. I chose Veronica's group because, of the four Westerburg teachers I observed, she was the one who most commonly invoked the requirements of the examination as an explanation for her teaching practices and so provided the most unambiguous message to students regarding the purpose of her classes. However, there were also more personal reasons for my choice. While Veronica is clearly a good and experienced teacher who cares a great deal about the young people in her classes she told me, "most of the [students with GCSE grade]

B's fall by the wayside". This statement troubled me more than any others I heard teachers say and I think that I wanted to look at the ways in which such an outcome could seem natural and unproblematic. Retrospectively, I suspect that the way I see her teaching style as the most different from my own was an unconscious motivation for my choice, since it makes my analysis less threatening to the identity work I do in constructing myself as a 'good mathematics teacher'.

I was a participant in the classes I observed. However, my position was an ambiguous one and my participation differed from that of either the students or the teacher. I sat with the students, obeyed instructions from the teacher, took notes (although of course very different ones to the other students), and was occasionally used by students to subvert the teacher's intentions. I also adopted some teacher-like practices: assisting students with their work, hanging out in the maths staff office, and even though I encouraged students to call me Heather I often found they addressed me as 'Miss'. I took detailed field-notes that I wrote up within a day of the experience. I then worked through these notes identifying and classifying incidents according to themes that emerged from the data. I do not see the process of these themes' emergence as one of my discovery of the intrinsic truth embedded within my data, as perhaps grounded theory approaches would suggest, but as one of creative invention on my part. Although this was a process of invention that often surprised me, what I noticed and how I understood it were influenced by my theoretical framework, so I will outline this before giving an account of the observations.

SITUATING PRACTICE: MEANING AND BECOMING

The weakness of the traditional, functionalist model of the transfer of skills within maths, highlighted by the mismatch between Veronica Sawyer's expectation for the lesson and the actual outcome of it embodied in Vicky, was famously challenged by Lave's (1988) study. Lave compared the near perfect performance of people she names 'just plain folks' on best buy calculations with their much weaker scores on a maths test covering the same numerical skills. Lave developed a different way of looking at learning to help her understand her findings, a perspective she calls situated cognition in which maths and learning are conceived of as social practices, taking place within communities, and learning is never context free. This is a significant shift for it recognises that although "people clearly do transfer ideas, feelings etc. from one context to another under all kinds of conditions...what they transfer is not always what we in education would like them to transfer" (Evans, 1998, p. 285). This approach displaces the power of maths; it is no longer seen as an absolute body of knowledge, but as something people do. Within this framework the opposition of abstract and concrete knowledge is viewed in terms of the socio-cultural practices within which the differently classified objects are learnt. The explanatory burden for problems experienced in teaching is shifted from the cognitive and the pedagogical to "issues of access, and...the transparency of the cultural environment with respect to the meaning of what is being learned" (Lave and Wenger, 1991, p. 104-5). If the 'ability' to perform a certain skill is tied to the context in which we are asked to perform it, this means that a variety of things impact on it, including the language and situation in which the task is framed, the physical environment, the actions of other learners and of the teacher, and the other experiences of the learner in which these are embedded. Evans (1998) work on adults' mathematical problem solving points to the complex interweaving of desire and discourse out of which individual performances are constituted. I concur that:

> In a given setting subjects in general *are positioned by* the practices which are at play in the setting *and* that a particular subject will *call up* a specific practice (or mix of practices) which may differ from those called up by other subjects, and which will provide the context for that subject's thinking and affect in that setting. (p. 274, original emphasis)

So the meanings of the practices for those doing them are complex mixes of the structural, the environmental and the psychic, and hence are intimately tied to the way we talk about ourselves, that is to our identity work.

It is this research that situates my approach in the analysis that follows. I am seeking the meanings that students and teachers give to what they are doing in their classroom communities of practice. Thus, it is not what maths is or is not that matters but what it is constructed as being and as not being; the stories we tell about it; the discourses through which it is constituted (Walkerdine, 1998). And also the subject positions that these make available to learners, the way in which maths becomes part of their identity work. I start where Lave and Wenger do with what people do.

SOCIAL PRACTICE IN A MATHS CLASSROOM

The lesson described earlier in this paper was typical of the way students in this maths class generally work individually on repetitive exercises that practice set procedures, when they are not being taught didactically. Mrs. Sawyer's decision to improve the group's skills by going back to basics exemplifies the way she feels that maths has to be done in a particular order. In another lesson she comments that they are going to do all the operations on polynomials except for division. She explains that it would be a waste to tackle division before the group have learnt more techniques. This discourse of a hierarchy of knowledge in which she inscribes mathematics, renders pointless the doing of topics out of order. However, it exists in awkward relationship with the way in which different syllabuses and different textbooks put topics in very different orders. The fact that division of polynomials has shifted in the new AS course from the second to the first year, and the fact that at least one student has already covered this topic, provide further contradictions.

Within this hierarchy of knowledge some topics are located near the top of the scale, as 'hard' and others near the bottom, as 'easy'. As the lesson described at the start further demonstrates Mrs. Sawyer often described the latter as 'babyish'; this association with a younger state being carried through by their being labelled as junior school or GCSE work. Such references draw on and fuel a series of parallel binaries in which 'baby' work is 'easy' and opposed to 'grown up' methods that are 'hard'. These are gendered, with the masculine 'hard' also opposing the feminine 'soft', and classed, because certain aspects of middle-class 'cultural capital' are commonly taken as signifiers of mathematical maturity and 'ability'.

In looking for meanings in maths classes many researchers have discussed the way that sense-making is absent from overly procedural, competitive maths lessons. Boaler commented on the maths classrooms that she had encountered:

> It seemed to me that in most of them, it was as if there were a kind of check-in desk just outside the classroom door labelled 'common sense', and as the pupils filed into the classroom, they left their common sense at the check-in desk saying 'Well we won't be needing *this* in here'. (1994, p. 554, original emphasis)

In a later study Boaler (1997) reported that many students, but particularly girls, found their 'quest for understanding' frustrated by the fast paced, repetition of techniques that dominates top set maths pedagogy. Veronica Sawyer's approach to the 'quest for understanding' is interesting. On a couple of occasions during exposition of demanding topic areas she substitutes belief for understanding, so for example her discursive framing of a topic shifts her from comments like "I know this is hard for an afternoon lesson" to "you've got to believe me, it's not magic, you've got to believe me, I'm not fooling you". Alternatively she often suggests that the quest must be deferred, "you've got to be patient with yourself when you're learning"; understanding will result from time, effort, and hard work. This construction of understanding is used as a rationale for stemming the flow of student questions. However, I would argue that the most important difference between the 'quest for understanding' of the girls in Boaler's study and the one that Mrs. Sawyer wants her class to pursue is its motivation. She makes clear that it is the examination that defines not only what understanding is needed, but also whether you have understood or not; an external authority replaces internally authorised sense-making.

The spectre of the examiner as a disciplinary presence in the classroom

As you would expect in all of the maths classes I observed there was some mention of the exam. However, the frequency and nature of such mentions was very varied. In Veronica Sawyer's class the exam itself was discussed often and the teacher used many opportunities to describe exam technique. However, it was largely through the constructed figure of 'the examiner' that the exam made its presence felt. This examiner is sometimes a hard taskmaster (male labels/pronouns are used here advisedly) rigorously insisting on one form of answer over another and at other times a doddery old man who may find messy work "confusing". The examiner guides the choice of methods as when Mrs. Sawyer instructs her students to score neatly through each term when expanding brackets so that the person marking your script can still see it and to write "comparing" when comparing coefficients in order "to show off to the examiner that you're a logical person". That the examiner is also the ultimate arbiter of right and wrong in the eyes of the students is clear from Imran and Saeed's comments to me during a lesson on inequalities. When I point out to Imran that he has used = instead of <, he says twice "you know what I mean" and then shifts to "the examiner will know what I mean". While Saeed, who gets the answer '$-x<-4$', wants to know how to get the answer at the back of the book. After I explain he wants to know (referring to his original solution) "will they mark that as right in the exam?" However, exams carry with them constant evaluation, not only against the requirements of the examiner, but also against each other. As Denscombe (2000, p. 370) found students recognised that exams offered "the prospects of success or failure which could be used as a 'measure of the person' on which to make comparisons with others". Next I look at the way this process of comparisons was played out and encouraged in the classroom.

How do I measure up? Competition within the classroom

During the ten hours of lessons that I watched there were two short tests. These tests are clearly linked to improving student performance in the exam. However, competition seems to be a motivation over and above this. In both cases students are required to declare their marks in front of the rest of the class. Veronica also sustains a competition between this group and her parallel first year class, whom she describes as "my other set, my decision maths set, that you're in competition with". When several members of the group provide her with the correct answer to the product of $4x$^3 and $3x$^2 she praises them with the words "you've already beaten my other group" (who had offered the incorrect solution of $12x$^6). They are similarly praised for naming the most basic numbers as "positive integers" rather than "counting numbers" as her other group suggested. And in fact while the students are working individually and in silence on some test questions she describes the errors made by her other group as her reason for doing this test adding to me (but clearly audible to the whole group), "I just wanted to see if this lot could beat them".

Veronica emphasises not only the differences between her groups but also the differences within them. For example, when I am recruiting students for interviews she several times encourages me to take some of them out of lessons explaining that there are a couple of people that I could take out now and it wouldn't matter that they had missed a lesson. She draws on two discourses to explain these differences: 'lack of preparation', which is used to explain how some are doing less well, and 'natural ability', which is used to explain how some are doing better. Although not explicitly invoked to explain failure, the use of a discourse of 'natural ability' to explain success, necessarily carries with it the implication that lack of 'natural ability' contributes to lack of success. Yasser is an example of how these positions are lived:

> Yasser is referred to by his teacher as "naturally able" and is clearly marked out as different. He is sometimes given different work to do and on one occasion is asked to teach the class his method for tackling quadratic inequalities. At first he tries to explain

his solution to her verbally, but this proves difficult so he suggests "shall I write it?", to which Mrs. Sawyer responds "please do". When Yasser writes up his solution there are many looks from students that combine amusement and bemusement. Imran declares "that is so complicated, I've never seen that in my life" and next to him AJ has his hand up, while Saeed says to his teacher "he's clever isn't he?", then adding "he should do further maths". She agrees with him, "he should but he's busy doing other things". Saeed asks her "why don't you encourage him to do further maths?", to which she responds "I've tried, it's his choice". Yasser has now completed writing up not just his original solution but also the graph that Mrs. Sawyer asked him for when his first approach appeared obscure. AJ asks "what is that?", and then repeats the question. Sanjay has a furrowed brow and his hand up. Then Mrs. Sawyer steps in and goes up to the board and explains the graphical method while leaving up Yasser's work because "it is worthy of honour". She further suggests that you could make sense of his diagram by putting numbers in "but you've done it theoretically like a good pure mathematician".

Yasser is constructed here as a mathematician. However, the manner of this construction as an esoteric being, a curiosity, "the spectre of mathematical 'genius'" (Bartholomew, 2000, p. 4), as *Other*, makes it more difficult for the rest of the group to share this position. In concluding, I examine what positions are available to them and the consequences for their identity work.

BECOMING A MATHEMATICIAN?

Learning "implies becoming a different person" (Lave and Wenger, 1991, p. 53) and the 16 students in this class are all engaged in making themselves, in a creative response to the constraints imposed by the situations in which they find themselves. I have argued in this paper that Mrs. Sawyer's classroom, in common with many maths classrooms, is one where the motivations, the reasons given for doing maths, promoted are not the intrinsic ones of pursuing a 'quest for understanding' or gaining enjoyment from it; that is not ones that make available a position as a mathematician. Thus in their identity work, in making sense of what they do and who they are, they look for other motivations and positionings. They become rule followers, good or bad students, 'naturally able', bottom or top of the class, among other things. But above all, they become exam passers, where this goal is officially about progression through the educational system and building a career. However, for students, exams are not just to get jobs; they have a powerful impact on young people's self-image. That this is not recognised within classroom discourses creates conflicts for learners, particularly those who previously saw themselves as enjoying maths, who often come to see the maths that is on offer as incompatible with their identity work:

> Where there is no cultural identity encompassing the activity in which newcomers participate and no field of mature practice for what is being learned, exchange value replaces the use value of increasing participation. The commoditization of learning engenders a fundamental contradiction between the use and the exchange values of the

outcomes of learning, which manifests itself in conflicts between learning to know and learning to display knowledge for evaluation. (Lave and Wenger, 1991, p. 112)

NOTES

1. All names of people and schools and colleges have been anonymised.

2. This is a qualification taken after compulsory schooling, usually between the ages of 16 and 18.

3. Identities are unstable, contradictory, and multiple. Identities are the way we have of talking about ourselves, and are constantly being produced in our actions and our interactions with others; identities are always in process and never attained. However, the word 'identity' suggests coherence and completeness so I have decided instead to use the phrase 'identity work'. Stuart Hall in his work (see Hall, 1996) imbues the phrase 'identification' with similar meanings.

4. Sixth form refers to the educational phase involving 16-19 year-olds; Further Education denotes all post-16 education.

5. The GCSE is usually taken at age 16 and pass grades go from A* down to G, it can be taken at the three levels of foundation, intermediate and higher.

ACKNOWLEDGEMENTS

My thanks go to Leone Burton and Dennis Atkinson for their comments on an earlier draft of this paper, and the Economic and Social Research Council for their funding.

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CERTAINTY BIAS AS AN INDICATOR OF PROBLEMS IN CONCEPTUAL CHANGE: THE CASE OF NUMBER LINE

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This paper presents an analysis of interaction between levels of answers and respective certainty estimations in a number concept test participated by pupils at upper secondary school. In the analysis four different profiles of interaction were found and they are explained by various pupils' characteristics (prior knowledge, achievement level in mathematics, gender and test effort). The results refer to significant differences in the sensitivity to the need for conceptual change and tolerance for ambiguity, which seem to be essential for a conceptual change.

INTRODUCTION

The crucial idea in the theory of conceptual change is the radical reconstruction of prior knowledge which is not adequately taken into account in traditional teaching. In educational contexts, mathematics is considered to form a hierarchical structure in which all new concepts logically follow from prior ones, which allows students to enrich their knowledge step by step. The transitions from the domain of natural numbers to the domains of more advanced ones, are often treated as a continuous growth of knowledge. From a cognitive point of view, however, they are better described as a radical reconstruction, because every enlargement requires fundamental changes in the previous thinking of numbers.

In previous research (Merenluoto & Lehtinen, in press; Merenluoto, 2001) about conceptual change needed in enlargements of the number concept, the majority of upper secondary school pupils had not restructured their prior beliefs on numbers in order to understand the concepts of rational or irrational numbers even on the preliminary level. The results refer to a mistaken transfer by the pupils from natural numbers to the domains of more advanced numbers. This suggests a low sensitivity by most of the pupils to the needed change in thinking the numbers. We explained these results with the theories on conceptual change (see Carey, 1985; Chi, Slotta, & de Leeuw, 1994; Vosniadou, 1994; 1999; Duit, 1995), which consider the relationship between the learners' prior knowledge and information to be learned as one of the most crucial factors in determining the quality of learning. In previous studies it has also been found that conceptual change involves not only change in specific beliefs and presumptions but also development of metaconceptual awareness and consideration of metacognitive effects (Vosniadou, Ioannides, Dimitrakopoulou & Papademetriou, 2001; Limòn, 2001).

Hence in the case of numbers, the major obstacle in understanding the advanced concepts is the quality of students' prior knowledge and beliefs on numbers. But there seem to be three distinctly different components in this obstacle, where the first component is cognitive, but the two others clearly metacognitive (see Flavell, 1987) and motivational (Pintrich, 1999).

From the cognitive viewpoint, the enlargements of the number concept require drastic changes in the very thinking of numbers. For example, the fundamental ideas of natural numbers, as the concept of a successor, are necessary for learning the notion of natural numbers. But in the domains of rational and real numbers the principle of a successor is not defined, but infinite successive division is possible. Thus some of these basic concepts are in serious conflict with the very character of both rational and real numbers (Sowder, 1992). Therefore in order to understand these concepts, a very profound change in thinking of numbers is necessary.

Secondly, small natural numbers and the concept of a successor seem to be among those special concepts which have a high unconditional certainty attached to them. This kind of certainty seems to be a result from innate cognitive mechanism relate to numeral reasoning principles (Gallistel, & Gelman, 1992) but also from the everyday experiences and the linguistic operations (Wittgenstein, 1969). It has a subjective nature with the feeling of self-evidence. Because these concepts seem self-evident, self-justifiable or self-explanatory, they easily lead to overconfidence (Fischbein, 1987). As such they might act as an obstacle for conceptual change or lead to mistakes and misunderstandings on more advanced domains of numbers. This is the case, when it does not even occur to the pupils that they need to rethink their knowledge and logic on numbers even though it would be necessary. In other words, the self-evident nature of this kind of certainty means that the students might have a low sensitivity to the need for a change.

Thirdly, because of the drastic nature of the needed change, the process seems to be related to pupils subjective experiences with mathematics (Merenluoto, 2001) and to their tolerance of ambiguity (Lehtinen, 1984; Stark, Mandl, Gruber & Renkl, 2002). Experiences in mathematics have been studied from many different viewpoints: In the analysis of *feeling of difficulty* in mathematics (Efklides, Akilina & Petropoulou, 1999) the results suggest that these kind of feelings form a system of their own, which is mainly influenced by performance and cognitive ability rather than affective factors. The certainty estimated by the boys were more typical than those of the girls, which seemed to me more context-related. Certainty experiences in mathematics have been studied also from the viewpoint of *self-efficacy, self-awareness, self-regulation and math anxiety* (Pajares, 1996; Schoenfeld, 1987). These studies are based on the assumptions that the personal confidence, which has earlier been experienced in mathematics has causal effects on the performance or certainty feelings later.

Thus we have a hypotheses that in process of a radical change in the thinking of numbers the students are forced to tolerance the ambiguity which comes from newly learned operations and characteristics of numbers while they do not yet fully understand the concepts.

The aim of this paper is to analyse the data from our previous research to find the factors referring to the sensitivity to the needed change in thinking of the numbers and to find the factors related to the tolerance of ambiguity.

METHOD

Subjects and procedure: The data used is from a number concept test, given to 537 students (mean age 17.2 years) from 24 randomly selected Finnish upper secondary schools (see Merenluoto & Lehtinen, in press). The students, who participated in the test after their first calculus course, were asked to estimate their certainty while answering the questions. In this estimation they were asked to use a scale from 1 to 5, where 1 meant that their answer was a wild guess, and 5 that they were absolutely sure, as sure as they know that 1+1=2. In this paper only the critical questions (table 1) pertaining to the density of the number line are discussed.

Scoring and variables: The performance in the tasks was measured with a 5-point scale, from 0 (no answer) to 4 (Table 1). The certainty scores were multiplied with 4/5 in order to set them to the same scale. *Certainty bias:* the task scores were subtracted from the certainty scores in each task. The negative values on this variable respond to uncertainty, the positive ones to overconfidence. *Test effort:* a percentage of answered items was calculated for every student. *Achievement level in mathematics:* students mark in mathematics was calculated as percentages from maximum. *Group position:* the group mean of achievement level in mathematics was subtracted from the respective pupils' mark in mathematics. The negative values refer to a group position below average, positive values to a position above average. *Gender:* there were 335 boys (62.4%) and 202 (37.6%)girls.

RESULTS

Identification of profiles: On the basis of interaction of the task scores (Table 1) with the respective certainty estimations, a cluster analysis was used and four different profiles were found. The general difference in the quality of conceptual change is obvious in the profiles of the answers (Fig. 1), where there is a significant difference between the tasks 2 - 3 compared to the tasks 4 - 5. This difference is due to the quality of conceptual change. Although there were question about the concept of infinite divisibility (and limit) in all the tasks, it is possible to answer at a high level without making any notable change in thinking of numbers in tasks 2-3. Whereas it requires a radical change in order to answer, that the "next" or "closest" is not defined in Q or R. The different level of task scores was obvious in the profiles (Fig. 1) but the appearance of the difference was clearer in the certainty bias profiles (Fig. 2).

TABLE 1. The critical questions pertaining to the number line and the scoring based on the level of answers.

The critical questions	Incorrect[1] (scored 1)	Superficial (scored 2-3)[2]	Correct (scored 4)
1. **Interval.** Define on which interval on the number line is the number, which has the approximate value: 5,01.	5 – 6	5.005-5.01499…	[5.005; 5.015[
2. **Density in Q.** How many rational numbers there are on the number line between the numbers 3/5 and 5/6? Why?	6	"several" "many" "infinite" [1]with or no explanations	There are infinite number of rational/ real numbers, It is always possible to add numbers between any two.
3. **Density in R.** How many real numbers there are on the number line between the numbers 0,99 and 1,00? Why?	None		
4. **Limit in Q.** Which fraction is the "next" after 3/5? Why?	4/5	"none" "all of them" [1]with or no explanations	The "next" or "the closest" number are not defined in Q or R, it's always possible to find numbers that a closer .
5. **Limit in R.** Which real number is "the closest" to 1,00? Why?	0,999…		

[1]The answer is based on the logic of whole numbers.
[2]The answers with no or meaningless explanations was scored as 2, answers with explanations like *"infinite, because it is possible to make the numbers more exact"* etc. were scored as 3.

<u>The factors behind the profiles</u> (table 2) indicate in the profiles the difference in pupils' prior achievement level in mathematics, his/her group position, test effort and certainty bias was significant. Whereas gender difference was significant between profiles 2 and 3 ($\chi 2$ (1) =14.34; p<.001).

TABLE 2. The means of variables behind the profiles and means of certainty bias and test effort.

Profile	N	Mark[1]	Group position[2]	Gender		Test effort[3]	Cert. bias[4]
				% of girls	% of boys		
Profile 1	135	54%	-.49	31	21	54%	-.06
Profile 2	161	61%	-.06	37	26	72%	.27
Profile 3	188	64%	.13	25	41	82%	.84
Profile 4	54	78%	1.08	7	12	94%	.34
All	538	62%	.01	100	100	74%	.40

Significant ($p<.00$) difference between profiles: [1]$F(3, 525)=17.5$, n.s. between profiles 2 and 3; [2]$F(3, 525)=16.2$, n.s. between profiles 2 and 3; [3]$F(3, 534)=112.9$; $p<.001$; [4]$F(3, 534)=53.2$;n.s. between profiles 2 and 4.

For the pupils in profile 1 (n =135; 25.1%) the quality of answers and the estimated certainty level was low. They identified numbers only by their superficial features as whole numbers, fractions and decimal numbers. The profile is characterised by answers based on the logic of whole numbers, where there was a negative bias in certainty except for the tasks 2 and 4 where the word "fraction" was used instead of "rational number". Their low level of answers suggest a low level of sensitivity for a need for change in thinking, which seems to be due to the serious deficiency in their prior knowledge about numbers. Their low level of test effort (Table 2) together with low estimations of certainty and low achievement level in mathematics suggest a low tolerance of ambiguity.

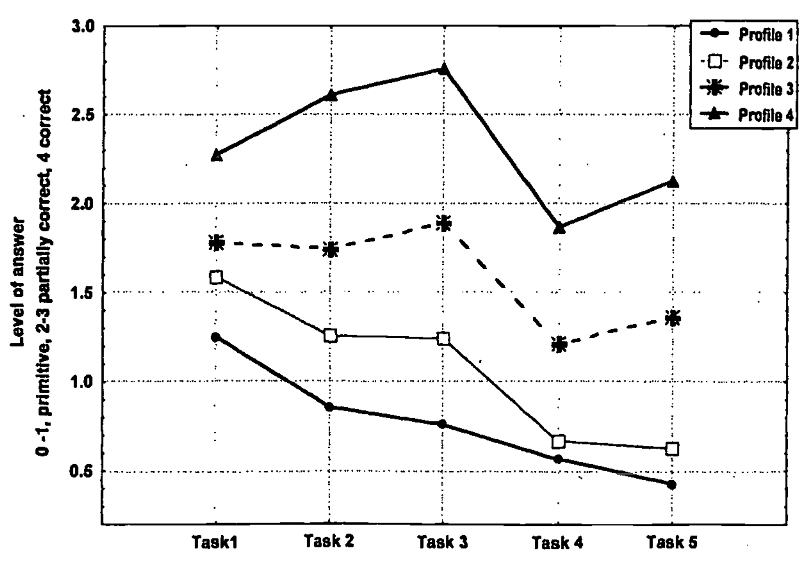

FIGURE 1. Profiles of the task scores, cluster means.

The profile 2 (n = 161; 29.9%), is characterised by a positive certainty bias in tasks which seemed familiar (tasks 1-2 and 4), but negative bias in task 3, where the set of numbers was referred to as "real numbers". They mainly used the logic of whole numbers in their answers and were sensitive to the difference in tasks 2-3 compared to the tasks 4-5 (Fig. 1), where the questions were very different (Table 1) compared to the ordinary questions they had been used to in school. This suggests a context-related tolerance of ambiguity. The level of answers refer to a low sensitivity to a needed conceptual change.

The profile 3 (n = 188; 34.9%), is characterised with systematic over-estimation of certainty. The levels of answers were significantly better than for the students in profile 2, but the difference in the certainty bias was still higher. This refers to a high context-independent tolerance of ambiguity, which is suggested also by their achievement level in mathematics and high test effort (Table 2). Their answers to the tasks, though they were better than in the previous profile, however, refer to a superficial level of conceptual change: to an enrichment kind of learning which in turn refers to a superficial level of sensitivity to the needed change. The difference between the profiles 2 and 3 has also a significant reference to the gender of the pupils.

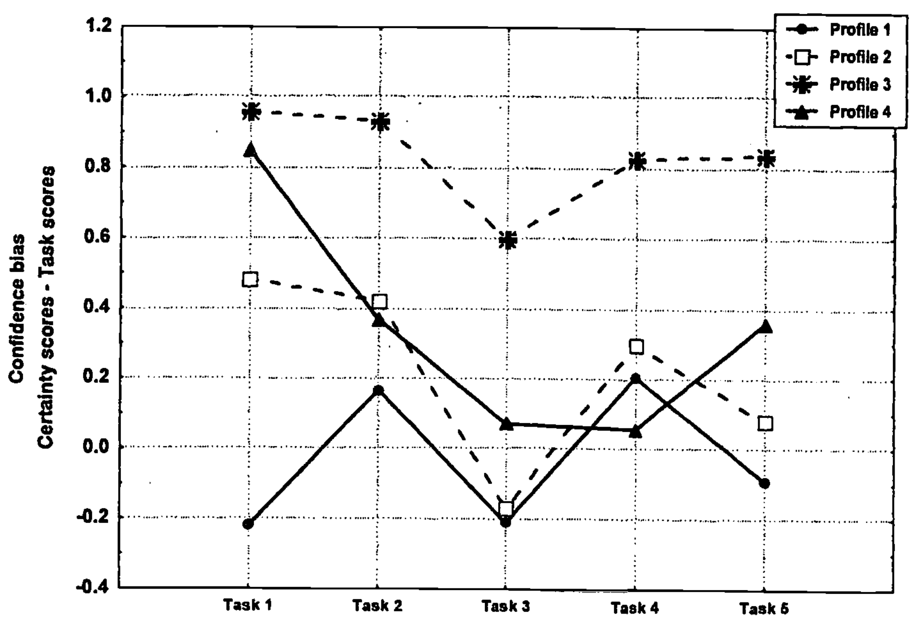

FIGURE 2. Certainty bias profiles in the tasks, cluster means.

The pupils in profile 4 (n = 54), who gave the highest level of answers to the questions, had a positive certainty bias in seemingly familiar tasks (2-3 and 5). But they were sensitive in their certainty estimations in more difficult tasks (4-5), where their explanations clearly referred to a more radical change in their thinking of numbers and the density of number line, compared to the pupils in the other clusters. This kind of change seemed to yield to a very significant difference in the certainty bias profiles between the profiles 4 and 3. The best answers had significantly lower certainty estimations because of the novelty of the ideas and radical nature of the change experienced. These students had a high sensitivity to the needed change, which was obvious in the quality of their explanations. Their achievement level in mathematics, their group position and high test effort refers to a high tolerance of ambiguity.

CONCLUSION

The results give a suggestion that the conceptual change is related to metacognitive and motivational aspects, which still needs further studies. The students in profile four had high sensitivity to the needed change combined with high tolerance of uncertainty, a combination which seems to be optimal for a conceptual change. These pupils had a quite high level of understanding of the density of the number line. Whereas the superficial level of sensitivity combined with the high tolerance of ambiguity seem to be restrictive to a more radical change and deeper understanding of the concepts. Their thinking of the density of the number line was based on operational thinking (making numbers more exact, adding decimals, etc.) without any references to the structural differences between the numbers. These students have an illusion of understanding and do not necessarily see any reason to strengthen their metaconceptual thinking. The majority of students (profiles 1 and 2) had serious problems in their prior formal understanding of numbers, which is an obstacle for their conceptual change also. Their thinking of the density of the number line was more or less based on thinking of whole numbers.

These results refer to the necessity to consider the metacognitive and motivational aspects in future research on conceptual change. They also suggest that process of conceptual change, altogether, is a complex and a gradual affair, which needs to be taken care of also when planning learning environments which support conceptual change.

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THE THEORY OF LIMITS AS AN OBSTACLE TO INFINITESIMAL ANALYSIS

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ABSTRACT

This paper reports on a study aimed at determining how students would respond to an introduction to intuitive infinitesimal ideas once they were granted that these constitute legitimate mathematical knowledge. During six meetings a group of four freshmen in a calculus course for physics students worked on the basic ideas of calculus, including the second fundamental theorem, with the support of CorelDraw zoom. Following a method of data collection used by Sierpisnka [1987], we asked the students to make a demonstration to the whole class. We report on the outcome and discuss the theoretical implication in terms of Bachelard's concept of epistemological obstacle.

INTRODUCTION

The pioneer work of Abraham Robinson [1966], granting that infinitesimals are a legitimate mathematical notion, generated a new branch of mathematics, called *non standard analysis* [Stroyan and Luxemburg, 1976, Cutland, 1988, Nelson, 1977]. For the sake of the readers who need a briefing on these ideas, we reprint the following excerpt, whose range and conciseness we have not been able to improve.

> "The concept of infinitesimal, of an "infinitely small" quantity, has met a variable fate along history. Banished by some, used in heuristic but circumspect ways by others, the infinitesimals until very recently, did not have a right of citizenship in mathematics, moreover after the 19th century analysts introduced in the differential and integral calculus, via the ε-δ, the cannon of rigor that came up to our days. Of course, the physicists and engineers persisted in their intuitive usage of infinitesimals but the mathematician knew that all this could (and should!) be replaced by a rigorous discourse evacuating all notion of an actual infinitely small" [Hodgson, 1994:157].

The endeavor of 19th-20th century mathematicians to get rid of infinitesimals has created: 1) an abyss between mathematics and its uses in other sciences and techniques [Harthong, 1983]; 2) the need to supply information to non-specialist mathematicians, introducing them to elementary and intuitive ideas about infinitesimals [Kossak, 1996]; 3) a problem for mathematics education, timidly tackled so far, in spite of appeals to bring infinitesimals back to school [Harnik,1986, Grattan-Guinness, 1991]. Keisler [1986] made an attempt to produce an infinitesimal

[1] Research supported by CNPq.

calculus textbook launching the idea of infinitely powerful microscopes and lunettes to "see" infinitesimal and infinite numbers. Stroyan [1998] renews such an attempt, in a recent software-supported textbook, using infinitesimals but without naming them so. Bell [1998] develops nilpotent infinitesimals, more suited to interpret some passages of Leibniz, where $(dx)^2$ is not just an infinitesimal of higher order, but it is actually zero. Some teaching experiments are reported by Tall [1980a, b, 1982].

The following example illustrates how strongly limit conceptions are taken as the only legitimate mathematical interpretation and how strongly infinitesimal conceptions in the students' culture refuse to die. Czarnocha et al [2001] investigate students' conceptions about the definite Cauchy-Riemann integral. First they made clear a genetic decomposition, making explicit their structured set of mental constructions about the integral. This genetic decomposition is infinitesimal-free. Then they interviewed students and finally they adjusted their genetic decomposition according to the data collected. The students' reported views are that "the limit of the Riemann sum is seen as the infinite sum of the rectangles of small width" or "of zero width" [p. 297]. However, the authors did not interpret this outcome as an emergence of an infinite-infinitesimal mental construction and insisted on a limit-conception interpretation: "rather than the limit of the sum of the areas of n rectangles the students state it as the sum of the limit of the areas of the rectangles" [ibid.].

Infinitesimal conceptions have been observed to emerge as *epistemological obstacles* in attempts to teach limits [Cornu, 1983, 1991, Sierpinska, 1987]. In these studies, infinitesimal conceptions were not asked for, but they emerged spontaneously from students' speeches. Our basic research question is then the following: what if, instead of waiting for the infinitesimal conceptions to emerge, we stimulated them? What if the students became aware of the abyss between mathematics and its applications produced by the, now unfounded, discrimination of infinitesimals? Will such an awareness stimulate the transposition of the obstacle towards the understanding of limits or will it create new obstacles? In what sense is there an obstacle?

THE SETTING AND METHOD

On order to boost the political dimensions of our research we chose to do the experiment in a regular calculus classroom. We chose a freshmen 2001-course for physics students[2] for whom the language of infinitesimals would be largely used in later courses given by the Physics Department. Since infinitesimals are not part of the calculus courses syllabuses, we expected that this choice would assure the legitimacy of the taught object so as to minimize the criticism from the colleagues from the Mathematics Department. We selected a group of four students willing to participate in the experiment and followed the method used by Sierpinska [1987]: during six videotaped encounters of two hours during class time but in a separate room, we

[2] We received stimulus and support from the responsible teacher, Prof. Dr. Miriam Godoy Penteado, Graduate Program on Mathematics Education, UNESP, Rio Claro, SP, Brazil.

introduced the students to the intuitive basic ideas and a brief history of infinitesimals. In the last meeting, we asked them to prepare a demonstration for the whole class. We did not interfere in their preparation. In our country, students do not generally come in contact with limits in high school, and by the time of our first meeting, the teacher had not introduced them. Since these physics students do not take analysis courses, we made no attempts at formalization, neither defining hyper-real numbers via classes of sequences of real numbers [Lindstrom, 1988], nor providing axioms to decide the truth of propositions [Tall, 1982, Keisler, 1986]. The four participants received bonus in the course and the written mid-term had a question that could be answered optionally by limits or infinitesimals.

Our presentations used a CorelDraw zoom in the following instances: to visualize the merging together of a curve and its tangent line at the tangency point P, to visualize that curve and tangent appear as parallel straight lines at $P+dP$, to visualize the merging together of the segments representing an infinitesimal arch, dx, and sin (dx), to visualize the difference between cos (dx) and 1, and to visualize the trapezium with parallel sides $f(x)$ and $f(x+dx)$ in integration. In all these cases we asked the students to foresee what would come out, before we showed them the zoom on the computer.

We introduced the sing $\approx$ to indicate "*infinitely close to*". The infinitesimal increment of the dependent variable was defined according to Leibniz and Robinson as $df = f(x+dx) - f(x)$ and called as *almost differential*. We maintained $f'(x)\, dx$ to the standard differential. The derivative was defined as what one gets from the infinitesimal quotient df / dx by neglecting the infinitesimal excess and retaining only the real part; it was interpreted as the constant of almost proportionality between df and dx, so that $f'(x)$ was equivalently defined by $df \approx f'(x)\, dx$. Several calculations were carried out. Here is an example:

$y = x^3 = f(x)$	$dy = dx(3x^2 + 3xdx + dx^2)$
$dy = (x+dx)^3 - x^3$	$\dfrac{dy}{dx} = 3x^2 + 3xdx + dx^2$
$dy = x^3 + 3x^2dx + 3xdx^2 + dx^3 - x^3$	
$dy = 3x^2dx + 3xdx^2 + dx^3$	$f'(x) = y' = re\left[\dfrac{dy}{dx}\right] =$
	$re[3x^2 + 3xdx + dx^2] = 3x^2$

The chain rule was proved as an immediate consequence of the definition (box 1). The definite integral was introduced as the *expression* of the area under a curve *in terms of* an infinite sum of infinitesimal parcels. Once the students believed that the sum of the infinitesimal variations dF due to an infinitesimal partition of the domain interval $[a, b]$ equals the variation $F(b)-F(a)$ in the whole interval, the second fundamental theorem of calculus was proved in one line (box 2) and expressed in the statement: *the area is the variation of a primitive*.

$h(x) = f(g(x)) = f(u)$ $u = g(x)$ $dh \approx f'(u)du \approx f'(g(x))g'(x)\,dx$ *QED.* *Box* 1	$dF \approx F'(x)dx = f(x)dx$ $F(b) - F(a) = \int_a^b dF \approx \int_a^b f(x)\,dx$ *QED.* *Box* 2

DATA OUTCOME AND COMMENTS

In a pilot study, we had asked the class of the year 2000 for a written report on the naught point nine classical question. We got 16 equalities, 16 "less than" and 5 "less than or equal to" answers, but only one evoked a limit conception, though in imprecise terms: "this number may be written as limit, then we should have $\lim_{x \to 1} x = 1$. So, among the properties of limits, this number may be considered as being 1, because it gets closer and closer to 1". The remaining justifications, either of = or <, did not differ qualitatively. They indicate that the notation has a strong appeal to concept images [Tall & Vinner, 1981] through categories such as: 1) **process**: "closer and closer to", "the number 0.999... tends to infinity and it becomes so close to 1 that..."; 2) **negligible difference**: "...becomes so close to 1 that one may say that this number has the same value as 1", "because experimentally, this number is rounded off"; 3) **the unthinkable**: "they are so, so close that... they may be considered as equal"; 4) **physical sense**: (from the 2001 study:) "depending on where you are going to use it, you may consider them as equal, but in fact they are not", "(used) in a rounding off, in a measure, you put 1 to facilitate understanding"; 5) **infinitesimal**: "because there is a gap of $1x10^{(-infinity)}$"; "it will always be less than 1 because in spite of tending to 1, a number will always fail to spot it as equal to 1"; (from the 2001 study:) "be as close as they can be and the difference as small as it can be, one is always greater". Every calculus teacher should, be aware that such are the spontaneous conceptions of their students if s/he is committed to teaching limits.

In the first 2001 meeting we asked whether the students had ever heard about infinitesimals and whether they recalled any word, phrase or figure related to them. We got answers such as: "I have heard of infinitesimals related to very small points. They are negligible points. Infinitely small points. They are related to infinite decimals. Small points that can be neglected in some calculations. I heard infinitesimal related to fractal, in the sense of something that tends to the uni-dimensional. It diminishes so much that it loses dimension. You may neglect them depending on your point of view, depending from where you look at them. (Pilot-2000 study:) It is something very small indeed, so small that you cannot imagine it, it tends to zero so small it is". There is nothing new here, except the notion of fractal that looks like a nice teaching device.

We then continuation we introduced the hyper-real line through the usual diagram and gave a provisional definition: An infinitesimal is a number smaller than any real positive number. We asked for examples. Here are discussion excerpts: "If you imagine a very small number, 0.000...1 it is always possible to put one more zero before the 1 and it will become smaller. It can be very close to zero but is not zero, so you can say that there are numbers that are even smaller than it". Nobody thought of zero nor of negative numbers. We asked whether zero would be an infinitesimal. Three students agreed, but one argued: "If an infinitesimal is positive, than it is greater than zero, not equal to zero. So I think that zero does not belong in this classification". The novelty here is the emergence of a concept image of an infinitesimal as a small amount of matter, like a particle, which would not make sense to think of as negative. Along these lines, Hanna & Jahnke ask: "What is the possible role of arguments from physics within mathematical proof, and how should this role be reflected in the classroom?" Hanna & Jahnke [1999:74].

After some calculations of derivatives, we asked the students how they justified dropping off infinitesimals. We got answers based on concept images such as: "The infinitesimal part is so small that it can be neglected. The infinitesimal in this case are not going to make much of a difference". But we also got answers indicating retention of the formal concept definition [Tall & Vinner, 1981]: "The infinitesimals do no exist in the reals. When you calculate the derivative you take only the real part". This kind of answer (the derivative is the real part *by definition*) was only received by the mathematics of 20th century and could not have been given by Newton and Leibniz to Berkeley, were they still alive by the time The Analyst was published. In the face of these two last answers, can we say that "we have the impression that students do not ascribe cognitive values to definitions, they seem to perceive them only as labels which are not relevant to mathematical work" [Furinghetti & Paola, 2000:345] ? Or is this conclusion only valid when definitions and images differ considerably, which *is the case* of limits, but *not* of infinitesimals?

Here are some excerpts from the students' discussion at the end of one of the meetings: "In physics we have to imagine the situation, we have to imagine what happens in a very small space, with tiny dimensions. The zoom helps. In the physics class the teacher also spoke about the zoom, taking an infinitely close view, but it was not clear for all the students". They seemed to feel privileged for being able to understand the teacher. "In the infinitesimals you see the reality of the thing; you keep the sense of the approximation. In the limits you approach the thing; you do not see what is behind. You have to believe. (In the infinitesimals) we understand the part we are neglecting, we see it. In the limit, you don't. In our calculus course, or even in any area, it would be more coherent to work with these ideas. It is interesting to see how this has developed, to show both sides. By studying the history you understand". We think that this remark requires no additional comments. We intend to check these students next year to see if their esteem for infinitesimals persists.

From the classroom demonstration, we collect the following excerpts, many of them enthusiastic. "This part of the subject matter is very interesting, a totally different universe. Sometimes you embark on a trip that leads nowhere. If you take a chronometer and record the time it takes for you to get home, you may notice that it will never leave zero". (Students from the class: Zeno? Zeno! Some laughed.) "You start perceiving that numbers are infinite, and this makes a big difference. You start thinking about this universe of which the infinitesimals are a part. There is even a special set, the hyper-reals, in which the infinitesimals are included". The students seemed to praise the existence of a "set" to lodge the infinitesimals; they felt that this made infinitesimals more legitimate. A student from the class asked: "Can you give an example of where you use such infinitesimals?" The answer: "You must be working with the hyper-reals. Consider extremely small particles in such a way that any variation could alter your result. If you are working with rays or with subatomic particles, maybe these infinitesimals will have to be included. If you are dealing with dilation, this infinitesimal could make a lot of difference. If you calculate the dilation coefficient of an iron bar, say the rails in a railroad, in the coupling of one bar to another, in the end it will make much of a difference in the dilation".

Here we can note the mingling of mathematical, continuous, infinitesimal conceptions with the physical, discrete, subatomic reality. The total dilation was conceived as the integration of infinitesimal dilations as if matter were continuous. It is reasonable to conclude that the whole research process led the students to spontaneously enlarge their concept images so as to incorporate elements from the microscopic physical world among infinitesimals. This inclusion has already been praised by a researcher in quantum mechanics:

> "We may regard the physicist who studies the macroscopic behavior of a phenomenon whose microscopic behavior is too complex for him, as a limited observer who cannot apprehend but the shadow of things. The microscopic behavior will be described by non-standard functions" [Harthong, 1983:1200].

On one hand, we may see that the inclusion may be praised as going in the direction recommended by Hanna & Jahnke [1999]. On the other hand, it may be criticized as leading to Bachelard's obstacle of quantitative knowledge: "magnitude as a property of [physical] extension" [Bachelard, 1980:211].

CONCLUDING REMARK: THE THEORETICAL FRAMEWORK

Bachelard's concept of epistemological obstacle has been evoked to describe the mode in which infinitesimal notions emerge in the learning of limits. The directive upon which we based our research is clearly indicated by the following "must": "The construction of pedagogical strategies for teaching students must then take such obstacles into account. It is not a question of avoiding them but, on the contrary, to lead the student to meet them and to overcome them" [Cornu, 1991:162]. There have been many misunderstandings about the concept of obstacle. Errors indicate obstacles but obstacles do not reduce to errors nor to mere difficulties. We rely on a thorough reading of Bachelard who states:

> "Errors are not only the effect of ignorance, of uncertainty, of chance, as espoused by empirist or behaviorist learning theories, but the effect of a previous piece of

knowledge which was interesting and successful, but which now is revealed as false or simply unadapted. Errors of this type are called obstacles" [Brousseau, 1997:82].

The difficulty resides in the "dialectical character of the process of overcoming an obstacle" [Brousseau, 1997:88] which stems from Bachelard: "In fact we know against a previous knowledge, destroying ill-formed knowledge, transcending what in the very spirit makes obstacle to spiritualization" [3]. [Bachelard, 1980:14]. The emphasis here should be on the verb "to transcend", in French "surmonter", which is generally used in philosophy to translate the German verb "aufheben", meaning to conserve-in-surpassing. Reading the excerpt from this dialectical perspective, the "destruction" acquires the meaning of transformation, of *becoming*, and the "ill-formed" refers to the surpassed basis of future knowledge.

It is falsifying the concept to say that the overcoming of an obstacle is a matter of making the original, insufficient, malformed knowledge disappear and replace it with new a concept which operates satisfactorily in the new domain. This distorted view of Bachelard's concept supports the commitment that pervaded the twentieth century teaching of advanced mathematics: erase infinitesimals and write limits. This uni-dimensional view is unable to think of limits and infinitesimals except in terms of "either…or…", whereas we say that both should be present. Robinson's non standard analysis is not a return to Leibniz; it is a theory posterior and subjected to Hilbert-Weierstrassian rigor.

If it is clear how infinitesimals play the role of obstacles in learning limits; it is not so clear how the theory of limits creates an obstacle, in the precise sense of Bachelard, to the learning of the now rigorous infinitesimal analysis and its use in Mathematics Education. We hope that this paper takes one step toward bringing this question to the fore.

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RELATION BETWEEN PROOF AND CONCEPTION: THE CASE OF PROOF FOR THE SUM OF TWO EVEN NUMBERS

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The aim of this research is to advance understanding of how mathematical knowledge functions in the proof construction, especially in its written outcome with a problem in algebra. The theoretical analysis allows us to explain some reasons of students' proofs and their tendency obtained by a case study. The first result is that the difficulty of constructions of mathematical proof is due not only to the algebraic competence or proof conception, but also to the mathematical knowledge.

INTRODUCTION

In this paper we report some finding from an analysis of students' proofs in algebra taking into account mathematical knowledge at stake. The aim of this research is to advance understanding of how knowledge functions in the proof construction, especially in its written outcome. This is to know which relation may exist between the mathematical knowledge involved and the nature of proof.

Our research question comes from the recent study about proof conceptions by Healy & Hoyles (1998, 2000). These authors found, from their grand survey in the Great Britain with a statistic method, students' proof conception in algebra that proofs constructed by students follow an empirical approach or a narrative style rather than a formal one although most students are aware of their limitations (2000, p.396). The authors discussed that this problem was due to the lack of algebraic understanding for the proof (2000, pp.425-426). We share the idea that this is one of the possible reasons, but we consider that this is not the unique one. We develop this point in this paper by focusing on the mathematical knowledge. We intend to evidence the role played by students' conception of the mathematical notion (not proof conception) in their proof construction and learning as well.

We take for our analysis the proof problem of sum of two even numbers already studied by Healy & Hoyles. We partially replicate this study (on a smaller scale) and show how one can interpret the data gathered.

THEORETICAL FRAMEWORK

Proof

To characterise or classify students' approaches to proving, some research results have already been presented: proof types of Balacheff (1987), proof schemes of Harel & Sowder (1998), characterisation by the structure of reasoning (Duval, 1991), "proofs that prove" and "proofs that explain" of Hanna (1989), etc. In this paper, we adopt as a theoretical framework, the proof types of Balacheff (1987) from "pragmatic proof" to "intellectual proof", more precisely four types of proof –

"naive empiricism", "crucial experiment", "generic example", and "thought experiment" – and "mathematical proof" (also called "formal proof"). We expect this choice to be relevant in order to characterise the relationship between the nature of proof and that of knowledge. In fact, to characterise these types of proof Balacheff takes into account not only the nature of the underlying rationality and the language level, but also the nature and the status of knowledge (1987, p.163), which is the most important point of view in our analysis.

Conception

To identify and differentiate students' knowledge of even number involved in their proofs, we adopt the notion of conception. Conception is a didactical tool for modelling the students' knowledge in problem solving situations. It reveals the plurality of the possible points of view on a same mathematical object, it differentiates the representations and related methods, and emphasises their more or less good adaptation at the resolution of such and such class of problems (Artigue, 1990, p.265). In our analysis, we pick up aspects as the operators which are used to solve a problem, and the language which is also important from the perspective of proof characterisation[1].

The conceptions of even numbers we can identify, considering operators appearing in schoolbooks or in students' proofs are following. From C_2, we can get out three operators by the concept of divisibility or multiplicity. C_3 is an algebraic one.

- C_1: Even numbers end by 0, 2, 4, 6, or 8.
- C_2: Even numbers can be divided by 2, or the result of the multiplication of a whole number by 2[2].
  - C_{21} : Even numbers can be divided in two identical parts.
  - C_{22} : Even numbers can be decomposed 2 by 2.
  - C_{23} : Even numbers have 2 as a factor.
- C_3 : Even number can be expressed by $a = 2p$ (p: whole number).

ANALYSIS OF THE POSSIBLE PROOFS FOR EACH CONCEPTION

By this analysis we intend to construct a framework for discussing the data obtained during the observation. We try to construct several different proofs from "pragmatic" to "intellectual" involving the conceptions we have presented. This analysis allows us to locate or to characterise students' proofs, and also to identify the relation between the conceptions of even number in the proof. Concerning the representations used to express proofs, we pay attention to the operational one[3], that is, the representations on which computations or transformations can be carried out. In fact, the natural language is often used in the formulations of proof, but we should often take care to distinguish it from the operational one.

List of proofs

C_1: Even numbers end by 0, 2, 4, 6, or 8.
Naive empiricism (numerical): $2 + 4 = 6$, $4 + 8 = 12$, $6 + 8 = 14$, $12 + 24 = 36$.

Crucial experiment (numerical): I take arbitrary two even numbers. 188+ 76 = 264.
Generic example (numerical): With two even numbers: 18 and 24. 18 + 24 = (10 + 8) + (20 + 4) = (10 + 20) + (8 + 4) = (10 + 20 + 10) + 2. We can do this for any two even numbers.
Thought experiment (natural language): Even numbers end by 0, 2, 4, 6, or 8. The last digit of number of sum of two numbers is calculated by the sum of their last digit of numbers. So the sum of two even numbers ends also by 0, 2, 4, 6, or 8.
Mathematical proof (numerical): The proof is same as that of thought experiment but with the table 1.

	0	2	4	6	8
0	0	2	4	6	8
2		4	6	8	0
4			8	0	2
6				2	4
8					6

Table 1: exhaustion

C_{21}: Even numbers can be divided in two identical parts
Naive empiricism (numerical): 4 + 8 = 12 = 6 + 6, 6 + 8 = 14 = 7 + 7, etc.
Crucial experiment (numerical): I take arbitrary two even numbers. 188 + 76 = 264 = 132 + 132.
Generic example (numerical): With two even numbers: 12 and 18. 12 + 18 = (6 + 6) + (9 + 9) = (6 + 9) + (6 + 9) = 15 + 15. We can do this for any two even numbers.
Generic example (graphical): ::: + ::::: = :::::::: (separate horizontally). We can do this for any two even numbers.
Thought experiment (natural language): Even numbers can be divided in two identical parts. So, if you add each divided part of two even numbers, the sum can be also expressed by two identical parts.
Mathematical proof (algebraic): $\forall$a, b: even number, $\exists$ p, q $\in$ **Z** s.t. a = p + p, b = q + q. a + b = (p + p) + (q + q) = (p + q) + (p + q).

C_{22}: Even numbers can be divided 2 by 2
Naive empiricism (numerical): 4 + 8 = 12 = 2 + 2 + 2 + 2 + 2 + 2, 6 + 8 = 14 = 2 + 2 + 2 + 2 + 2 + 2 + 2, etc.
Crucial experiment (numerical): I take arbitrary two even numbers. 188 + 76 = 264 = 2 + 2 + ...
Generic example (numerical): With two even numbers: 4 and 8. 4 + 8 = (2 + 2) + (2 + 2 + 2 + 2) = 2 + 2 + 2 + 2 + 2 + 2. We can do this for any two even numbers.
Generic example (graphical): ::: + ::::: = :::::::: (separate vertically). We can do this for any two even numbers.
Thought experiment (natural language): Even numbers can be divided 2 by 2. So, if you add two numbers divided 2 by 2, the sum can be also expressed 2 by 2.
Mathematical proof (algebraic): $\forall$a, b: even number, $\exists$ p, q $\in$ **Z** s.t. a = 2 + 2 + … + 2 (p terms), b = 2 + 2 + … + 2 (q terms). a + b = (2 + 2 + … + 2) + (2 + 2 + … + 2) = 2 + 2 + … + 2 (p + q terms).

C_{23}: Even numbers have 2 as a factor
Naive empiricism (numerical): 4 + 8 = 12 = 2 × 6, 6 + 8 = 14 = 2 × 7, etc.
Crucial experiment (numerical): I take arbitrary two even numbers. 188 + 76 = 264 = 2 × 132.

Generic example (numerical): With two even numbers: 12 and 32. $12 + 32 = 2 \times 6 + 2 \times 16 = 2 \times (6 + 16) = 2 \times 22$. We can do this for any two even numbers.
Thought experiment (natural language): Even numbers have 2 as a factor. So, if you add two numbers having 2 as a factor, the sum have also 2 as a factor.
Mathematical proof (algebraic): $\forall$ a, b: even number, $\exists$ p, q $\in \mathbf{Z}$ s.t. $a = 2 \times p$, $b = 2 \times q$. $a + b = 2 \times p + 2 \times q = 2 \times (p + q)$.

C_3 : Even number can be expressed by $a = 2p$ (p : whole number).
Mathematical proof (algebraic): $\forall$ a, b: even number, $\exists$ p, q $\in \mathbf{Z}$ s.t. $a = 2p$, $b = 2q$. $a + b = 2p + 2q = 2 (p + q)$.

What we can expect from this analysis

Distance between the conceptions: C_1 tests all the cases, C_2 and C_3 show structure

We can point out some differences between C_1, C_2, and C_3 by analysing each type of proof from a mathematical point of view. C_1 pays attention to numbers or the representation expressed by the decimal system[4] and observes all the sums of last digit of numbers between 0, 2, 4, 6, and 8 (exhaustive). In this case, the structure of even numbers or their sum is not evidenced. On the contrary, C_2 and C_3 do not focus on specific numbers, but the structure of even numbers, of their transformations, and of their sums, and the proofs are constructed by showing them.

Whereas we gave a proof based on a generic example with C_1, this example is not for sums of numbers chosen among {0, 2, 4, 6, 8}. In fact, C_1-proofs except naive empiricism and crucial experiment can be separated in two phases. The generic example showing the structures intends to establish the first phase "the last digit of sum of two numbers can be calculated by the sum of each last digit of numbers". The second phase consists of showing that the sums of two digits between {0, 2, 4, 6, 8} end by one of the same digits. Thus C_1 cannot give generic example for a whole proof, although C_2 can. This division in two phases also shows a gap between pragmatic proofs having only one phase and more intellectual proofs. In fact, not limiting at this example of two even numbers, it's not easy for the students to shift from a mere judgement to eliciting the structure of a mathematical notion.

Besides, we can also find that C_2 and C_3 have a very close nature. In fact, C_{21} divide into two identical parts ($p + p$), C_{22} divides 2 by 2 ($2 + 2 + \ldots + 2$ (p terms)), and C_{23} has 2 as a factor ($2p$). And these "2" appearing in C_2 can be seen as "2" of "$2p$" in C_3. Thus we consider that there is not a big obstacle in passing from C_2 to C_3. On the contrary, C_1 where only the last digits are important is very different from C_2 and C_3. From these points of view, we can find that the distance between C_2 and C_3 is smaller than between C_1 and C_2.

Three elements relying on each other: conception, representation, and proof type

Among the different proof types we have presented, the construction of some proofs is a little probable. For example, C_2-naive empiricism and C_2-crucial experiment. Almost everybody would use C_1 when they are asked to judge whether the given number is even or not (ex. Is the number 243992 even?). It's enough to verify the last digit of number and very easy and simple. On the other hand, judging a large number by C_2 requires a great and hard work. As the naive empiricism and the crucial experiment require this simple judgement, if the students have a proof conception of these types, it's obvious that they take C_1. Thus, we could state that more relevant or effective conception exists according to the choice of a proof type.

On the contrary, we can also state that the choice of a proof type depends on the "available conception"[5]. As C_2 already shows the structure of even numbers, this conception facilitates the construction of a generic example. And it's very obvious that nothing but a mathematical proof can be constructed with C_3.

As concerns the language, the representation used in a C_1-proof is likely to be numerical since C_1 focus is always on the last digit expressed by the decimal system. Thus, even if students have algebraic representation as a modelling tool, it will be difficult for them to produce an algebraic mathematical proof if they have not available a relevant conception (in our case, C_2 or C_3). In other words, if the available conception for students is only C_1, it is unlikely that they will construct an algebraic proof. On the other hand, the possible representations with C_2 could be numerical, graphical, algebraic, and the natural language, because C_2 doesn't always focus on the digits like C_1. And, only the algebraic representation is used for C_3. This key role of language is not new. However what is remarkable is that it's not always the algebraic representation which is necessary to construct a mathematical proof. The most relevant one for a mathematical proof depends on the mobilised conception. Therefore, if the intention of teacher is to make students to construct an algebraic proof, he has to design a situation which "disqualifies" C_1 but favour C_3.

The graphical representation that appears to be a rather easy way to present the structure of a number and a given operation, is only plausible under C_{21} and C_{22}. We suggest that this representation may be used with the intention to make more "visible" the structure (like "proof that explain" of Hanna (1989)). On the contrary, one can remark that the representation impacts the choice of a proof type. Let us take C_2 as an example. If one use only the numerical one, the possible types of proof are naive empiricism, crucial experiment, and generic example. With natural language, only thought experiment, with graphical one, generic example, and with algebraic one, mathematical proof.

THE RESULTS OF A CASE STUDY

Observation

To get data about proofs constructed by students, we have proposed a questionnaire to 37 students of 9th grade (aged almost 14 years) from Grenoble area in France. The students were just asked to make a proof for the sum of two even numbers: "Is the following statement is true or false? Prove your answer. «When you add any two even numbers, your answer is always even»". We didn't present examples of proofs to students like in the study made by Healy & Hoyles (2000), because the aim of our observation is to get some data in France and to characterise their tendency and their possible reasons. In France, the proof learning begins progressively from explanation or justification at the 6th grade and is taught mainly in relation to deductive reasoning at the 8th grade in geometry.

Results

We have classified students' proofs depending on the conceptions and the types of proofs (table 2). The criteria of this classification are following:

	C_1	C_{22}	C_{23}	Other
Naive empiricism	12			
Crucial experiment	8			
Generic example	1	4		
Thought experiment			1	
Mathematical proof	3		2	
Other			2	4

Table 2: Frequencies for conceptions and proofs

Naive empiricism: proofs based on few cases, only some sums are shown.
Crucial experiment: these proofs consist of a statement that describes the use of large numbers like "with two large numbers", "take arbitrary two numbers", or large numbers explicitly used comparing to others, even if it is not stated.
Generic example: these proofs consist of just one case that is specifically analysed and this analysis attempts to show the structure of even numbers and their treatment.
Thought experiment: such proofs attempt to show the structure of even numbers and their transformation or computation with natural language.

These proofs can not always reach a complete achievement. Some proofs in which we were not able to identify a related conception, are classified to the column "other". And the line "other" of proof type is for the case where the statement "the sum is even" is taken as a hypothesis (circular argument), or where students evidence their conceptions but don't show any idea of a proof. We couldn't find C_{21}-proof or C_3-proof. As regards the operational representation, only one student used the natural language (C_{23}-thought experiment). Very few students used the algebraic representation (2 C_{23}-mathematical proofs and 1 C_{23}-other). No student used the graphical one. And all the rest took the numerical one (hence 33/37). Whereas the observation method was not same, we couldn't remark, as Healy & Hoyles (1998, 2000), that students used narrative argument for their own approach.

Much more than half of the students (24/37) produced a C_1-proof, most of them (20/24) based on a naive empiricism or a crucial experiment. And proofs of these types were only based on C_1. We consider two mutual reasons of these frequencies following our theoretical analysis. First, proof conceptions of students remain in the types of naive empiricism or crucial experiment, like Healy & Hoyles stated, so they take C_1 that is useful for these types. The proof types precede the conceptions. Second, the conception C_2 is not available to students, so they take C_1 which lead to these types of proof. The conceptions precede the proof types. In this second reason, it is not same what they "know" about a tool as they can use it as own tools. For example, Maud's proof (figure 1) shows two conception: "the even numbers go two by two" (C_{22}) and "they always end by same digit" (C_1). But the conception used in the given example is only the later. With these two reasons, we cannot explain why they cannot give intellectual proof. We consider that this is due to the different characters between the proof types. As we have mentioned in our analysis of the possible proofs, there is a gap in the proof structure between the first two types and the others: the former has one phase and the latter has two phases. Among these reasons, none relates to the lack of algebraic knowledge, that is, even if students had algebra available, they could not construct an algebraic mathematical proof with C_1.

Only one seems a proof based on a generic example (figure 1). We have classified this proof as a generic example, whereas the treatment of last digits was not shown. In this proof the second phase (exhaustive sum) was implicit. Maud would think the second phase was too evident. On the contrary, all C_1-mathematical proofs were proof by exhaustion and the first phase was always implicit, that is, students would think the first phase was too evident. This point gives evidence of difference what must be proved between students.

> The statement is true, because the even numbers go two by two; they always end by the same digit: 2, or 4, or 6, or, 8, or 0.
> ex: 16 + 18 = 34
> even digit
> even digit

Figure 1: Maud's proof

C_{22}-proofs were always based on a generic example (4/4), and none of them were naive empiricism or crucial experiment. A reason would be that C_{22} is relevant to showing the structure which leads to a generic example and useless for naive empiricism and crucial experiment. However, none of these students could construct a mathematical proof. Three reasons may be offered to understand this. First, the lack of algebraic knowledge as a modelling tool. Second, their proof was always of the type "generic example" and C_{22} is relevant in this case. Third, a lack of method for applying algebra to the expression "$2 + 2 + \ldots + 2$". In fact, students are not used to an expression like "$a = 2 + 2 + \ldots + 2$ (p terms)". So, we could also suggest that C_{22}-mathematical proof is little probable as a student's proof.

Concerning C_{23}-proofs, while the total number of these proofs is not great (5/37), any of them were not based on naive empiricism, nor crucial experiment, nor generic example. For the former two types of proof, the reason would be same as C_{22}: C_{23} is useless for verifying whether a number is even or not. But why no generic example? We consider that this fact distinguishes the nature of C_{23} from that of C_{22}. The reason would be that algebra as a modelling tool is easy to apply with C_{23}, or the generic example of C_{23} is not enough to persuade for students.

CONCLUSION

In this paper, we have presented a study of the relation between proof and conception, in the case of the problem taken from Healy and Hoyles. In the mathematical analysis of possible proofs, we have shown that the proof character (exhaustion, demonstration evidencing a structure, or another types of proof) would be changed by several conceptions which are possible with a mathematical notion.

In the case study, we got the data of students' proofs and their tendency, and explained them with the results of the theoretical analysis. This would be one response to the results obtained by the study of Healy & Hoyles (1998, 2000). For example, it would be not only the problem of algebra (representation) or of proof conception (proof type) that students could not construct mathematical proof, but also that of conception on a mathematical notion (in our case even number).

For the perspective of mathematical proof learning, our study also showed the necessity to design a situation where students could mobilise more pertinent conception or which enable to shift from a conception to more pertinent one. The mathematical knowledge has a crucial role there.

NOTES

1. This point is similar to the two elements of the formal definition proposed by Balacheff (2000) who models a conception with a quadruplet (P, R, L, $\sum$) in which: P is a set of problems; R is a set of operators; L is a representation system; $\sum$ is a control structure.
2. We didn't separate "divided by 2" and "multiplied by 2", because they have the same sub-conceptions.
3. In the French literature, Duval (1995) calls this representation "semiotic register".
4. C_1 is not applicable if numbers are given in the trinary system or odd number system.
5. "Available conception" means in this paper the conception or the operator that is not only known but also which can be used by students.

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INVESTIGATING THE INTERPRETATION OF ECONOMICS GRAPHS[1]

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Several studies in cognitive psychology have investigated the development of mathematical knowledge in cultural practices. For example, some authors have investigated situations where people deal with economic issues, which refer to quantitative relations and mathematical concepts. Print media use graphs to give information about economic topics, e.g., variations of the rates in inflation, and wages. This study investigates the ways in which economists and business people, who subscribe to magazines and newspapers, interpret such graphs dealing with economics. The aim of the study was to identify how the background of the interviewees and the specific aspects of the graphs influenced the interpretative situation. Analysis of the data raised issues related to the teaching of graphing.

INTRODUCTION

Several research studies in cognitive psychology have investigated the development of mathematical knowledge in daily professional activities (Schliemann, 1995). Generally speaking, three aspects could be noted in the majority of those studies. First of all, the participants have limited school experience. Secondly, those studies approach specific mathematical concepts and procedures (e.g., arithmetical operations, area calculation, etc). Finally, these studies indicate that subjects competently solve mathematical problems in their work practices.

Similarly, this paper reports a study that investigated cognitive processes pertaining to the utilization of mathematical knowledge among professionals in an everyday activity. However, in contrast to the studies described above this paper approaches a topic that has been little explored within the Psychology of Mathematical Education: the interpretation of graphs in print media. In addition, the present study investigated the activity of interpretation among professionals with high levels of schooling.

Carraher, Schliemann & Nemirovsky (1995) argue that individuals with limited school experience do not have a general difficulty in working with symbolic representations because everyone uses symbols whenever they think and communicate. On the one hand, these authors remark that in interpretative situations a graph could be defined as an instrument to express the ideas of the person who interprets it. On the other hand, they acknowledge that a person with a limited schooling level would not work out all the issues involved in a graph.

[1] CNPq – Brazilian governmental institution, funded this study. I thank Luciano Meira (Federal University of Pernambuco) for his supervision in this study, and Janet Ainley (University of Warwick) for comments on drafts and for contributing to the ongoing discussion of graphing.

In the light of such discussions, this paper investigates how individuals with specialized competence could interpret media graphs. Would their interpretations be refined? Could professionals with a background in the utilization of graphs realize a more complete interpretative approach?

THE INTERPRETATION OF PRINT MEDIA'S GRAPHS

Even in the 'strict sense', the word 'interpretation' means more than the perception of information. The interpretation of graphs is not an isolated activity. It should be considered as part of a Data Handling process that is a human activity in which people can explore information and construct knowledge.

Therefore, interpretation is more complex than observation. The person needs to make relationship between data and needs to make inferences about the information (Shaughnessy et al., 1996).

Referring to the relations between data and people, Gal (forthcoming) differentiates two kinds of context. When people read newspapers or watch TV, they should be called 'data consumers'. On the other hand, when they interpret and report their own findings and conclusions, they could be denominated 'data producers'. In spite of this differentiation, a same person might be consumer and/or producer, depending on his/her context.

The context of use of graphs is an important aspect in understanding interpretative processes. For example, readers of print media and students in the classroom have qualitatively different contexts for the interpretation of graphs. Consequently, different meanings could be attributed to interpretation when it involves specific participants and particular settings.

In contemporary society, graphs are frequently used as sources of information in the news media. In this context, the graphs illustrate journalistic arguments for different topics, and could emphasize and/or disguise data (Meira, 1997; Ainley, 2000).

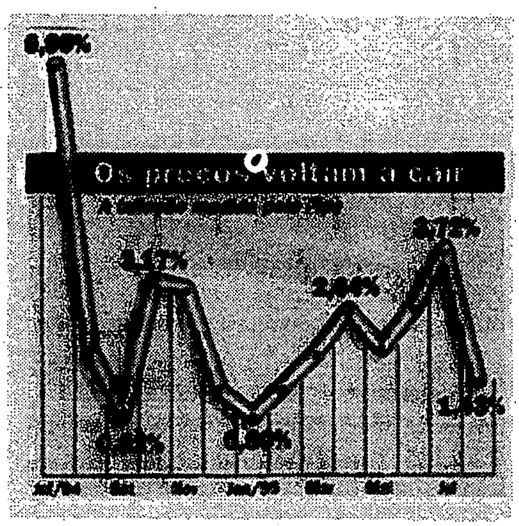

Figure 1: graph reprinted from a Brazilian weekly magazine.

Figure 1 provides an example of a media graph that was published in a report about the consequences of Real Plan (Brazilian government's current economic policy). The title is: *"Prices have fallen again; Inflation measured by FIPE"* (Economics Research Foundation in Brazil). The graph displays information related to a period of 14 months, but it presents just 7 indices that correspond to peaks and troughs. The choices of figures presented and the omission of some months labelled were deliberately used to emphasise the main theme of report.

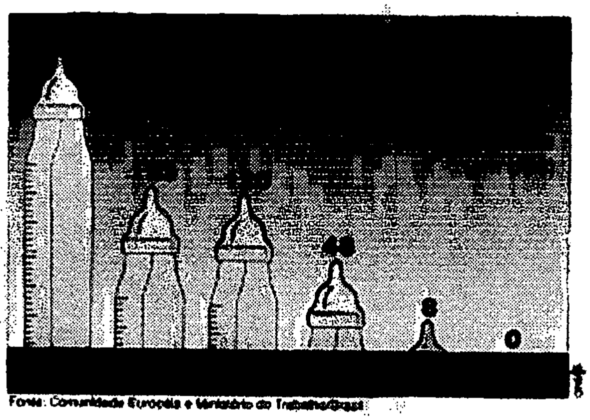

Figure 2: graph reprinted from a Brazilian weekly magazine: "the value of children: monthly benefits for two children in some countries (US$)".

Another example of a media graph is shown in Figure 2. On the top of the 'feeding bottles' there are figures for each country, and on the bottom there are the names of countries. Belgium's bar is a complete feeding bottle. The graph presents partial pictures, which symbolize France, Germany, and Italy. Brazil's value is just shown through the feeding bottle's tip. Finally, there is not any piece of a picture for United States' zero value.

The feeding bottles used in graph were 'adornments' used to illustrate the topic (Meira, 1997). On the one hand, those visual resources could be a distraction from the interpretation process. On the other hand, the feeding bottles, as extra-mathematical components, might be a symbolic source related to "the value of children".

In the following sections we present the design, results, analyses and conclusions of a study which investigated the processes of interpretation of such media graphs among specialised professionals. This study aimed specifically to explore how the interpretations of media graphs could be influenced by specialized background of readers.

ECONOMISTS AND BUSINESS PEOPLE INTERPRETING GRAPHS

The final research design of this study was based on the analyses of pilot interviews with several professionals (e.g. secondary school teachers, engineers, researchers). The analyses of pilot interviews mainly provide reflections about the relationship between the background of interviewees and their familiarity with the theme of graph. We decided to focus on economic topics because these were frequently shown in the media. The economic theme was also very important during the period of data collect, because recently Brazil had moved from high inflation levels after a governmental economic plan.

Among several specialized professions related to economics issues we chose research economists and businessmen/women. The economists have a 'data producer' routine, which include production, analysis and reporting of economics data. Those economists frequently construct, use and interpret graphs. The businessmen/women also produce and analyse data related to their work. However, rarely they construct and use graphs in their work routines.

Two groups of eight interviewees were formed. All participants were daily readers of print media, and subscribed to at least one newspaper and one magazine. Therefore, all of them were 'data consumers'. In addition to specific professional routines, these groups were differentiated by academic background and age (See Table 1).

Groups / Averages	Age	Years after of first degree	Postgraduate courses
Businessmen/women	37	13 (different courses)	12,5%
Economists	47	23 (economics)	100%

Table 1: Background of two research groups.

Each participant interpreted seven graphs, which were published by two national magazines and one local newspaper. This paper only refers to data from two graphs (see Figures 1 and 2).

In interview, two types of questions were asked. The questions called 'general' were asked at the beginning and at the end. After of first question, the researcher asked more specific questions, which attempted to investigate interviewees' understanding related to particular aspects of each graph presented.

ANALYSIS OF INTERVIEWS

The interpretations related to the first general question were called 'initial approaches', and the answers resulting from other questions were called 'specific approaches'.

Initial approaches

The analyses of 'initial approaches' demonstrated that there was a tendency towards discursive and extensive comments. We could distinguish three categories of 'initial approaches'. The first one was characterised by comments which related to data that came from the graph. The majority of interviewees demonstrated this kind of strategy (59%). The following exchange is an example of this category:

Researcher: What could you notice from the reading of this graph? [See figure 2]

Businesswoman: [after reading the title, subtitle and labels] It is in dollar. Isn't it? ...Right...It means that in the United States...Here there isn't any benefit for two children!

Researcher: The graphs shows that, isn't it?

Businesswoman: It is… At least… I don't know if I am wrongly interpreting. [Reread the title]… It is for two children, it is not for one, either for three. For two… [Reread the names of countries and corresponding figures]… I didn't really know this aspect!

In her initial approach, the Businesswoman tried to establish relations between the data shown by the graph. We can note that her surprise at values for the United States provoked a rereading of the graph.

Another type of discursive strategy was characterized by interviewees talking extensively about the theme, but basing their analyses on their previous knowledge and experiences rather than on quantitative information presented in the graph. Approximately one quarter of the interviewees (26%) were placed in this category.

A third type of strategy identified was that interviewees approached the quantitative information more directly. They made approximations and estimates, and/or they made written calculations and used calculators. The frequency of these strategies was low in 'initial approaches' (15%). The following extract shows an example, in which the interviewee utilized a formula to compare the relations between the values of benefits in each country in figure 2.

Researcher: What could you notice from the reading of this graph?

Economist: …Have you cut something here? Because this graph is not explicative (…) I understand that (…) the benefits in France, Germany, Belgium, and Italy, are very, very much higher than the benefits in Brazil. In the United States there is not benefits [for children]. You can realize the following calculation [using a calculator, talking aloud] 222 ÷ 8 = 2775%. It means that the benefits in Belgium are 2275% more than the benefits in Brazil. Or the benefits in Belgium are 28.7 times higher than in Brazil. The benefits in Italy are 6 times higher than in Brazil.

His procedure provided a more accurate approach to the quantitative information from the graph. Afterwards, he explained his strategy that could be represented by following expression:

$$\frac{\text{Higher value} - \text{Least value}}{\text{Least value}} = W \times 100 = \text{comparative percentage}$$

Specific Approaches

The analyses of interviewees' answers to specific questions only generated two categories: comments from the graphs' data (50%), and approximations and estimations, and/or written calculations (50%).

However, the analyses of frequencies of interpretations developed from each specific question revealed particular tendencies. For instance, all the interviewees utilized the strategy 'comments' to answer the question: 'what would the number 8 mean?'(Figure 2).

Researcher: What would the number 8 mean?

Businesswoman: So... The benefit would be for one child? But, here it is US$8, isn't it? It shows US$8, but this value is an average, isn't it? Because, I know that on the commercial sector [my] the employees earn 7.5 per each child [she referred to Brazilian money = Reais]. It is very few!

Researcher: On the commercial sector is?

Businesswoman: On the commercial sector are 7.60 [Reais] per child. Here, it is showed US$8 per two children, isn't it? ... Per two children. I also think that it is the average of all the professions. I have heard that they really pay very well (referring to European countries).

The Businesswoman drew on her experiences as employer, and her general knowledge about social policies, when she was answering the specific question. In spite of these references to her expectations about the figures, she analysed the data from graph.

In contrast to the frequency of answers from the benefits' question, nobody developed extensive comments from the inflation's question: 'Which period had the faster speed of increase of inflation, in Jan-April/1995 or in May-July/1995?' (Figure 1). The analyses of protocols indicated that all interviewees answered through approximations and estimative, and/or calculations. The following extract gives an example this strategy.

Researcher: If we will compare these two peaks [Jan-April/1995 and May-July/1995], which is the faster increase?

Businessman: Well... If I will trust the drawing... Because it does not have number for that I can calculate [Referring to May of 1995]... Is that from May to July? Its inclination is bigger!

Researcher: If will you trust the drawing?

Businessman: Well... Looking to the drawing... It depends on... Would you like to know how much increasing between here and there? [Referring to May-July/1995] Or would you like to know the inclination?

Researcher: The speed.

Businessman: The speed here is bigger [May-July/1995].

Researcher: Do you see it just because the inclination?

Businessman – Exactly! I know that the speed is derived to the distance here. Then you look to the inclination to know what is more quickly increasing.

The Businessman's approach was more strictly related to the data from the graph. His calculations were associated with specific features from the graph.

This specific question had some peculiarities. Firstly, the question asked was connected with the comparison between two periods. In general, the other questions involved comparing two particular values. Secondly, the graph did not present all the numerical values for the calculation asked. The interviewee had to estimate the percentage of May/1995. Thirdly, the concept of speed could not be applied for percentage values of inflation. The suitable procedure should be the comparison between the average increasing percentages on each period. But, this calculation was not possible because the graph did not present all monthly percentages.

We researched the real values in each month displayed by the inflation graph, and we found that the period of highest increasing was between May and July of 1995. In spite of the absence of all necessary numerical values, 25% of interviewees correctly answered the question. But, the more frequent answer was the period between January and April (44%).

The variability of specific questions and of the levels of complexity could be possible causes of particular tendencies of interpretation of interviewees. It could be argued that the specific question related to the benefit graph was more open than the specific question for the inflation graph. This argument might be simplistic, because, for example, the same structure of question was used with the interviewees when they were interpreting another economics graph. In that case, 31% of the interviewees made strategies of approximations, and estimative. Thus, we believe that the analysis of the whole body of data is important to understand these particular results in specific questions.

A second cause could be connected with the type of graph: the benefits graph might be simpler than the inflation graph, partly because it presents 'adornments' that could make it most accessible. However, our analyses did not show any relation between pictures shown and 'facilities' or 'distractions'. In general, the feeding bottles were an initial attraction, but that it did not 'help' or 'confuse' the interpretation process.

On the contrary, real interpretation could be more difficult for the benefits graph than for the inflation graph because it requires the reader to know other information related to several areas. When the researcher asked about the necessity of other data, the majority of the interviewees affirmed that this was necessary, for example to know something about the cost of living in each country.

CONCLUSIONS

Our initial hypothesis was that familiarity with the use of graphs and academic background could be important elements which would influence the interpretations. However, analysis of the data collected from the whole study does not show any particular patterns suggesting differences between the two groups of interviewees in the ways that they interpreted the graphs.

Firstly, we could emphasis that the familiarity with the theme of graphs is not itself a facilitative aspect. The familiarity needs to be immersed in meaningful relationships etween the interpreter and the graph. In other words, the importance of familiarity is not a pre-established aspect that independently happens.

Secondly, the academic qualification of the interviewees is one part of their background. The economists and businessmen/women are also citizens, consumers, fathers and mothers, electors and so on. From multi-background, interviewees brought their believes, desires, and knowledge about everyday situations to their interpretation. Therefore, the interpretation of graphs demands more than a specific competence for perception of information.

On the other hand, we could remark that the generic mobilisation of knowledge is not enough. Even 'data producers' like economists also need to engage in a context, in which the mathematical relationships could become explicit. In our study, the specific questions prompted the interviewees to approach the graphs in more detail, becoming aware of the mathematical relations involved. For example, our analyses demonstrate that specific questions provoked a decrease in generic approaches.

In the contemporary society, the school is a context in which processes of teaching and learning related to graphs are developed. Specifically from our data analysis, we may infer that it is important to consider the planning of didactic sequences with graphs, which provide the mobilisation several types of knowledge, and that construct bridges between previous experiences and the formal aspects of mathematics present in graphs.

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MATHEMATICS EDUCATION AND CRITICAL CONSCIOUSNESS

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This paper discusses mathematics education as a contribution to the development of democratic competence. As a feature of democratic competence, the concept of critical consciousness is examined and some results of a study involving students' work within a First Calculus Course are presented.

EDUCATION FOR CITIZENSHIP

As Constant (1998) states, «the job of citizen cannot be improvised». Thus the citizen must be educated. Educational institutions are most often indicated as being the place of such education, but the meaning and content of education for citizenship have been understood in different ways. For some, education for citizenship takes place in a specific academic subject, in which students learn about the rights and obligations of the citizen, the rules which establish who is authorised to take collective decisions and which processes are to be used. For others, this type of education is transversal: it is to be found across the board of academic subjects and involves the development of capacities which are indispensable to self-determination and autonomy (Hall *et al.*, 1998). The work presented in this paper falls within the latter context.

CRITICAL CONSCIOUSNESS

If the democratic citizen is a presupposition of democracy, democratic competence is also an indispensable condition for the democratic citizen to behave as such. In this paper it is considered that democratic competence has, essentially, three components: critical consciousness, sustained and sustainable action, and co-operation.

According to Freire (1975a), critical consciousness is the ability that each individual must have to «understand the reality which surrounds one and situate oneself within the social context in order to intervene in this context in a conscious and creative manner» (p.33).

It is critical consciousness that allows us to understand that the political, social and economic conditions in which we live are not immutable, and that we can participate in changing them, that we can participate in the making of history. But in order to do this, we must begin by understanding what is happening with ourselves. Thus a primary component of critical consciousness is the capacity to identify the *dispositions* (Skovsmose, 1994) which shape our convictions, our attitudes, our behaviour – it is the awareness of our own system of values, it is the capacity to find a justification for our actions and judge the rationality of them.

To recognise that systems of values and codes of conduct which we thought to be a personal creation, are, in fact, culturally induced can be liberating and can constitute a stimulus for change and a factor of understanding of others (Brookfield, 1987).

Critical consciousness involves, therefore, a reflective dimension – critical reflection – which results in a change in the way we see ourselves, the way we see the world and the way we see ourselves in the world. Or, as Freire (1975b) said:

> "(...) people begin to understand, critically, how they *are* in the world *with which* and *in which* they find themselves" (p.102).

In the words of Broughton (cited by Brookfield, 1987, p.18), imagining and exploring alternatives involves «the capacity to generate mentally a structure of possibilities standing beyond the empirically known world of the here and now». Creating and exploring alternatives is another aspect of critical consciousness that leads to methodical doubt – to «reflective scepticism» (Brookfield, 1987, p.21). It is this doubt which questions, for example, whether the fact of a given social structure having remained unaltered for a certain period of time is synonymous with its worthiness and inevitability. It is this scepticism which is revealed in the calling to question of information sources or even universal rules.

THE PEDAGOGICAL PROJECT

The pedagogical project, from which the small sample of the students' work analysed here originated, is a curricular development project whose aim is to develop democratic competence within the context of mathematics education at the same time, obviously, as acquiring mathematical knowledge and skills.

The students involved were in the first year of a degree course in Management and the work was undertaken in Calculus classes. The author of this paper was both the teacher and the researcher.

In what concerns the mathematical content it was decided to keep to the items originally stipulated in the Calculus syllabus. Mathematical applications and real problem solving were selected to provide a context in which to approach mathematical concepts. Applications and problems were chosen so as to encourage reflection and to elicit sustained opinions about topics which affect Portuguese and/or world society. Ethics and the exercising of power were, then, the backdrop to the topics chosen: ecological issues – global warming, the treatment of waste, the preservation of biodiversity; problems of personal responsibility – drug use, AIDS; population increase and the exhaustion of natural resources; problems of ethics in companies. Co-operative group work was preferred as *modus faciendi*. The students participated in the organisation of the course in the following aspects: definition of the assessment model and choice of topics for the piece of work each group had to develop and present. The students' receptivity to the type of work being developed was evaluated by gathering their impressions/opinions at three different points in the semester: at the end of the fourth, ninth and fifteenth weeks.

ABOUT THE RESEARCH PROJECT

The duration of the research project was one semester - the length of the course module in which it was undertaken. A qualitative/interpretative methodology was used. Two types of data were collected: (1) students' written reports; (2) students' impressions/opinions regarding the work they were carrying out. The work assignments collected for the sample came from five groups of students, randomly selected from the ten groups formed in the class.

The research project was developed around the following questions: (a) to what extent does mathematics education contribute to the development of a democratic competence? (b) in which ways can mathematics education and democracy be connected through mathematical contents? (c) how are the nature of problem situations and the nature of classroom culture relevant to the ways in which democratic competence is acquired?

This paper presents the analysis of a sample of data collected. It addresses a particular piece of an essay produced by a group of students on one of the activities proposed.

ANALYSING A SAMPLE OF EMPIRICAL DATA

The sample analysed here comes from an assignment which was set as follows:

> In the 1960's the population of the Antarctic Blue Whale was reduced to such a low level that the rapid extinction of the species was forecast. However, under the influence of an International Commission for the Protection of Whales, measures were taken which permitted the population to increase to around 10,000 in just 10 years. By 1994, their numbers had reached around 50,000.
>
> We assume that the mathematical model which expresses the population variation of the Antarctic Blue Whale between two consecutive years is the function h(x), analytically defined by:
>
> $$h(x)= 0{,}000002(-2x^3+303x^2-600x)$$
>
> where x represents the population of the current year, in thousands of whales [1].
>
> Write an essay analysing the problem of whale hunting in economic, ecological and political terms.

The students from group C began by carrying out an analytical study of the function h(x). They then identified the minimum viable population level, the carrying capacity and the maximum sustainable yield, thus giving ecological content to the study. Next they attempted to construct a graph and it was here that the first problem arose.

Students: To make a graph of the function, we used the Excel programme and obtained the following graph (fig. 1).

Due to the scale, the graph does not show the values found in the study of the function. So we decided to construct a new graph, with the actual population varying between zero and three thousand (fig.2).

By ascribing values between zero and three to the population, the graph already shows that the function is negative between zero and two, and the minimum found for x=1 can be seen perfectly.

This critical attitude of the students regarding the product of the computer stands out; not only for the critical attitude in itself, but also for the confidence they reveal in their own work. Students often trust the efficiency of technology above their own and it is not uncommon to see them re-doing a correct piece of work so that it will agree with the results given by the computer. The students quoted above, however, are convinced of the correctness of the results of their study and try to identify the discrepancy between the graph supplied by the computer and their own expectations about the behaviour of the function.

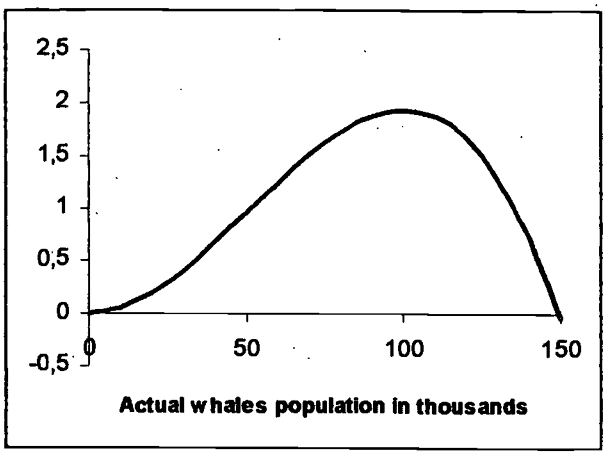

Figure 1

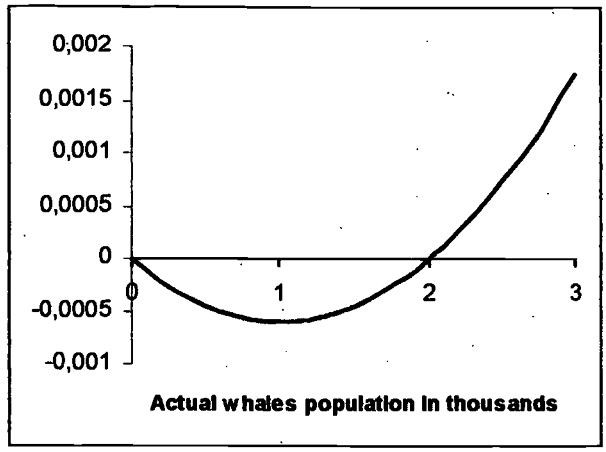

Figure 2

Further on, the students introduce new information, combining it with the values supplied about the actual population size:

Students: At the beginning of the 20th century, large regions of whales were discovered in the Antarctic. Each summer, numerous whale-hunters and factory ships headed South. Thousands of whales were killed and processed in the floating factories.

Whaling was at its height between 1925 and 1960. In the 1960's, the number of blue whales was so small (we do not know the exact number) that the International Commission for the protection of whales took its first protective measures. Due to these measures, within 10 years the population reached 10,000 and in the beginning of the 1990's it stood at 50,000, that is, in 20 years it had grown five times.

And, surprisingly, the criticism of the numbers obtained:

Students: However, if we consider the reproduction function, instead of the variation function, we do not get these results.

Let us see how they reached this conclusion:

Students: As the population variation function between two consecutive years is h(x) = f(x)-x where x is the population in a given year and f(x) is the population in the following year, we can determine f(x):f(x)=h(x)+x.

In the case of the blue whale it will be f(x) =0,000002(-2x3+303x2–600x)+x.

Reconstructing the function $f(x)$ is a simple operation. Attributing values to x and obtaining the corresponding values $f(x)$ is a trivial activity. Less so is the manipulation of the function so as to produce the values of the population in an interval of time, since this involves creating a recursive sequence in which the first term, u_1, is $f(a)$, with a chosen randomly, the second term, u_2, is $f(f(a))$, that is $f(u_1)$, … , $u_n=f(u_{n-1})$.

The results of the work of these students are shown below:

Students: The values we obtained from Excel were the following:

Year	Actual	1 year later
1	10	10,0446
2	10,0446	10,08963
3	10,0896345	10,13511
4	10,1351096	10,18103
5	10,1810318	10,22741

Year	Actual	1 year later
17	10,769428	10,82179
18	10,8217927	10,8747066
19	10,8747066	10,9281779
20	10,9281779	10,9822153
21	10,9822153	11,0368275

As can be seen, if the population variation function were that, the population twenty years after reaching 10 thousand (year 1) would be around 11,000 (year 21). The value of 50 thousand whales could only be reached 178 years later, as can be seen from the spreadsheet calculations:

Year	Actual	1 year later
175	46,2732692	47,11899
176	47,1189937	47,98944
177	47,9894376	48,88538
178	48,8853838	49,80763
179	49,8076271	50,75697

In assigning the piece of work on the blue whale, the researcher had been less rigorous than the students and had not proven that the true numbers concerning the blue whale populations and the values obtained with the proposed function actually coincided. Tomastik was consulted again, where it was verified that the values obtained by scientists regarding the carrying capacity and the maximum sustainable yield did indeed coincide with the values obtained by applying the function. The minimum viable level was unknown. But what had led the author to suggest a model of population variation that gave such different results to those observed empirically? The researcher had an idea of what may have happened, but she was not the only one.

In fact, during the discussion of the assignments in the classroom, and in the face of group C's discovery, a student from group G suggested that there could have been a mistake and the function should perhaps have been:

$$h(x) = 0{,}00002(-2x^3 + 303x^2 - 600x)$$

that is to say, an extra zero had been introduced in the factor in evidence.

This student thus showed a perception of the effects of multiplying by 10, the multiplicative parameter, on the behaviour of the function. Group C understood and accepted the proposed alteration and immediately went ahead with the modification of their calculations. And there, after 20 years, was the five-times-as-big population:

Year	*Actual*	*1 year later*
1	10	10,446
2	10,446	10,93631
3	10,9363144	11,47755
4	11,477552	12,07765
5	12,0776512	12,74622

Year	*Actual*	*1 year later*
16	28,1191655	31,68396
17	31,6839628	36,114961
18	36,114961	41,7014054
19	41,7014054	48,8386105
20	48,8386105	58,0473056

Having accepted the idea that the function needed to be altered, which aspects of the study of the function should be re-examined? And which characteristics maintained? The discussion was taken up again in the different groups.

Coming back to group C's report, the students also analysed the problem from the economic point of view:

Students: Whale-hunting is a highly lucrative business (...)

And they juxtaposed this with an ecological vision:

Students: Yet despite generating wealth, this activity threatens biodiversity and the balance of the ecosystem, as the extinction of the whales would break the food chains (...)

However the local situation is not forgotten:

Students: (...) in zones like the Azores, it was an activity with a long tradition and, as it was artisan in character, it would hardly have put the survival of the whale population in danger (...)

Finally, they concluded:

Students: (...) the countries which do not condemn and therefore do not apply laws and coercive measures to combat the capture of whales (...) should change their attitude, indeed they should be conscious of the true implications of this problem and transmit to their citizens the message that they should not compromise the future of generations to come.

CONCLUDING REMARKS

From the strictly mathematical point of view, the students developed skills in the analytical study of functions and acquired competences for the interpretation of reality, in view of the relevant aspects of the population variation function. But besides the skills and competences they applied in the analysis of the problem it is the actual situation – in the broad sense – which guides the students towards certain procedures, leading to a better understanding of the mathematical concepts involved. In fact there is a reciprocal influence: mathematical knowledge helps the understanding of reality, but reality 'demands' certain mathematical procedures and gives sense to the concepts.

The analysis of the problems posed by whaling cannot be reduced to the «*conscientization*» (Freire, 1975a, 1975b) of the established reality. The knowledge of the dynamics of populations constitutes a prospective knowledge which should orientate practices. These should not be subordinated to economic interests, but rather be governed by an ethic of responsibility. It is in the balanced management of ecosystems that these students focus their attention and the maintaining of biodiversity as a value in itself is as important to them as the warning of ruptures which could affect the quality of life of future generations, if not their actual survival.

It is also noteworthy that, for these students, there is no room for dogmas or for "untouchable" experts. Things are not what they are because "that's what it says in the book" or because "teachers say it is so" [2]. With all the naturalness of those who have learned to think about things, rather than submit themselves to a supposed truth, they detect a contradiction, propose an explanation and correct the author of a book and the teacher who placed blind trust in it; and they do it in an environment and context in which this has become a legitimate, permitted and even structuring process.

As regards group C's use of the computer, some observations would seem to be pertinent. The group used the computer with two different aims in mind: firstly, to create a clearer, well-presented graph of the function; secondly, to avoid having to carry out fastidious calculations themselves. The first objective was initially thwarted because the first graph did not faithfully represent the behaviour of the function. And this is, in fact, one of the negative aspects of a purely technological approach, when the computer users have no idea or expectation of what they should obtain. In the case under analysis, the students had previously carried out a complete study of the function. That is why they proceeded with a localised study to produce an amplification of a part of the curve representing the function. On the other hand, the huge potential of the computer is shown in the work that the students did to verify the affirmations about the size of the whale population. It is unthinkable that without the computer the students could have taken their work in this direction, so great being the number of operations they would have had to do. The same could be said regarding the work carried out in the classroom, i.e. the alteration of the function's parameters. Once again, it is evident that technology should be considered a vehicle of cognitive development, a tool which promotes autonomy, an instrument which can stimulate an investigative attitude (Moreira, 1989).

The «reflective scepticism» shown here in questioning the sources of information was, at other times, also evident in the contestation of universal rules.

Adopting an idea, adhering to a cause, assuming a practice is not something which comes from a spontaneous enthusiasm, but from a period of questioning, analysis and reflection during which its value or legitimacy is established.

In such social and political dimensions, mathematics education has a crucial role to assume but the conditions for the development of citizenship arising from mathematics classrooms have to be created. That is a job that cannot be improvised either.

NOTES

[1] Numerical data and function taken from Tomastik (1994).

[2] The expressions in inverted commas occur frequently, in other contexts, in the discourse of students to justify procedures or affirmations.

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QUADRATIC FUNCTION AND IMAGERY: ALICE'S CASE

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Abstract: The present paper is part of a more extensive study whose essential objective is to identify and analyse mental images that are constructed by students during mathematics learning process. In particular, it is one's intention to analyse the role that these images play in the realization of a mathematical task and to understand the origin of their development.

In this work, two mental images are analysed; these images are evoked by a 10th grade student while solving a task on the roots of a quadratic function. The origin of the identified images will be discussed while considering the use of a graphic calculator.

INTRODUCTION

The concept of function, considered by many investigators as one of the most important mathematics concepts, has acquired a central role in mathematics school curriculum (e.g. Ponte, 1984; Dubinsky & Harel, 1992). In spite the different didactics approaches suggested to introduce this concept - some years ago, this was done using a more algebraic perspective and, now, it is taking place using a perspective that values global understanding of the concept, considering all of its representations, with an emphasis on the graphic one, and making the most of the new technological potential (e.g., Eisenberg, 1992; Sfard, 1992) - it still seems to maintain its complexity in what respects the teaching-learning process (e.g., Sierpinska, 1992).

In fact, there is evidence that highlights the proliferation of mental images developed by students (e.g., Bakar and Tall, 1991; Resnick, Schwarz & Hershowitz, 1994; Ruiz Higueras et al., 1994) during the concept's learning process. These images and/or conceptions - rich probably due to the multiplicity of representations that the concept involves and the level of abstraction that those representations require for their understanding - might be idiosyncratic or be directly related to the approach context used (e.g., Eisenberg, 1992; Monk, 1992).

However, students' imagery does not appear exclusively connected to this concept and many investigators have dedicated their work to its study because they believe that it plays an important role in the learning process of mathematics (e.g., Clements, 1981, 1982), in the activity of "making sense of mathematics" (e.g., Dörfler, 1991, 1995; Presmeg, 1992; Reynolds and Wheatley, 1992; Solano and Presmeg, 1995) or in the development of mathematical reasoning at all levels (e.g., Thompson, 1996).

In this article, a broad perspective of imagery will be used - in which an image is "much more than a mental figure" (Thompson, 1996) - associated to the theory[1] of mathematics meaning construction proposed by Dörfler (1989, 1991, 1995).

Dörfler elaborates his theoretical perspective on the construction of mathematical meaning based on the notion of the image schemata. According to this author, an image schema is a set of cognitive interactions with, and cognitive manipulations of, a concrete carrier (an object as model, a material model or just an imagined one, a drawing, a graph or other representation of a mathematical concept...) that allows an individual to obtain a personal meaning of the concept in question. The concrete carrier does not represent the concept; it only serves as a referent for the individual - it is the cognitive activity developed with and on the referent that allows for the concept to be "present cognitively and mentally" (1991, pp.21) to the individual. In this way, the same carrier may serve different image schemata corresponding to different concepts and, conversely, the same concept may allow for various image schemata based on different carriers - it all depends on what constitutes the main focus of the individual's attention and on the properties, relationships or transformations that this person constructs.

As a cognitive process, the image schemata is idiosyncratic and can not be shared by the individual and his/her speakers; only the carrier can be used in the communicative process and it is here - the communicative process - that subjective deviations are corrected (negotiation of meanings) and socially normalised carriers are supplied (Dörfler, 1991).

The mathematical meaning, according to Dörfler (1991, 1995), has a predominant holistic aspect that corresponds to the pertinent image schemata for the respective concept[2]. Each mathematical concept may allow for, as previously stated, various image schemata, based on different concrete carriers, and each adequate image schemata constitutes, for the individual, a mathematical meaning of the respective concept. This meaning, which derives from the set of image schemata and the respective concrete carrier, serves as a basis for the individual's memory and other cognitive functions such as, for example, argumentation and logical inference (Dörfler, 1991).

In what concerns the specific context of mathematical learning, this author defends that "the cognitive manipulation of mathematical concepts is highly facilitated by the mental construction and availability of adequate image schemata" (Dörfler, 1991, pp.20), in view of which the teaching of mathematics should privilege this

[1] A theory essentially based on the notion of Johnson's (1987) and Lakoff's (1987) image schemata.

[2] The author refutes the idea according to which the mathematical meaning of a concept is constructed accumulatively from basic particles whose meaning is predefined or previously understood (the Bourbakist school is an example of this construction).

construction and availability. As it is not possible to construct pertinent image schemata without adequate concrete carriers and, also considering that these are acquired through appropriate fields of experience, Dörfler suggests that mathematics teaching should constitute, in its entirety, a good field for experience. Not only will this situation offer an accessible set of normalised concrete carriers - as potential referents for mathematical concepts - but, also, it will basically orient individual construction and appropriate development of image schemata, without which mathematics learning only restricts itself to "the formal, rule-guided manipulation of symbols without referents" (1991, pp.22). He also defends, so that an adjustment of individual schemata can take place to socially normalised patterns, the continued use of image schemata that explicitly have, in a conscious form for the individual's understanding, the way in which they were constructed. For the conscious construction of image schemas in this context, he proposes protocols of actions and processes (1989, 1991).

In this present piece of work, part of an interview will be analysed. It seems to illustrate the existence of idiosyncratic images that interfere, in a decisive way, with the solution of a mathematical task. In particular, the way that two mental images are used is analysed; these images were evoked by a 10th grade student while solving a task on the roots of a quadratic function. The idiosyncrasy of one of the identified images is discussed, thus trying to establish its origin. The influence of the use of graphic calculators in the development of that image will be speculated.

DATA PRESENTATION AND ANALYSIS

The excerpt that is presented is part of a more extensive interview elaborated for a presently on-going investigation project.

The student, Alice (15 years old), is a 10th grade urban secondary school student. She likes maths and she does not easily give up when she finds tasks difficult.

The interview, which is semi-structured and task-based, consists of essentially two moments: (a) a moment[3] during which the student tries to explain and illustrate how many roots a polynomial function of the third degree has; (b) a second moment in which the student argues that a quadratic function has always two roots. In the present article, the arguments presented by the student during the second moment will be analysed (the first moment was reported elsewhere).

The analysis of this moment in the interview reveals the existence, in the student's way of thinking, of two images of "roots of a function": a visual image associated to the

[3] This first moment consists of two phases inserted by the analysis of the roots of the quadratic function; this is, the analysis of the quadratic function roots appears during the explanation of the cubic function roots: the student tries to explain that a cubic function has one, two or three roots (first moment). This explanation is interrupted so that the root of a quadratic function may be analysed (second moment); the cubic function is approached again to systemize the roots (conclusion of the first moment).

graphic representation of a function - *'the roots are the points of intersection of the graph with the xx axis'*; a non-visual image where she tries to establish connection between specific mathematical vocabulary - *'if it is of the third degree, it has three roots, if the second, it has two'*. These two images coexist and are evoked throughout the interview.

Roost of a quadratic function[4]: Alice's arguments

(...)

Interviewer: *So, the second degree has always two roots?*

Alice: *Hum. Yes.*

Interviewer: *Can you give me an example?*

The student draws the example in figure 1 - she begins by representing the axis, she draws without hesitation, from left to right, and highlights the points of intersection of the graph with the *xx* axis.

At this moment the investigator, trying to explore the stability of the idea 'that a quadratic function always has two roots', suggests the application, to this parabola, of a translation associated to a vertical vector. The student accepts the suggestion and draws the new situation (figure 2).

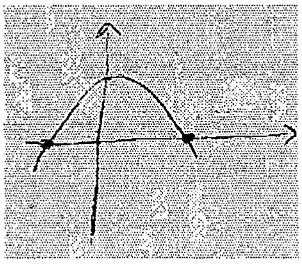

Figure 1. Second degree function with two roots.

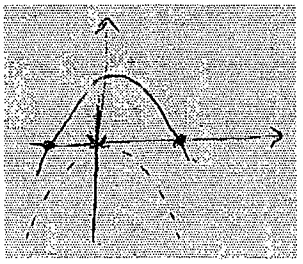

Figure 2. Translation applied.

Interviewer: *How many roots are there?*

Alice: *Also two... there are two; it's zero here* [she indicates the origin of the coordinates] *b'cause it's tangent...two.*

Interviewer: *And what if we apply another translation associated to a vector of greater length...*

Alice: *Down more?*

Interviewer: *Yes. How many roots will it have?*

[4] During the first part of the interview, when explaining the roots of a cubic function, the student arrives at a conflicting point where, on the one hand, she affirms that a cubic function has one, two or three roots while, on the other, she states that 'if is of the third degree, it has three, if of the second, it has two'.

Alice: *Ahh, it'll probably have the same* [she pauses a few seconds - she looks at the paper and, with the pencil close but above the paper, she makes some gestures above and below the *yy* axis; she lightly sketches a point on the *yy* axis and draws a part of what would be the new parabola, she does this so lightly that you almost can not see it (figure 3)]. *But I don't know if it's still a second degree. Is it?*

Interviewer: *What do you think?*

Alice: *The form is, ... of a quadratic function.* [She pauses] *Hum. I don't think so. It's zero only here* [she indicates the vertex of the third parabola that she did not actually draw][5] *I don't think so* [pause] *but these* [referring to the second and third parabolas] *come from this one* [she indicates the first parabola], *they're written from this one and this is a quadratic function* [she points to the first*], this one... I think... is too* [indicating the second] *and this one...logically should be* [she pauses for a few seconds during which she looks at the paper. She shakes her head negatively and state]...*Maybe it is.* [She giggles].

The interview continues to explore the possibilities in determining the roots of a function, when it is defined by an equation and not by its graph, but the student is unable to do so. During the dialogue, another graph appears (figure 4) and the student is once again asked to analyse this function's roots.

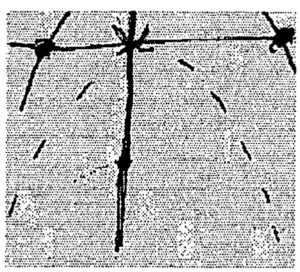

Figure 3. Enlargement of part of the third parabola.

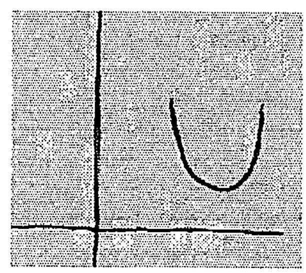

Figure 4. Another example of parabola.

Alice: (...) *there's always root, but...* [she pauses for a few seconds] *we can adjust here the graph* [she moves her open hand along the *yy* axes] *we make a translation...if we put the vertex here*...[she indicates the *xx* axes]...*it'll have a double root.*

The student's hand movement seems to imply that it is possible 'to adjust the axis' in such a way as to make the quadratic function always have roots, as if it were possible

[5] It is notorious to see how much the student wants the quadratic function to have roots to the point where the parabola's vertex is considered a root.

to physically hold the parabola and place it in a new more adequate position, in relation to the axis, so that the graphic could intersect the *xx* axis.

The student continues to think about the parabola and tries to find an argument that will allow her to accept the fact that a quadratic function may not have roots without contradicting her non-visual image. She tries to explain how, even after having stated that a quadratic has always two roots (and she continues to believe it is thus), it is possible to obtain a graph that does not intersect the *xx* axes.

Alice: *It could if...it can be written through this one here, ...if it is the result of a translation of a function...that has, that has...roots. This one is written through this one here above...*[she points to figure 2, the second parabola]*...it will be written...if it's the result of a translation, I think so.*

What the student seems to be trying to explain is that a quadratic may not have roots if it is the result of a translation applied to a quadratic that does; in other words, if the expression that defines it can be written from the expression of the original quadratic, associating to it the corresponding translation calculations. For this student, a translation does not seem to be an application that transforms a parabola into another parabola but, rather, something that allows for the moving of the parabola to another place: the parabola is always the same one, and since it is possible to put it in a convenient place so that it can have two points of intersection with the *xx* axis (or one tangent point), then any quadratic has, always, two roots.

DISCUSSION

The following discussion is based on a preliminary analysis of the data.

This episode seems to reveal the student's preference for her visual reasoning component, in this specific task, given that every time she is asked to supply an example, she does so using the graphic representation. This preference, reported, for example, by Norman (1992) but contrary to what other investigators have given accounts of, may be related to the fact that the graphic component was initially the most explored representation within the study of functions in the classroom (even though the quadratic function had been systematically focussed as a whole, which allowed for the interaction of the various representations) or, still, the continued use of the graphic calculator and, some times, of the computer.

The two evoked images of the 'roots of a function' - a visual one associated to graphic representation of a function, *'the roots are the points of graph intersection with xx axis'*; and, the other, a non-visual one related to specific mathematical vocabulary terms, *'if it is of the third degree, it has three roots, if of the second, it has two'* - coexist throughout the task appearing to take on a similar level of credibility for the student.

The visual image, *'the roots are the points of graph intersection with xx axis'*, is a standard image introduced by the teacher (and, also, by school textbooks) to help with graphic interpretation of a function.

The non-visual image, *'if it is of the third degree, it has three roots, if of the second, two'*, seems to be idiosyncratic given that no other subject involved in the study has referred to it. It also appears to have assumed a role of 'uncontrollable image', in Presmeg's (1992) and Aspinwall's et al. (1997) sense, given that it is maintained even when confronted with evidence to the contrary, and it overlaps the visual image not letting the student irrefutably accept the fact that a quadratic function may not have roots.

The origin of this non-visual image seems to be associated to a third one that interferes, in a direct and determinant way, with the task's resolution. For the student, a translation is not an application but 'a vehicle of transportation' that allows for the moving of the parabola in such a way that it may preserve one of its characteristics: to have always two roots.

This third identified image - one of translation seen as a 'vehicle of transportation' - might be connected to the kind of, and writing used in, some exploratory tasks, in the study of function families, using graphic calculators; for example: "Use your graphic calculator to study a and b parameter variation effect in functions of type $f(x) = ax^2 + b$. What happens to the graph when parameter b is altered?" Answers to a proposal of this type usually appear as, 'the function graph has suffered a translation associated to the vector...' or 'the parabola suffered a displacement of ... units'. In this specific case, phrases such as 'the function graph suffered ...' or 'the parabola suffered...' may lead students to develop images such as the one evoked by Alice. (This point needs further investigation.)

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ON THE PATH TO PROPORTIONAL REASONING

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Four seventh grade students participated in a constructivist teaching experiment in which manipulatives within a computer microworld were used to solve fractional reasoning tasks followed by tasks that involve concepts of rate, ratio and proportionality. Through a retrospective analysis of video tapes, their thinking processes were analyzed from the perspective of the types of cognitive schemes of operation used as they engaged in the given problem situations. One result of the study indicates that the modifications of the students' available schemes of operation when solving the fractional reasoning tasks formed a basis for the cognitive schemes of operation used in their solutions of tasks involving rate, ratio and proportionality.

This paper focuses on Michael, one of four seventh grade students who participated in a study involving a constructivist teaching experiment as proposed by Steffe and Thompson (2000). In this study manipulatives within a computer microworld were used to solve fractional reasoning tasks and tasks that involve concepts of rate, ratio and proportionality.

BACKGROUND AND RATIONALE

It is commonly accepted among mathematics teachers and educators that reasoning involving the concepts of rate, ratio and proportionality proves to be difficult for most students. Because of the critical role that proportional reasoning plays in a student's mathematical development, Lesh, Post and Behr (1988) describe it as a watershed concept. That is, they refer to it as a cornerstone of higher mathematics and the capstone of elementary concepts. Thus, it is logical that there has been considerable research centered around these concepts that are embedded within multiplicative reasoning. The results of a study by Lamon (1993) indicate that before students proceed in using the formal representation of proportions and in using the cross-multiply-and-divide algorithm, they should be given extensive experience in exploring tasks involving multiplicative situations. One of the findings of Tourniaire and Pulos' (1985) review of the literature was that further research was needed in exploring the elementary proportional strategies and the problem context with its relationship to proportional reasoning acquisition. Studies by Thompson (1994) and Behr, Harel, Post & Lesh (1992) can be seen to be directed towards answering this call. Kaput and West's (1994) research can also be viewed as an answer to this call. The results of Kaput and West's (ibid.) study is used extensively in the study on which this paper is based.

The work in these former studies has included attempts to interpret the nature of concepts involved in proportional reasoning from the *adult's* perspective. The research in the present study (Nabors, 2000) is an attempt to clarify the nature of the *students'* concepts while engaged in such reasoning through producing cognitive models of students' mathematical reasoning from the students' perspective, thereby enriching former research. The cognitive schemes used to describe the mental operations and constructs of the students are based primarily on work by Steffe (1992, 1994), Sáenz-Ludlow (1990) and Olive (1996).

METHODOLOGY AND THEORETICAL PERSPECTIVES

A constructivist teaching experiment as proposed by Steffe andThompson (2000) was considered to be an appropriate methodology for the study because this type of teaching experiment was originally designed to formulate explanations and models of students' mental constructions based on intensive interaction between the researcher and the students. The theoretical underpinning of such a methodology is that of constructivism as perceived by Ernst von Glasersfeld (1995). In this epistemological perspective, knowledge is viewed as not being passively received but actively built up by the cognizing subject.

Four seventh grade students were selected from two general math classes through the use of a written pretest and a 20 minute interview. (Nabors, 2000) The students' responses were categorized according to Kaput and West's (1994) levels of proportional reasoning without instruction: (1) Coordinated build-up/build-down processes, (2) Abbreviated build-up/build-down processes using multiplication and division, and (3) Unit factor approaches.

In this study, these levels were interpreted through the use of the scheme theory mentioned previously. During the study the students used computer technology available in the computer microworld referred to as "TIMA: Bars"(Olive, 1996). In this microworld, the manipulable objects are rectangular regions that the student can make by clicking and dragging the mouse. The computer microworld provides a setting for the task situations that helps to discourage the students' use of school algorithms in their problem solutions. The physical hardware used to record the data collection consisted of the following equipment: a camcorder, a computer, a converter, a digital video mixer, a VCR and two television monitors. This technology enabled the researcher to work unassisted while collecting video and audio data.

THE CASE OF MICHAEL

The objective of the pretest and the subsequent interview is to explore the level of proportional reasoning each student seems to possess at the beginning of the experiment. The pretest is composed of seven problem situations covering the topics of money exchange, recipe formulations, mixtures, magnification and work which are consistent with those found in former research in which proportionality is being investigated among middle school students.

Pretest Results.

Michael's written results of the pretest indicates that he possesses Level I (Kaput and West, 1994) proportional reasoning operations. That is, he used additive reasoning in his attempts to solve the tasks. Therefore, in an effort to further investigate his potential proportional reasoning, he is given a problem in which a hook is placed on one pole and he is asked to indicate where a hook should be placed on a second pole so that there would be a proportional relationship between the hooks and poles. Michael understands that there is a need to maintain a constant ratio between the hooks and poles. Thus, in spite of the additive nature of his responses to most of the tasks, there are two important insights that his responses on the pretest illustrate. They involve his awareness for comparing units for the calculations in the tasks involving recipes and in his recognizing the need for maintaining a constant ratio in the pole problem during the interview. Due to the importance former research gives to *units* in the area of rate, ratio and proportion, (Lamon, 1993; Kaput and West, 1994), these insights lead to the categorization of Michael as being one of the two stronger reasoners in the study. That is, he had available schemes of operation that characterize Kaput and West's (1994) Level II proportional reasoning before instruction.

Part I: Fractional Reasoning.

In Part I of the teaching sessions, Michael was asked to solve tasks involving fractional reasoning by using manipulatives within the computer microworld: Tima: Bars. (A bias going into the study was that the fractional reasoning involved in the tasks in Part I would be sufficient for the reasoning involved in the tasks in Part II of the study.)

Michael quickly learned to use the actions of the microworld. The first task involved the construction of a bar that would be 3/4ths of a given bar. Michael partitioned the given bar into 4 equal parts and used the computer to dis-embed 3 of the four parts to get the new bar.

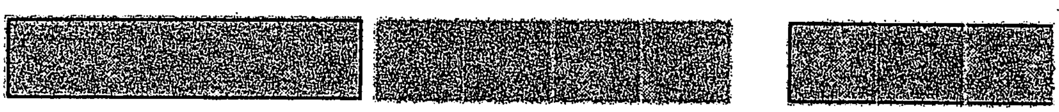

Figure 1

This was accomplished with no hesitation and illustrated, at the very least, the use of a partitive fraction scheme. "The partitive fraction scheme is regarded as an initial fraction scheme since it is here that the child begins using standard fraction words while working with partitive units." (Tzur, 1995, p. 135) Michael knew he could iterate 1/4 three times to get 3/4 or iterate it four times to get the whole. This task was also given to show that the whole is not destroyed through this operation. Thus, a whole-to-part concept (Sáenz-Ludlow, 1990) could be shown or promoted through this task. The next task presented Michael with strong perturbations:
The given bar is 5/7 of another bar. Make the other bar.

This task was given to test for and/or promote reversible operations. That is, in order to make the required bar, one needs to be able to 'see' or think of the other bar as being made of 7 sevenths. Then one needs to see the given bar as being made up of 5 of those sevenths.

M: The other bar is 5/7 of this one? (A typical response.)
T: No, the given bar is 5/7 of another bar.

Michael first partitioned the given bar into 5 vertical pieces and then partitioned the same bar into 7 vertical pieces. It is possible that he was trying to see the other bar as if it were superimposed on this bar; trying to see both bars at the same time. The resulting partition did not consist of equal parts, so he erased the bar. Then he partitioned a new bar into 7 pieces, copied it and pulled out 2 pieces of the new bar and joined it to the copied bar. Realizing that it did not give him what he wanted, he erased everything and made a new bar and pondered it for several minutes.

T: Can you picture both bars in your mind... at the same time?
M: Yeah.
T: Which one is bigger?
M: The second one.

Michael then made and partitioned a bar into 5 parts. Next he made a bar that *appeared* to be two sections longer. I interpret these actions to indicate that he knew the second bar would be two parts longer, but couldn't make the connection of how to make it *exactly* the right amount longer. So I gave him another problem with numbers representing more familiar 'parts of wholes'; that is, the given bar is 3/4ths of the second bar. Again, he tried to follow the same procedures and finally said: "I don't know..". It appears that he was unable to make a modification of his existing schemes in order to solve the task. I went back to more familiar parts of wholes and instructed him to make a bar from the given bar knowing that the given bar is one half of the bar he is to make. At this point Michael copied the given bar twice and joined the two new bars to get a correct second bar. To get the second bar in this case does not require any kind of partitioning of the given bar. He just doubled the first bar by copying and joining. He had to realize that two halves make a whole. I then asked him to make a second bar given that the first bar is 2/3rds of the second bar. This time, without hesitation, he copied the given bar, partitioned the given bar into 2 equal pieces, pulled out one of the sections of the given bar and joined it to the copy to represent the requested second bar. (See next page, figure 2.)

M: Is that it?

Michael had constructed a correct solution, but because he still asked if it were correct, I was not sure that an accommodation, a modification of a conceptual structure in response to a perturbation, had been made. Next, he attempted the task dealing with 3/4ths. He made a bar, partitioned it into 4 parts, pulled out one part and joined it to a copy of the given bar and turned to me as if for approval. I read the problem again, and without realizing it, emphasized the 3 in 3/4. Michael said: "Oh." He then partitioned the given bar into 3 parts, copied the bar, pulled out one part and joined it to the copied bar. He commented: "Now, I get it." At this point an accommodation had taken place because he was able to assimilate the situation into his existing scheme of adding a part of the given parts to make a whole composite unit.

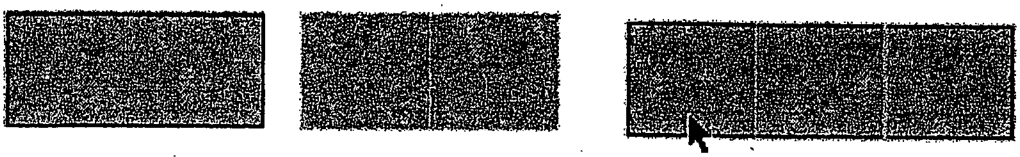

Figure 2

I hypothesize that Michael had modified his part-to-whole scheme to include viewing the whole given bar as a part of another bar. Up to this point he had viewed a whole bar as a whole composite unit from which he could pull out a part. In order to check to see if learning, "a permanent modification of a conceptual structure in response to perturbation" (Steffe and Wiegel,1996, p. 496), had taken place, I gave the following task:

A given bar is 5/3rds of another bar. Make the other bar.

With an extremely surprised, puzzled look, he asked:

M: Five thirds?!

T: Yes.

This requires a huge conceptual leap in that not only are reversible operations involved, but the given bar is greater than a whole. With little hesitation though, Michael made a bar and partitioned it into 5 parts, pulled out 3 parts and turned to me as if for verification. (See figure 3.)

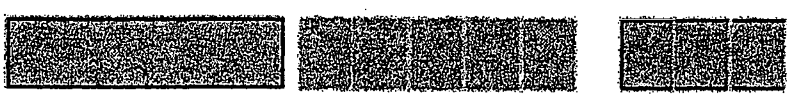

Figure 3

It is my hypothesis that Michael's correct solution to this task indicates that he reorganized his iterative fraction scheme to include reversible operations. He reasoned that 5/3 could be obtained by iterating 1/3 five times and 1/3 iterated three times would be 3/3.

The previous exercises were given primarily to determine the availability of reversible operations with fractions. It appears that Michael did not have such operations available at first, but was led to develop them through reverting back to a situation in which assimilation and accommodation could take place. According to Glasersfeld (1995, pp. 62-63), "...cognitive assimilation comes about when a cognizing organism <u>fits</u> an experience into existing sensorimotor or conceptual structures it already has." In this case, Michael was familiar with doubling and halving and with parts of wholes. I interpret his responses to indicate that he possessed a *unit fraction scheme* (Sáenz-Ludlow, 1990) which allowed him to consider both whole-to-part and part-to-whole relations. I hypothesize that his reversible operations were basically brought forth through the use of his partitive fraction scheme. But more than this, he was dealing with fractions such as 5/3 which represents the meaning of fractions beyond the part-of-a-whole concept. Thus, his partitive fraction scheme was modified to include a unit fraction scheme and an iterative fraction scheme which allowed him to extend his fractional reasoning to include operating with fractions representing more than a whole.

Michael was able to solve the remaining tasks in Part I of the study tasks without significant perturbations. He (as the other students) did experience difficulty in naming the different fractional parts of consecutively constructed bars. For example, the task of making a bar, constructing a bar that is 1/4 of the given bar and then constructing another bar that is 2/3 of the one just made, caused perturbations in terms of naming the fractional value of the new parts. He viewed each newly constructed bar as an entity unto itself and not as a fractional part of the previous bar, etc. I hypothesize that Michael was working in his zone of potential construction (Steffe, 1992) because he could name the fractional parts when reminded after each construction, that each new bar was constructed from a previous bar.

Part II: Word Problems.

The word problems presented in Part II of the study are generally considered, from a school math perspective, to involve reasoning that involves concepts of ratio, rate and proportion. An analysis of Michael's attempts to solve one such problem in terms of the schemes he used in his solutions is given here.

Work Problem: Adam can mow the grass of a rectangular field in four days. Jim can mow the same field 12 days. How many days would it take Adam and Jim, working together, to cut the grass? (Task adapted from Abramovich,1996).

Michael immediately made a bar and then made two copies of it. (See figure 4.) Each bar represented the field of grass to be cut or the whole job. One bar was partitioned vertically into 4 parts with each part representing the amount of grass Adam could cut in one day. The second one was partitioned vertically into 12 parts with each part representing the amount of grass Jim could cut in one day. In both cases, he viewed the whole field (the whole bar) as a composite unit or whole. In the first solution Michael set up a three-to-one units coordination scheme, a 3-to-1 ratio, between the parts of the bars. This indicated to him that it would take Jim 3 days to cut the same amount of grass that Adam could cut in one day. Next he made a one-to-one coordination in which he matched one day's work of Adam with one day's work of Jim, three consecutive times. This 'used up' three of the vertical parts in the bar representing Adam's work. The last part of Adam's bar is recognized as the part of the field that Jim cut, thereby indicating that it takes three days to cut the field when working together.

Michael's solution to the work problem involves a 3-to-1 units correspondence scheme to indicate the amount of work performed by one person in each day. From the perspective of school math, it might be said that he illustrated the concept of rate when he talked about the amount of work per day. From the perspective of this study, rate was defined as: a reflected abstracted constant ratio (Thompson,1994). I hypothesize

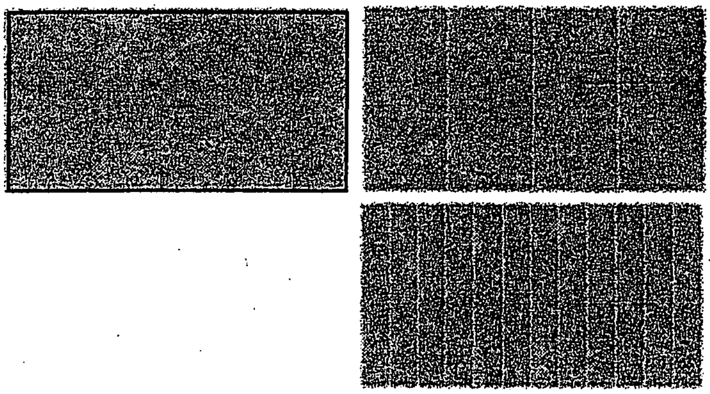

Figure 4

that Michael was viewing the 3-to-1 ratios as static ratios involving a comparison of invariant quantities. That is, he had yet to view the constancy of the ratio from the viewpoint that the ratio would remain constant regardless of variation of quantities being compared. His reasoning did not demonstrate this level of abstraction.

DISCUSSION.

Michael left the experiment possessing more sophisticated fractional reasoning operations available to him than when he entered the experiment. My conjecture is that one of the reasons he was able to conceptualize problem situations that had previously involved constraints for him on the pretest is because he was first met at his level of reasoning and understanding. Next he was placed within mathematical situations which promoted the reorganization of his present schemes of operation to form operations that allowed for solutions requiring more sophisticated schemes of operation. His solving of *all* of the written tasks on the post test, which were similar to the tasks on the pretest, appears to be a byproduct of the increased sophistication of his available operations. With this being said, it is my hypothesis that Michael had still not demonstrated proportional reasoning from a Piagetian perspective. For Piaget (Inhelder, B. and Piaget, J., 1958), proportional reasoning involves a relationship between two relationships, a second order relationship. This entails a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons. For Michael (as with the other participants) proportional reasoning was still at the stage of involving the comparison of static ratios.

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Gender and Written Mathematical Communication

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Abstract

This research study deals with the connection between two central topics in mathematics: communication and gender. The study examines the connection between gender and achievement, communication types and communication quality. The findings are based on analysis of 164 9th grade students' answers to problems that demanded communication of reasoning, explanations and justifications. The results of the study challenge pervious research, as not only were no gender differences were found in achievement, but also no tendency was found for girls to communicate verbally more than boys. These findings demand further study on the value of verbalized communication, particularly before policy makers modify high stake tests that are meant to be in the interests of implementing affirmative action for the advancement of girls in mathematics education.

Background

In research literature dealing with gender differences in mathematics, it is agreed that no gap exists in achievement at the elementary school level. However, gaps in favor of boys, begin to manifest in middle school. These gaps grow as the students grow older and are most prominent at the very high levels of achievement (Leder, 1990). In Israel, gender differences in mathematical achievement have been found at all ages (Amit & Movshovitz-Hadar, 1989; Ministry of Education, 1996;TIMSS-R, 2000).

The source of differences in test results is not unequivocal and depends on the content and structure of the exam and type of items. It has been found that boys surpass girls in exam items having to do with geometry and problem solving, particularly those given within the context of everyday life. Boys have higher achievement when the problem solving demands strategies that have not been taught in school. The gaps, in favor of the boys, become more significant as the degree of difficulty in the exam items is greater. The girls surpass the boys' in exam items dealing with calculations, algebraic manipulations and in problems having to do with abstract mathematics. Girls have higher achievement also when solving exam items similar to those appearing in text books (Seegers & Boekaers, 1996; Wang & Lane, 1996; O'Neil & McPeek, 1993). Despite the above, in an analysis by Hyde, Fennema & Lamon (1990), no gender differences were found at middle school classes in anything relating to verbal problem-solving.

In psychology, differences between boys and girls in verbal ability are regarded as an established fact. For example, in tests for verbal ability, it was found that already at the age of 10-11, the girls reached higher achievement than the boys. Those gaps were found in the varying aspects of verbal ability: vocabulary, reading comprehension, writing and speech flow (Halpern, 1986). Despite the above, in an analysis by Hyde & Linn, (1988), it was found that the intensity of the differences in verbal ability was so small that it may be considered nil.

Pomplun & Capps, (1999) argue that some of the gender differences stem from the test structure itself. Hence, boys have higher achievement on multiple-choice tests while girls have higher achievement in open-ended problems. What makes this argument stronger is that the American SAT exams where there has been a gap for many years in favor of the boys, but when a verbal element of essay writing was added to the test, the gap between the boys and the girls decreased significantly (Willington & Cole, 1997).

Mathematical tasks based on open-ended problems have the potential for the students to show a variety of mathematical comprehension and communication levels. Even regular problems, taken from textbooks, may "turn into" open-ended problems by adding a demand to explain and justify the answer, and therefore cause the students to develop and use mathematical communication. (Cai, Jakabcsin & Lane, 1996). The information obtained from the process of writing and its product comprise a diagnostic tool for the teacher and makes it possible to identify misunderstandings and assessing the degree of the student's comprehension of materials that have been studied (Pugalee, 1998).

We assumed that verbal ability may improve mathematical communication and problem solving, particularly where verbal reasoning is required. Based on this assumption and on research literature, we hypothesized that we shall find significant difference in achievement and communication between boys and girls; this hypothesis, however, has not been confirmed.

Methodology

This research study examined gender differences in achievement and in mathematical communication modes which students used in their answers to problems that required explanation, argument or justification. In particular, we examined the connection between gender and:

1) level of achievement (correctness of answers);
2) the type of written mathematical communication;
3) the quality of written mathematical communication

The type of written communication referred to the representation through which the students chose to explain their answers. This is based on the perception that the mode of representation is the external expression that reflects the solution processes of mathematical thinking. The quality of communication referred to soundness of the mathematical justification, that is, whether the justification is based on correct mathematical argumentation and whether it consistently and completely supports the solution presented by the student (Cia, Magone, Wang & Lane, 1996).

Setting and instruments

The research instrument comprised of three problems from a regional test in mathematics to given 5,928 ninth graders in 73 schools in the southern district of Israel in February 2000. The test items were put together by the Ministry of

Education – as part of evaluation of a 3-year intervention project. The students being tested were asked to answer 46 problems referring to varying areas within the ninth grade mathematics curriculum. About half of the items were multiple-choice or short constructed response ones, and half of them were open-ended problems (to different degrees of "openness").

For this research, three problems, whose solutions the students were asked to explain and/or justify, were chosen. This requirement obligated the students to select a communication method suitable for expressing mathematical ideas and arguments. In the instructions on how to answer, the students were told they could use any method they wished for justifying their answers.

The first problem dealt with an optimization situation: finding the most worthwhile condition for purchasing a product offered by two companies. The students were asked to mark the correct answer and then to justify their choice. The problem was a non-routine one and was new to the students.

The second problem dealt with rate of change. This problem seems to be a routine open-ended problem, like those appearing in ninth grade textbooks. But it was new to the students to be asked not only to give a verbal answer, but also to explain their answer.

The third problem was in geometry and dealt with the relation between area and circumference. This problem was hypothetical – the students were asked to deal not only with the given situation, but also with a hypothetical change, a process that requires deep understanding of the subject. The problem was formulated as a multiple-choice question in which the students were to choose the correct answer and, afterward, to justify their choice. The integration between the closed items and the demand to explain and justify the answers, turns the problems into open-ended problems and makes it possible to examine mathematical communication.

It is important to note that neither the teacher nor the students being knew, at the time of the exam, that part of the tests would be researched. This fact increases the authenticity of the answers.

Method of Analysis

The analysis of the students' answers was according to three criteria: the correctness of the answer, the type of communication and the quality of mathematical communication.

- The correctness of answers: The students' answers were checked with regard to their correctness, regardless of the type of explanation or its quality. The scale included: right answer, wrong answer and no answer.
- Types of communication which the students chose, were sorted according to their representations: verbal, numerical, algebraic, diagramatic.
- Quality of mathematical communication: This ranking reflected the quality of the explanation or reasoning given for the answer. The score was on a scale of: 0 = no explanation up to 3 – complete explanation that communicates effectively

and presents supporting arguments which are logically sound and complete. This ranking method is based on Cai, Jakabcsin and Lane (1996).

The above method was tried out in a pilot study, and underwent refinement. The emphasis on checking communication quality was on the nature and quality of mathematical communication and not on the verbal/linguistic ability of the student. There are likely to be situations in which the explanation is impressively formulated but not supported by mathematical knowledge. In such a situation, the communication quality will be ranked as low quality. On the other hand, it is likely that a mathematical explanation is correct while not formulated properly because of spelling mistakes and/or language errors. In this case, the communication quality will be ranked as high.

Results

Comparison of Test Scores

For the sake of generalization, the test scores of the study population were compared with the scores of the entire examinee population, as rendered by the Ministry of Education. There was no significant difference between the mean score of the two populations; Study population mean score 67.08; Examinee population mean sore 65.19; t-test result $t(163) = 0.99$

There were no significant differences between the girls' achievement and the boys' achievement in this test as indicated in Table 1.

Table 1: Gender differences in test mean scores

	Boys N = 83	Girls N = 81	
Mean score in entire test	67.02	67.11	$t(164) = -0.02^a$
Mean score in test (without the 3 problems being analyzed)	67.90	68.20	$t(164) = -0.07^a$

[a] n.s.

Gender and correctness of answers:

Answer correctness was divided into three categories: correct answer, wrong answer and no answer. An χ^2 test was implemented to the connection examine between gender and correctness of answers: No significant connection was found in any of the problems between gender and answer correctness. The distribution of answer correctness and the results of the χ^2 test are presented in Table 2.

Table 2: Distribution of answer correctness in each problem by gender:

	Optimization Problem		Rate of change Problem		Area-circumference Problem	
	Boys	Girls	Boys	Girls	Boys	Girls
Correct Answer	40	36	64	58	19	17
Wrong answer	27	24	6	4	44	50
No answer	16	21	13	19	20	14
	$\chi^2(2) = 1.041^a$		$\chi^2(2) = 1.805^a$		$\chi^2(2) = 1.543^a$	

[a] n.s.

Gender and type of written communication:

In the optimization and in the rate of change problems, no significant connection was found between the type of communication mode and gender. In the problems of area-circumference, the distribution of the different communication types and the few students who used algebraic representations, statistically prevented the use of a χ^2 test, but it may be seen that the distributions of communication types for boys and girls are alike. The distribution and the result of the χ^2 test are presented in Table 3.

Table 3: Distribution of communication types by gender

	Optimization Problem		Rate of change Problem		Area-circumference Problem	
	Boys	Girls	Boys	Girls	Boys	Girls
Numerical	41	35	27	23	3	3
Algebraic	8	10	7	8	3	2
Verbal	15	10	31	26	33	38
Diagramatic					14	12
	$X^2(2) = 1.022$[a] N= 120		$X^2(2) = 0.301$[a] N= 122		N= 108	

[a] n.s.

Gender and the quality of written communication

In the problem of optimization and in problem on area-circumference, there was no significant connection between gender and the quality of mathematical communication. In the problem on the rate of change, there was a significant connection, favoring girls, between gender and the quality of mathematical communication. The distribution of the quality of written mathematical communication by gender and the results of the χ^2 test are presented in Table 4.

Table 4: Distribution of communication qualities by gender:

	Optimization Problem		Rate of change Problem		Area-circumference Problem	
	Boys	Girls	Boys	Girls	Boys	Girls
Good Quality	26	26	30	40	4	6
Medium Quality	27	21	30	10	32	35
Poor Quality	11	9	9	5	17	14
	$\chi^2(2) = 0.42$[a] N= 120		$X^2(2) = 11.71$* N= 122		$\chi^2(2) = 0.67$[a] N= 108	

[a] n.s. $*p < 0.003$

Discussion and Recommendations

Reform in math education puts emphasis on writing in mathematics as a communication instrument meant to develop mathematical reasoning and for assessment (NCTM, 1989, 2000). This study examined the quality and type of written mathematical communication modes of 164 ninth-grade students from five high schools in southern Israel. The analysis was derived from answers to problems within a regional exam. The first problem was a non-routine optimization problem. The second was a routine rate of change problem in which a requirement for justification was added. The third problem was a non-routine one in geometry, in which one chosen answer had to be justified. We selected open-ended problems, requiring explanation and justification, because they enable students to manifest a variety of levels in mathematical comprehension and communication.

Specifically, in this paper we present connections between gender and (1) correctness of answers, (2) type of mathematical communication that referred to representations and (3) quality of mathematical communication that referred to reasoning.

Some of the results of this study go in line with previous research findings, while some challenge previous ones. In the current study, there were no gender differences found at the achievement level on the entire exam, nor in answer correctness to problems that were investigated and analyzed. These findings go well together with the general trends closing the gender gap (Hyde & Linn, 1988), but contradict other studies where gender differences were indeed found (O'Neil & McPeek, 1993; TIMSS-R, 2000). A possible explanation for this result may be attributed to the nature of the population being studied. The gender gaps reported in the research literature are not only age-dependent, but are also dependent on learning level: Among college bound students, gaps appear already in the 7th grade, while among students who are not college bound, these gaps appear in the 11th grade (Hyde et al, 1990). Students in this study were 9th-graders learning in heterogeneous classes, of which about 40% will be bound for academic studies (college bound) and about 60% for non-academic studies. Based on previous research, it is reasonable to assume that among the majority of the study population, the differences will arise at a later stage.

Surprisingly, in the current research, no connection was found between gender and the type of communication mode, particularly in the verbal mode. Yet in previous research as well as tests, that explicitly required verbalization, a verbal advantage had been found among girls (Amit & Movshovitz-Hadar, 1989; Leder, 1990; Halpern, 1986; Amit, 1988; O'Neil & McPeek, 1993;). It may have been expected that these differences would be reflected in the choice of communication type, and that girls would tend to use verbal representations more than boys. However, in the three problems investigated, no connection was found between gender and type of representation that were chosen as a means of communication.

A possible explanation for this rests on a trend of closing the gap in verbal ability in favor of the girls, to the point where it may be said that it does not exist (Hyde & Linn, 1988). However, if verbal ability indeed still exists, another possible explanation is that despite there being girls with higher verbal ability, they do not tend to use this ability when there is no explicit demand for it. That is, girls do not naturally transfer the relative advantage they have in verbal ability for the area of mathematical learning.

This assumption was further supported by the analysis results, where the girls surpassed the boys in the quality of communication for the rate of change problem. In the optimization and area-circumference problems, no connections were found between the quality of written mathematical communication and gender. The rate of change problem was distinct from the other two, in that it was completely open-ended, without an interim state of multiple-choice or constructed response that was in the others. In order to be ranked into "communication of good quality", the students were asked to answer the problem fully verbally, to reach verbal conclusions and then to justify them well. It is likely that in a problem of this type, where there is an explicit demand for verbalization, the girls' advantage in verbal ability was expressed. Girls produced fuller explanations of their reasoning, and therefore their quality of communication was graded higher. These findings are similar to the ones reported by Lane, Wang & Magone (1996).

Closing Recommendation

This research study challenges a number of conventions regarding the gender differences in mathematics education. No evidence was found in support of previous widespread reports, that at the age of 15, boys' achievement surpasses that of girls. This study reinforces some others that claim the closing of the gender gap. However, the intriguing finding is that no tendency for girls to choose communication modes based on verbal representations was found, although girls have an advantage over boys in verbal ability.

Those who determine education policy, who are interested in advancing girls' achievement in mathematics education, tend to integrate more and more verbal communication opportunities into the tests with the assumption that this will comprise a reformation preference for girls. This assumption, which finds no support in the above study, needs to be considered cautiously, and further research in this direction is required.

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WHAT FACTORS INFLUENCE THE FORMATION OF TEACHERS' PROFESSIONAL COMMUNITIES AND WHY SHOULD WE CARE?

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We describe a large-scale urban school initiative aimed at teachers' professional development with the goal of increasing teachers' mathematics content knowledge and helping them improve their practice. Believing that the formation of professional communities of teachers is crucial in supporting teachers trying to implement changes in their practice, the initiative provided opportunities for the formation of site-based supportive communities. Professional communities developed at some sites and not at others. We describe the factors, institutional and individual, in the formation of teachers' professional communities.

The administration and teachers working in a large urban school district in the United States have embarked on a massive reform of mathematics instruction in the district's schools. This reform includes extensive professional development from a local university. The coursework, site-based support, and daily shared professional development time is intended to facilitate teachers' sustained growth in content knowledge and practice. Somewhat serendipitously, this reform effort has offered extensive opportunities for research on teacher change. One aspect of our research, which we report on here, focused on the nature of the site-based communities in which these teachers participate. What aspects of these teachers' professional lives influenced the development and nature of these professional communities?

THEORETICAL PERSPECTIVE

We see this question as important because we adopt an interpretive approach that views the practices of the teachers as situated within their professional communities and in the schools and school district in which they work. In adopting the situative perspective, we focus on the reflexive nature of the settings of teacher learning and teachers' different kinds of knowing (Cobb & McClain, 2001; Putnam & Borko, 2000). Our working hypothesis is that teachers' practices develop within a community and that local contexts can play a crucial role in teachers' beliefs, knowledge, and conceptions of effective pedagogy (cf. Franke & Kazemi, 2001; Kyriakides, 1998; McLaughlin & Talbert, 2001). We see the development of a teacher's identity as a mathematics teacher as situated within communities. Other studies (Adler, 1996; DeFranco & McGirney-Burelle, 2001; Talbert & Perry, 1994) suggest that departmental and school-based communities in which teachers participated were a fundamental part of their development of identity as teachers of mathematics. Franke & Kazemi (2001) suggest that the existence of a professional community may be even more critical in supporting experienced teachers learning to teach in new ways.

THE SETTING

Our study took place in a large urban school district. The superintendent had developed a broad-based plan to increase student learning in literacy and mathematics, and had in fact been able to obtain large amounts of funding from private agencies to carry out this unique plan. One component of the plan was to assure that all students in grades four through six were taught mathematics by teachers prepared to do so. To begin this work, the eight lowest performing schools in the district were allowed to hire 32 additional teachers to their staffs, with the understanding that these teachers would teach only mathematics (three classes each 90 minutes long), that they would have 60-90 minutes each day together for professional development time, and that they would take university coursework that would help them obtain a deeper understanding and ability to teach appropriate mathematics. For the most part, teachers who sought out these positions did so because they enjoyed teaching mathematics. All teachers were experienced; about half were bilingual in Spanish and had a credential for teaching bilingual children.

The university mathematics specialist professional development program for the teachers consisted of 12 semester hours of coursework designed specifically for the needs of the teachers. The 6 semester hours of mathematics were taught by experienced instructors with master's degrees in mathematics and familiarity with the elementary school mathematics. The 6 hours of mathematical pedagogy courses were taught by two teachers-in-residence (TnR) who were master teachers from the school district with prior experience in high-poverty culturally diverse schools and prior experience as providers of professional development. The pedagogy courses focused on children's thinking about mathematics, teaching second language students, curriculum implementation, and pedagogy.

As a result of their extensive coursework as a cohort, we would expect these teachers to have theoretically shared patterns of discourse developed over time. This 'community of discourse' would have been further supported by their interactions with the two teachers-in-residence who, in addition to teaching courses, provided extensive on-site assistance in pedagogy. The teachers-in-residence each worked with four schools, visiting them each week to plan with teachers, to observe, and to discuss what had been observed. These meetings were often focused on a need that had been identified by the teacher. The period of time designated each day for professional development provided an opportunity for reflection with the teacher-in-residence. Also, the university mathematics instructors worked closely with the teachers-in-residence and frequently visited teachers in their classrooms and assisted them with the many problems inherent in teaching from new textbook materials to poorly prepared students, many with little understanding of English.

In our analysis we use the construct of a teachers' professional community as defined by Secada & Adajian (1997). According to this definition, a *professional community* is a group of people with a shared purpose and a common base of technical knowledge. Its members are accountable to each other in achieving their goals and their practice is open to review in what is known as deprivatization of practice. Secada & Adajian further described what distinguishes *teachers*' professional community by operationalizing this construct along four dimensions: (1) shared sense of purpose, (2) co-ordinated effort to improve students' (mathematics) learning, (3) collaborative professional learning, and (4) collective control over important decisions affecting the mathematics program.

In this report, we focus on two of the eight schools participating in the first year of the mathematics reform initiative because they provide examples of the factors affecting the formation of teachers' professional communities. Our data corpus included field notes of the TnRs and researchers and informal interviews with teachers. It also includes the written reflections of the teachers as they responded to journal prompts and a survey. We documented changes in teachers' mathematical content knowledge as measured by pre- and post-intervention instruments and their self-reporting of confidence levels for various content areas.

THE CASE OF TWO ELEMENTARY SCHOOLS

We will call the two schools Harbor View and Palm. Harbor View had five mathematics teachers in this program, Palm had four. Both schools had a high percentage of second language students. At Harbor View 63.3% were English learners and at Palm 73.3% were English learners. However, Harbor View students were Spanish-speaking (62.5%) whereas the Palm students were more multi-lingual (only 44.5% were Spanish-speaking). Three of five mathematics teachers at Harbor View and two of the four Palm mathematics teachers had teaching credentials for teaching bilingual students. Mathematics teachers at Harbor View averaged 14.9 years of teaching experience; Palm teachers averaged 9.75 years teaching. Four of the five teachers at Harbor View and two of the four teachers at Palm had prior professional development. The professional development time given to these teachers was left for the teachers to structure, except when they were visited by university or school staff.

There were no significant differences in the two groups of teachers' reported confidence levels in teaching different content areas at the beginning of the school year. For the purposes of this discussion, we characterized teachers as weak, sufficient, or strong in mathematics content knowledge. Harbor View had two mathematically strong teachers and three were sufficiently strong in mathematics. Palm had two mathematically strong and two mathematically weak teachers.

We found that, according to Secada's & Adajian's (1997) description, the teachers at Harbor View aligned well along the four dimensions of a teachers' professional community while the teachers at Palm did not. The eight sites varied in the extent to which the teachers formed professional communities, with these two schools representing, to some extent, the extremes. We discuss them along each of the four dimensions.

Shared sense of purpose

Harbor View teachers developed a shared sense of purpose. Their reflections express a collective desire for their students to be prepared for later life. They discussed instilling children with the desire to learn and giving Latino children an opportunity in later-life situations. Their discussions revolved around what they could do as teachers to provide students with the opportunity to learn. At Palm, teachers espoused a goal of having the students understand mathematics. In their reflections, their stated goals are more diverse, such as getting the students to work independently and enabling students to solve problems in novel situations. They did not share common goals nor did they discuss how individual goals were to be met.

Co-ordinated effort to improve students' mathematical learning

The two sites differed in their use of time together. Harbor View teachers made a conscious effort to work together. They arranged their physical office space to facilitate their working together by arranging their desks to face one another in a large circle. The Palm teachers displayed no conscious effort to work together, and arranged their desks so that two of them faced a wall and two faced each other.

The TnR and the Harbor View teachers reported using their professional development time planning together prior to teaching and reporting on successes and failures of their previous classes and what modifications could be made for revisiting the topic. They also looked at and discussed videotapes of each others' teaching. As the year progressed they spent time looking ahead to prioritize topics with standardized tests and with the next year's curriculum in mind. The university mathematics instructor visiting the site reported that the teachers needed less assistance with mathematics content and spent their time discussing teaching mathematics.

In contrast, the teachers at Palm reported using their professional development time to report on successes and failures and learning content from each other or from the university mathematics instructors. They did not discuss teaching solutions to failed lessons. The TnR made several references in her field notes to her efforts to move the professional development time discussions during her visits beyond a discussion of what the students cannot or were not willing to do to what the teachers could change in their practice.

Collaborative professional learning

Collaborative professional learning describes how well and closely the teachers work together to learn about and improve their instructional practices *as related to mathematics*. Hargreaves (1994) distinguished between a collaborative culture and contrived congeniality. Collaborative culture is spontaneous, voluntary, development oriented, pervasive across space and time, and unpredictable. At Harbor View, teachers initiated discussions, informally in smaller groups, and sought opportunities to work together outside of the space of their professional development time to improve their instructional practice. The teachers observed one another's practice, both by visiting each other's classrooms and viewing videotapes with the videotaped teacher, and toward helping one another improve on teaching practices.

Contrived congeniality, as defined by Hargreaves, makes working together a matter of compulsion such as in mandatory peer coaching and when teachers are "persuaded to work together to implement the mandates of others". At Palm, teachers spent little time together except when mandated by the presence of on-site assistance. The discussions at Palm tended not to focus on planning but on sharing and reporting of experiences and on students' shortcomings. They did, however, report having helped each other learning mathematical content. Furthermore, the Palm teachers expressed a view that they were implementing district mandates and reported feeling that their "hands were tied".

Collective control over important decisions affecting the mathematics program

Harbor View teachers made grade-level decisions on content to be taught. They were willing to deviate from the adopted curriculum as needed to meet their shared goal of preparing underachieving students for their later life experiences, including students' success in school in subsequent years.

Palm teachers felt a lack of control over decisions affecting the mathematics program. The vice-principal and the principal both sought out the TnR on separate occasions to express concerns about the mathematics instruction being given to students, and particularly about students' need for direct instruction. The TnR was told that parents and classroom teachers requested that more homework be assigned. In a subsequent meeting with the TnR, the mathematics teachers expressed a desire to give more homework but frustration that they were limited by the "no adaptations, no supplementing" rule of the district mathematics office. Palm teachers predominantly adhered to the curriculum as given until one teacher made the decision to use materials from the same curriculum two grade levels below the grade level of the students she was teaching.

One mathematics teacher at Palm faced the problem of working in the classroom of a teacher who had classroom rules and procedures that were not aligned with those of the mathematics teacher. In October, the TnR and the mathematics teachers met with the classroom teacher and the principal to talk about ways to make the rules and procedures ones that both teachers would find suitable. They had some common ground but were unable to align them. There were constant problems with students--in February a mathematics teacher at Palm reported that a student had threatened to kill her. There was no action taken against this student though, by school district policy, the student should have been immediately suspended.

In sum, Harbor View teachers articulate a shared sense of purpose, demonstrate a co-ordinated effort to improve students' mathematical learning, collaborate to improve their practices, and exercise collective control in decision making. Palm teachers could not.

DISCUSSION & CONCLUSIONS

The shared professional development time was intended to allow for co-operative planning amongst teachers at a site. Hargreaves (1994) described a three-year initiative intended to develop co-operative planning among school staff by providing additional preparation time. Yet, among the findings cited were that "...increased preparation time did not necessarily enhance the association, community, and collegiality among teachers. Time itself was not a sufficient condition for collegiality and community." (p. 131). So, why, under what could be identified as ideal conditions for the formation of communities that support teacher growth, did some groups of teachers form professional communities at their site and others not?

We identified several factors that seem to account for the differences in development and strength of teachers' professional communities at the school sites:

- the relationship the mathematics teachers had with the school administration and other classroom teachers,
- the respect for and access to the knowledge of other mathematics teachers,
- the presence or absence of a teacher leader,
- the mathematical content knowledge of the teachers and their reported comfort level when teaching mathematics, and
- the teachers' familiarity with the culture and language of the student body.

We discuss how these factors are related to the dimensions that distinguish teachers' professional community. One factor we identified as critical in the formation of teachers' professional communities was the nature of the relationship with the school administration and other classroom teachers within a school. In our eight schools, we saw a difference between the relationship between teachers and the administration at schools where the mathematics teachers were drawn from the site versus recruited from another site. Although these two school sites had all new staff, the Harbor View principal hired first, thus taking and being able to choose and select the best combination of teachers applying for the positions, whereas Palm teachers were selected last by their principal. Additionally, the shared professional development time was scheduled for the beginning of the day at Harbor View, reflecting the principal's commitment to the initiative and its success. The professional development time at Palm was shorter because of other responsibilities given to these teachers, and was scheduled for the end of their day. The Palm school administrator's questioning of the mathematics teachers' teaching and the classroom teachers' wanting to assign more mathematics work contributed to the lack of collective control Palm mathematics teachers had over important decisions related to the teaching of mathematics.

A second factor was the respect for and access to the knowledge of other mathematics specialists both at their sites and beyond. "Knowing what others know, what they can do, and how they contribute to an enterprise" is one of Wenger's (1997) indicators of a community of practice. A Harbor View teacher said, "I feel the strongest support [for change in my practice] came from the daily staff development we have at Harbor View. . .We have a level of trust to be able to discuss what went wrong and what could be done to correct a lesson." At Harbor View, each mathematics teacher has an area of expertise and others knew and respected and took advantage of the expertise of others. In contrast, Palm teachers did not share their expertise. Technology was also a factor in the quality of communication with those outside of school site. Harbor View teachers regularly used e-mail and the electronic bulletin board for the mathematics and pedagogy class. Palm teachers did not have access to e-mail and the Web. These constraints affected the degree of collaboration possible.

The presence or absence of a teacher leader was another key factor in the formation of a teaching professional community. The teacher leader can be a positive or a negative force and can play a fundamental role in shaping the shared sense of purpose. In this case, a Harbor View teacher with her experience with the California Mathematics Project led the co-ordinated effort to improve students' mathematical knowledge. No one stood out as a teacher leader at Palm.

The fourth factor we identified was the teachers' content knowledge and reported comfort level around teaching the content. This affected their collective control over decisions. Harbor View teachers felt empowered to alter the curriculum. Palm teachers did not feel that they could. Though two teachers at Palm were strong in mathematics content knowledge, two were quite weak. Their conversation around improving practice was limited by their mathematical understandings. This contributed to their feeling a lack of control over decisions affecting the mathematics program.

Last but not least, familiarity with the culture and language of the student body was a factor in two ways. In Palm's case, a multi-lingual environment and vast cultural differences in the student body affected teachers' co-ordinated efforts and shared sense of purpose, whereas Harbor View teachers shared a vision of empowering Latino children and were able to begin working together toward meeting their goals. Palm teachers struggled with classroom management issues late into the year. As documented in studies of beginning teachers, until issues of classroom management have been resolved a teacher feels as if he or she cannot move on to other considerations, such as design of instruction. Their image of 'self as a teacher' needed resolution before the focus could be turned outward (Kagan, 1992).

Given our hypothesis that teachers' practice develops within community, we examined factors that seemed to affect the development and strength of teachers' professional communities at their sites. These factors were related to both the institutional setting and the individuals. As teachers participated in the larger professional community of mathematics specialists and the school communities, endeavouring to teach in new ways, their settings and changing identities co-contributed to the nature of the site-based professional communities.

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A STUDY ON THE ROLES OF "*OTHERS*" IN LEARNING: FROM THE CASE OF *CHARACTERS METHOD*, A NEW WAY TO LEARN MATHEMATICS

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*In this paper, the social aspect of mathematics learning, especially the concept of "**Others**" is examined. The three types of "Others" in the learning environment are identified, and a framework of learning, "Reflexive Writing Activity", which consciously emphasizes the existence of "others", is presented. Moreover, one type of learning method in this framework, "Characters Method" has put into practice for some implications. In conclusion, Reflexive Writing Activity is not only the effective learning method, but also the aims of mathematics learning which is helpful for both knowledge-understanding and the rich-fruitful learning environment.*

INTRODUCTION

From the script of TIMSS video study, one characteristic of Japanese mathematics lessons is described as follows.

> Problem solving comes first, followed by a time in which students share the solution methods they have generated, and jointly work to develop explicit understandings of the underlying mathematical concepts.

In this script, we can find the importance of "*students' sharing*" or "*jointly work*", which might be the same ideas to social affairs or so-called Cooperative Learning. In the previous studies, Bishop(1985) points out the importance of "social construction of meaning". NCTM(1989) asserts "Mathematics as Communication", and Artzt & Newman(1990) examines "Cooperative Learning". Yackel et al.(1990) indicates the "Student-Student Interaction". It seems that mathematics learning is not just personal affairs any more, but the social activities which are conducted not only by the learner her/himself but with others. In this paper, social aspect of mathematics learning is focused on, and the roles of "*others*" in the learning environment is examined. In order that, one of learning methods, "Characters Method", which makes students much easier to be aware of "*others*", has put into practice, and some implications are discussed.

THEORETICAL FRAMEWORK

Lave & Wenger(1991) regards learning as "increasing participation in communities of practice", or "participation in social practice"(p.49). It says;

> Activities, tasks, functions, and understandings do not exist in isolation; they are part of broader systems of relations in which they have meaning. These systems of relations

arise out of and are reproduced and developed within social communities, which are in part system of relations among persons. The person is defined by as well as defines these relations. (p.53)

Legitimate peripheral participation is far more than just a process of learning on the part of newcomers. It is a reciprocal relation between persons and practice. This means that the move of learners toward full participation in a community of practice does not take place in a static context. The practice itself is in motion. Since activity and the participation of individuals involved in it, their knowledge, and their perspectives are mutually constructive, change is a fundamental property of communities of practice and their activities. (pp.116-117)

From the same standing point, Saeki(1995) mentions the concept of *others*. It defines two kinds of *others*. One is the *others* who relate to the person her/himself just as mere classmates. In this case, the *others*' learning never affect to the person her/himself. Of course they might be friends, but learning mathematics is totally the personal affair and nobody never relate to others' learning. Saeki(1995) describes such learning environment as "**They**-World", because every relation between classmates in their learning is the third personal. She/he is in the same classroom, but what I am learning is not her/his business and what she/he is learning is not my business. In this paper, such kind of *others* are regarded as "***Third Personal Others***". Students tend to be *Third Personal Others* under the environment of knowledge-transmit type classes.

The other concept of "others" is in the environment of situated learning. As Lave & Wenger(1991) mentions; "*Activities, tasks, functions, and understandings arise out of and are reproduced and developed within social communities, which are in part system of relations among persons* ", ***others*** in such learning environment play very important roles to the learning of a person her/himself. Saeki(1995) describes such learning environment as "**You**-World", because every relation between students in their learning is the second personal. What you learn is very important for my learning, and what I learn is also important for your learning. Everyone is in the same learning community and appreciates each other. Student-student interaction, or cooperative learning will be the integral part of such learning community. In this paper, such kind of *others* are regarded as "***Second Personal Others***".

Further more, Hirabayashi & Shigematsu(1987) explains the concept of metacognition on the analogy of person, "Inner Teacher", which means it works as if there is a teacher who warn to the student within the inside of her/himself. This analogy seems to be very effective to understand such internal operations. Not only metacognition but reflective thinking, metaknowledge, or affective issue are also essential internal operations which play important roles in learning mathematics. Since such internal operations can be regarded as a person who is in the inside of the learner her/himself, the term of "***First Personal Other***" is provided in this paper. Another way to say, *First Personal Other* are another self, who watches the learning activities of the self and makes some internal operations.

It is obvious that the ideal learning environment makes classmates to be the *Second Personal Others* rather than the *Third Personal Others*. It is also important to notice that the *Second Personal Others* are not just classmates but the essential source of each student's learning. Moreover, *First Personal Other* should also plays an important role in the ideal learning activities. We need to pay more attention to such internal operations. After all, it is crucial that learners should be conscious of both *Second Personal Others* and *First Personal Other*, and put them to practical use for fruitful mathematics learning.

METHOD

Ninomiya(2001) proposes a framework of *Reflexive Writing Activities*, which is a kind of writing that has reflexive interaction with both learner and class activities. Student writes *Reflexive Writing* from the viewpoint of either her/himself, *Second Personal Others* or *First Personal Other.* Although *Reflexive Writing* is basically the reflection of learner's own learning, she/he can never stops writing just the answer or her/his own solution. Students need to write more. Since students need to reflect their own solutions, some reflective *internal operations* are needed and students are encouraged to show them. However, sometimes it is hard for students to distinguish metacognitive or other internal affairs from cognitive operations, so the concept of *First Personal Other* is introduced to the students as *another self.* Students are encouraged to watch their own learning processes from the view of *another self*, and have *another self* make some comments toward their own learning as if they were told by teacher. Moreover, the ideas or comments from *Second Personal Others* are also important for students' learning. Because such ideas or comments are integral part of learning, students are also encouraged to show them.

In this way, *Reflexive Writing* is formed with (1)student's own answer or solution, (2)ideas or comments from *Second Personal Others*, and (3)comments from *First Personal Other*. The important point is that *Reflexive Writing* never ends with only one single statement. For example, when a student writes her/his own solution, she/he may also add to write some other ideas from *Second Personal Others* afterward, and compare with his/her own. Making comparison, she/he may be aware of something which are metacognitive or other internal affairs', then she/he may be able to add some more comments from *First Personal Other*. Further, because of such comments, she/he can foster her/his own idea, and may get another solution. In such way, writing activity and student's learning may develop their mutual interaction, and the nature of their relation is reflexive. Moreover, since every student is an autonomous participant in each class activity, her/his description is not a copy of the blackboard but her/his own learning process. However, watching the student's *Reflexive Writing*, we can figure out not only how she/he promote her/his own learning, but also how the whole class activity progresses. *Reflexive Writing* is a reproduction of the class activity, which is produced from each student's viewpoint. From this point of view, *Reflexive Writing* is the reflection of each class activity, and the nature of their relation is also reflexive. Summarizing, there are reflexive natures both in the relation between *Reflexive Writing* vs. student's own learning, and *Reflexive Writing* vs. the whole class activity.

One of the best ways to promote *Reflexive Writing Activities* is "*Characters Method*". In this method, some characters such as persons, animals, etc. are used on purpose, in order to let students be aware of the existence of *Second Personal Others* and *First Personal Other.* Each character becomes either student her/himself, *Second Personal Others,* or *First Personal Other* (another self). Because students can directly see the subjects of each comment, or they can distinguish who is talking which comment, "***others***" are easily intervened into the students' own learning environment. During the instruction of *Characters Method,* teacher never force students to use characters, but just show how to use characters on her/his blackboard writing. Most of the students spontaneously imitate their teacher's way, because they love to study in such a way. Although there is not special instruction for "*Others*", students learn the importance of *Others* by themselves. The example of *Reflexive Writing* is shown as Fig.3 in the following paragraph.

RESULT

In this paper, 4^{th} grade class (18 boys and 19 girls), in which students have been encouraged to learn with *Characters Method,* is investigated, and a case of one hour period class is examined. The topic of the class is "folding a piece of paper". The outline of the class is shown in Fig.1, and the description on the blackboard is as in Fig.2. Also, two typical cases of *Reflexive Writing* are shown as in Fig.3 and Fig.4.

In this class period, the time of Reflexive Writing is set up twice, as after the Introduction and after the Wrapping up. After the Introduction, students are encouraged to write the task and the first impression. Then, they start their own investigation, or problem solving. Solving the task, they also write some Reflexive Writing for the reflexive interaction with First Personal Other. When the class discussion starts, they are encouraged to write some Reflexive Writing for the reflexive interaction with Second Personal Others, as well as with First Personal Other. During this period, they are expected to present their own ideas, to discuss each other, to write some Reflexive Writing with both First and Second Personal Others, and think again through their Reflexive Writings. Finally, they are encouraged to write some Final Remarks after the Wrapping up. Although there is very little time to devote only to Reflexive Writing Activity, students may write anything they want at any time during the class, and their Reflexive Writing become so fruitful.

DISCUSSION

First of all, a brief description of the students, Miku and Yoji, is presented, and their Reflexive Writings are investigated based on their learning behaviors.

Introduction (5 min.) presenting the task

Teacher picked up a piece of paper and begin to fold it.

T: How many lines are there when I fold once?
S: One line.
T: How about twice?
S: Three lines.
T: How about 3 times?
S: Seven.
T: 4 times, 5 times, 6 times, 7 times------, so how many lines will be there when I fold 7 times?

Reflexive Writing activity (2 min.)

Writing the perspective of solving the task

Problem solving by each students (10 min.)

Each student try folding a piece of paper.

Confirming the task (6 min.)

T: Don't you have any trouble?
S: It has become harder to fold. It is impossible to fold 7 times.
T: How can we manage it?

Presenting Yoji's idea (6 min.)

# of folding:	1	2	3	4	5	6	7
# of lines:	1	3	7	15	31	63	127
	2	4	8	16	32	64	

Yoji: Adding the next number to the #of lines, we get the next # of lines.Added number have become twice as big as previous one.

Presenting Yuri's idea (4 min.)

T: Where are these added numbers come from?
Yuri: They are the numbers of smell rectangles (rooms) when folfing.

Comparison their ideas (5 min.)

T: How about you, Yoji? Is this same to what you have thought?
Yoji: No, this is different. I have folded the paper till I have got 16. Then, I found the number become twice as big as previous one when I fold one more time.

Presenting Ayu's idea (2 min.)

Ayu: The number of lines is just 1 smaller than the number of the room.

Summarizing students' ideas (5 min.)

T: What are the numbers which have become twice as big?
S: Number of the rooms. The # of the room at the previous stage.
T: # of rooms (previous stage) $\times 2 - 1 =$ # of lines

Wrapping up (3 min.)

Teacher briefly summarized today's class, again.

Reflexive Writing activity (4 min.)

Presenting several student's writing (1min)

Two students read their own brief remarks.

Fig.1 The Outline of the Class

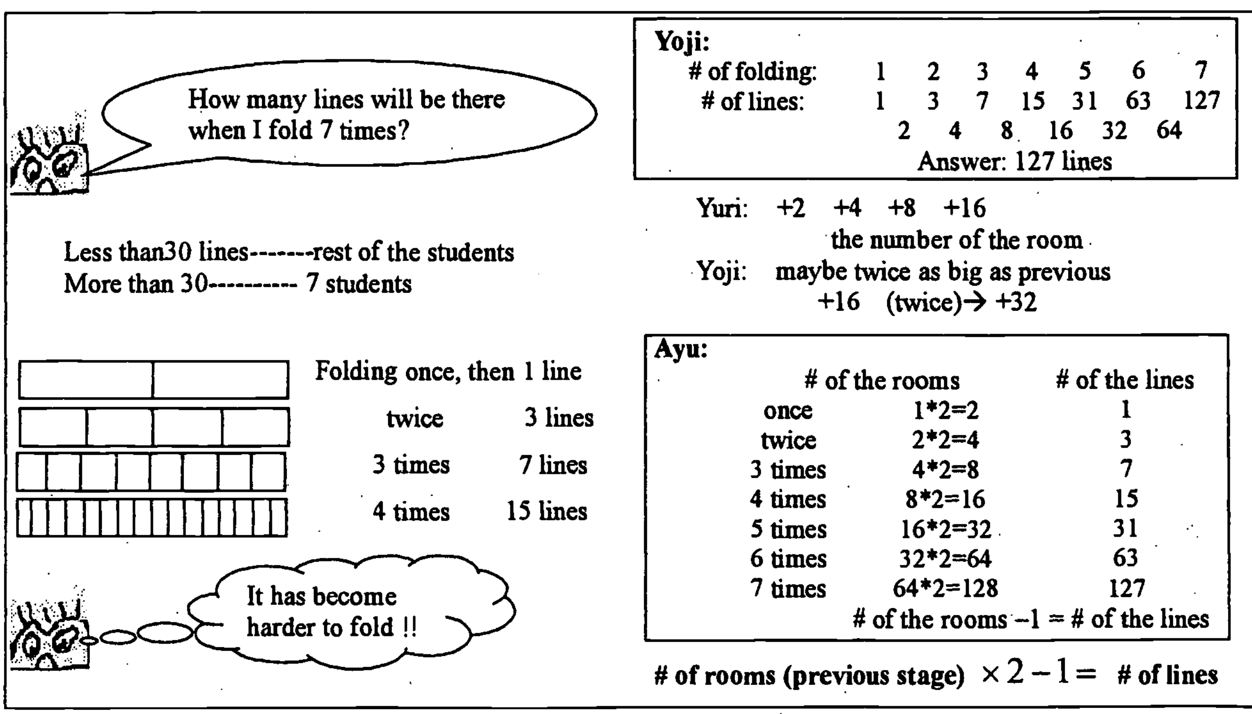

Fig.2 The Description on the Blackboard

1302

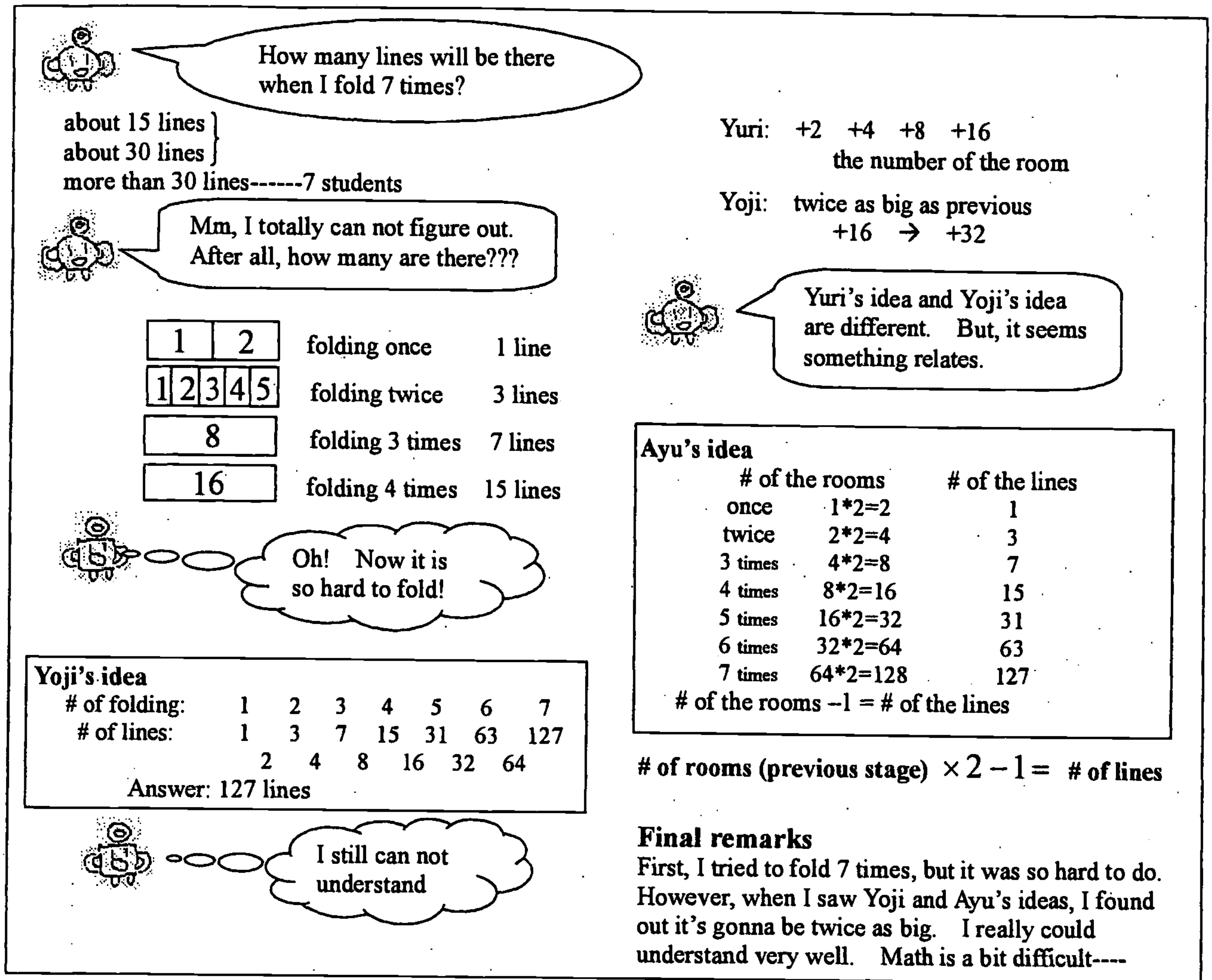

Fig.3 Miku's Reflexive Writing

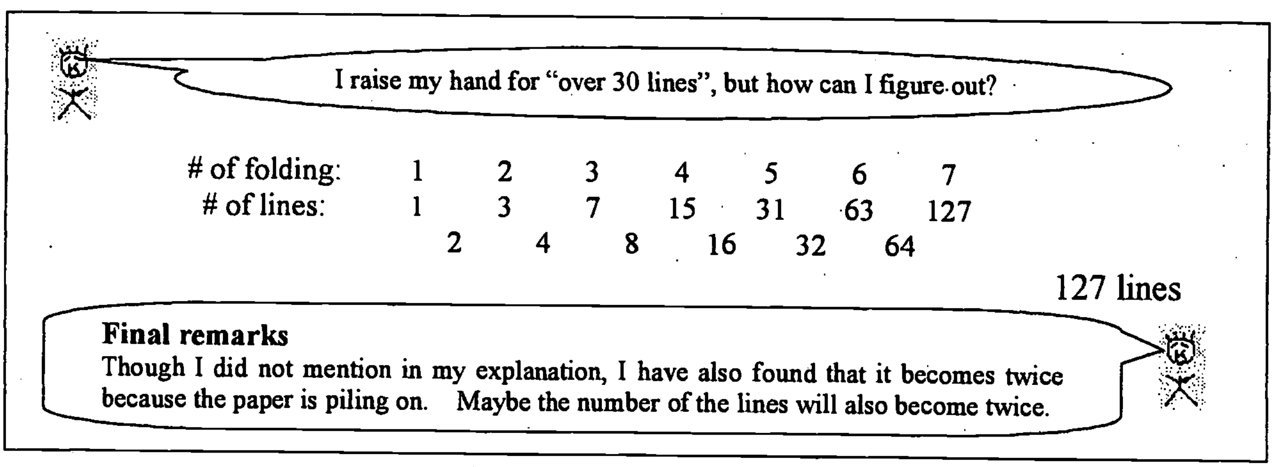

Fig.4 Yoji's Reflexive Writing

Miku is a student whose math ability is not so high. She used to have no confidence for mathematics, but when she knows Reflexive Writing Activity and start learning with this method, she has become to be confident of. Because she surely understood the importance of Reflexive Writing Activity and started learning consciously with this method, she has been able to understand mathematics. Although she sometimes has hard time to understand classes, she has flexible and accepting attitude toward *Others*, and try to learn from every classmate.

In contrast, Yoji is a student whose math ability is very high and has strong confidence in his math ability. However, he believes that the value of learning is only the "correct answer", and he never appreciates the "process" of the problem solving. He also values his rapidity of solving, and believes that the rapidity shows his high quality of math ability. Therefore, he understands neither the importance of Reflexive Writing Activity nor the importance of interactions with classmates. He is a type of the student who rejects "*Others*" in his learning process, because he seems to believe that he is the smartest student in his class and his solution is the best.

The biggest difference between Miku's and Yoji's writing is whether there are ideas or comments from *Second Personal Others*. This is due to the difference of how they appreciate Reflexive Writing Activity. Miku tries to deepen her ideas or understanding through the class discussion, by means of her Reflexive Writing. As mentioned above, her math ability is not high enough to solve the task in her alone. In fact, she made a mistake in her diagram of "folding twice", and her First Personal Other expresses that she could not understand Yoji's idea. However, her Reflexive Writing promotes her understanding, and it indicates the progress of her learning until she finally understands the idea and solves the problem. She could learn from Yoji's, Yuri's, and Ayu's ideas through her own Reflexive descriptions. The comments from First Personal Other arrange her process of learning, and they clarify her own understanding. Finally, her learning becomes so rich and fruitful.

In contrast, Yoji indicates only his own ideas or comments, and never mention *others*; which means there is no comment from Second Personal Others and almost nothing from First Personal Other. Even though there are some characters in his writing, these comments are regarded as his own. Because of his rejecting attitude toward *Others*, he has missed some important ideas, although he was the first student who could get the correct answer in this class. He found the pattern in the number of *lines*, as the difference of sequence is powers of 2. He could get the right answer so quick, but because he values only the "correct answer" and never appreciates the "process", he has not deepened his investigation more. He persisted in his own idea of "the powers of 2". Because this is a very strong strategy for solving this task, he was very satisfied with getting this strategy and never examined the reason why "the powers of 2". Because of his rejecting attitude, he could not accept Yuri's idea and he missed the meaning of *the difference of sequence*, even though there was a description on the blackboard.

Comparing Miku's and Yoji's learning, there might be little difference on understanding the problems; however, the results of their learning are so different. Even though Reflexive Writing Activity is one of the learning methods, Reflexive Writing itself must be regarded also as the aim of math learning. Compare to Yoji, Miku has greater ability to express in Reflexive ways. Although it may seem that their difference is whether they know how to use Reflexive Writing, we ought to regard it also as the "difference of the learning abilities", which means that this is not only the evidence of her power in representation but the evidence of her learning ability. Because Miku has improved on her ability to express in Reflexive ways since she began to learn in Characters Method, her knowledge or understanding of mathematics have been also improved, shown as in Table 1.

	Miku's score	Mean	SD	Miku's Standard score
1st semester	87.5	71.5	22.1	0.72
2nd semester	92.0	79.8	10.7	1.14

Table 1 Miku's Achievement in "Knowledge-Understanding"

Table 1 indicates not only Miku's achievement but also the development of whole class learning environment since they started learning with Characters Method. The change of mean is the improvement of their knowledge-understanding score, whereas the change of standard deviation(SD) implies that those who was low score could improve their score a lot. It might be true that Characters Method is helpful in the improvement of knowledge or understanding, as well as in the rich and fruitful learning environment.

In conclusion, *Reflexive Writing Activity* is not only the effective learning method, but also the aims of mathematics learning which is helpful for both knowledge-understanding and the rich-fruitful learning environment.

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REPRESENTING NUMERICAL DATA: THE INFLUENCE OF SAMPLE SIZE

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Abstract

Twenty secondary school students (in Grades 9 & 11) were given two datsets to represent graphically – one with 10 pieces of numerical data, and one with 30. Students were more likely to represent the large dataset in an organised form than the small dataset. The more mathematically able students found it easier to organise the data than their less able counterparts. Grade level had no effect. Possible explanations for the results are explored and the implications for teaching and the curriculum are discussed.

Introduction

"A picture is worth a thousand words"

This old saying might explain why many people find it worthwhile to use graphs to represent data. However, just as the capacity of a picture to convey the meaning of "a thousand words" depends on the technical ability of the artist, so too, the capacity of a graph to communicate messages depends on the ability of the drawer of the graph to represent the data appropriately. This paper concerns the ability of secondary students to represent numerical data in a graph, and explores factors which assist students to organise the data before drawing the graph. The term *numerical data* refers here to counts or measures such as the number of pencils a student has, whereas *categorical data* refers to categories such as eye colour.

The ability to draw an organised graph is one in a suite of skills expected of all students according to recent curriculum documents. The Australian Numeracy Benchmarks (Curriculum Corporation, 2000) include the ability of primary school students to organise, summarise, and display information in graphs. Similarly, the National Council of Teachers of Mathematics Standards (NCTM, 2000) has highlighted the need for students at all levels to organise and represent data.

Research into children's ability to draw graphs has included the development of a framework for statistical thinking by Jones, Thornton, Langrall, Perry, & Putt (2000) (Framework). The third construct in the Framework - *Representing Data* - is the main issue under consideration in this study, and incorporates constructing representations that exhibit different organisations of the data. As with the other constructs, four levels of thinking have been proposed for this construct. The levels are defined by statements describing students' data displays in terms of the validity of a display when asked to complete a partial graph, and the degree of reorganisation of data when asked to produce a display. The evidence obtained in this study relates to the latter – the degree of reorganisation of data shown in the display.

According to the Framework, at Level 1 the student produces an idiosyncratic display that does not represent the data set. At Level 2, the student produces a display that represents the data but does not attempt to reorganise the data. At Level 3, the student not only produces a display that represents the data but also shows some attempt to reorganise the data. At Level 4, the student produces multiple valid displays, some of which reorganise the data.

Research in the area of students' representation of data is not extensive, however a small number of studies provide some background for this study. Lehrer and Schauble (2000) investigated the process of data organisation with elementary school children in grades 1, 2, 4 and 5. They examined how these children developed and justified models to categorise (by grade level) drawings made by children in the same grade levels as themselves. Their results suggest that, at higher grades, children use more sophisticated strategies for organising data.

Nisbet (1999) examined the representations of categorical data generated by teacher-education students. The majority (99%) drew representations of the data showing some reorganisation of the data. However, the data was categorical, not numerical. Nisbet, Jones, Langrall & Mooney (submitted) analysed children's representations of categorical and numerical data. The study revealed that numerical data was significantly harder for children to organise and represent than categorical data. Children beyond Grade 1 can make connections between organizing and representing data when the data are categorical but generally not when the data are numerical. Whereas 60% exhibited Level 3 thinking with categorical data by reorganising the data, only 20% exhibited Level 3 thinking with numerical data. Two of the three Level-3 thinkers produced a tally table while the third drew a pictograph.

Another study (Nisbet, 2001) found that teacher-education students had similar difficulties with organising numerical data. All could produce an organised graph from categorical data, but only 19% could produce an organised graph from numerical data. For the latter, the majority of students merely drew separate bars for each individual piece of data without organising the data into numerical categories.

Why do more students find it difficult to represent numerical data in an organised way, compared to categorical data? It could be that the way to organise categorical data is obvious, but less obvious for numerical data. Maybe the need for organisation is not perceived to be great when there are only 10 items in the dataset. Perhaps, if the dataset was made larger, then the students would be more likely to see the need to organise the data, and subsequently draw an organised graph based on numerical categories. This proposition was the motivation behind the current study.

This study was therefore designed to test the hypothesis that if students were presented with two data sets, one small (say 10 items) and the other significantly larger (say 30 items), then the students would be more likely to draw an organised graph of the larger dataset. It was decided to conduct this investigation with secondary students as the statistical thinking of this band of the age/grade spectrum had not been investigated by the researcher. Students in Grades 9 and 11 were given the task of drawing two graphs – one for a small data set, and a second for a larger set. The Framework was used to evaluate the graphs produced by the students' organisations and representations of the data, and an interview protocol was employed to ascertain the extent of prompting required before students realised that the larger set needed to be organised before it could be represented meaningfully.

Method

Participants

A sample of 20 students at a suburban secondary school was drawn from Grades 9 and 11 – eight students in Grade 9, six students from the Grade 11 Mathematics A classes, and six students from the Grade 11 Mathematics B classes. At the school, all Grade 9 students do a common mathematics course, but Grade 11 students can select either Mathematics A ("life-skills mathematics") or Mathematics B, a higher-level course which includes algebra and calculus. The latter however is recommended only for students who achieved highly in Grade 10 mathematics. Mathematics B students in Grade 11 are more mathematically able than their Mathematics A counterparts. The students were selected for this study by the Head of Mathematics such that a spread of achievement (high, medium & low) was included in all three groups.

Tasks

Participants were given two tasks. The first required them to draw a graph representing the information in the following scenario.

> *Ten students were asked about the number of novels they read during the term. These are their answers.*
> *NUMBER OF NOVELS: 5, 4, 1, 7, 5, 0, 3, 4, 5, 6*
> *Draw a graph which represents this data.*

After the students had drawn their graphs, they were asked the following questions:

> *(i) What sort of graph did you draw? and*
> *(ii) Why did you draw it that way?*

The second task required the students to draw a graph representing this information.

> *Thirty students were asked about the number of CDs they bought during the year. These are their answers.*
> *NUMBER OF CDs: 2, 4, 2, 7, 5, 0, 3, 4, 5, 1, 5, 4, 1, 7, 5, 0, 3, 4, 5, 6, 3, 4, 8, 6, 3, 2, 3, 4, 5, 6*
> *Draw a graph which represents this data.*

If a student had drawn an organised graph successfully, he/she was asked:

> *(i) What sort of graph did you draw? and*
> *(ii) Why did you draw it that way?*

If the student was experiencing difficulty in working out what to do, a protocol comprising a series of prompts was available, and the extent of prompting necessary for the student to embark on the correct course of action was noted.

> *1. How many students bought no CDs? How many bought 1? Does that help with your graph?*
> *2. Could you fill in a table of values like this?(Show blank table of values as in Table 1)*

3. Could you draw a graph with this table of values? (Show completed table of values.)

Table 1: Prompt No. 2 – Blank table of values (No. of CDs)

No. of CDs									
No. of people									

If those prompts were not sufficient, then the student would be shown a graph of the data set (Figure 1) and asked the following question.

4. Here's a graph drawn by someone else. What does it tell you?

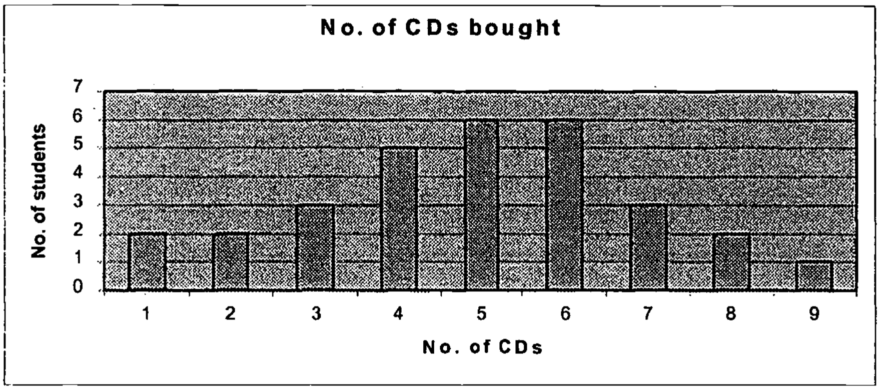

Figure 1: Graph shown in Prompt 4.

Each student was interviewed individually away from any class distractions, and all interviews were audio-taped. The researcher also kept brief notes of the interviews. Sheets of graph paper, rulers, pens and pencils were supplied by the researcher, and all graphs drawn by the students were collected by the researcher for analysis.

Results

Overview of results

1. The effect of size of data set: With the small data set (10 items) most students drew graphs showing no organisation of the data. However, increasing the size of the data set to 30, lead more students to organise the data and draw a representation based on number of CDs rather than individual measures.
2. The effect of mathematics ability: Most Grade 11 students in the higher ability group (Mathematics B) were able to organise and represent the data in an organised way without any prompting. However, only one student in lower ability group (Mathematics A) completed Task 2 without any prompting. There was no similar ability effect for the students in Grade 9.
3. The effect of Grade level: Grade level had no effect on performance at organising and representing numerical data.

Results in detail

Increasing the size of data set lead more students to organise the data and draw a representation based on categories rather than individual measures. In Task 1, which had only 10 items of data the majority of participants (95%) did not organise the data, but without much hesitation, drew a bar graph based on the individual pieces of data – one bar for each person in the dataset, showing how many novels read (Figure 2).

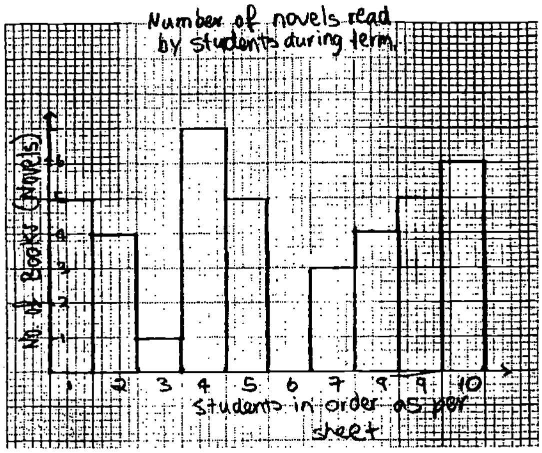

Figure 2: Typical bar graph drawn for data in Task 1

According to the Framework, one student produced a response at Level 1 (Idiosyncratic), 18 students produced responses at Level 2 (Transitional), and one student produced a response at Level 3 (Quantitative). The response patterns were fairly uniform across the three groups of students.

(See Table 2)

Table 2: Numbers of students and level of thinking in Task 1 by group.

Mathematics	Level of thinking (representing data)			Total
group	1 (Idiosyncratic)	2 (Transitional)	3 (Quantitative)	students
Grade 9	1	7	0	8
Grade 11A	0	6	0	6
Grade 11B	0	5	1	6
Total students	1	18	1	20

Only one student organised the data into categories, e.g. the number of novels read. The dependent variable in her graph was the number of people who read that many.

The pattern of responses was quite different for Task 2, which had 30 items of data (number of CDs bought by 30 people). In response to this task, 10 of the participants (50%) organised the data into categories without any prompting and drew graphs based on the how many CDs people bought. The other 10 participants required varying degrees of prompting before they realised how the data could be organised first and then represented graphically. Table 3 shows the number of students requiring prompts to represent the data in Task 2 in an organised fashion.

Table 3: Number of students requiring prompts for Task 2 by group.

Mathematics	Number of prompts			Total
group	0	1	2	students
Grade 9	4	0	4	8
Grade 11A	1	1	4	6
Grade 11B	5	1	0	6
Total students	10	2	8	20

There was no significant effect of mathematics group, nor of Grade level. However, there was a significant difference between the Grade 11 Mathematics A students and the Grade 11 Mathematics B students [$\chi^2(2, N = 20) = 6.67$, $p < .05$) indicating an effect of mathematics ability. Students in Mathematics B produced organised data representations with fewer prompts than students in Mathematics A. Interestingly, there was no similar ability effect for students in Grade 9. Of the four Grade 9 students who produced an organised graph without any prompts, two were A students (high achievement) and two were D students (low achievement). Further, the other four students who required two prompts to assist in organising the data were spread across the achievement spectrum – their results were A, B, C, and E.

Discussion

The effect of dataset size seems to imply that when students see the need to organise data they are more inclined to draw an organised graph. The nature of statistics vis-a-vis data handling comes to the fore when students are faced with a large dataset. Unfortunately, not all students understand how to organise the data in such situations.

What factors come into play in determining how well students are able to organise the data? This study showed that there was an ability effect demonstrated, but it evident in Grade 11 students but not Grade 9 students. The ability effect is quite understandable, but why only Grade 11 students? Either the levels of achievement allocated by the school for the Grade 9 students are not valid measures of their true mathematical ability, or the ability effect is more to do with mathematical processes such as categorising than passing standard mathematics exams. The study concluded that there was no age effect – Grade 11 students did no better than Grade 9 students. So it appears that performance at drawing organised graphs is related to some extent to a general mathematics ability rather than age.

Another factor relevant to some students' difficulties in organising numerical data may be the mathematics curriculum in its various forms – the formal syllabus, the school work program, the intentions of the teacher, and students' experiences in the classroom. Hopefully secondary students would have had extensive experience in collecting, organising and representing data over seven years of primary mathematics. The current state-wide Mathematics syllabus, which has been in official use since 1987, incorporates statistics from Grade 3 onwards. Topics include collecting, organising, and representing data with various forms of graphs – picture graphs, bar graphs, line graphs, histograms and circle graphs. However, most of the examples shown in the support booklets involve categorical data, and very few involve numerical data. With such an emphasis on categorical data in the syllabus, it would be of no surprise if teachers had a similar emphasis, giving students more experience with situations involving categorical data compared to numerical data.

Another syllabus-related issue is the approach taken by teachers in the teaching of statistics. At one end of the spectrum of teaching approaches is a *mechanistic* approach (Ernest, 1989) which implies teaching rules and formulae (e.g., for finding mean) and using data out of a text book. At the other end of the spectrum is a *dynamic* approach (Russell & Friel, 1989) in which students investigate an issue of interest to them by collecting, analysing and representing primary data. It could be that during the students' school careers, most of their teachers have used the mechanistic approach in preference to the dynamic approach. In the mechanistic approach, teachers would have given the students secondary data that are already grouped. Hence the students wouldn't have had to think through the reorganisation step, thus missing a crucial stage of the data-reduction process. The author contends that if more teachers used a dynamic approach, and if an expanded range of techniques for data organisation was specified in the school curriculum and taken up by teachers, then students' would find organising and representing data easier.

A number of other issues did arise from the study. The first issue concerns the participants' choice of direction of axes in the process of constructing the graphs. Many hesitated over which variable was the independent variable and which was the dependent variable, especially in Task 2. In Task 1, most chose "students" as the independent variable and put it on the horizontal axis. Then they chose "the number of novels read" as the dependent variable and put it on the vertical axis, thus producing a series of vertical bars – one bar for each observation. (See Figure 1.) However, in Task 2, when faced with data that had to be grouped according to how many CDs were bought, many participants had difficulty in determining that "the number of CDs" was the independent variable and "the number of students buying that number of CDs" was the dependent variable. This apparent reversal of axes caused some difficulty for the number of students.

Perhaps this difficulty is associated with an inherent complication in the data-reduction process. In collecting the raw data, such as how many CDs people bought during the year, "the number of CDs" is the dependent variable – it *depends* on who you ask! However, when organising the data, "the number of CDs" becomes the dependent variable, and "the number of students" becomes the dependent variable: the number of students varies according to how many CDs they bought! The data have been transformed by the organisation process. Related to this complication was the choice between horizontal or vertical bars in drawing their bar graphs. For most, it was determined by the labels they gave the axes. If they wrote "number of CDs" on the horizontal axis, then the bars drawn were vertical. If, however, they wrote "number of CDs" on the vertical axis, then the bars drawn were horizontal.

Another observed difficulty was coping with zero – zero novels or zero CDs. Firstly, some students did not realise initially that zero was a legitimate piece of data, and that they had to allow for it in their organisation of the data. (In the Task 2 scenario, two people bought zero CDs.) Secondly, in representing the fact that two people bought zero CDs, the bar can't be located at the intersection of the two axes – the vertical axis has to be located away from zero on the x-axis.

The last significant issue noted during the interviews was how helpful the table of values was to most of those who had difficulty initially in working out *how* to organise the data. Seeing a blank table of values seemed to "turn on the light" for them. They immediately knew what to do, filled in the table, and drew the graph.

A number of implications for teaching arise out of the results of this study. Firstly, it is clear that the formal syllabus needs to distinguish between categorical and numerical data, and to place more emphasis on situations involving numerical data. Categorical variables have their place in the early Grades but beyond the middle-primary Grades, they should take a lesser role. Teachers then need give students greater exposure to organising and representing numerical data. Further, teachers need to highlight the process of organising and summarising data as a prerequisite to representing the data. Use of the term *summarising* as a key word could be useful for some students – it conveys one of the purposes of statistics! Secondly, the examples used by teachers should involve large datasets, with sample sizes of the order of 30 rather than 10, so that students are challenged to think about organising the data. Teachers should not waste too much time on drawing graphs based on individual measures; they should get onto organising the data e.g. rank ordering, , grouping, and tabulating. During this procedure it would be beneficial for teachers to offer *scaffolding* to the students (Wood, Bruner & Ross, 1976) by talking through the processes and their reasoning behind the steps.

In conclusion, it is worth reiterating that a *dynamic* approach to teaching statistics would be the means of integrating all of these curriculum and teaching issues to improve students' skills in representing numerical data.

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WHAT DO SUCCESSFUL BLACK SOUTH AFRICAN STUDENTS CONSIDER ENABLED THEM TO BE SUCCESSFUL

Pentecost. M. Nkhoma

Technikon North West

Black South African students have a poor success rate in school leaving mathematics examinations. Attempts have been made to shift teachers' practices from teacher-centredness to learner-centredness, in Black schools, in order to improve the situation. The attempts have failed to yield the desired results. This paper reports on the findings of a study that aimed at learning from students and teachers in Black schools, as to what classroom practices lead to success in school mathematics, in their impoverished context.

INTRODUCTION

An analysis by Blankely (1994) in Howie (1998:27) reveals that only 1 in 312 Black students entering the school system leaves with physical science and mathematics as final year subjects. In comparison, the corresponding figures for the other racial groups are: 1 in 5.2 Whites; 1 in 6.2 Indians, and 1 in 45.9 Coloureds.

Although the above quoted analysis is now several years old, in my experience as an educator in South Africa, there has been little improvement if any in the schooling of Blacks in South Africa. The poor performance of Black students has resulted in their inability to pursue careers that require one to have passed the school leaving mathematics examination. Tertiary institutions in South Africa have not yet found any other acceptable yardstick to use in selecting students. Whether school leaving examinations are a good reflection or not of students' capability, nevertheless, they still continue to be the only instrument used for selection.

IN SEEKING A SOLUTION

Fingers have been pointed in all sorts of directions to demand an account for why Black students do not do well in mathematics. It is acknowledged that through apartheid policies there has been poor human and material resourcing amongst Blacks. One area that gained prominence in mathematics education circles in South Africa, and with the new democratic government, is teachers' classroom practices (more specifically, the methodology of teaching).

Even before the democratically elected government came to power in 1994, attempts had been made to try and improve classroom practice amongst Blacks. This is because what goes on in the classroom is believed to influence greatly the learning gains of students.

When it was impossible to convince the apartheid government to improve the lot of disadvantaged Black communities, Research and Development units in Mathematics and/Science Education based at universities and independent NGOs in the 1980s, intervened in the teaching and learning of mathematics and science. A National Audit by Arnott and Kubheka (1997:54) revealed that, there were at least 36 NGO programmes in Mathematics and Science aimed at teacher up-grading throughout the country. These programmes are essentially designed to meet the need for in-service training of teachers already in the field. The aim of the NGOs is to shift teachers' classroom practices from **teacher-centredness** to **learner-centredness**.

Studies and observations on the effectiveness of the intervention programmes in Black schools have yielded findings including:

1) There were no sustainable changes in the teachers' classroom practices that is, from teacher-centred to learner-centred (Harvey:1999).
2) There are problems if classroom practices are seen not to be directly related to improving results in the existing examinations (Kitto: 1994).

3) What is possible in the developed world with adequate supply of human and material resources may not be possible in Black South Africa (Brodie:1998).
4)In the absence of support it might be wise to limit educational reform to developing the quality of existing teacher-centred teaching methods through improved resourcing, rather than attempting a radical shift in underpinning pedagogy (Harvey:1999).

The findings on the effectiveness of intervention programmes in Black South African schools including those summarised above gave rise to my study.

PURPOSE OF STUDY

The study focussed on Black students, in tertiary institutions, who have been successful in school mathematics, and their teachers. The aim of study was to learn what classroom practices students and teachers find enabling and which lead to success with reference to the South African Curriculum. Secondly, my study attempted to learn from students and teachers how those practices are carried out in a manner that is meaningful to them in their own social contexts, and why it is meaningful to them. This is against a background of intervention programmes that simply labelled teachers' classroom practices as "teacher-centred" therefore bad, and in need of changing. A shift to learner-centred classroom practices by teachers is accepted, by the interventionists, as one way of improving the success rate of Black South African students.

FINDINGS FROM THE STUDY

In the study successful Black students were asked what classroom practices helped (enabled) them and which particular teachers have been effective in teaching them mathematics.

Data gathering was preceded by a questionnaire completed by 480 students from two tertiary institutions. This was followed by individual interviews of a few selected students. From the student interviews, 14 students and 14 teachers were finally selected. Five of the students were taught in Black rural schools whilst the rest (9 students) attended schools in townships. Townships are residences formerly designated for Blacks close to the cities. Generally, they have better infrastructure when compared to the Black rural schools. Following the students' interviews, teachers were first interviewed and then observed teaching.

The study followed a grounded theory approach, where categories and themes emerged from students and teachers own responses. The following practices were cited by students and teachers as enabling: Extra classes; Friendly to us, open to us, created a good environment; Provided extra resources; Working in groups; Preparedness in class; Used Practical examples; Availability; Encouragement / Motivation; Active participation; Language used in class; Homework; Tests; Competition. Other than these practices above, my study also established the challenges that students and teachers face in the teaching and learning of mathematics. This throws some light as to why the suggested practices are critical to students' success. For this paper, I have chosen only to elaborate on two categories namely, extra classes and active participation as a sample of my findings.

Extra classes

The extra classes were conducted at different times depending on the circumstances of teachers and students. For those that took place in the school, only two teachers conducted them in the morning before classes began, one teacher created space in the school timetable to have extra classes. The rest were conducted in the afternoon during study time or free periods. Other extra classes are those conducted on Saturdays and during school holidays. The duration of the classes ranges from one to two hours. The frequency of classes was from 2 to 4 times a week.

Activities in extra classes are different from those in normal classes, between 08:00 and 14:00.The morning sessions are dominated by teaching and some class work whilst the afternoons are dominated by group work and discussion. This description of activities in the different sessions does not mean that there is no discourse in the morning sessions. My class observations showed that the majority of morning sessions had students asking questions and teachers responding. The discourse, in the afternoon sessions, is between teachers and students and mostly between students themselves. Few observed classes had vigorous discussions in morning. This somehow explains the sweeping generalisations about classroom practices in Black schools, because if class observation is done only in the morning, and for a short time, one would simple claim that meaningful dialogue between students and teachers hardly takes place in Black schools. There are however some setbacks to this practice in that not all the students are keen on these extra sessions and that, as my study has documented, they are mainly common with grade 12 students in the year when they write school leaving examinations.

For reasons why the extra classes were critical to success in school mathematics both teachers and students frequently cited the following points, which are generally learner-centred:

- Morning sessions do not allow much time for discourse as they are short and teachers hurry to cover the heavy syllabi
- Students struggle with concepts and extra classes provide students with an opportunity to catch up where they feel left out. Students struggle because they lack some basics which were supposed to have been learnt in earlier grades
- Provides students with an opportunity to teach each other
- Provides an opportunity for discussion and where the teacher simply supervises as students practice sums
- Students are exposed to different approaches to manipulations of mathematics, especially for extra classes conducted during holidays and weekends where the teachers are different. Here, students meet others from different schools. In some cases it is the very same teacher that takes students during the week
- Students have more practice of problems/sums as they learn from each other
- Provides opportunities to handle difficult tasks

From the above discussion it is obvious that opportunities get created in Black schools, with some teachers, where teachers elicit their pupils understanding and then develop concepts from there.

Active participation

Other than giving pupils a lot of tasks, in the form of homework or class work, some teachers promoted active participation by asking students to come to the board during morning sessions. Students volunteer or are randomly asked to come to the board to do sums. This was usually after a homework had been assigned the previous day. There was only one case where coming to the board followed after students had been exposed to a particular mathematical concept.

The chalkboard is usually divided into four parts or less. Whilst the students are writing solutions the teachers took the opportunity to check whether students have done the homework. After students finish the writing of solutions on the chalk board, the teacher together with students discuss the solutions, line by line. One teacher, even showed the allocation of marks for each solution. Where there were problems or misconceptions the teachers in my study embarked on re-teaching by going back to some basic mathematical concept(s) that throw light into the task at hand. In the discussions that ensured as solutions are checked students actively participated. In the mathematical discourse, that occurs, teachers were able to elicit pupils meanings and understanding and then building upon them. An ideal situation would have been where a teacher elicits the meanings and understandings of students in a one to one basis but I found this practice to be one of the closest ways that teachers could get to pupils' meanings and then building upon them, given the large numbers and other social constraints in Black schools.

DISCUSSION

There is no doubt/debate about the significance of learner-centredness. Numerous studies have shown learning gains, on the part of students, where classroom practices leaned towards learner-centredness. However, an argument arises when the status of learner-centredness in teaching and learning is raised to the level of a panacea, especially in Black South African schools. Not only that, but a problem also arises when it is presented as the only way in which teachers' classroom practices may developed (improved) in order to lead Blacks learners to success in South African school mathematics. This ignores, as my data has revealed, that rich experiences for learners can also be provided, by teachers, in a way that enables students to succeed in school mathematics in Black South Africa. These classroom practices that can be casually characterised as teacher-centred and therefore bad for any learning to take place, by those who have not taken a closer look at them.

At the centre of the argument is a lack of an interpretation or clarity, of how the concept of learner-centredness can be meaningfully implemented in Black South African secondary schools. Several studies in South Africa cited earlier have revealed that there have been no sustainable changes in Black teachers' classroom practices when attempts were made to impart the concept of learner-centredness.

Although the same South African studies record teachers showing an interest and appreciation of the concept of leaner-centredness, however, teachers later resort to the same old practices. This dilemma of non-sustainability is not unique to Black South Africa, which can be classified as part of a developing world. A review of existing literature reveals a need, amongst researchers and education practitioners elsewhere, to define and elaborate on the meaning and implications for practice of successful "progressive" classroom practices such as learner-centredness, in the developing world. I must, again, state here that learning gains associated with progressive practices are supported by research in the developed world, for example, Boaler (1997) compared two schools in UK and found that there was an improvement in learning gains amongst students whose teachers implemented practices that leaned towards learner-centredness.

CONCLUSION

It is not beneficial to stereo-type classroom practices into, simply, teacher-centred therefore bad, and learner centred therefore good. Even classroom practices that appear learner-centred, in how they are organised, for example students working in groups, can be a failure depending on what goes on in there. On the other hand rich experiences can be provided in practices that appear teacher-centred. Dewey (1902/1975) in Liping Ma(1999: 153) sums it all up when he states that,

> "...it is easier to see the conditions in their separateness, to insist upon one at the expense of the other, to make antagonists of them, than to discover a reality to which they belong."

My study has begun to lay a base to which learner-centredness and teacher-centredness belong to in the Black South African situation.

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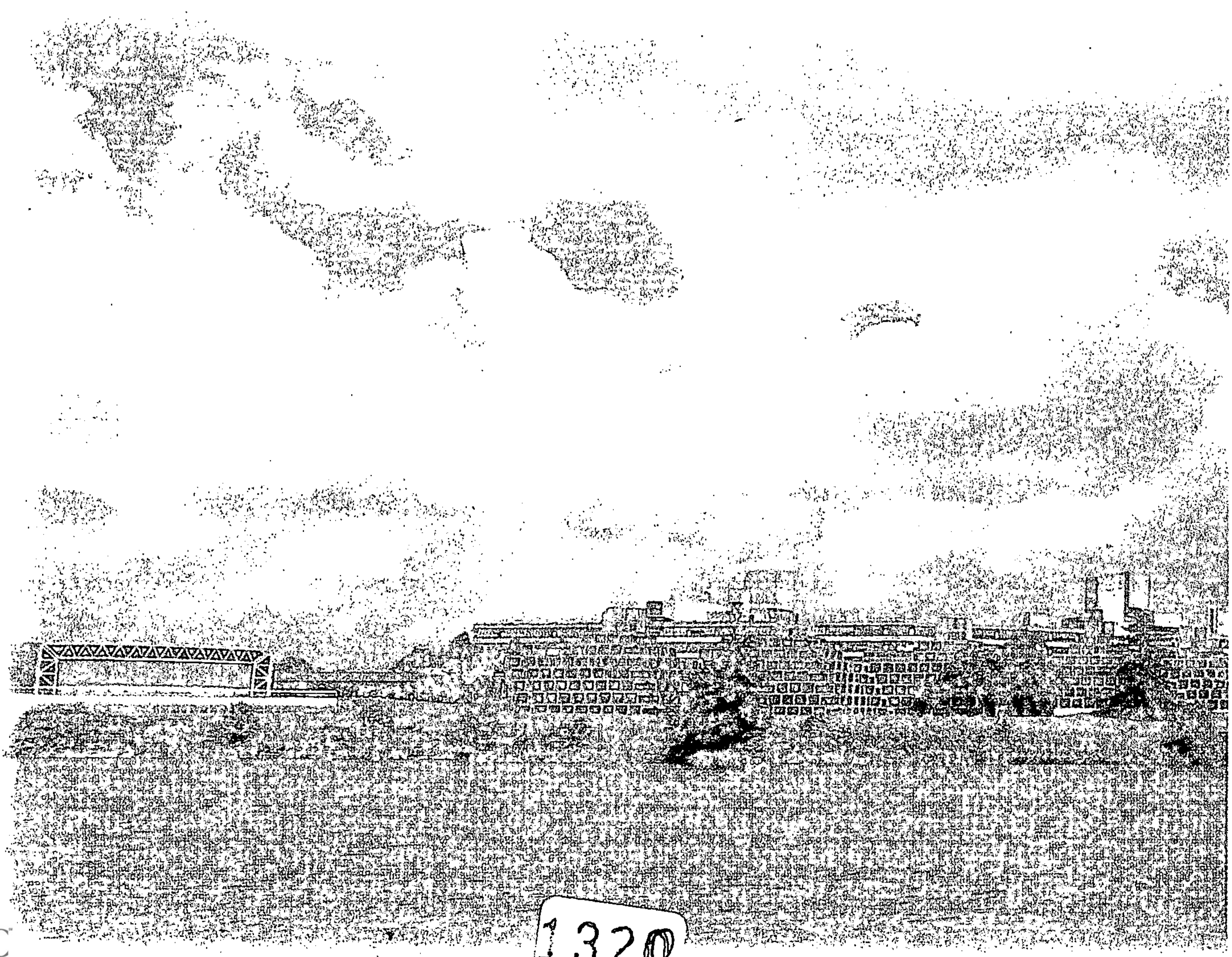

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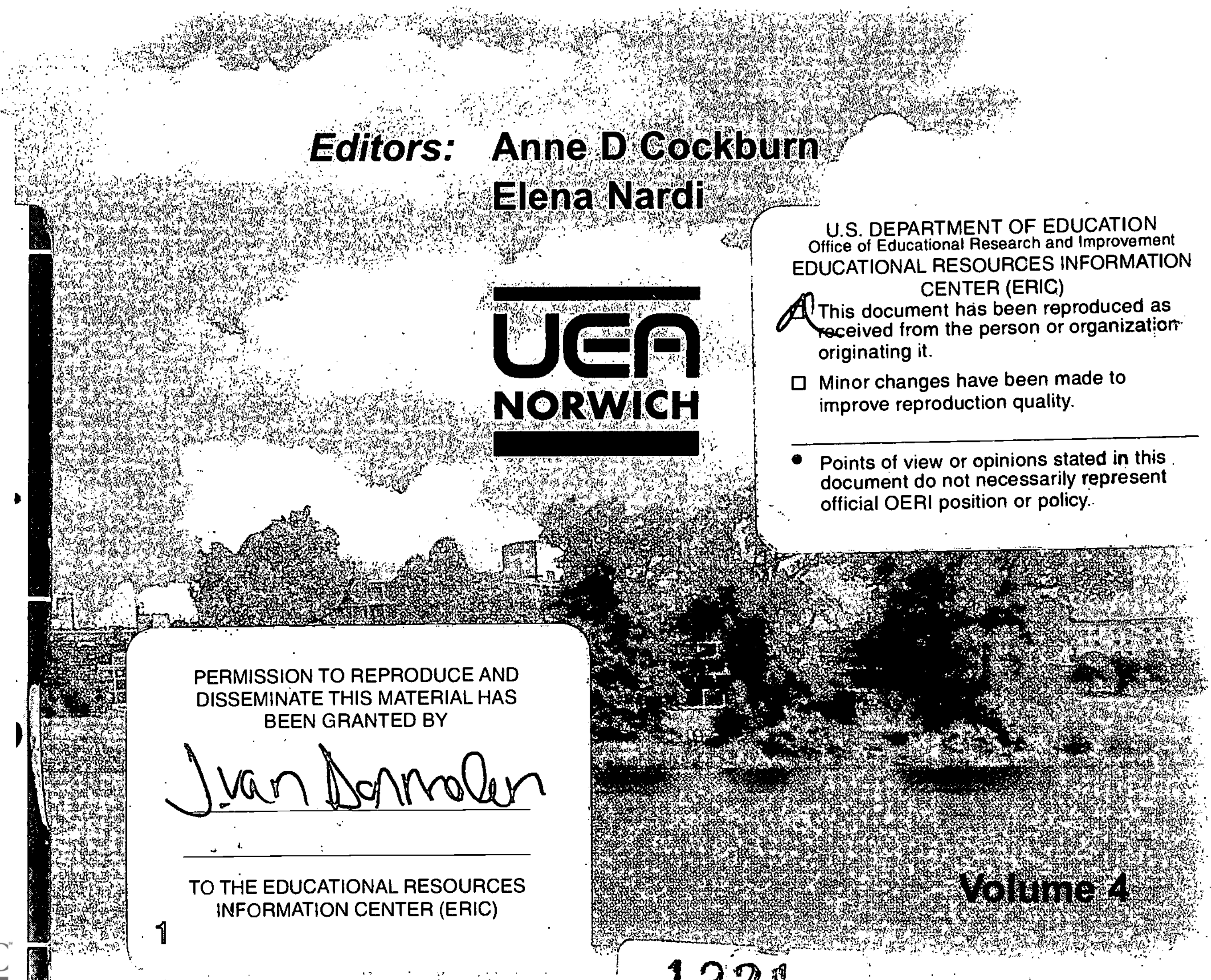

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INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

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21 - 26 July 2002
Norwich, UK

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School of Education and Professional Development
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Proceedings of the 26th Conference of the International group for the Psychology c
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Volume 4

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Fax: + 44 1603 593446
Website: www.uea.ac.uk/edu
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ISBN number 0-9539983-6-3

Design of the PME26 logo: Steve Giles

Printed by the University of East Anglia, Norwich, UK

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RESEARCH REPORTS

THE CONSTRUCTION OF COMMENSURATE FRACTIONS

John Olive

The University of Georgia

This research took place in the context of a 3-year constructivist teaching experiment. The report indicates how Joe and Patricia, two 9-year old children, were able to construct what we are calling a commensurate fractional scheme, whereby composite fractions (e.g. 6/24) of a composite unit can be renamed as fractional quantities in their simplest form (and vice-versa). During the second year of the experiment both children constructed an iterative fractional scheme with which they could produce both common and improper fractions as iterations of a unit fraction. They also constructed an equi-partitioning scheme for composite units (a "partitive division" scheme). Commensurate fractions emerged through a coordination of these two schemes and recursive partitioning.

THE CONSTRUCTIVIST TEACHING EXPERIMENT[1]

The research reported in this paper is part of an on-going retrospective analysis of videotaped data from a three-year constructivist teaching experiment with 12 children (Steffe & Olive, 1990; Steffe, 1998). A team of researchers began working with the children at the beginning of their third-grade and continued through the end of their fifth grade year in a rural elementary school in the southern United States.

More than 600 video-taped teaching episodes were conducted during the three years of the teaching experiment. Pairs of children worked with a teacher/researcher using specially designed computer tools (TIMA) (Olive, 2000a). The major hypothesis to be tested was that children could reorganize their whole number knowledge to build schemes for working with fractional quantities and numbers (the rational numbers of arithmetic) in meaningful ways. This reorganization hypothesis (Olive, 1999) contrasted with the prevailing assumption that whole number knowledge is a "barrier" or "interferes" with rational number knowledge (Behr et al., 1984; Streefland, 1993).

Previously Reported Results.

In my report for PME-25 (Olive, 2000b) I presented evidence for Joe's construction of an *iterative unit fractional scheme* that enabled him to construct common and improper fractions as iterations of a unit fraction of a designated whole. I indicated in that paper how Joe used his multiplicative reasoning with whole numbers and his *equi-partitioning scheme* to establish fractions of composite units.

Subjects of this Report.

During the second half of Joe's second year in the teaching experiment he was partnered with Patricia. We hypothesized that she would make a good partner for Joe based on our analysis of Patricia's first year in the teaching experiment. We hypothesized that she had constructed at least an *Explicitly Nested Number Sequence* (Steffe and Cobb, 1988) and possibly had constructed iterable composite units (Olive, 1999) prior to our work with her in the second year. Patricia was able to establish recursive partitioning operations that led to her construction of composite fractions not in simplest form. Patricia provided evidence of being able to project a partition into the elements of a partitioned stick (using our TIMA: Sticks software – see Figure 1 below) and maintain the relations between the different levels of partitioning. Joe evidenced a similar ability while working with Patricia during the episodes that took place in April and May of year 2. The teacher/researcher working with these two children during year 2 was a doctoral research assistant named Azita.

THE CONSTRUCTION OF COMMENSURATE FRACTIONS

We choose to use the term "commensurate fractions" rather than "equivalent fractions" so as not to imply that the children had constructed equivalence classes for rational numbers, a far more abstract mathematical construct than that with which the children and the researchers were grappling in this experiment. Commensurate fractions are fractional numbers that provide measures for the same quantity.

Provoking Recursive Partitioning by Taking a Fraction of a Fraction.

In a teaching episode that took place in April of the second year, we introduced composition of fractions as a problem situation that might bring forth recursive partitioning and an awareness of the inverse relation between the resulting fraction and the original whole. The context was established using our computer environment TIMA: Sticks to represent pizzas that could be cut into so many slices. In the course of the session, Joe explained how he worked out 1/2 of 1/3 of a pizza as 1/6 of the pizza: "It'll be two, umm two of those (pointing to the half of one third) in each one (pointing to the 3 parts of the pizza stick), and just count them up and it'll be six." This explanation explicitly indicates recursive partitioning: mentally inserting a partition into the results of a prior partition in order to solve a non-partitioning problem. In this same episode Patricia spontaneously named twice 3/4 as 6/4. Joe renamed 9/4 as "2 whole pizzas and a fourth left over." These responses indicated that both children were comfortable with fractions greater than one and could produce them through iteration of non-unit fractions.

Generating a Fractional Number Sequence for Twelfths.

The theme of baking pizzas continued in the next teaching episode that was conducted three days later. Our goal was to provoke the children into thinking about different fractional names for quantities of pizza, based on the number of slices in a pizza. The children chose to make pizza with 12 slices (sticks partitioned into 12 parts). Azita asked them to name all the different fractions of that pizza. Patricia began by naming 1/12, 2/12, etc. all the way to 12/12. Joe spontaneously went beyond the whole (13/12, 14/12 etc.). Patricia then realized that this naming process could go on indefinitely and chimed in with "infinity twelfths"! Twelfths had become units of a fractional number sequence for these two children that was on a par with their whole number sequence. They reasoned with fractions now using their whole number operations. Joe found the stick that was 1/3 of the original stick by finding a 4-stick because "three times 4 is 12." He was, however, able to rename the stick as a 4/12-stick when asked what he meant by "4". Being able to switch back and forth between reasoning with whole numbers and naming the fractional number is further indication that the children's fractional and whole number sequences had achieved similar levels of abstraction.

Later in this same episode Patricia offered the 3-stick as another possible fraction of the original 12-stick. Joe renamed this stick "one fourth" when asked by Azita because "Three four times...to make a 12-stick." In demonstrating what he meant by this statement, Joe attempted to repeat the 3-stick four times to make a stick the same length as the original pizza stick. He inadvertently clicked one extra time when repeating the 3-stick, creating a stick that was five iterations of the 3-stick. Both children named this new stick as 5/4 of the pizza stick. Patricia explained that 3/12 was one fourth based on the classroom procedure she had been taught for reducing fractions:

> We did this in math. What's it called? You reduce. If you reduce 3/12. Like, how many... You divide by 3. Three will go into 3 and 3 will go into 12. So 3 divided by 3 is 1 and 3 divided by 12 is 4. So it's reduced to 1/4. (Patricia drew the numerals on the table with her finger as she talked).

Joe's way of demonstrating that the 3-stick was 1/4 of the 12-stick was to iterate the 3-stick four times to make a stick the same length as the 12-stick (an application of his iterative unit fractional scheme). Repeating the 3-stick one time too many was serendipitous for us as it gave evidence that both children could reason with their commensurate fractions beyond the original whole. Patricia counted in triplets to establish the 15-part stick as 5/4 of the original 12-stick. Later in this same episode, Joe indicated that he might be able to reason with commensurate fractions that were not unit fractions of the original whole. When they were asked to find a twelfths fraction that was the same as 3/4 of the pizza stick, Joe appeared to solve the problem by finding what 3/4 of 12 would be (9) and then looking for the stick that was composed of 9 parts (9 twelfths). It is unclear how Joe found 3/4 of 12. He may have remembered that 1/4 was 3 and then multiplied 3 by 3. If so, this would indicate a decomposition strategy that was lacking in previous episodes.

Finding Commensurate Fractions of a 24-Part Pizza Stick.

In the next teaching episode one week later, we decided to use a 24-part pizza stick with which the children could work. Both children quickly named fractions in terms of twenty-fourths. Joe spontaneously offered a half of the 24-part stick and correctly renamed it as 12/24. The encouraging surprise came when Joe established 1/4 of the 24-part stick by pulling 6 parts out of his 12/24-stick (see Figure 1). This indicates that he regarded each part of the 12-part stick as still being 1/24 of the original stick.

He maintained his 1/4-relation through the intervening half-stick. I hypothesize that he knew implicitly that 1/2 of 1/2 was 1/4 of the original whole.

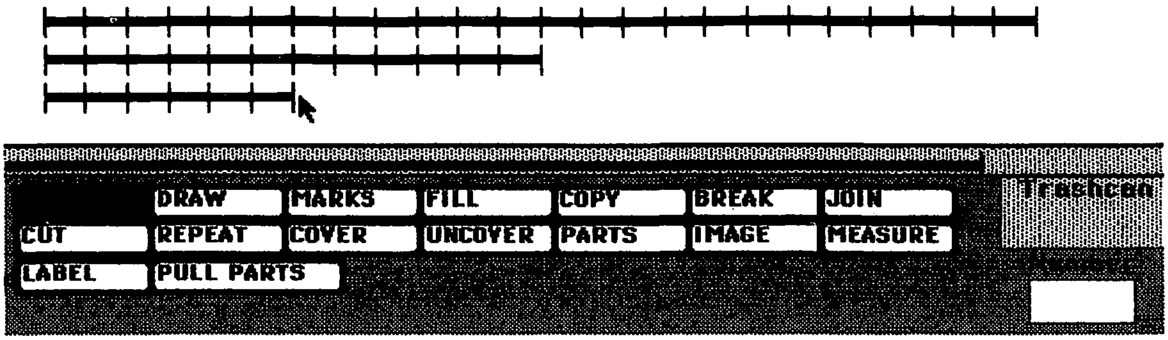

Figure 1: Pulling out 6 parts from a 12/24-stick in TIMA: Sticks

Joe was also able to spontaneously rename the 1/4-stick as 6/24, again indicating that he was relating the 6 parts he pulled out of the 12-part stick back to the original 24-part stick. Patricia agreed with Joe that one fourth was 6/24 because "Six times four is 24." Joe demonstrated the one-fourth relation by iterating the 6/24-stick four times to make a stick the same length as the original 24-part stick, an application of his iterative unit fractional scheme. In the continuation of this episode, however, Joe (J) appears to associate twelfths with his 12-part stick rather than with two parts out of the 24-part stick. The first protocol begins as Azita (A) asks Patricia (P) to think of a fraction.

Protocol I: Establishing 1/12 of a 24-part pizza

A: Now, Patricia, can you think of another you would like to do?

P: A twelfth.

A: You want to do a twelfth.

P: (to herself) A twelfth of 24, let's see.

J: (Looks quizzically at P) I already did that one.

A: Which one?

J: Twelve. It's right there (points to his 12-part stick).

A: (To Joe) What's a twelfth of 24?

J: Twelve (he has his head on his arms). A half of 24.

P: I don't know if it's called that, but this is what I meant.

(Patricia pulls out a 2-part stick from the 24-part stick. Joe sees this and then won't look at the screen. She repeats the 2-part stick 12 times making a stick the same as the 24-part stick and then lines them up one above the other.)

A: So what did you make?

P: I pulled 2 out like this (she pulls 2 parts out of the 24-part stick) and I repeated it 12 times and it made a 24-stick.

J: Yep, yep.

A: So what is that (pointing to the 2-part stick)? What fraction of the whole is that?

P: One twelth of 24.

A: That's 1/12 of 24. That's really good.

(Azita asks Patricia to get rid of the two 24-part sticks and pull the 2-part stick underneath the 12-part and 6-part sticks.)

A: (Pointing to the 2-part stick) Is there another name for that guy, Joe?

J: (After 5 seconds) 2/24.

A: That's really good. Why is it 2/24?

J: There's only 2 of those twenty-fourths, umm stick.

Patricia apparently chose 1/12 as a fraction to make before she knew how much of the 24-part stick she would need. She may have chosen 12 as a possible divisor and then figured out that she would need 2 parts twelve times to make 24. Joe associated a twelfth with his 12-stick rather than as a fraction of the 24-stick. He seemed to be responding to his interpretation of Patricia's problem as to make a 12-stick from the 24-stick, and knew that this would be half of the 24-stick and that he had already done that one. When he saw Patricia pull out the 2-part stick, he realized his mistake and was able to accept the stick as not only 1/12 but also 2/24 of the original stick. In the second Protocol, both Joe and Patricia confirm their construction of a scheme for generating commensurate fractions for unit fractions of a 24-part whole.

Protocol II: Establishing a commensurate unit fractional scheme

(Next problem: Joe pulls an 8-part stick out of the 24-part whole.)

A: What are you trying to make?

J: A third... of 24.

A: Why is it a third?

J: Because, umm, it goes into 24 three times.

A: What fraction of the 24-part whole is that?

J: 8/24.

A: O.K.

J: Or a third.

A: Or a third. Excellent! You're next (to Patricia). Joe, how many 1/24 do we have in 12/24?

J: 12.

A: How many 2/24 do we have in 4/24?

J: Two.

In the first part of this protocol Joe used his *equi-partitioning scheme for composite units* to explain why an 8-stick would be 1/3 of a 24-stick: "Because, umm, it goes into 24 three times." He had partitioned his composite unit 24 into 3 equal parts, with 8 parts each. He also knew that this one third was also 8/24. He used the two names as referring to the same quantity. They were *commensurate fractions* for Joe.

In responding to Azita's question that there would be two 2/24 in 4/24, Joe was reasoning with 24ths as composite units in much the same way as he reasoned multiplicatively with whole numbers. This is another indication that Joe had constructed a *fractional number sequence* (Olive, 2000b) at a level of abstraction similar to his explicitly nested whole number sequence.

While Joe was responding to Azita's final question in the above protocol, Patricia pulled three parts out of the 24-part stick. Joe immediately named this 3-part stick "one eighth." Patricia agreed with Joe and demonstrated the one-eighth relation by repeating the 3-part stick 8 times to make a stick commensurate with the original 24-part stick. Joe offered "3/24" as another name for the fraction "because it's three of those 24ths." Azita then asked Joe if he could make 1/6 of the original stick. Joe was about to pull out 4 parts when Azita asked him to stop. She covered the four parts he had selected and asked the children for another name for 1/6. Both responded with "4/24". Joe commented "I already knew that. If I didn't know that I wouldn't know how many to pull parts." This comment indicates that he was aware of his numerical operations and their results ahead of his actions in the microworld.

The above episode indicates that both children could now produce unit fractions *commensurate* with composite fractional parts of a partitioned whole, as long as the number of partitions was factorable. The episode continued with Azita asking the children to produce non-unit fractions of the 24/24. Protocol III begins with the problem of making 3/4 of the 24-part stick.

Protocol III: Establishing commensurate fractions for common (non-unit) fractions

A: Can you make me 3/4 of the whole?

J & P: Three…fourths.

P: Oh! I know.

J: I've got it! I've got it!

(Patricia is counting along the 8-part stick. She then moves the 6/24-stick to the middle of the screen.)

A: Patricia, do you know the answer?

J: Yes, I know. Tell me the answer first! Tell me the answer (to P).

P: Three of these, three of these (waiving the 6/24-stick around). Six, six, six.

J: Well, what is that? Tell us how long it will be.

P: Oh! Three times 6 -- 18. 18/24.

J: (Claps his hands) Yeah!

A: How did you know that? How did you know that was 18/24?

Both: Because 6 times 3 is 18.

In contrast to the preceding teaching episode just one week prior to this one, both children now had an immediate strategy for finding 3/4 of the 24-part stick. Patricia realized that it would be three of the 6/24-stick because that stick was 1/4 of the 24-

part stick. Her explanation that it will be "Six, six, six" and her response to Joe's request for how long that would be: "Three times 6 -- 18. 18/24." indicate that Patricia could use her iterative unit fractional scheme to generate composite units with commensurate fractions, thus extending her commensurate unit fractional scheme into a scheme for generating fractions commensurate with common fractions. Joe was asking for the length of Patricia's iterated 6-part stick in terms of 24ths. Joe affirmed her result, indicating that he had already worked it out.

In the continuation of this episode, Azita asked Joe to find 5/8 of the whole pizza stick. Joe immediately responded with "15" and verified his response by iterating the 3/24-stick five times. This is further evidence that Joe had extended his commensurate fractional scheme to include common fractions by decomposing the quantity 5/8 into five of 1/8 of 24. It is also evident that he was working with the 3/24 as a composite unit of 3 units. In response to Azita's request for another name for the 5/8, Joe exclaimed "Oh! It's 15/24."

This episode indicates that both children could iterate a composite-unit fraction (6/24) three times to construct a non-unit fraction (18/24) as the quantity 3/4. Further, they had constructed the operations necessary to *transform* a partitive fraction such as 3/24 (3 parts out of 24) into the quantitative relation 1/8 (of 24/24) and use the transformed fraction to create the quantity 5/8 of 24/24 by iterating the 3/24 five times. These are the constitutive operations of a *commensurate fractional scheme*. This episode also indicates that Joe was now able to decompose and recompose non-unit fractions. He saw 5/8 (of 24/24) as 5 of 1/8 of (24/24), and substituted the commensurate fraction 3/24 for the 1/8, thus obtaining 5 of 3/24 to give him 15/24.

DISCUSSION

Rather than interfering with their construction of commensurate fractions, the children's whole number multiplicative schemes were instrumental in the construction of their fractional schemes. The decision (on our part) to use partitioned sticks as the referent unit for the children's fractional reasoning created situations which were assimilated into both their multiplicative schemes and their iterative fractional schemes. This dual assimilation provided the children with powerful ways of operating whereby the same fractional quantities could be named in different ways. Their iterative unit fractional schemes enabled them to interpret 3/4 as three of 1/4; their multiplicative schemes enabled them to find 1/4 of the 24-part stick as 6/24 because "six times four is 24." Combining these two constructions provided the children with the insight that 3/4 was three of 6/24, and that would be 18/24 "Because 6 times 3 is 18." Using their multiplicative operations with whole numbers in this way is very different from Patricia's learned classroom procedure: "Three divided by 3 is one and 3 divided by 12 is 4 – so it's reduced to 1/4."

Unit fractions were now unit items on a par with the children's whole number units and the children could apply all the operations and complex unit structures (units of units of a unit) of their explicitly nested number sequence to these fractional units.

This transformation of their iterative unit fractional schemes is necessary for constructing commensurate fractions that generate quantitative equivalence.

NOTES

[1] The teaching experiment was supported by grant number RED-8954678 from the National Science Foundation (NSF) in the United States. The retrospective analyses are being conducted under grant number REC-9814853, also from the NSF. Both grants are co-directed by Dr. Leslie P. Steffe and Dr. John Olive of the University of Georgia. All results and opinions in this paper are solely those of the author.

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HOW MUCH DOES CABRI DO THE WORK FOR THE STUDENTS?

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ABSTRACT

This report will discuss potentialities and pitfalls of a piece of technology (Cabri-Géomètre), as used by secondary school students in open geometry problems. A case study will illustrate in what way and how much the software can do the work for the students. The conflict between a perceptual-numerical way of using the software and a more theoretical-general one can be solved thanks to the intervention of the teacher, who leads the students to different schemes of use of the technology involved.

INTRODUCTION

"The computer by itself cannot fundamentally change either what is learned or how, and issues of learning and teaching are dependent on more than the simple presence of the computer in the learning situation" (Noss & Hoyles, 1996, p.52). This means that using new technologies in the classroom implies the redefinition of content, of methods and of the role of the teacher (Bottino & Chiappini, 1995).

Our research is focused on what the computer makes possible for mathematical meaning-making. This issue can be investigated from different perspectives, taking into account the interaction between the students and the technology, and the role of the teacher in this interaction. In relation to this, many recent works in the literature highlight a crucial question: what does it mean to say that a technology does the work for the students? (Hershkowitz & Kieran, 2001).

In this paper we will tackle this question as regards one particular piece of technology, that is Cabri-Géomètre. Cabri is a dynamic geometry microworld generally recognised as a powerful technological tool, which provides a useful mediation in many different directions (Arcavi & Hadas, 2000; Arzarello et al., 1998; Gardiner et al., 1999; Healy, 2000; Laborde, 1998; Mariotti, in press; Olivero & Robutti, 2001). This report, starting from the potentialities and pitfalls of using Cabri in open geometry problems (Arsac et al., 1988), first analyses in which way and how much this software can work for the students, and then points out the fundamental role of the teacher in the process of meaning construction.

In doing so, we take on board the perspective of the instrumental approach (Verillion & Rabardel, 1995), as elaborated by (Mariotti, in press). According to this, any technical device has a double interpretation: on the one hand it has been constructed according to a specific knowledge which assures the accomplishment of specific goals, and on the other hand, there is a user who makes his/her own use of the device.

In this perspective, it is important to highlight the distinction between *artefact*, which is "the particular object with its intrinsic characteristics, designed and realised for purpose of accomplishing a particular task" (Mariotti, in press), and *instrument*, that is "the artefact and the modalities of its use, as elaborated by a particular user" (Mariotti, in press). "For a given individual, the artefact at the outset, does not have an instrumental value. It becomes an instrument through a process, or genesis, by the construction of *personal schemes*" (Artigue, 2001, p.4, *italics is ours*), or *schemes of use*, which function as organisers of the activity of the user. As different and co-ordinated schemes of use are successively elaborated, the relationship between user and artefact evolves, in a process called *instrumental genesis*, which can be directed either towards the artefact or towards the subject. In the second case, the genesis "leads to the development or the appropriation of schemes of instrumented action which progressively constitute into techniques which permit an effective response to given tasks" (Artigue, 2001, p.4).

AIM

We consider Cabri as an artefact, which students use to solve a particular task. In using this artefact, an instrumental genesis takes place, and students construct meanings from the activity they carry out with the software. However the meanings constructed may not be the useful ones directed towards the solution of the given problem. "Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artefact is a component) but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher." (Mariotti, in press).

This *evolution* may take different forms, according to the educational aims which are at stake. An evolution from a "mechanistic-algorithmic" way of using the technology, to one that is "led by students' search for meaning" has been observed(Hershkowitz & Kieran, 2001) in another environment, i.e. the symbolic-graphic calculators. In our case, concerning geometry problem solving with Cabri, this evolution can be described in terms of a shift from a *perceptual-numerical* use to a *theoretical-general* one. A *perceptual-numerical* use of the software involves reading either qualitative or quantitative information about the figures on the screen, based on perception only, doing manipulations and taking measurements. A *theoretical-general* use implies a (re)-interpretation of what happens on the screen in terms of a theory. These two ways of approaching the software can be considered two different *schemes of use*.

In the next sections, we will present and discuss a situation of geometry problem solving in Cabri, in which the students use Cabri in the two ways mentioned above and are able to evolve from the one to the other thanks to the teacher's intervention in the small group discussion.

METHODOLOGY

The example we discuss below is taken from a research project which aimed to develop and implement an integrated approach to new technologies in the classroom. Teachers and researchers developed and implemented classroom activities in an attempt to integrate geometry teaching and Cabri at secondary school level (15-18 year old students), with a particular focus on conjecturing and proving. All the classroom teachers involved in the project participate in a Mathematics Education Research Group [1] at the University of Turin as *teachers-researchers* (in the sense of Arzarello & Bartolini Bussi, 1998).

In the classroom sessions the students were presented with an open problem and were asked to work in pairs at the computer (with Cabri), trying to formulate conjectures and proofs, in a 2-hour session. Classroom observations were carried out. Two observers were usually present in the different classrooms and observed one pair of students. The data collected are field-notes and videotapes of the work of this pair, both in Cabri and with paper and pencil.

The observed students, in the protocol presented below, are 15 years old and attend 4 mathematics classes per week. They are medium achievers; they have used Cabri some times over the year before this activity [2].

"WHY AREN'T YOU PARALLEL?"

The problem given to the students is "The axis of a quadrilateral":

You are given a quadrilateral ABCD. Construct the perpendicular bisectors of its sides: a of AB, b of BC, c of CD, d of DA. H is the intersection point of a and b, M of b and c, L of c and d, K of a and d. Investigate how HMLK changes in relation to ABCD. Prove your conjectures.

The students draw a quadrilateral and they drag it until they get a parallelogram.

[...]

542 Paola: (looking at Figure 1) so if ABCD is a parallelogram, HKLM is a parallelogram. **Hypothesis**: AB parallel and congruent to DC and AD parallel and congruent to BC. **Thesis**: KL parallel and congruent to HM, KH parallel and congruent to LM.

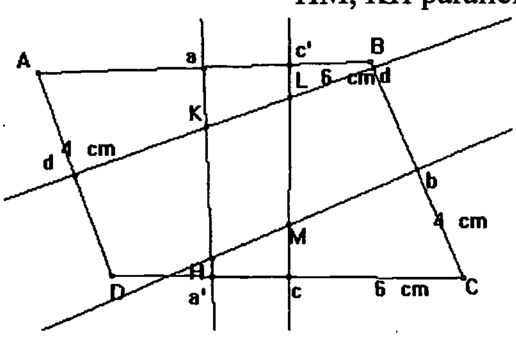

Figure 1

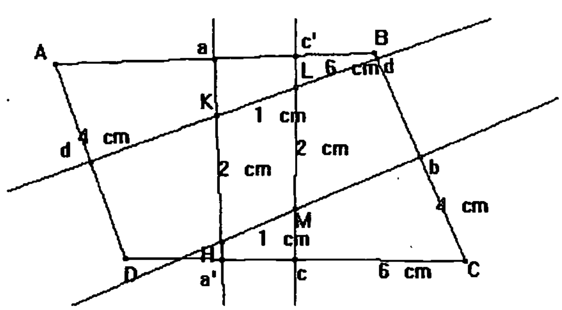

Figure 2

In the first phase, the students observe the Cabri figure while dragging it and formulate a conjecture. The use of Cabri, at this stage, is perceptual, because the students get the idea of the parallelism of some sides of the quadrilaterals only by observing the constructed figure.

543 Paola: but you can see that it's... wrong.

544 Giulia: no, it's right, try with measures!...you must define the segments.

545 Paola takes measurements of the sides of HKLM (Figure 2).

546 Paola: **look at the measurements, it's right! End of the story!** The proof is finished!

At this point the students search for a validation of their conjecture within Cabri, through the use of measures (544) [3]. According to Paola, once they have checked that the measures of the sides of HKLM correspond to (one of) the requirements of the thesis (which was previously formulated), the proof is finished (546). Cabri is now used in a numerical way, as the students focus on some quantities related to the figure. A first scheme of use appears: the perceptual-numerical one.

547 Giulia: well...in order to prove they are congruent we need to prove that the figures... [...]

549 Giulia: are congruent!...of course...and then parallel, I have no idea how we can prove they are parallel! [...]

551 Paola uses 'parallel line' and constructs the line parallel to d and going through M.

552 Paola: they are not parallel...ok, they are not parallel.

553 Paola deletes the line.

554 Paola: **why aren't you parallel?...**let's see if the sides of the initial figure (ABCD) are parallel...I'm worried about that!

Now the students want to prove the conjecture (547-549). The construction of the proof shows a strong link with the Cabri environment. Paola uses the construction of parallel lines in order to check whether HKLM is a parallelogram, that is in order to check if the thesis is correct (551). This test tells her that lines d and b are not parallel. In this case the checking does not support the conjecture previously formulated within Cabri. This provokes Paola to ask "Why aren't you parallel?"[4] (554). She seems to rely on her conjecture and try to understand the answer given by the software, which contradicts what previously seen and formulated. This episode provokes a chain reaction: since the thesis appears to be wrong, the students go and check if the hypothesis is correct, i.e. if ABCD is a parallelogram (554). Their reasoning is mathematically correct, but it is not supported by a correct interpretation of the particular construction in Cabri. The way they use Cabri is perceptual-numerical, not connected with the theory: as a consequence, the conflict grows in the students' minds.

555 Paola draws the line parallel to AD and going through B.

556 Paola: actually...it's parallel...

557 Giulia: well, but the other one as well...

558 Paola draws the line parallel to AB and going through C.

559 Paola: look. **It's not...parallel**

560 Giulia: **then there's nothing to prove...it's all wrong!**

561 Paola: I have no idea!

562 Giulia: neither do I!

563 Paola: those are not parallel, the sides are not parallel...there is still one thing missing: the angles! Let's measure the angles!

564 Paola measures the angles A, B, C and D.

565 Paola: **They are not congruent...it's all wrong!**

566 Paola deletes measures.

They check the parallelism of the opposite sides of ABCD (i.e. the hypothesis), and, by looking at the Cabri drawing, they realise the sides are not parallel, so they conclude that neither the hypothesis holds. For them the end of the story is that "it's all wrong" (560). As a last resort, they check the congruence of the angles of ABCD, that is they check another property that should hold for ABCD to be a parallelogram. Using Cabri in a numerical way, they get the same result: "they are not congruent" (565), because their measures are different. The students trust constructions (parallel sides) and measures (equal angles), so they get to the conclusion that their conjecture must be wrong: "It's all wrong!" (565). The Cabri figure does not correspond to the geometric properties a parallelogram should have from a theoretical point of view. In this case the use of measures and constructions in Cabri works against the validation of the conjecture. At this point the students are stuck, because they are presented with a conflict between a conjecture, discovered in Cabri, and a refutation of this conjecture, discovered in Cabri too. Will the software work for the students and help them to solve the conflict?

WHAT'S NEXT?

The answer to the previous question seems to be negative. The conflict can be solved only by a change in the way of looking at the situation. This change rarely happens if the students are not supported by the teacher, as happens in this case. In fact, the students call the teacher to understand how to deal with a figure that is supposed to be a parallelogram, but it is not a parallelogram in Cabri.

567 Giulia: miss! [...]

575 Teacher: you had a conjecture, didn't you? Your conjecture was: if ABCD is a parallelogram, then HKLM...

576 Giulia: but the problem is that ABCD is not a parallelogram!

577 Teacher: but can you transform ABCD into a parallelogram?

578 Giulia: I need to make the opposite sides parallel to each other!

579 Teacher: can you do that?

580 Paola: yes...but...it is no more the initial figure

581 Giulia: you must construct another figure!

582 Paola: I mean, you need to move this line parallel to this (BC)

583 Teacher: (she takes the mouse) you get a figure which has got parallel sides…this is OK!

584 Paola: but they are not parallel…shall we **do as if they were parallel?**

585 Teacher: yes. You formulated a conjecture saying that **if** ABCD is a parallelogram **then** HKLM is a parallelogram. OK? Now, can you prove that? If you can construct a proof, then what you're saying must be true.

586 Paola: eh…we can try... **if we do as if that was a parallelogram, we'll make it!**

The figure observed by the students is still the same, but it changes status and it is now seen as a *generic object* (in the sense of Balacheff, 1999), which has the mathematical properties of a parallelogram, even if on the screen it does not look like a parallelogram: "shall we do as if they were parallel?" (584). The students need to construct a proof, as the teacher says: "If you can construct a proof, then what you're saying must be true" (585). A new scheme of use is developed through the intervention of the teacher: the use of Cabri in a theoretical-general way. While in the previous excerpts the words written in bold indicated facts (equal, parallel), in this last part of the protocol the words in bold indicate theoretical assumptions (do as if they were parallel; if … then; as if that was a parallelogram). These words are indicators of a change in the scheme of use.

FINAL CONSIDERATIONS

How much did Cabri work for the students? In the first part of the protocol, the students stick to Cabri wanting to understand what is happening: *why aren't you parallel?* (554). They search for a 'unity': they want all the answers gathered in different ways (conjecturing, measuring the sides, constructing parallel lines) be the same and coherent. This solution path presents many similarities with others in the literature, for example with what is shown by (Hershkowitz & Kieran, 2001) in a problem situation involving the use of graphic calculators. The students want to make the technology work for them and do not abandon the technological device even if the figure produced cannot be justified mathematically. They develop a first scheme of use, which relies on a perceptual-numerical interpretation of the software's feedback. The artefact (Cabri) is transformed into an instrument, which however is not the one which serves to accomplish the goal of the problem situation.

The technology works for the students if there is an evolution from a perceptual-numerical use to a theoretical-general one, that is if there is a change in terms of schemes of use. A new instrument needs to be constructed by the students, based on a theoretical-general way of 'reading' the Cabri figures. The role of the teacher is crucial in developing this new scheme of use. On the one hand, the students use the artefact in order to accomplish a task and some meanings emerge; on the other hand the teacher uses the artefact to direct the students in the construction of meanings which are mathematically consistent. Further research will analyse the role of the teacher in the framework of *semiotic mediation* as presented by (Mariotti, in press).

From a didactical point of view, this classroom situation gives the opportunity to introduce the concept of *generic object*, by reproducing in a concrete case what happened in history. In fact, the dialectic between perceptual-numerical and theoretical-general approaches has a long history. For example,Locke considered the generic triangle as the one which is not oblique, nor rectangle, nor equilateral, nor isosceles, nor scalene. Berkeley defined the generic triangle as the one which has no particular properties and he does not assume any particular hypothesis about it. And he specifies that even if the drawing of a particular triangle is used when constructing a proof, its particular properties should not be used in the construction of the proof. Kant solved the debate assuming that the generic triangle is a triangle which is constructed by imagination in the pure intuition, whom a drawing is a representation. From the didactical point of view, it is crucial that the teacher shows how to "see that it is not" (Lolli, p. 139), that is to see that a proof does not depend on the particular graphical representations being used. In this respect, Cabri may support students because of the multitude of representations it produces. However, students need to be able to give those representations the correct theoretical status (as either hypothesis or thesis) in order to fulfil the aim of proving. The software does not work for the students if a particular *scheme of use* is not developed by students, through the interaction with the teacher.

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1 In this group teachers and researchers collaborate in the development, implementation and evaluation of new materials for the classroom.

2 They use Cabri I-7, because this was the version available in the school at the moment of the experiment (2000).

3 The numbers refer to the lines in the transcript.

4 This situation is very similar to other situations reported in the literature, as for example *"So why aren't they meeting?"*, reported by (Hershkowitz & Kieran, 2001), and commented with: "They now pay attention to the hard numerical evidence that there was a single point of intersection. Their inability to match this evidence and the graphical representatives they had obtained became very clear and waited to be resolved and explained" (p.103).

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TEACHING AND LEARNING ABOUT MEASUREMENT: RESPONSES TO THE "COUNT ME INTO MEASUREMENT" PROGRAM

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A program to teach measurement of length, area, volume and mass in grades kindergarten to three is described. This program, "Count Me Into Measurement", was introduced into 38 elementary schools across New South Wales in 2000. The implementation was evaluated and in this paper the teachers' reactions to the program and their assessments of student learning are analysed. Overall, the teacher's reactions were positive; they found the Program easy to use, they particularly liked the expanded lesson notes and they felt the Program improved their teaching of measurement. Teachers' assessments of student learning showed increased knowledge of specific outcomes as a result of using the Program.

INTRODUCTION

Although measurement is a fundamental aspect of everyday mathematics, there is some evidence that measurement concepts are not well understood by students (Bragg & Outhred, 2001; Hart, 1989; Lokan, Ford & Greenwood, 1996; Outhred, 1993). For example, an analysis of elementary school students' knowledge of length measurement found that many students measured from 1 on a ruler or counted marks or spaces; few appeared to perceive the units as linear (Bragg & Outhred, 2001).

Students may not realise that there are major differences between measurement and number concepts, in particular, that measurement involves subdivision of a continuous quantity and is inexact, and that the same quantity can be measured using different units, both informal and formal. For the quantities, length, area and volume, spatial organization of these units is crucial, while for quantities, such as mass, time and temperature, spatial organization is not important.

However, students may consider that measurement is essentially a process of counting or calculating units, and may not realise how the actions, of exhaustively aligning, covering or packing lengths, areas and volumes respectively, are related to subdivision of quantity. The spatial structure of the iteration of identical units for these quantities does not seem to be emphasised in textbooks and curriculum documents, yet it would seem to be central to understanding and linking these two processes of unit enumeration and subdivision of quantity.

Emphasising the spatial structure of the unit iteration may assist students to link concrete, pictorial and symbolic representations of measurement concepts. For example, moving one length unit and marking the end of each move (without gaps or overlap) creates a linear scale analogous to that of a ruler. Although many students can use rulers, few of them understand how the scale is constructed (Bragg & Outhred, 2000) and many of them do not seem to understand linear measurement in such a way that they could generalize the procedures to practical problems. One possible explanation for students' reliance on procedures might be that teachers rely largely on worksheets and textbook exercises that focus on techniques of measuring. Thus, abstract concepts, such as knowledge of linear scales and the relationships between quantities and also measurement units are not developed.

Studies by Outhred and Mitchelmore (2000) and Battista, Clements Arnoff, Battista, and Borrow (1998) have found that the structure of a unit covering of squares is not well understood by students. Their findings suggest that students will not be able to subdivide a region into equal parts if they cannot visualize the unit structure for area—a rectangular array (tessellation) of squares, partitioned into rows and columns with equal numbers of units in each row and in each column. However, teachers may not emphasise the tessellation structure because they conceive area measurement to be a process of covering and counting, rather than of subdividing of a region (Outhred & McPhail, 2000). In their descriptions of teaching area measurement, the teachers did not mention any structural features of a covering (e.g., no gaps or overlaps, congruent units, rows and columns). An emphasis on counting suggests that area is a set of discrete points rather than a region and may hinder students' development of abstract representations of area units, in particular, rows and columns as composite units.

Structural knowledge of unit packings of rectangular containers has also found to be difficult for students, especially linking the structure of packed cubic units to the dimensions of the container (Battista, 1999; Hart, 1989). Early experiences of volume are often limited to filling containers with water (capacity). Such an approach avoids completely the structural problems of packing. The two situations appear to be very different, and students may not see that both involve measurement of volume. As with area, volume measurement seems to involve the construction of composite units (in this case, layers or sections), but the process is more complex because students have to coordinate three dimensions and diagrams cannot show the layer structure clearly. Campbell, Watson, and Collis (1990) found that elementary students counted the number of individual cubes in pictured regular rectangular prisms and that many students counted only the visible cubes.

The Count Me Into Measurement Program

Many teachers may not be familiar with or confident of fundamental measurement processes and how these can be designed as sequences of instructional activities. To assist them, the New South Wales Department of Education and Training (NSW DET) has commissioned the "Count Me Into Measurement" (CMIM) project (Outhred and McPhail, 2000) to develop materials to support the teaching of measurement concepts in the elementary school. The CMIM materials are based on the assumption that students' understanding of the spatial organization of the units develop from practical activities involving estimation and measurement, followed by discussion or recording of the measurement process. The materials provide a conceptual framework to elaborate significant measurement concepts and processes and to provide a sequence of similar conceptual levels for length, area, and volume. The first three levels are:

- **Level 1:** Identification of the attribute includes directly comparing and ordering quantities.
- **Level 2:** Informal measurement includes finding the number of units to cover, pack, or fill a given quantity without overlapping or leaving gaps; knowing that the number of units used gives a measurement of quantity; using these measurements to compare quantities and realizing that a quantity is unchanged if it is rearranged (the principle of conservation)
- **Level 3:** Unit structure includes replicating a single unit to cover, pack, or fill a given quantity, either by drawing or visualizing the unit structure; and realizing that the larger the unit, the fewer units will be needed.

While the first two levels are common to other suggested measurement teaching sequences, the third level does not seem to have been included previously. These levels provide a conceptual sequence for teaching the topics, length, area and volume—students are expected to progress through these levels for each topic but they are not expected to be at the same level in each topic. A sequence for mass that partially fits the framework has also been devised. Formal units are introduced in Level 3 and the Program is currently being extended to include additional levels for Grades 4 to 6. Accompanying the framework are suggested activities that are sufficiently open-ended to promote different approaches to measuring. Teachers can exploit this diversity of approaches to focus student attention on fundamental measurement principles and processes. Example lesson plans for some activities are also provided to scaffold teachers' knowledge of key questions they might ask to develop students' knowledge of measurement processes.

This paper presents the results of an evaluation of the introduction of the first three levels of the CMIM Program in the second half of the year 2000.

METHODOLOGY

Each of the 40 districts in NSW was invited to participate in the initial trial of CMIM and 38 agreed to take part. In each district, the mathematics consultant nominated one teacher who would act as a facilitator for their school to implement the CMIM Program in Kindergarten to Grade 3. These facilitators all attended one training day in Sydney. Facilitators were provided with eleven release days to train and assist a minimum of three teachers from their schools to trial the Program with their students. In addition to teaching ten lessons across two of the four topics (length, area, volume and mass), each teacher was asked to complete an open-ended questionnaire about their responses to the CMIM materials and to assess five students before and after teaching each topic. Facilitators also completed the questionnaire, which sought feedback about:

- The CMIM document itself.
- The facilitators' and teachers' approach to teaching measurement.

The final sample comprised 36 facilitators and 118 teachers, 154 in all. The teachers who returned completed questionnaires included 33 Kindergarten or Year K/1, 44 Year 1 or Year 1/2, 30 Year 2 or Year 2/3, 11 Year 3 or Year 3/4.

The questionnaire responses were scanned for commonly occurring themes, issues and concerns. These responses were formulated into categories and a coding system was devised for each question; these categories were then coded. A response to a question might encompass one or more categories. The written responses were also analysed to identify any insights, issues or suggestions.

The student assessments were complex to analyse and only assessments for which common outcomes had been assessed at both the beginning and end of the implementation were included. Thus, the numbers of students varied for each topic and level. Although, the data for all the classes were combined for a topic, the results for Level 1 mainly included Kindergarten and Grade 1 students while Level 3 mainly comprised students from Grades 2 and 3. Fewer students were assessed on Level 3 than on the earlier levels.

RESULTS AND DISCUSSION

The responses of both facilitators and teachers have been combined in reporting the results in this paper and for simplicity both are referred to as teachers. In general, the facilitators' responses were more positive and detailed than those of the teachers, perhaps because the facilitators were more familiar with the document and had a broader perspective as a result of their professional development role.

The CMIM program

The responses clearly indicated that both facilitators and teachers were satisfied with the CMIM document. The majority of the respondents (57%) commented positively on the organisation (clearly set out, well-organised, sequential) of the document and many (42%) on its ease of use (easy to follow, user friendly). However, simply providing the lesson ideas and the knowledge and strategies that teachers should look for did not seem to be sufficient. Teachers valued the expanded lesson notes that gave examples of questions that teachers might ask—about a quarter of both facilitators and teachers commented on the helpfulness of these: "Within the expanded lesson notes, the section entitled "Questioning, Comments and Discussion" is important as it is through this and teacher knowledge that the children can potentially think on a deeper level and in addition become more aware of, and able to use and understand specific mathematical terminology" (School 17).

The expanded lesson notes also emphasised the mathematical language associated with measurement and some teachers had not realised that their students did not know this, "The language or lack of it was an eye opener and emphasised the necessity of doing these lessons" (School 10). Since many measurement terms (e.g., length, area, row, column) are commonly used in everyday life, teachers may assume that students understand the mathematical meaning of the words. Often misunderstandings of the terms only become apparent when teachers ask students to explain their reasoning.

Requests for more expanded lesson notes and lesson ideas were common responses from teachers, presumably because of time constraints—teachers expressed a need for a "pick-up and teach" document (School 5)—but also because the expanded lesson notes were seen to be a "good model" (School 38) and provided guidance about the mathematical concepts underpinning the activities. About a fifth of the teachers mentioned that using the document had encouraged them to reflect on how to teach measurement or had clarified their own concepts of measurement: "Yes, it gave measurement a sequence. Something that I didn't understand." (School 38); "Yes, taught concepts I'm not confident in teaching. Made me use the correct vocabulary" (School 20).

Even teachers who are confident about teaching measurement may have limited knowledge of basic measurement processes. In the small-scale study mentioned earlier (Outhred & McPhail, 2000) in which teachers described their teaching of area measurement, even when teachers were specifically asked about skills and understandings, they did not mention important area concepts. Most teachers construed area to be a covering. Few used the term "surface" and only one mentioned that the same units should be used for comparison and that an appropriate sized unit should be considered.

The main change that teachers recommended to improve the CMIM document was to clarify the student assessment. Teachers suggested the inclusion of either proformas or work samples to provide explicit assessment guidelines. They thought that annotated work samples would help to clarify differences between the levels, as well emphasising what they should focus on in their assessments.

Approaches to teaching measurement

Almost all respondents (96%) agreed that the document had helped them to teach measurement. The majority of responses reinforced comments that had been made about the strengths of the document—the teachers approved of structured, sequenced material that provided information about what they were expected to teach and that students were expected to learn. For example, "The clear progression of activities, the activities themselves, the framework seems so logical although I'd never really thought out those divisions of learning." (School 34); "Yes, it made me realise that I take some things for granted. For example, I may have assumed that a particular child knew something that they didn't yet, or I may have underestimated some children's abilities and been surprised with their response or interpretation" (School 23).

Student engagement would seem to be fundamental to learning and many teachers (21%) mentioned student interest in the activities, often associated with the use of concrete materials. About a third of all respondents referred positively to the "hands-on" activities—although some teachers said they taught measurement using such an approach, others did not. The responses of students to the CMIM activities appeared to prompt several teachers to reconsider their methods of teaching while others mentioned that it increased their confidence in teaching measurement: "It made the class teacher reflect on the present style of teaching concepts and gave them practical examples of how to achieve new concepts in a motivational way." (School 27); "Yes, it gave us a different view of teaching measurement, more learning by doing with concrete materials and students were accountable for learning records." (School 21)

A number of teachers (14%) commented that they liked the Program's emphasis on students discussing, drawing and writing explanations of their mathematical thinking. Such recording appeared new to many teachers and they seemed to find this feature of the document informative: "I especially liked the idea of students recording what they had learnt so (1) they thought about it and (2) I was aware of their strengths and needs" (School 21). One teacher's assessment of student writing made her aware that "the lower ability children were always focussing on irrelevant factors when writing about their experience. Eg It was fun, I used pink counters, Pink is my favourite colour" (School 15). Teachers who are aware of the problems of such students would seem to be in a good position to offer assistance.

The main concerns raised by teachers involved resource and time issues because the activities were designed as small group ones in which the students undertook practical tasks using a variety of equipment. As one teacher commented "practical groupings could be difficult without assistance if I had to teach a larger group as there is a lot of equipment involved and many children need guidance" (School 3). Others were concerned about the preparation of resources; "My only concern would be the time factor. Whilst these lessons were great, in reality if a teacher had to prepare all the resources before teaching especially volume/mass strands I don't think many would or could afford the time" (School 1).

The time and resource issues are important ones to address because many teachers will not continue to incorporate practical measurement activities into their classrooms if they find the management and organization of groups and materials too demanding.

Student learning

The results indicated a large change in student achievement over the period of the project. The only length outcome that more than 50% of students had mastered at the beginning of the study was to directly compare and order lengths (Level 1); only about 10% could replicate a given unit to measure a line (Level 3). By the end of the project more than 80% of students appeared to have mastered the first two levels, that is, they could measure, compare and order lengths using informal units, as well as by direct comparison. The results for Level 3 had also increased dramatically—about 50% of the students assessed at Level 3 were considered to have mastered the outcomes for this level.

The results for area indicated that area concepts were much less well known than those for length. Only about 20% of the students assessed could directly compare and order areas, as well as systematically cover areas with identical units and compare them. The results for Level 3 were similar to those for length; few students appeared to have any knowledge of the structure of a rectangular tessellation at the beginning of the project. After the CMIM lessons the assessments show increased performance on all levels with teachers reporting that about half the students perceived the structure of a unit tessellation in terms of rows and columns. The results for volume were consistent with length and area—teacher reports of students' learning increased markedly.

However, these results would have to interpreted with caution, as the teachers were likely to be biased towards finding an improvement in student learning because they had spent much time and effort in implementing the program. Nevertheless, the data on student learning provided some evidence that the sequencing of the levels and activities was sound.

CONCLUSION

Overall, the first implementation of the CMIM Program appeared to have been successful. Teachers approved of the Program and felt that it assisted their teaching of measurement. They liked the open-ended, practical nature of the activities although there was evidence that they may not continue such teaching without support because of the time and effort that was required in terms of resources, time and organization.

A number of teachers reported that they did not feel confident about the measurement concepts they were expected to teach, nor the sequence in which these concepts should be taught. Many teachers appeared to rely on commonly published textbooks for teaching measurement. In effect, publishers are interpreting the mathematics curriculum for teachers. In textbooks, concepts are often fragmented, for example, a worksheet on length may be followed by ones on area, mass, and volume, interspersed with pages presenting number and space concepts. Several teachers commented that they found that teaching a sequence of five lessons on one topic as part the CMIM Program consolidated students' learning.

Teachers may not realise the extent to which the complexity of measurement increases with the dimensionality of the units. Anecdotal comments on the training day indicated that some teachers considered volume concepts to be easier than length or area ones, suggesting that these teachers may have focussed on filling containers with liquid (capacity) rather than volume as the packing of an interior space, or as displacement. There was also some evidence from work samples that were sent back with the questionnaires that a few teachers may not have realised the importance of constructing the unit iteration to reinforce the structural properties of measurement to students.

Building connections between a linear scale, two-dimensional array structure, and three-dimensional packing are important if students are to understand relationships among measurement concepts. These concepts are also closely linked to repeated addition and array multiplication. The difficulty of applying multiplication skills in a meaningful way has been documented for concepts of area and volume (Hart, 1989).

She found that many students continue until well into secondary school to determine both area and volume measures by counting, rather than by multiplying. Counting does not generalise to fractional dimensions or to formulae. Students require instructional tasks that assist them to construct links between the measurement unit and the spatial structure of the unit iteration, as well as to repeated addition and multiplication. Teachers' knowledge of measurement processes may also need to be developed so that they realise the fundamental differences between the enumeration of discrete and continuous quantities.

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AN ACTIVITY FOR CONSTRUCTING A DEFINITION

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The notion of definition is central in mathematics. We notice differences and analogies between axiomatic definitions in education (which often come at the beginning of a lesson) and definitions in mathematical research (which come generally at the end of a research process). The core of this paper concerns the activity for constructing a definition (called definition-construction). We aim to study a situation of definition-construction and to bring up conclusions about the nature and functions of definitions (constructed by students).

THEORETICAL FRAMEWORK AND EXISTING RESEARCH

In the usual mathematical activity of a researcher, a dialectic exists between the concept in construction as well as its definition which is constructed, too. According to Kahane (1999, p11):

> la transposition didactique est de règle en sciences: (...) l'exposé d'un sujet prend pour point de départ un aboutissement historique et réécrit l'histoire à l'envers. En mathématique, ce point de départ est une définition. [1]

Besides, Lakatos recommends a heuristic approach, in opposition to the usual deductivist one and underlines the dialectic between the construction of definition and the construction of concept within the framework of problems resolution. He develops two concepts in his thesis (1961): *zero-definitions* and *proof-generated definitions*. Thereby two functions of definition appear in a problem-situation: on one hand, *zero-definitions*, so called alluding to their place at the start of the investigation, are initial and tentative definitions and "the different choices of zero-definitions do not affect the domain of the proof" (p.71,ibid). They must be a little vague, and their heuristic rules correspond to Popper's remark [2]. This notion of *zero-definition* does not thwart Vygotsky's idea about language and verbal definitions; indeed Vygotsky (1962) studies the capacity to use language as a problem-solving tool and accounts for the importance of the naming process: the word guides and determines the course of action. Furthermore, Vergnaud (1991) describes three functions of language: communication, representation and contribution to the conceptualisation. On the other hand, Lakatos underlines the importance of the relation between proof and definition, and presents a new concept: the proof-generated concepts (i.e. generated by proof: it consists in establishing the domain of validity of a primitive conjecture) and their definitions, the *proof-generated definitions*. According to Pimm (1993), this notion

> ... seems particularly problematic in terms of teaching mathematics, because of needing to perceive the definition as a tool custom-made to do a particular job that cannot be known by those trying to learn it, certainly not with an order of presentation that seems to require definitions to come first (p.272).

We keep this idea of concept generated by proof as a possible situation for constructing a definition. We know that a lot of difficulties exist in teaching about understanding a concept with a formal definition. Annie and John Selden [3] focus their attention on the role of examining examples and non-examples in order to help students to understand definitions and ask how to help them to understand newly-defined concepts. Furthermore, we retain Vinner's hypothesis (1991,p79), who notes that "the ability to construct a formal definition is for us a possible indication of deep understanding" and explains the "scaffolding metaphor" which presents the role of a definition as a moment of concept formation. Vinner assumes that "to acquire a concept means to form a *concept image* for it (...) but the moment the image is formed, the definition becomes dispensable" (p.69,ibid) and proposes some interplay between definition and image. We suppose that *concept image* and *concept definition* are necessary to analyze an activity of *definition-construction*. Moreover, we share Vergnaud's idea (1991,p.135): "un concept ne peut être réduit à sa définition; c'est à travers des situations et des problèmes à résoudre qu'un concept acquiert du sens" [4]

An activity for constructing a definition

For the purpose of this article, to construct a definition is an activity which could concern three types of problems. P1 = the request of a definition (starting from given examples and counterexamples); P2 = a problem-situation whose resolution passes by the construction of an object (or a concept) and its definition (alluded to by Lakatos); P3= a situation of modelling. We will retain several aspects concerning the definition of a mathematical object : characterization of this object, naming process, relations between definition and proof.

About the naming process, two aspects emerge : the importance of the denomination (when the mathematical concept is of interest in that it can be used usefully) and, on the other hand, a denomination allows two mathematicians to speak about the same thing. And the expression "good definition" is frequently used, it means "precise definition" according to the mathematical accuracy and the arbitrary character of definitions, as they are presented in axiomatic form.

To consider an activity of *definition-construction* requires a change of point of view (relatively to handbooks) which consists in accepting the provisional status of a definition, the multiplicity of the definitions of the same object (thus equivalent definitions), the dialectic between definition and properties and the operational aspect of a definition in a proof. We will explain our point of view about *definition-construction* and possible analysis with the presentation of an activity (type P1).

PRESENTATION OF THE ACTIVITY

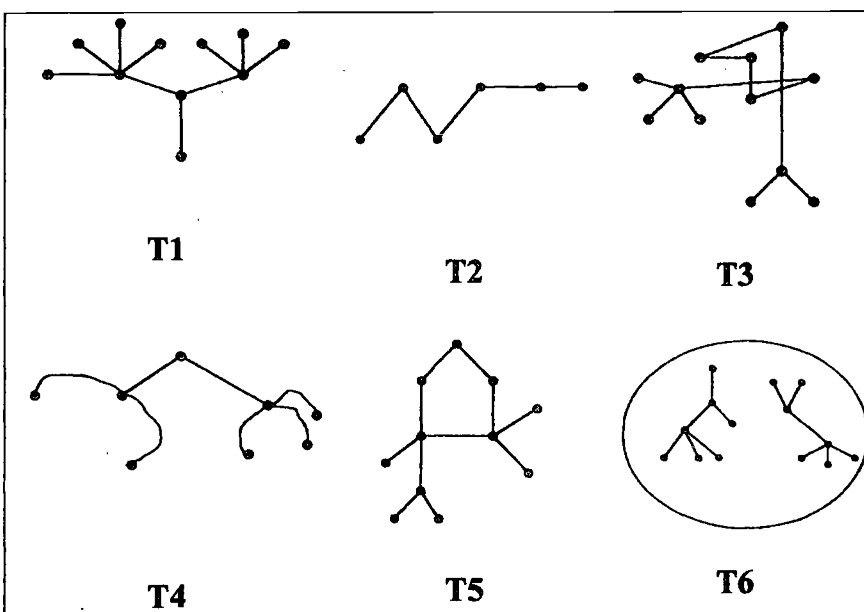
For the activity of *definition-construction* the mathematical object that was selected was the tree, for several reasons: in France, it is a familiar object in teaching, used in the handbooks as a tool of representation (it is recommended by the official secondary syllabus), however it is absent as a mathematical object, hence there is no institutionalised definition (before University). An experimentation by Balmand (2001) proved that definitions and properties of this mathematical object are unknown to French teachers (discrete mathematics are not learnt in France before University). However, the tree is a "natural tool" of representation and resolution, which can be used in restricted fields (combinatory, probabilities).

From a mathematical point of view, it is an accessible object (by its representations) but hard to theorize and to define owing to the difficult concept of connectivity. Let us notice that the students did know neither the word 'connected' nor the concept 'connectivity' before our activity. Moreover, the tree has equivalent definitions which are different in nature. We don't claim to construct all the aspects of the mathematical concept 'tree' but some of them, and we assume that the construction of a mathematical concept is required for knowing and mastering this concept.

We chose to call it "thingummy" (a neutral name, whose semantic meaning is attached to nothing) to avoid students connecting too quickly with the meaning of tree as it is used in probability (otherwise we assume that it would stall the situation). Etymologically, to define means to delimit (one defines an object compared to another in order to find out a criteria of recognition) and we believe the construction of a definition is possible starting from examples and counterexamples.

Presentation

First, 4 examples and 2 counterexamples were proposed to students, with the question



1- How could you define the mathematical object 'thingummy', knowing that :

T1,T2,T3 and T4 are representations of 'thingummy' and T5,T6 are not representations of it ?

Figure 1 : examples and counterexamples of "thingummy" (first question)

We think that a definition is not a finished product in itself, so we proposed a second question (when the students think they have done with the definition): "**2-** Exercise : Let G be a graph (i.e. a collection of dots and lines between two dots) connected (i.e. in only one piece). Prove that G admits a spanning tree (i.e. a tree with same vertex set than G)" [5]

Analysis

There are a lot of mathematical definitions of "tree": let G be a graph on n vertices. Then G is a tree if and only if one of the following equivalent assertion holds:

Definitions	Nature of the definition
Def1 - G is connected without cycle.	Perceptive
Def2- Between any two vertices of G, there exists a unique path.	Perceptive
Def3- G has no cycle and n-1 edges	combinative (counting)
Def4- G is a connected graph with n-1 edges	combinative (counting)
Def5- G has no cycle and if we add a new edge then we create a unique cycle (which means that G is a maximal acyclic graph).	Dynamical definition: requires action on the object
Def6- G is connected and if we remove any vertex v then G-v is disconnected (which means that G is a minimal connected graph).	Dynamical definition
Def7- A tree is a vertex (basis) or a tree T for which we add a new vertex adjacent to only one vertex of T (induction step).	Constructive, inductive and dynamical definition

Table 1 : mathematical definitions of 'tree'

The exercise has two aims. First, it allows students to return to the definition, and second, to use (and perhaps reconstruct) the definition. To achieve the proof means to use a definition to plan an overall structure of the proof [6]. Two definitions of a tree are particularly "effective" to write the proof. We propose the main ideas of these proofs:

Resolution with <u>inductive definition</u> : we search the tree directly and we avoid cycles ; we choose a vertex and we connect it (by edges) to its neighbours which are not yet connected (it is possible because the graph is connected) and so one.

Resolution with <u>connected minimal definition</u> : the idea is to remove some edges of the given graph in order to obtain a tree. We search if an edge exists of which suppression does not disconnected the graph: if such an edge does not exist, we have a tree (with def6). If such an edge exists, we remove it and we search a new one to remove. When we can not remove any edge, we have a tree whose same set of vertices than G.

RESULTS

Devolution of the problem

The students were not reluctant to do the first question. We note that to define is not an activity strictly reserved for mathematics and this can support the devolution of the problem. Moreover the examples and counterexamples allow a process of comparison. Lastly, the students feel they have enough elements. In order to explain the main results, we will study three representative groups (called group 1, 2 and 3).

Defining-methods: defining by genus and differentia

Aristotle's defining method by genus and differentia [7] is to indicate what specific object a word means ('thingummy' here), to take a bigger class (graph) within which that object falls, and then to try and see what distinguishes it from the rest of that class. For Aristotle, a definition is a discourse according to specific rules (about language, syntactic and semantic rules). The chosen activity of *definition-construction* is by genus and differentia, so called because the students try to distinguish the examples from the counterexamples. But their regulations are not specifically like Aristotle's rules. Let us describe some of the main students' regulations and conceptions.

The two expressions "sufficient definition" and "minimal definition" were often used. These search for a sufficient definition proves that they seek a sufficient condition allowing the recognition and/or the construction of the mathematical object i.e. they evaluate up to what point their definition does not relate to too large a class of objects. When the students talked about 'minimal definition', it was not a matter of minimal sufficient condition, but short sentence. Moreover, the two properties 'connected' and 'acyclic' were mixed-up (conceptually and at the linguistic level) as testified by this extract (the student does not distinguish 'connected' and 'acyclic') :

Yohan: a unique path: that leads to only one condition. Instead of having two conditions to check on each figure each time.

The wish to define the "vague" terms of their definition was expressed by the students rather quickly when they defined 'thingummy' as a "set of dots and lines without cycle and in only one piece". This last property was "fuzzy" for them and some of them worked on the definition of connectivity to redefine "in only one piece" by "a path exists between two unspecified dots of the figure". We assume that when the students have given a name to a property ("in only one piece" for connectivity and "closed polygon" or "circuit" for cyclic), that allows them to work on this property (cf. Vygotsky) and to overstep simply verbalising the representation.

Produced definitions

Vygotsky notes the deep discrepancy between the formation of a concept and its verbal definition. With the examples and counterexamples, the students described the representation of 'thingummy': "in only one piece, without cycle". We notice that this *zero-definition* was as narrow as to coincide with def1 (table1). But the students did not agree to accept this assertion as a mathematical definition (although it could be a "good" one for a researcher because it gives the structure of the mathematical object) in accordance with one of their conceptions: a mathematical definition should be specified. So they questioned the mathematical object, its properties and its representation. In the table 2, *unfinished proof-generated definition* concerns the *proof-generated definition* and means that it was unfinished in terms of its form and its use. We notice that def5 (table1) was not constructed by the students: this is an inappropriate exercise to make it worthwhile.

Students' statement	Group	*Reference definition*	*Production phase*	*The role in exercise*	*Our status stand point*
In only one piece without cycle	1,2,3	Def 1	1	Idea of the proof (students'understood' the proof but did not know how to say it.)	Zero-definition
Inductive def.	2,3	Def 7	1	Proof	Definition
A unique path	1,2,3	Def 2	1	Idea of the proof	Definition
n dots and (n-1) lines	1	Def 3 or 4	2	Proof	Property
	3	Def 4	They add 'connected' (end of 2)	Usable only for a small graph in order to obtain a tree.	Definition
Minimal connected graph	1,3	Def 6	2	Unused	Unfinished proof-generated definitions
Remove a line it disconnects	3	Def 6	2	Unused	

Table 2 : description of students' statements

Functions of produced definitions

The research of a definition was guided here by functions of definition : to communicate, to recognize, to build and to prove.

Definition in order to communicate means to explain to another persons what a tree is i.e. the properties and the construction of the mathematical object.

Definition in order to build is alluded to by these students (group3):

Arnaud: here is the definition to create a thingummy, and there the definition to know if we have a thingummy or not.

Yohan: but to check, I prefer the idea of the path. If one wants to check whether the figure is a thingummy or not, one takes the first definition (a unique path), and if one wants to build a thingummy, one takes the inductive definition.

Group3 spoke about "good definition". For us it means algorithmic definition,

Yohan: It is a good mathematical definition when one can make a program [...] One can make a data-processing program which checks the thingummy, if it gives the good result each time, that means that the definition is good, inevitably since the computer does not think.

and/or it allows the *recognition* of the object :

Fabien: lets consider one must give a definition. If I, who don't know the thingummy, I have it explain to me by somebody, I inevitably see what it is.

The last function of a definition was *to prove*: the interest of equivalent definitions (multiplicity of the possibilities in a demonstration) is alluded to by group1 and we find the connection between a definition and the function which it will be able to have, in particular in a proof (operational definition) :

Arnaud: And then finally, especially the inductive definition, that facilitated the second exercise to us.

Yohan: Yes, one can choose a definition to prove. In fact a few moments ago, it was not useful. If you want to draw a tree, you don't need to know that.

DISCUSSION AND CONCLUSION

We raised some properties in the protocols, for example: "when one adds a dot to a tree, one always has a tree". Another property emerges at the time of the proof: "when a line is removed, that disconnects". Why did these "potential" definitions not emerge under the status of definition? Which leads us back to the question: which criteria allow a definition to be recognized (perceived) as such? Let us notice perhaps that these two characterization properties of the tree alluded to above contain a dynamic aspect in opposition to the static representation of the tree available to the students' experience. Moreover, we already mentioned the gap between definition and characterization properties which can be one of the causes of the non-emergence of these definitions as such. This gap means that a definition is enough, what can appear later only represents properties, as testified by this extract (group2):

Vincent: That will be included in the first definition (...) I have the impression that we have finished our work.

Angelique: It will be the same thing. Actually it all boils to giving properties. Finally they are not definitions, but properties for me.

We would like to stress the following : group1 has not reconstructed an appropriate definition in order to solve the exercise because the constructed definition (first question) has a form and a content that institutionalised it. Moreover, the definition represents a common knowledge for the resolution of a problem (Balacheff,1987).

It's possible to make students construct a definition of an object which is accessible by its representations. It was through examples, counterexamples and the produced definition that the students were able to build their *concept images*. We assume that a definition is not a finished product, so the necessity of the exercise. It appears that this activity of *definition-construction* could be a part of a process of concept acquisition. We would like to study the dialectic between the formation of a mathematical concept and its definition (with *definition-constructions*' activities) or more precisely the role of a *definition-construction* in learning. And also, the students' conceptions on mathematical definitions could be an obstacle to the concept formation. These conceptions concern the dynamic and static points of view, the functions of a definition and the gap (and the relations) between a definition and characterization properties.

NOTES

1. "the didactical transposition is the rule in sciences : (...) the presentation of a subject takes as a starting point a historical result and rewrites the history in reverse. In mathematics, this starting point is a definition".
2. "we are always conscious that our terms are a little vague... and we reach precision not by reducing their penumbra of vagueness, but rather by keeping well within it, by carefully phasing our sentences in such a way that the possible shades of meaning of our terms do not matter. That is how to avoid quarrelling about words." (p19. Popper (1945) *The Open Society*, Vol.II – London ; Henley : Routledge & Kegan Paul)
3. See columns of research sampler (www.maa.org).
4. "a concept cannot be restricted to its definition (...) it is through situations and problems to be solved that a concept acquires meaning".
5. A graph is made up of dots (vertex/vertices) connected by lines (edges).
6. For the possible ways of operating with definitions in doing proofs, see Moore.
7. *Topiques & Seconds Analytiques* (in Aristote, *Organon*, trad. J.Tricot, Paris, Vrin, Ed.1965).

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EXPLORATIONS OF RANDOM MIXTURES IN A 2D SPACE CONTINUUM

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This paper focuses on how children express their ideas for randomness in two-dimensional continuous space, through tools for directing and redirecting the simulated movement of balls. It reports the findings of a study in which children aged between 6 and 8 years old engaged with a game-like environment to construct for themselves random behaviour by making spatial representations of sample space. In response to a range of tasks, the children manipulated the sample space in ways that generated corresponding outcomes in the game. We present some case studies of children's activities, which illustrate how the medium mediates the children's understanding of chance events.

INTRODUCTION

The literature on randomness overflows with references to children's and adults' incompetence in dealing with judgements of chance (for just one of many references, see Kahneman et al., 1982, who set out a series of heuristics that, because of their inherent bias, have been termed by other authors as "misconceptions"). Piaget and Inhelder's (1951, translated 1975) work represents a seminal effort to investigate the origin of chance. They devised a number of experiments, which yielded 'chance' outcomes and they used these for probing the conceptual development of children from pre-school ages to adolescence. One of their experiments concerned a tilt box, where the child was given a rectangular box, resting on a transversal pivot, allowing seesawing. They concluded that young children fail to understand the random nature of mixture. Their main argument is that a fundamental property of operations is that they are reversible, so that even when a child is a concrete operational thinker, randomness is still a strange idea that does not fit into the way they normally see the world. Further, they maintained that the idea of chance is not acquired before the stage of concrete operations.

However, Fischbein (1975) has reported how children have intuitions for relative frequency from a very young age, and this leads us to search for other evidence that even very young children have emergent cognitive resources for making sense of randomness. In research on intuitions and fairness, Pratt & Noss (1998) found out how their subjects make sense of dice situations and they show how existing intuitions about fairness, often based on actual outcomes, are co-ordinated with new meanings, derive from interacting with the microworld. Papert (1996) has argued that educators should look in pragmatic ways for connections between pieces of knowledge and that, on a theoretical level, the metaphor of learning by construction leads to a range of interesting questions about the connectivity of knowledge.

In terms of knowledge about stochastic phenomena, Pratt (2000) has also reported on how children (aged 10/11) were observed to use four separable resources for articulating randomness namely: unsteerability, irregularity, unpredictability and fairness. Children constructed meanings for randomness in the context of a computational microworld, visualising how the random behaviour of objects actually worked. In this paper we describe the activities of even younger children (aged between 6 and 8) who were expressing their ideas for randomness in two-dimensional continuous space, through tools for directing and redirecting the simulated movement of bouncing balls.

THE SOFTWARE

The game was written in a rule-based system, called 'Pathways'[1] and was designed to afford children the opportunity to talk and think about probability in the context of quasi-concrete manipulations. The game is shown in Figure 1.

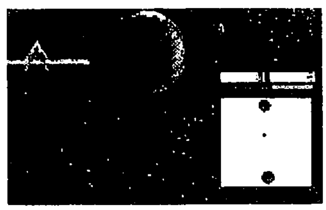

Figure 1: The game

In the right-hand corner is a square. A small ball bounces around the walls of the square and occasionally collides with and bounces off other balls. In Figure 1, one red and one blue ball is depicted (they are the other two larger balls inside the rectangle), but the child can control the number, size and position of the balls. Each time there is a collision with a red ball, one point is added to the red score (just above the rectangle), and the "space kid" (the triangular creature) moves one step up the screen. Similarly, collisions with the blue balls add one point to the blue score and move the space kid one step down the screen. Whilst individual collisions can be seen as single trials in a stochastic experiment the totality of these movements gives an aggregated view of the long-term probability. The game itself is defined in terms of simple iconically represented rules, which are designed so that the children can easily change the nature of the game itself (see Goldstein and Pratt, 2001, for a fuller description of the Pathways environment).

The task was given to the children to construct for themselves representations of the sample space. They constructed for themselves random behaviour by making spatial representations of sample space in a game-like environment. In order to

[1] This research was funded by the European Union as part of the *Playground Project*, Grant No. 29329. See http://www.ioe.ac.uk/playground/

explore children's connections between fairness and randomness, we began with a situation in which the children had to try to make the space kid move around a centre line in order to construct a 'fair sample space'. Later, we asked children to make changes to the two-dimensional continuum space to make the space kid move upwards or downwards from the centre line in ways that represented other global outcomes (such as being likely to go off the top of the screen).

METHODOLOGY

The findings reported here are part of a broader study that adopted a strategy of iterative design, in which the computer-based tool was developed alongside the gathering of evidence for children's use of the tool. The children were interviewed before and during their interaction with the computer: all interviews of each iteration were videotaped and transcribed. The role of the researcher was that of participant observer, interacting with the children in order to probe the reasons behind their answers and their actions. In the final iteration, 22 children, aged between 6.5 and 8 years, were involved. The children worked with the software individually for between 2 and 3 hours. The semi-structured interview began with children expressing their ideas about Piaget & Inhelder's tilt task. The children were then asked to inspect the rules of the game through the Pathways tools, and finally worked with a range of game-oriented problematic situations.

EXPRESSIONS OF RANDOM MIXTURE

The results from the interviews on the tilting task were consistent with those of Piaget and Inhelder. Most of the children responded to the task by expressing what the order of the colours should be when the balls move to the other side of the tilt box. For example, Jane said, 'The reds will move here, then the yellows, the greens here and the oranges here'. The children envisaged that the balls would finish up in an organised manner, rather than being mixed up in the tilting process.

In this context, the children were generally unable to express notions of random mixture. In contrast, they seemed able to express the notion of random mixture in two-dimensional space continuum environment. The computer environment supported several ways in which the idea of mixture could be expressed. The children changed the position, size or number of the red and blue stable balls in the sample space, or they described in words the 'uncontrolled' continual movement of the white ball. We describe four categories of children's strategies below: haphazard movement, complex movement, symmetry of placement and equal size of balls.

Haphazard movement

The children's awareness of a lack of pattern and lack of controls over the movement within the two-dimensional space is evocative of the local resources for making sense of randomness as reported by Pratt (2000). Lucy (age 7 years and 8 months) characterised the white ball's movement and mixture as follows: 'It goes right-left, up, down and on the balls. We don't know where it goes. It moves in the yellow square, where it wants to go'. Mixture for Lucy meant to move where the ball wants to go, without any obvious pattern or pre-ordained positions. Children were using similar internal resources or intuitions as one might expect in a more conventional study of randomness, and so lends credence to the idea that these very young children were indeed constructing meaning for a random mixture, in a way that was not evident in Piaget & Inhelder's experiment.

Complex movement

Paul (age 6 years and 10 months) expressed his idea of random mixture in a different way. He was influenced by the manipulations of the task, and he attempted to make all the balls bounce around, collecting points.

Paul: Ok now! Ah! I know what to do!

He changes the balls and the speed again.

Researcher: The blues have more points than the reds.

Paul: Do you know what to do? We can take out all the white balls and give a speed to the red and blue balls and when the blue touches to a blue or a red touches to a red ball to get two points otherwise to get one point.

Paul gave movement to the red and blue balls and decided to change the rules under which points were scored and the mixture of the balls was made more complex.

Symmetry of placement

Symmetrical teams

Most of the children tried to achieve a fair game by placing the balls symmetrically. They didn't care so much where each ball was placed, but were concerned about the positioning of a team of the balls. This is very obvious to Brian (age 6 years and 6 months), who not only separated the two colours, but also put lines between the two teams and constructed new rules.

Researcher: Can you do changes in the lottery machine?

Brian: I want to make a line... I will blow this ball... I know, I will do something (Figure 2).

He separates the two colours and he constructs a new object.

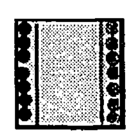

Figure 2

Researcher: What do you want to do?

Brian: I need a rule 'When the ball touches me, I take all the messages from all the red balls and I give them to the red scorer.' I need the same to happen to the blue balls.

The children saw each colour as one team. For example, Jane (6 years and 7 months) expressed randomness by first separating the two colours to create two different teams of equal numbers.

Jane: Because I will put them near to each other in the middle. So, when the ball goes to touch the one, it will touch also the other that's near it. So, it will touch both of them and we will have equal points. We do not know where it (*the white ball*) will go, but if it touches one ball it will touch the other as well.

Researcher: That sounds a nice idea...

Jane: ...I will put the two lines in the middle. Now, I need another two balls. I'll get the magic wand to get more balls. It's the good fairy that gave it to us...

Researcher: Do you know how many balls do we have?

Jane: Yes...they are equal. I know they are equal, but I don't know how many balls I have (Figure 3).

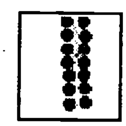

Figure 3

Researcher: How do you know that they are equal?

Jane: I copied one red and one blue each time. It doesn't matter actually how many they are. They are equal.

This excerpt emphasizes that what mattered to Jane was the equality of the teams, rather than precisely how many there were. She seemed to equate equality with fairness, and presumably fairness in some way with randomness.

Making a pattern

The pattern was very useful for Lucy to create a random environment.

Researcher: Can you arrange the balls in a way in order the space kid to move near the yellow line?

Lucy: I will then leave only two balls...or...I will make a pattern in order not to move so up or so much down… I will copy some more balls (Figure 4).

She destroys balls and keeps only one red and one blue. Then she copies some to make a pattern.

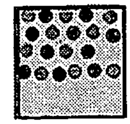

Figure 4

Researcher: How does the pattern work?

Lucy: They are going to have equal numbers. It (the white ball) will move up, on the edges…the ball will get the same points. I will also copy another white ball to move quickly…They became rows.

She starts the game.

Researcher: What happens? Where does the white balls go?

Lucy: It goes everywhere…around the balls. They have equal numbers now! I got one ball and another. I made a row and then another row and I made the white ball to move in a way and now they are going to have the same numbers.

The logic behind it was for one colour to be near the other, so that when the white ball was going to touch one colour it would touch the other as well. The pattern was also used as a way, as she described, to have equal number of balls, without counting.

Making circles

The circle was made to trap the white ball in order to touch the balls in the circle the same number of times, and sometimes it turned out to be a start for a symmetrical development, as well. Anne (age 6 years and 6 months) here started by having the white ball in a circle and then constructed another random situation by copying more balls.

Anne: …I'm going to make all the balls have the same size. I'll do another arrangement.

Researcher: So, what are you doing now? (Figure 5)

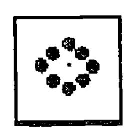

Figure 5

Anne: I'll make more copies of them.

She switches the game on.

Researcher: Oh…does it work?

Anne: Yeah…It keeps going up, down, up, down….

Researcher: Great!

Anne: I'll make more copies…

Researcher: What's the arrangement now?

Anne: That one (the blue ball) is facing that one (the red ball) and that one is facing that one and so on… I've got also a better idea! They (the red balls) will face a blue one. There! (Figure 6)

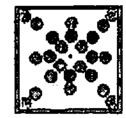

Figure 6

Researcher: What did you do?

Anne: The blue ones are facing the red ones and the red ones the blue ones.

Researcher: Ok! What number will you have here (on the scorers)?

Anne: I don't know, I'll try it out!

She starts the game.

Anne started here by constructing a circle and developed that into a symmetrical picture.

Equal size of balls

The spatial environment played a major role in helping Jane to look at whether two balls were equal in size or not. As she said, one of the balls 'is bigger and it (*the white ball*) will touch the most of the time, because the ball takes up more space in the yellow square'. She looked at the effects of the game and she used the global event, the movement of the space kid, to see whether her environment was fair.

Jane: I think the red will win.

Researcher: Why is that?

Jane: I think I made it a little bigger than the other… We can open the game and if the scorers are the same that means that they have the same size, otherwise the one is bigger than the other (Figure 7).

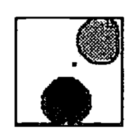

Figure 7

Researcher: What about the space kid?

Jane: If it is as now that means our balls have the same size...

Jane was making a connection between the spatial appearance of the sample space and the possible outcome from the game in the longer term.

CONCLUSIONS

The study seems to indicate interesting differences between the way that children responded to Piaget & Inhelder's experiment and how children in this non-conventional context were able to express ideas for random mixture. A main difference from the Piagetian study was perhaps that, in this study, the children constructed randomness, not just as a cognitive act or thought experiment, but in a quasi-concrete way. Children's thinking moved from finding outcomes and describing the random behaviour to *constructing* a random behaviour. The study indicates that children have various cognitive resources for constructing randomness, beyond those that might be expected from classical experiments. We believe that a possible reason for this is that the tool offers them the opportunity to use these resources for random mixture in a two-dimensional continuum. Moreover, children's culture involves many experiences of random movement, and the context for such experiences is changing from playing board games towards playing video games. We conjecture that cognitive resources for making sense of random mixture may be more likely to find a means of expression in continuous two-dimensional movement than in more conventional contexts that involve discrete number.

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FUNDAMENTAL CYCLES OF COGNITIVE GROWTH

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Over recent years, various theories have arisen to explain and predict cognitive development in mathematics education. We focus on an underlying theme that recurs throughout such theories: a fundamental cycle of growth in the learning of specific concepts, which we frame within broader global theories of individual cognitive growth. Our purpose is to use our experience in such areas as the SOLO Model, van Hiele levels, process-object encapsulation and symbols as process/concept, to raise the debate beyond a simple comparison of detail in different theories to move to address fundamental questions in learning. In particular, a focus of research on fundamental learning cycles provides an empirical basis from which important questions concerning the learning of mathematics can and should be addressed.

The named author is the presenter of this paper. This paper was written as a joint paper by John Pegg and David Tall. It is presented here as a single authored paper to comply with a newly-implemented rule restricting the number of papers that can be accepted by a given co-author.

INTRODUCTION

We here focus on two kinds of theory of cognitive growth:

- **global theories of long-term growth** of the individual, such as the stage-theory of Piaget (e.g., Piaget and Garcia, 1983).
- **local theories of conceptual growth** such as the action-process-object-schema theory of Dubinsky (Czarnocha et al, 1999) or the unistructural-multistructural-relational-extended abstract sequence of SOLO Model (Biggs & Collis, 1982, 1991; Pegg, 1992).

Some theories (such as that of Piaget, the SOLO Model, or more broadly, the enactive-iconic-symbolic theory of Bruner, 1966) incorporate both aspects. Others such as the embodied theory of Lakoff and Nunez (2000) or the situated learning of Lave and Wenger (1990) paint in broader brush-strokes, featuring the underlying biological or social structures involved.

Our focus is on local theories, formulated within a global framework whereby the cycle of learning in a specific conceptual area is related to the overall cognitive structures available to the individual.

A range of global longitudinal theories each begin with physical interaction with the world and, through the use of language and symbols, become increasingly abstract.

Table 1 shows four of these theoretical developments. It is not our purpose to make a detailed stage-by-stage comparison of these theories here. Although reports on comparisons between SOLO and van Hiele can be found in Pegg and Davey (1998), and between SOLO and Piaget and SOLO and Bruner in Biggs and Collis (1982).

Piaget Stages	van Hiele Levels Hoffer,1981	SOLO Modes	Bruner Modes
Sensori Motor	I Recognition	Sensori Motor	Enactive
Preoperational	II Analysis	Ikonic	Iconic
Concrete Operational	III Ordering	Concrete Symbolic	Symbolic
Formal Operational	IV Deduction	Formal	
	V Rigour	Post-formal	

Table 1: Global stages of cognitive development

What stands out from such theories is the gradual biological development of the individual, growing from dependence on sensory perception through physical interaction and on, through the use of language and symbols, to increasingly sophisticated modes of thought. SOLO offers a valuable viewpoint as it explicitly nests each mode within the next, so that an increasing repertoire of more sophisticated modes of operation become available to the learner. At the same time, all modes attained remain available to be used as appropriate. As we go on to discuss fundamental cycles in conceptual learning, we therefore need to take account of the development of modes of thinking available to the individual.

LOCAL CYCLES OF DEVELOPMENT

We now turn to the core of our study: the cycles of development that occur within a range of different theories. These have been developed for differing purposes. The SOLO Model, for instance, is concerned with assessment of performance through observed learning outcomes. Other theories, such as those of Davis (1984), Dubinsky (Czarnocha et al, 1999), Sfard (1991), and Gray and Tall (1994, 2001) are concerned with the sequence in which the concepts are constructed by the individual (see Tall et al, 2000, for a further analysis of these theories).

SOLO of Biggs & Collis	Davis	APOS of Dubinsky	Gray & Tall
			[Base Objects]
Unistructural	Procedure (VMS)[1]	Action	Procedure
Multistructural	Integrated Process	Process	Process
Relational	Entity	Object	Procept
Unistructural		Schema	

Table 2: Local cycles of cognitive development

[1] Davis used the term 'visually moderated sequence' for a step-by-step procedure.

As can be seen from table 2, there are strong family resemblances between these cycles of development. Although a deeper analysis of the work of individual authors will reveal discrepancies in detail, there are also insights that arise as a result of comparing one theory with another as assembled in table 3.

First, Gray and Tall (2001) note the role that objects operated upon play in concept formation which focus on actions of those objects. These *base objects* may be perceptual or conceptual, provided that the individual conceptualises them as entities. Gray and Tall see the configuration of the base objects (such as a set divided into three equal pieces and two of these being selected) as an embodied representation of the encapsulated abstract concept 2/3.

Second, Dubinsky's notion of action begins at a more primitive level than the notion of procedure, formulated in the following terms:

> **Action**. A transformation is considered to be an action when it is a reaction to stimuli which the subject perceives as external. This means that the individual requires complete and understandable instructions giving precise details on steps to take in connection with the concept. (Czarnocha et al, 1999)

The SOLO Model is in the Piagetian tradition. Here humans are seen as actively constructing their world as a result of processes of interaction between a person and his or her social and physical environment. The resulting development can be identified through 'logically' sequential, qualitatively different, levels each representing a coherent viewpoint of the world. This pattern of thought is revealed through the underlying cognitive structure of what the person says, writes or does. Growth in understanding is seen as an active change in patterns of thinking. Hence, cognitions are seen to be structures that are rules for processing information.

The central task associated with SOLO codings, and one that can be very demanding, is to analyse the pattern of responses provided by individuals and groups. In doing this, it is important to distinguish between the content and the cognitive structure. There is a tension here. The pattern of responses has to be sensitive to (i) those aspects that are cultural or individual and primarily associated with the subject content, and (ii) the reasoning structures of a student's thoughts that are part of our human heritage and true for all individuals.

The unistructural level of SOLO focuses on the appropriate domain but uses relevant data seen by the students as a single entity. This notion seems to encompass the levels of base object, action and procedure. On the other hand, the distinct SOLO level of 'multistructural' is not explicitly mentioned in any of the other theories. Analysing the Gray and Tall distinction between procedure and process, we find that it is possible for students to have more than one procedure to carry out a process, but yet may not have a flexible conception of process itself. For instance, in the analysis of Ali and Tall (1996) considering Malaysian students' flexibility in having different procedures to differentiate a given formula in calculus, there are examples of students having different procedures available for a given derivative, and yet their thinking remains procedural, rather than conceptual. Thus the procedure-process-procept spectrum naturally expands to procedure-multiprocedure-process-procept.

The relational level of SOLO readily equates to the process level of the process-object encapsulation theories. Here the learner has an overview of the elements or procedures. The data known to the student are able to be woven into an overall mosaic of relationships. The whole has become a coherent structure with no inconsistency within the system known by the student.

The fourth level titled unistructural relates to a combined object-schema level. This level highlights the two possibilities of cognitive growth that exist within SOLO when development occurs past the relational level. This is summarised diagrammatically in figure 1. If the nature or abstractness of the thinking is the same as that identified in the previous three levels, i.e., U_1 M_1 R_1, then the next level is a new unistructural level, U_2. It is distinguished by the person seeing as a single concise entity what was previously an integration of several aspects.

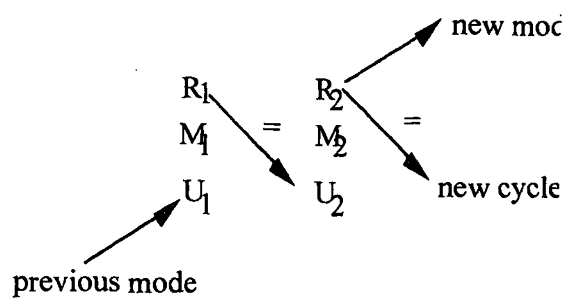

Figure 1: At least two cycles within concrete symbolic mode.

If, on the other hand, the new response represents a qualitatively different, more abstract way of thinking, then the response can be coded outside of the current mode and within the next acquired mode. This new level can be described as a new unistructural level, a U_1, in the next acquired mode. In the case of moving from the concrete symbolic mode into the formal mode, this new unistructural level can be written as U_{1F} and represents the start of a new Fundamental cycle. It is this cycle that has most to offer students in their late secondary and early tertiary education.

There is support from a range of papers in the literature to see the object-schema level as a coherent whole (which may be subdivided as appropriate). The first is Skemp's varifocal theory, which sees the duality of concepts and certain types of schema, depending on whether it is seen as a whole (concept) or in detail as a structure of relationships (schema). The second, in part, is Dubinsky's own perception that objects can be formed not only in terms of encapsulation of processes, but also of encapsulation of schemas (Czarnocha et al, 1999).

An in-depth discussion of the relationships between process, object and schema is given in Tall & Barnard (2001). This discussion shows a correspondence between these local theories and a fundamental cycle of conceptual development, see table 3.

The first stage involves some kind of action on one or more base objects with the focus of attention of the individual either on the objects, or on the actions. These can lead to different kinds of learning, one which focuses on the nature and properties of objects, the other on the nature and properties of the actions, involving symbols being introduced to represent the actions. Isolated actions are consolidated into (step-by-step) procedures, possibly with several procedures available to carry out essentially the same activity. The activity may then be seen as a single process that may be carried out by different procedures. At this stage, with the support of some kind of symbolism, the individual may construct a mental object that is both a schema within itself and becomes manipulable within a wider schema of activities.

SOLO	Davis	APOS	Gray & Tall	**Fundamental Cycle**	
			Base Object(s)	Base Object(s)	
Unistructural	VMS Procedure	Action	Procedure	Isolated Actions Procedure	
Multistructural			[Multi-Procedure]	Multi-Procedure	
Relational	Process	Process	Process	Process	
Unistructural (Extended Abstract)	Entity	Object Schema	Procept	Entity Schema	

Table 3: The fundamental cycle of conceptual construction

Example: Fractions

In the case of the fraction concept, the base objects may be a single object (a cake) or a collection of objects (the children in the class). The action is to divide the object (or separate the objects) into equal parts, by some kind of division or sharing process, then selecting a given number of the parts. Different actions (e.g., divide into 3 parts and take 2 or divide into 6 parts and take 4) lead to the same output. Different techniques for carrying out the activity may lead to procedures, then multi-procedures to carry out the action.

When the child realises that different procedures have the same output and focus on the effect of the procedures, then this moves to the process level where different procedures have the same effect (Watson & Tall, 2002). The introduction of the symbols 2/3 or 4/6 for the two distinct actions leads to the notion of equivalence of fractions that have the same effect. The notion of equivalent fraction is a schema in itself (as in Skemp's varifocal theory) which, conceived as a single entity, may become an element within the wider schema of arithmetic of fractions.

To expand on the above let us consider a particular problem asked of some several hundred students in the age range ten-to-fifteen years. This question was asked in mathematics lessons along with several other fraction-related questions. Students were asked how much would each person receive if nine apple pies were divide between sixteen people so that each person receives the same amount. In analysing the student responses we can identify an initial fundamental cycle followed by several components of a second fundamental cycle. In terms of SOLO modes all responses were within the concrete symbolic mode. Early cycle responses, which attempt to deal with the problem, focus on the need (the action) to cut the pies equitably, usually in halves. However, a problem emerged for some students when their action of cutting did not result in each person gaining an equal amount: "cut the pies into halves and you have two pieces left over" (for these students there was one pie too many). These students did not resolve this issue.

The next category of response also has the cutting of the pies fairly as its focus, but the problem is seen in two parts with a new procedure employed to deal with the left over pie. "Cut one pie into 16 pieces and the rest into halves." The third category of response considers the effect of the cutting. The response moves on past the focus on equitable cutting to provide a summative view on how much each person would receive. "Cut each apple pie into sixteen pieces and each person gets 9 pieces each." Missing from this response is any use of standard fraction notation.

The next level of response, a new unistructural level occurs as the response provided is more succinct and fraction notation is employed to express the different parts of the problem. "I would cut 8 pies in half and then cut the last pie into 16 pieces. So everyone gets 1/2 and 1/16 of a pie each." In this response students use fraction notation freely in their written language. This is the culmination of the first fundamental cycle and also represents the first element in a new cycle. This new Fundamental cycle represents a development of fractions as numbers.

The second level in this new cycle is on combining the fractions identified previously. The method employed is equivalent fractions whereby all fractions are rewritten with the same denominators before combining. "Cut 8 pies in half and give one half to each person. Then cut the last pie into 16 pieces and give one piece to each person. 1/2 + 1/16 = 8/16 + 1/16 = 9/16 each person ends up with 9/16 of a pie." Here, work with fractions takes on a more systematic process.

This was the highest level of response identified from the responses received. One could predict that the next level is when students can approach the task with considerable versatility. They can use a number of written approaches and are usually able to provide the answer efficiently in verbal form. These options of response were not available to the students in this study.

THE BIOLOGICAL BASIS OF THE FUNDAMENTAL CYCLE

The biological basis of this phenomenon was discussed in Tall (1999), in which the stimulation of neurons places them on alert so that they will fire more easily for a while. They then react to a lower level of stimulation and, if this occurs, the link becomes even easier to make, until the neurons concerned fire together as a unit in a more subtle pattern. This *long-term potentiation* of neuronal links builds sophisticated structures that act in consort, which are both complex (because they have many connections) but also simple (because many neurons fire as one unit).

However, this highly subtle mental development has a consequence. If the individual becomes aware that certain sequences of activity tend to occur, the individual can operate with them at a higher level. Thus, whilst individuals may need initially to be externally guided at the action level before contemplating procedures, then processes and then symbolise the activities to build a sophisticated mental schema, others are aware of the overall scheme of things. Thus a natural attribute of the fundamental cycle is the possibility of seeing the kind of outcome that might be possible and have a 'top-down' view of the need to build actions into processes into schemas to construct the wider vision in greater detail.

Here the different modes of operation may be of great advantage. For instance, an embodied combination of SOLO's sensori motor and ikonic modes may enable the individual to gain an insight into the overall plan *before* building the more powerful symbolic and deductive modes of thought. On the other hand, an embodied approach which may work for some (natural) thinkers (using the language of Pinto, 1998) may be less effective with (formal) thinkers who have highly developed symbolic and deductive cognitive structures.

CONCLUSION

A primary goal of teaching should be to stimulate cognitive development in students. Such development as described by these fundamental learning cycles is not inevitable. Ways to stimulate growth, to assist with the reorganisation of earlier levels need to be explored. Important questions about strategies appropriate for different levels or even if it is true that *all* students pass through all levels in sequence. Research into such questions is sparse. Nevertheless, the notion of fundamental cycles of learning does provide intriguing potential for research.

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MANIPULATIVE HELP IN VERBAL SHARING OUT OF CONTINUOUS AND DISCRETE WHOLES PROBLEMS SOLVING

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Abstract. *This is the report for the study of a teaching program consisting on activities related to the use of fractions. The study was conducted on a small group of second grade primary education students at a state school. Students had received no previous teaching of fractions. In this investigation, we will show the feasibility of manipulative material and of the collective game as a type of tool to assist the student in building an initial understanding of sharing out (although it can not be take as the only tool). A questionnaire was given to the students as well, both before and after the teaching program. In order to prove that the results reported in the final questionnaire were accurate, various individual interviews were carried out on three children that participated in the teaching program.*

Introduction

In Mexico[1], like in other countries, fractions make up one of the most difficult areas in the teaching of math at the basic educational levels (particularly regarding primary and secondary school). Therefore, many students who complete their secondary education and go on to a higher level find themselves with limited knowledge of fractions, even though they have studied fractions since primary school. This problem has become evident from the results obtained from some recent studies (Fredenthal, 1983; Pitkethly and Hunting, 1996; Kieren, 1980, 1998; Valdemoros, 1993; Figueras, 1998; among others), and from the teaching experience of those that participated in this study. It is possible that one of the many factors that fall into this process is the teaching method, which is considered not only determinant but also a process through which the child acquires knowledge.

As we face this problem of complexity when students acquire fraction knowledge, we must ask: What teaching activities can a teacher design in order to promote or facilitate a rich understanding of fractions among students? This question has served to delimit and concentrate on the problems presented in this study, which focus on how the systematic school teaching influences a child's previous knowledge[2] regarding the sharing out and elementary notions of fractions.

[1] Mexico: Fractions are first taught in the third grade of primary school and the didactic strategies are taught between the third and sixth grades. During this time, a broad semantic overhaul of the relationship between equivalences and order is developed along with an elementary introduction of addition and subtraction.

[2] Previous knowledge is made up of an organized set of notions, social representations, and hypotheses and/or theories that put into play the understanding of scholastic components and that function as true, assimilated instruments of such components.

Our hypothesis is set out as follows: *The manipulative material will provide the third grade primary school student with assistance in order to build a formal understanding of the sharing out upon his previous knowledge.*

Therefore, the objective of the present investigation is to identify and analyze the effects on students in the third year of primary school when solving sharing out problems in continuous[3] and discrete[4] models with the help of manipulative materials.

Theoretical Framework

Freudenthal (1983) states that the dividing into equal parts is a link to the mental construction of all types of magnitudes. While observing children from seven to eight years old, he realized that they are capable of estimating the half or one third of an irregular shaped area. Freudenthal considers this ability to divide as an important component in the mental area object.

On the other hand, Kieren (1983, 1984, 1988) recognizes that the division, the equivalence, and the formation of a divisible unit (constructive mechanisms) constitutes the cognitive basis for fraction language of the part-whole relationship. In their studies, researchers like Kieren, Nelson and Smith (1985), Figueras (1996), Piaget and Inhelder (1966), Hunting and Korbosky (1990), Pitkethy and Hunting (1996), and Pothier and Sawada (1989) have separated the strategies children use to solve partition and sharing out problems.

Freudenthal (1983) states that the formation of mental objects should come prior to the acquisition of concepts because the mental objects lay the foundations for the acquisition of concepts. The phenomenon are first seen as mental objects and then are transformed into concepts. Using specific material, the mental objects are formed of phenomena and not of concepts. Comparing mental objects will lead to the concept. The specific material holds a temporary significance because, once a student has formed long-lasting and permanent mental objects, the specific material is no longer necessary.

Researchers like Behr and Post (1988), Behr, Wachsmuth and Post (1998), Kaplan, Yamamoto and Ginsburg (1989), Bezuk and Cramer (1989), and other researchers, have used manipulative material when teaching fractions. They have suggested as well that it be used with primary school children and have given teachers both orientations and suggestions for correct usage.

Likewise, Kamii (1985) introduces three teaching principles: autonomy, daily situations, and collective games. She points out that if the teacher considers autonomy as the main purpose of teaching, he must foster the exchange and coordination of the students' points of view, allow them to express their thoughts freely, to make their own decisions, encourage them to think on their own instead of

[3] Continuous models: Fractions are a way to relate one part to the whole and are put into context with geometric figures in that the fraction symbolizes a part of the unit (Hart, K. 1981).

[4] Discrete models: Fractions are a way to relate a subset of a set in which it is included and it is put into context in sharing out situations.

reciting correct answers, let them debate his answers. Moreover, he suggests that everyday situations happening in the classroom should be taken advantage of by the teacher, since they are great opportunities for arithmetic. In the same manner, Kamii says (1985, 1994) that collective games are an essential part of constructive teaching. Games provide a way of organized playing through which, children are encouraged to think in numerical combinations and remember them. They also allow children to develop their ability to govern themselves, foster social interaction, and provide feedback among participants.

Methodological Aspects

In the Mexican school, as mentioned in the beginning of this text, the teaching of fractions starts officially in the third grade of primary education; however, at the time of this study, we became aware of the fact that this subject had already been taught to third graders. Consequently, this study is carried out on a second grade class during the last period of the school year (may – June). This period was chosen because it would be certain that there be no previous teaching of fractions. This study is of a qualitative kind, due to the fact that an analysis of the progress of a small group of children in their natural environment (the classroom) is enclosed in this study, in order to recognize the effects produced in their thinking during the process and termination of the application of a brief teaching program focused on sharing out.

The study began with an exploratory exam given to 28 students, with the purpose of getting information about the previous knowledge on fractions the children had. Moreover, the questionnaire allowed the selection of 10 children who participated in the study and whose spontaneous and intuitive knowledge on the subject was recognized before the start of the teaching program.

The questionnaire included 13 tasks organized in five blocks, with sharing out situations represented by pictograms[5]. In the first block, all are distributed continuously among several objects or several people. In the three following blocks, all are distributed discretely among several people, the result of the sharing out may be less than one or more than one. The last block focuses on the comparison of different sharing outs and on the reconnaissance of equivalence. The following is a task from the last block: From this cookie drawing, Julie will eat one fourth, and Michael will eat two eighths. Who will eat more, Rosa or Luis?

The study's second instrument was the teaching program, with a constructivist focus, which basically considers that children learn and develop as they build accurate concepts of their study subjects. This building includes the child's active attitude, his availability, his previous knowledge in an interactive situation where the teacher's role is that of a guide and mediator between the child and the culture (Solé and Coll, 1999).

[5] Pictograms: drawing representations of qualitative and quantitative aspects of a certain problematic situation involving the use of fractions. Operations can also be made with the use of pictograms (Valdemoros, 1993).

The didactics considered for the teaching program is Kamii's proposal (1985), presents three teaching principles: autonomy, everyday situations, and collective games. From this point of view, the purpose of the teaching program was to create a proper environment where the child could adequately develop the activities proposed in the work sessions and that would allow him to acquire experiences in which he could establish different types of relations that would help him building basic knowledge on the subject of fractions.

The tasks that make up the teaching program were designed taking into account the results obtained from the initial questionnaire (the child's previous knowledge on the learning content) and the purposes of our research. The activities included are related to the context of everyday situations and collective games.

It was carried out in twelve 45 minute sessions[6] every three days. The activities developed in the work sessions were: covering a figure's surface with other equal figures, building a figure with other equal figures, figure-folding to obtain two, four, and eight equal parts, partition of figures in two, four, and eight equal parts, comparison of a figure's fractional parts, identification of a figure's fractional parts, writing the fraction that represents each part of the fractioned unit in halves, thirds, quarters, and eighths, the game used for the exchange of ideas and coordination of points of view for the debating of answers as a feedback process of acquired knowledge, and the solving of verbal sharing out problems.

In said activities several manipulative materials were used, some were acquired, others adapted and others were designed. The manipulative materials were used in the first ten work sessions but not in the last two. Below, we enumerate some of them.

Pattern Blocks is made of prisms in different colors, shapes, and sizes that can fit together to build other figures. It was used in activities to cover and build figures. Geometric figures made from sheets of paper in different colors and sizes (squares, rectangles, and circles), they were used to carry out activities of partition in half, quarters, and eighths. They were also used to represent objects such as: chocolates, cakes, cookies, jellies, etc., in sharing out situations.

The "figure resaque" was taken from the Montessori didactic material called "metallic resaque", with several modifications. It was used for comparisons of fractional parts, which fraction is larger than or equal to another.

A domino set was designed in plastic, with the characteristics of a common domino. The difference lies on the fact that it is related to the fractions: ½, 1/3, ¼, 1/5, 1/6, and 1/8, and it was used to bring about the exchange of ideas and coordination of points of view among the children, in order for them to debate their answers as a feedback process of acquired knowledge from previous activities.

[6] Sessions: the number of sessions developed in teaching is equal to the amount of time which is officially dedicated to these topics.

After applying the teaching program, the initial questionnaire was applied again to the ten children who participated in the working sessions. The period of time which went by between the first and second application was of five weeks, which makes the solution of the second one based on the first one impossible. The purpose was the assessment of the advances achieved during the teaching program. Based on the results obtained in the last questionnaire and the participation of the teaching program, three students were chosen to carry out an interview with each one of them.

The purpose was to verify if the given results were solid in the last evaluation, as well as to search for an answer to some situations manifested in said evaluation, which weren't fully appreciated. Likewise, the interviews allowed the strengthening if the manipulative material is required by the student when developing his/her resolution strategies of the given tasks. The interviews were semi structured (according Cohen and Manion, 1990), due to the fact that there was a previous protocol and the needed material for each case. The general nature of the interviews may be considered as an evaluation instrument, which means that it was carried out *a priori* of the last questionnaire application so as to verify that the answers to the tasks presented by the students in said evaluation were sound. The purpose of this interview evaluation is different from Piaget's exploratory one (1978).

The interview's design consisted of the presentation of five semi structured tasks. Pictograms were not shown on them so that the children could solve the problems using different strategies. The three first tasks are presented in sharing out situations, the result of the first and the third ones are less than one and the result of the second one is greater than one. The fourth task presents sharing out situations in which relations of order and equivalence between ½ and ¼ of a whole are established. The last task establishes the equivalence between ½ and 2/4 of a whole. Next, we present the fourth task: from a whole pie, Mrs. Linda bought a half and Mrs. Rosa bought a fourth. ¿Who bought more pie?

In order to validate this study, the results obtained in the questionnaire were considered. Such questionnaire included the one of the interviews and the observations of the research responsible and the one of another observant. The compiled data in the questionnaire , the observations in the teaching sessions, and the observations and data obtained in the interviews were compared in the previous analysis.

Results

In the analysis of the initial questionnaire, the children resorted to their previous knowledge (acquired from experiences of their family and the society) in order to solve the tasks. They did not come across with any obstacles to carry out equitable partitions in two and four parts of a continuous quantity. Nevertheless, most of them presented problems when fractioning in three equal parts a continuous whole, manifesting non-equitable partitions; despite the fact that the task's sentence specifies that "all the persons must get the same". They also had problems when solving sharing out activities of a discreet whole among *x* number of persons.

The children who participated in this evaluation showed greater difficulty in the activities when establishing the relation of order and equivalence between the obtained parts in two different situations of sharing out. It is safe to state that the few experiences which with the second grade students count, in regards of the equal division of continuous wholes limit their skills to develop appropriate sharing out strategies. These students also showed lack of vocabulary in order to name the fractional part in the distributed whole, which reveals that the cultural contents of their previous knowledge was really low.

While developing the teaching program, teamwork qualified children to exchange their ideas, carry out decision-making, discuss their answers, as well as accept their own mistakes and their knowledge feedback. The manipulative material was required by the children so they can solve problems; sometimes it was used as a feedback support when they had doubts of what they were doing or when they wanted to prove their answers.

The individual work brought about development in discovery and creativity skills when the children solved the tasks by themselves.

The result analysis given by the second application of the questionnaire, confirms the advances achieved by the children within the teaching program. The difference between both evaluations is self-explanatory, not only due to the number of right answers per student, but also because of the processes of significance, representation, and partition which they develop in order to solve the tasks.

Regarding partitions, the children carried out equitable partitions in two, three, four, and eight parts of continuous and discrete. Wholes in matters of sharing out, they made exhaustive equitable situations of sharing out in two, three and four equal parts. They established the order and equivalence relations between two different situations of sharing out. The students used symbolic fraction expressions to name the fractional part which came as a result of distributing the whole.

The capacity shown by the children when trying to solve these tasks by means of the appropriate procedure in the second application of the questionnaire, may be considered as a background for a development in their mental structures. To seek and find the answer to a problematic situation by themselves is a sign of improvement. With the aforementioned information, we may state that the teaching program fostered the usage of more appropriate partition and sharing out strategies.

After assessing the results obtained in the last questionnaire, we carried out the interview of three children: Manuel, Ana y Lizeth in order to enquire about the answers to some situations which manifested themselves in the evaluation and which were not possible to analyze in depth.

Manuel solved the tasks mentally, providing the correct answer for each and everyone of the tasks given to him. We believe that the child effortlessly constructed the mental objects related to the fraction, which allowed him to do without the manipulative material. He also learned the meaning of fraction as related to with sharing out. This was evident when he mentally manipulated the fractional parts of a unit obtained by using a partition. He also established the order and equivalence relations between two different situations of sharing out made in one same unit.

Ana had to use the manipulative material to represent each problematic situation in order to visualize it first and then continue to construct the mental strategy. The educational assistant helped her in developing the correct process for each activity in order to get the correct answer. She was able to identify the existing relation between the numerator and the parts given to a person. She also found the correspondence between the denominator and the parts into which the unit was divided. The generalization of the concept was made in an intuitive fashion. But she was not able to recognize the equivalence relations between two different situations of sharing out done for one same unit, despite using the manipulative material to develop strategies. There's no doubt she will require more teaching time to attain a solid learning of the relations of equivalence between fractions.

Lizeth used pictograms to develop her strategies and to solve the tasks. The representations of the problematic situations by means of pictograms, made getting the right answers easier. She also managed to mentally establish the relations of equivalence of two different situations of sharing out made for a single unit . Such pictographic representations proved to be a good aid when explaining her thought.

Conclusions

This teaching program fostered and developed the basic semantic contents related with fractions, helping the children to construct their own knowledge based on daily life experiences. During the development of the teaching program , the students successfully handled diverse geometric shapes with the aid of manipulative material, involving several modes of sharing out in: halves, fourths and eighths. In the final stage of the teaching program when solving sharing out problems, the students were able to use the technical-symbolic language with clear support in diverse significance processes.

By using the manipulative material, the children managed to establish the relations of order and equivalence between fraction couples. The manipulative material, geometric shapes, divided into halves, thirds and eighths were useful when the child wanted to identify and write the fraction represented by each fractional part of the unit. The educational assistant, "domino", fostered debate and exchange of ideas between the students for them to have a feedback in their acquired knowledge.

After building some basic fraction notions, the students broadened their knowledge when solving sharing out problems using different strategies without the aid of manipulative material. This research shows that the study of fractions in primary schools may be taught in a very propitious fashion through partition, sharing out and equivalence activities for which the manipulative material have proven to be extremely useful.

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THE ROLE OF MATHEMATICS EXPERIENCES IN FORMING PRE-SERVICE ELEMENTARY TEACHERS' VIEWS OF MATHEMATICS

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Abstract. *The purpose of the study was to examine pre-service elementary teachers' views of mathematics and the experiences that influence their development. A model was created in order to be able to investigate students' experiences and views in a fruitful way. The study was carried out in a mathematics methods course during the academic year 1999-2000 in a Finnish university. The study shows the importance of finding out and taking into account students' views of mathematics during their mathematics studies. In addition, the data lead to the conclusion that it is possible to influence students' views of mathematics during their studies.*

Introduction

Over a period of several years pre-service elementary teachers have many experiences that are linked to mathematics and its teaching and learning. The experiences shape their views of mathematics and thus influence their ability to receive new knowledge during their studies. In addition, their views of mathematics influence their teaching at school in the future and therefore also their pupils' views of mathematics. (e.g. Ernest 1989) The results of prior research show that pre-service elementary teachers' views of mathematics are not the best possible at the beginning of their studies from the point of view of their future career. Their beliefs and knowledge about school mathematics seem to be restricted. In addition, many of the students have a poor attitude toward mathematics and some of them are even afraid of it. (e.g. Ball 1990)

This paper addresses a major problem that has been noticed in mathematics education in many countries: that mathematics education programmes appear to have only a limited affect on pre-service elementary teachers' capacity and willingness to change their teaching (e.g. Hill 2000). There have been many attempts to develop mathematics studies in order to bring about permanent changes in students' views of mathematics and their teaching practice. Some studies have paid attention to students' school memories (e.g. Trujillo & Hadfield 1999), others to increasing students' knowledge (e.g. Graeber 1999), to improving their attitudes (e.g. Ellsworth & Buss 2000) and to diversifying their beliefs (e.g. Vacc & Bright 1999). Despite all of these studies, the problem of effectiveness still remains unsolved. The study reported in this paper was designed to contribute to the solution of this problem (Pietilä 2002).

The view of mathematics

The purpose of this study was to examine pre-service elementary teachers' experiences in mathematics and their influence on students' views of mathematics. The view of mathematics develops with exposure to different experiences with mathematics in interaction with affective, cognitive and conative factors (e.g. Op 't Eynde, De Corte & Verschaffel 1999). Emotions, beliefs, conceptions and attitudes work as a kind of regulating mechanism in the formation of one's view of mathematics. In addition, learning demands cognitive aptitudes, like understanding, identification, thinking, evaluation and reasoning as well as conscious goal-oriented aspiration and activities. On the other hand a student's view of mathematics also influences his or her understanding, decisions, affective reactions and actions, for example in different mathematics-related learning situations (Schoenfeld 1985). In addition, self-concept is of great significance in the formation of the view of mathematics (cf. McLeod 1992).

The view of mathematics is therefore defined here to be a combination of knowledge (e.g. Shulman 1986), beliefs and conceptions (e.g. Thompson 1992) as well as attitudes and emotions (e.g. McLeod 1992) that develop with exposure to different experiences with mathematics. The view of mathematics consists of two parts: knowledge, beliefs, conceptions, attitudes and emotions about

1) oneself as a learner and teacher of mathematics, and

2) mathematics and its teaching and learning.

Both parts can be divided into smaller parts. The second part includes, among other things, views about how teaching should be organized and what the roles of teacher and students are.

Mathematics experiences are of central importance in the formation and development of the view of mathematics. A model for personal experiential knowing made by Malinen (2000) was modified to model the formation of the view of mathematics. The view of mathematics consists of a hard core, which contains the persons' most fundamental views, and a protective belt, which contains more flexible views (cf. Green 1971: the psychological centrality of beliefs; Kaplan 1991: deep and surface beliefs). Mathematics experiences need to be penetrated to the hard core in order to change the view of mathematics in an essential way.

Former experiences (or rather the situations that are linked with them) are marked in the model so that their arrows reach the hard core of the view of mathematics (at the bottom of figure 1). This is based on the reasoning that home, school, friends, myths about mathematics, and temporary posts as teachers have influenced a student's view of mathematics before his or her studies. One example of the mechanisms of this influence is the fact that students' study of mathematics at school is usually textbook-centred. It is then natural that mathematics is merely considered to be computation. Likewise for example the myth that boys are better in mathematics may have influenced one's view of mathematics.

Experiences during studies (or rather the situations that are linked with them) are marked in the model (at the top of figure 1) so, that their arrows reach only the border of the protective belt. Student can have meaningful experiences in mathematics methods course, other studies and practice teaching. In addition, temporary posts as a teacher can be significant even at this stage. For example, a student can receive guidance in the mathematics methods course on how to use manipulatives in his or her teaching. Students also have an opportunity to use temporary posts as a means of applying what they have learned and can reflect on its usefulness.

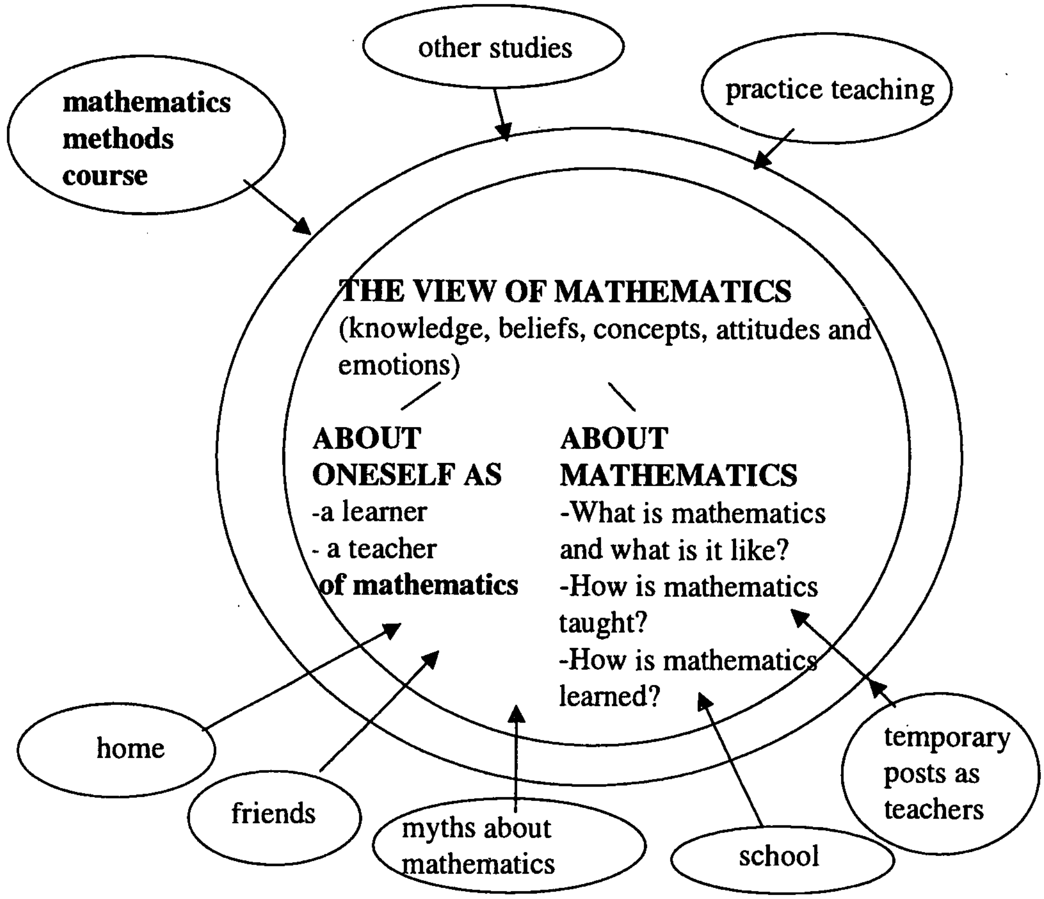

Figure 1. A model for the formation of the view of mathematics

The study reported in this paper was designed to determine 1) how pre-service elementary students viewed mathematics at the beginning of the mathematics methods course and what experiences had shaped their views, 2) what kinds of experiences students find meaningful in shaping their views of mathematics during their first year of university study, and 3) the main changes in the students' views of mathematics during their first year of study.

Attempts were made to influence students' views of mathematics using a variety of research-based methods during the mathematics methods course (4 credits). It was considered important to focus on the students' views at the beginning of their studies in order to be able to arrange the most beneficial learning environment (cf. Malinen 2000). In addition, it was thought significant that students pondered their views of mathematics, which demanded a safe atmosphere (e.g. Stuart & Thurlow 2000). The studies were designed so that students could try out as much as possible of what they had learned. It was assumed that exploring and learning the use of manipulatives would help students to understand elementary school mathematics more deeply (cf.

Quinn 1998). At the same time it was assumed that students would learn to teach for understanding (cf. Graeber 1999). In addition, combining theory and practice was thought to be able to help students to consider the learned topics useful (cf. Hill 2000).

Methodology

A research method based on phenomenological starting points was used in methodological solutions, but problems related to interpretation and understanding were also examined from the viewpoint of hermeneutics (cf. Giorgi 1997). The research material was gathered from the students (N=80) in the form of written homework as a normal part of their study. The questions were made as broad as possible and they were designed to obtain information about certain themes. Students wrote five different letters during their mathematics studies that dealt for example with their experiences of mathematics at school and at the mathematics methods course. In addition, eight students were interviewed approximately one year after the end of the mathematics methods course in order to increase the trustworthiness of the study.

The research material was analysed using following steps: 1) The material was read through twice in order to get a sense of the whole. In addition, it was important to 'bracket', which means to put aside things that we know in order to experience students' experiences freshly. 2) The texts were divided into meaning units (themes of the study). 3) The students were grouped based on their situation in the beginning of their studies. 4) The research material was analysed one research question at a time. 5) The contents of the meaning units were identified and recorded. 6) The common features and structures for different groups were recorded. (cf. Giorgi 1997)

Interviewed students read analyses of their views of mathematics at the beginning of their studies and checked their correctness (member check). An effort was made to increase the validity of the study by means of triangulation, prolonged engagement, peer debriefing and referential adequacy. (Lincoln & Guba 1985)

Results

The students were divided into four groups based on the answers given at the beginning of their studies. Students' views of themselves as learners of mathematics were considered as criteria for this grouping because they influenced very holistically both their descriptions of their experiences and their views of mathematics. The groups were categorized according to their views of mathematics as follows: 1) Mathematics is challenging problem solving (13%), 2) Mathematics is important and usually pleasant (36%), 3) Mathematics is one subject among others (20%) and 4) Mathematics is difficult and unpleasant (31%). This data led to the conclusion that only approximately half of the students are interested in and/or enthusiastic about mathematics and about studying it at beginning of their studies. Some students were even afraid of mathematics (c.f. Trujillo & Hadfield 2000).

Mathematics is a bugbear. Mathematics is just the thing that I have always felt difficult. When I was small others were afraid of bugbears while I was afraid of mathematics. That bugbear *has kept me in his possession until these days. (a letter written at the beginning of the studies, group 4)*

In addition, all students but the students in group 1 had a very narrow view of mathematics and its teaching. Most of them thought that mathematics is merely computation and that it is based on rules and procedures that should be memorized (cf. Ball 1990).

I think that mathematics is a tool that gives means, formulas and instructions for solving computational problems. (a letter written at the beginning of the studies, group 2)

In addition students' knowledge of mathematics was compartmentalized, usually very superficial and not based on understanding (cf. Hill 2000). Thus it became important to find out students' views of mathematics at the beginning of their studies and to pay attention to them during the studies.

Mathematics studies helped students to question and redefine their views of mathematics and its teaching and learning. Students felt that their view of mathematics became better organized and they knew more about what they think about mathematics (cf. Llinares & al. 2000). They noticed that mathematics could be taught using many different methods that had not been used when they were in school. They realized that it is important to understand what they learn and not just to memorize. In addition, they understood that it is important to actively involve pupils in the learning process.

A new insight to teaching and learning mathematics was that one should try to figure things out by oneself instead of trying to memorize rules. (a letter written at the end of the studies, group 2)

The studies also challenged students to re-evaluate their relation to mathematics. Their views of mathematics became more positive and they became enthusiastic about teaching it (cf. Raymond & Santos 1995).

Mathematics has actually surprised me. As a subject it is really many-sided, at its best an impossibly inspiring and attractive subject. (a letter written at the end of the studies, group 3)

In addition, as they learned more about the subject and its methodology their views of mathematics became broader and more accurate. The studies gave the students an opportunity to understand elementary school mathematics more deeply, for example by using manipulatives (cf. Quinn 1998).

The belief that not everybody can understand mathematics has vanished and I noticed during the mathematics methods course that I realized many things that I had not understood at school. (a letter written at the end of the studies, group 4)

Students reported many factors that were important from the point of view of the change in their views of mathematics. The mathematics methods course, practice teaching, holding temporary teaching positions and other studies combined to make it possible for the students to consider their studies to be meaningful. Students thought that it was important that the mathematics studies gave them the opportunity to try to explore different things by themselves. The close relationship between theory and practice helped students to see that the studies were useful (cf. Ellsworth & Buss 2000).

It is important that we did things instead of just talking about them. We handled the materials and tried them out. If you can't use them in the course why should you try them later? (a letter written at the end of the studies, group 3)

Pondering and talking about own experiences and views, finding models for their own teaching and understanding the learned topics were also experienced to be important. In addition, a positive learning atmosphere was considered to be very essential (cf. Stuart & Thurlow 2000). Conversations helped students to understand their past and improve their attitude.

The fact that I have been obliged to ponder on my past experiences as a learner of mathematics has helped me to disprove the myth: "I don't understand anything about mathematics."... We have discussed fears and attitudes toward mathematics. It has been rich and has changed my attitudes. (a letter written at the end of the studies, group 4)

The students had an opportunity to use practice teaching and temporary posts as a means of applying what they had learned. Students' experiences were mainly positive and they recognized the usefulness of what they had learned (cf. Stuart & Thurlow 2000). Students noticed that they could explain so that pupils understood. In addition, the pupils were enthusiastic about the new teaching methods.

I could test my own ability to demonstrate and concretize when I helped students. It was enlightening to realize how much concretization helped students to understand. Some of them seemed to understand some basic things for the first time. (a letter written after practice teaching, group 4)

Other studies helped students to see mathematics studies in a larger context, although their significance for enhancing their views of mathematics seemed to be small. Students noticed during their studies that they needed many skills in order to develop into good mathematics teachers. They realized they needed knowledge about both the subject and pedagogy. On the other hand the learning of new skills and knowledge increased their self-confidence as teachers (cf. Raymond & Santos 1995).

I noticed during the mathematics methods course how many facilities and skills I actually needed in order to develop into a good mathematics teacher. We treated mathematics broadly during the lectures and especially during group meetings, and I became aware of many new facets. (a letter written at the end of the studies, group 3)

Conclusion

Many positive changes took place in students' views of mathematics during their first study year. The depth and the durability of the change can still only be guessed. Some idea of this was gained through the interviews that were carried out one year after the mathematics studies had ended. Four students had participated in practice teaching during their second study year, and their views of mathematics had became even more positive because they had become more confident as mathematics teachers. Four students had not studied anything that could be linked to mathematics. Their views had remained almost unchanged, becoming perhaps a bit more realistic.

It was important on the point of view of the stability of the change that students developed the ability to recognize and reflect on their views of mathematics (cf. Raymond & Santos 1995). In addition, the awakening of students' responsibility gave the researcher faith in their willingness to change their teaching. The students realized that most of the pupils' understanding and enthusiasm depend on teacher's teaching methods. The stability of the change could probably be increased by combining theory and practice even more (cf. Hill 2000). It would be important for the students to try out what they learn as much as possible in practice (with students) during the mathematics methods course. By doing so, they would have the opportunity to become more convinced of what they learn. In addition, the material would be easier to internalize and remember.

The model developed for researching the students' views of mathematics seemed to be functional. Based on the changes in the students' views of mathematics, it was possible to conclude that knowledge, beliefs, conceptions, attitudes and emotions affect the formation of their views of mathematics. In addition, the division of the view of mathematics into two parts was justified because students' views of themselves as teachers had such a strong influence on their views of mathematics as a whole. In addition, the graphic model of the view of mathematics seemed to be functional and useful. It provided an opportunity to portray their experiences and their views of mathematics in a compact picture.

It would be interesting to follow the students' views of mathematics when they obtain posts as teachers. In addition, it would be interesting to study what experiences have the most influence on their views of mathematics, for example, the school's curriculum, colleagues, size of the class, textbook, teaching materials, in-service training and so on.

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MENTAL REPRESENTATIONS IN ELEMENTARY ARITHMETIC

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This paper discusses emerging themes about the nature of embodied objects and symbolic procepts in the context of elementary arithmetic. Drawing upon the mental representations and strategies children use whilst executing mental addition and subtraction the paper also aims to illustrate how the nature of base objects may be determined and used within elementary arithmetic. The paper indicates how the nature of base object may change in the development of arithmetical concepts and illustrates how for some it may be a stepping stone to more sophisticated thinking whilst for others it acts as epistemological obstacle preventing the attainment of such thinking.

INTRODUCTION

The notion of numerical concepts being formed from actions with physical objects forms the background for the conceived cognitive development of simple arithmetic (see for example Piaget, 1985; Gray & Tall, 1994). Encapsulation (or reification) theories suggest that a cognitive shift takes place between carrying out actions and the formation of numerical concept. How this process takes place, however, remains the subject of theory and open to debate. More recently, Gray and Tall (2001) suggested that a simple switch of viewpoint where theorised encapsulation (or reification) of a process as a mental object may be linked to a corresponding embodied configuration of the objects acted upon (the base objects) may reveal some powerful insight into the different ways which individuals construct mathematical concepts. Our effort to gain an insight into how this is happening has taken a route to consider mental representations.

This study considers the association between different kinds of mental representations projected by 8-11 year olds and the construction of arithmetical concepts. Different kinds of mental representations are identified based on what objects and actions are mentally represented when children are presented with an addition or subtraction. Our purpose is to compare the meaning embodied in the objects and their configurations as denoted by the children's mental representations with process-object abstraction.

Earlier research by Gray, Pitta, Pinto and Tall (1999) indicated that students who consider the descriptive qualities of the embodied object remain at a more primitive level whereas, those that rely on the more intrinsic qualities of the object such as the mathematical symbolism and its relationship to other objects move to a more sophisticated level.

THEORETICAL BACKGROUND

Encapsulation (or Reification) of a procedure

Piaget believed that learning as well as performing mathematics was a matter of active thinking and operating on the environment. This activity was strongly linked to 'physical experience' which "consists of acting on objects to discover the properties of the objects themselves... not from the physical properties of particular objects, but from the actual actions (or more precisely their co-ordinations) carried out by the child on the objects" (Piaget, 1973, p.80). His focus was on the way in which actions and operations became thematized objects of thought or assimilation (Piaget, 1985, p.49). Interiorising processes into mental objects is seen as a fundamental way of constructing mathematical objects; dynamic actions conceived of as conceptualised entities is now frequently associated with notions such as encapsulation (Dubinsky, 1991) and reification (Sfard, 1991). In the context of number what is clear is that there is a growing sophistication in the nature of the entities operated on, from physical objects to mental operations with the number symbols themselves. This development is manifest in an increasing detachment from immediate experience, the evolution of different aspects of counting and a change in the form of unit counted (Steffe, Richards, von Glasersfeld & Cobb, 1983).

In the context of elementary arithmetic the nature of the unit counted may be seen to be analogous with the nature of the object operated on. Tall, Thomas, Davis, Gray and Simpson (2000) draw the distinction between "perceived objects" and "conceived objects". The first is based upon perceptual information where the focus is upon specific physical manifestations of the object. The second occurs when the focus shifts from the physical manifestations to the actions/process performed on them. Such a distinction has been used by Gray and Tall (2001) to distinguish between the notions of embodied object and symbolic procept. The former "begins with the mental conception of a physical object in the world as perceived through the senses" (p.66) and can "only be constructed mentally by building on the human acts of perception and reflection (p.67)". Though they see an increasing sophistication in the notion of embodied objects, Gray and Tall see a significant distinction between embodied objects such as a triangle and a graph on the one hand and the symbols of arithmetic on the other. The latter

> "act as pivots between processes and concepts in the notion of procepts and provide a conscious link between the conscious focus on imagery (including symbols) for thinking and unconscious interiorised operations for carrying out mathematical processes"
>
> (Gray & Tall, 2001, p.67)

Within the field of elementary arithmetic the theorised encapsulation or reification of a process as a mental object is often linked to a corresponding embodied configuration of the objects acted upon which we have indicated to be the base object. Counting processes operate on physical objects. Thus the seemingly abstract concept of number already has a primitive existence in the physical configurations of base objects. In the context of addition, for example, the base objects are initially physical objects, then they become figural objects but later these become redundant as they are subsumed within a counting process which itself can be compressed into the concept of sum.

> This emphasises a proceptual structure that consists of a theory of related procepts, including the base objects on which the processes act, the symbols as process and concept, and the concept image on which the processes act. (Gray & Tall, 2001, p.68)

This papers attempts to consider the way in which children's explanation of the way in which they solve elementary number tasks may give us insight into the nature of the object that is being used. We feel that the distinction is essentially one that is to be made in terms of the use of external or internal representations. The latter suggests the notion of mental representation and in the sense that is discussed within this paper these mental representations may be embodied objects or symbolic procepts. Clearly a mental conception of a finger is an embodied object and of course the actual physical object 'finger' is not. Both may be referred to as base objects. However, we suggest that the 'threeness' associated with the display of three fingers is embodied. Clearly then, the mental representations of numerical objects reported by Seron, Pesenti, Noel, Deloch, and Cornet (1992) may also be seen as embodied objects. Their subjects reported seeing simple digits or numbers, numbers transformed into patterns as found on a dice, numbers with colour and numbers as on a number line. They suggest that quantity directly represented by "patterns of dots, or other things such as the alignment of apples or a bar of chocolate (p.168) may be deemed to be analogical. In our sense these objects are embodied — they arise initially in perception but they can carry mental ideas. Thus the dots on the dice may carry the idea of five.

In this study one portion of the complex structure of mathematics is isolated, that of mental additions and subtractions to 20. By considering children's mental representations and more specifically the object of the representation whether embodied object or symbolic procept, it tries to make sense of the influence that they have on children's successful understanding.

METHOD

The research was conducted in a "typical" primary school in the English Midlands. Sixteen children aged 8-11 years old, representing the extremes of numerical achievement in each of the four years - thus two at each extreme of each year - were presented verbally with mental additions and subtractions to 20. The children's numerical achievement was measured by criterion based test results available in the school and a numerical component which formed part of a larger study of which this paper is part.

For these combinations children were reminded that the object of the interview was for the interviewer to get a sense of the approach the child used to solve particular items and to gain some indication of what was happening in the child's head while doing these mental additions and subtractions.

Classification of the Results

To classify children's representations we have adopted some of Steffe's *et al* (1983) classifications for two reasons, since it is a finely-grained analysis of counting units and provides valuable insight into numerical development.

Automatic Responses: These were identified whenever a child indicated that "I just knew it" and no overt or covert actions appeared to be associated with the response. Here the base objects will be the inputs, for example 3 and 5, although it is difficult to discern whether an individual who knows facts projects the use of a mentally embodied object or the use of a symbolic procept (see abstract representations below).

Perceptual Representation: Strategies are applied with the support of physical items, for example fingers. Physical items are the base objects. The embodiment is the mental association made between the configuration of the fingers, as sets to count, and the quantity counted.

Verbal Representations: Here the number word is taken as a substitute for countable items. Typical examples include:

> Five, counted in my head three, four, five. (Y3-, 3+2)
>
> 9, 8, 7,... just said that to myself. (Y6-, 9-7)

Here the base objects are mental embodied objects. The number words, are embodied as mental conceptions of the number counted.

Figural Representations: Here the counting process is taking place in the absence of actual items but is associated with visual or verbal analogues of the items:

> I saw a line of numbers. It was one, two, three, ...13, 14, 15. After 10 the numbers got bigger. I counted from 1 until I got to 8. (Y3-, 3+5)

The mental conceptions of fingers are embodied objects.

Abstract Representations: The use of the term is based on the notion of Steffe *et al.* (1983) but in the current context not only did a child not require to construct countable units but the symbol was identified as the object of thought. The classification was most frequently associated with derived facts. A typical response could be:

> [Said to myself] the difference between six and seven is one, so two sixes are 12 and 12 plus 1 is 13. (Y4+,7+6)

This category suggests the use of symbolic procepts.

All responses were videotaped and supported by field notes. The results are established from analysis of the video transcriptions.

RESULTS

Analysis of Results

Children's combinations to twenty

As illustrated in Table 1 (which illustrates the percentage use of each representation) perceptual and verbal counting responses were given more often by low achievers whereas automatic and abstract responses were given more often by high achievers. Very few figural responses featured in either of the two groups.

	Automatic	Abstract	Perceptual	Verbal Counting	Figural
High achievers	64	29	4	2	1
Low achievers	10	8	61	17	4

Table1: Representations of Mental Arithmetic: Number combinations to twenty

Low achievers' combinations to twenty

Perceptual representations featured strongly amongst 'low achievers' and this may have been associated with vocal or sub-vocal counting. Most frequently fingers, these perceptual units, became objects of thought which were sequentially tagged in the counting procedure. Tagging was mostly overt in that children looked directly at their fingers, tagging those on one hand with those on the other, or, if the number was relatively large, tagging through touching the desk or even the nose. On some occasion motor acts were used as substitute for tagging. At times the counting was not associated with any obvious tagging. Children would "feel" movements in their fingers without any obvious sign that they were doing so. When this happened, it seemed that the younger the child or the more difficult the sum the more exaggerated this movement was. Therefore, while a Year 4 'low achiever' said:

7, 8, 9, 10, 11 and I was counting on my fingers. (Y4-, 4+7)

a Y6 child said

I was counting in my head 6, 7, 8, 9 but had it as a finger feeling. (Y6-, 6+3)

We conjecture that the need to use perceptual items amongst the 'low achievers' reinforces the evidence that symbols need to be concretised.

There were few instances where low achievers made reference to figural representations. These may be seen in the context of analogical representations. In the instances associate with number combinations to 20 these figural representations resembled the number forms of Seron *et al* (1992) in that they were based on the number line:

I saw the number line in my head and saw a lot of numbers in my head on a number line. There are lots of numbers but it depends on how high the numbers are. High means how big the numbers are. This time it was three plus four. (Y3-, 3+4)

The smaller combinations were often associated with verbal counting and the numbers themselves served as countable objects.

I was thinking of it. Add. It is 11. I said 10, 11 to myself. (Y6-, 9-2)

5. I counted in my head 3, 4, 5. (Y3-, 3+2)

Often the verbal tone or double counting accompanied the counting act, although this sometimes led to errors:

Got to 13 to add up to 17...13, 14, 15, 16, 17. 13. One 14, two 15, three 16, four 17, five ...said all in the mind. (Y6-, 17-13)

In all the above examples there is an intrinsic similarity, 'low achievers' use a lengthy counting procedure as if it is a generalisable process. Irrespective of whether the object is a concrete finger or a mental number line or a number word they are essentially doing the same thing. Low achievers seem to be trapped in the only process that one is able to do with base units that are physical — counting. The overemphasis on embodied objects and on surface characteristic of objects and processes does not allow children to see the power of symbolic procepts. Therefore, things are becoming increasingly difficult as numbers are getting bigger.

High achievers' combinations to twenty

High achievers' responses were mainly 'automatic' and 'abstract', the first category indicating an almost immediate response whereas in the second the symbols dominated the children's thinking.

The symbols were frequently associated with the input combination or with the final solution.

I saw the eight then the two and then I see them altogether. (Y5-, 8-2)

The tendency to see the input symbols and/or the final output was relatively common amongst those 'high achievers' who reported seeing symbols though other characteristics were also apparent.

I saw a picture of 9-2, black. (Y6+, 9-2)

I saw 9 and 7 flash, not as a sum but just 9 and 7. I then saw 2 much stronger. (Y6+, 9-7)

'High achievers' also reported symbolism associated with the formation of derived facts:

I said [to myself] seven and seven and take away one. Told you thirteen. (Y3+, 7+6)

I *saw* 13-2 and *thought* I would split it up into different parts to make it easier. This stood out. Saw 13-3 then 10. Then I saw 10-2 this stood out. Then I saw 8. 13-3 and 10-2 stayed at the same time. I saw 8 on its own. (Y6+, 13-5)

In the second example the child seems to be oscillating between visual representations "saw" and verbal representations "thought". This relationship between visual and verbal mental representations was one that clearly gave the child the power to complete the combinations. The visual image of the high achiever generates the idea and then it is put aside so that she can focus on the relational characteristics. The symbols are thought generators and not the mental embodied objects used to carry out an action. Of course the fact that their mental representations are dominated by symbols, which can either be seen or talked about, gives them this flexibility.

'High achievers' are not only able to filter out the surface characteristics of embodied objects and the lengthy procedures but they are also able to avoid "difficult" numerical combinations by transforming the question to a more manageable one, or one that is derived from an easier known fact:

> I said to myself the difference between 6 and 7 is 1 so 2X6=12, 12+1=13. (Y4+, 7+6)

'High achievers' seem to be able to filter out surface characteristics from the embodied objects, condense the counting procedure or most often omit it altogether and carry out their mental processes in a truly abstract fashion. Their emphasis lies on the more intrinsic qualities of the mathematical symbolism, process to do and concepts to know. Another strength of the mathematical symbolism is that it can be seen and also talked about. It is very rare that an embodied object can be interpreted as a process and a concept or carried in the mind as a visual sign or a verbal word.

DISCUSSION

Hearing the combinations triggered the 'low achievers' to carry out a procedure; in this instance an overwhelming desire to count. It could be hypothesised that in their failure to recall combinations the representation that the 'low achievers' used, evoked either physical objects or mental conceptions of these objects. In the first instance, their mental representations were embodied as general representations of the number sequence. This general mental representation was retrieved either as physical objects or as an embodied object, such as the number line. Qualitative differences in the nature of the base objects were determined through this real or imaginary difference. Thus the 'specificity' of the number sequence is not only identified through the inclusion of the numbers that need to be counted but also through the different objects used. Therefore, it seems that the choices the children make (consciously or subconsciously) focus on the nature of the counting procedure and representation (mental or physical) they need to support this procedure. Increasing procedural efficiency may determine both the counting strategy, for example, count up as opposed to count back, and the nature of the base unit to be used. However, within the three categories of representations identified, physical, figural and verbal we may see the gradual shift in the nature of the base unit from perceptual to a mentally embodied object.

In contrast categories identified for 'high achievers' illustrated extensive use of the retrieval of known facts, either to give an automatic response or in order to proceed to a derived fact. Clearly the nature of the entities operated upon have changed. Now they signify conceptual entities that appear to exist "independently of the child's actual or represented motor activity" (Cobb, p.168). The 'high achievers' are performing the operations of addition and subtraction on symbolic procepts. It is conjectured that underpinning this approach is the power that emerges from their representational flexibility. They seem to be carrying out a 'search' for the most appropriate number fact that can be used. They retrieve it and either present it as the answer or manipulate it in order to reach an answer. The former is not easy to qualify, the latter is the essence of proceptual thinking. The symbolic procept acts seamlessly to switch between a mental concept to manipulate to a process to carry out.

Essentially the role of base objects may be seen as a stepping stone to higher order concepts. However, they may have specific meanings for some individuals that act as epistemological obstacles that prevent a hierarchical development that is essential to progress. These differences would seem to become apparent very early within the child's mathematical development.

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THE INFLUENCE OF SOCIAL ISSUES ON THE RE-CONSTRUCTION OF MATHEMATICAL NORMS

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This report centers on the diversity of interpretations of the norms of mathematical practice. In particular, it focuses on one such norm and in how a group of students changes its interpretation after a whole class discussion in which issues of social positioning and power are clearly in play. We argue that an understanding of these social issues is necessary in order to develop learning environments that are more sensitive and hence equitable to all learners.

INTRODUCTION

The frequent lack of cultural and social affinity in the mathematics classroom requires an understanding of what factors lead the learner to re-construct his/her first interpretations of certain norms. The presence of different meanings for the classroom norms and the obstacles to negotiate these meanings are to be taken into account when analysing shifts in students' participation trajectories. An understanding of these shifts in participation seems necessary towards the development of equitable learning environments. In this report we center on one episode from a recent research (Planas, 2001) that illustrates the diversity of interpretations for the norm of the "context of a problem" in a problem-solving situation. Relevant excerpts from classroom transcripts will be used to uncover the construction of social meanings behind the multiple ways of interpreting this norm.

THEORETICAL FRAMEWORK

The idea of a system of meanings has been broadly used to characterize the so-called 'culture of the mathematics classroom' (Voigt, 1998). In particular, the normative meanings, that is the obligations and rules of how to act, conform an important part of this culture. The research of Yackel and Cobb (1996) has pointed to the diversity of interpretations of classroom norms and its impact on the teaching and learning of mathematics. However, recently some researchers have questioned an approach essentially based on the cultural issues of the classroom. There has been a call to emphasize the 'social' in the socio-cultural paradigm of research in mathematics education (see, e.g., Lerman, 2000). As it is argued by Morgan (1998), the study of the norms from a socio-critical perspective is still in its beginning stages. There is a need to look at how norms can be interpreted differently as well as how different interpretations may be related to different values given to participants (Abreu, 2000). Abreu states that the tasks which need to be solved in the classroom can be made difficult or easy depending on how participants interpret the legitimate norms.

RESEARCH QUESTIONS AND METHOD

In this report we address the following questions:

a) Given a norm, can it be interpreted differently by the various participants?
b) How do different interpretations of a norm influence each other?

The research was conducted in three classes located in three urban high schools in Barcelona, Spain[1]. In these schools there is an increasing number of students socially at risk. They are students belonging to ethnic minorities, first or second generation immigrants. Many local families do not want their children to attend such schools due to the high percentage of immigrant population from Magreb, Pakistan, India, South America as well as the local gypsy students. The enrollment in these schools is therefore lower than usual.

A total of 36 lessons were videotaped, tape-recorded and transcribed. Non-participant observer methods, field notes and individual interviews of students were also used. There was a triangulation of methods for obtaining the data and a triangulation of perspectives in analyzing them. The video-recordings and the interviews' transcripts were discussed by the teacher, the observer and an external researcher in regular meetings. The lesson we document here is embedded in a problem-solving unit. The nine students were asked to work in small groups. This was followed by a whole class discussion in which the three groups shared their approaches. They were given the following problem:

Here you have the population and area of two neighbourhoods in your town. Discuss in which of these places people live more spaciously:*

Neighbourhood 1 (N1)	*Neighbourhood 2 (N2)*
135,570 inhabitants	297,930 inhabitants
14 km^2	3 km^2

(*Neighbourhoods 1 and 2 were given their real names and were known to the students)

RE-CONSTRUCTIONS OF ONE MATHEMATICAL NORM

The question 'where do they live more spaciously' created important differences in the interpretation of what was required to solve the problem. Some students decided that there were data missing while some others were comfortable with solving the problem as it was. This led to different interpretations of the norm "context of the problem". One group (GA) took the problem as a typical school task and solved it in an algorithmic way. They divided the population by the area and reached the conclusion that "they live 10.25 times better at N1 than at N2." Clara, this group's leader said, "we've solved similar problems before; you have to make some divisions, that's all. The more neighbourhoods, the more divisions."

Another group (GB) also used the concept of density. They divided the population by the area but then expressed some concern as to whether the answers were reasonable. For example for Neighbourhood 1, the division led to 9,683.57 persons per square kilometer. They agreed that it should be 9,684 p/km^2 because "you cannot have 0.57 people." This group also concluded that people would "live much better in N1" but most of their time on the problem was spent on their concern on what to do with these decimal answers for people.

The third group (GC) engaged in an animated discussion about the wording of the problem. Their concern centered on a need to know more about the living conditions in the two neighbourhoods. Albert, one of the group members, said, "well, it doesn't say anything about the people. We need to know if they live in flats or houses. Some of them live very spaciously and some of them live very badly, right?" And Lordwin, another student in this group said "it depends on how many children there are in each family, then we write the number of children there, we'd better write it again." Group C took their task as that of rewriting the problem to fit more closely their own lived experience.

All the groups offered reasonable explanations without making incorrect uses of mathematical operations or procedures. The difference was in their understanding of the norm of the context suggested by the problem. Groups A and B followed academic procedures and used the concept of density. Neither of these groups questioned the wording of the problem. Group A took it as typical school problem, while Group B showed concern for the real-world context and was aware that one cannot have decimal numbers as answers about people. Group C took the problem as a real situation for which more information was needed. These neighbourhoods were real to them. Their approach to the task reminds us of critical pedagogy contexts in which students would be encouraged to use mathematics to make sense of their world (Skovsmose, 1994). The teacher in this case, however, did not seem interested in pursuing such an approach. The other two groups were surprised at Group C's questioning of the problem statement. The whole class discussion that followed the group work shows how the social positioning of the students may have influenced their subsequent approach to the problem.

As each group faced comments from the class and was confronted with alternative norms, first approaches were reaffirmed or changed. This allows us to study not only the various interpretations of the same norm but also how dynamic these interpretations are. We only look at the changes that occured during the whole class discussion, though other possible meanings for the appropriate context of the problem may have emerged in the small groups. Groups A and B insisted on their first approach. The notion of density was still considered the key factor. Group C's change was highly significant. The change was apparently radical but we wonder whether this change was more the result of pressure from members of the other groups than from having reached internal conviction that the other approaches were more valid. Group C ended up rewriting the problem as follows:

In N1, houses have gardens and swimming pools. Families have one children. In N2, there are small houses and skyscrapers. Families have many children. But the important thing is that there are 135,570 people and 14 km^2 in N1, and 297,930 people and 3 km^2 in N2. Where do people live more spaciously?

The first part of the problem (until the sentence starting with "But") is what they had at the end of their initial group work. The second part is what they added to conform to the whole-class discussion. The critical context was somehow combined with the school context. All throughout the discussion, however, comments from this group indicate that they were not really satisfied with their revised approach. For example, Lordwin kept on mumbling that they needed to know how many children there were, and said, "now you have a number but you don't know anything about the kind of houses they live." An indication of the pressure they felt is given by Albert's comment " if we don't do as we are told, the teacher is going to get angry." The group was reluctant to give in, "we don't need to cross out our problem, do we? Just say ours is the same as Clara's (Group A leader)." In the next section we focus on the case of Albert, the leader of group C, to explore the possible role of social positioning in classroom dynamics.

EXPLORING ISSUES OF SOCIAL POSITIONING

Why did group C change their initial approach to the problem and ended up with one that somehow conformed to the expected school context? We argue that an understanding of the social positioning of students in the class may shed some light on this question. The other two groups' reluctance to accept Group C's strategy, and this group's decision to change their approach must be analysed taking into account the valorizations that were made explicit between the first attempts and the re-constructed contexts that appeared some minutes later. This expresses not only a normative difference between the three groups, but also interpersonal relations of power between individuals. For example, if we focus on Albert, we see a student who seems concerned about the teacher's reaction ("if we don't do as we are told, the teacher is going to get angry"). While the teacher showed concern about Albert's involvement, he was also somewhat sarcastic in his reaction to that group's attempt to rewrite the problem ("how cute of you..."). We also see how other students reacted rather negatively to Albert's suggestions. For example, Clara (GA) got very upset at him, "if you don't know what to answer, just shut up!", and Bernat (GB) was extremely concerned about Albert making them waste time. At a certain point, many participants were frowning at Albert. Table 1 summarizes some of the negative valorizations given to this student as well as his reaction.

<table>
<tr><td>Teacher: "How cute of you! If someday I put you in charge of writing a problem, please, remind me of your sense of humour… You are confused about what it means to think about a problem, but don't worry, don't be discouraged about the problem, it's very normal that you find it difficult"</td><td>Clara: "You must be joking as usual! Ask for help if you have no idea of how to solve the problem… if you don't know what to answer, just shut up!"</td><td>Bernat: "Don't make us waste our time! Stop clowning around! You are always trying to get the teacher's attention by saying silly things! This problem has nothing to do with swimming pools and gardens!"</td></tr>
<tr><td colspan="3"><u>Albert's responses</u>: (utterances and speech defects such as stuttering)

(to Teacher) (stumbling) "We didn't know it was forbidden to improve the problem, we thought we had to understand the situation, if…if we had known it was forb… forbidden, we would have never tried to change it, teacher, we promise"
(to Clara) (without looking at her eyes) "I'm not joking, I'm saying the same as you are but in a diff… different way"
(to Bernat) (puzzled) "I'm not a clown, you are always putting… putting me down! We didn't mean to make you waste your time, don't be so hard on me!"</td></tr>
</table>

Table 1: Some negative valorizations given to Albert

Albert is a 16 years old student who usually interprets the problems by himself without having to wait for the teacher to interpret them. His responses during an interview that same day seem to indicate that he was not really willing to change his group's approach but felt like he had no choice. The following excerpt from the interview shows to what extent he is aware of issues of power and status that took place in the class:

Interviewer: What do you think about the other groups' strategies?

Albert: What Clara and Bernat said is okay, but what I said is also okay.

Interviewer: Why do you think they didn't like your ideas?

Albert: I don't like looking for the most difficult way to solve a problem and they do.

Interviewer: What about the teacher? He didn't seem to like your ideas either...

Albert: I don't care if they [Clara and Bernat] know a lot of math because I also know a lot of things. But they don't want to know what I know... and neither does the teacher.

(...)

Interviewer: Did you like what they told you?

Albert: I'm sure they wouldn't talk to me like this if I was not repeating this grade for the second time.

Different people in front of similar circumstances may feel in very different ways. Individual differences are then an important issue in the understanding of certain re-construction processes of the mathematical norms. The same circumstances can encourage some students and discourage others. In other words, the processes of re-constructing normative meanings refer to both the characteristics of the context and those of the individual. Students like Albert easily feel upset and do not seem to struggle with such difficulties. During the interview, he reflects on how his role in the classroom would be different if he was not repeating the year. The data obtained reveals that we need a better understanding of the interaction of cultural, social and emotional factors.

When students participate in the mathematics classroom and get involved in certain practices, they do so in ways that other participants may perceive as appropriate or unappropriate. To what extent these perceptions of correctness influence the ongoing patterns of participation? Do some students have less opportunities to negotiate their meanings and patterns of participation? Do unequal opportunities have to do with status as well as with the distance to the legitimate norms? Although the results of the study do not provide complete answers to these questions, they suggest the kind of research that is needed. Cases like that of Albert illustrate the need to go deeper into the 'social'. We strongly believe that if we want any educational act to be positive both for the individuals and their communities, it will be necessary to consider the impact of the classroom social context on the learning processes.

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[1] The project is partially funded by Fundació Propedagògic.

ON HEROES AND THE COLLAPSE OF NARRATIVES: A CONTRIBUTION TO THE STUDY OF SYMBOLIC THINKING*

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This paper is dedicated to Raymond Duval on the occasion of his retirement

KEY WORDS: Husserl's philosophy of language, signs and meaning, semiotics.

INTRODUCTION AND THEORETICAL FRAMEWORK

The use of symbols in mathematics raises two different problems. The first one is linked to the mode of designation of the objects of discourse. The second one corresponds to the operations that are carried out on the symbols designating the objects. Although these problems are related, they are underpinned by different semiotic and cognitive demands. As far as algebra is concerned, the designation of objects of discourse requires a substantial reduction of vocabulary (Duval, in press). Indeed, while natural language accounts for a large set of words allowing one to *describe* objects (e.g. the next figure, the small rectangle), algebraic symbolism requires that these objects be designated using combinations of a *few characters* (*viz.* 0, 1, 2, ..., x, y, $\sqrt{\ }$ and the like). In previous papers (Radford 2001a, in press-a), I reported the tremendous difficulties that Grade 8 students had in finding a symbolic expression for the rank of the figure that follows the figure of rank 'n' in a pattern. Although the students could refer to the objects of discourse using more or less accurate descriptions in natural language, it took a long time before they could, with the teacher's help, figure out the expression 'n+1'. Algebraic language does not include adjectives, adverbs and other linguistic terms that prove to be crucial in natural language-based communication.

In addition to this, even if the students reported in my previous research could start designating simple objects with a few characters, they were not able to operate with symbols. They could not recognize that (n+1) +n, (n+n)+1 and 2n+1 referred to a same state of affairs. The problem is not merely the students' impossibility to operate with the unknown. As a matter of fact, at the same time that they were struggling with the generalization of patterns they were able to easily solve equations like 14+2e = 2+4e (see Radford, in press-b). The problem is related to the students' mode of designation of objects through algebraic symbolism.

* This article is part of a research program funded by the Social Sciences and Humanities Research Council of Canada.

Indeed, in the designation of objects, the way signs stand for something else is related to the individuals' intentions as they hermeneutically unfold against the background of the contextual activity. In the designative act, intentions come to occupy the space between the intended object and the signs 'representing' it. In doing so, intentions lend life to the marks constituting the corporeal dimension of the signs (e.g. alphanumeric marks) and the marks then become signs that *express* something, and what they express is their *meaning*. The possibility to operate with the unknown thereby appears linked to the type of meaning that symbols carry.

Intentions occur in contextual experiences that Husserl called *noesis*. The conceptual content of such experiences he termed *noema*. Thus, noema corresponds to the way objects are grasped and become known by the individuals while noesis relates to the modes of cultural categorial experiences accounting for the way objects become attended and disclosed (Husserl, 1931).

Pursuing my investigation on students' semiotic processes of meaning construction and symbol use, in this paper I want to address the question of how the students' symbolic expressions are intended to convey meaning when the students proceed with the designation of objects and operate with the designating signs in a typical short story-problem. Within the sketched theoretical framework, the research question will be addressed in terms of the *manner* (the noesis) in which students use signs to *express* particular features (the noema) of the objects of discourse. After briefly commenting on the methodology, I will suggest a distinction between story-problems and symbolic narratives. This distinction will allow us to provide an interpretation of some 'nonsensical' symbolic expressions elaborated by novice students. I will then discuss the concept of nominalization whose theoretical interest is not simply to account for the introduction of unknowns in a problem. I intend it as a theoretical tool to examine how symbolic expressions become endowed with meaning in this limbo where we have neither fully left the original story (told in natural language), nor have fully entered into the symbolic narrative (told in symbols). The last section presents a short discussion concerning the problem of abstract or formal use of signs in obtaining the equation associated with the story-problem.

METHODOLOGY

The data presented in this paper comes from my longitudinal classroom-based research program involving 4 classes of Grade 9 students. The classroom activities were designed to be carried out co-operatively by the students according to a small-group (2 or 3 students) working format and were usually followed by general discussions conducted by the teacher. Due to space constraints, I shall mention excerpts of the video-taped word-problem solving activity of 3 small groups of only one of the 4 classes. Transcriptions from the video-tapes were analyzed using the *qsr N5* software for interpretative, qualitative research (details in Radford 2000).

The mathematical activity was based on the following short story: "Kelly has 2 more candies than Manuel. Josée has 5 more candies than Manuel. All together they have 37 candies. »[1]. The same story was used to generate *three* problems involving transformations in the algebraic expression of the data. In problem 1, the students were asked to designate Manuel's number of candies by x, to elaborate a symbolic expression for Kelly and Josée, and, then, to write and solve an equation corresponding to the short story. Problems 2 and 3 included similar questions. The difference was that, in Problem 2, the students were asked to designate Kelly's number of candies by x while in Problem 3 the students were asked to designate Josée's number of candies by x.

RESULTS AND DISCUSSION

From Heroes to *Objectivities* [2]

One of the difficulties in dealing with problems involving comparative phrases like "Kelly has 2 more candies than Manuel" is being able to derive non-comparative, assertive phrases of the type: "A (or B) *has* C". If, say, Manuel has 4 candies, the assertive phrase would take the form «Kelly (Subject) has (Verb) 6 (Adjective) candies (Noun)». In the case of algebra, the adjective is not known (one does not know how many candies A has). As a result, the adjective has to be referred to *in some way*. In using a letter like 'x' (or another device) a new semiotic space is opened. In this space, the story problem has to be re-told, leading to what has been usually termed (although in a rather simplistic way) the 'translation' of the problem into an equation. I prefer here to use the term *symbolic narrative* since what is 'translated' still tells us a story but in mathematical symbols. Although there are similarities in the story problem and the symbolic narrative, the personages change. This change is best characterized as a noematic shift that brings forward certain parts of the story while putting others in the background. The 'heroes' –so to speak– of the re-told story are no longer Kelly, Manuel or Josée, but numerical relationships between the amount of candies that constitute the objectivities expressed in the new semiotic space (i.e. the symbolic-algebraic one).

Difficulties in accomplishing this noematic change or shift of attention may become an obstacle in the learning of algebra. Let us show an example in which we can see the students of Group 1 trying to produce symbolic expressions without achieving the aforementioned noematic change.

Signs as marks in narrative acts

In this group, a (wrong) calculation with comparative phrases led the students to conclude as follows:

> Stacey: Kelly has 2 more candies than Manuel. Josée has 5 more candies than Manuel. Together, they [Kelly and Josée] have 7 more candies than Manuel.

Instead of transforming comparative phrases into assertive ones, the students changed the comparative phrase into an adverbial form ('more'), something that allowed them to rank the heroes in the story problem according to the number of candies that each one had:

> Stacey: Josée has 5 [*more*]. Josée has more, Kelly is second, Manuel third. Okay, so you put x that represents... no x that represents 7, okay? 12 [*as the result of 7+5*], 9 [*seen as the result of 7+2*] but I don't know how to find... [...]. He [*Manuel*] has 7 less than these two put together (*she writes*) x–7 [...] This [37] has to equal x-7 (*suggesting 37=x-7 or x-7=37*).

The transformation of comparative phrases into assertive ones is related to the possibility of explicitly taking into account the unknown amount of candies. However, the clear introduction of a letter for the designation of such an unknown amount of candies does not fix the problem. This is shown in the excerpt below (Line 2). When the teacher came to see the students' work, he realized that the students had not taken into account x as the amount of Manuel's candies. Trying to help, he said:

1. Teacher: Manuel is x.
2. Stacey: Yeah. Josée has 5 more candies than Manuel and the 3 together have 37 candies.
3. Teacher: Here, they are asking you to write the algebraic expression for the number of candies represented by Kelly. So, if he is x, she is what? That's what you have to figure out.
4. Stacey: (*while looking at the teacher, she says*) x-2.

Although the teacher's utterance took an elliptical form (Manuel *is* x), it was an attempt to cause the students to focus on Manuel's amount of candies. His attempt to shift the noematic content, nonetheless, was countered by a phrase (Line 2) that amounts to a monotone answer "Yeah, yeah, we know that".

Constructing a symbolic narrative for the story-problem requires a new approach: while the story-problem unfolds according to a left-to-right lineal reading (with eventual flashbacks) the starting point in the symbolic narrative does not have a permanent location. In the symbolic narrative, the *order* of discourse (to borrow Foucault's term) is different and the thematic character is about other things.

What, then, is the role of symbols in the students' previous symbolic expressions? We shall now see that the students' signs constitute short scripts recounting salient parts of the *original* story. Let us take a closer look at Stacey's algebraic expressions ("x-7", "x-2"). Each one of them is made up of three signs: the signs in the second one are: 'x', '-' and '2'. Their meaning, of course, is not the one required in the practice of algebra. We cannot say, however, that the expression is meaningless. The expression "x-2", which is polyphonic in tone in that it merges the teacher's voice (Line 1) and Stacey's understanding of it (Line 4), might be read as telling us that Manuel has a certain amount of candies ('x') and that he has two ('2') less ('-') candies than Kelly. Thus the sign '-' is *not* performing a subtraction on the unknown x but is an orienting mark of a short script about the story-problem. In a similar vein, the sign '7' in the expression "x-7" does not translate merely as "x minus 7". As indicated by Stacey's utterances, the number 7 comes to form part of the symbolic expression with an imported meaning so that each symbol in the equation tells us a part of the original story.

Later, the teacher came to inspect the group's work. He said:

Teacher: x is Manuel, right?

Caroline: Yes.

Stacey: (*interrupting*) So, x minus…

Teacher: (*continuing his utterance*) Kelly has 2 more candies than Manuel. Let's suppose that Manuel has 20 candies, how many candies would Kelly have?

Stacey: 22?

Teacher: 22. *(He looks at Caroline).* If Manuel had 30 candies, how many …

Stacey: (*interrupting*) 32.

Teacher: *(He looks at Jessica).* Therefore, um, what did they do to find Kelly?

Stacey: You put the 2.

Teacher: (*correcting*) You add 2.

Stacey: (*having understood how to algebraically express the relationships, says, referring to Josée*) There you add 5. […] So, it's x+5. (*The students write 'x+2' and 'x+5'.*)

Teacher: Then, this (*indicating the question about the equation for the problem on the page*) would be equal to what? This is an equation so it has to equal something. (*The teacher is called by another group and feeling that the students are on the right track he leaves.*)

Caroline: (*adding the 3 algebraic expressions*) So, if I have 3x+7, *(she looks at Stacey)* 3x+7? […] That means 3, no, 3x+7. This equals 37?

Stacey: (*recognizing the number 7, says*) I don't believe that! 3x+7 is equal to 37! … oh!

We see how using the elliptic formula 'x *is* Manuel' and through a calculation on numbers (which functions here as the ground of *noesis*, i.e. the meaning-conferring act), the teacher shifts the students' attention to the relationships between amounts of candies. What is important, though, is not that the students could write the sought symbolic expressions. The important point is the emergence of a kind of awareness that, in the symbolic expressions, the heroes, without being thrown away, are put in the background and predications in the symbolic narrative are done about other things, about *objectivities*. Perhaps elliptical formulas based on the verb 'to be' of the kind "x is Manuel" are not the best way to forge the distance between the story-problem and the symbolic narrative. And, perhaps, the use of the verb 'to have' would have been more suitable in terms of the goal of the activity (of course, we became aware of this only after the activity was analyzed). Nevertheless, in the classroom context, the choice of the elliptical formulas allowed the students to start moving into the realm of algebraic symbols and to begin learning the incredible amount of meaning that these phrases encompass despite the dramatically limited number of signs they use.

Nominalization

Groups 2 and 3 did not face the same difficulties as Group 1. For instance in Group 2, we find Anik saying:

Anik : Okay. … Manuel is going to be the variable x. *(she points to the paper)* like… like if they want… find the equation there… the equation for Kelly is… because … uh, she has 2 more than Manuel. Manuel has… has the amount x. So x+2 because we don't know, x is how many Manuel has. Right? So, she [Kelly] has … *(she points to the paper)* has like whatever Manuel has +2.

We see how the comparative phrase was transformed into an assertive one ("she has like whatever Manuel has +2"). By introducing the letter x (in "Manuel is going to be the variable x" and "Manuel has … has the amount x"), Anik (first using the verb 'to be' and then the verb 'to have') opens the door that leads to the symbolic narrative. We can see, despite the final reformulation at the end of her utterance, how the heroes start fading away. The insertion of x as a designation of Manuel's number of candies, allows room for a *nominalization*, that is, a process in which something becomes enabled to function as the subject or the object of a verb. In saying "whatever Manuel has", the expression can now become the noun in the assertive phrase "Kelly has (noun) +2". It is indeed interesting to notice that, without help, Group 1 could not offer nominalizations. Groups 2 and 3, in contrast, did offer clear instances of nominalizations. Here is an example, taken from Group 3, concerning Problem 3 (where x designated Kelly's number of candies).

1. Michelle: Kelly... (*inaudible*) ... There the x is all moved around. They're trying to trick us. So if Kelly has 2 more candies than Manuel, then Manuel has 2 candies less than Kelly, right? […] But now that Kelly is x, minus 2 ...
2. Jessy: (*interrupting*) Yeah, yeah.
3. Michelle: I'm thinking… Josée has 5 more candies than Manuel. So Manuel has x-2. Then Josée has 5 more than that, right? So x-2 in brackets ... +5."

Line 1 indicates a change of meaning. Although the sentences "Kelly has 2 more candies than Manuel" and "Manuel has 2 candies less than Kelly" refer to the same state of affairs, the meaning is not the same (as in Frege's famous example also discussed by Husserl: *the victor at Jena* versus *the vanquished at Waterloo*). The meaning changes because of differences in the way of attending the object –the noematic content is not the same. In the last part of Line 1 and the first part of Line 3, Michelle establishes Manuel's amount of candies. The insertion of the sign 'x' allows for a *first nominalization* which makes possible the phrase 'Manuel has x-2' (an important hybrid phrase where meaning is lent from the story problem to the symbolic narrative). In the second part of Line 3, the attention is focused on 'x-2' only. Instead of seeing this expression as expressing any of its various possible meanings (e.g. 'the amount of Kelly's candies minus two' or 'Manuel's amount of candies'), Michelle proceeds to a subtle and fundamental suspension of these by using the deictic 'that'. In doing so, a *second nominalization* is produced: the referent is formally nominalized and can thereby become the noun of the verb 'to have' in "Josée *has* 5 more than *that*". As mentioned in the Introduction, the theoretical interest of nominalizations is to inform us how symbolic expressions become endowed with meaning in this limbo where we have neither fully left the original story, nor have fully entered into the symbolic narrative. In particular, nominalizations make it possible to see how high-order meanings are made available for further predication. Let us now discuss the didactic problem of the operations with signs necessary to obtain the equation.

The collapse of narratives

This is an excerpt from Group 2 during their discussion about Problem 1:

1. Anik: Yeah ... Well guys ... *(She takes the papers)* what we're trying to do is to put [*the symbolic expressions*] with the people, okay? Kelly has 2 more than Manuel. Manuel has x. Plus 2 is what Kelly has. [She] has whatever he has +2. Okay. That's going to be x+2, that's in brackets, plus x+5. That'll be what Josée has, plus x, that'll be what Manuel has [*she intends the expression '(x+2)+(x+5)+x'*].
2. Luc: Equal to what? 30? 37? (*Chantal writes 2x+5x+x.*)
3. Anik: (*looking at Chantal's symbolic expression*) 2x, I don't think so.
4. Chantal: Why not?
5. Anik: *(She points to the paper)* because there you are about to do 2 times x.
6. Chantal: No.
7. Anik: Here we're doing 2+x. (*Anik writes (x+2)+(x+5)+x*).
8. Luc: (*looking at Anik's expression*) You group them together, you group all the x's *(Chantal erases what she had written.)*
9. Anik: (*Talking to Luc*) No, no!
10. Luc: Yeah! You group all the x's.
11. Anik: No! Wait guys! Wait! *(She points to the paper).*
12. Luc: Oh my God!
13. Anik: I just want to explain it to you. Guys, here! He has ... she has x+2, right?

In line 2, Chantal uses a syntax based on the criterion of juxtaposition of signs. The sentence is structured in the manner of a narrative where signs become encoded as *key terms* – much as pictographic signs used by Mesopotamian scribes did in the proto-literate periods ca. 3300-2900 BC where, e.g., a set of pictograms representing "sheep" "two" and "temple" may mean "two sheep delivered to (or received from) the temple" (see Radford 2001b, 28-33). The expression 2x does not mean twice x or two times x. For Chantal, 2x does convey the idea that Kelly has 2 more candies than Manuel, and this is why she is surprised (line 6) that Anik could have interpreted it in a different way. But the previous dialogue shows another feature of the students' struggle with algebraic symbolic language. In line 8, Luc proceeds to collect similar terms. This action is radically opposed by Anik. Why? The reason is that the collection of similar terms means a rupture with their original meaning. All the efforts that were made at the level of the designation of objects to build the symbolic narrative have to be put into brackets. The whole symbolic narrative now has to collapse. There is no corresponding segment in the story-problem that could be correlated with the result of the collection of similar terms, i.e. with 3x+7. Anik's desperate effort not to lose track of the narrative meaning is clear in line 13.

CONCLUDING REMARKS

Focusing on a story problem, in this article, I dealt with two main points: (1) the designation of the objects of discourse in the construction of symbolic narratives and the meaning of symbolic sentences, and (2) some of the problems arising in the operations that are carried out with signs that recount the symbolic narrative. As for the first point, the analysis of some key lines in the students' dialogue suggests that the students' success in constructing the symbolic narrative depends on their ability to move across different layers of noematic content. We have indeed seen the interplay between the various meanings and the dynamics required to enrich, shift, and abandon these meanings as well as the role played therein by nominalizations. As for the second point, the classroom observations intimate how difficult it may be to tackle what I termed the *collapse of narratives*. The constitution of meaning after such a collapse deserves more research. While Russell (1976, p. 218) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention, a noematic change: the focus becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl 1961, p. 79), which led him to talk about signs having a *game signification*. I think that the richness of Husserl's metaphor resides in its stressing the cultural, conventional role of rules. Since convention and arbitrariness are two different things, the weakness of the metaphor is that it does not help us to see the rationale behind its conventional nature.

Notes

1. Story-problems of this kind have been investigated in depth by Bednarz and Janvier (1994) in terms of the effect that different kinds of comparative relationships (e.g. additive vs multiplicative comparisons) have on the students' strategies.
2. Husserl (1961, 44) coined the term *objectivity (Gegenständlichkeit, obectité, objectidad)* to refer not necessarily to an individual thing but also to complex things, categories and states of affairs as they become the referent in sentences.

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Definitions and Images for the Definite Integral Concept

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Definitions and images, as well as the relation between them of the definite integral concept, were examined in 41 English high school students. A questionnaire was designed to explore the cognitive schemes for the definite integral concept that are evoked by the students. One question aimed to check whether the students knew to define the concept of definite integral. Five others were designed to categorize how students worked with the concept of definite integral and how this related to the definition. The results show that only 7 students out of 41 of our sample knew the definition.

All mathematical concepts except the primitive ones have definitions. Many of them are introduced to high school or college students. However, the students do not necessarily use the definition to decide whether a given idea is or is not an example of the concept. In most cases, they decide on the basis of their concept image, that is, all the mental pictures, properties and processes associated with the concept in their mind. (Tall & Vinner, 1981; Rasslan & Vinner, 1997).

The concept of the definite integral is a central in the calculus. In many countries, including the UK, it is taught in the last two years of school to students aged approximately 16–18. The students in this study followed a curriculum based on the School Mathematics Project A-level. In the current version of the textbook (SMP, 1997), integration is introduced through activities to estimate the area between a graph and the x-axis using pictures and numerical methods. After this experience the notion of integral is defined as follows (in the form of a description rather than a formal Riemann sum):

> The symbol $\int_a^b f(x)\,dx$ denotes the *precise* value of the area under the graph of f between $x = a$ and $x = b$.
> It is known as the integral of y with respect to x over the interval from a to b.
> The integral can be found *approximately* by various numerical methods.

Figure 1: The definition of the integral concept (SMP, 1997, p.143).

This is followed by ten pages of experience with numerical approximations including functions with positive and negative values before algebraic integration is introduced in the next chapter. Here the student is encouraged to build up the relationship between polynomials and their (definite) integrals, before the fundamental theorem is introduced using local straightness as a visual form of derivative in the following terms:

> For any differentiable function f, $\int_a^b f'(x)\ dx = f(b) - f(a)$.

Figure 2: The fundamental theorem of calculus (SMP, 1997, p. 169).

Several studies have highlighted difficulties with the integral concept. Tall (1993) remarked on conflicts and contradictions that arise as students study the calculus. For instance, in the 'function of a function rule' $\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$, the student is told not to cancel the dx—it has no separate meaning. Later in $\int f(x)\,dx$, its meaning changes to 'with respect to x'. Orton (1980) observed student difficulty with the integral $\int_a^b f(x)\,dx$ when $f(x)$ is negative or b is less than a. Mundy (1984) reported student problems with integrals slightly beyond their experience, such as $\int_{-3}^{3}|x+2|\,dx$.

Students also experience difficulties with communication. Rather than responding conceptually, they may exhibit 'pseudo-conceptual behaviour' by using minimal effort to respond in a way thy hope will satisfy the teacher (Vinner, 1997; Rasslan & Vinner, 1997). Examples of such responses will arise in this study.

The empirical data collection is based on a questionnaire designed to seek the students' definition of the definite integral and to categorise their responses to selected problems for their use of the definition or image. The research questions are as follows:

- What definitions of the definite integral are given by high school students?
- What images of the definite integral do students use in various problems?
- What misconceptions do they exhibit relating to the definite integral?

METHOD

Sample

Our sample comprised 41 students in four classes of final year ('upper sixth form') English high school students. All these students had access to graphical calculators and had encountered all the concepts on the test. The average A-level score of the school is 20.2, which is above the national average of 18.5.

The Questionnaire

The Questionnaire in figure 3 was administered to all subjects in the sample. Questions 1 to 5 were designed to examine aspects of the respondents' concept images revealed through *doing* integration, whereas Question 6 was designed to examine their definitions. Question 1 was designed to examine how students apply the definite integral to integrals when the integrand becomes infinite. Question 3 examines whether the students understand an integral when the function changes its sign.

Questions 2, 4, and 5 test the student using functions that are not simple formulae. They are more easily answered by drawing the graph and calculating the area directly from the picture. Are the students able to see the integral as 'the *precise* value of the area under the graph' as given in the definition in the text, or do they feel a need to carry out symbolic integration attempting to extend the techniques at their disposal?

1. Find, if you can: (a) $\int_0^6 \frac{1}{(x-4)^{2/3}}\,dx$ (b) $\int_{-1}^{2} \frac{1}{x^4}\,dx$.

If you can, please explain the sign of the answer.

2. The function $f(x) = x - [x]$ is given. Find the area directly below the graph and above the x--axis between $x = 0$ and $x = 3$.

3. Find the area bounded between the function $y = \sin x$ and x-axis over $[0, 2\pi]$.

4. The following function is given: $f(x) = \begin{cases} 2x, & x \le \frac{1}{2} \\ 2x-2, & x \ge \frac{1}{2} \end{cases}$. Find: $\int_0^1 f(x)\,dx$.

5. The following function is given: $f(x) = 1 - |x-1|$. Find $\int_0^2 f(x)\,dx$.

6. In your opinion what is $\int_a^b f(x)\,dx$ (the definite integral of the function f in the interval $[a, b]$).

Figure 3. The Questionnaire

Procedure

The questionnaire was administered to the students in their classes. They were not asked to fill in their names, only their background information. It took them 40-50 minutes at most to complete the questionnaire. All the questions in the questionnaire were analysed in detail by the two authors in order to determine the answer categories.

RESULTS

The Definition Categories

We categorized the student's answers according to methods described elsewhere (Vinner & Dreyfus, 1989; Rasslan & Vinner, 1997). We illustrate each category with a number of sample responses.

Question 6: The definition of $\int_a^b f(x)dx$

Category I: The area between the graph and the x-axis between $x = a$ and $x = b$. (4/41).
Example: The area between the x- axis and the graph $f(x)$ between the limits $x = a$ and $x = b$.

Category II: A procedure of calculation; $\int_a^b f(x)\,dx = F(b) - F(a)$. (3/41).
Example: $\left(\int fx \text{ where } x = a\right) - \left(\int fx \text{ where } x = b\right)$.

Category III: Students substitute specific formula in the definite integral. (3/41).
Example: $\left[x^2/2 + c\right]_a^b = (b^2/2 + c) - (a^2/2 + c)$.

Category IV: Answers based on pseudo-conceptual mode of thinking or wrong answers. (5/41).
Examples: 1. $[f(x)]_a^b$. 2. $f(x)$ increases.

Category V: No answer or missing answers. (26/41).

In the above categorization only seven students out of 41 (categories I, II) gave a definition of the definite integral concept. The three category III students show that they can *use* the concept of integration in specific cases. The majority are in category IV and V, with five erroneous responses and twenty six not responding. The students were not directed to memorise definitions and the majority do not appear to be able (or willing) to explain the definition of the definite integral.

The Concept Images. Questions 1-5

Various aspects of the definite integral concept, as conceived by the students, were expressed in their answers to questions 1 to 5. Some of these aspects are given below:

Question 1.a

Category "Zero": Students with correct theory. (0/41).

Category I: The definite integral is the area between the function and x-axis in $[a, b]$. (7/41)

Category I_a: As above with the correct calculation of the definite integral. (3/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = \left[3(x-4)^{1/3}\right]_0^6 = 3.78 - -4.76 = 8.54$. The graph is all above the x-axis from 0 to 6, and so the sign will be positive.

Category I_b: As above, but only giving the final answer without showing any working.(3/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = 8.542$. The area indicated by the integral is above the x-axis.

Category I_c: As above, with wrong calculation of the definite integral. (1/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx \Rightarrow (x-4)^{-2/3} \Rightarrow (x-4)^{1/3} \Rightarrow \left[3(6-4)^{1/3}\right] - \left[3(0-4)^{1/3}\right] = 2 - - 4 = 6$.

If the sign is positive, the area calculated it from the x-axis and above (i.e. for positive values of y).

Category II: Answers without explanation: the definite integral is a procedure of calculation. (27/41)

Category II_a: As above, but with right calculation of the definite integral. (8/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = [3(x-4)^{1/3}]_0^6 = 3.78 - -4.76 = 8.54$.

Category II_b: As above, but with wrong calculation. Incorrect use of algorithms. (19/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = \left[\ln\ (x-4)^{2/3}\right]_0^6 = (\ln(x-4)^{2/3}) - (\ln(0-4)^{2/3}) = 0.46$.

Category III: Wrong explanations, which based on pseudo-conceptual mode of thinking. (2/41)

Example: $\int_0^6 \frac{1}{(x-4)^{2/3}}dx = 4000$. The graph has their area above x-axis, so the sign is true.

Category IV: No answer. (5/41)

From the above categorisation only 3 students out of 41 (category I_a) give evidence that they understand the definite integral concept. For eleven students of our sample (categories I_b, II_a) we cannot claim it but we cannot claim the opposite. The remaining 27 students (categories I_c, II_b, III, IV), do not seem to be able to handle the concept of definite integral with an infinite discontinuity.

Question 1. b

The categories of this question were intended to be the same as in question 1. a. This was true in the majority of the cases (see Table 1). However, the three students adjudged correct in Question 1.a gave a faulty answer to question 1.b which diverges, indicating that they may have simply ignored the question of convergence in both cases.

Category	"zero"	I_a	I_b	I_c	II_a	II_b	III	IV
Distribution	0	0	0	5	0	26	4	6

Table 1: Distribution (Number of respondents) of the Categories to Question 1.b (N=41)

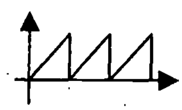
When we analyzed Question 1.a we added category "zero" which refers to the correct theory. It turns out that no one of our sample knew the correct theory according to the improper integrals. This fact is true also for Question 1.b.

Question 2

Category I: Numerical answer. (14/41)

Category I_a: As above with indication of knowledge how to calculate the area correctly (the student draw a graph of the function and calculate the area). (9/41)

Example: 0.5 + 0.5 + 0.5 = 1.5 (with right graph).



Category I_b: Right answer without explanation. (5/41)

Example: 1.5.

Category II: Wrong answers based on overgeneralization of symbolic method. (13/41)

Example: $\int_0^3 (x-[x])\,dx = \left[\frac{1}{2}x^2 - \frac{1}{2}(x)^2\right]_0^3 = 1.5$.

Category III: Incorrect use of algorithms. (4/41)

Examples: 1. Area = $[1-[x]]_0^3$. 2. 0.5.

In the first example the students uses differentiation and not integration. On the other hand the student does not know what to do with [x], so leaves it as it is.

Category IV: No answer. (10/41)

From the above it turns out that for only 9 students out of 41 (category I_a) give evidence that they know how to find the area between the graph of the function and the *x*-axis. As for other 5 students (category II), we cannot claim that, but we cannot claim the opposite. The remaining 27 students of our sample (categories I_b, III, IV), either make errors or do not respond.

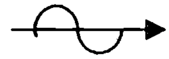
Question 3

Category I: The area is the definite integral. (5/41)

Example: area = $\int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} \sin x\,dx = [-\cos x]_0^{\pi} + [-\cos x]_{\pi}^{2\pi} = 2 + 2 = 4$.

Category II: Right answer without reasoning. (1/41). Example: 4.

Category III: Positive/negative area means positive/negative y respectively. The area is 0 because the positive area and the negative area cancel each other. (11/41)

Examples: 1. 0. (See figure). 2. Area = 0 units^2. 

Category IV: The area is the definite integral $\int_a^b f(x)dx$. The student does not refer to the sign of the function in $[a, b]$. (15/41)

Example: $\int_0^{2\pi} \sin x \, dx = [-\cos 2\pi] - [\cos 0] = 2.$

Category V: Pseudo-conceptual or seemingly nonsensical answers. (3/41)

Example: $\int_0^{180} \sin x \, dx = \left[\dfrac{\cos x^2}{2}\right]_0^{180} = 114.54.$

Category VI: No answer. (6/41)

From the above categorization, 5 students out of 41 (category I) may be claimed to understand the definite integral concept. For one student we cannot claim that, but we cannot claim the opposite. The remaining 35 students of our sample (categories III, IV, V, VI), either make errors or do not respond.

In our analysis of the results in Question 6, we hypothesised that the students not necessarily know how to calculate the area when the function change its sign. From the above categorization, it follows also that 15 students out of 41 (category IV) do not explicitly evoke a change in sign when it occurs in a given interval $[a, b]$.

Question 4

Category I: Numerical answer. (24/41)

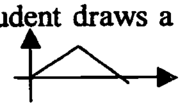
Category I_a: As above with right integration and calculation. (17/41)

Example: $\int_0^{1/2} 2x dx + \int_{1/2}^{1} (2-2x)dx = [x^2]_0^{1/2} + [2x - x^2]_{1/2}^{1} = 0.25 + 0.25 = 0.5.$

Category I_b: As above with wrong integration / calculation. (7/41)

Example: $\int_0^{1/2} 2x \, dx = [x^2]_0^{1/2} = 0.25.$

Category II: Visual answers: the student draws a correct graph of the function and calculates the triangle area. (4/41)



Example: $0.5 \cdot 1/2 + 0.5 \cdot 1/2 = 0.5$.

Category III: Answers without explanations. (2/41). Examples: 1. 0.25 + 0.25 = 0.5. 2. 0.5.

Category IV: No answers. (11/41)

For 21 students out of 41 (categories I_a, II) we can claim that they are able to apply the definite integral to a split domain function. For 2 other students (category III) we cannot claim that, but we cannot claim the opposite. The remaining 18 students (categories I_b, IV), either make errors or do not reply.

Question 5

The categories of this question were supposed to be the same as in question 4. This was true in the majority of the cases (see Table 2), however, the results show that for only 2 students out of 41 (categories I_a, II) we claim that they know the application for the definite integral for such a special function. For 6 students of our sample (category III) we cannot claim that but we cannot claim the opposite. About the rest (categories I_b, and those who did not answer the question) 33 students of our sample, we can claim that they do not know to apply the definite integral for the above function.

Category	I_A	I_b	II	III	IV
Question 4	17	7	4	2	11
Question 5	0	14	2	6	19

Table 2: Distribution (Number of respondents) of the Categories to Questions 4 and 5 (N=41).

When we analyzed the tasks in Questions 2, 4 and 5 we suggested that the students might be expected to give answers involving visualization (categories I, IV, IV respectively). Table 3 provides this information about our sample. It turns out that the visualization thinking is very weak in our sample.

Question	question 2	question 4	question 5
Number	14	4	2

Table 3: Distribution (respondents) with correct visualization answers to Questions 2, 4, 5. (N = 41)

When we analyzed Question 6 we mentioned that it is interesting to compare the results there to the results of other questions; especially the 26 students who did not respond (category IV) in Question 6. Table 5 provides information about our sample.

Question	1.a	1.b	2	3	4	5
Number	0	0	5	8	9	0

Table 4: Distribution (of respondents) of students who did not answer Question 6 and answered correctly Questions 1-5. (N = 26)

Table 4 shows the responses of those who did not answer Question 6 but gave correct answers to Questions 1-5. The conclusion is that these students know what to do but they do not know to explain at the general level. However, our conclusion is not true when they face improper integrals or definite integral of modulus function (Questions 1 and 5).

Discussion

One of the goals of this study was to expose some common images of the definite integral of a function held by A-level high school students. This has a direct implication for teaching. If one wants to teach the definite integral of a function to a group similar to our sample, it is important to know the starting point of the students (Rasslan & Vinner, 1997). Taking into account the difficulties mentioned in this study and also in Tall (1993) at least some doubts should be raised whether the given approach to the definite integral is the most effective way for teaching such a concept. If improper integrals, definite integrals of more general functions such as the modulus function or the integer-value function, are needed, we suggest that they should be introduced as cases extended the students' previous experience. The pool of examples introduced to the students should include a variety of examples and students should be encouraged to express their ideas in ways which help them to build a more insightful concept. A similar conclusion was mentioned by Rasslan and Vinner (1997) according to other concepts, such as even/odd function.

The strategy applied by the School Mathematics Project is to introduce conceptual ideas through class discussion and then to experience them in use. The using of the ideas is a major part of the activity in a manner reminiscent of the 'tool-object dialectic' (Douady, 1986). The concept definition is essentially an incidental part of the process which is far more concerned in practice with developing experience and images of the concepts themselves. Students learn implicitly what they do. Ferrini-Mundy & Guardard (1992) have already illustrated that students who essentially practice routines in High School Calculus learn procedural techniques which may even be prejudicial to later developments at College. Here we have investigated students whose examination results are above average who are following a curriculum intended to be more experiential and conceptual. The majority do not write meaningfully about the definition of definite integral, and have difficulty interpreting problems calculating areas and definite integrals in wider contexts.

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QUANTITATIVE JUSTIFICATION OF SOLO CYCLES

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More and more research into student understanding is being undertaken, in which analysis of responses often involves the use of taxonomies to describe hierarchies. Recent use of the SOLO Taxonomy has identified that there can be cycles of levels within each mode in the hierarchy. Most recent research has involved qualitative description of these levels and identification of cycles based on subjective decisions. This report demonstrates how Threshold Values produced by Rasch Analysis can be used, as a quantitative measure, to justify the cycles of SOLO levels. The data used is from a study into statistical understanding, undertaken with Australian school students aged 12 to 18, in which the SOLO taxonomy was used as the framework for developing a hierarchy of student responses to open-ended questions.

INTRODUCTION

As more qualitative research is undertaken in the area of student understanding, hierarchies, expressed in levels, are increasingly being used as descriptive and assessment tools. Structure of the Observed Learning Outcome (SOLO), which has proved useful in a variety of disciplines, is one such taxonomy. A need was identified for some means of quantitatively justifying the cycles of levels that were observed, and discussed qualitatively, in more recent applications of the taxonomy. After outlining background information about the SOLO Taxonomy, the context of the study and the Rasch analysis used, 'threshold values' are defined. Discussion follows on how these threshold values can be used as a quantitative justification of cycles of levels. Some implications for future research are presented after conclusions have been drawn.

SOLO

SOLO originators, Biggs and Collis (1991, pp. 62-65), in a more recent description of the taxonomy, identify 5 basic levels in a learning cycle; prestructural, unistructural, multistructural, relational and extended abstract. Also identified were modes of representation within which students operate, ranging from sensorimotor, through ikonic, concrete-symbolic and formal to post formal. The learning cycle is repeated for each successive mode, the unistructural, multistructural and relational levels being within the mode, and prestructural and extended abstract levels belonging to the adjoining modes. However, more recently, researchers in a variety of disciplines are identifying more than one cycle of levels within each mode. For example, Lawrie, Pegg and Gutierez (2000) in geometry and Reading (2001) in data collection.

Most studies rely on qualitative justification of the levels in student responses. The levels are described, and cycles of levels identified, based purely on the researcher's subjective interpretations of student responses. This report demonstrates the use of Rasch analysis as a quantitative approach to justifying the cycles of levels identified.

CONTEXT

Discussion here is based on data from a study where 180 Australian students, aged 12 to 18, were given open-ended questions on statistical understanding covering four focus areas; Question 1 in the area of data collection; Question 2 in data representation; Question 3 in data reduction; and Question 4 in interpretation and inference. Each question had two parts. Question 1 Part I presented students with a general data collection question while Part II asked a more specific question, where part of the collection procedure had already been specified. Part II, with some prompting, was included to delve more deeply into the understanding of data collection. For Questions 2, 3 and 4, Part I and Part II asked similar questions except that Part I presented the data in a raw form while Part II presented graphed data. Two parts were included in these three questions to determine whether data presentation in raw form or as a graph influenced the level of understanding.

Student responses, including explanations of 'why' the particular response was given, were analysed to establish a hierarchy of eight levels, from Level 1 to Level 8. So few responses were recorded at Level 8 that this level does not appear in the following discussion. The SOLO Taxonomy was then used as a framework for describing these levels and dividing them into three separate cycles. Level 1 and Level 2 fell within the ikonic mode. The first cycle in the concrete-symbolic mode consisted of Levels 3, 4 and 5 and the second cycle, Levels 6, 7 and 8. For more detail about this study see Reading (1996), with specific discussions of each of the four questions in Reading (2001,1999,1998) and Reading and Pegg (1996), respectively. Rasch analysis, used to produce a measure of statistical understanding for each student, also produced threshold values for the various levels within each question.

RASCH ANALYSIS

The Rasch model was used to produce an estimate of statistical understanding on a logit scale, for each student, which incorporates the information from the questions covering all four focus areas (Reading, 1996, pp. 93-94). Masters' (1982, pp.157-

158) partial credit Rasch model for polychotomous data (where responses can be coded according to an increasing, or decreasing, degree of 'correctness') considers the individual difficulty of each successive step from one level to the next in the question, using a formula for calculating the probability of a student responding at a particular level of a particular question. The advantage of the partial credit model is that the parameters are separable, making it possible to produce sufficient statistics for person ability (understanding) and for step difficulty within each question. Masters (1982, pp. 163-166) used a maximum likelihood procedure to estimate the parameters, overall ability for each student and difficulty for each question, in the model. In the study, appropriate measures indicated that the questions were all consistent in testing the same underlying construct, 'statistical understanding', and that the data fitted the Rasch model well (Reading, 1996, p170). Although the study reported on measures of overall understanding, question difficulty and threshold values (Reading, 1996, pp. 170-190), this report only discusses the thresholds values.

THRESHOLD VALUES

Threshold values, produced by the Rasch analysis for the various levels in each question, were used to justify the arrangement of levels within the cycles described in the study (Reading, 1996, pp. 178-181). These values are estimates of the score a student would need to attain a 50% chance of having his or her response coded at that level. For example, the threshold value of 0.15, for Level 5 of Question 2 Part I, means that a response from a student with an estimate of statistical understanding of 0.15 has a 50% chance of being coded at Level 5 for that particular question.

The results are presented graphically in Figures 1 to 4. No threshold value is shown for Level 1 because there is no information about performance at a level below this to allow estimation of the better understanding needed to be able to be coded as Level 1. Each graph shows the threshold values for Part I and Part II of a question. In each graph, the trends as indicated by the shape of the graph, for Part I and Part II are similar. This reinforces the authenticity of the trend. Following is a discussion using these threshold values to justify the cycles of levels.

CYCLE JUSTIFICATION

The study provided insufficient responses at Levels 1 and 2 to make detailed judgements about SOLO levels, or cycles of levels, within the ikonic mode and so discussion is centred on the two cycles of levels within the concrete-symbolic mode. These corresponded to Levels 3, 4 and 5 within the first cycle, and Levels 6, 7 and 8 within the second. As previously mentioned, there were too few responses at Level 8 to justify including the threshold values in the figures presented.

It was anticipated that once a student is able to function within a cycle there is not as great a change in the level of understanding needed to be able to progress from one level to the next within that cycle. This would be indicated by similar threshold values for the levels within a cycle, for example in the move from Level 3 to Level 4 and the move from Level 4 to Level 5. Then, with the move into a new cycle, presumably a complex transition for many students, the threshold values should be greater. Thus greater jumps in threshold values, indicated by steeper segments of the graph, would demonstrate movement from one cycle to the next. For example, with the Level 3,4,5 cycle, steeper segments would be expected from Level 2 to Level 3, moving into the cycle, and then from Level 5 to Level 6 moving out of that cycle into the next.

RESULTS

The threshold values for three of the questions (figures 2, 3 and 4) tend to confirm that cycles, indicated by flatter sections of the graph, exist but those for Question 1 (figure 1) suggest that all movements from one level to the next were reasonably complex for students, with no clear indication of cycles. Next is a consideration of the threshold levels in more detail.

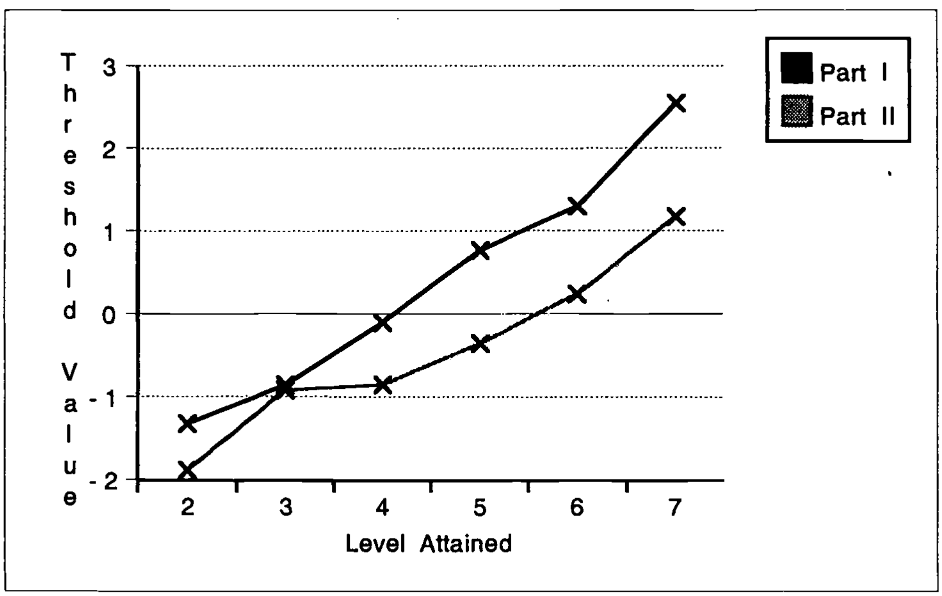

Figure 1 - Thresholds to Attain Levels - Question 1- Data Collection

As indicated earlier, the similarity of shape of the graphs for Part I and Part II in each question confirms the trends indicated, but what of the gap between the graphs for Part I and Part II? The threshold values for Part II of Question 1 are consistently lower, except for Level 2, than for Part I. This is not unexpected as some prompting as to the data collection details was given in the second part of the question.

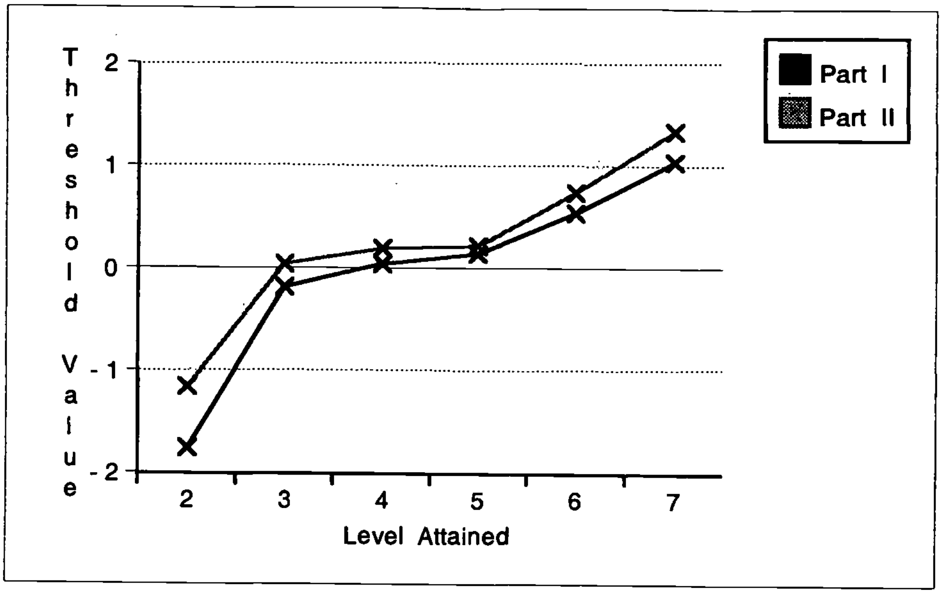

Figure 2 - Thresholds to Attain Levels -Question 2-Data Tabulation and Representation

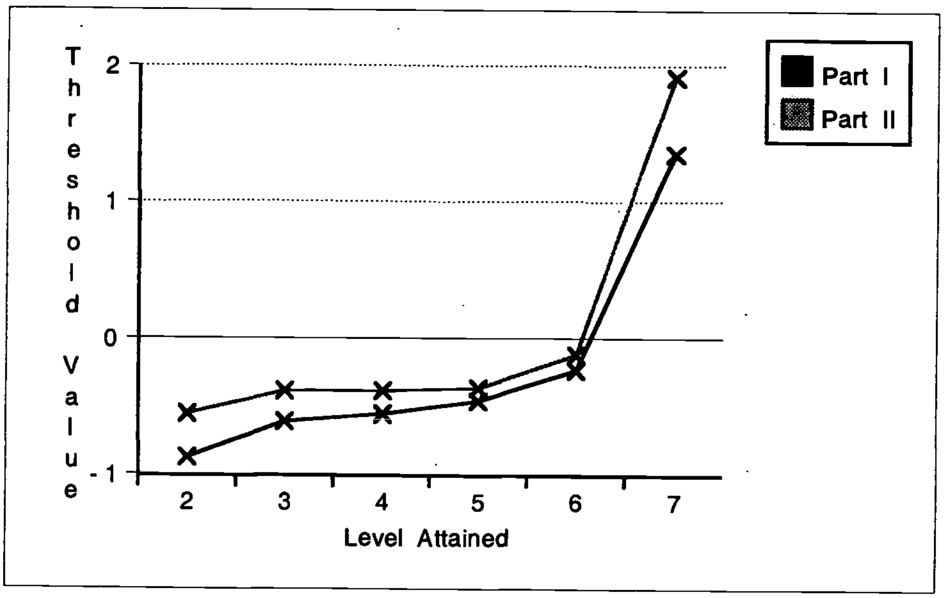

Figure 3 - Thresholds to Attain Levels - Question 3 – Data Reduction

For the other three questions (figures 2, 3, and 4) the threshold values for Part I and Part II are sufficiently similar to suggest that there is little difference between the results whether the data is presented in a raw form or in a graph. For Questions 2 and 3, the Part II thresholds are slightly higher suggesting that, for the tasks of data tabulation and representation and reduction, to respond at a certain level may require a little more understanding when the data are presented as a graph rather than in raw form. However, the differences between the values for Part I and Part II are only slight. For Question 4, Part I and Part II values are almost identical, suggesting that for interpretation and inference the form of presentation of the data does not appear to affect the level of understanding required to present responses at specific levels. How then does the shape of these graphs aid in the justification of cycles of SOLO levels?

For Question 1 (figure 1) there is a steady increase in the threshold value required to attain each of the levels. The threshold values are not necessarily supportive of the cycles identified by the study, although the Part II results do show some flattening out of the change in threshold vales moving between Levels 3, 4 and 5. It is possible that these students find the progression through levels, even within cycles, difficult in the data collection focus area because in the Australian Secondary schooling curriculum there are fewer opportunities to engage in meaningful activities for data collection than the other three focus areas.

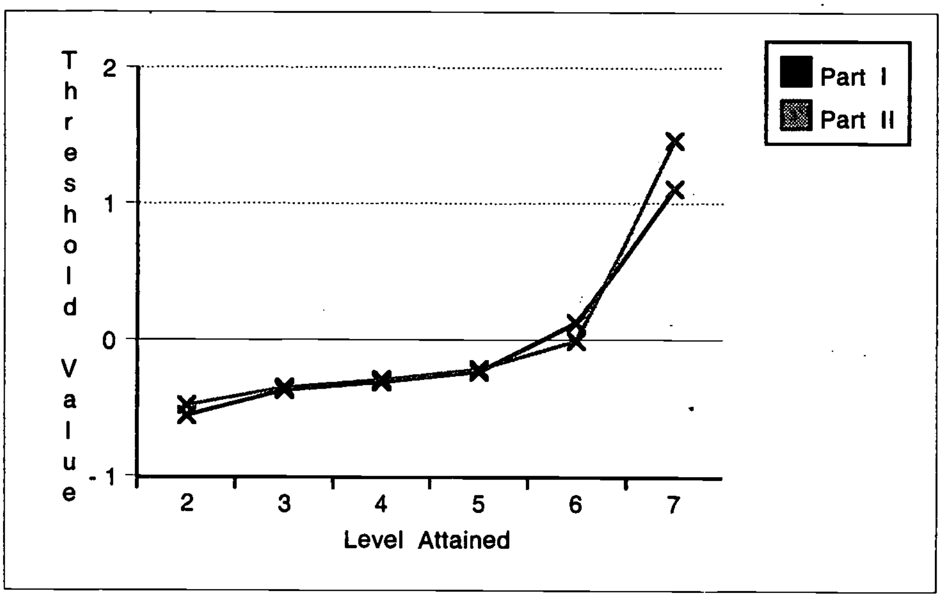

Figure 4 - Thresholds to Attain Levels - Question 4 – Interpretation and Inference

For the other three questions (figures 2, 3 and 4) there is a levelling off of threshold values within one cycle, especially apparent for the first cycle in the concrete-symbolic mode. For each question the threshold values moving from Level 3 through Level 4 to Level 5 only show a slight increase. This suggests that once a student can respond at Level 3 it is likely that they will experience little increase in complexity when responding at Level 4 or Level 5. Also, for each question, the step into the cycle from Level 2 to Level 3 and the step out of the cycle, from Level 5 to Level 6, are steep suggesting movement between two different cycles. These results help to confirm that the three Levels, 3, 4 and 5, are in fact one cycle within a mode, in this case the first cycle identified in the concrete-symbolic mode.

In these three questions, the justification of the second cycle is not as clear. The threshold for Level 6, the entry level for the second cycle, is steep enough to indicate that the first cycle is complete and a new cycle is being entered. However, the increase in threshold value from Level 6 to Level 7, is still quite high in Question 2 and significantly higher in Questions 3 and 4. Although this suggests that the Level 6 and Level 7 are not within one cycle, it should be remembered that the second cycle actually had three Levels 6, 7 and 8 and that with limited responses at Level 7 and virtually none at Level 8, there is probably not sufficient data to confirm or deny the existence of the second cycle at this stage.

CONCLUSION

The use of threshold values has proved a useful tool in quantitatively justifying the previously qualitative arrangement of SOLO levels into cycles. Small differences in threshold values between levels indicate a movement between levels within a cycle while a large difference indicates a move from one cycle into the next. The similarity of threshold values for Levels 3, 4 and 5 suggests strongly the existence of a cycle of levels, at least for three of the focus areas. The jumps in threshold values from Levels 2 to 3, into the cycle, and out of the cycle, from Level 5 to 6, add to the justification of identification of a cycle of levels here. This is the first cycle within the concrete-symbolic mode as identified earlier.

The second cycle, coded qualitatively in the study, is not so clearly identifiable from the threshold values available. The increased thresholds for Level 6 demonstrate the entry point for the second cycle but the move into the next level in the cycle, Level 7, does not show the slight increase in threshold value, as expected. In fact, the threshold values show that in all questions, except Question 2, the step into Level 7 required even more understanding than the step into Level 6. However, as there were fewer responses at this upper end of the hierarchy, further research may help to clarify the existence and character of this second cycle in the concrete-symbolic mode.

IMPLICATIONS

Implications for future research of this quantitative analysis exist both within the narrower context of this particular study and within the broader context of the use of the SOLO Taxonomy. From this study, there are still many unanswered questions. Analysing responses from students younger than 12 would provide more information to describe the levels, and perhaps cycles of levels, within the ikonic mode. Similarly, responses from students over 18 would help to clarify the second cycle in the concrete-symbolic mode. This would involve new studies at the Primary and Tertiary level of education. It is also necessary to delve more deeply into analysis of responses in the data collection area to clarify the cycle structure.

In the broader context, these threshold values should prove useful to those who want to justify cycles of levels identified when using the SOLO Taxonomy. Researchers should be aware, though, that this would not form part of the process when using the qualitative descriptions of the hierarchy to assess individual students. The relevance of this tool would be in larger studies when hierarchies are first being created and justification for the identified cycles of levels is needed.

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DESCRIBING YOUNG CHILDREN'S DEDUCTIVE REASONING

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This paper reports results related to the development of a consistent descriptive language for research on mathematical reasoning. Ways of reasoning deductively are highlighted, using examples drawn from observations of young students. One-step deductions versus multi-step deductions, known versus hypothetical premises, and single versus multiple premises, are used to distinguish different ways of reasoning.

This paper reports results related to the development of a consistent descriptive language for research on mathematical reasoning. These results arose out of a long term research project (the PRISM project [1]) aimed at elaborating and clarifying previous models and terminology for describing reasoning. The model now in use for this research project describes reasoning across five dimensions: need, target, kind of reasoning, formulation and formality; and has been used to describe reasoning of students of all ages (Reid 1995a,b, 1997, 1998, in press). In this paper one dimension, ways of reasoning, will be highlighted, using results drawn from observations of students aged about seven years.

THE MODEL

The PRISM project took as its beginning point a model for reasoning outlined by Reid (1995a, 1996b). It includes four dimensions for describing reasoning. *Need* includes the needs to explain and to verify mathematical statements and to explore to discover new statements. This dimension of the model was inspired by the work of Bell (1976) and de Villiers (1990). *Kind of reasoning* includes reasoning deductively, inductively and by analogy, and was inspired by the work of Polya (1954/1990). *Formulation* refers to the degree of awareness the reasoner has of their own reasoning. *Formality* refers to the degree to which the expression of the reasoning conforms to the requirements of mathematical style. The work of Lakatos (1978) and Blum & Kirsch (1991) inspired this dimension. One of the refinements of this model that has resulted from the PRISM project is the addition of a fifth dimension, *target*, that describe who the reasoning is for: a teacher, a peer or oneself.

This model of reasoning is compatible with those being developed by others. For example, Sowder and Harel (1998) have outlined a model describing what they call "proof schemes". In terms of the PRISM model it offers additional detail concerning kinds of reasoning but is limited to those kinds of reasoning related specifically to the need to verify.

THE CONTEXT

The episodes of mathematical activity that will be described here were recorded as grade two students (aged seven or eight years) worked in small groups at their classroom mathematics centre, one of five learning centres that the students moved through on a weekly rotation. The regular classroom teacher supervised the other centres while another teacher, working as a research assistant in the class, supervised and interacted with the students at the mathematics centre. Mathematics centre activities included playing games, reading and discussing stories, and engaging in geometric activities with pattern blocks and geoboards.

The research assistant recorded the activities at the mathematics centre on both video and audio tape, and produced summaries of the students' mathematical activity based on these recordings and her own observations. The author re-viewed the tapes, discussed the students' mathematical activity with the research assistant and occasionally visited the classroom as a participant/observer.

The results below evolved through an enactivist research methodology (Reid, 1996a). In keeping with the enactivist position that all learning is structure determined, the learning of individual researchers concerning the data is acknowledged as being determined by the structure of the researchers. At the same time, the researchers' structures co-emerge with their environments, which include physical artefacts (e.g., video tapes) that constrain the researchers' learnings, and other human beings whose structures impose additional constraints, if the researchers wish to maintain communication with them. Enactivist research methodology seeks to build on this co-emergence, through the mechanism of multiple perspectives.

Perspectives can be multiple in several ways. Several observers will inevitably observe from their own perspectives, each of which has something to offer. One researcher re-viewing a video tape on several occasions does so with different perspectives each time. Data represented on video tape, audio tape or in transcript affords further perspectives. All of these are ways of obtaining additional perspectives can be involved in enactivist research. In the case of the results reported here, multiple perspectives arose from the individual structures of the research assistant and the author, from multiple viewing of tapes, and from re-presentations of the data as video tape, audio tape and written summaries.

RESULTS

One of the aims of the PRISM project is to explore how the model must be adjusted when used to describe the reasoning of students of different ages. In the case of students in the first years of schooling, the dimensions of need and target require no adjustment. The dimensions of formulation and formality are not of much interest for this age as all the reasoning observed was unformulated and hence informal. It was found to be necessary to refine the description of deductive reasoning, in order to capture any hint of such reasoning present in the students' mathematical activity. For this reason the results described here focus on describing this dimension.

To describe deductive reasoning precisely it was found to be necessary to distinguish between one-step deductions and multi-step deductions. In addition the number and nature of the premises of the deduction were found to be useful for distinguishing kinds of deductive reasoning. The interaction of these two criteria and the descriptive categories arising from them is summarised in Table 1.

Premise(s)	One-step	Multi-step
One	Specialisation	[Not observed]
Two or more	Simple one-step deductive reasoning	Simple multi-step deductive reasoning
Two or more, at least one Hypothetical	Hypothetical one-step deductive reasoning	Hypothetical multi-step deductive reasoning

Table 1: Kinds of deductive reasoning observed

Specialisation

The most common type of deductive reasoning observed in the grade two classroom was *specialisation*. Specialisation is determining something about a specific situation by applying a general rule that applies to it. Specialisations can involve simple or complex general rules. Several specialisations from a general rule involving multiple attributes occurred when the children were playing Set. The game of Set involves determining if three cards satisfy the conditions that define a "Set." In the standard game those conditions are that the three cards be identical, or all different, in each of four attributes: colour, number, shading, and shape. In this case the children played a variation in which the striped shading was removed from the deck, and the open shading was treated as a fourth colour: white. The teacher defined a Set as three cards that were different colours, shapes, and numbers.

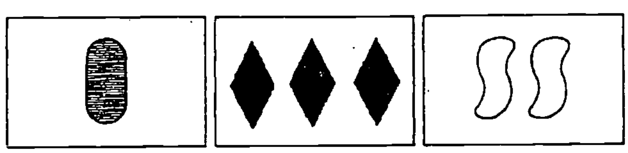

Figure 1: 1 red oval, 3 green diamonds and 2 white squiggles make a Set. (In this game variant a set is defined as three cards of different colour, shape and number.)

After playing for a while, the teacher asked the children to explain why 1 red oval, 3 green diamonds and 2 white squiggles are a Set (See Figure 1). Cynthia replied, "They're all...different. They're all different shapes. They're all different colours." Alison added, "They're all different numbers." And Jared summed up, "Different colour, number and shape." Jared states the general rule that they are using to identify Sets, which he is applying to this specific case.

Simple deductive reasoning

Simple deductive reasoning is deducing a conclusion from two or more established premises. It can be one-step or multi-step, of which one-step is more common.

When a grade two student makes a simple one-step deduction it is not likely to be clearly stated. Following the reasoning can be difficult. For example, consider this statement made by Maurice when playing the game Mastermind with the teacher (See Figure 2).

"It's blue. Cause if there's three there. I changed the blue and I only got two."

					Score
Guess 1	blue	orange	yellow	green	3 white pegs
Guess 2	green	brown	yellow	orange	2 white pegs
Target (Hidden)	green	red	blue	yellow	

Figure 2: The Mastermind board as it appeared when Maurice made his simple one-step deduction. (The object is to guess the colours and order in a four colour pattern picked by one's opponent. A white scoring peg indicates that one of the pegs in the guess is the right colour but in the wrong place.)

The teacher had asked Maurice if he knew anything new after receiving the two white pegs for his second guess. Maurice's response can be re-expressed as:

It's blue. Cause...	Blue is correct because:
if there's three there	1. Three of the colours in my first guess are correct
I changed the blue	2. And the only relevant change I made in the colours from my first guess to my second guess was leaving out blue
and I only got two	3. And only two colours in my second guess are correct

He has taken three statements about the situation and concluded a fourth statement that follows logically from them.

While simple one-step deductions are the building blocks of proving, they need to be assembled into chains to make a proof. Reasoning with chains of deductions is called simple multi-step deductive reasoning. It is difficult to observe in early elementary classrooms, both because it is relatively rare and because young students rarely articulate their reasoning. It is sometimes possible, however, to conclude that multi-step deductive reasoning has taken place by observing the conclusions students come to and the information they had to work with.

For example, when Maurice was playing Tic Tac Drop, a computer game, he made the comment, "If he don't put it next to me, I won," immediately after placing his first marker (Marked as O1 in Figure 3).

1	2	3	4	5	6	7
	X4	**O3**	**O1**	**O5**	**X2**	

Figure 3: Tic Tac Drop. The board after Maurice's game with the computer. The object of the game is to get three of one's markers in a row. Markers can only be added directly above the other markers in a column. O represents Maurice's markers. The numbers indicate the order in which they were placed.

On a previous day Maurice had described a general rule he used when playing a related game, Connect Four, at home. That rule was: *If you have three markers in a row with both ends free, then you can win.* It depends on the winning condition of Connect Four, which is getting four in a row. Because the winning condition for Tic Tac Drop is different, Maurice's general rule for Connect Four is not directly applicable, so he cannot be specialising from it. While we cannot know for sure how Maurice came to his conclusion that he would win if the computer did not place a marker next to his, it seems plausible that he deduced this new general rule either from the strategies he already knew from playing Connect Four or from analysing the new situation in terms of general features (e.g., two in a row with free ends). In both cases he would have had to make a chain of deductions to get from what he knew to his conclusion. As it happened, the computer played its marker in column six, and Maurice won.

Because of the emphasis on arithmetic in early elementary mathematics, the context in which students are most likely to evidence simple multi-step deductive reasoning is in the course of solving problems involving arithmetic. The following example occurred when the teacher posed the following problem to Maurice and Saul:

> First there were 8 cookies. The children got four each. Then some more children came in. Then they got 2 each. How many people came in?

Both Saul and Maurice reread the question several times, which was written on a piece of paper. They tried to solve it with paper and pencil only. Saul circled the number eight in the question. Another boy, Ira, interrupted and stated, "Very easy!" but Saul disagreed, "No! Eight take away two is six. — First there were eight cookies. Then the children got four each. Oh! Let me see. First there were 8 cookies... — Four people!"

The teacher asked Saul, "Did four people come in? How many people were there at the beginning?"

"There were two."

"You think there were two people in the beginning and then four people came in?"

"No, then two more people came in. That made four. Because eight, — let me think about that again. — Um, this is hard. — I think there were four or six people."

The teacher asked Saul how he was going to figure out if there were four or six people, "Are there any parts that you can figure out?"

Saul drew a diagram on the bottom of the page showing eight cookies, in two rows of four. He circled the two groups of four. Then he drew a line through each group of four, creating four groups of 2. "There were four people in there." He said, and then he reread the question again silently.

Maurice agreed that there were four people. The teacher asked once again how many people came in. Maurice replied, "Two."

To solve this problem Saul and Maurice had to first determine the number of children originally present, then the number of children present after the new ones arrived, and then the difference between the two numbers. Each one of these requires a one-step deduction. It is the linking of them together that makes this an example of multi-step deductive reasoning.

Hypothetical deductive reasoning

Thus far the deductions we have seen involve reasoning from something that is known. In mathematics proofs however it is often necessary to reason from a hypothesis, something that is not known to be the case, either to show that it cannot be the case (as in a proof by contradiction) or to show that if it were the case for one number it would also be true for the next number (as in a proof by mathematical induction). Such reasoning, because it involves a hypothesis, is called hypothetical deductive reasoning. Although hypothetical deductive reasoning is often thought to be more difficult than simple deduction from known statements, it can be observed in the reasoning of early elementary school students, in both one-step and multi-step forms.

An example of a hypothetical multi-step deduction occurred during a game of Mastermind (See Figure 4). After giving Kyla two white pegs for her third guess, the teacher asked her which one she thought might have been in the right place. Kyla pointed to the blue in the first row and then changed her mind. "I never got a black one right there [pointing to the blue in the second turn]." She then indicated that green could not be correct either in the first try. "Cause on this one [turn three] I didn't get a black." After that Kyla stated that orange on turn one must be in the correct spot but then realised it can not be. "Cause I got a black one right here — no! Oh my! It's yellow." Kyla's reasoning includes three hypotheses: That blue is in position three, that green is in position four, and that orange is in position two. Having arrived at contractions from each of these hypotheses in turn she concludes that the one remaining case (yellow in position one) must be correct.

					Score
Guess 1	yellow	orange	blue	green	1 white peg, 1 black peg
Guess 2	brown	green	blue	orange	2 white pegs
Guess 3	blue	orange	yellow	green	2 white pegs
Target (Hidden)	yellow	red	orange	brown	

Figure 4: The Mastermind board after Kyla's third guess. (The object is to guess the colours and order in a four colour pattern picked by one's opponent. A white scoring peg indicates that one of the colours in the guess is the right colour but in the wrong place. A black peg indicates one of the colours is in the right place.)

SUMMARY

Describing the reasoning of young children is difficult. Quite often they do not articulate their thinking clearly, and they tend to make use of many implicit assumptions (Anderson, Chinn, Chang, Waggoner, & Yi, 1997). This makes it all the more important for researchers investigating children's reasoning to be precise in their descriptions of that reasoning. For example, while of the reasoning described here is deductive, but there is a significant difference in sophistication between a specialisation and a multi-step hypothetical deduction. Omitting to note that difference would be a serious weakness in a description of a child's mathematical activity. The distinctions I have made in this paper, along with the terminology I and others have outlined elsewhere, should help provide researchers with the tools they need to observe and describe children's reasoning more precisely.

NOTES

1. The PRISM project is funded by the Social Sciences and Humanities Research Council of Canada, grant #410-98-0085.

Thanks to R. Hudson and B. Dowden for all their help.

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REASONING AND PROOF IN GEOMETRY: PREREQUISITES OF KNOWLEDGE ACQUISITION IN SECONDARY SCHOOL STUDENTS[1]

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Proof is regarded to be an important aspect in the mathematics classroom. Nonetheless, it is difficult for students to give mathematical proofs. Our research aims at identifying important dimensions of proving abilities. In a study with 669 grade seven students, we investigated how mathematical knowledge, the ability to evaluate correct and incorrect proofs, and scientific reasoning influence the students' performance in proving. Our results give evidence that these aspects contribute significantly to students' abilities.

1. Theoretical Framework

Proof and logical argumentation are important topics in mathematics as a science, and mathematics may even be regarded a proving science. The role of proof in the school curriculum did not always reflect that importance. In the 1970s and 1980s, there was an intensive discussion whether proofs should be part of the mathematics curriculum. Mathematics educators criticized that proving in the classroom emphasized formal aspects but disregarded mathematical understanding (Hanna, 1983). This is still a consensus among many mathematics educators, but its consequences have been revised. Proof is considered as an important topic of the mathematics curriculum (NCTM, 2000) and as an essential aspect of mathematical competence. Nonetheless, proof is not used as a synonym for formal proof. In particular, researchers such as Hanna and Jahnke (1993), Hersh (1993), Moore (1994), Hoyles, (1997), Harel and Sowder (1998) have pointed out that proving spans a broad range of formal and informal arguments. Understanding and generating proofs is an important component of mathematical competence, and mathematical argumentation has been identified as an essential element of higher order mathematical competence in the TIMS study (cf. Baumert, Lehmann et al., 1997). The current discussion emphasizes the development of proof concepts (Boero, 1999), the continuum from exploration to proof (NCTM, 2000) and the role of reasoning and argumentation in finding a proof (Reiss, Klieme & Heinze, 2001). On the background of the PISA findings (Deutsches PISA-Konsortium [German PISA Consortium], 2002) the individual prerequisites for secondary school students' performance in proof tasks are of particular interest.

The *Standards and Principles for School Mathematics* (NCTM, 2000) call for a "focus on learning to reason and construct proofs as part of understanding mathematics so that all students

[1] This research is funded by the Deutsche Forschungsgemeinschaft (RE1247/4).

- recognize reasoning and proof as essential and powerful parts of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use types of reasoning and methods of proof as appropriate."

The approach is applied to all stages of education from preschool to grade 12. In the early years informal inductive elements are underlined whereas the formal deductive elements become more important for older students.

This standard is the basis for our own work (Reiss & Thomas, 2000; Reiss, Klieme & Heinze, 2001; Reiss & Renkl, in press). With respect to the aspects described in this standard, we assessed students' performance on mathematical proofs. We concentrated on

- students' abilities for proving (i.e. the knowledge of mathematical propositions and concepts, its application to simple situations, understanding a proof, the ability to argue mathematically),
- their knowledge of proof methods and their evaluation of the correctness of proofs (cf. Healy & Hoyles, 1998),
- their abilities in the domain of scientific reasoning.

2. Design of the Study

The aspects described above were investigated in a study with students at the end of grade 7 and in the beginning of grade 8. In grade 7, students get an intensive instruction on geometry proofs and argumentation, which is continued in grade 8. The study was performed between January 2001 and February 2002. We will report on data of a pre-test, which was given to them in June 2001. The results presented here are based on the data of 669 students (363 female students, 306 male students).

We developed a test for the assessment of their abilities to prove which consisted of six items on basic qualifications and seven items on justification and reasoning. For assessing their knowledge of proof methods and their evaluation of the correctness of proofs we adapted a test constructed by Healy and Hoyles (1998). The students had to evaluate four solutions for a proof task: two incorrect ones (empirical, circular) and two correct ones (narrative, formal). In order to assess abilities in the domain of scientific reasoning we presented tasks which consisted of two parts, namely a reduction of a given problem space (Klahr & Dunbar, 1988) and ordering of information given. The test items were presented to the students as paper and pencil questionnaires in a classroom situation.

3. Results

The students performed quite well in terms of knowledge of mathematical propositions and concepts, and its application to simple situations. We observed more difficulties concerning their understanding of proofs and their abilities to reason mathematically. The variation within these components was considerable. Whereas most of the students had a basic knowledge of mathematical propositions and concepts (M=7.4, s=2.5, maximum number of points: 12), they scored lower with respect to items asking for mathematical argumentation (M=5.2, s=3.7, maximum number of points: 14).

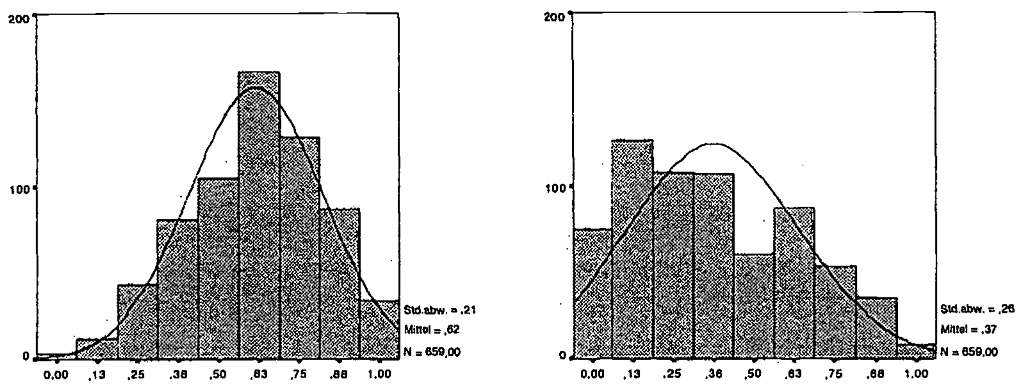

Figure 1: Distribution of norm scores for basic mathematical knowledge (left) and mathematical argumentation (right)

When constructing the test, we applied Klieme's (2000) model of mathematical competencies. The data show a close connection between this model and the actual students' performance. The following table provides information on the students' performance (table 1). According to the achievement we grouped the students into a lower third, a middle third, and an upper third and compared their performance with respect to the levels of competency.

In level of competency I, which was represented by five items, there is no formal mathematical reasoning required, just the application of concepts and rules as well as elementary inferences. In level II (three items) the students must have a sound knowledge of geometrical concepts and facts. They must be able to give a correct justification to given geometrical problems and to find an appropriate notation. In level III (four items) the students must be able to order their arguments in a meaningful way. A prerequisite for this is autonomous and creative problem solving and reasoning. Table 1 shows that the lower third of the students did not have any correct solutions in level III tasks whereas the upper third managed to solve 85% of the level I problems and 89% of the level II problems. Nearly all the other numbers are gradual increases or decreases between these extremes. This can be regarded as an internal validation of the test.

	Level of competency I Simple application of rules and elementary reasoning M=0.69	Level of competency II Argumentation and justification (one step) M=0.56	Level of competency III Argumentation and justification (several steps) M=0.24
Lower third of the students (N=238)	51%	22%	0%
Middle third of the students (N=225)	72%	61%	18%
Upper third of the students (N=206)	85%	89%	50%

Table 1: Percentage of correct solutions

It should be mentioned that we found large differences between the 27 classes involved in the study. Their average scores ranged from very low (M=5.7) to appropriate (M=17.7) given a maximum of 26 points for the 13 items. In the three classes showing the lowest average scores the students achieved less than half of the possible points (with the exception of only one person). Accordingly, these classes lack of high achieving students, whereas in the classes with the highest average scores there was a wide variation of achievement levels. The high achieving classes solved the simple tasks in a good or even very good manner, the low achieving classes performed satisfactorily on these items. However, we found differences at tasks, in which argumentation skills were needed for the solution. The students of the low achieving classes hardly had any correct solutions with respect to these items. The responses show that the students had the declarative knowledge to solve these items, but they were not able to apply it correctly. These differences might be caused by teaching styles (cf. Baumert, Lehmann et al., 1997). We will investigate these differences in an ongoing video study.

Concerning the students' knowledge of proof methods and their evaluation of the correctness of proofs, it can be seen that students have difficulties in identifying incorrect solutions as incorrect. It is significantly easier for them to classify correct solutions as correct ($p<.001$). Moreover, for the students it is easier to evaluate proofs than to formulate proofs by themselves. Concerning norm scores we found an average score of M=0.67 (s=0.33) for the knowledge of proof methods and the evaluation of the correctness of proofs. The norm score for the test items of level II and level III is M=0.37 (s=0.26). This difference is highly significant. Comparing the lower third, medium third, and upper third with respect to the achievement in the test on mathematical performance, there are highly significant differences concerning the students' knowledge of proof methods and their evaluation of the correctness of proofs.

In the domain of scientific reasoning we found that students are often guided by plausible arguments even if these arguments are not logically consistent. A high proportion of the students could not solve tasks in which plausibility was not a hint for the solution (cf. table 2). If students could not solve a plausible task, they could not solve tasks presented in an unusual context either. On the other hand, if they could solve plausible tasks they had a high probability to solve tasks presented in an unusual context.

	Tasks with a unusual context not solved	Tasks with a unusual context partly solved	Tasks with a unusual context solved
Plausible tasks not solved	163	22	11
Plausible tasks partly solved	111	54	43
Plausible tasks solved	80	56	119

Table 2: Distribution of solutions in items with unusual and with plausible contexts

We expected a correlation between abilities in the domain of scientific reasoning and the achievements in the area of justification and proving (cf. table 3). This expectation was confirmed.

	Level of competency I Simple application of rules and elementary reasoning M=0.68	Level of competency II Argumentation and justification (one step) M=0.56	Level of competency III Argumentation and justification (several steps) M=0.24
No formal strategy available (N=354)	67%	54%	20%
Rudimentary formal strategy available (N=132)	66%	55%	21%
Complete formal strategy available (N=173)	74%	60%	33%

Table 3: Percentage of each level of competence dependent on the availability of formal solution strategies in tasks concerning scientific reasoning

The data show that students who are more successful using formal strategies (namely students who do not need a plausible context but are able to solve problems in an unusual context) have higher scores with respect to items of competency level II and competency level III.

Discussion

There are a number of abilities, which may enable students to perform proofs. In our study there is evidence that the following abilities play an important role in this context:

Basic knowledge of mathematical facts and argumentation:

The students need an appropriate level of knowledge about simple mathematical propositions and concepts. Most students have those basic abilities, although the level was quite different in the various classes investigated. Moreover, students need higher order skills in order to apply their knowledge in a proof context. This is more difficult for the students, thus only a few of them are able to apply their knowledge. These findings are in accordance with results of Klieme, Neubrand and Lütge (2001) within the PISA study. Moreover, our data could be validated by Klieme's (2000) model of mathematical competency.

Knowledge of proof methods and evaluation of the correctness of proofs:

It was more difficult for the students to identify incorrect solutions as faulty than to identify correct solutions as being correct. This confirms findings of a former study with grade 13 students (Reiss, Klieme & Heinze, 2001) and findings of Healy and Hoyles (1998) as well as Küchemann and Hoyles (2001). Moreover, we found a correlation between this knowledge and the achievement measures on basic mathematical knowledge of facts and argumentation.

Scientific reasoning:

Plausibility may be regarded as guiding principle for scientific reasoning. Many students rely on plausible argumentation when confronted with reasoning tasks. The plausibility of propositions is more important for their argumentation than facts, which determine that a line of thought is logical or not logical. If students show a high level of scientific reasoning they usually show good achievement in proof-related tasks. Accordingly, scientific reasoning seems to be closely related to mathematical abilities.

These three aspects will only partly explain the interindividual variance in argumentation, reasoning, and proof. We could identify significant differences between mathematics classrooms. Accordingly, the differences in the students' performance cannot be explained by individual prerequisites alone. This is in accordance with Bronfenbrenner's theory of ecological development. Bronfenbrenner (1979) suggests, to take into account the context of teaching and learning, which includes the teacher and the social context. Our findings suggest that a significant portion of achievement differences in the mathematics classroom will be explained by these variables.

Our future research (and the second part of the study described here) will concentrate on the influence of teachers and school variables in mathematics education with respect to argumentation and proof. Instruments will be questionnaires concerning the teaching styles, which are presented to students and teachers, questionnaires concerning the mathematical beliefs of students and teachers, and video observations of mathematics classrooms.

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APPROACHING GRAPHS WITH MOTION EXPERIENCES

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ABSTRACT

In this paper we report on a study about students' activity in interpreting their body motion graphs on a symbolic-graphic calculator connected with a motion sensor. This is done in the context of introducing functions, graphs and modelling at secondary school level, in order to teach algebra with a semantic approach (in which symbols are used not only in a pure syntactic way). We analyse the ways in which students build a meaning for the graphs, comparing them with the ways used by students who did not take part in the same experience. Finally, we try to interpret these differences, by comparing language and gestures of the two students' groups.

INTRODUCTION

In literature it is well-known that students have many difficulties in interpreting graphs, particularly those in which one variable is time-dependent, as for example space-time or velocity-time graphs. Researchers have identified some ways in which students misinterpret graphs. One is the *graph-as-picture* interpretation, in which students expect the graph to be a picture of the phenomenon described. In kinematics, this can result in the student interpreting a graph of space versus time as if it were a road map, with the horizontal axis representing one direction of the motion rather than representing the passage of time (Berg & Phillips, 1994; Clement, 1989). Another common misinterpretation is the *slope/height* confusion, in which students use the height of the graph at one point, when they should use the slope of the line tangent to the graph at a point, and vice versa (Thornton & Sokoloff, 1990).

Mathematics and Physics Education research tells us that microcomputer-based laboratory instruction with motion sensors is quite successful in supporting students to learn to interpret graphs, because of the help of the real-time display of graphs, together with the kinaesthetic experience of motion, that give students control over the data. Generally, all the experiences in which students can interact with a tool to create phenomena, help them to understand the mathematics connected with those phenomena (see e.g. Noble & Nemirowsky, 1995).

The ongoing research project we are involved in, is concerned with the students' cognitive behaviour when using a symbolic-graphic calculator, connected with a motion sensor. Secondary school students are first engaged with experiencing various body motions. Second, they are asked to interpret the graphs and tables containing the data related to their motion, using natural language first, then in terms of increasing/decreasing functions, and finally in terms of slope, calculated in some points (Arzarello & Robutti, 2001). The general research problem is the study of the students' transitions from the perceptual level [1] to the theoretical one and backwards and the mediation of technology in these transitions.

In our investigation, we obtained the same results witnessed in literature, as regards the comparison of students who experience body motion with sensors and calculators (we will call them the *experimental group* of students) with students who do not participate in the experience and have to solve the same problem situation (we will call them the *control group* of students). Generally the students of the experimental group interpret the motion graph correctly, while the students of the control group reveal the kind of misinterpretations described in the introduction.

Our attention is focused on the *ways* the students of the experimental group construct the meaning of the graph they observe on the screen after the motion experience. In doing so, we analyse the students' language and gestures, which provide information about the students' cognitive activities, and compare them with those of the students of the control group, who observe the graph only on the paper, after a brief introduction made by the teacher. A crucial point seems to be the transition *static-dynamic* and backwards, namely when the graph is seen as a pure graphical sign, or is conceived as a time-dependent representation [2].

Our hypothesis is that, for the construction of a correct meaning for a graph, it is fundamental to make some transitions from a static to a dynamic interpretation of it and backwards. Two main theoretical ingredients are at the basis of this study, namely: the transformational reasoning (Simon, 1996) and the mental times (Arzarello, et al., in press).

METHODOLOGY

The activity described in this paper is part of a long-term teaching experiment carried out in a classroom of 25 first year students (the first set) of a scientifically oriented high school, [3] (9^{th} grade) during the school-year 2000-01 in Italy. This activity lasted three hours, in which the students worked in *small groups* (three-four pupils each). Each small group used two technological tools: the CBR (Calculator Based Ranger, a motion detector which collects space-time data in real time) and the symbolic-graphic calculator TI92, for realising a motion in front of the sensor and then interpreting the space-time graph on the calculator. The small group activity was followed by a final classroom discussion, directed by one of the authors. Previous activities, specifically oriented to make students familiarise with the motion sensor and the gathering of the data in the calculator, have been submitted to the students belonging to the experimental group. At the moment of the experiment, the pupils did not know any algebra rules, the linear function and its representation, the equation of uniform or accelerated motions and the definition of velocity (in a few words, they had not been taught algebra, nor yet kinematics).

The students of the control group are those of another classroom (20 pupils in the same initial conditions as the first set), in the same school. Working in *small groups*, they do the same kind of activity as the experimental group of students, without making the motion experiment and without using technology. They have to analyse the space-time graph of a body motion printed on a sheet of paper, and to answer the same questions as those of the first set, after an explanation led by the teacher about the way this graph has been obtained.

During the activity of both groups (experimental and control), four people were present in the classroom: two teachers (one of mathematics and one of physics), to observe the small groups; one University student, to videotape one small group and the final discussion, and one researcher, to observe the groups and guide the final discussion. The data consist of video-tapes and some written notes.

THE OBSERVED STUDENTS

In the experimental group of students we observed a small group of four pupils, three boys and one girl: Filippo, Gabriele, Fabio and Giulia. They are all medium achievers, but with different features: Filippo is a reserved, studious and thoughtful pupil, Gabriele is a discontinuous student, who prefers to go on with a personal rhythm, Fabio is a clever and intuitive boy, while Giulia is a determinate and studious girl, active in a collaborative group.

In the control group of students we observed a small group of three boys: Filippo, Francesco and Giorgio. Filippo is a meticolous and respectful pupil who obtains good results, Francesco is an extrovert and expansive boy who overcame some initial difficulties with good will, Giorgio is a student with discontinuous results.

THE ACTIVITY

The activity proposed to the experimental group of students is the following [4]:

> *Walk or run in the corridor in order to make a uniform motion; when you arrive at the red line, come back with the same motion. The CBR will record your position with respect to time and will collect the data in a graph and in a table. The data are expressed in seconds [s] and in meters [m] respectively. Each 1/10 s a couple of data (time and position) are collected.*
>
> *Describe the kind of motion you made in the corridor.* Using the graph and the table, describe how space changes with respect to time (increase, decrease, ...). Analyse the graph (Is it like a line? Is it like a curve? Does that curve increase? Does that curve decrease?). Consider the ratio $m = (s_2 - s_1) / (t_2 - t_1)$ and use it to describe mathematically the graph of your motion (t_1 and t_2 are two subsequent time data and s_1 and s_2 are two subsequent position data).

PROTOCOLS

Experimental group of students

In the experimental group of students, each small group makes the experience of motion in the corridor. In the observed small group, the student who is running in the corridor (from the sensor to a red line and back) is Fabio, while the other members are looking at the calculator. After the motion, they are all around a desk, to answer the questions. In this first excerpt, the students are exploring and describing the shape of the graph (Fig. 1) and trying to connect it with the motion.

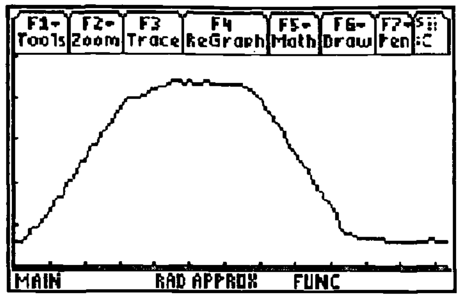

Fig. 1 - The graph obtained by the observed small group in the motion experience.

74	**Gabriele:**	*So, we started from a point and we made a motion...*[...]
78	**Gabriele:**	*And then, at the end, this line* [he is moving his pen on the horizontal stretch of the graph from left to right]
79	**Fabio:**	*Here* [he is pointing at the starting point of the horizontal stretch], *here is when I got to the red line* [...]
81	**Gabriele:**	*Here you stopped* [he is marking the same point on the graph with his pen]
82	**Giulia:**	*...You came back* [she is pointing at the negative slope stretch] *and then you stopped again* [...]
91	**Fabio:**	*The one who ran, uhm...* [...]
94	**Fabio:**	*...He tried to take the steps always in the same...in the same time interval* [...]
112	**Giulia:**	*He tried to keep the same pace there and back* [...]
116	**Gabriele:**	*Walking always the same distance*
117	**Fabio:**	*...During the whole, during the whole time in which I walked, I tried to, to make, to keep always the same velocity* [...]
123	**Giulia:**	*Constant velocity!* [she is writing it]

The students go back and forth from the graph on the screen to the description of the motion experience. The way pupils are looking at the graph changes over time. At the beginning, they use the *deictic* function of language (Radford, 2000), to indicate some parts of the graph (e.g. #78 "*this line*"). This kind of language reveals a static approach to the graph. Two observations are important now: first, the students observe the graph dividing it into different parts, as already noticed (e.g. Monk & Nemirovsky, 1994), and they try to interpret these parts first separately and then together. Second, the *cognitive pivot* (Arzarello & Robutti, 2001) to interpret the graph is given by an event of the motion experience: Fabio's arrival to the red line (#79). So, the deictic function is used together with the *generative action* function of the language (Radford, 2000), in order to link the graph with the motion experience (#79 "*when I got*"; #81 "*Here you stopped*"; #82 "*You came back and then you stopped again*"; ...). The generative action function, together with some adverbs (e.g. "*when*") reveals a dynamic interpretation of the graph. This evolution from static to dynamic indicates the presence of a transformational reasoning: "... the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated" (Simon, 1996, p. 201). At #91 a new phase starts, with new generative action functions. There is a mathematisation process, in which the language evolves and becomes more specific through consecutive refinements. The steps taken in the same time interval (#94), in fact, become, for Giulia (#112) "*the same pace there and back*" and then, for Gabriele, "*the same distance*" (#116). The students find a sort of motion rule: the presence of the adverb "*always*" (#94, #116, #117) reveals the generalisation of this regularity in the sentences listed above. This word, as observed by (Radford, 2001): "underpin the *generative functions of language*, that is, the functions that make it possible to describe procedures and actions that potentially can be carried out reiteratively" (p. 83).

In our opinion, there is something more in the information carried out by this term: the transition from the timed experience to a never-ending process, through the de-timed rule describing the graph. So, there is a new transition, from a dynamic to a static interpretation of the graph, to produce a time-independent rule. This new static viewpoint is not to be intended at the same level as the first static one, but at a higher one, due to the presence of a new concept: velocity. So, the uniform motion becomes a motion with "*constant velocity*" (#123). The basis of this transition is a transformational reasoning, as in the previous excerpt. The mathematisation process in the interpretation of the graph evolves towards correct ideas, as noticed by other researchers (e.g. Monk & Nemirovsky, 1994). This evolution, in our opinion, passes from static to dynamic stages and backwards. All the previous observations refer to the *physical time* (Varela, 1999), while also the students' *inner times* (Guala & Boero, 1999) are involved in this evolution. In fact, the students act in *the time of past experience*, when they recall the kind of motion, and in the *contemporaneity time*, when they connect the motion with the graph (Arzarello et al., in press). Here the inner times of the students are very closed together, so the interaction among them is more productive than the situation of different inner times.

439	**Fabio:**	*...I stopped and when I stopped...*
440	**Giulia:**	*It goes on* [she is pointing at the curve], *doesn't it?*
445	**Gabriele:**	*If you always go on at the same velocity, they* [time and space] *both increase* [...]
448	**Gabriele:**	*Yeah, it* [the curve] *goes on in this way and then it goes down* [with his pen he traces the consecutive slanted sides of an "open triangle"]
449	**Filippo:**	*Yeah, it is not a straight line. They are two half lines* [...]
452	**Giulia:**	*Because if you did not, if you did not lose, let's say, time for turning on yourself* [...]
459	**Fabio:**	*I should have walked in a more uniform way, I did not walk...*

In this excerpt, initially, the word "*when*" and the generative action function mark again a series of timed sentences (#439). Then, at #440, we can observe a dynamic reading of the graph (when Giulia says that the curve "*goes on*" although Fabio stopped), which marks a transformational reasoning. At this point a new function of language enters the scene (#445, #452): it is the *logic function*, which appears in hypothetical expressions of the form: "if... then..." and reveals a conjecture or a deduction. The pupils evolve toward a more mathematical description of the graph in terms of increasing and decreasing functions. In doing so, the motion experience is linked to the shape of the graph (#452, #459): the students are in the time of past experience.

Control group of students

The excerpts listed below refer to the observed small group belonging to the control group of students. They have to describe the motion, referring to the graph of Fig. 1.

11	**Francesco:**	*It* [the graph] *is uphill, therefore he is accelerating* [...]
19	**Giorgio:**	*However he is going on at the same velocity* [...]
22	**Francesco:**	*Then he suddenly slows down* [he is moving his finger twice on the second slanted stretch] *and...* [...]
26	**Francesco:**	*Let's see: there is an increase of velocity and then, I don't know*
36	**Francesco:**	*There is a moment* [he is pointing at the horizontal stretch] *in which the velocity is constant* [...]
42	**Francesco:**	*Sorry, but here* [he is pointing at the starting point of the decreasing slanted stretch] *is he going fast?* [he is moving his finger on this stretch] [...]
46	**Giorgio:**	*Then here* [he points at the first slanted stretch several times] *he is covering more space*
47	**Filippo:**	*But here too* [with his pen he is pointing at the second slanted stretch] *theoretically* [...]
59	**Filippo:**	*And then here* [at the starting point of the second slanted stretch] *he is stopping in theory, because, hum...*
60	**Giorgio:**	*No* [he is pointing at the higher horizontal stretch], *he continues to go on at the same velocity* [...]
64	**Francesco:**	*It is here* [he points at the starting point of the graph]. *Namely he starts here and then he comes back here* [at the final point of the graph] [...]
88	**Giorgio:**	*This* [the first slanted stretch] *and this* [the first horizontal stretch] *are a straight line, all equal straight lines!* [...]
96	**Francesco:**	*It is a curve* [Filippo is writing it] *which starts increasing and ends decreasing*
97	**Filippo:**	*Increasing...then acceleration* [...]
99	**Francesco:**	*...And ends decreasing*
100	**Filippo:**	*Slowing down!*

The students misinterpret the graph, with a confusion between distance and velocity (slope/height mistake). For example, at #11 they think that the ascending part means an increasing velocity; at #19 the first horizontal part represents a constant velocity. From #42 on, the misinterpretation continues: for example at #46, where pupils refer to a velocity. In all the dialogue there are many deictic functions of language and many pointing gestures, with the role of indicators of the graph (e.g. the word "*here*", often used with a gesture, in #42, #46, #47, #59, #64, #88). Here, language and gestures are a kind of communication indicating a static interpretation of the graph which does not become dynamic. These students do not come to a correct interpretation of the motion represented in the graph, because they are not able to link the graph itself to a real motion. We think that they do not make a transformational reasoning, because they did not experience the motion, nor use the technology. Furthermore, these students' inner times are not so closed together, as those of the previous small group.

CONCLUSIONS

It would be too simple concluding that the motion experience, together with the use of a proper technology offering a powerful mediation for learning, is sufficient to construct a right meaning for a graph. The two observed small groups of students offer a significantly different approach to the activity, but it is also important to describe the general approaches of the two classrooms (the experimental group and the control group). In the first classroom, all the small groups were able to describe the graph in terms of motion, while in the second classroom, only one small group reached a correct interpretation, while all the others (5 small groups, for a total of 16 students) produced the misinterpretations described in the literature [5].

Our suggestion is that both the motion experiment and the mediation of technology (calculator and CBR), together offer a support to make the transitions from a static to a dynamic interpretation of the graph and backwards. These transitions help the students to evolve towards a correct meaning for the graph, linking it to the phenomenon which it describes. This fact appears crucial, insofar the technology used in this experiment supports both the use of transformational reasoning and the synchronisation of students' inner times. Along with Nemirowsky, when he says that "learning graphing entails the enrichment of a broad range of experimental domains, involving the refinement of visual, kinaesthetic and narrative resources" (Nemirovsky et al., 1998), we believe that this kind of activities is important for Mathematics Education research and practice, because they foster the passage from perceptual to theoretical thinking.

NOTES

1. As perceptual level, we mean using perception (e.g. seeing and touching, while moving). As theoretical level, we mean producing a conjecture in a conditional form and validating it with a proof.
2. It is important to underline that not only a graph of a time-dependent phenomenon, but also a graph of time-independent one, or a geometric figure, or a function and so on, can be conceived in a dynamical way.

3. It is a Liceo Scientifico. These students attend five mathematics classes and three physics classes per week.

4. The words written in italic type are not present in the activity of the control group of students: they are substituted by an oral description of the motion.

5. It was only after a discussion led by one of us, during which the actual motion was carried out, that these students were able to understand the shape of the graph.

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COGNITIVE TENDENCIES AND THE INTERACTION BETWEEN SEMANTICS AND ALGEBRAIC SYNTAX IN THE PRODUCTION OF SYNTACTIC ERRORS

Eugenio Filloy, Teresa Rojano and Armando Solares, Cinvestav, MEXICO

In Filloy and Rojano (1989) we introduced the use of concrete models for the teaching of solving linear equations and studied the abstraction processes that take place when such models are put in work by 12-13 year olds. In this paper we discuss M and V cases where, on the face of elements provided by the same "concrete" model for the operation of the unknown, the evolution paths of their use for the resolution of more and more complex equation modes are dissimilar (in fact, antagonic). However, in spite of this antagonism there is a common tendency to abbreviate the processes involved and this generates (in both cases) a number of well known algebra learning obstacles and syntactic errors.

The literature about algebraic errors in the learning of algebra is mainly focused in its syntactic component: Matz (1982), Kirshner (1987), Drohuard (1992). Few works like Booth (1984) and Bell (1996) situate this problematic component in a more general context, for instance, problem solving. In this paper we analyse the interaction between semantics and algebraic syntax as a source of syntactic errors, when this ineraction takes place in teaching processes that involve concrete modeling. We argue that such analysis provides a perspective that allows us to give different explanations of the presence of some typical algebraic syntax errors.

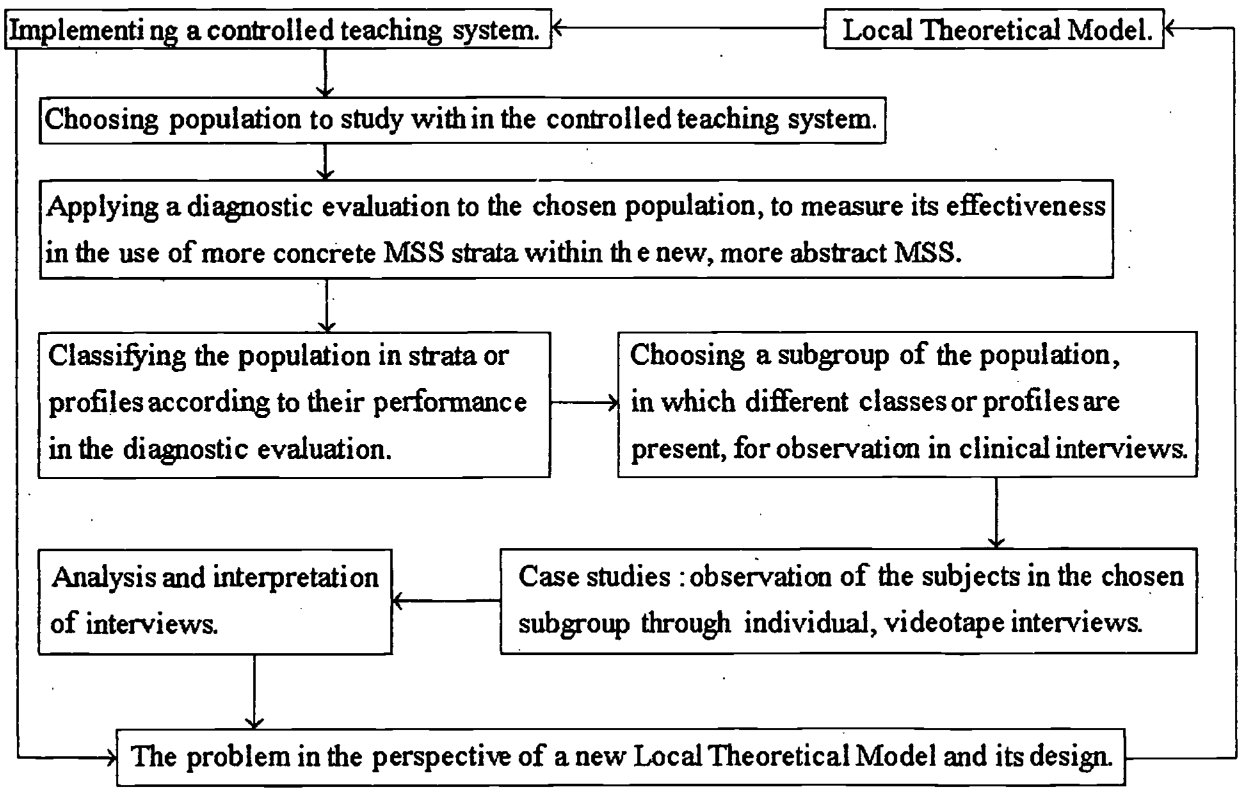
Undertaking a semantic introduction to new algebraic concepts, objects, and operations implies selecting a concrete situation (i.e., a situation which in some context is familiar to the learner) in which such objects and operations can be modeled. With this approach it is possible to resort to previous knowledge, in order to accomplish the attainment of new knowledge. This is one of the driving principles of modeling, the strengths and weaknesses of which become manifest at the time a specific model is put into operation (see Filloy/Rojano, 2001, for a more detailed description). In the cases we report here (V and M cases) the concrete situation in question is a geometric model that was used as a semantic introduction to the operation of the unknown for the resolution of the first non-arithmetical equations. In this model, the translation of the proposed equation into equalities between quantities or magnitudes in a more 'concrete' situation permits to find out the numnerical value of the unknown in the context of area comparison. The use of such a geometric model, then, presupposes a good handling of operations with areas. This handling, as can be verified in V and M's interviews, is a requirement that was covered in both cases. When this study was carried out, M and V showed to be highly proficient at school maths. They found no difficulties in handling the model during the instruction phase aimed at modeling the first non-arithmetic equations (equations of the form $Ax+B=Cx$, where A, B, and C are given positive integers, and $C>A$). It is in the transition to more complex modes of equations that modeling and actions in the model, in turn became more and more complex. In contrast to previous explanatios given with regards pupils' syntax errors, within the modeling realm it is possible to formulate explanations grounded on the nature of the model and the sort of cognitive tendencies displayed by the subjects.

THEORETICAL AND METHODOLOGICAL FRAMEWORK

In Filloy (1990) we introduced the methodological framework of **local theoretical models** in which the object of study is brought into focus through four inter-related components:

(1) **Teaching models** together with (2) **models of cognitive processes**, both related to (3) **models of formal competence** that simulate competent performance of the ideal user of a Mathematical Sign System (MSS) and (4) **communication models** to describe rules of communicative competence, production of texts, texts decoding, and contextual clarification.

The following scheme describes the rationale of the case study:



Results from the diagnostic test located V and M in the category of students with high proficiency in a) solving arithmetic linear equations; b) solving arithmetic word problems; and c) numeric skills. Before the interview, V and M had not been introduced to the learning of algebra. Once V and M got to solve the first non-arithmetic equations by means of the geometric model, they were faced with more complex modes of such type of equations as it is shown in the lists of the interview items below. It can be noticed that the list of items differ from one case to another, due to the specific characteristics shown either by the "semantic" case or by the "syntactic" case during the interview.

THE INTERVIEW ITEMS

We will write **IM.n** and **IV.n** for the **nth** item of Sequence I Series in the interview.

IV.1	$x+2=2x$	IV.17	$5x=3+2x$	IM.1	$x+2=2x$	IM.17	$3+2x=5x$
IV.2	$2x-14=4x$	IV.18	$6x+15=9x$	IM.2	$x+5=2x$	IM.18	$2x+3=5x$
IV.3	$2x+3=5x$	IV.19	$4x-3x=7$	IM.3	$2x+4=4x$	IM.19	$5x=3+2x$
IV.4	$x+5=2x$	IV.20	$4x+25=9x$	IM.4	$3x+8=7x$	IM.20	$5x=3+2x$
IV.5	$2x+3=8x$	IV.21	$7x+2=3x+6$	IM.5	$3x+8=6x$	IM.21	$6x+15=9x$
IV.6	$7x+6=8x$	IV.22	$13x+20=x+647$	IM.6	$2x+4=4x$	IM.22	$7x+2=3x+6$
IV.7	$8x+56=15x$	IV.23	$8x+30=5x+9$	IM.7	$x+2=2x$	IM.23	$13x+20=x+164$
IV.8	$6x+144=18x$	IV.24	$8x+986=12x+647$	IM.8	$7x+15=2x$	IM.24	$13x+12=x+144$
IV.9	$2x+3=5x$	IV.25	$7x+10=4x+4$	IM.9	$7x+15=8x$	IM.25	$10x-18=4x$
IV.10	$3x=4+2x$	IV.26	$9x+33=5x+17$	IM.10	$5x+12=9x$	IM.26	$10x-18=4x+6$
IV.11	$8x=5x+36$	IV.27	$9x+33=5x-17$	IM.11	$15x+13=16x$	IM.27	$7x-20=5x+30$
IV.12	$9x+90=19x$	IV.28	$5x-25=11x+3$	IM.12	$38x+72=56x$	IM.28	$10x-20=5x+30$
IV.13	$x+25=6x$	IV.29	$8x-10=6x-4$	IM.13	$129x+51=231x$		
IV.14	$x+5=2x$	IV.30	$23x-7=14x+2$	IM.14	$37x+852=250x$		
IV.15	$3+2x=5x$	IV.31	$18x-41=9x-5$	IM.15	$x+5=2x$		
IV.16	$5x=2x+3$	IV.32	$19x-3=4x$	IM.16	$2x+3=5x$		

ABBREVIATION PROCESSES

The development of the use of the concrete model is not uniform, it depends on the individual student's tendency to choose a particular approach (Filloy, 1991; Filloy/Sutherland, 1996).

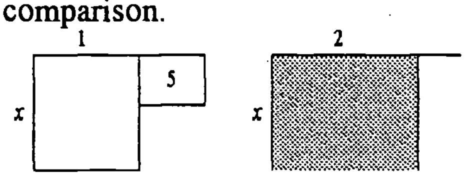
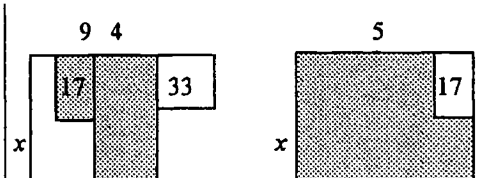
Two extreme cases were detected in the interview: In one case (V case) with an operative tendency, the development anchored to the use of the model context even when the equation types required very complicated modelling procedures. This is the *semantic cognitive tendency* case. In the other case (M case) there was a constant search for the syntactic elements present in the actions on the model as they were repeated in equation after equation and in type after type. The subject broke away from the semantics of the model with a more abstract languaje through the creation of personal codes, belonging neither to the model nor to algebra. This is the *syntactic cognitive tendency* case.

Notwithstanding the bias that the subject's own tendency introduces in his or her use of the concrete model for the resolution of the new equations, there exists, as can be observed from these extreme cases, a common tendency. This tendency consists in abbreviating both the translation processes from the equation to the model and the actions performed in the model (or in the equation itself). In the following subsections, we discuss the abbrevation processes related to both cognitive tendencies, the semantic and the syntactic tendencies.

A semantic tendency

V, shows distinctive evolution lines with regards the abbrevation processes: on the one hand, 1) there are those of stability, progress and generalization of the graphic abbreviation, with the intervention of anticipatory mechanisms regarding the actions, and on the other 2) the transference and discrimination of strategies for area comparison, for each mode of equation.

V's resolution examples:

IV.14 *x+5=2x* Areas comparison	The comparison of areas and the writing (and verbalization) of the simplified equation is done at the same time - without difficulty - without help.	*Obs.: In the areas comparison V draws some more lines in the incomplete figure. This stage and the written form of the simplified equation are almost mixed up, due to the speed with which V completes them. From Vs' manner of writing the simplified equation, making very clear the coefficient 1 of 'x', it could be said that this equation is very close to the context of the model(1x represents an area). Very probably V solves this equation through a specific fact.*
Construction of the simplified equation	V: "We take away this piece of land (referring to the rectangle of 1 x 'x') and we are left with this which should be one(referring to the remaining front)": V only draws some additional lines for the comparison.  V: "One by 'x' should be equal to five". At the same time writes, making clear the coefficient 1 of 'x' : $1x = 5$	
IV.27 $9x+33=5x-17$ Construction of the simplified equation	 V: "Four plus seventeen". She writes 4×17. V: "...Times 'x'?".	*Obs.: Mistaken operation between terms with different geometric dimensions, at the moment of constructing the simplified equation.*

Also, in M's performance with the model, abbreviation processes are observed, but in this case such processes tend to not merely automate the actions in the 'concrete' model (as is observed in V's case), but to extract the syntactic elements present in these actions, so they can, in turn, be modeled in a more abstract language and thus make it possible to detach herself from the semantics of the more concrete model.

A syntactic tendency

In M's case, her very concern about finding syntactic rules for the resolution of the new equations leads to a reflection on the actions that are always performed, one case after the other, and this presupposes (as in V's case) an abbreviation process of such actions. The search for syntactic rules is carried out, in this case, by describing the actions in a more abstract language: such a description requires a synthetic version of the actions.

This synthesis of the actions is achieved through an abbreviation of the resolution process, both in the concrete model and in her translation into the more abstract language. During this translation process, graphs are created (arrows as personal codes) that do not belong to either algebra or to the context of the 'concrete' model. These personal codes allow M to understand, orient, and **represent** the operations outside of the model, although it should be pointed out that such extra-model representations have the defect of not being adequate ways of representing the result of the operations. Such a representation appears in terms of the operations that led to a simplified equation, without making it clear that it (the simplified equation itself) is a final state of those operations. This situation (having in mind just the operations), very frequently leads to an aberrant operation between terms of different degrees.

M's resolution examples:

IM11 $15x + 13 = 16x$	$15x + 13 = 16x$ $15x + 13 = 16x - 15x$ M. "Sixteen x minus fifteen x is one; then, one plus thirteen is fourteen"	*Obs.: The fact of not registering 'x' as a part of the result of subtracting terms in 'x' (15x and 16x), leads M. to effect an aberrant operation between terms of different degrees: one (x) plus thirteen.*
IM13 $129x + 51 = 231x$	$129x + 51 = 231x$ $129x + 51 = 231x -$ $129x + 51 = 231x - 129x = 102x$	*Obs.: M develops her own code of signals in order to leave written traces of those actions that have already been done, as well as those to be performed and of intermediate states in the resolution of the equation:*

	M: "Therefore 'x' is equal to two", the answer is wrong.	*a) Indicating, by means of arrows, the transfer of terms, through the corresponding inverse operation, from one side of the equation to the other.* *b) Indicating, by means of arrows, the equality between terms in this case, such equalization to tantamount to the so-called simplified equation.* *c) Indicating, by means of chains of equalities, both operations between terms and the results of such operations.*

Different Tendencies

The antagonism of these two tendencies (V's and M's) becomes clear by merely observing their respective interviews. However, from their comparative analysis a couple of remarks regarding the aspects that are common to both cases, deserve to be emphazised. On the one hand, 1) it can be seen that in spite of the aforesaid antagonism, there is a common tendency to abbreviate the processes (following, in each case, its' own execution path), and on the other hand, 2) during such abbreviation processes a number of obstacles and errors, also common, are generated, and can be considered as typical in the later syntactic handling of symbolic algebra. In one case (V's), the abbreviating tendency consists of trying to lighten the operations performed in the model, while staying within it. To this end, attention must be given to the actions (translation, comparison, etc.) that are performed again and again. This reflection, in turn, leads to an abbreviation process of such actions. It is through this abbreviation that some parts of the concrete model are lost. On the one hand, 1) the "bottom of the terrains", (the linear dimension which corresponds to the unknown), is a situation that leads to hiding the operation of the unknown. On the other hand, 2) the area condition of the constant term is also lost, as well as its' operative handling. This provokes a tendency of performing the addition of 'x' with the terms of degree zero, resulting in the aberrant operation that has been formerly pointed out between terms of different degrees.

SYNTACTIC ERRORS

The generation of the same type of syntactic errors in the two cases discussed here is not accidental and can be explained from a more general level of analysis. In teaching by means of models there is the danger that the main virtue of any concrete model (namely, that of seeking support in previous knowledge) becomes the main obstructor in the acquisition of such new knowledge. In the cases of the children interviewed, whom are left to develop by themselves the use of the geometric model, it happens that the model component which tends to abbreviate, and therefore, to hide the operation of the unknown, will persist in both cases. In cases such as Vs', subjects possessing a strong semantic tendency have this happen due to the fact that the automation of actions in the model weakens the presence of the unknown throughout the procedure. In cases such as Ms', this tendency is due to the effects of the creation of personal codes, to register intermediate states of the originally proposed equation. Corrections, in each case, are of a local nature and according to the subjects' tendency. Thus, when there exists a leaning towards staying in the model, the correction of the syntactic abberration mentioned above is performed in the model itself, for only the model semantics can make such an aberration evident. On the other hand, in the case of a syntactic tendency, the correction takes place currently with other events in the syntax, namely, an essential modification of the notions of equation and the unknown.

GENERAL DISCUSSION

By way of conclusion, the interaction between semantics and algebraic syntax which takes place along the abstraction processes of operations performed with algebraic objects (that have been given meanings and senses within the context of a concrete model) is modulated when learning the algebraic language by tendencies in the subject and by features of the specific model being used. There are, however, some aspects of such an interaction that remain constant when changing the subject's tendency factor, or the type of model. These essential aspects in the relationship between semantics and algebraic syntax reflect, in turn, essential aspects of another interaction, the one occurring between the two basic model components: the reduction to the concrete, and the detachment from the semantics of the concrete. The transference of the problem, semantics vs. algebraic syntax to a level of model actions allows one to close the existing teaching gaps between these two algebra domains. The analysis of this interaction between semantic and syntax at this new level, points to the necessity of intervening through the teaching model at key moments at the beginning of algebraic language use.

In a forthcoming paper we will indicate how this dialectic semantics/syntax; the theoretical description of the relationship between the deep and superficial forms; the generative and transformational aspects to describe the grammar (see Kirschner, D. 1987 and Drouhard, J. P. 1992) of algebraic mathematical sign system syntax, can be linked with the explanation of why errors are committed when following a rule is needed to utilize one or more previous and competently used rules.

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The authors would like to thank:

Centro Escolar Hermanos Revueltas – for the use of their classrooms and workspaces, the cooperation of the teachers, and not lastly, their wonderful students in the development of this project.

Consejo Nacional de Ciencia y Tecnología for the founding to carry out new stages of the research program "Acquisition of Algebraic Language" in the framework of the project "The incorporation of new technologies in the school culture" (grant number G 263385).

DEVELOPMENT OF MATHEMATICS TEACHERS' PROFICIENCY IN DISCUSSION ORCHESTRATION

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Abstract:

This paper describes a study in progress that examines development of the teacher's proficiency in managing a whole-class discussion in the context of her experience of teaching students in an inquiry-based learning environment. Discussion segments of two lessons by one teacher with the same class, which were were condacted a year and a half apart, are analyzed. We suggest elements of a structure for discussion orchestration. Using these elements for the micro-analysis of the lesson segments we demonstrate a possible evolution of teacher proficiency.

BACKGROUND

The study presented in this paper focuses on the development of the teacher's proficiency in managing a whole-class mathematics discussion. This type of teachers' endowment is central for many reform-oriented curricula that establish, strengthen and broaden conjecturing, exploration, and investigation procedures in the mathematics classroom (e.g., Ball, 1992; Lampert, 1990; Lampert, 2001; Schifter, 1996; Yerushalmy, Chazan & Gordon, 1990). Teachers are no longer considered the central source of mathematical truth, but rather act to support students by creating problem-solving situations, to acknowledge the value of students' ideas and to respond to them. As the nature of the mathematical task moves towards exploration, the structure of a whole-class discussion changes as well.

In such a clussroom students and their teacher share their roles in the classroom, i.e., the teachers and the students listen to each other, reflect on and clarify others ideas, perform representation translations, and provide explanations (Lampert, 2001; Forman & Ansell, 2001). We believe in the possibility of (almost) symmetrical roles of teachers and students in classroom discussion, however we assume that sharing roles between a teacher and the students depends heavily on the teacher's proficiency. We assume that a teacher without an experience in orchestrating a classroom discussion would be more dominant and less open to students' needs. This study stems from the standpoint that teaching provides many important opportunities for professional development (e.g., Ma, 1999) and examines how discussion orchestration changes through teaching experience.

In their analysis of the discussion orchestration Forman & Ansell (2001) demonstrated that I-R-E (initiation-response-evaluation) social participation structure of the traditional classroom suggested in the 1970s (e.g., Mehan, 1979) does not reflect the structure of discussion orchestration. A discussion can contain several different patterns of interaction, so that there may be several possibilities with respect to IRE structure (e.g., it may not appear at all, or only sometimes). Variations in discussion structure are closely connected to the variations in roles of students and teachers in such a discussion. In inquiry-based discussions there are shifts toward teachers' reflectivity and flexibility. In this study we try to detail these shifts in terms of teacher's discussion actions.

Research on mathematics teachers' education demonstrates the importance of flexibility, deepness and connectedness of teachers' mathematical knowledge (e.g., Ball, 1992; Lampert, 1990; Schifter, 1996). Ball (1992) establishes teacher knowledge about the nature and discourse of mathematics as essential part of teachers subject-matter understanding. Jaworski (1992) defines the essence of mathematics teaching as being placed in three domains: the management-of-learning, sensitivity-to-students, and mathematical-challenge. This study tries to analyze development of teachers' proficiency of managing the whole-class student-sensitive discussion through presenting students with challenging mathematical questions.

THE PURPOSE AND THE QUESTIONS

The purpose of the current study is ***to characterize teacher's actions*** in the inquiry-based classroom discussion and ***to analyze changes in the teacher's actions*** over time in order to identify what kinds of teacher behavior makes discussion more effective, reflective and flexible. In this paper we focus on two questions:
(1) What are the main types of teacher actions in the inquiry-based classroom discussion?
(2) How do these actions change in the course of teaching practice?

METHOD

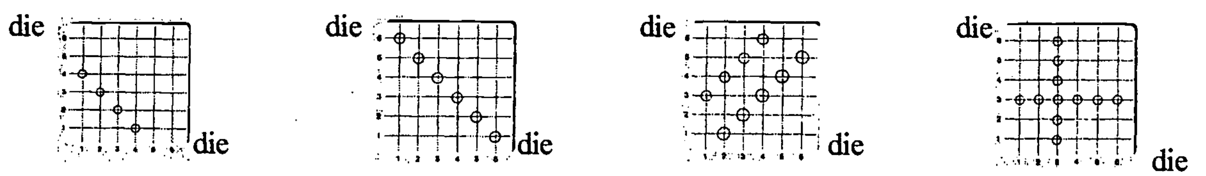
The teacher: Shelly -- the first author of this paper -- was a beginning teacher, having graduated with her B.Ed. three years before the experiment in her classroom started. During first three years of her teaching career she took part in a professional development program for elementary school mathematics teachers. The course was focused on inquiry-based learning of mathematics. During the second and the third years of the professional development program she joined a development team that specialized in materials for an inquiry-based mathematics classroom (Fridlander, 1997; Fridlander & Rota, 1996). At the end of the professional development program, Shelly started experimental implementation of the new learning materials. Our study considers Shelly's teaching experiment (Cobb, Wood & Yackel, 1990) during two years and the development of Shelly's proficiency through teaching.

The data: Overall about 20 of Shelly's lessons were videotaped during the two years of the experiment for the purpose of the formative evaluation of the teaching materials. Thus at that time she was not aware of the possibility of using these data for the purposes of the current study. For the purposes of this study, three of 20 lessons with the same group of students and very similar in their structure were chosen. This paper presents analysis of teachers discussion actions during these three lessons and analyzes changes in the teacher's actions using 10-minutes segments from two of these three lessons, which were 1.5 years apart. The first lesson took place when the students were in the middle of the 2nd grade (January, 1993), the second lesson took place when the students were at the end of the 3rd grade (May, 1994).

Lesson structure: As noted above, the two lessons have very similar structure including three main stages: Introduction, Inquiry, Summary-discussion. In the very short introduction stage Shelly presented students with the problem. During the inquiry stage the students worked in small groups and Shelly helped them to progress in their investigation when the help was needed. The last stage of the lesson, which is under investigation in this paper, is the whole-class summary-discussion. During this stage the students from the small groups shared results of their investigations with the whole class, they made conjectures, discussed them, compared different solutions of the problem and tried to come to a conclusion. The teachers' role in this setting was helping students make progress in their presentations, generalize the results and finish the lesson with shared mathematical meaning.Mathematical tasks for both of the lessons included series of questions that were under students' investigation. Below we present mathematical tasks that were at the focus of the discussion in the segments of the lessons considered in this paper.

Task for the segment from the first lesson: In this task (from Dice activity: Fridlander & Rota, 1996) the children investigated the relationship between a graph and the common property of the points in the graph. In this paper we refer to the part of the Discussion that was dedicated to the following questions (Figure 1).

Write down the result marked by the points on the boards below. Try to describe all the points marked.

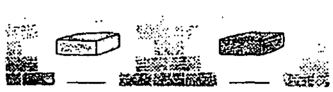


The marked points are.... The marked points are.... The marked points are.... The marked points are....

Figure 1: Task for the segment from the first lesson

Task for the segment from the second lesson: In this task the children dealt with an unknown number of matches that satisfy a given condition (from Matches activity: Fridlander & Rota, 1996, see Figure 2).

There are 21 matches in the two boxes. How many matches are hiding in each box?



i) In the white box there are **5 matches more** than in the dark box.

ii) In the white box there are **3 matches less** than in the dark box.

iii) In the white box there are **half** as many matches as in the dark box.

Fig 2: Task for lesson 2

DATA ANALYSIS AND RESULTS

We would like to be precise in the description of Shelly's flexibility. Thus, first we present main types of the teacher's actions in her classroom and then analyze the changes that were found in Shelly's discussion orchestration.

Main types of teacher's discussion actions: During *whole-class discussion* Shelly performed different actions that composed her discussion orchestration. While students' learning activities were mainly focused on solving a mathematical problem and were aimed at constructing students' personal mathematical meaning, Shelly's activity was mainly focused on solving teaching problems and on supporting students' construction of the mathematical meaning. Based on Forman & Ansell's (2001) statement that both the teachers and the students are "involved in revoicing each other and listening to, reflecting on, clarifying, expanding, translating, evaluating and integrating each other's explanations", we tried to zoom in the transcripts of the discussions and to identify precisely main teachers' and students' discussion actions. We first defined three main classes of the discussion actions: class of *Stimulating Initiation* including actions that begin discussion of a new mathematical question; class of *Stimulating Reply* including actions that stimulate continuatoin of a discussion and are connected to prior utterances; and class of *Summary Reply* including actions that finish discussion of a particular question. Second, based on our analysis of the transcripts of the three lessons, we defined possible teaching discussion actions. We (the two authors of the paper) defined these categories of actions independently, discussed the terminology and agreed about the following categories to be used in the further analysis. Note, that in this paper we present the part of the study that analyses teacher's discussion actions only.

Questioning: In this category we included utterances in which Shelly presented students with a problematic situation and invited students to find a solution to this situation. In other words, posing a problem, starting a new stage of the discussion, changing a problem under consideration were all attributed to the questioning category. A teacher may ask a question that was planned or may refer to (reflect on) a student's conjecture, statement, or difficulty. Thus, questioning category was subdivided into different sub-categories according to their purposes in the course of the discussion (e.g., Opening questions, like "Who is ready to present his solution?" Promoting questions, like "And then what did you do?" Clarification questions, like: "Why did you take [this numbers] one and two?" or "What can you tell about the points on this diagonal line?" or "Let's check whether this results is correct."). Note here that the teacher's utterance was categorized as questioning without relation to its semantic structure. Mainly questioning appeared at the two stimulating stages. Note also that in our segments "Stimulating Initiation" appeared in questionning form only.

Translating a representation: Teacher's action in which she performs symbolic or graphical representations of students' utterance, like writing and drawing on the board were considered as performing translation between representations. In our segment this category was usually accompanied by constructing a logical chain.

Constructing a logical chain: These teacher's actions include a chain of the type "*if – then*". Initially, this category was considered a complex explanation but at the later stage of the analysis complex explanations were subdivided into translating representations and constructing logical chains.

Repeating students' utterance: A teacher repeats exactly what a student said. This action was performed for different purposes, e.g., to continue discussion, to stress the student's idea, to invite students to think about the correctness or the conjecture. Thus according to the purposes of the repetitions they were subdivided into other subcategories.

Hinting: In some situations, when Shelly felt that the discussion was stuck, she tried to help students to move forward in their reasoning mainly by means of *connections* with other similar cases in which they applied a similar problem-solving strategy or algorithm. For example: "Think how I usually write this".

Stating a fact: In this category we included teacher's statements of mathematical or metamathematical facts.

Providing feedback: Teacher's reflective evaluation of students' solutions was included in this category of teachers' actions.

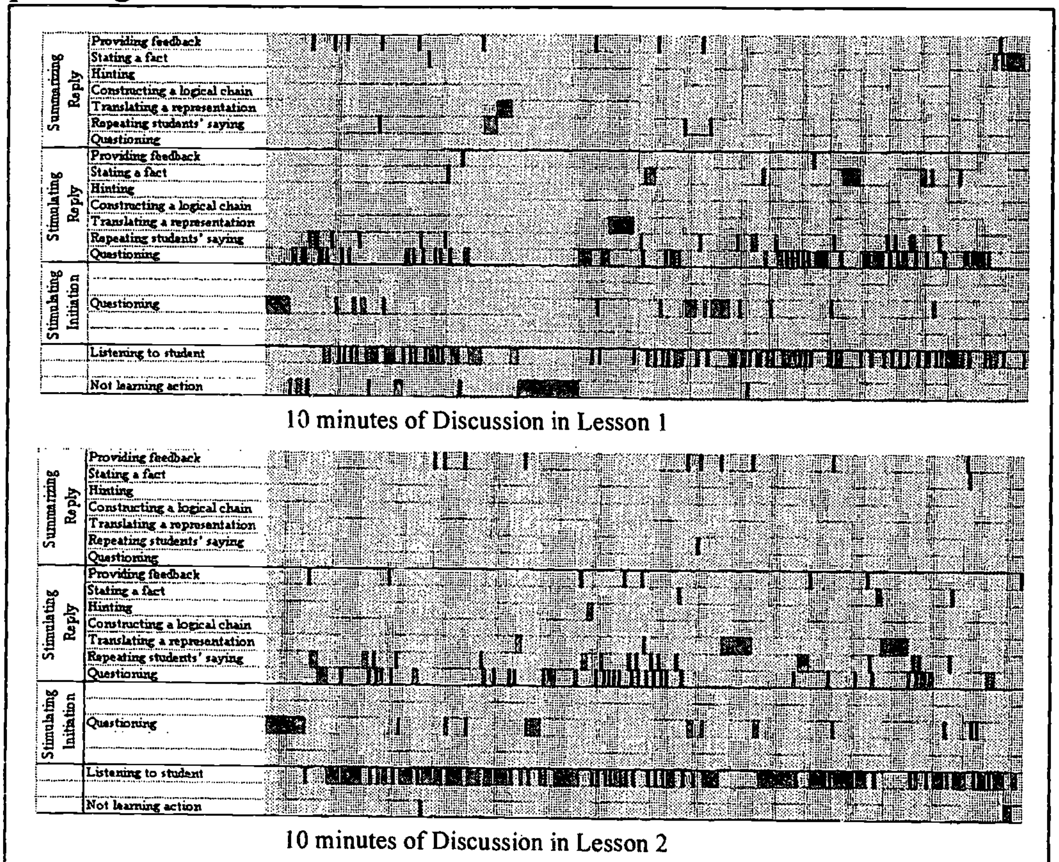
Figure 3 demonstrates the coded transcript of the segment of discussion in the second lesson.

Time	Name	Utterance	Coding
0:16:05	Shelly	O.K. [Ben is raising his hand] Ben, how did you do it?"	Summary reply: F*eedback* Stimulating initiation *Questioning*
0:16:07	Ben	I took the all twenty one matches. I put two in one side and after every two in one side I put two in the ather side, two in one side and one in the ather. I did so untill I had in one box seven and in the ather fourteen.	Listenning to students
0:16:23	Shelly	Why you took two and one two and one?	Stimulating reply: *Questioning*
0:16:27	Ben	Because two divided by one equals one and half of two is one.	Listenning to students
0:16:33	Shelly	O.K. Half of two is one. Did you understand what Ben did?	Stimulating reply: *Feedback* Stimulating reply: *Repeating* Stimulating reply: *Questioning*
0:16:37	Asaf	Two divided by one equals one	Listenning to students
0:16:39	Shelly	Two divided by one equals two [Shelly is drawing on the board while she is talking to the class] What Ben said was that he took.. half of two is one, right?. So, every time he took two and put them here he put one there, add two here and one there until he had no matches. $\boxed{2+2+\square}$ $\boxed{1+1+\square}$ =2 Matches How many did you have here? [Shelly is pointing to the writing on the board]	Stimulating reply: *Stating a fact* Stimulating reply: *Translating a representation* Stimulating reply: *Questioning*

Figure 3: Example of the coded transcript (from discussion in lesson 2)

Changes in teachers' actions over time: After defining the categories we turned back to the transcripts and coded teachers' actions in the two transcript segments (each 10 minutes long) independently (see example in Figure 3). Ninety-three percent of discussion actions (of the total number of teaching discussion actions) were coded identically by the two coders. The other seven percent were discussed in order to achieve an agreement on the coding. We applied the Shoenfeld (1985) analysis schemes to solving the teaching problems. Figure 4 presents micro time-line analysis of Shelly's discussion actions embedded in the transcripts. The scale along the horizontal axis indicates time. Ten minutes of each segment were divided into 300 units of two seconds. The labels along the vertical axis indicate different teachers actions at the different stages of the discussion.

Figure 4 provides visual support for our suggestion about development of teacher's proficiency in managing a whole-class discussion. We may see that during the second lesson Shelly was listennig to the students more while the students were more active participants of the discussion. We may see that Shelly is less active at the stage of stimulating initiation thus we may suggest that students had more possibilities to initiate discussion. We see only a few summarizing actions that Shelly performed during the second lesson discussion. The nature of the actions at this stage also changed. At the stage of summarizing reply during the first lesson Shelly repeated students' utterances, translated representations and stated facts whereas in the second lesson she mostly provided (positive) feedback to students' actions. At the stage of stimulation reply during the second lesson Shelly provided fewer hints and asked fewer questions than in the first lesson. At the same time Shelly repeated students' statements, and provided more feedback as stimulating reply. In her reflective analysis of the lessons Shelly points out that during the second lesson she felt more secure in basing the discussion on students' conjectures whereas during the first lesson she felt she needed to proceed according to her initial planning.

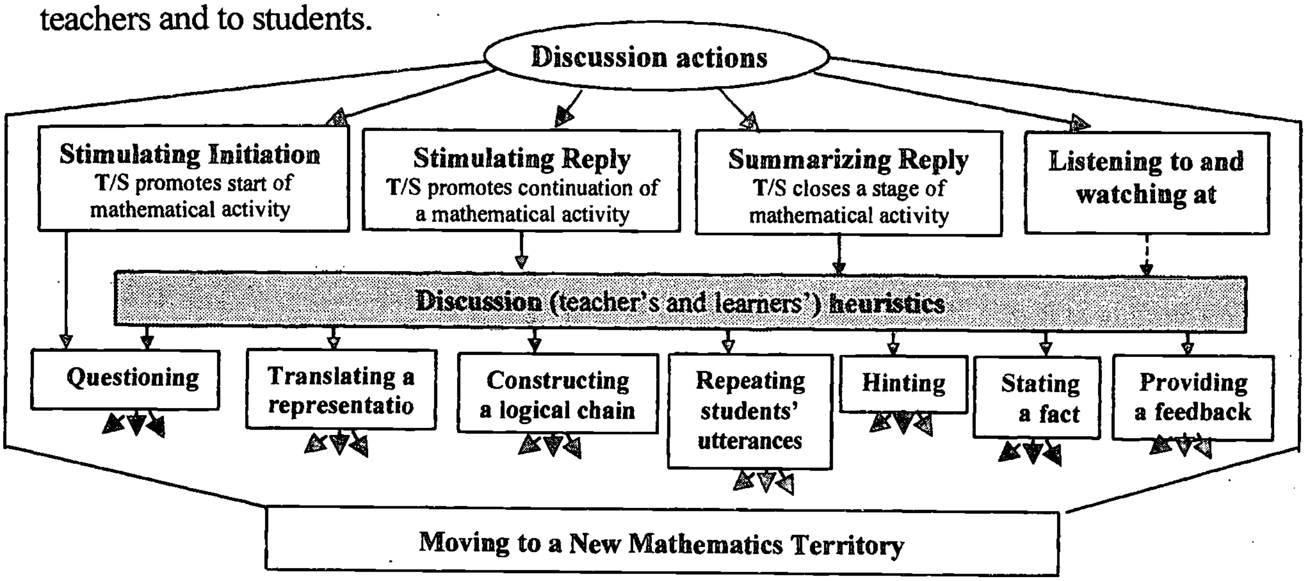


Fugure 4: Discussion Diagrams

Overall we found that Shelly was much more flexible during the second lesson; she was able to change her initial plans in terms of ideas the students presented, and was able to show more trust in her students. This changes in flesibility we describe in terms of discussion actions that were differently distributed between and within the three main classes of actions.

DISCUSSION

We defined four large categories of teachers discussion actions that we refered to as classes of actions. Two of these classes were subdivided into seven categories of teacher's actions (see Figure 5). Our categories are consistent with Lampert's (2001) latest description of management of a whole-class discussion in which she divided her lesson into episodes according to the teacher's roles in the discussion and with Forman and Ansell's (2001) social participation structure of the classroom. In terms of Lampert all these teacher's actions were aimed to lead students into new mathematics territory (see Figure 5). As noted earlier, we consider teacher's and students' roles in the inquiry based classroom to be almost symmetrical, thus in Figure 5 all the actions are attributed both to teachers and to students.



Fugure 5: Discussion actions

We provided a microanalysis of the structure of the discussions in two lessons by the same teacher with the same students one-and-a-half years apart. These micro-pieces of the discussion actions were integrated into the whole picture for the further analysis. During the first lesson the teacher was at the beginning of her experience of managing the whole-class inquiry-based discussions. Development of the teacher's proficiency at managing the whole-class discussion developed in the course of her teaching experience without any professional-development intervention. Thus we consider our findings to be an indication of the act of teaching being an opportunity for teachers' professional development. The type of analysis that we suggested in this paper provides additional evidence of how teachers may orchestrate classroom discussion. We tried to describe what it mean that the teachers become more reflective and more flexible when orchestrating the whole-class discussion.

In this study we applied Shoenfeld (1985) schemes of analysis of solving mathematical problems to solving teaching problem (in terms of Lampert, 2001). Futhermore we would like to pose the following analogy between mathematical problem solving and teaching problem solving. As Shoenfeld (1985) found that one may effectively teach algorithms whereas it is difficult to teach the heuristic for when to apply the proper algorithm. This use of heuristics distingueshes the expert problem-solver from the novice. We suppose that teacher actions may be teachable; however, it may be difficult to teach a teacher when to apply a particular teaching action. In this sense we would like to speculate that teacher actions are of heuristic nature, i.e. are not describable by algorithms. At the same time we suggest that our categories may serve as benchmarks in planning and analysing discussions and may be useful in different professional development programs for mathematics teachers that focus on the issue of whole-class inquiry-based discussion.

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SOLUTION OF WORD PROBLEMS THROUGH A NUMERICAL APPROACH. EVIDENCES ON THE DETACHMENT OF THE ARITHMETICAL USE OF THE UNKNOWN AND THE CONSTRUCTION OF ITS ALGEBRAIC SENSE BY PRE-UNIVERSITY STUDENTS

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In this document, a didactical proposal based in a method to solve arithmetic – algebraic word problems which uses a numerical approach is showed. The method serves as a mediator between the arithmetical and algebraic methods revealing three different uses of the unknown, namely: the "arithmetical unknown", the "numerical unknown", and the "algebraic unknown." Evidences obtained in a case study are shown, and such evidences present the need to make pre-university students face problems conveying to the use of a "numerical unknown." Then, students can achieve the detachment of the arithmetical use of the unknown, construct its algebraic sense and move forward in the competence of solving word problems through algebra.

This investigation is part of a more than 10 years project in which a problem solving approach to pre-university algebra it has been fostered. Some others results can be observed in (Rubio, 1990; Filloy & Rubio, 1993; Rubio, 1994; Filloy, Rojano & Rubio, 2001).The evidences presented hereby are linked to the performance of 15-16 year-old students in tests, clinic interviews and their work in the classroom by using a didactical proposal based in a successive explorations method to solve word problems where numerical values are assigned to the unknown. The purpose of the use of numbers is that they serve the student to: **a)** facilitate the analysis of some families of problems; **b)** detach the sense of the arithmetical use of the unknown that students tend to use when solving word problems; and **c)** advance in the construction of the algebraic use of the unknown through the use of a method that allows the student to give sense to the operation of the unknown of a problem, following a process in which it is operated with a numerical representation of the unknown in the first place, and then with a letter, as specified by the algebraic method to solve problems. It is considered that an unknown has an algebraic use sense when it is operated with the letter it represents.

The numerical approach used by the successive explorations method does not have as central objective that one student must solve a problem with numbers, but to facilitate its analysis, help to construct and give sense to the operations between the elements of the problem, and achieve its representation and solution algebraically. The method was conceived as a didactical device from which the student is detached as it acquires greater competence in more abstract uses of the algebraic language. In fact, it can contribute with elements to close the gap existing between the syntactical development of the algebraic language – to which one generally tends in teaching – and the semantic one.

REFERENTIAL AND THEORETICAL FRAMEWORK

In order to structure the didactical proposal, central aspects of the false position method were taken into account (Bruño, 1939; Vallejo, 1835), as well as the stages established by Piaget (1979) in relation to the assimilation process of the real facts to the mathematical-logical structures in the development of theories constructed from the physical experiences verified by such theories. Besides, the successive explorations method in which the didactical proposal is based, the trend observed in many students regarding the spontaneous use of non-algebraic methods were taken into account, such as the trial-and-error method to undertake several families of arithmetic/algebraic word problems.

The numerical approach of the method is connected from an epistemological point of view with: a) the two first stages described by Piaget (1979) and; b) the first indications about the actions to be made in order to apply the false position method (*Regula Falsi)*. Piaget argues that those stages precede the data translation to the system equations (stage 3). The first one, the establishment of facts or data from the real world is not independent of mathematics modeling such as: classification, relationships, correspondences, measurement, etc. The second one is oriented to the search of an explicative model of the facts or data of the actual world starting from the construction of an intuitive and qualitative scheme. The numerical interpretation of the method captures both stages and its main purpose is to facilitate the settlement of the relationships between the elements of the problem, before translating the data, and its relationships to the equation representing the problem.

A critical historical analysis of the use of the *Regula Falsi* to solve problems in old text books was useful to incorporate its central ideas to the phases of the successive explorations method, which central aspect is the analytical intention of the use of numerical values to unchain the solution of a problem. The study of some of the problems solved with the *Regula Falsi* led us to conclude that its importance is due to the fact that it has an objective very similar to that of the algebraic method, which is to facilitate the analysis of problems and treat all of them in a similar way, as if they were the same problem (different to the arithmetical method in which each problem is analyzed case by case, in a particular way). The *Regula Falsi* (e.g. Bruño, 1939) starts by considering the problem as solved, but instead of using a letter to represent to unknown as in the algebraic method, a numerical value is proposed as a possible solution, using it later to make all the operations indicated in the problem; finally the *Regula Falsi* ends with the settlement of a proportion that leads to the solution of the problem. This last aspect was not incorporated in the explorations method.

METHODOLOGY

The general project contemplated in its first stage an initial study during one school year in which the use of the method was observed in the classroom with 15-16 year-old students and through the clinic interview of one girl student of the same age and school, but from another group. The main objective of this study was to improve the teaching model based on the successive explorations method. In the second stage of the project, a Case Study was carried out composed by six students whose clinical interviews were videotaped. Such students were chosen among 35 (15-16 years old) through a classification in classes based on their performance in three tests linked to three areas of competence: arithmetical problems; algebraic problems and solution of equations.

THE STUDY

The empirical investigation was focused in the study of meaning processes given by pre-university students to the unknown when they face some families of arithmetic / algebraic problems. The case study tried to find evidences regarding the changes of the use that students give to the unknown when facing problems which nature is "more arithmetic" than algebraic or vice-versa. The empirical work revealed the need to distinguish three different senses of use of the unknown: "arithmetical unknown" when given an arithmetic use, that is, when not operating with it; the "numerical unknown" when it is assigned a numerical value and when operations involving this value are accomplished and the "algebraic unknown" when it is represented with a letter and it is operated with the latter.

It was observed that in order to construct the algebraic meaning of the unknown it is required that the student is successively detached from the senses of the arithmetic and numerical uses of the unknown, considering that there is a detachment in the sense of the arithmetical use, when in the solution of a problem it is used a representation of it (either a number or a letter). The study proved that in order to propitiate such a detachment, it is required to face the student to problems which arithmetical solutions are complex or need a competence in arithmetic at an expert level. On the other hand, the detachment from the numerical use of the unknown is induced by the whole structure of the successive explorations method, where one starts operating with the "numerical unknown" and ends with the algebraic solution of one equation where the letter representing the unknown of the problem is used.

Briefly, in the study we try to evidence that the use of a "numerical unknown" can propitiate that the student detaches from its arithmetical use and begins the construction of the "algebraic unknown". It is thought that this is due since the "numerical unknown" facilitates the analysis of problems, where the relationships between their elements are established between numbers and where it can be seen that the numerical relationships between the elements of the problem not only have more semantic load than the algebraic ones, but also that one can use the unknown in them, although such unknown is "numerical", within an analysis process "moving forward" as in the algebraic method where one operates with a letter.

The observation of the solution processes in the case study allowed us to obtain evidences about the moments and situations conveying the student to leave the arithmetical use of the unknown and those situations in which the student constructs a pre-algebraic or algebraic use. The latter was achieved by facing the student to problems of such nature that their arithmetic solution is complex, but by also giving him/her a way that leads him/her to the algebraic use. In the study can be seen that the use of a "numerical unknown" helps to fulfill the aforesaid since it facilitates the analysis of the problems and the settlement of relationships between their elements by operating first with numbers and then with the "algebraic unknown" when solving the equation obtained in the last phase of the successive explorations method.

It is also evidenced that in order to achieve that one student be competent in the algebraic method it is not enough that he/she detaches from the arithmetic and algebraic uses of the unknown, and that even he/she represents a problem with an equation, but that it is also required that he/she progressively advances in the competence to solve algebraically the equations that arose by following the phases of the successive explorations method and then, progressively, of the algebraic method.

THE SUCCESSIVE EXPLORATIONS METHOD

The method consists in six phases, each one with an objective of their own. The first three are basically pre-algebraic and have as paradigm the false position method (Bruño, 1939; Vallejo, 1835). With them, it is intended that the student: a) recognizes that in the problematic situation exists the presence of something unknown that can be determined by considering the limitations of the problem; b) uses a "numerical unknown" to settle the relationships between the elements of the problem through a numerical operations; and c) compares two amounts representing the same thing in the problem in order to construct the sense of the algebraic use of the equal sign (Kieran, 1981). The objective of the other phases is the construction and solution of the equation representing the problem.

To represent its phases we will use an episode of Maribel's interview, a student that was not competent in the solution of problems with the arithmetical method, but that was competent in the solution of some families of problems with the algebraic method and others with the successive explorations method. It was observed that she was using this method when she was not able to understand the relationships between the elements of the problem and, regarding this, Maribel says in one part of the interview: "....when I find it difficult to understand things...I start to use ...to use the method that I took... thus I find out what is happening in the problem...".

The episode begins by showing an algebraic problem, the Problem of the Bottles: *"350lt. of liquor must be bottled in 500 containers, some of them of 3/4 lt. and others of 1/2 lt. How many bottles of each type were used?"*

Phase 1 demands that the unknown must be explicit. This is illustrated in the next:

Maribel: ...Yes...now I am taking out...which are our unknown quantities...[She writes one below another of the unknown quantities: No. of 3/4 lt. bottles and No. of 1/2 lt. bottles].

Phase 2 asks for the assignation of a numerical value to the unknown as a hypothetical solution of the problem, which allows the generation of the problem's operations. Let's see this:

Maribel: ...Now I am going to...suppose...a value...to see if from there...I can go to one equation.

Interviewer: ...What are you going to do?

Maribel: There are 500 containers...I guess we have 200 here... [She uses this number as a "numerical unknown" and points out where she wrote "No. of 1/2 lt. bottles].

The use of a "numerical unknown" is now helpful to Maribel to construct and give sense to the relationships between the elements of the problem as follows:

Maribel: ...I multiply 200 containers by 1/2lt...and I have ...100 litres...[This represents the litres in the 200 bottles]. Then she adds: ... I have 500 containers...then...there must be here...300 of...[She points out the other unknown: No. of 3/4 lt. bottles]

Interviewer: That's it...How did you get it? ...

Maribel: I got it from...subtracting 200 from 500...then multiply 300 containers by 3/4 lt. ...that would be...75lt...[This value representing the litres in 300 3/4 lt. bottles, but she forgot multiply by 4].

Phase 3 prepares the path of the equation. It requires the comparison between two amounts representing the same thing in the problem. In the interview,

Maribel: ...They are 175 litres...which is not the same to 350... [Maribel compares two amounts representing the total of litres to be bottled in 500 containers].

Interviewer: ...Isn't it the same?

Maribel: ...No...then I get from there my comparison ... [She writes: 175lt. = 350lt? and below each of these numbers their meaning in words.]

Phase 4 indicates that it is necessary to work backwards starting from the comparison: A=B? recovering the operations made with the purpose of getting a "numerical pattern". Let's illustrate it:

Interviewer: Now...What do you do?

Maribel: ...I recover operations...the 175 came out...of the addition of 100 plus 75...the 100 came out of...200...which is the value I supposed by...1/2 ... and the 75 came out of ...300 by...3/4...[While speaking, she wrote the operations and obtained the numerical expression: $200\times1/2+300\times3/4=350?$]. After reading what she wrote Maribel adds: But the 300 came out of...the subtraction of 200...minus...well...500 minus 200... [When saying this, the student wrote the numerical pattern: $\underline{200}\times1/2+(500-\underline{200})\times3/4=350?$]

Phase 5 requires the use of a letter in order to represent the unknown, as well as the construction of the equation of the problem from the numerical pattern obtained in the previous phase, as follows:

Maribel: In order to get the equation...we say that "x" is equal to the 1/2lt. bottles that...are used...and we substitute the supposed value... [We can see that the letter is used here as a name]

Interviewer: That is…the "x" is the same as the supposed value?...

Maribel: You can say so, yes, because…with the "x" we are giving…a supposed value

Interviewer: The "x" is two hundred?

Maribel: No…

Interviewer: So? ... the "x"...What is it?...

Maribel: The "x" is the exact 1/2 lt. bottles used to bottle one part of the 350 litres …

When saying the abovementioned and writing: $1/2\ x + (500 - x) \times 3/4 = 350$, Maribel is detached from the sense of the numerical use of the unknown that fosters the successive explorations method.

Phase 6 asks to solve algebraically the equation obtained. In the interview:

Interviewer: …What do you do to find out the value of "x"? She says:

Maribel: The "x"…I solve the "x" to know its value…that is to say…I solve the equation

THE "ARITHMETICAL UNKNOWN" AND "NUMERICAL UNKNOWN" IN EDGAR'S INTERVIEW

The use of an "arithmetical unknown". This student was chosen because he showed competence to solve some families of problems with the arithmetical method and others with the algebraic one. Besides, it was observed that even though he didn't use the successive exploration method's phases systematically; he supported himself above all in the numerical exploration to unchain the analysis of complex problems. An episode of Edgar's interview is shown as follows, where we can see the solution of a problem, through the arithmetical method and the arithmetic use of the unknown, ("arithmetical unknown").

The episode begins showing the Perfume Problem: *"How many millilitres of perfume essence must be added to 60 millilitres of alcohol to get a lotion with 70% of perfume?*. Once Edgar has read it he says:

Edgar: …We have 60ml. of alcohol…and we need it to have 70% of perfume…then the other 30% missing here…it is supposed to be covered by alcohol…then…because there are 60ml…they are divided by 3… [Edgar writes: 10% — 20ml] …each 10% of the mix…they will be 20ml…then…a 30% is 60ml….a 10% will be…20ml… Then…I need them to be 70% of perfume…I multiply 20 by 7 …it will be 140….millilitres…of perfume [This is the value of the arithmetical unknown. It is observed that Edgar does not operate with such unknown].

The use of a "numerical unknown". In the following episode of the interview it is observed how Edgar detaches from the arithmetical use he was giving to the unknown in the Perfume Problem presented to him a few minutes earlier by posing him a problem which complexity leads him to use one "numerical unknown" to facilitate its analysis and where he uses numbers with this representation of the unknown to obtain the relationships between the elements of the problem. Then, the student supports himself on the numerical representation of the problem to give an algebraic sense to the unknown and to get the equation representing the problem.

The episode begins by presenting the student the Problem of the Perfumes' Factory: *"In one perfumes' factory are two containers, one with 18% concentration of Chanel and the other with 43% concentration of Chanel. How many millimetres from each container must be used if a 12ml. bottle is needed and which must have a concentration of 36% of pure Chanel?"*

In the study of cases, it was observed that when the student always expressed explicitly the unknown quantities of the problem, it was the first signal that he/she was not going to use an "arithmetical unknown" to solve it. This is evidenced when Edgar draws three containers and says:

Edgar: ... How many millilitres...will be taken out from each container to fill this one...Right?...[He points out the 12ml. bottle]...once again we do not know how many millilitres are here...[He points out the containers with 18% and 43% of perfume]. Then, Edgar adds: ...We have to pour from both containers ...but they are in different concentrations...then...they are 12ml. ...let us suppose...if we pour half of it there will be ...18% from the first one and 43% from the second one ...then ...out of this 12ml....we will take 6ml. of each container ...right?...

It can be observed that at the moment of supposing a numerical value, Edgar detaches from the sense of the arithmetical use of the unknown using this value ("numerical unknown") to get the relationships between the elements of the problem:

Edgar: Yes ...half and half...then...of the first container, from the six millilitres ["numerical unknown" value] we took ... we know that...one 1.08 [He multiplies the "numerical unknown" value by 0.18] ...will be perfume and from the second container ... 2.58ml. [He multiplies 6 by 0.43] ...will be perfume...then ...once putting together what was taken from both containers ...there will be ...3.66ml. of perfume...in the 12ml. bottle we are going to fill...

Interviewer: Is there anything else you have to do?

Edgar: Now...we have to see if this ... [He points out the 3.66ml. of perfume] is the 36% of this amount... [He points out the 12ml. bottle]...now...we will get the 36% of 12...we see that...it is 4.32...which is incorrect...because it must be 3.66 ... [3.66ml. is compared to 4.32ml. which are the two amounts representing the same in the problem, phase 3 of the method].

After doing this, Edgar assigns other values to the "numerical unknown" and gets the pattern: $\underline{8}\times 18/100 + (12-\underline{8}) \times 43/100=12\times 36/100$ and as of such pattern he constructs the following equation: $x\times 8/100 + (12-x) \times 43/100)=12\times 36/100$. It is observed that in a first moment, the "x" is used as a name, but already being under tension with an algebraic use since the letter arose from a "numerical unknown" which has been operated. The tension ends with the algebraic solution of the equation.

CONCLUSIONS

As it can be observed in the episodes of Maribel's and Edgar's interviews, the numerical approach to solve algebraic-arithmetic word problems proposed can be used as a bridge between the arithmetic method which is tended to be used even in the university level (Ursini & Trigueros, 1997) and the algebraic method. The use of the successive explorations method revealed several aspects: a) the need to distinguish and characterize three senses of use of the unknown ("arithmetic unknown", "numerical unknown" and "algebraic unknown"); b) the need to pose problems propitiating the student to detach him/herself from the use of an "arithmetical unknown"; c) the possibility to achieve this by using one "numerical unknown", since this unknown is less abstract than the algebraic one; d) the possibility to unchain the analysis of some families of complex problems by making easier the settlement of the relationships between the elements of the problem with the use of numbers instead of letters; and e) the possibility to prepare the construction of the algebraic use of the unknown and of the algebraic method by connecting them to the syntax of the algebraic language.

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DIDACTICAL REFLECTIONS ON PROPORTIONALITY IN THE CABRI ENVIRONMENT BASED ON A PREVIOUS EXPERIENCE WITH BASIC EDUCATION STUDENTS

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Abstract

In this paper, the authors propose constructions in the Cabri-Géomètre environment to approach activities of proportionality. The proposal builds upon a prior experience with students in México who had completed elementary education. Those students worked on the topics of ratio and proportion through teaching models that had been designed based on recognizing some cognitive components. We attempt to translate our reflections to the role that dynamical geometry can play in the teaching of notions such as ratio and proportion.

Introduction: Proportionality in basic education research

The themes of ratio and proportion are fundamental in school teaching. Hence, these topics have been subjected to study by different educational researchers. In spite of advances in this area of educational research, it is necessary to go deeply into its study, especially in relation to the implementation of new technologies in the mathematics classroom.

Piaget explored proportional thinking. He conducted longitudinal studies of the stages of cognitive development through the stage of *formal operations*. Specifically, Piaget's findings led to understanding the foundations of the treatment of the ratio and proportion themes. Piaget (1978) pointed out that individuals could construct the scheme of qualitative proportionality when they understand that an increment in an independent variable yields the same result as a decrement in the dependent variable. That is, when subjects realize that an element of counterbalance is required.

Other research studies focused on the needs of instruction. For instance, Hart (1988) pointed out that proportional thinking is present in the adolescent and that its most advanced level is accomplished once the adolescent has constructed certain concepts. For Hart, some levels of generalization, such as the handling of ratios or ways of generating equivalences, occur when multiplicative strategies are used. Moreover, Freudenthal (1983) and Streefland (1984, 1990 and 1991) combined didactical aspects with the mathematical reflection about ratio and proportion. These researcher reports allow us to contemplate their contributions to these fields. Based on the cited studies, it is possible to create an instruction design under a constructivist approach.

Proportionality and curriculum: The contribution of the new technologies

It is necessary to analyze and classify the areas of knowledge, mathematical and transversal to the discipline, on which the understanding of the concepts of ratio and proportion exercises a direct or indirect influence. Thus, it is very important to focus the theoretical reflections about those concepts from a curricular and formative viewpoint. In the light of that type of analysis, we can justify the importance of teaching the topics of ratio and proportion by reasons such as the following (Fiol & Fortuny, 1990):

1. From the intermediate courses of elementary education (grades 4-6) throughout lower secondary education (grades 7-9), proportionality is the core for the unification of trends of notions in the learning of mathematics. Such notions include, among others: fraction and rational number, changing units of measurement and scale, distribution problems, percents, probabilities, graphs of linear functions, geometric theorems and similarity of figures, and the number π and the golden ratio.
2. Many concepts in physics and chemistry are associated to relations of proportionality. These concepts include nouns such as velocity, acceleration, density, and dilatation, and the formulation of laws such as Ohm's law, Hooke's law, or Proust's law.
3. The notion of proportionality is also present at the end of elementary school in the programs of social sciences or geography under the form of population density, birthrate, reading of maps, and so on.
4. Proportionality also appears, as we have cited above, outside the realm of the sciences in specialties such as genetic epistemology (Inhelder & Piaget, 1955) and in the studies related to cognitive development.

In this same trend, other researchers such as Lesh, Post, and Behr (1988) emphasized that the teaching of ratio and proportion starts in elementary school and serves as a basis for the understanding of other concepts. Very often, it is necessary to resort to the recognition of similar patterns or structural similarity in different situations. Many of those later concepts are critical in fundamental knowledge of the sciences and mathematics, as well as for the solution of daily life problems.

The increment of the number of educative projects that include the use of a technological component is impressive. This situation motivates teachers and researchers to include new technological devices in their teaching activities. Hence, the translation of our reflection to the role that these devices can play in the teaching of notions such as ratio and proportion is immediate (Lupiáñez & Moreno, 2001). In this paper, we introduce some activities in the Cabri-Géomètre environment of dynamical geometry. The designing of these activities was based on observations in a previous experience (which is described below) with students of elementary education.

A previous experience with elementary school students

The didactical program of the prior educational experiment initiated in January and finished in June of 2000. The problem situations that formed part of this proposal were associated to teaching model[1] (Ruiz, 2000), which were experimented at different moments, as required by the development itself of the research study

[1] Figueras, Filloy, & Valdemoros (1987). A teaching model comprises meanings—in the technical as well as in common language—, didactical treatments, specific modes of representation, and the relationships among them.

implementation. From two to seven working sessions were carried out per month, depending on the progress shown by the students and on factors external to the program implemented by the researcher. Each session lasted from one hour and a half to three hours; including the complementary activities, 25 sessions were necessary. The scheduling of the sessions is shown in Table 1 below.

Table 1. Schedule for the work sessions of the teaching program.

Month	Model(s) used	No. of sessions per model or complementary activity	Total of sessions per month
January	"Designing dance halls" "The world of Snow White and the seven dwarfs"	2 2	4
February	Complementary activity 1: "Size and shape" "The world of Snow White and the seven dwarfs" Complementary activity 2: "Double and half"	1 5 1	7
March	"Making photo frames" "The great problem of the footprint" Complementary activity 3: "Equivalent fractions"	2 1 1	4
April	"Establishing proportions" "Making photo frames" and "The world of Snow White and the seven dwarfs" Complementary activity 4: "Review of the concept of ratio"	3 2	5
May	"Soccer competition" "Building your own football soccer field"	2 1	3
June	"The picture of your team"	2	2
Total of sessions		25	25

The teaching proposal departed from qualitative aspects. Qualitative aspects are those based on linguistic recognitions for the elaboration of comparison categories, such as "big" or "small." The intuitive component, which is the piece of information based strongly on experience, on practice, on the senses, is a part of the qualitative aspects. This approach was used so that the students could give meaning or signification to the concepts of ratio and proportion, and could make the transition from the qualitative thinking to the quantitative thinking gradually, without completely abandoning the qualitative aspects. This transition process was traversed by means of different situations and at different moments. Similarly, Kieren (1988) emphasized the transition from the concrete to the abstract: he pointed out that the intuitive component is not completely abandoned.

After that, an ordering appeared: when students made comparisons, they used the phrases "greater than" and "less than." In the matter, Piaget (1978) pointed out that when the idea of order appears in the transition from the qualitative to the quantitative thinking, the concept of quantity is not yet present—what he called

intensive quantifications. Later, the pupils used measurements when doing comparisons. First, they contrasted parts of the body with objects, they superimposed one figure over another, and then they used a measuring instrument (conventional or unconventional). In terms of Freudenthal (1983), comparers can be classified in two modalities: direct or indirect. The direct modality of comparing occurs when an object is superimposed over other object, while the indirect modality occurs when there are two objects (A and B) and a third element (C) to make the comparison.
Thus, a great step forward to come close to quantification was that students started to measure using natural numbers when doing comparisons. Measuring became a significant prerequisite for the use of multiplicative operators.
At a later moment in the development of the didactical experience, students established relations among magnitudes, worked in the set of natural numbers, and used fractional expressions as well. Thus, they started to work in the field of rational numbers in an elementary way. Eventually, students characterized "ratio" as the relation between two quantities, expressed as the quotient of one divided by the other, and "proportion," as the equality between two ratios. This outcome coincides with the definitions given by Hart (1988).
Independently of the model under treatment, we gave special attention to the different modes of representation that students used when they were confronted to the proposed situations. We attempted that students could use any of the following three modes of representation indistinctively: pictorial, tabular, and numerical. For the purpose of illustration, in the following section we describe one of the models used; it was composed of two activities and was developed through several sessions with a sixth grade class of elementary education.

Activities of the model: "Soccer competition" and "Building your own football soccer field"

The first activity is to determine the ratios of different measurements of three football soccer fields in relation to the authorized measurements of a football soccer field, some values being given. That is, in the activity of the *Soccer competition,* students have to find out the measurements of three soccer fields given the length of one of them and the measurements for an official one, so that the four fields are proportional to each other.
A maxima problem is posed in the second activity: to construct the greatest possible soccer field in the playground of the students' school so that it is proportional to an official soccer field. For this activity, *Building your own football soccer field,* students know the measurements of an official football soccer field and the actual measurements of the playground in their school.
In the two tasks, students determine proportions by working with the ratios they find. The dynamics of work varies as each activity develops: students are required to work individually or collectively. However, they always have worksheets where to make computations and write down their results.
The analysis of the activities allowed observing that the students could remember the authorized length and width measurements of a soccer field. They used a process of

halving these measurements, wrote the obtained quantities on the corresponding sides of rectangles they had drawn, and tabulated those same values. By contrasting with the measurements of their playground, they realized that it was possible to lay out on it different soccer fields proportional to an official one. Then they read the directions for this activity carefully and noticed that in addition to being proportional to an official soccer field, the one to be built in the playground of their school should be the greatest possible. They knew that the required values could not be greater than the measurements of the playground. After several trials, they found out that the greatest soccer field would be one with dimensions equal to the fifth part of an official one.

For their arguments, students made use of the three registers of representation mentioned above: they placed the obtained data in tabular form and drew rectangles. Students asserted that the rectangles they drew were similar because the ratios of their measurements were proportional. They handled ratio as a relation between magnitudes, using fractional expressions, and handled proportions as equality relations of ratios. They also used equivalent fractions to determine proportionality. They did not use the rule of 3: this rule was not taught by the researcher during the didactical sequence, neither by the teacher because it is not included in the official program of school mathematics.

Activities of proportionality in Cabri-Géomètre

Students in the last grades of elementary school have potentiality for the acquisition and handling of the notions of ratio and proportion, in a context coherent with their prior mathematical knowledge. This potentiality showed clearly in the light of results obtained throughout the development of the teaching proposal. Our proposal provides the setting for the activities described in the next paragraphs. Our aim is that students complement their learning with the observation and manipulation of representations made available by the dynamical geometry environment of Cabri-Géomètre. However, some of the tasks are better adapted for a continuation of work at the beginning of lower secondary education (grades 7-9), not for the level of elementary education (grades 1-6). We assume that students in both levels of education have acquired the basic skills for handling this software.

Recognition of patterns. In this first activity, students are shown a family of rectangles. These rectangles can be moved and superposed without deformation in the screen. Students are asked to pair rectangles of the "same shape," no matter if these are of different size. Although this activity is clearly related to similarity of figures, the term "similarity" is not used.

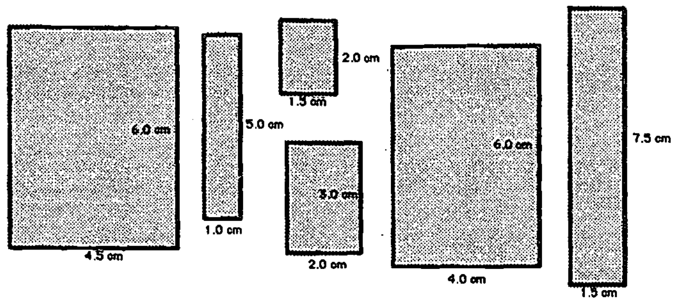

Figure 1.

The objective is that students arrange the rectangles according to their visual perception, and once they have done this, the measurements of the rectangles can be shown. After discussing whether there is a relation in the simple cases, such a relation can be generalized to all cases.

First measures. Computation of ratios. The objective of this activity is that students get to know the basic tools of the Cabri-Géomètre software and use them to work with ratios. For this purpose, students can draw line segments, measure them, and find out the ratios between their dimensions. Starting with these simple manipulations, important aspects of the system are brought into operation: taking measurements, manipulating units of measurement, and the possibilities of computation.

Construction of proportional and unproportional figures. The purpose of this activity is to introduce a criterion for the construction of rectangles proportional to each other. The rectangles must be constructed so that they share one vertex and their diagonals through it lie on the same straight line (Figure 2). If we move the vertex F on that straight line and keep record of the different measurements in a table, we can verify that the resulting proportion does not change, obtaining thus a whole family of similar rectangles. Likewise, in Figure 3 we can check that by adding the same constant to the dimensions of the sides of a rectangle, we do not necessarily obtain a rectangle proportional to the original one. After developing this task, it is now possible to confront the activity of the *Soccer competition* by implementing these criteria.

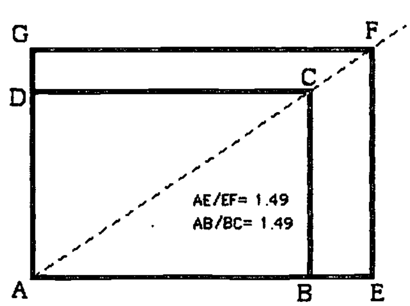

Figure 2.

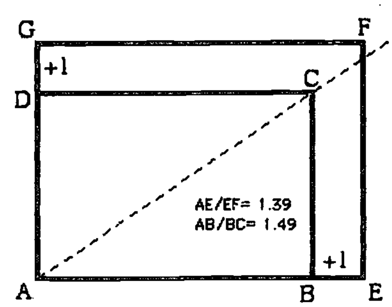

Figure 3.

Another interesting method for finding a rectangle proportional to a given one is by using the center and the diagonals of the latter one. If the new rectangle has the same center and its produced diagonals coincide with the extended diagonals of the first one, both rectangles are similar, that is, the sizes of their sides are proportional.

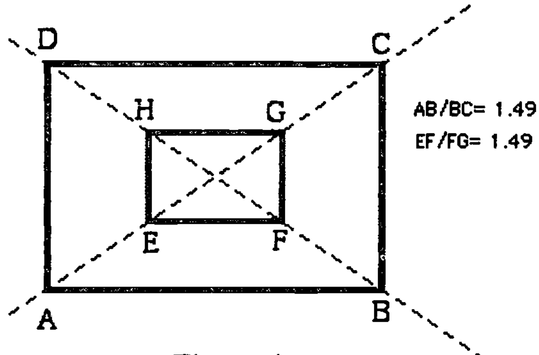

Figure 4.

Moreover, this type of constructions yields a variety of figures that remain equal to each other although proportions change, and many new possibilities of study arise.

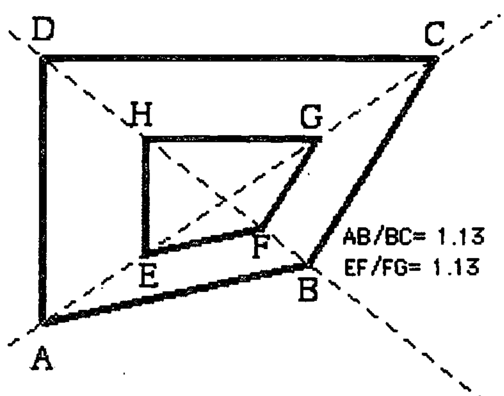

Figure 5.

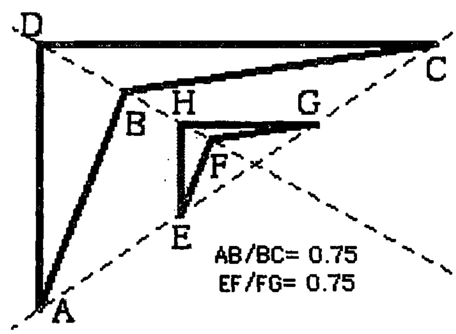

Figure 6.

Similarity and homothesis. In the first courses of lower secondary education, it is possible to introduce the invariance of similarity of figures under homothesis. This topic becomes especially easy and at the same time very illustrative in the Cabri-Géomètre environment: it is very interesting to see how moving the point *P* generates families of similar figures (see Figure 7).

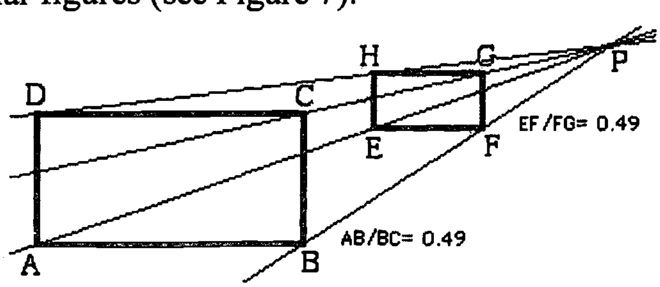

Figure 7.

Conclusions

There are different activities for enriching and widening students' knowledge about the notions of ratio and proportion. The representation and visual verification of Tales theorem, the study of linear functions, and the computation and graphic representation of constants like π or the golden ratio can be used for those purposes. Additionally, some of those activities can serve as potent elements for motivation.

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A PRACTITIONER MODEL OF THE SUCCESSFUL USE OF COMPUTER-BASED TOOLS AND RESOURCES TO SUPPORT MATHEMATICS TEACHING AND LEARNING

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This study analyses the pedagogical ideas underpinning teachers' accounts of the successful use of computer-based tools and resources to support the teaching and learning of mathematics. These accounts were elicited through group interviews with the mathematics departments in seven English secondary schools, conducted in the first half of 2000. The central themes are organised to form a pedagogical model, capable of informing use of such technologies in classroom teaching, and of generating theoretical conjectures for future research.

RESEARCH INTO COMPUTER USE IN SCHOOL MATHEMATICS

Although there has long been interest in the potential of computer-based tools and resources in school mathematics, their use has only recently become more established in mainstream practice. An indication of the extent of such use is provided by the recent international surveys (Beaton et al. 1996; Mullis et al., 2000). In the first, England was the system with easily the largest proportion of students reporting computer use. In the second, England was joined by Singapore, the system showing by far the most marked increase over the intervening period. Typically across systems, however, computer use remains low, and its growth slow. Against this background, it is not surprising that focused studies of computer use in school mathematics have largely examined explicitly innovative situations, usually linked to development projects of some type.

More recently -and very predominantly in the United States- studies have started to examine the evolution of computer use beyond situations of innovation, in circumstances where it has become more ordinary than innovatory, more established in operation than consciously under development. Surveys have provided a simple profile of the forms of computer use taking place, and of pedagogical orientations associated with them (Becker et al., 1999; Niederhauser & Stoddart, 2001). However, the reports of these surveys themselves highlight some of the limitations of employing preconceived models and questionnaire items to capture teachers' perspectives and practices. Case studies based on interview and observation have offered a useful complement to such surveys, providing more naturalistic and holistic accounts of the perspectives and practices of individual teachers (Myhre, 1998; Doerr & Zangor, 2000). However, the focus on individuals restricts the capacity for analysis across cases, and thus for building theories and models with stronger potential for transferability and generalisability.

DESIGN OF A STUDY OF TEACHER CONCEPTIONS OF ICT USE

The study to be reported here develops this line of enquiry into how teachers conceive their incorporation of use of computer tools and resources into mainstream mathematics teaching. It complements previous studies by extending such research to England, where the use of computer-based tools and resources, commonly referred to as information and communications technology (ICT), has been a component of the national curriculum for over a decade; and by following a somewhat different theoretical orientation, and employing correspondingly different methods.

Two intellectual traditions have influenced the theoretical orientation of this study. The first is a naturalistic tradition of research into teaching which has insisted on the necessity of attending to teacher thinking, not just as an important component of the phenomenon under investigation, but as a crucial source of constructs with which to build more potent theories (Brown & McIntyre, 1993; Cooper & McIntyre, 1996). The second tradition is a cultural psychological one concerned with collective systems and inter-subjective processes of thought, which emphasises not only the social dimension of thought, but the distinctive forms of everyday thinking (Moscovici, 1981; Wertsch, 1991). The concern here is to explore the content and form of 'common sense' thinking, focusing on public norms over personal variants.

The contemporary circumstances of teaching accentuate this stance. In England, the subject department acts as a basic unit within secondary schools for the formal organisation of teaching and the informal association of teachers. Recent educational reforms, too, have led to departments playing an active part in mediating between national regulations and classroom teaching, through the development of schemes of work. Hence, the unit of observation for this study was the school department, and the concern with shared systems of ideas.

The study drew on group interviews held with the mathematics departments in seven non-selective schools working in an established partnership with the university. Access to ICT in these departments varied considerably, but all made some use of it in their teaching. The interviews proceeded from a positive stance, with the main prompt requesting examples of ICT use which participants felt had been successful in supporting teaching and learning. The audio-taped sessions were transcribed and edited, and the resulting transcripts imported into a computer database to facilitate a recursive process of thematic organisation through constant comparison. This led to the construction of prototypical categories, grouping together related material. The goal was to identify well developed themes running across transcripts, and relationships between them.

THEMATIC ANALYSIS OF CONCEPTIONS OF SUCCESS

Initial analysis focused on the *success themes* which were explicit or implicit in teacher accounts. It appeared that these themes could be conceptualised in terms of three related priorities, concerned with securing and enhancing the participation of students in classroom work, the pace and productivity of such work, and the progression in learning arising from it. Two quotations will be used to illustrate these success themes. The choice of these particular quotations reflects an emphasis on the part of the teachers. Although they referred to a range of ICT tools and resources, it was the use of various forms of calculator or computer graphing which attracted a disproportionate amount of comment. Equally, these particular quotations illustrate how these references to use of graphing technology covered the full age and ability range at secondary level: from the academically lowest class in the youngest age cohort; to the academically highest course in the oldest age cohort:

> We've used spreadsheets in Year 7 and 8, to enable them to look at handling data, because they can quickly get tables and produce charts that are much better quality than those that they can produce themselves. I've got the bottom set in Year 7 and it can take them the whole lesson to draw a bar chart. So it's particularly successful from that point of view... because they don't have to draw all the axes so much, and it doesn't take them so long to develop the ideas because they're not having to spend a whole lesson drawing something. They can draw twenty graphs in a lesson and actually see connections, rather than spend twenty minutes drawing the axes and then twenty minutes talking and then twenty minutes drawing all the graph. [VC]

> It saves a lot of time as well with the Further Maths and the graphing that we did. It would have taken forever to actually plot all the points and see what happens when you transform certain shapes. Whereas it was done in a flash and they could see and they learnt an awful lot. So then they were ready and they'd accepted it because they'd seen it happening... Whereas it would have taken many lessons if we'd actually plotted all these graphs, they'd have just got bored by it. So that definitely helped, just kept the pace going. [MC]

Both these quotations appeal to comparisons with similar activities conducted without access to computer graphing tools. Issues of participation are framed in terms of avoidance of disengagement ('twenty minutes talking') and demotivation ('just g[etting] bored by it') on the part of students. Issues of pace and productivity are framed in terms of time saved and pace maintained ('it doesn't take them so long'; 'it saves a lot of time', 'it would have taken many lessons', 'just kept the pace going') and work produced ('they can draw twenty graphs in a lesson'). Issues of progression are framed in terms of ideas being formed ('actually see connections'; 'see what happens when you transform certain shapes') and embraced ('they'd accepted it because they'd seen it happening'), and in more general terms of development and learning ('to develop the ideas'; they learnt an awful lot').

THEMATIC ANALYSIS OF CONCEPTIONS OF PROCESS

These success themes provided a useful basis for further analysis to identify the technological affordances and mediating processes which teachers saw as underpinning these forms of success. From this further analysis, ten *process themes* emerged as prominent across the departmental interviews. These will now be outlined and sparingly illustrated (due to limitations of space).

The theme of *Ambience enhanced* associates ICT use with change, difference or variety in working ambience. This was sometimes a matter of change of working location -from ordinary classroom to computer classroom- and correspondingly of work organisation. Where technology use was seen as something of a break from 'routine', such motivational effects could be attributed to the novelty of the situation, or to the conferment of privilege. Typically, too, work in the computer classroom involved students interacting directly with a machine; but also, given the normal situation in which the number of class members was around double the number of machines available, students often worked not individually but in pairs. But even in an ordinary classroom, with teacher-led activities directed at the whole class, an interactive style of ICT use created a distinctive feel:

> I've only this week been using a program, just a short one, for the students to input a number, which it then outputs following a rule, so they can try and spot the rule by choosing different inputs and seeing what the outputs are. They quite enjoy seeing things done in a different way. [VC]

It seems that it was such exploratory, interactive styles which led to ICT use sometimes being characterised -in a wholly unpejorative way- as 'playing around' and the devices themselves as 'toys'. Teachers talked of the 'variety', of the 'difference', even of the 'other dimensions', associated with ICT use, and related these to securing and enhancing student participation in classroom activities:

The theme of *Restraints alleviated* associates ICT use with the alleviation or mitigation of factors restraining or inhibiting the participation of students in classroom work. One important sub-theme (related to the 'play' sub-theme of Ambience enhanced) concerned the capacity of ICT use to prevent 'work' becoming 'drudgery'. A further sub-theme focused on the way in which ICT use is often associated with a reduction -or removal- of the writing demands -physical and intellectual- of much conventional classwork; demands which may deter or challenge some students. Another specific area where weakness in student capacity was seen as alleviated by access to technology was the drawing of graphs: A final sub-theme concerns the way in which ICT use changes the status of mistakes, not only by facilitating their correction, but by removing evidence of them which might attract unwelcome -and demotivating- attention from the teacher:

> They don't seem to mind getting things wrong either. Whereas, if they got it wrong in their book, particularly children who are already feeling a little bit, lacking confidence let's say, mathematically. To get something wrong in their book often results in 'Oh, I'm not doing maths.' But if they get it wrong on the screen, okay and they just go back to the drawing board and do it again. And they will do it over and over again. / If it is deleted on the screen and it is not there anymore, it is forgotten. / And you don't write all over it. [LC]

The theme of *Tinkering assisted* focuses on how this provisionality of many ICT results assists forms of tinkering to improve them. The most basic of these forms is self correction by students, typically following on from a direct check of a result, sometimes made vivid by the technology, occasionally prompted by an evaluative feedback mechanism. Whereas such 'checks' are applied to what are intended as definitive solutions to a given problem, the provisionality of ICT facilitates 'trials' of more tentative solutions, and hence a corresponding shift in strategic logic. The idea of 'trial and improvement' strategies formed part of the national curriculum being followed by these schools, and teachers made reference to the way in which use of ICT supported such strategies. More generally, they pointed to the way in which use of ICT afforded a more experimental approach to tasks (related also to the invisible mistakes sub-theme of Restraints alleviated):

> They are more prepared to have a stab at something and get it wrong because not everyone can see it's wrong and they'll keep trying until they can get it right. [VC]

Arguably, it is awareness of this tinkering style of ICT usage which underpins earlier comments about students 'playing on a computer' or 'playing around on the calculator', under the Ambience enhanced theme.

The theme of *Motivation improved* associates ICT use with the motivation of students towards classroom work. Teachers commented on what students 'love', 'like' and 'enjoy' in relation to using ICT; likewise on what 'motivates them', on what they 'respond well to', and on what they are 'quite taken by'. One further sub-theme suggests that by assisting or permitting students to display -and to be seen to display- greater capability, use of ICT can help build their self-confidence. Across more extended sections of transcript, this idea was associated with the earlier themes of Restraints alleviated and Tinkering assisted:

> I think they can then start to feel more positive about themselves and their work, because often they can just get worse and worse with their untidiness because they know they can't do it so they can't be bothered to try. If they see what ICT can help them do then they might be encouraged to try it a little bit more themselves and then try to improve their own work to that standard. [VC]

The theme of *Engagement intensified* associates ICT use with deeper and stronger student engagement in classroom work. Clearly this theme is closely related to Motivation improved. Teachers noted students being 'more prepared to have a stab', to 'put in more effort', to 'keep trying'. The following quotation draws together many of the ideas associated with this theme -good behaviour, attention to classwork, and a degree of independence and persistence in it:

> They are working more, they're not wasting so much time, they actually do get on with it, the majority of children will get on with it, rather than do one question, turn around and I've got to tell them what to do so they are learning something. [CC]

The theme of *Routine facilitated* associates ICT use with facilitation of relatively routine components of classroom activity, allowing them to be carried out more quickly and reliably, with greater ease, and to higher quality. In this respect, aspects of Routine facilitated underpin some of the phenomena already noted under the theme of Restraints alleviated, and there was a clear intertwining of these themes in some sections of transcript. What characterises Restraints alleviated, however, is a focus on particular factors inhibiting the participation of persons, whereas Routine facilitated focuses on factors supporting the execution of tasks. The sub-themes of speed and ease were the most prominent. Such use of ICT was seen as particularly important in supporting the more extended project tasks which older students tackled as assessed coursework for the GCSE examination. An example illustrates how use of ICT tools facilitated routine components of such tasks, highlighting the removal of important constraints on the strategies which students could pursue:

> I've just worked this year using draw commands in Word for a bit of coursework called 'Square Moves' where they have to move counters around, and part of the coursework is to show their strategy for moving the counters, and I did it with a low ability group, and we got far better results I feel, because I showed them how to draw circles, and get the sizes right, fill them in different colours, and then once they'd set up the basic grid they could copy and paste the whole grid however many times they wanted, and then they just moved the individual counters around, and as a result of that they did far more work than with the old style of always drawing them out. [TC]

The theme of *Activity effected* associates ICT use with securing and enhancing the pace ('rather than someone poring over a page... they were able to do this in no time') and productivity ('far better results', 'far more work') of classroom activity as a whole. The following quotation illustrates the linkage of Routine facilitated to the pace aspect of Activity effected:

> We've got a graph plotting package which isn't particularly sophisticated... but it makes it nice and easy for the kids to use... Actually drawing a graph and seeing it on the screen, they can very quickly see what's happened to the graph. So using IT in that respect, it makes a significant difference in the depth of understanding and the speed in which it takes to learn skills... You may get there in two lessons rather than three, so you can gain a lesson. [TC]

The theme of *Features accentuated* associates ICT use with the provision of vivid images and striking effects through which features of mathematical constructs -or relations between them- are accentuated:

> The actual immediacy of it and also the fact that it's living, it moves. [VC]

This 'immediacy' of response was one of the important sub-themes to emerge. Likewise the power of 'visual' representations, and the dynamic of actions and their effects:

> So you quickly see what is happening to the gradient and what is happening to the intercept. [LC]

The theme of *Attention raised* associates ICT use with reducing or removing the need for attention to subsidiary tasks, and with avoiding or overcoming related obstacles, so as to better focus students' attention on overarching ideas and processes.

The following quotations bring out the way in which processes already alluded to under the head of Restraints alleviated and Routine effected clear and smooth the way so as to raise and focus students' attention:

> The key thing about the calculators or any ICT applications being able to take away the drudgery out of doing the calculations, so that you can start to access a higher learning point without the problems of making mistakes along the way, clouding the issue. [SC]

The theme of *Ideas established* associates ICT use with the formation and consolidation of ideas through their being 'seen', 'understood', 'accepted' and 'remembered' by students. The following quotation illustrates the association of both Features accentuated and Attention raised with Ideas established:

> When we're using Coypu [graphing package] for shifting curves, and that shows them very easily what altering the equation does to the curve, and that does help them tremendously. / ...It's a very, very difficult topic and so you can just actually get them to draw various curves, and show them what happens when you start altering the equation... Just the immediacy of it, actually means that it hangs together better, because if you see these results and then spend another 15 minutes drawing a graph, the whole thing just feels nothing much, so it's just not better efficiency, but also it is actually sounder for the brain really, if it can see things more immediately. [TC]

CONCLUSION

This study has elicited from practitioners a model of the successful use of ICT to support classroom teaching and learning in mathematics. The model is organised as a system of themes identifying key processes and critical states. Four themes depend most directly on exploiting affordances of ICT: *Ambience enhanced* in changing the general form and feel of classroom activity; *Tinkering assisted* in helping to correct errors and experiment with possibilities in carrying out tasks; *Routine facilitated* in enabling subordinate tasks to be carried out easily, rapidly and reliably; and *Features accentuated* in providing vivid images and striking effects which highlight properties and relations. Three further themes depend in turn on these processes: *Restraints alleviated* in mitigating factors inhibiting student participation such as the laboriousness of tasks, the requirement for -and the demands imposed by- pencil-and-paper presentation, and vulnerability to mistakes being exposed; *Motivation improved* in generating student enjoyment and interest, and building student confidence; and *Attention raised* in creating the conditions for students to focus on overarching issues. Three final themes depend again on preceding processes: *Engagement intensified* in securing the commitment, persistence and initiative of students in classroom activity; *Activity effected* in maintaining the pace and productivity of students within classroom activity; *Ideas established* in supporting the development of student understanding and capability through classroom activity.

This model is intended to serve as a helpful abstraction, providing a generic scheme for conceptualising use of ICT to support classroom teaching and learning in mathematics. In any actual instance of classroom activity, of course, certain components of the system may assume more prominence than others, and those operative components of the scheme must be filled out more concretely according to the particular circumstances. Nor is the model a deterministic one. Rather it highlights key processes and critical states which require active -and reactive- planning and management on the part of the teacher for ICT use to successfully support teaching and learning. Finally, the model is a tentative one. It is based only on teachers' somewhat decontextualised accounts of successful practice, elicited through group interview; rather than on more strongly contextualised accounts of specific instances of practice, supported by examination of actual classroom events. Nevertheless, earlier variants of this model have already proved their worth in subsequent work with teachers, in helping them to articulate their conceptions of how particular forms of ICT use support teaching and learning.

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PROCESSES OF PROOF WITH THE USE OF TECHNOLOGY: DISCOVERY, GENERALIZATION AND VALIDATION IN A COMPUTER MICROWORLD

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Our general concern is to investigate the role that new technologies play in the development of processes of proof. We use Balacheff's work (1987, 1999) to talk of pragmatic proofs vs. intellectual and formal proofs. We present part of a case-study from a larger research where students used a Logo-based microworld for the exploration of infinite sequences and series, to illustrate how some of the elements that computer microworld explorations, activities and visual means bring, can lead to a process of discovery and acceptance of mathematical results, and create stepping-stones (pragmatic proofs) in the development of mathematical proofs.

INTRODUCTION AND THEORETICAL FRAMEWORK

We are interested in investigating the role that new technologies play in the stages of proving in mathematics education. Research in this direction is often related to geometry and the influence of dynamic geometry systems on the learning of mathematical proof (Hoyles & Jones, 1998; Villiers, 1998, Balacheff, 1999). However, in this paper we draw data from a different kind of study where students used a Logo-based microworld for the exploration of infinite sequences and series. We aim to illustrate how the environment and its tools gave students enough means of mathematical exploration and expression through which they could discover patterns, make generalizations and conjectures, and validate their results.

We begin with a brief discussion on the function of proof as a means of understanding, and the role of technology for developing a sense of proof and intuition before a formal proof is presented. We then present a brief summary of some important concepts developed by Balacheff (1987, 1999) that will serve as part of our theoretical framework. Finally, we present sample data from our study to illustrate the ways in which computational experiences can function as a means of creating understanding and acceptance of mathematical results, thus constituting processes of proof.

The construction of meanings through proofs, and the role of technology

The function of proof as "a key tool for the promotion of understanding" has been stressed by Hanna (1995). While Hanna defends the value of formal proof, she also makes a distinction between the function of proof in mathematics and that in mathematics education:

> while in mathematical practice the main function of proof is justification and verification, its main function in mathematics education is surely that of explanation … It might be a calculation, a visual demonstration, a guided discussion observing proper rules of

argumentation, a preformal proof, an informal proof, or even a proof that conforms to strict norms of rigour ... Clearly the challenge is to have [students] understand *why* [results] are true. (p. 47)

Other studies have highlighted features that are making clear the complex field of mathematical proofs: the role played by empirical evidence in contrast with deductive arguments (Chazan, 1993); the difference between argumentative reasoning and deductive reasoning (Duval, 1991); and students' proof schemata (Harel, 1996).

With the advent of new technologies, the empirical methods of mathematics research have been revitalized. Mathematicians themselves are recognizing that technology is changing the way we approach proving. Thurston (1994), who analyses the nature of proof and of mathematics itself, also emphasizes that it is a search for *understanding* which is at the basis of the exploration and logical processes leading to a proof; and he advocates the use of computers for exploration and discovery of mathematical ideas, giving priority to what he calls "humanly understandable" proofs over formal proofs.

Other authors and researchers have investigated the role of computer-based explorations and visualization to develop a sense of proof and give intuition for formal proof (for instance, by building up an overall picture of the relationships involved), considering these important complementary elements to mathematical proofs (e.g. Barwise & Etchemendy, 1991). Among those are Cuoco & Goldenberg (1992) who consider that the activity of constructing proofs involves a research technique where conjectures arise through the combination of experimentation and deduction.

One of the difficulties with proving in school mathematics is that concepts which expert mathematicians regard as intuitive are not intuitive to students since intuition depends on previous experience (Tall, 1991). To overcome this, Tall (op.cit., p.118) considers that:

by introducing suitably complicated visualizations of mathematical ideas it is possible to give a much broader picture of the ways in which concepts may be realized, thus giving much more powerful intuitions than in a traditional approach. ... intuition and rigour need not be at odds with each other. By providing a suitably powerful context, intuition naturally leads into the rigour of mathematical proof.

The components of proving

Balacheff (1987, 1999) has introduced certain ideas that we consider important for our analysis. He distinguishes between *pragmatic proofs* and *intellectual proofs*; emphasizing the role of language in the passage from the former to the latter. Pragmatic proofs are those based on effective action carried out on the representations of mathematical objects. They lead to practical knowledge that the subject can use to establish the validity of a proposition. Intellectual proofs demand that such knowledge is reflected upon, and their production necessarily requires the use of language that expresses (detached from the actions) the objects, their properties and their relationships. In other words, pragmatic proofs are based on action, while the use of a functional language (which includes a specific vocabulary and symbolism) and a "mental experience" (where actions are interiorised) characterize the transition to the intellectual ones. The transition from pragmatic proofs to intellectual proofs culminating in mathematical proof, involves three components: the *knowledge* or *levels-of-action component* (the nature of knowledge: knowledge in terms of practices — "*savoir-faire*"; knowledge as object; and theoretical knowledge); the *language* or *formulation component* (ostentation, familiar language, functional language, formal language); and the *validation component* (the types of rationale underlying the produced proofs: from pragmatic, to intellectual, to mathematical proofs).

COMPONENTS OF PROOF IN A LOGO MICROWORLD

Here, we present an example to illustrate how certain computational activities constitute pragmatic proofs, with progress on the level of the first component (knowledge), and some progress in the other components. Although we cannot provide evidence on how computational activities can lead to the development of more rigorous intellectual proofs, we believe that progress achieved in earlier stages constitute suitably powerful experiences that create familiarity and understanding of the problem (an intuition) necessary for transition to the latter (cf. Tall, 1991; see above). It is likely one of the kind of experiences incorporating exploratory computer and visual activities considered useful and advocated by some of the researchers reviewed above (e.g. Tall, 1991; Thurston, 1994; Cuoco & Goldenberg, 1992), and which provides elements of proof as "explanatory" (Hanna, 1995).

This example (from a case-study of a pair of high-school students) is taken from a larger research that investigated the mediating role of a Logo microworld designed for the exploration and study of infinite sequences and series, in students' conceptions of infinity and infinite processes. That study involved detailed case studies of 4 pairs of students working and interacting with the microworld. (Students worked in pairs to facilitate the sharing and discussion of ideas —simultaneously providing the researcher with insights into their thinking processes— and give them independence from the instructor). Each pair of students, previously instructed in Logo programming, worked with one computer for five 3-hour sessions. To facilitate the analysis of students' experiences, we worked with only one pair of students at a time, using a clinical interview style. The role of the researcher was that of a participant observer, suggesting the field of work (the initial procedures and activities), as well as new ideas for exploration when needed, yet allowing students to be in control of the explorations, giving them freedom to explore and express their ideas. Students were informally interviewed throughout the sessions but formal interviews were conducted at the beginning and end of the study.

As part of the microworld activities, students explored sequences such as $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$, and $\{1/n\}$, $\{1/n^{1.1}\}$ and $\{1/n^2\}$, and the sequences of their corresponding partial sums. They wrote Logo procedures to construct visual models of those sequences, using Logo's turtle geometry: spirals (Fig.1) (where each successive "piece" of the spiral represents a term of the sequence, so that the total length of the spiral corresponds to that of the sum of the terms, i.e. the corresponding series), bar graphs (Fig.2), staircases, and straight lines (where again, the total length represents the corresponding series or partial sum of the terms of the sequence).

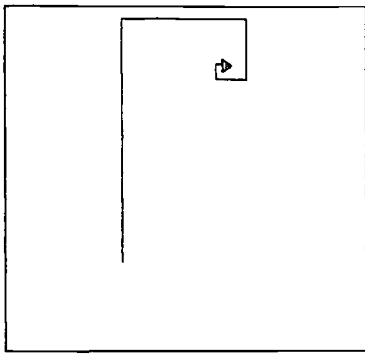

Fig.1: Spiral model for the sequence {1/2ⁿ}

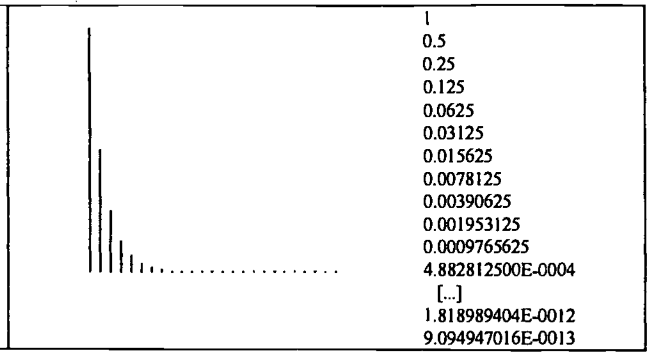

Fig.2: Bar graph model for the sequence {1/2ⁿ} with numeric output

For example, one of the initial Logo procedures used in these activities was of the form:

```
TO DRAWING :L
IF :L < 1 [STOP]
MODEL
DRAWING (FUNCTION :L )
END
```

Where MODEL described the steps for building a model such as a spiral (FD :L RT 90) or a bar graph (FD :L JUMP), and FUNCTION described the operation on :L at each step (e.g. :L/2).

Thus, the different visual models for the same sequence provided different perspectives of the same process. As the explorations progressed students added Logo commands to their procedures that would allow them to carry out a complementary analysis of the numerical values (to further analyse their behaviour, and the apparent limits, if any existed or appeared to exist). Through the observation of the visual (and numeric) behaviour of the models, students were able to explore the convergence, and the type of convergence, or divergence, of a sequence and its corresponding series.

The microworld was designed to simultaneously provide its users with insights into a range of infinity-related ideas, and offer the researcher a window (cf Noss & Hoyles, 1996) into the users' thinking processes. The microworld provided a means for students to actively construct and explore different types of representations (symbolic, visual and numeric) of infinite sequences via programming activities. In general, the computer setting provided an opportunity to analyse and discuss in conceptual (and concrete) terms the meaning of a mathematical situation. For example, drawing a geometric figure using the computer necessitated an analysis of the geometric structure under study and an analysis of the relationship between the visual and symbolic representations.

What is relevant for us here, is that the exploratory setting of the microworld activities allowed students to engage in an active process of discovery of the properties and characteristics of the processes under study. That is, the environment seems to have provided a language for asking questions, as well as tools for exploring these questions. In many cases students found what seemed like patterns and properties, which led them to formulate and test conjectures, as well as articulate relationships and build generalizations.

There were several levels in which the microworld activities functioned as stepping-stones towards a proof. At a first level, students made observations and discoveries situated within the medium of the microworld. We have described elsewhere (e.g. Sacristán, 1997) how students were able to discover patterns related to the processes under study but situated within the context of the microworld environment. By playing in, interacting with, and working within the microworld, they could express themselves through the tools, activities and forms of symbolism built into the environment. At a second level, some students were able to abstract and articulate their findings in a way that could be taken beyond the medium in which the findings were constructed; consciously exploiting the tools of the microworld for discovery, exploration, and "pragmatic proof" (Balacheff, op.cit.) of mathematical relationships or "theorems".

Pragmatic proofs: discovery, generalization and validation in the microworld

Through the sample data below, taken from one of the case-studies (a pair of 16-17 year-old boys: Manuel and Jesús), we aim to illustrate, in particular, how students made predictions and then tested their validity by using all the available tools in the microworld.

During the second worksession of explorations, as described above, of sequences of the type $\{(1/k)^n\}$, these students discovered that the series of the type $\sum_{n=1}^{\infty}\frac{1}{k^n}$, where the integer[1] $k>1$, converge to $\frac{1}{k-1}$. These students had been exploring and comparing the sequences $\{1/2^n\}$, $\{1/3^n\}$ and began to discover a pattern in the behaviour of the corresponding series: Manuel observed that as they increased the denominator value k in the sequences of the type $\left\{\frac{1}{k^n}\right\}_n$, then the limit of the corresponding series was smaller and in fact seemed to have as value $\frac{1}{k-1}$. They explicitly constructed a generalization for this mathematical result (which later they would call proudly "their theorem") and used it to *predict* the probable behaviour of other sequences and series of the same type:

Manuel: Look, if you subtract 1 from the number that is the base in the denominator, and you divide 1 by that number, then that is the number to which it will approach. If we do it with 3: 3 minus 1 is 2, and it tends to a half... So if it was $1/2000^N$, the sum must approach 1/1999...

Jesús: Yes, the bigger the base in the denominator, the smaller the limit.

[1] Manuel and Jesús seemed to consider k implicitly as a positive integer larger than 1, although they did not make this condition explicit.

Manuel: But now we have a method for knowing to where it approaches.... We saw that $1/2^N$ became small very quickly, but the case of $1/3^N$ decreases much much more quickly. And we saw that its series didn't tend to 1 like the previous one, that it approached a half, so we noticed a more or less regular behaviour, so if we wanted to know to how much the series of $1/2000^N$ would be we would only have to reduce it by a number, and it would tend to 1/1999.

Manuel and Jesús then employed the medium and its tools to test out their predictions, looking for a proof of their conjecture. They began by changing the sequence generating function to $1/4^N$, predicting that the corresponding series would tend to 1/3. They used all the resources available to explore this sequence and its series, looking at all the available visual models (the Spiral, Stairs, Bar Graph and Line models). They were amazed at how quickly the values of the sequence decreased. From the rapidly decreasing behaviour of this sequence they deduced that the corresponding series converged, which they visually verified, using the Line model. Although the visual explorations were enough to convince the students of the validity of their conjecture, they complemented these with a numeric exploration of the partial sums (they built a procedure which computed the value of a partial sum). They observed that the 20th partial sum printed out to be 0.3333333333, confirming further their hypothesis. A final test of their conjecture was carried out by exploring the sequence $\{1/13^n\}$, through visual and numeric representations — which showed the much more rapid decrease of this sequence — and again verified that the corresponding series tended to the predicted value of 1/12.

For Manuel and Jesús there was now no doubt that the conjecture was true, but Manuel did worry that this mathematical generalization would not hold if the value of k was infinite (what Balacheff, 1987, 1999, has described as a *crucial experience*), and he looked for favourable arguments:

Manuel: Listen... there might be a contradiction in our assumption: if we did one over infinity... Ah! but infinity minus any number is still infinity..., so we are right! It tends to zero. If one over infinity tends to zero, then also one over infinity-minus-one, because infinity minus one is infinity, then it also tends to zero. And here we have that the bigger the base of the denominator gets, the smaller the series.

It is worth noting that most students, including a pair of younger students (two 14 year-old girls, one of them called Consuelo) discovered the rule for the behaviour of the series of the type $\sum \frac{1}{k^n}$, which they tested and then generalized. Manuel and Jesús were more experienced mathematically which was reflected in the way they expressed the rule, but the younger students also constructed the generalization within the context of the activity — what Noss & Hoyles (1996) have called a *situated abstraction* — expressing it relative to the inputs used by the procedure (e.g. the scale):

Consuelo: So the sum of the bars for 1/3 it's one half [the scale], and for 1/4 it would be 1/3 [of the scale], and for a fifth: 1/4 [of the scale], and so on.

DISCUSSION AND CONCLUDING REMARKS

The possibility of working with many cases (different sequences of the same type), and use diverse resources (different visual models and complementary types of representations: visual and numeric), provided students with a means to (i) infer their own generalization through the discovery of a pattern, and (ii) to validate and confirm their predictions and generalization (becoming convinced of the general validity of their conjecture). In this sense, the microworld became a mathematical laboratory. The results were not formally proven, and the students were aware of this, but the process of repeatedly observing different variations, cases, and situations, was enough to convince the students of the validity (or in other cases falseness) of their conjectures[2], constituting pragmatic proofs. Elsewhere, we called these experiences "situated proofs" (Moreno & Sacristán, 1995; Sacristán, 1997). These are pragmatic proofs that result from the combined use of all the elements available in the microworld in an attempt to confirm conjectures. Situated proofs are experiences that lead students to discover and make sense of a mathematical relationship, convincing them of its validity. In the same way as situated abstractions, these experiences are dependent on the tools of the medium (hence the term "situated"). They are therefore not yet detached enough from the representations of the objects and the actions to constitute an intellectual proof.

Nevertheless, progress is shown for all three components. The fact that students spontaneously tested their conjectures (adjusting them as a result of the explorations), in order to convince themselves (and others) of their validity, shows progress on the level of the *validation component*. On the level of the *knowledge component,* the formulation of predictions or conjectures involved a process of reflection and analysis on the part of the students, as they had to, for instance, evaluate the role and relationship of the variables involved. Also, the Logo-based activities allow students to construct certain forms of symbolism[3] to express and interact within the medium (situated abstractions); this is progress on the level of the *language component.* Thus, these experiences were often powerful enough to act, at least, as explanatory proofs (Hanna, 1995). They are pragmatic proofs that can be thought of as the collection of activities that build meaning for a theorem before a formal proof is presented.

Thus, the use of technological tools can provide a field of exploration and mathematical experimentation in which it is possible to deal with mathematical content through the representations provided by the medium. Proofs on the pragmatic level can become more powerful and enrich students' mathematical experiences by allowing them to discover results and formulate probable conjectures. In this sense, we consider that this type of

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[2] Exploration through visual and numeric representations, observation of the behaviour of the process under study, and the structure of the code, are all elements that students used to convince themselves of the convergence or divergence of a process, and/or of the existence of a limit.

[3] This contrasts with findings in another of our studies using Cabri-Géomètre, where geometrical objects are manipulated, and there is a development of geometric visualisation, but the use of the symbolic geometric language is very limited.

microworld activities can strengthen the foundations for the passage to intellectual proofs. However, more research is needed to find activities that follow or are based upon computational activities, which will lead to the transition towards intellectual and formal proofs; in agreement with Hanna (1995), the role of the teacher is likely to be crucial.

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INTEGRATION OF DOMAINS OF KNOWLEDGE IN MATHEMATICS TEACHERS' PRACTICE

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This study is a part of an ongoing research that attempts to characterise the relationship between mathematics teachers' knowledge and practice. Here we are focusing on the identification of the domains that are integrated in distinct moments of teachers' practice, in particular in the introduction of Thales theorem. Data were obtained through audiotapes of several semi-structured interviews, observations and videotapes. Although the teachers of our study had a similar background and experience, the analysis of data showed differences in the way of introducing the theorem, in the characteristics of the domains integrated in the previous decisions and in the changes related with the initial planning. The study contributes to highlight the complexity of the teachers' professional practice.

INTRODUCTION

Over the last decades, the relation between mathematics teachers' knowledge and practice has been studied from different perspectives. Within the cognitive perspective, Leinhardt and her colleagues have shown that teachers have a background that provides them resources for confronting the different teaching situations (Leinhardt et al., 1991). Other researchers have pointed out several ways in which teachers' knowledge and conceptions influence what the teachers are able to do in their classrooms (Lloyd and Wilson, 1998, Sherin, 2000). Furthermore, analytical models have been created to characterise 'the decisions and actions of teachers as they teach' (Schoenfeld, 2000, p.243), or to examine teachers' instructional practice with respect their underlying cognitions (Artzt and Armour-Thomas, 1999). The latter have tried to understand the role of the cognitions in the instructional practice by examining knowledge, beliefs and goals in different stages of teaching. For them, these components form a network of overarching cognition that controls the instructional behaviour of teachers in the classroom, assuming that differential instructional practice is related to differences in teacher cognitions.

FRAMEWORK AND AIMS

Our work draws upon the literature described just above and is situated in the line of researches that attempt to characterise the relationship between mathematics teachers' knowledge with different moments of their practice. In this study, we consider teachers' professional knowledge as 'a personal construction generated through the professional activity in which several domains of knowledge are integrated' (Garcia, 1997). This knowledge is developed over a long time and in different context, being a 'situated knowledge' (Brown et al, 1989). We assume that cognitive integration of the different domains of knowledge (pedagogical content knowledge, subject-matter knowledge) takes place in the different situations of professional practice.

In addition, we consider professional practice as 'the set of activities that the teacher generates when he carries out the tasks that define the teaching of mathematics, and the justifications that he provides about these activities' (Llinares, 1999, p. 110). Following this author, we consider that the professional practice is not limited to classroom events. It is conceptualised from a wide perspective that includes the relationships between peers, work in groups, etc., and the articulation, justification and communication of the decisions and actions. Therefore, although our study is built on cognitive perspectives, we try taking into account socio-cultural aspects.

Although we assume that teachers' practice is 'a conglomerate that cannot be understood by looking at parts split off from the whole' (Simon & Tzur, p.254), in our research project, we have focused on the teaching/learning process management of two secondary mathematics teachers' in a set of lessons to describe and characterise the professional practice. These lessons were related to the unit 'Thales theorem and similarity'. On other occasions within the above mentioned research project, we have studied what happens in mathematical classes when mathematical content appears different to that scheduled by the teacher (Escudero y Sánchez, 1999). The present study is focused on the identification of the domains that are integrated in distinct moments of the practice, (in particular in the introduction of Thales theorem). We are trying to characterise these domains, studying their role in the changes in relation to the initial planning.

Since we investigated the teachers' knowledge and practice in relation to Thales Theorem and similarity in our study, it is important to make explicit the framework that guide this inquiry. Our framework incorporates the consideration of the similarity in a 'intrafigural' way based on the properties of similar figures or as a geometrical transformation. We also consider the different modes of representation used (real world situations, manipulative, figurative, natural language, symbolic representations), and the type of tasks and the demanded activity in these tasks (calculation, construction or demonstration).

METHODOLOGY

Participants and Site

The research participants were two experienced Secondary teachers, Ismael and Juan (the names are pseudonymous). There was not a specific preparation as regards the design or implementation of the unit. Both prepared their units independently. The pupils belonged to two classes without any special characteristics (3rd and 4th years of Compulsory Secondary Education, 14-15 and 15-16 years of age, respectively). The schools were situated in the outskirts of the city.

The unit

The development of a new curriculum for Secondary level (12-16 years) in Spain commenced in the early 1990s. This reform has emphasised the teaching of Geometry. Objectives in this section include 'knowledge and use of basic geometrical properties: Pythagorean theorem, Thales configurations, similar figures and elementary notions about trigonometry (Junta de Andalucía, 1993, p.133). Supporting documents encourage teachers to include these objectives in their planning.

Instruments

Data were obtained through audiotapes of several semi-structured interviews, observations and videotapes. We videotaped each of the teachers teaching the whole unit of their own design (eight lessons and nine lessons respectively). During the initial interview, the teachers were asked to explain their lesson plans and to describe how they meant to develop the lessons. Previous and post lesson interviews were carried out. The field notes taken down during the observation provided information about classroom events that might not show on the videotape. Transcriptions were made of the audio part of the videotapes for analysis.

Data analysis

We analysed the previous interviews in terms of their goals, lesson plans, selected problems, etc. Thus, we obtained the teachers' agenda. The observation notes and the videotapes were used to categorise the teachers' instructional practice. Firstly, we identified 'segments' in the lessons (presentation segments, monitored practice, homework check, etc), considered as parts of a lesson with an objective, and perfectly differentiated by teacher and students (Leinhardt, 1989).

In the different segments, we characterised the actions developed by the teacher to reach the objectives, taking into account the grade of intervention of the teacher with respect the pupils and the type of data used. We identified different actions in the presentation segments: utilisation of an example/problem for reaching a definition, a property, a theorem, with the constant intervention of the teacher and pupils; explanation for reaching a definition or property, with minimum or nil intervention from the pupils, among others. In these cases, the data were whole numbers. The whole set of actions provided us with the structure of the segments. By integrating the data of the different instruments it was possible to know the reasons and justifications that made the teachers adopted a specific structure. This allowed us to identify and characterise the underlying domains of knowledge in the structure.

Through our analysis we were able to discern differences in teachers' instructional practice. We used the integration of domains as a way of better understanding these differences. In particular, within the segments identified in the unit, here we are going to focus on a presentation segment identified in the two teachers, whose objective was to introduce the Thales theorem. Firstly, we are going to present the structure we have identified. Afterwards, we will characterise the domains of knowledge that have influenced the adoption of this structure, and the aspects of these domains that were responsible for the changes.

RESULTS

The two teachers in our study had a similar background and experience. Nevertheless, we were able to observe differences in the structure of the presentation segments of Thales theorem and in the characteristics of the domains related with the previous decision of adopting these structures.

For Ismael, the unit of 'Thales theorem and similarity' was important because 'it helps to give sense and to understand the concept of numerical proportionality'. In particular, the theorem is considered as a context to visualise the numerical proportion in order to be able to connect 'the abstract numerical view with graphical images'. This unit was developed throughout 9 consecutive fifty-minute lessons. In these lessons, we identified 7 presentation segments; 3 monitored practice and 1 homework check, which were used alternately. In particular, the presentation segment of Thales theorem occupied the first place and was developed in 80 minutes (almost two lessons).

This segment belonged to a group of three segments, which structure was composed by the following actions: utilisation of an example or problem / explanation for reaching the definition or formula / definition / explanation for completing. This structure was identified in those cases in which the introduction of the definition or property was based in contents previously introduced by the teacher in other units. In the introduction of Thales theorem, former contents were numerical proportion and parallel lines cut off intersect lines. The teacher wanted the pupils to identify these contents in some proposed problems.

In the first action, utilisation of an example, Ismael proposed three problems previously selected in his planning. Through these problems, Ismael wanted 'to collect pupils' previous ideas that are related to the theorem' (initial interviews). The three problems were real world situations that included figural representations, pointing out the aspects of projection and homothety. They were proposed together to small groups of pupils. The teacher had justified the importance of this way of working by saying that 'the pupils work together sharing meanings and communicating their ideas' (initial interview).

In the following action, explanation for reaching the definition, the teacher pointed out the ideas that he had collected through the interaction with the groups of pupils, which he considered important for the introduction of the theorem. Ismael had mentioned that: 'after the work in groups, I am going to collect the important theoretical ideas, clarifying the difficulties' (initial interview). Ismael conducted a whole-class discussion, writing these ideas on the blackboard. He ended with the definition of the Thales theorem. Finally, through the explanation for completing, the teacher introduced other aspects that allowed him to connect with the monitored practice. He justified the introduction of this contents by saying: 'I would try to show several Thales configurations in relation to the position of data and unknown', adding 'the consequence of Thales theorem diminishes the difficulty that the pupils may have with a specific type of configurations' (initial interviews).

Globally considered the characteristics of the domains of knowledge that were decisive as regards the previous decisions of adopting this structure were related with his pedagogical content knowledge. In particular, the importance of establishing relationships between the earlier knowledge and the new concept by means of several problems previously selected, and working in small groups as a way of communicating and sharing ideas, followed by his intervention as 'organiser'. His knowledge of specific difficulties about the content played a more specific role, influencing his decision of including an explanation after the definition of the theorem.

Nevertheless, regarding the structure identified in the classroom, we appreciated variations in relation to the agenda. The teacher had indicated that his way of working 'was working in small groups/ setting out common grounds/ recollecting the important ideas and clarifying the difficulties to reach the formal definition' (initial interviews). Why did the setting out common grounds disappear? It was related to the difficulties that pupils had with the content. The teacher justified this omission by saying: 'I meant to start the explanation when the pupils had advanced more, but I had to intervene because of their difficulties' (Post-lesson interview). We want to point out two ideas. Firstly, the way in which Ismael understood the 'setting out common grounds', focused on his own intervention of collecting and clarifying the pupils' ideas that he considered important. This was coherent with the action 'explanation for reaching the definition' identified in our analysis.

Secondly, the important role of the difficulties in the change of action. Ismael related these difficulties to specific characteristics of the content. In particular, in the post-interview he pointed out two aspects, which were important for him. The first aspect refers to the difficulties related to the contents he wanted the pupils to identify in the proposed problems: 'with the first two problems, the pupils had difficulties with the proportion they had to identify' (post-lesson interview). The second aspect refers to the characteristics of the real world situation proposed in the problems. Initially, Ismael emphasised the use of real world situations and the importance of recognising the proportion in different figural representations, therefore, all three problems were real world situations with different geometric configurations. The problems showed both aspects projection and homothety. Two of the problems established a relationship between two data and the other was a missing value problem.

Nevertheless, the pupils' understanding of the enunciate of the problems was complicated by the real situations. Ismael concluded that this was a cause of difficulties: 'I realised that the enunciate of the problems was too long compared to what these pupils were habituated to' (post-lesson interview)'. This allowed him to think about changing the real world situations for other easier ones (even though these situations were no-real world situations) in the future. For Ismael, the role of real world situations was motivation. Nevertheless, he did not doubt of maintaining the use of several problems with variations in geometrical configurations. This aspect, and the importance that Ismael gave to the role of Thales theorem in the unit, was closely related to his subject matter knowledge.

All of this shows the role played by the perception in the classroom of the difficulties with the syntax of the real situations and the specific aspects of the content. Why did he have included these problems? As we have mentioned before, his decisions were related with his subject matter knowledge, but they were also influenced by his knowledge of other students. The teacher justified his election by saying: 'I had proposed these problems to older pupils. I think I overvaluated their capacity of abstraction' (post-lesson interview). The knowledge of students of Ismael was situated in other pupils.

For Juan, the inclusion of the Thales theorem in the unit was justified because it is used later on to establish the criteria of similar triangles, and also because of his previous experience 'I have done it like this for years' (initial interviews). This unit was developed in 8 fifty-minute lessons. We identified 11 presentation, 7 monitored practice and 4 homework check segments. The presentation segment of Thales theorem took about 20 minutes and was the second of the unit. It belonged to a group of 4 presentation segments with the following structure: utilisation of an example or problem/ definition/ explanation for completing. This structure was linked to the teacher's introduction of contents whenever he supposed the pupils had obtained some knowledge of the matter in previous courses ('the pupils have learned about numerical proportionality of previous courses' (initial interviews)). On this basis, Juan thought to introduce the concepts of ratio and proportion of segments, which he considered previous to approach the theorem, in the same unit.

In the first action, utilisation of an example, Juan developed the content in sequential steps by means of instructions to the whole group, intercalating questions and asking for the results. Juan was indicating to the pupils that made a verification of properties of parallel lines cut off intersect lines to equal and distinct distances by means of measurements carried out with drawing tools. The examples to be used, though previously selected by the teacher, were not known to the pupils at the beginning.

Juan justified this way of introducing the theorem because he considered that the pupils knew the geometrical elements that intervene in the examples to be used (parallel lines, intersect lines, segments) so that they will be able to follow his instructions. Furthermore, by assigning a numerical value to the segment pupils could connect with the arithmetic proportion ('already known'). During the initial interview Juan explained that the pupils 'more easily believed things' after having carried out the verification by them.

The enunciate of the theorem (by means of a transparency) collected all mathematical aspects that were sequentially emphasised in the performed verifications, showing a 'typical' configuration of the theorem, and emphasising the projection aspect. Finally, in the explanation for completing, the teacher stressed the generality of the lines in the drawing and placed emphasis on the general character of the theorem by specifically mentioned to the pupils that 'the things that we have done with these specific cases are going to be always true' (video transcriptions).

Globally considered, the domains of knowledge that intervened in the adoption of this structure were related to his knowledge of school mathematics as a teaching/learning object and the knowledge about learning processes, considered as a part of pedagogical content knowledge. Especially, as regards the use of a verification as a procedure of validation and the use of sequential steps that the pupils were carried out 'because they learn more' (post lesson interview). However, Juan specifically mentioned the low level and lack of motivation of these pupils in the initial interview. We think that this aspect was also integrated in the previous decision of using verification. Juan mentioned the characteristics of the general case only in the content of explanation.

This teacher only presented few variations as regards of his way of working compared to his agenda. In the case of Juan, difficulties played a different role in comparison to the case of Ismael. By dividing the content into sequential steps and giving a certain type of instructions to the pupils, few difficulties came up in the classroom that the teacher considered worth while to take into account with respect to his objective of introducing the theorem. Pupils' difficulties with the lack of precision at measuring the segments, which had an effect in the results of the verifications, were not relevant aspects from the perspective of his subject matter knowledge in relation with the introduction of the theorem. The algorithmic view, which was appreciated in the sequential steps, underlined the decision of including the theorem, in the sense that it was considered as a previous step towards the later introduction of the concept of similar triangles. Along with this, it was made evident the influence of the school culture, which traditionally assigns an important role to theorem.

CONCLUSIONS

This study is a part of an ongoing research into the relationships between teacher's knowledge and practice. The main objective of this paper was the identification and characterisation of domains of knowledge that intervened in this relationship. The objective of the presentation segments of both teachers was the introduction of the Thales theorem. Pedagogical content knowledge and subject matter knowledge were integrated in their decisions. Nevertheless, our study has shown that these teachers adopted different structures, being the initial decisions of adopting these structures linked to different characteristics of the domains of knowledge, which were integrated in a very different way.

Both teachers realised the importance of the pupils' difficulties in the adopted decisions. But, whereas Ismael was very sensitive to the difficulties, which came up during the interaction, Juan minimised the importance of these difficulties. He tried to avoid the difficulties globally in his previous decisions about his planning, admitting the verification as an easier procedure for the introduction of the content. The characteristics of the subject matter knowledge related to Thales theorem made that Ismael gave a great importance to the establishment of numerical/geometrical connections by means of the selected problems. Juan's algorithm approach of the contents was associated to a more general view of school mathematics.

There are many issues in our research that require further investigation. A better characterisation of domains of knowledge and its role in the different moments of professional practice can help to improve our knowledge about the complexities of mathematics teaching.

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AFFECT AND MATHEMATICS LEARNING

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Beliefs, attitudes, emotions and values are the categories of affect used in mathematics education research. All these categories are founded on mental background systems which control thinking, learning and acting. To gain a deeper insight in the mechanisms of the mental systems and their forecasting character, results of neuroscience are used. By distinguishing between an explicit conscious memory system, which stores experiences about emotions, and an implicit emotional memory system, which operates unconsciously, it is possible to understand some emotional reactions in a better way.

1. Research in Mathematics Education on Affect

For a long time mathematics learning has been mainly seen as a problem of cognition. Researchers investigated the specificity of mathematical terms and concepts, steps of concept building, the influence of representations on the learning process, and so on. Consequences of research were theoretical models describing the learning process. The consideration of the social conditions in classroom learning led to an important extension. Mathematics learning was seen as a social process which controls the cognitive processes of the individual learners.

During the last decades there has been an increasing interest in affect and its influence on mathematics learning. The first step was to use the psychological "attitude concept" to explain differences in the students' mathematics performance. "Attitudes toward mathematics" (seen as a (learned) predisposition to respond in a consistent way with respect to a special object) was identified as a factor influencing the learning process and, as a consequence, the mathematics outcome. The most influential measure to investigate attitudes toward mathematics was the Fennema-Sherman Mathematics Attitude Scale (Hart, 1989; McLeod, 1994).

One consequence of this concept was the development of a "generic model for relating affect and outcome" (Fennema, 1989). In this model affect is a consequence of external influences like gender, social class etc. which influence learning activities and the mathematical test results. (For a further development of this model, see Evans, 2000).

A second very important concept in research on affect is the "belief concept" which stresses students' knowledge about mathematics in a very broad sense. The belief concept includes beliefs about mathematics, about the self, about mathematics teaching and about the social context (McLeod, 1992, 1994).

Furinghetti and Pehkonen (2000) describe the function of beliefs in the following way:

> (a) beliefs form a background system regulating our perception, thinking and actions; and therefore, (b) beliefs act as indicators for teaching and learning. Moreover, (c) beliefs can be seen as an inertial force that may work against change, and as a consequence, (d) beliefs have a forecasting character. (p. 8 - 9)

Research on attitudes and beliefs is based on questionnaires as instruments and statistical methods for evaluation. Qualitative methods especially in connection with research on nonroutine problem solving processes and higher-order thinking skills led to the interest in more intense affective reactions called "emotions". Goldin (2000) describes a model of affective states on different steps of a nonroutine problem solving process. These affective states can evoke special heuristic problem solving strategies.

The categories *beliefs - attitudes - emotions* are used by McLeod (1992) to reconstruct the research on the affective domain in mathematics eduction. To better distinguish these categories, McLeod (1989) uses the dimensions *intensity - direction - stability*. Longitudinal studies on mathematics learning processes and affect motivate DeBellis and Goldin (1997) to introduce a fourth affective category "values/morals/ethics". This new category emphasizes the appraisal aspect of affect.

A summary of the discussion of affect in research of mathematics education is provided by a quotation from Goldin (2001) which describes the categories of affective representations:

> (1) emotions (rapidly changing states of feeling, mild to very intense, that are usually local or embedded in context),
>
> (2) attitudes (moderately stable predispositions toward ways of feeling in classes of situations, involving a balance of affect and cognition),
>
> (3) beliefs (internal representations to which the holder attributes truth, validity, or applicability, usually stable and higly cognitive, may be highly structured).
>
> (4) values, ethics, and morals (deeply-held preferences), possibly characterized as "personal truth," stable, highly affective as well as cognitive, may also be highly structured). (p. 3)

The categories of affect in research on mathematics are on the one hand a consequence of research methods and on the other hand motivated by observed phenomena. The concepts don't describe the interaction between affect and cognition in a sufficient way to understand some affective reactions; especially, the relationship between conscious and unconscious is not really discussed.

2 Affect and Cognition -- Some Results of Neuroscience

First some remarks:

1) The use of terms "affect" and "emotion" is not uniform in neurosciences (Ciompi, 1999; Roth, 2001). Often they are used synonymously as generic terms for affective-emotional basic conditions.

2) The brain is the organ in which is represented all the knowledge about the world outside but also about inside of the body.

3) All systems of the brain are the result of evolution with the aim of helping the individual to survive. Emotion and cognition are also part of the brain system and therefore a result of evolution (Ciompi/Wimmer, 1996; Damasio, 1999; LeDoux, 1998; Roth, 2001).

4) A special brain system of an individual is a consequence of the evolution as well as of the ontogenetic development of the individual (Ciompi, 1999).

5) First of all we have to note that all processes on the neuronal level are unconscious. But some of these processes lead to conscious results. We are aware only of these conscious parts of the processes.

Emotion and cognition are both subsystems of the brain system. They are located in different parts of the brain (Damasio 1999; LeDoux 1998; Roth 2001) but there are connections between both systems which allow interactions. A very important consequence of these two different systems of concepts is the fact that we have to distinguish between "feeling" and "knowing that we have a "feeling" (Damasio, 1999; 26) or "emotional reactions" and "conscious emotional experience" (LeDoux, 1998; 296).

The emotion system has also connections to the arousal systems, therefore emotional reactions are often combined with "body reactions" (LeDoux, 1998). For this paper the interaction between affect and cognition are of particular interest:

* The emotion system influences memory processes. On the one hand, the emotion system is involved in the storage process. The system works like an appraisal filter. On the other hand, the results of the retrieval process are emotionally related (Roth, 2001).

* The emotions and consciousness interact: "While conscious control over emotions is weak, emotions can flood consciousness" (LeDoux, 1998; 19). That means that emotions have more influence on the cognition than the cognition has on emotions.

* LeDoux (1998) uses the concept of working memory to represent "the here and the now in the brain." In the working memory inputs from a sensory system meet inputs from the emotional and arousal system and information from the long-term memory.

3. Reconstructing the Theme by Results of Neuroscience

The most important research concepts of affect are beliefs and attitudes. Both are seen as the stable mental background which regulates perceptions, thinking and actions of a person (Furinghetti & Pekhonen, 2000). This stable mental background system is a consequence of all the processes of an individual to a special object, person or idea. Törner and Grigutsch (1994) speak therefore of a "mathematical world view" of a person; Ciompi (1999) of "one´s own truth".

Research methods for investigating beliefs and attitudes are in the most cases questionnaires; in some cases interviews. Both methods rely on the memory of the interviewees.

If we look at the results of neuroscience in relation to the memory, we have to distinguish between two memory systems with repect to emotions: the implicit emotional memory and the explicit memory about emotions (LeDoux, 1998). The implicit emotional memory operates unconsciously, is strongly connected to arousal systems and may often lead to bodily reactions.

The explicit memory about emotional situations contains all the conscious knowledge about emotional situations, emotional reactions to objects, persons and ideas etc.. The most important consequence is the fact that this memory system is part of the cognitive memory and there is no distinction between a remembrance of an emotion and a remembrance of a cognitive content (LeDoux, 1998).

The fact that the memory about emotions is a cognitive memory has some important consequences:

1) We have knowledge about our feelings and their origin. This knowledge is stored in memory systems as cognitive knowledge. Therefore we have the possibility of investigating knowledge about affect using cognition. Research methods used in cognition research (questionnaires and interviews) can help us to explore affect.
2) The memory about emotions is open for "rational" manipulations. That means we are able to think about our emotional remembrances. But we have to note all verbal statements are controlled by the cognition.
3) A very important control authority to all verbal statements is the "feeling of one´s own worth". Therefore we try to give our emotional remembrances a meaning which doesn´t destroy the feeling of one´s own worth. In this sense persons have in interviews a tendency to trivialize their own weaknesses. This trivialization can, for instance, take place by the expression of opinions that particular mathematical contents are unimportant if the person is unable to cope with these kinds of problems (Wagneder, 1998).
4) Research on emotion and memory suggests that humans "construct" their memories in a way that they are able to live with this memory. A part of this

process is that we forget unpleasant facts more easily than pleasant ones. And our memory has suppression mechanisms to handle unpleasant remembrances (Roth, 2001).

5) Group processes have an important influence on verbal statements concerning contents which are emotionally coloured. Humans are able to "learn" emotions in group processes (Ciompi, 1999; Damasio, 1999). These processes can lead to common shared emotions concerning special contents. Especially, shared value systems can influence the contents of the memory. In this sense we have to see the habitus concept (Gates, 2001) or the socially constructed "feeling rules" (Ulich & Kapfhammer, 1991) (Feeling rules mean that the group says what a member of the group has to feel in a particular situation) .

These reflections on the two memory systems give us a better insight into the meaning of the belief and attitude concepts. Both concepts are based in the outcomes of the cognitive memory system. This system contains not only our knowledge of cognitive contents but also the conscious knowledge of emotions. Therefore we are able by questionnaires and interviews to get information on cognitive as well as on emotional facts. The difference between the both concepts lies more in the kind of questions than in the quality of the memory. The restrictions of the remembrance of emotional situations and the handling of emotions by the memory also restricts the meaningfulness of the concepts. Belief and attitudes as a mental background system do not completely regulate our perception, thinking and actions, because the conscious controls on our emotions are only weak (LeDoux, 1998). Therefore we have learning situations, which are not ascertainable by the belief – attitude concept. In this sense it is not surprising that qualitative research methods led to a new affective category.

Studies on the problem solving process, especially with nonroutine problems, suggest that there exist emotional outbreaks during the problem solving process. These emotional reactions could be from mild to very intense, and are local and embedded in a special contextual situation. Goldin (2000) developed a model which describes states of the problem solving processes and the possible affective-cognitive reactions to the demands of this state. All these reactions are situational, caused by the demands of the problem solving process. The model describes two possible results, dependent on whether the problem solving process leads to success or not.

Interpreting these results by the concepts of neuroscience we have to note that there is a second memory in connection with emotions. This memory (LeDoux calls it emotional memory) operates unconsciously. Its evolutionary function is to help an individual to survive. Therefore it often works in dangerous situations, is strongly connected with arousal and bodily systems, and leads to reactions which are not controlled by cognition (LeDoux, 1998). This emotional system is always activated if the appraisal of a situation leads to the result an activation is necessary. This appraisal of a situation is not conscious.

If we analyze the model of Goldin (2000) there are states in the problem solving process which can activate the emotional memory system. This activation can be observed by the researcher. The most well-known signs of an activation of the system are bodily reactions. To interpret bodily reactions is a qualification which is evolutionarily rooted in humans and therefore a valuable research method to evaluate emotional situations. As discussed above, these emotional reactions underlie only a weak control by the cognition (LeDoux, 1998) and are therefore especially valuable. Goldin describes the feelings by words which we can connect with the concerning expressions of the face, shaking of the hand, quavering of the voice, and so on. These signs of the body also allow us to recognize all changes of the emotional state.

It is important to note that emotional situations are not only stored in the memory which contains the knowlege about feelings they are also stored in the emotional memory and influence therefore further reactions to a special situation.

Activation of the emotional memory can lead in some cases to very intense emotional reactions. Reports on learning situations in adult education in mathematics tell from "learning blockades" (Lindenskov, 1996; Schlöglmann, 1999; Wedege, 1998). To elucidate these problems I will give an example which was reported on the internet discussion group "adult numeracy" by Bonnie Fortini:

> Has anyone run in to a case like the student I have had who seems unable to do any math that has unknowns or variables in it? She is mid 40s, very bright, English major going on to a Masters program. She can do all sorts of computations including fractions, percentages, ratios, and word problems are some of her favorite things to do. But as soon as you give her something like $4 + 2x - 6 + 5x= 95$, she is totally frozen. She can't get past go when trying to combine like terms, and reacts physically (anxiety, tears, etc.).

Such an emotional reaction is not understandable within the conceptual frame which is used by research in mathematics education. This reaction is a traumatic reaction. The distinction between emotions used in mathematics education research is the stability. This is not a short sign of frustation described in papers to emotional reactions in problem solving processes. The mental system which caused this intense emotional reaction is stable, operating always if the relevant content is recognized.

LeDoux (1998) writes about such reactions:

> In traumatic situations, implicit and explicit systems function in parallel. Later, if you are exposed to stimuli, that where present during the trauma, both systems will most likely be reactivated. (p. 202)

The specificity of such emotional reactions is that there is no chance to help on the cognitive level. Such problems are not solvable in classroom situations. The emotion floods the cognition and the whole cognitive system is blocked and not able to be reached by "rational" arguments.

During the last year there has been an increasing interest in teachers' values and beliefs and the consequences for the classroom practice. Some researchers see a strong connection between teachers' values and beliefs and the classroom situation. But classroom observations and case studies suggest that there are differences between beliefs and values expressed in interviews and the real classroom practice (Skott, 2000). Bishop, FitzSimons, Seah, and Clarkson (2001) describe the situation in the following way:

> It is also recognized that there are differences between the values that are officially planned and those espoused by teachers, as well as between teachers` espoused beliefs and their actual classroom practices – due in part to differential positionings as interview subjects and as teachers. (p. 170)

We note that beliefs as well as values expressed in interviews are products of cognition. Both use the explicit memory as background. In the classroom reality implicit unconscious systems are also activated and these systems influence all actions in complex situations. We have in the classroom situation on the one side the cognitively controlled values and beliefs and on the other side the unconscious from the implicit emotional memory system influenced actions. It depends on the local situation which system dominates the concrete action. Especially in critical situations, the implicit system is more active because the implicit system leads to actions which allow survival independently of which values and beliefs are formulated in interviews.

Summarizing the results of this paper: Neuroscientifical results helps to prove the adequacy of the theoretical concepts used in research of mathematics education. Furthermore they can help us to understand these concepts in a better way. However, it is not the idea of this paper to say that research questions of mathematics education can only be solved by neuroscience.

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TESTING AND FOURTH GRADE TEACHING

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Advocates of testing often maintain that tests can provoke teachers to adopt new instructional practices. The purpose of this paper is share results from a large scale study designed to investigate the teaching practices of a group of fourth grade teachers who are all involved in state mandated testing programs. We found that while teachers are aware of the test and have made some instructional changes in terms of specific teaching strategies, the changes that have been made tend to focus mostly on strategies and techniques such as the use of small group instruction or manipulatives rather than changes in, for example, the nature of the discourse that takes place in the classroom

Introduction and Framework: In the last two decades, more and more states within the United States have either introduced or extended their testing of children (Editorial Projects in Education, 2001). This trend, however, is not limited to the United States (Niss, 1996; Keitel, and Kilpatrick, 1998 as cited in Abrantes, 2001; Firestone and Mayrowetz, 2000; Abrantes, 2001). Some advocates of tests consider them to be part of a broader effort to raise educational standards and make educators accountable for reaching them. They see testing as a way to use the authority of the state to improve teaching and learning and enhance equity by holding all children accountable to the same high standards (O'Day & Smith, 1993). Some maintain that a test that is well designed can prompt teachers to revise their practices because teachers will inevitably "teach to the test", and that can be good if the test is well designed. Others maintain that content that is emphasized on tests gets emphasized in class, and that untested content either falls out of the curriculum or gets put off until the end of the year (Corbett & Wilson, 1991; McNeil, 2000). The types of items that are placed on the test are also claimed to influence the types of problems teachers use in class. The argument is that by including items that require students to solve more complex types of problems, teachers will be more likely to provide students with the opportunity to do the same in class. One reason given for the great interest in various forms of performance assessment and portfolios in the 1990s was the hope that tasks requiring students to show their work and explain their answers would promote inquiry-oriented instructional approaches (Resnick & Resnick, 1992; Rothman, 1995). Currently many tests combine conventional, multiple- choice formats with other formats intended to measure higher order thinking and problem solving abilities. However, even when tests employ formats where students construct responses, some of the same risks that are typical of the more traditional tests have been found to occur (Smith, 1996; Stecher & Barron, 1999).

Some opponents believe that extensive testing will encourage measurement of less relevant skills, and reinforce traditional approaches to teaching (McNeil, 2000). There are also those who believe that the effects of state tests have been overstated and that any modest changes in teaching exist alongside what has been conventional practice (Wilson & Floden, 2001). Regardless of the format, the evidence that testing promotes instructional change remains unconvincing or inconclusive at best (Newmann, Bryk, & Nagaoka, 2001; Smith, 1996).

In this research[1], we were interested in learning more about teachers reactions to a standardized state test designed to encourage them to implement more student-centered instructional practices that reflect state and national standards (c.f. NCTM, 2000), and what kinds of practices they were actually implementing in their classrooms. Would the teachers, for instance, adopt practices that are associated with reforms and found to be more likely to help students to develop a deeper understanding of mathematics (Davis, 1984; Cobb, Wood, Yackel & McNeal,1993; NCTM, 2000; Klein and Tirosh, 2000; Schorr, 2000)? Would they adopt specific strategies, like using more manipulatives? Would they simply ignore the test and not make any changes? Most importantly, we were also interested in how the changes that they might report actually manifested themselves in the context of their actual classroom instruction. More specifically, in this research, we were interested in gaining a deeper understanding of how teachers feel that the fourth grade test is impacting their teaching, and also in learning more about their actual mathematics teaching. This paper presents data regarding the actual practices of a group of teachers and their perceptions of how the test has influenced their teaching. We looked for evidence that teachers were incorporating approaches that would provide opportunities for students to learn mathematics as they were engaged in meaningful mathematical activity. Our approach in this research was to observe a sample of 4th grade mathematics teachers from across the state, and interview them about their practice and their reactions to the test (The test that we focus on is New Jersey's fourth grade Elementary School Performance Assessment (ESPA) that has been in place since 1999.) which include any changes that the test has prompted in their teaching. This paper describes the results. The particular codes and methods of analysis will be further described below. Our data suggest that the teachers we interviewed and observed are adopting new procedures as part of their instructional practice but not changing their basic approach to teaching mathematics. For example, they tend to continue to assign tasks that reinforce a procedural view of

[1] A portion of this paper was presented, in a preliminary format, at the annual meeting of the American Educational Research Association, April 2001 and at PME NA, October 2001. Neither presentation focused on both the test data and the observation data. The work on this paper was supported by two grants from the National Science Foundation. The opinions presented here are those of the authors and are not necessarily shared by NSF, Rutgers University or Rider University.

mathematics, and classroom discourse does not tend to encourage students to defend and justify solutions.

Methods and Procedures: This paper focuses on observational and interview data from the first two years of a three-year multi-method study of testing and teaching in New Jersey that combines a statewide survey with a more intensive observation and interview study of a smaller sample of teachers.

Sample: The observation study focuses on 63 teachers. The sample was chosen to be representative in terms of both district wealth and geographic spread—i.e., north-south, east-west (Firestone et al., 2001). Almost all teachers taught fourth grade. That grade was chosen because it is the elementary grade tested in New Jersey.

Actual Observations: Fifty-eight teachers were observed for two math lessons and five teachers were observed once for a total of 121 classroom observations. The classroom researcher kept a running record of the events in the classroom, focusing on the activities of the teacher as well as capturing the activities of students, all problem activities and explorations, the materials used, the questions that were posed, the responses that were given—whether by students or teachers, the overall atmosphere of the classroom environment, and any other aspects of the class that they were able to gather. (For further information on sample selection, see Schorr and Firestone, 2001; Firestone, et.al. 2001)

Interviews: At the conclusion of each lesson, the teachers were asked to respond to a series of open-ended questions about the observed lesson. For example, they were asked: What were you trying to accomplish for today's lesson? What concept or ideas were you focusing on? What, if anything, would you change about today's lesson, and why? Why did you do this, or how did you feel about that (referring to a particular instance where for example, students explained mathematical ideas to each other or to the teacher, or with regard to a particular event or activity).

Teachers were also asked how state testing affected their teaching. Sample questions included: What kinds of things do you generally do to help your student get ready for the Elementary School Performance Assessment (ESPA)? Considering either the ESPA or the Content Standards, how, if at all, has that affected the topics you teach? How have you changed the teaching strategies you use in response to the ESPA and/or the Content Standards?

Coding: While observations were underway, researchers conducted detailed analyses of records of classroom observations, and adapted several pre-existing coding schemes to be used for coding the classroom data. These were based on the works of Stein, Smith, Henningsen, and Silver (2000); (Stigler & Hiebert, 1997, 1999); and Davis, Wagner, and Shafer, 1997). These codes were selected because they reflected ideas about effective mathematics instruction as indicated in national and state standards. They included attention to the mathematical discourse that emerged, the opportunity for conceptual understanding to take place, the nature of student conjectures, the opportunities students had to share ideas and defend and justify solutions, etc. They were also chosen because we felt that they would supply information on the nature and use of reported strategies (i.e. manipulatives, small group instruction, use of different types of problems and activities, questioning strategies, etc.).

A preliminary coding scheme was tried out on approximately six observations before being agreed upon. A sheet of code definitions was created and a training session was held for coders involved in the activity. Ultimately, a coding instrument was developed which incorporated 18 dimensions, along with detailed descriptors of each coding category.

Two individuals independently coded each observation—at least one coder was an experienced mathematics education researcher. The other coder also had extensive experience in elementary education. After independent coding, raters sought to reconcile their differences and were successful in all but 2 of the 108 cases. In those two cases, another mathematics education researcher discussed differences with the raters and helped them to reach agreement.

Interview data were transcribed and entered into a qualitative data analysis software package. Interviews were sorted by question. Responses were analyzed in clusters, as there was considerable overlap in responses given to individual questions. Within each cluster, responses to specific questions on test preparation practices were reviewed and coded according to emergent themes. Responses were counted within each code. Interviews from 58 of the 63 teachers were available for analysis.

Results and Discussion: Some teachers reported that they liked the changes they had to make in response to ESPA. Nine made general comments to the effect that the test is forcing teachers "to evaluate their teaching style". Several made more specific comments that the presence of ESPA was encouraging them to use alternative teaching methodologies like manipulatives or have children respond to more open-ended questions. Many teachers reported that ESPA is encouraging them to implement more inquiry-oriented instructional practice. One teacher explained, "It's become my philosophy to teach them the concepts before, just, you know, ramming these rote facts down their throats."

In the interviews, teachers mentioned four general changes: having students explain their thought processes, using manipulatives, problem solving and working on students' writing. Forty three percent talked about trying to get students to explain their thinking in more detail. According to one teacher, the part "that I guess I really didn't do a lot of before is really get the students to start to learn how to explain their thinking, to explain what they were doing. Sometimes they do it in writing; sometimes they do it to a partner; sometimes they do it to me." One strategy to encourage student explanation is the use of more open-ended questions on tests and in class, and was mentioned by 33%. Fourteen percent talked about using more "how" and "why" questions in their whole-group teaching. One described this as working on "critical thinking skills" instead of "feeding them the answer". Several also talked about using small-group instruction so students would explain their work to each other.

Another theme involved using manipulatives, and was mentioned by 45%. The ESPA has questions that involve at least written or pictorial descriptions of manipulatives. Many teachers felt that students who are more familiar with some of the current manipulatives could therefore better respond to those questions.

A third theme was a greater emphasis on problem solving (mentioned by 38%), though the actual meaning of "problem solving" was not always clear. For example, some teachers noted that they actually give students a set of strategies (i.e. draw a picture, think of a simpler problem, work backwards). Another teacher said, "We do a lot of work with problem-solving skills, just the basic skills of how you read a problem, how do you find the question, how do you find the information that you need, how do you check to see whether your solution is logical and can solve it a couple of different ways." This emphasis on word problems reflects in part the use of open-ended problems on the ESPA.

Finally, 40% said that they emphasize "writing" to prepare their students for the ESPA. One teacher said that she now had her students "write all the time for all subjects." Some teachers used "writing" to have their students explain their line of thinking in mathematics. In fact 12 teachers said they had students keep journals in math as well as other subjects.

The observations confirm that teachers are making some changes. Manipulatives were used in about 60% of all observed lessons. Similarly, students worked in groups for at least a portion of the time, in almost 65% of all observed lessons and in almost half of all cases, teachers made an effort to connect the lessons to the students' real life experiences.

The adoption of specific strategies was not necessarily accompanied by a change in overall approach to teaching mathematics, however. For example, while manipulatives were used extensively, they were used in a non-algorithmic manner in less than 19% of all observed lessons. This essentially means that the manipulatives were used in ways that did not foster the development of conceptual understanding. In fact, in almost two thirds of the lessons where manipulatives were used, they were used in a very procedural manner, where the teacher generally told the students exactly what to do with the materials, and the students did it as best they could.

Other times, teachers used manipulatives to demonstrate a particular procedure to the class. In many of the lessons—while many teachers had students physically touch concrete manipulatives, there often was little or no opportunity for the students to develop their own solutions to the problem or consider the relationship between the problem activity and the concrete (or alternative) representations.

Beyond looking at the use of specific practices and materials, we also examined the mathematical tasks students were asked to perform. We categorized tasks as memorization only, doing procedures where the focus was on producing correct answers rather than developing mathematical understanding, doing procedures to develop a deeper understanding of mathematical concepts or ideas, or doing a mathematical task that requires complex and non-algorithmic thinking (Stein and Smith, 1998). Only 3% of all observed lessons involved situations where students were required to do non-algorithmic thinking.

We also examined whether tasks involved practice or non-practice activities. With practice tasks, the teacher demonstrates or develops a procedure, such as long division, and then assigns a number of similar problems on which students are to repeat the same procedure (Stigler, 1999). Alternatively, in a non-practice task the student may be required to invent a new solution method, analyze a mathematical situation, or generate a proof. Practice tasks predominated, constituting almost 80 % of the observed lessons.

We also found that classroom discourse did not foster substantive conversations amongst students. Many teachers reported that they were interested in having students explain their reasoning. They also said they were interested in having students find and understand multiple strategies for solving problems, however, they rarely insisted on such activity. For instance, one code documented whether or not the teacher encouraged students to reflect on the reasonableness of their responses. In almost 80% of all cases the teacher rarely asked students whether their answers were reasonable. If a student gave an incorrect response, another student provided, or was asked to provide, a correct answer, but there was little discussion of an appropriate strategy to solve the problem. In an additional 15% of all cases, the teacher may have asked students if they checked whether their answers were reasonable, but did not promote discussion that emphasized conceptual understanding.

When students *were* provided with opportunities to talk about their answers or strategies, they usually simply stated answers to problems, and did not elaborate on their solutions. When a student was asked to share his solution, often he would respond with a numerical answer such as "5" or a procedure such as "you should add". Students were rarely asked to explain how they got their answer, or how they arrived at their particular strategy. In fact, students only explained their responses or solution strategies in a way that went beyond the execution of procedures in six percent of the observed classes. Sometimes teachers would ask for an explanation for using a particular operation but would not encourage students to expand upon their answers, or move beyond simplistic responses.

Conclusions: The teachers involved in this research have indicated that they have been motivated to change their styles of teaching as a result of the ESPA test. Indeed, our observations confirm that they do incorporate many of the strategies and techniques that they reported in our interviews (such as small group instruction and the use of manipulatives). This research does not and cannot document just when these strategies first became part of their practice; we can only note that the teachers attribute the implementation of many of them to the test. This study provides evidence that the teaching practices that we noted in our observations, however, are not focused on the more conceptually oriented aspects of instruction. Perhaps with appropriate support, teachers who are ready and willing to make changes in their teaching will be able to incorporate practices that will enable children to have access to mathematical instruction that fosters the growth of mathematical thinking.

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STUDENTS' CONCEPTIONS OF AN ACCEPTABLE GEOMETRIC DEFINITION

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The current study focuses on students' conceptions of a mathematical definition. The research instrument consisted of written questionnaires and group activities. These activities aimed at eliciting considerations and argumentation surrounding students' decision-making process related to the acceptance of a statement as a definition of a well-known mathematical concept. The focal concept for this study was a square, because of students' familiarity with it. Data was collected from students' responses to written questionnaires and videotaped observations of their group activities. The findings suggest three main perspectives underlying students' conceptions of an acceptable mathematical definition.

PROLOGUE

We invite the reader to consider the following statements:

1. A square is a rectangle with four equal sides;
2. A square is a parallelogram with diagonals that are equal and perpendicular;
3. A square is a polygon with four sides, in which all sides are equal, and all angles are equal.

Which of the statements would you accept as a definition of a square? Which of the statements that you accept, do you prefer the most? Why?

The following three excerpts of 4 outstanding 12th-grade students debating over these questions convey the spirit of our study, which we report hereafter.

Excerpt 1:

Erez:	[refers to statement 3] It's correct, but it **is not** a definition.
Yoav:	It's correct, and **it is** a definition.
Erez:	It has too many details.
Yoav:	Too many details, but it is still a definition.
Omer:	What does "too many details" have to do with that?

Excerpt 2:

Erez:	I don't accept statements (a) and (b) [refers to statement 1 above, in which a square is described as a special rectangle, and to another statement where a square is described as a special rhombus].
Yoav:	I don't either.
Omer:	Why?
Erez:	Because you need to know what a rhombus is, and you need to know what a rectangle is.

Omer:	So what?
Erez:	It is not acceptable to base a definition on other concepts.
Yoav:	In fact a square is a special rectangle or a special rhombus, so you can define it using these concepts.
Erez:	There is no doubt that it is correct, it is correct. But a definition, according to its nature, should be based on the lowest base.

Excerpt 3:

Omer:	Yoav, draw a square.
Yoav:	[sketches a square] o.k.
Omer:	Erez, draw a square.
Erez:	[looking at Yoav's square] You mean he didn't draw it with the diagonals?
Omer:	Exactly.
Erez:	What's the connection?
...	
Mike:	When you draw a square you see this [draws a square with his finger on the table], like he [Omer] told you, you see sides, you don't see diagonals that are perpendicular. A square is first of all 4 sides. You don't refer to the diagonals. The diagonals are a property of the square.
Erez:	So are equal sides and right angles, they are also properties.
Omer:	No, they are a definition.
Mike:	Right, sides are sides. They build the square. Sides and angles build the square. Diagonals don't build the square.

The current study is part of a larger one. A different portion of it, sharing some aspects of the theoretical background, appears in Shir & Zaslavsky (2001).

CONCEPTUAL FRAMEWORK

Definitions play a central role in mathematics and mathematics education. Recent studies on the notion of definition (e.g., Winicki-Landman & Leikin, 2000; Furinghetti & Paola, 2000; de Villiers, 1998; Borasi, 1992, 1987; Linchevsky, Vinner & Karsenty, 1992;) differ mainly with respect to the population under investigation (students vs. in-service or pre-service teachers) and the features of a definition that are focal to the study. The vast majority of these studies investigate how the participants view the minimal aspect of a definition. Others look into the arbitrariness aspect that is associated with the freedom to choose a definition among equivalent statements as well of the advantages of certain choices over others. In addition, Borasi (1992) studied gradual refinement processes directed towards reaching a valid definition. Our study looks into students' ways of thinking about mathematical definitions and their processes of reaching an agreement on the necessity of a broad range of features of mathematical definitions.

Features of mathematical definitions

There are various features of a definition that charaterize a mathematical definition (Vinner, 1991; Borasi, 1992; van Dormolen & Zaslavsky, 1999). For some of these features there is a consensus regarding whether they are strictly necessary or just preferable, while for others there is no such consensus. The following three features of a definition are commonly accepted as crutial, thus, a mathematical definition must be: *non-contradicting* (i.e., all the conditions of a definition should co-exist), *unambiguous* (i.e., having only one interpretation), and *logically equivalent* to any other definition of the same concept. In addition, there are some features of mathematical definitions that are necessary only when applicable: A mathematical definition must be *invariant* under change of representation. In addition, when possible and appropriate, definitions should be *hierarchical* (i.e., based on basic or previously defined concepts, in a non-circular manner).

As mentioned above, a possible feature of a definition is heirarchy. We find it reasonable and useful, when applicable, to consider the level of heirarchy of a definition for concepts that are hierarchical in nature. For example, the 3 statements in the prologue are all hierarchical, yet they differ with respect to what we call *level of hierarchy*. Accordingly, statement 1 is based on a *rectangle* (thus we consider it the 1st and highest level of heirarchy), statement 2 is based on a *parallelogram* (thus we consider it the 2nd level), statement 3 is based on a *polygon* (which we consider the 4th level). Note that a definition of a square based on a *quadrangle* would constitute the 3rd level. We may continue in this way to even higher levels of hierarchy. The hierarchy of polygons to which we relate is rather common and concurs with the hierarchical classification of quadrangles opposed to the partition classification that de Villiers (1994, 1998) discusses.

Unlike the features described above, there are features of definitions upon which there is no consensus regarding their ultimate need. For example, it is not unanimously agreed upon whether a mathematical definition must be *minimal* (i.e., economical, with no superfluous conditions or information). While Hershkowitz (1990), Winicki-Landman and Leikin (2000), Vinner (1991) and Borasi (1987) claim that minimality is an ultimate requirement of a definition; others (e.g., de Villiers, 1998; Pimm, 1993) recognize the role of context with respect to the minimality requirement.

Another kind of distinction can be made between different types of definitions. A definition can be either *procedural* – by genesis, or *structural* - by a common property (Leron, 1988; Pimm, 1993)*. When geometric concepts are involved, we distinguish between a structural definition that relies on a property of certain parts of the object, and a property that is common to all, and only to, the points that constitute the object (i.e., definitions that are stated in terms of loci).

In our study we focus on definitions of a square that differ along the following dimensions: minimality, type (structural vs. procedural), and levels of hierarchy.

* Definitions can also be recursive (e.g., $n! = n(n-1)!$), or axiomatical (e.g., the definition of the Natural Numbers), however, our study does not address these types of definitions.

THE STUDY

The aim of the current study was to investigate students' conceptions of a mathematical definition. Thus, we decided to focus on alternative definitions of a familiar and well-known concept – a square.

Four 12th-grade top-level students participated in the study. The participants took part in three consecutive meetings dealing with alternative ways for defining a square. At the first meeting each student received a written questionnaire that contained eight equivalent statements (see Table 1), and was asked to reply to it individually. In the second meeting the four students were grouped together and were requested to relate to the same task and to try to reach an agreement. In the third and last meeting, the students were asked to reply again to the original written questionnaire individually. Table 1 presents the eight statements in the questionnaire, and their characterizing features.

A SQUARE IS:	Minimal	Type	Level of Hierarchy
(a) A rectangle with four equal sides	No	Structural	1
(b) A rhombus with a right angle.	Yes	Structural	1
(c) A parallelogram with diagonals that are equal, and perpendicular.	Yes	Structural	2
(d) A quadrangle in which all sides are equal and all angles are 90°.	No	Structural	3
(e) A quadrangle with diagonals that are equal, perpendicular, and bisect each other.	Yes	Structural	3
(f) A polygon with four sides, in which all sides are equal, and all angles are equal.	No	Structural	4
(g) The locus of points for which the sum of their distances from two given perpendicular lines is constant.	Yes	Structural (points)	Not Applicable
(h) An object that can be constructed (in the Euclidean Plane) as follows: Draw a segment; from each edge erect a perpendicular to the segment, in the same length as the segment (both in the same direction). Connect the other 2 edges of the perpendiculars by a segment. The 4 segments form a quadrangle that is a square.	Yes	Procedural	Not Applicable

Table 1: The Statements in the Questionnaire and their Characterizing Features

In constructing the different statements for the research instrument, we attention was given to several features. As mentioned earlier, the statements differ from each other with respect to three main constructs: minimality, type (procedural vs. structural), and hierarchy. The hierarchical statements differed from one another with respect to the level of hierarchy of the defining concept.

FINDINGS

Students' written responses to the questionnaire included 152 arguments: 108 arguments justifying the acceptance and 44 arguments justifying the rejection of a statement as a possible definition. The written arguments, as well as the arguments that were raised during the group discussion, were classified according to what seemed to be their underlying perspective: *mathematics*, *receptiveness*, or *figurative*.

Table 2 presents the distribution of types of arguments that the students used to support their decisions.

Underlying Perspective	**Reasons for Acceptance:** The statement is ...	**N**	**Reasons for Rejection:** The statement is ...	**N**
Mathematics	Correct (constitutes a necessary and sufficient condition for the concept)	25 (23%)	Incorrect (doesn't describe the concept)	1 (2%)
	Equivalent to a known definition	9 (8%)	Not structural	8 (18%)
	Useful	14 (13%)	Not useful	2 (5%)
	Minimal	1 (1%)		
Receptiveness	Simple or clear	30 (28%)	Complicated	11 (25%)
	Based on basic concepts	14 (13%)	Based on concepts that are not basic	19 (43%)
	Short	3 (3%)	Long	1 (2%)
	Captures unique features	6 (5%)		
	Elegant	3 (3%)		
	Familiar	2 (2%)		
Figurative	Based on properties of integral parts of a square	1 (1%)	Based on properties of latent parts of a square	2 (5%)
Total		108 (100%)		44 (100%)

Table 2: Arguments for Accepting or Rejecting a Statement as Definition of a Square

By a *mathematics perspective* we refer mainly to arguments in which logical considerations were involved (e.g., evaluating a statement on the grounds of its correctness, namely, on whether it is both a necessary and sufficient condition for a square). Other considerations related to the extent to which the statement is useful in mathematics (e.g., for proving or for classifying examples and non-examples) or to the legitimacy of using definitions that are not in the form of a conditional statement (e.g., procedural). Excerpt 1 in the Prologue is an example of students' employing mathematical considerations in their discussion.

By a *receptiveness perspective* we refer to arguments that focus on the communicative nature of a definition. According to this perspective a statement was evaluated based mainly on its clarity and whether it is comprehensible, within reach to those who deal with it, captures the essence of the concept, and based on accessible concepts (see Excerpt 2).

The last category of arguments, which we call a *figurative perspective*, has to do with the way the participants perceive a geometric object and its different components. According to this perspective, there is a distinction between the parts that seem integral to a geometrical object (such as the sides and angles of a polygon) and those that are often hidden (such as the diagonals of a polygon). In accordance with the figurative perspective, the latent parts are not equally considered integral parts of the object. Arguments of this category focused on the issue of whether it is legitimate to define a figural concept by properties of its latent parts (e.g., congruence of its diagonals). Thus, this perspective is characterized by the reluctance to accept statements that are based on properties of latent parts of a square (see Excerpt 3).

Note that about half the arguments (45%) for accepting a statement were based on mathematical arguments, while there was less mathematical support (25%) for rejecting a statement. On the other hand, receptiveness considerations played a major role both in accepting (54%) as well as in rejecting a statement (70%).

The group discussion (in the second meeting) yielded more insight to students' conceptions regarding what a (good) mathematical definition is. During this discussion the students found out that they don't agree on what conditions a mathematical definition should satisfy. One of the questions that they raised dealt with the correctness of the statement, as in the following two excerpts:

Mike: All the statements can serve as definitions.
Erez: All the statements are correct, there is no doubt about that.
Omer: No, (h) can't, because a definition should be more abstract and (h) is... a...
Erez: (h) is an instruction how to construct a square.
Mike: No way, (h) is too long.
Yoav: (h) constitutes construction instructions, it's not...
Erez: [interrupts] It's a description of how to build a square.
Omer: So, our question is whether we agree to accept it as a definition, [writes while speaking: w-e do-n't ac-ce-pt...] we don't accept statement (h) as definition, because?
Erez: because it's not a definition, it's an instruction regarding how to build.

The next excerpt occurred after reaching a consensus regarding Statement (h). In this part the students are discussing Statement (c):

Yoav:	[refers to Statement (c)] We don't accept it. It doesn't mean that it is not correct. We just don't accept it.
Erez:	It is obviously correct, all the statements are correct, so should we accept them all as a definition?!
Omer:	If it is correct, why can't it be a definition?
Erez:	[turns to Omer] Is statement (h) correct? Is it correct? Is statement (h) correct or isn't it?
Mike:	Statement (h) is correct.
Erez:	[turns to Omer] So why didn't you accept statement (h) as a definition?
Mike:	[turns to Omer] Yes, why didn't you?
Erez:	[turns to Omer] Give yourself an answer for that.
Omer:	Because it isn't abstract.
Erez:	This statement [refers to Statement (c) (see Excerpt 3 in the Prologue);] is also correct, but I am not willing to accept it, in the same way that statement (h) is correct and I was not willing to accept it.

Eventually, Erez succeeded to convince all the others that correctness of a statement is not a sufficient condition for a mathematical definition, and that there are many other considerations that should be involved in the decision process.

Other interesting questions, which were raised and discussed by the group of four included: Is it legitimate to define a concept based on properties of its latent parts (e.g., equal diagonals)? (as in Excerpt 3 in the Prologue); On what kind of concepts can a definition be based? (as in Excerpt 2 in the Prologue); Must a definition be minimal? (as in Excerpt 1 in the Prologue); Should the term (i.e., the concept name) indicate the meaning of the concept?

There were only 2 statements on which there was a unanimous agreement between all four students in their pre and post written responses, as well as in their final group decision. For Statement (d) they all agreed that it is acceptable as a definition, while for Statement (h), they agreed that it was unacceptable. Interestingly, in the latter case they expressed no doubts at all, while in the former case they had a stage in the second session in which they reconsidered their views.

The group discussions proved particularly influential with respect to the views held by the participants regarding the remaining 6 statements. Thus, for each of these statements at least one student changed his standpoint as a result of the interaction with his peers. Interestingly, students continued to think about the issues they had discussed together even after the second session was completed. Consequently, there were 3 occurrences of a student changing the standpoint that he held at the end of the group discussion, and responding differently in the third session (i.e., in the second time the written questionnaire was administered).

CONCLUDING REMARKS

Our findings suggest that the kind of task that was designed and implemented within the study has the potential of creating a rich and stimulating learning environment, in which the learners are motivated to interact meaningfully on their own, taking an active part in genuine mathematical inquiry surrounding different features of a mathematical definition, and engaging in argumentation and justification, in the sense that Yackel (2000) discusses and in the spirit of NCTM (2000). This activity also proved valuable as a research tool aimed at identifying students' conceptions of a mathematical definition and its roles, and tracing the changes in their thoughts as a result of interactions with each other. Presumably, such activity may enable students to develop a view of mathematics as a humanistic discipline in which there is room for various opinions (e.g., Borasi, 1992). In fact, at the beginning of the study the students were convinced that a textbook definition is unquestionable, while at the end they became aware of the arbitrariness of the choice of definition and of their right to question a given definition and suggest alternative ones that are supported by valid considerations.

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BELIEF RESEARCH AND THE ACTUAL AND VIRTUAL COMMUNITIES OF A NOVICE TEACHER'S PRACTICE

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So far, few attempts have been made to develop socially oriented understandings of the relationship between teacher beliefs and classroom practices. Two such attempts are outlined and a third presented, which is based on an empirical study of a novice teacher. This study suggests understanding the belief-practice relationship from the perspective of the multiple actual and virtual communities of a teacher's practice.

Research in mathematics education has recently moved towards a greater focus on the social contexts of the teaching-learning processes (Lerman 2001; Bishop, 2001). This *social turn* in mathematics education (Lerman, 2001) is apparent for instance from the increasing emphasis on the communicative and interactive aspects of learning and from the inclusion of situated perspectives on knowing.

There are, however, fields in which the social emphases are still conspicuously absent. One of them is research on the relationship between teachers' beliefs and the classroom practices. Belief enactment, then, is still considered a highly individual endeavour, and beliefs are often considered an explanatory principle in relation to the classroom interactions (cf. Skott, in press). More specifically, if - from an observer's perspective - there is apparent compatibility between the beliefs espoused in research interviews and the classroom practices, there is little more to explain. If no such compatibility is found, an argument for the lack of impact is made, for instance using highly individualistic and condemning explanations of teacher inconsistency, or capitalising on the methodological and conceptual problems of beliefs, deeming the classroom practices dependent on beliefs residing at other levels of consciousness, than those espoused in research interviews or questionnaires. In this understanding, beliefs are seen as a main determinant and the teacher as the sole agent of the mathematics classroom. It is indeed *the teacher's* practice.

One socially oriented reaction to this is Hoyles' (1992) and Lerman's (2001) claim that beliefs are contextualised. Drawing on the work of Lave and Wenger (1991), Hoyles introduces the notion of *situated beliefs*. In this understanding, situations are co-producers of beliefs, and mismatch between beliefs espoused outside the classroom and the practices within it is only to be expected. Slightly modifying this argument, Lerman claims that although there may be 'family resemblance' between the views of school mathematics expressed in research interviews and those held in the mathematics classroom, they are qualitatively different entities (2001, p. 36). Another socially oriented reaction to the individual emphasis in the larger part of belief research, is to adopt an interactionistic perspective. Skott (2001) points to the classroom interactions as a source of changes in the objects and motives of the teacher's activity. When the intention of facilitating mathematical learning is submerged other motives of the teacher's activity (managing the classroom, building students' self confidence, taking their family backgrounds into consideration, etc.), there may not be compatibility between the teacher's explicit priorities and his/her contributions to the classroom interactions. However, this is neither to be seen as teacher inconsistency (as traditional belief research claims), nor as belief change within the situation at hand (as Hoyles suggests). It is to be perceived as a situation in which the teacher becomes involved in activities not immediately related to mathematics teaching: (s)he is 'playing another game than that of teaching mathematics' (ibid. p. 24).

This paper is based on the premise that the extremely individual perspective often adopted in research on the belief-practice relationship is too limited, and that a *social turn* is needed also in this field. However, the above attempts to include a social perspective appear to need an elaboration. Hoyles' use of the notion of situatedness does not indicate specific ways in which different situations structure beliefs, and the notion seems to be conceived merely in terms of time and setting. Consequently, it becomes an explanatory principle for otherwise inexplicable differences between beliefs and classroom practices, ironically somewhat in the same sense as teacher beliefs have become an explanatory principle in more traditional belief research. Skott, on the other hand, includes activity as a constituting element of situatedness, but does not account for how the objects of the teacher's activity relates to contexts broader than the immediate interaction. More specifically, it is beyond the approach taken how for instance a set of dominant educational priorities in a certain school is brought to the classroom by the pupils and the teacher, and how they become part of the set of mutual expectations that frame the interactions.

The intention of this paper is to suggest an approach to research on belief-practice relationships that takes it seriously that a social perspective on teachers' beliefs is needed, while maintaining that beliefs are significant. I shall refer to a study initially framed within traditional belief research to make the point.

THE STUDY

The study described in this section aimed to contribute to an understanding of the relationship between novice teachers' beliefs and the classroom practices. Larry, a 29-year-old teacher from Copenhagen, was one of 4 teachers for primary and lower secondary school (1-10) selected for the study upon their graduation from college.

Larry had responded to a questionnaire sent to those Danish student teachers in their fourth and final year of pre-service education, who specialised in mathematics. The questionnaire included open and closed items on the students' views of teaching-learning processes in mathematics and their priorities of different aspects of mathematics as a school subject. Also, it asked questions about the students' own educational experiences with mathematics, their reasons for specialising in it at college, and their confidence as beginning mathematics teachers.

11 student teachers were selected for interviews out of 115 respondents to the questionnaire. Larry was interviewed immediately after the holidays following his graduation, i.e. within the first week of his teaching career. A semi-structured,

qualitative approach was used (Kvale, 1996) in which he was invited to elaborate on his priorities in relation to school mathematics and to describe significant educational experiences with the subject. Also, he was asked to comment on three sets of written materials: (a) a grade 5 student, Mathias, proudly presenting a false conjecture about the area and perimeter of rectangles; (b) an experienced teacher describing the different roles of investigative activities for high and low ability students; (c) five 12-17 line transcripts from imaginary interviews with teachers presenting different visions of school mathematics. (I shall return to (a) later).

Larry's mathematics classes were videotaped for 2½ weeks six months after his graduation. The videotapes and transcripts were analysed from the perspective of congruence and conflict with the value judgements and educational priorities Larry expressed in the questionnaire and the interview. How, for instance, do the interactions look, when viewed from the perspective of his proclaimed view of student learning? A special focus was on how he organised and orchestrated classroom communication both in whole class and small group settings. In a final interview, Larry was asked to comment on 5 video clips selected as exemplary for the interactions when viewed from this perspective.

Larry at the time of his graduation

Larry was selected for the study because he seemed strongly committed to teaching, and because his views - as presented in the questionnaire and the interview - reflected significant aspects of current reform initiatives in mathematics education (e.g. NCTM, 2000). These views, he claimed, were mainly the outcome of his pre-service education. He described learning as individual knowledge construction especially related to investigative activities with manipulatives, and he saw teaching as facilitating learning in a variety of ways, not merely as explicating concepts and procedures. However, Larry also emphasised more traditional elements of school mathematics, especially student command over basic skills. This lead to a perceived conflict between the amounts of time needed for investigations and the pressure to cover the traditional syllabus. Further, he considered these two objectives irreconcilable: at any one time you have to do one *or* the other. These priorities - including the perceived conflict between them - was apparent in his response to the questionnaire, but even more so in the first interview. As a teacher, you have

> "to see how your individual students understand - of course you can't *see* that - but you can see how the individual student works and see how they try to move on and see what works for them. [...] Actually, I'd like to allow the students to have more time. Maybe if I can inspire them in different ways, they find out on their own [...], but I also know that we don't always have the time to do that [...] Sometimes you just have to lecture ... all the other stuff isn't possible all the time. [...] maybe 70% of the time they have to work traditionally, and then they may experiment in the last 30%". (The first interview)

Larry was concerned with how to differentiate his teaching according to the students' ability levels, an issue that was important in his pre-service education. His view of the issue was apparent from his reaction to the transcript on Mathias, cf. the previous section. In the transcript, Mathias works with three other students to find the area and perimeter of rooms they have measured and to present their findings to the rest of the class. At the end Mathias exclaims with some excitement: 'Wow, look at this, the longer the perimeter, the larger the area!' Larry was informed that as he did not know Mathias and the class, he was expected only to provide an immediate reaction to the situation. He said he would praise Mathias for the observation, but challenge him to find counter examples. Asked if other students would be involved he said:

> "No, this is Mathias' observation, so it's OK that he continues with it ... it's also a way to differentiate between the students, because he is ready for it, but maybe the others would find it boring [...] If some of the others see what he is doing and are caught by it, then they can continue working on it, but I wouldn't impose it on them". (The first interview)

In summary, Larry claims that student investigations are required but isolates them from the bulk of his instruction dominated by more traditional tasks. This 'island'-approach is combined with a view of differentiated teaching, which individualises student activity and misses significant opportunities for communal learning.

Larry meeting the school

Larry works at Mellemvang, a conservative private school 30 km from the city. To exemplify the school's traditional educational priorities, he mentions that marks are given from grade 1; standardised tests are used frequently, and the results published within the school, so that marks may be compared between students, classes, and teachers; and students are streamed according to ability in grades 8, 9, and 10. These practices, alien to those of most Danish schools, are the result of decisions made by the management. However, they are endorsed by the parents, who are interested in their children's education, but focussed on traditional values like good behaviour and skill acquisition. At times, their interest does become a trying experience.

Already in the first interview, Larry is worried about the extent to which his own educational priorities would clash with those of the school:

> "... somehow it [the management] doesn't fit very well with the ways I think, [...] I have to strike a balance between the expectations of the parents and the school management and the ways I want to work [...]" (The first interview).

Also, Larry considers his colleagues an obstacle to change. He explains that they used the management as an excuse for not taking any initiatives themselves:

> "...several of my colleagues have said, if we've discussed doing things differently: 'Good idea, but you have to remember that at any time you'll be held responsible by the parents and the management, and it's important that you can [...] show that you've been through all that is in the curriculum'." (The first visit to Mellemvang)

Larry is very much aware that he is deemed responsible for the students' learning. At Mellemvang, teaching is a highly individual endeavour:

> "It is the individual teacher who makes all the decisions. There is no teamwork. This means that the experiences I am to gain from older colleagues have to be collected in a hurry during the break, while I'm eating my lunch, and doing a whole lot of other stuff

that needs to be taken care of - so obviously: It isn't much. [...] And I miss that a lot. [...] It would be easier if some of my colleagues would ask 'I'm going to try this out, do you want to join in'. I would say yes immediately. But nobody asks" (The first visit to Mellemvang)

It is apparent from the initial interviews that Larry considered himself in opposition to the dominant priorities of the school. He soon developed an 'I-they' relationship, not an 'I-we' relationship, to the colleagues, the management, and the parents.

Larry meeting the classroom

Larry teaches mathematics in grade 5. He calls it 'my little luxury', as there are only 15 students, there are very few disciplinary and social problems, and most of the students do well in mathematics. The only major difficulty is that the school's competitive assessment practices have lead two girls to attach a self-imposed stigma on their mathematical activity, due to their relatively poor performances.

On the day when the class begins on the new chapter on integers, Larry starts off by correcting the students' homework in a fast and insistent manner in a whole class setting. He then introduces the notion of perfect squares by writing '$2^2 = 2*2 = 4$' on the board, and asks what 3^2 may mean. A student suggests that the result is 6. Larry, turns it down, saying "Now you have multiplied by 2". He writes '$3^2 = 3*3 = 9$', and another student suggests that the two numbers be the same. Larry confirms this and writes the students' homework - a number routine tasks from the textbook - on the board. The students start working on these tasks individually, Larry giving fairly direct instructions to the ones who find it difficult. By the end of the lesson, Larry writes the perfect squares from 1 to 196 on the board and asks for a connection between the numbers. After a few seconds, a boy exclaims: 'Between each number there is an increase of two'. Larry praises him for the observation, and asks, if they think this pattern continues. They claim that it does. Larry agrees and gives an explanation, but one that few students understand. Immediately afterwards, a student interrupts and claims to have found another pattern: 'If you multiply something//two numbers with 5s in the end [as the last digit] the result is always something with a 5'. All of a sudden, other students present other conjectures in a very spontaneous fashion. Larry chooses to close the issue and not pursue the questions any further.

This first lesson on integers is exemplary for the atmosphere in Larry's classroom. First, Larry often dominates whole class discussions in the sense that he uses the textbook to delineate a narrow focus for the students' activity. Second, he is fairly explicit in his support to the students. Third, the students often present conjectures on patterns or relationships, either spontaneously or after being prompted. To sum up, Larry uses fairly traditional modes of interaction with the students in large parts of the lessons, but an atmosphere has evolved that encourages and values student contributions. Larry's response to these contributions is generally to take over the conjecture or observation, both when it concerns a standard result and when this is not the case. The following episode exemplifies the latter situation.

In the next lesson Larry brings in a box of centicubes, i.e. plastic cubes of 1×1×1 centimetres that may be put together and used for instance when teaching place value. The students are to make squares and cubes out of centicubes in order to relate integer concepts of perfect squares and cubes with their geometric representations. Larry suggests that the students build just a skeleton of a corner and its adjacent sides to illustrate a cube (cf. fig. 1). The students begin and soon this happens:

Kasper: I came to think ... I found that when you make one of these, you don't use 27 centicubes [shows a skeleton of a 9×9×9 cube].

Larry: Well how many do you use?

Kasper: You only use 25.

Larry: Because//

Kasper: //even though 3 by 9 is 27 ... because you make like this [takes off two rods of 8 centicubes each]. Now there are 9 centicubes.

Larry: Yes.

Kasper: And then these two, they are 8 each [shows the two rods of 8]. And together ... that is 25.

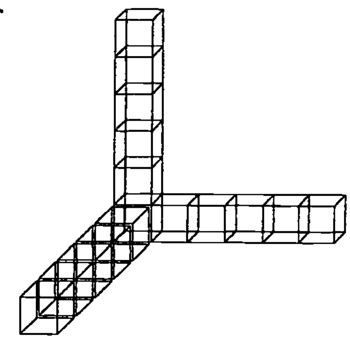

Fig. 1: A 6×6×6 corner of the type Kasper shows

Larry: And that is 25, yes, and that is simply because//

Kasper: [Is trying to remain in control of his proposal] //but Max he thought that the 8×8×8-cube would be 24...

Larry: Yes, well that is because this one [picks up Kasper's cube, and points to the centicube in the corner] //

Kasper: //it counts as three.

Larry: It belongs to - take good care now - it belongs to this side, to this side, and to this one [Larry takes over and continues his explanation].

In this episode, Kasper presents his observation that fewer cubes are needed than expected. Larry praises him for it, but takes over the conjecture, preventing Kasper to remain in control. Commenting on the episode in the interview, Larry explained that he wanted to give other students access to Kasper's observation, but that few of them did understand. He did not consider this a problem and referred to need to differentiate his teaching. Also he did not pursue the observation any further, because he was uncertain where it would take him, and because

> "there are certain things we have to cover in the course of the year, and that's why we spend a lot of time doing textbook tasks. We also have to do stuff like this, but it is obvious that I have to limit it and consider, how much they are to do of one and the other. Obviously I could return to this later, but I am not sure I'd do it. I don't think I would." (The final interview)

The above episodes from Larry's classroom significantly resemble his school mathematical priorities as espoused in the initial interviews and in the questionnaire. The atmosphere invites student conjectures, which may be pursued by the originator, but never by the whole class, cf. Larry's concern for differentiation. In most cases, Larry takes over the students' observations and conjectures and closes the investigative opportunities without involving the students. This latter characteristic of his classroom is a result of his initial intention of ensuring enough time for more traditional activities, but is fuelled by the pressure he is under at Mellemvang to ensure that the students perform well in next test.

DISCUSSION AND CONCLUSIONS

As mentioned earlier, two attempts have been made to describe belief-practice relationships socially. One emphasises how the energising element of the teacher's activity emerges from the interactions with specific groups of students and directs her further contributions to the discourse; the other uses situatedness - in terms of time and setting - as an explanatory principle for discrepancies between beliefs and practice. The study of Larry suggests combining these attempts by using the notion of communities of practice (Wenger 1998).

Wenger claims that developing a practice requires

> "the formation of a community whose members can engage with one another and thus acknowledge each other as participants. As a consequence, practice entails the negotiation of ways of being a person in that context. [...] In this sense, the formation of a community of practice is also the negotiation of identities " (1998, p. 149).

From this perspective, there is no *community of **teaching** practice* at Mellemvang. The teachers' do not jointly relate to the same children, do not discuss ways of handling classrooms in general, do not to challenge the educational priorities of the management, and do not participate in joint efforts to develop or make different use of teaching-learning materials. In short, they do not mutually engage in practices related to facilitating student learning. However, this delineation of the teachers' engagement signifies important aspects of a *community of **teachers'** practice*, albeit in negative terms. Other aspects of this practice are how to behave in the staff room, how to prepare for the school festival, how to put a request to the management or how to prepare for the next PTA-meeting. These negatively and positively defined fields of practice are those in which the teachers 'acknowledge each other as participants', in which they jointly engage, and which constitute their indentities at Mellemvang. At times Larry struggles hard to become part of this community.

However, the community of teachers' practice is not a source of Larry's identity as a facilitator of learning. Repeatedly, he referred to his experiences with mathematics and mathematics teaching and learning at college, both when he was satisfied with his own teaching and when he was not. In spite of the physical and temporal distance between Larry's pre-service education and his teaching activity, the former does constitute a context for the formation of his identity as pertaining to the latter, not only while he was still at college, but also at the beginning of his teaching career. In other terms, when Larry's classroom activity is primarily directed at facilitating learning his participatory framework, the social aspect of his professional identity, is linked to his pre-service education. Specifically, what matters most for the ways in which he invites student contributions, for his use of manipulatives to support their learning and for the ways he deals with problems of student differentiation, is his participation in the educational discourse of his pre-service training. Upon his graduation, then, Larry is related to the virtual and probably fading community of teaching practice established and defined in the course of his pre-service education, the main source of his beliefs.

The moral of this story concerns how the role of teachers' beliefs in relation to practice may be conceived. The need for a contextualisation of beliefs, i.e. for a *social turn* in research on belief-practice relationships, neither requires one to refrain from dealing with beliefs as a set of relatively stable school mathematical priorities, nor does it force one to rely on vague descriptions of a 'family resemblance' between beliefs and practice. I suggest - in line with Lave and Wenger (1992, p. 32 ff.) - to include activity as a constituting element of situatedness, and to acknowledge the simultaneous existence of multiple, possibly conflicting, actual and virtual communities of a teacher's practice. Each of these may play a role when different objects of the teachers' activity emerge in the course of the classroom interaction. From this perspective, the focus of classroom research on teachers' beliefs is not to state congruence or conflict between beliefs and practice, but to disentangle the ways in which - from the teachers' perspective - the multiple communities interact and frame the emergence of different objects of his or her activity.

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DESIGNING TASKS TO EXPLORE DRAGGING WITHIN SOFT CONSTRUCTIONS USING CABRI-GEOMETRE.

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Tasks were designed to study the relationship between students' actions and reasoning in Cabri geometry problems. In task-based interviews with two UK students of differing geometric experience, I recorded the simultaneous words, actions and visual results whose interrelation expressed their problem solving approach. This data was analysed using Arzarello's (1998) classification of dragging modes, and relationship with transitions between inductive and deductive reasoning. This indicated that these students spontaneously used dragging modes related to soft construction and inductive reasoning, sometimes also finding deductive explanations. Implications of these results for future task design include the importance of promoting construction as a means of gaining control of a figure for future enquiry.

AIMS

In this study I design and evaluate tasks that exploit and promote distinctive heuristics identified in the research concerning learning with Cabri-géomètre. Cabri is a dynamic geometry environment characterised by: direct manipulation of screen geometric "drawings", construction tools based on Euclidean geometry, and - its key design feature - the invariance of geometric dependencies under dynamic variation of their initial objects (*dragging*). Any software medium changes not only the accessibility of mathematical objects of study, but also their nature, the way that they are acted upon and how meaning is made of them. We can therefore distinguish between the geometry of the curriculum, and the meanings constructed through Cabri.

A review of case studies of children's work with Cabri illustrates the variation in children's geometric knowledge and in the foci of researchers' interest across different countries (Hölzl, 1996; Arzarello, 1998b; Jahn, 2000 Healy, 2000). The background to my study is the UK curriculum, which traditionally has little in the way of extended problem solving or explicit teaching in deductive geometric reasoning., Given appropriately designed tasks, could such students find that dragging heuristics supported deductive geometric learning?

DRAGGING AS MEDIATOR BETWEEN CONCRETE AND ABSTRACT

The functional uses made of Cabri – the actual sequences of drags and clicks – are of interest in pedagogic research because they are both indicative of the learner's understanding, and formative in making meaning. The computer provides a *window* that enables reconstruction of the learner's conceptions from her actions in a context that is both *exploratory* and *expressive* (Noss et al., 1997). Cabri users recursively exploit the dual exploratory/expressive dialectic by developing heuristics of dragging in which geometrical conceptions, already reorganised by the Cabri medium, are made accessible at a perceptual level to permit further reorganisation (Arzarello, 2000).

Noss et al. (1997) identify a *situated abstraction:* a reorganised mental concept which does not replace a concrete visual representation but simultaneously derives from it and gives meaning to it through actions. Hancock (1995) finds a similar two-way interaction in his notion of *transparency*: a computer modelling tool embodies mathematics in a way that makes it easier to "see" its subjects; but equally the subjects are a background against which the mathematics can be seen. He traces a phenomenological-epistemological progression from *naïve transparency*, as learners use Cabri to express what they already know, to *opacity* as they struggle with the tool and with their geometric understanding, to *co-ordinated transparency* as the two are reconciled. The progression from opacity is the hardest, and of most interest to teaching.

Different modes of dragging

Recent literature has distinguished modes of dragging according to how they are used and their mathematical nature. This emergent classification has been summarised and extended by Arzarello (1998b), who focuses on the interplay of perception, movement and thought, and the distinction between what is taken as given and what is sought.

Wandering dragging: describes the often random dragging that is used when exploring a construction, seeing the dynamic relationships and searching for an interesting feature. In this mode there is only one direction of progress: from visual/ physical to conceptual; and successful use puts a high premium on what students perceive and how they interpret it.

The dragging test: dragging part of a construction to verify that a desired regularity is maintained. This powerful and direct Cabri representation of the mathematical idea of generality is the most clearly related to inductive and deductive geometric reasoning: affording both empirical justification and an sequence of actions that expresses relationships between the initial objects and the conclusion. However, the necessary co-ordination of a set of figures with an appropriate independent variable and a construction sequence leading to an interpretable phenomenon is often a complex task.

Lieu muet *dragging*. Whereas the dragging test starts with a set of conditions on which to verify a hypothesis, *lieu muet* dragging seeks to establish a locus for which the hypothesis is satisfied. Arzarello (1998a) relates this mode of dragging to logical *abduction*: seeing what rule you have the case of. This mode of dragging is popular when students learn with Cabri (Jahn, 2000): it is a heuristic that is suggested by the medium rather than a representation of a traditional geometry heuristic.

Bounded dragging: dragging a point along a perceived trajectory to explore what happens to a feature of interest. The difference from the dragging test is that the trajectory is not actually constructed, so the domain of the dragable point remains implicit and is independent of the history of construction.

Use of dragging within constructions

The dragging test has a natural mathematical status in verifying a conjecture formed as an if-then statement. Students, however, are found to prefer the other forms of dragging, and may rarely perform a drag test without the mediation of other modes. Hölzl (1996) first describes the common heuristic of dropping a construction constraint then varying by dragging until a visual solution is found. Working on a task without simultaneously concentrating on all conditions allows the user to construct representations of the constraints in the most natural way without considering the order of dependence necessary for a linked construction. Healy (2000) identifies similar heuristics in what she calls *soft constructions*. Both Hölzl and Healy show how students' soft methods could have led to complete geometric solutions, but raise the question of how to support students in this final stage of reasoning. The starting point for this study is Healy's finding that students had difficulty in making connections between deductive reasoning and apparently equivalent Cabri construction sequences, and her recommendation for more consideration of soft construction methods in task design.

Such research has been started by Arzarello (1998a, 2000) in the context of students who have had explicit instruction in modes of dragging. He traces their shifts from exploration to explanation, or from *ascending* to *descending* control, and finds that dragging, especially *lieu muet dragging* is a crucial mediator. The actions of clicking and dragging taken as non-verbal language have both a *deictic* and a *generative action* function in triggering and supporting the students' transition to abstract generalisation. Students' verbal reasoning changes from broken oral narrative associated with exploration using wandering dragging, through abductions describing *lieu muet* dragging, to the more sequential linear sentences of canonical mathematical proof, which may still use the vocabulary of Cabri actions. One crucial transition appears to be between seeing a *lieu muet* locus as a dynamic trajectory resulting from dragging to a geometric object that could be constructed in it own right (Jahn, 2000).

In summary, distinctions between the geometry of the curriculum and Cabri geometry become evident in differing approaches to the activities and logic of problem solving. Difficulties such as functional dependence, and the co-ordination of several constraints arise in both geometries, but Cabri can offer modes of action and of expressing meaning that are intermediary in finding a solution. Tasks have been most successful when learners have a clear idea of the type of endpoint required, either in Cabri or in geometrical terms, and have explicitly been taught some Cabri heuristics. The focus of study should be the interplay between the actions and words accompanying a task, which characterises the modes of dragging, and then between these modes and developing control of the problem solving and proving strategies.

METHODOLOGY

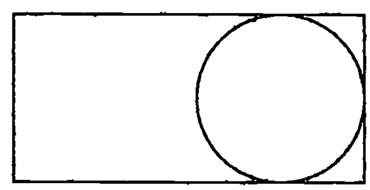
The study includes task-based interviews with two contrasting subjects: a six-year old, Daisy, with no experience of either Cabri or Euclidean geometry, and a nineteen year old, Helen, working on a task developed during a mathematical reasoning course using Cabri. Daisy's 45 minute interview was interactive; in Helen's case I took the role of non-participant observer for most of the hour interview, asking her to type responses into comment boxes onscreen, but finally asked her to talk through her progress and intentions. The data collected were recordings of what was said or written annotated with the simultaneous Cabri actions and gestures, and the screen's appearance. This data was analysed using the classifications of dragging and reasoning modes as described above.

Daisy's task – Circles and Rectangles

The aim of this interview was to investigate designing a Cabri task involving only elementary properties of shape but accessible to different modes of dragging and reasoning. Daisy played with Cabri circles and triangles, and I then gave her a previously constructed rectangle, which she dragged, from tall/thin to short/fat, then aligned horizontally with traditional proportions.

Cathy: If you have a rectangle, can you make a circle that touches all four of its sides?

Daisy constructs a circle, placing the centre near the centre point of the rectangle and pulling out the radius. "No – it would have to be an oval," stops dragging and draws an ellipse in the air. Changes the rectangle to look square, "Now you can..." and fits her circle to it.



"And if I pull it now, it won't be touching ": she extends the square horizontally into a rectangle.

"If we were trying to move it across we would have to move it up too because if we didn't move up it would change into an oval. Say we had a square the sides of the square are all the same size. If we had a rectangle they are not all the same size. If we have to pull it across ... to be a circle it has to be the same length everywhere ... and those were against the edge so if we move it up it's going to go off the top. "

Daisy's initial response is visual – she imagines a curved shape inside a rectangle and recognises that it would be an oval i.e. not a circle. Her focus on the special case of a square appears to be motivated by getting some solution on screen. She then starts to reason about continuity from a square via one-way enlargement, a mode of reasoning afforded by my prior construction of the dragable rectangle. She makes her argument verbal, firstly in transformational terms, describing the effects of hypothetically dragging the circle ("if we were trying to move it .."), and finally by relating the geometric properties of a rectangle, square and circle (" it has to have the same length

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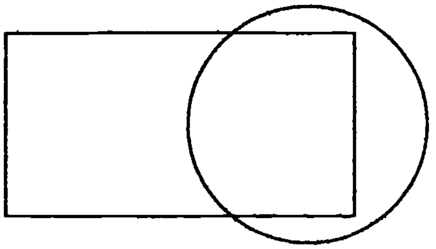
everywhere"). Her use of "Say we had... If we had..." indicates that this is an abstract generalisation, although still phrased in concrete dynamic terms ("go off the top").

Cathy: Can you make a circle that touches all four corners?

Daisy: "No, only if it's a square". She drags back to a square and pulls the circle out to touch all four corners. "It's on the outside now".

She drags out a rectangle and starts to make the circle bigger. Pauses. She moves the centre of the circle fairly haphazardly and reattaches the two right hand corners. She increases the radius again, and again moves the centre to attach the corners. Eventually: "Yes I think you can ... its got to be in the middle."



She drags the centre of the circle onto the middle of the rectangle, adjusts the radius, points and checks: "One, two, three, four corners, yes you can."

Daisy initially intended to use her previous heuristic of arguing from a square to a rectangle to demonstrate impossibility, but the perceptual results of dragging intervene to change her reasoning. She realises that she no longer has to keep the circle inside the horizontal and vertical dimensions of the rectangle. Hancock's description of *opacity* seems apt as Daisy struggles for another way to relate the two shapes. She has lost the *naïve transparency* of her earlier interactions with the geometric objects, and no longer uses any of her knowledge of horizontal and vertical symmetry. Eventually by repeating the actions of varying the centre she establishes a horizontal locus for the centre of the circle - *lieu muet* dragging to meet relaxed constraints - and this suggests the centre of the rectangle. Finally she notices as a geometric phenomenon that it is equidistant from all four corners.

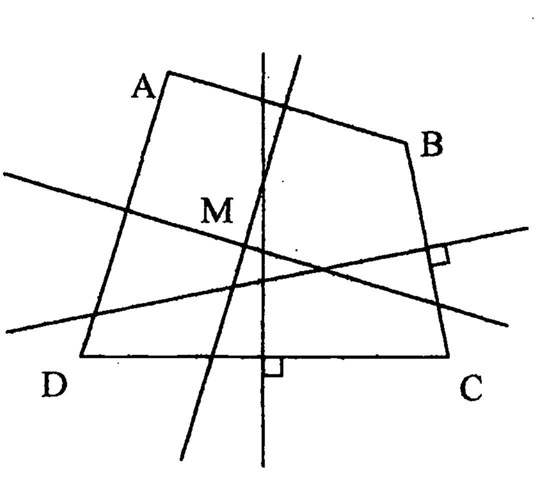
Helen's task - Perpendicular bisectors

Draw a quadrilateral ABCD. Construct the perpendicular bisectors of each side. Label the four points (M, N, P, Q, say) at the intersection of the perpendicular bisectors from adjacent sides. If you move ABCD, what happens to the inner quadrilateral MNPQ?

1. Investigate the relationship between the internal angle at A and the internal angle at M.

2. For what types of external quadrilateral is MNPQ a parallelogram?

3. Find a quadrilateral ABCD that is similar to its inner quadrilateral MNPQ (i.e. an enlargement/reduction).



This construction is also described in Arzarello (2000) with students exploring the degenerate case, but this version starts with angle relationships which are familiar to UK students, and has structured goals that leave open further enquiry. Helen started the task by *wandering* dragging, noticing that one inner point was invariant when she dragged A. She used the causal metaphor of dragging to describe this geometrical relationship: changing A changes only two outer sides, which changes only two inner sides, so only three of the inner points. She stated that the inner angles at A and M add to 180.

Helen's next step was intuitive (and she couldn't later explain it)- she constructed the diagonals of the inner and outer quadrilaterals, and it helped her see a way to make MNPQ a parallelogram: "It would be a parallelogram on the outside because the intersections of the diagonals must be at the same point. That's the only way; if I move A away, the centres don't coincide." Her reasoning accompanying this *lieu muet* dragging seems to be purely inductive; arising from the observation that as she dragged ABCD to move the intersections of each pair of diagonals together, both MNPQ and ABCD became parallelograms.

Helen again moved on without explanation to the third question. Like many of the students she did not use the geometric meaning of "similar", but also tried to give ABCD and MNPQ the same orientation: "Can't find exactly similar – must be a rotation, because to get the same angles … need a square or a rectangle .. and then it disappears." She spent some time explaining to herself why this degeneracy happened. In doing so she went back to the relationship of the angles at A and M, and wrote a justification of this invoking the two right angles in the quadrilateral with diagonal AM. Although the degenerate case was a deviation from the set questions, considering the role of the perpendicular bisectors helped to give insight into the whole task. Helen then wrote a deductive explanation of her second result: "For MNPQ to be a parallelogram, its opposite sides must be parallel. The sides of MNPQ are the perpendicular bisectors of the sides of ABCD so for them to be parallel, the sides of ABCD must be parallel". Helen then tried to manipulate the quadrilateral ABCD while keeping it a parallelogram, an example of *bounded* dragging, but she had great difficulty co-ordinating this and finally gave up. While her perceptual aim was clear in this episode, she was less clear in her purpose within the task: "I'm looking for it to be the same … similar".

Helen's task gives an example of progress towards a deductive argument in a more challenging geometric context. Although the task was structured to build up a sequence of reasoning, she actually synthesised all her reasoning at the end of the sequence of exploratory tasks. Unlike Daisy, Helen's developing *reasoning* was not immediately associated with her concurrent screen actions, which led her simply to visual answers, and so it is less clear whether dragging played a specific mediating role or that she simply had inspiration. However it was noticeable that she built up a body of empirical knowledge through the soft dragging modes before attempting any explanations at all. The construction of the diagonals, for example, was a temporary stage of her reasoning and after making her explanation she later deleted them. It was nevertheless important because it gave her *wandering dragging* a clear visual endpoint, and allowed her to progress to finding the *lieu muet* that achieved this condition.

CONCLUSIONS

My aim was to design tasks that asked for elements of deductive geometric reasoning but were accessible to students with little experience of such problems. I followed the suggestions of Healy (2000) that task design in this context should give equal consideration to soft construction methods, and of Arzarello (2000) that the focus of observation should be the interplay between the physical/visual aspects of different modes of dragging and the language of argument. Both Daisy and Helen spontaneously made purposeful use of the *wandering, lieu muet*, and *bounded* dragging modes that are characteristic of soft constructions. These modes were always associated with the purpose of exploring the next situation, finding the next answer and making progress with the task. Equally it is clear that they accompanied inductive reasoning and making abductions, i.e. establishing and expressing necessary conditions for a solution. In Daisy's case these lead to a contradiction and she finds a deductive argument, but Helen's explanation remains at the stage of finding what is necessary. This is what Arzarello characterises as gaining *ascending* control of the problem. Helen does start to move towards *descending* control, trying to express the reverse argument by varying a general parallelogram ABCD to see what happens. This suggests that her reasoning at the end of her investigation is still closely related to her perception and expression in Cabri – she does not look to see that the logical connectives are all equivalences, but starts another dynamic experiment.

In the final stages of their task both subjects carry out painfully slow dragging as they try to co-ordinate their physical actions with a number of perceptual constraints, and the experience is frustrating. Much of this could be avoided by using construction e.g. fixing the radial point as a rectangle vertex, or ABCD as a parallelogram, but Helen and Daisy did not attempt to do so. It seems that the most important heuristic is finding the methods that allow us to control the variation that we want to perceive. The soft dragging methods above have the benefit of being very accessible, allowing the user at the same time to explore the task and start to make sense of it, but they also have their limitations when the user has more sophisticated experiments in mind. It is perhaps at this stage that Arzarello's students benefited from their teaching in Cabri techniques, and were ready to start again with a fresh construction that would enable them to vary what was now of interest. *Lieu muet* dragging was found to phenomenalise a locus of conditions, thereby mediating the transition from ascending to descending control, but the next stage would be to make explicit this locus as the starting point for experiments and the given for deductive reasoning. This study suggests that it may be helpful not just to exploit the potential of soft dragging methods in making sense of and establishing a context for a theorem, but also to include support for reorganising the problem and expressing the context as a construction which can be dragged in the desired way.

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TEACHERS' BELIEFS ABOUT GENDER DIFFERENCES IN MATHEMATICS: *'GIRLS OR BOYS?'* SCALE

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The focus of the paper is to examine teachers' beliefs about the differences of boys and girls (aged 13-15 years) as learners of mathematics. For this purpose a rating scale was developed with a new type of response format: 'Girls or Boys?'. A sample of teachers of mathematics (N=204) were asked to classify a list of characteristics as being more frequent among girls or among boys in their mathematics classes. Factor analysis revealed six main dimensions indicating beliefs in gender differences and one secondary factor suggesting a core belief dimension 'Mathematics as a gendered domain'. The teachers' scores on the six belief dimensions revealed the following highly believed gender differences: girls avoid using intelligence and boys attain most of the teacher attention.

INTRODUCTION

Since the early 1970's there has been an increasing research activity in the field of gender and mathematics education especially in the English-speaking Western nations (Leder, 1992; Leder, Forgasz & Solar, 1996; Lubienski & Bowen, 2000). Research on affect and mathematics has focused on the affective responses of students rather than those of teachers (McLeod, 1994). Identifying classroom behaviours that influence gender differences in learning, and patterns in how students choose to study mathematics has been difficult (Fennema, 1995). Teachers' knowledge of, and beliefs about, mathematics have been studied from the perspective of cognitive science, but this perspective is less used in studies concerned with gender (Fennema & Hart, 1994). Studies that deal with the mental processes of teachers might give insight into why teachers interact with boys and girls the way they do. The body of literature available regarding gender issues related to teachers' beliefs does not give conclusive evidence that teachers believe that mathematics is more appropriate for males than females (Fennema, 1990; Li, 1999).

The acquisition of beliefs or their modification is a major issue in the activity of teaching. As Green (1971, p.42) further points out, beliefs are always gathered as parts of a belief system. Therefore it is more important to explore the nature of sets of beliefs or belief systems than to examine the nature of a belief alone.

The problem is, that in the educational literature and among researchers, there is no common definition for the concept "belief", nor a clear distinction between beliefs, conceptions and knowledge (Pajares, 1992; Pehkonen & Furinghetti, 2001). Thompson (1992) distinguishes knowledge from belief systems on the basis of the possibility of objective evaluation of validity.

The distinction between the terms "belief" and "attitude" is also problematic. The instruments to asses or measure beliefs are those used in the research of attitudes. Whereas attitude refers to a person's favorable or unfavorable evaluation of an object, beliefs represent the information he has about the object. One distinction that has been repeatedly proposed is the theory of three components of attitudes; affect, cognition, and conation, and beliefs belonging to the category of cognition. Fishbein and Ajzen (1975) reserved the term "attitude" for one of the categories, namely affect. Beliefs are on the border between cognition and affect. The latter, affect, is often more or less emphasised in a teacher's belief concerning gender and especially gender and mathematics.

METHOD

The aim of the study was to examine teachers' beliefs about the differences of boys and girls as learners of mathematics. A new scale was developed for this study. There are two research questions:

Do Finnish mathematics teachers believe in gender differences in mathematics?

What are the components of teachers' beliefs about gender differences in pupil's behaviours in mathematics learning situations?

Instrument

The study was a survey, a belief inventory, and quantitative methods of analysis were employed. The instrument of the study was a belief questionnaire with 55 structured items and some open questions, in which the teachers were asked to characterize, in their own words, boys and girls as mathematics learners. In this paper I approach the belief components, which I call belief dimensions, by factor analysing the answers to the structured items. It was anticipated that starting with a large number of items would be necessary in order to find the most relevant ones. The 55 statements of student characteristics were grouped under the following headings: A) Girls and boys in math-class, B) Girls' and boys' attitudes, C) Girls' and boys' abilities and cognitive skills, D) Upper secondary mathematics choices and career choices, and E) The situation of gender equity in school. This grouping of the statements was only intended to support the teacher in answering.

The topics of the items were mostly found in the literature about gender issues. Some of the items were adopted and modified from earlier studies (e.g. Maccoby & Jacklin, 1974; Leder, 1992; Brusselmans-Dehairs & Henry, 1994). Some items arouse from my experience as a mathematics teacher and reflections upon my beliefs and gender dependent teaching practices. The topics were discussed with a couple of experienced in-service mathematics teachers. The instrument was examined by some ten mathematics teacher educators and researchers. The first version of the questionnaire was tested with a group of fifteen pre-service secondary mathematics teachers. All the groups mentioned above gave valuable feedback and helped in developing the items.

The *Girls or Boys?* scale

According to Fishbein and Ajzen (1975, p.12) with respect any object–attribute association, people may differ in their belief strength, in terms of the perceived likelihood that the object has the attribute in question:

> we recommend that "belief strength," or more simply, "belief," be measured by a procedure which places the subject along a dimension of subjective probability involving an object and some related attribute. (Fishbein & Ajzen 1975, p.12)

The three commonly mentioned rating scales for verbalized attitudes are Thurstone, Likert and Osgood scales (see e.g. Nunnally 1967; Henerson, Morris & Fitz-Gibbon 1978 p. 82; Keats 1997). The Likert method of constructing and applying attitude scales is by far the most common. The items have alternatives e.g. "strongly agree, agree, undecided, disagree, strongly disagree", which are scored 2,1,0,-1, and –2, respectively. The alternative "undecided" has been shown to lead to anomalous results and should possibly not be used (Keats 1997).

Forgasz, Leder, and Gardner (1996) have reexamined the widely used Fennema-Sherman *Mathematics as a male domain* scale and they found evidence that several items in the scale may no longer be valid. For example it is nowadays not obvious what can be referred from disagreement with the item: "Girls can do just as well as boys in mathematics." Are girls doing better or are girls doing worse?

In this study it was aimed to develop a new scale, and to take into account the remarks mentioned above. A Likert-type rating scale with a response format suitable for comparisons and measurements of differences was constructed. The statements in the questionnaire were of the type: "*X finds mathematics difficult.*" For each statement, the respondent had to select the subject X out of the following five alternatives:

G usually a girl

g a girl more often than a boy

± a girl as often as a boy

b a boy more often than a girl

B usually a boy

The principle of this *Girls or Boys?* scale is that each item counts for two Likert-type items a) and b) which are opposite to each other and in sum-scales either of them is reversed (Table 1). The scores for the *Girls or Boys?* items are counted as the average of the two scores a and b* of the corresponding Likert items (score b* is a reversed one): G = -2, g = -1, ± = 0, b = 1, and B = 2.

When a *Girls or Boys?* item needs to be reversed for sum-scales, it means that the Likert item to be reversed is item a) instead of item b). For certain this *Girls or Boys?* scale does make only an ordered scale, but like Likert scales it will be used in this study as an interval scale.

Table 1. Two agreement-items make together one *Girls or Boys*? item: an example.

Girls or Boys?	Likert item **a)**		Likert item **b)**	a b* average		
X finds mathematics difficult	X, who finds mathematics difficult, is **a boy**	a	X, who finds mathematics difficult, is **a girl**	(b)	b*	a+b* / 2
usually a girl	strongly disagree	-2	strongly agree	(2)	-2	-2
a girl more often	disagree	-1	agree	(1)	-1	-1
no difference	disagree (or undecided)	-1	disagree (or undecided)	(-1)	1	0
a boy more often	agree	1	disagree	(-1)	1	1
usually a boy	strongly agree	2	strongly disagree	(-2)	2	2

*reversed score

The scale of this study meets the requirements of summative models. In the summative scaling of people respect to psychological traits it is assumed only that individual items are monotonically related to underlying traits and that a summation of item scores is approximately linearly related to the trait (Nunnally 1967, p. 604). The response format was based on probabilistic interpretations: it was assumed that e.g. the answer 'usually a girl' means that the respondent assigns a probability of about 90 % (i.e. from 80 % to 100 %) to X being a girl and respectively, a probability about 10 % to X being a boy. The percentages in Table 2 are only hypothethical and used in order to justify reversing and summing up the item scores. The scores (from –2 to 2) are linearly related to the middle of the probability intervals (10 %, 30, % ...).

Table 2. Probability interpretations of response categories.

X finds mathematics difficult

Probabilities		Girls or Boys?	Score
X is a boy	X is a girl	X is	
0-20 %	80-100 %	usually a girl	-2
20-40 %	60-80 %	girls more often	-1
40-60 %	40-60 %	no difference	0
60-80 %	20-40 %	boys more often	1
80-100 %	0-20 %	X is usually a boy	2

Participants and design of the study

The study was a survey. The test participants were Finnish mathematics teachers from a sample of 150 randomly chosen schools for grades 7-9 (13-15 year olds). In each school one female and one male mathematics teacher, if available, were asked to answer to a questionnaire. This was carried out in February 2000. The response rate was approximately 69%. One year later, in May 2001, ten of the respondents were interviewed.

Implementation

The goal of the factor analysis was to detect structure in teachers' beliefs about gender differences in mathematics. In the choice of the analysis method, principal component analysis vs. classical factor analysis was considered. In the former it is assumed that all variability in an item variable should be used in the analysis. In the latter only the variability in an item that it has in common with the other items is used, and it is assumed that the remaining variance of an item is its unique variance (Harman 1976, p.15). Furthermore, the theoretical background of gender beliefs did not suggest the latent factors to be uncorrelated, which directed the choice to an oblique rather than to an orthogonal reference system.

I started with the principal component analysis and then compared it to a classical factor analysis in which I extracted the results by the principal axis method with oblique rotation. Further I factor-analyzed the data with hierarchical principal axis method to divide the variability in the items orthogonaly into that due from shared or common variance (secondary factors) and unique variance due to the clusters of similar item variables.

It was not aimed to use all the items of the questionnaire but to choose the most relevant ones for the belief structure. The amount of items was reduced based on low communality (<0,35) in principal components and classical factor analysis (both methods yielded similar results). After three reiterations 31 items were left for the final analysis. Cattell's scree test and Kaiser criterion (Harman 1976, p.163) were used to determine the number of factors that best describe the data. The former supported five to seven factors and the latter nine factors. The six factor solution was chosen since it appeared to be very interpretable. Moreover the seven and eight factor models would not have raised markedly the accountability except on only one of the item variables. The six factors accounted for 47 % of the total variance.

RESULTS AND DISCUSSION

Both the principal components model and the hierarchical principal axis model with oblique factors represented similar "clusters" of item variables. Each of the obtained six factors determined a sum-scale of the items that loaded highest on that factor. These new variables, called belief dimensions, each representing one component in the belief structure, were taken to measure the direction and intensity of the beliefs about gender differences.

A positive value on a belief dimension indicated that the teacher associated the characteristics of the dimension to a boy more often than to a girl. A negative value indicated that a girl more often was mentioned having the characteristics. Value 0 was the score for no difference between boys and girls. A value 0 or near 0 was also obtained if the respondent gave contradictory answers e.g. "a girl more often" to *X are capable of higher mathematical thinking*" and "a boy more often" to *There are more mathematically talented among X*.

The mean score was negative for *Avoid using intelligence* -0.59 (sd 0.37) indicating a belief in a trait typical for girls. A positive mean score, indicating a feature addressed more often to a boy than to a girl, was found for *Teacher attention* 0.53 (sd 0.49), *Expectations of success* 0.44 (sd 0.37), and *Talent* 0.22 (sd 0.37). Smaller average differences were found on the dimensions *Lack of equity* 0.09 (sd 0.37) and *Work-orientation* –0.12 (sd 0.41). All these means differed statistically from the 'no difference' value zero.

The hierarchical principal axis analysis gave one secondary factor. All item variables of the primary factors *Avoid using intelligence* and *Talent* and one variable of the factor *Expectations of success* loaded on this secondary factor. Of these the item *"X is innately mathematically more talented"* had the highest loading. These results can be interpreted to reflect a core belief dimension *"Mathematics as a gendered domain"*. These three central belief dimensions are represented in Table 3.

Table 3. Examples of belief dimensions and their items.

Belief dimension	Number of Items	Cronbach's alpha
Avoid using intelligence	7	0.75
X's success in mathematics is more due to painstaking practice than to understanding. *X's success in mathematics is based on the use of intelligence and power of deduction. X leans on rote learning and does not even try to understand. X is better at routine tasks than at problem solving. *X can solve unfamiliar tasks. *X can solve by reasoning. *X can solve spatial problems.		
Talent	4	0.71
X passes the extended mathematics course more easily. X is innately mathematically more talented. X is capable of higher mathematical thinking. There are more mathematically talented among X.		
Teacher attention	5	0.65
I have to ask X to behave himself/herself during lesson. X interrupts unduly. *I should interact more often with X. *X is a silent hard worker. X constantly asks for teacher's help.		

*items marked with an asterisk were reversed

The validity of the scales refer to the validity of reported verbalized beliefs, which may not correlate highly with the behaviour pertaining to the belief. The validity is limited to what individuals know about their beliefs and are willing to relate. After a year had passed the reliability of the empirical data was examined by interviewing ten of the respondent. The persistence of the item responses was about 95 %. Also the interpretations of the item contents were confirmed in the interviews.

The instrument developed, the questionnaire with a new answering scale, seemed to be feasible and practical in measuring beliefs about gender differences. According to the interviews it was answered without much effort and maybe frankly as well. It still remains open how sufficient the verbal expressions alone are as evidence of beliefs.

The responding rate (69%) supported representativeness. The results and findings discussed in this paper can be generalized to include the wider population of all mathematics teachers in lower secondary schools in Finland.

The most emergent result of this study was the teachers' belief in girls employing inferior cognitive skills. The results of the factor analysis did not show any general belief factor that would affect all kinds of beliefs measured by the items. Nevertheless the results suggested a core belief dimension "Mathematics as a gendered domain". Is this a "primary belief" as Green (1971, p.44) defines i.e. a belief for which a person can give no further reason?

The results of this study are alarming if these beliefs of mathematics teachers - boys use their brain and girls are just hard workers - reinforce and sustain these differences.

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USING EYE-TRACKING TECHNOLOGY TO DETERMINE THE BEST USE OF VIDEO WITH PROSPECTIVE AND PRACTICING TEACHERS

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We employed eye-tracking technology and individual interviews to investigate the differences among prospective elementary school teachers, practicing teachers, and mathematics educators as they observed and analyzed a videotape of a mathematics teaching episode involving a researcher and a fifth-grade student thinking about fractions. Results indicate that, relatively speaking, mathematics educators attend more closely to the student, teachers attend more closely to the interviewer, and prospective teachers attend more to the mathematical content than do the mathematics educators and teachers. We discuss these differences in terms of an expert-novice continuum and conclude by discussing implications for video use in teacher development and enhancement.

Videotapes are widely used in the preparation of prospective elementary school teachers and in the professional development of practicing teachers because they can provide rich contexts for considering issues of learning and teaching. Although widely used, there is little research to guide mathematics educators as they determine effective ways to incorporate the use of videotapes in teacher development and enhancement. At a more basic level, we know little about that to which people attend while viewing videotapes, or how one's background and training affect the sophistication with which one interprets videotapes. The purpose of this study was to address these issues by investigating differences among three groups; prospective elementary school teachers, practicing teachers, and mathematics educators, as they viewed a mathematics teaching episode involving a researcher and a fifth-grade student. We employed a methodology whereby reactions of participants were gauged though analysis of point-of-gaze and level of attention, measured and recorded with sophisticated eye-tracking technology. Additional data was collected through interviews of the participants. The study was part of a larger project aimed at helping prospective elementary teachers better understand the depth of knowledge necessary for teaching elementary school mathematics, and was intended to inform the development of early field experiences that would meet this goal.

Early studies contrasting expert and novice reasoning (e.g., Larkin, McDermott, P. Simon, & A. Simon, 1980) have indicated that experts are more likely than novices to focus on relevant problem features that lead them to qualitatively better conclusions. Experts have access to a large body of organized knowledge that can be quickly accessed and used to guide problem interpretation and solution. In a major study of the qualities exhibited by experts, Dreyfus and Dreyfus (1986) outlined five stages of development of expertise: Novices are relatively inflexible in their reasoning, advanced beginners have experiences that allow them to build cases that complement their book knowledge; the competent performer can articulate goals and find means to achieve them; the proficient performer recognizes similarities between different situations and are more holistic in their approach to finding solutions; and experts share this holistic approach by responding effortlessly and appropriately to the situation encountered. Livingston and Borko (1990) noted that teaching mathematics has characteristics of expertise found in other domains, that is, expert and novice mathematics teachers differ in their perceptions, information processing, knowledge structures, and decision making. Indeed, in their own comparison of review lessons of expert and novice teachers, they found that explanations of expert teachers were conceptually linked and less procedural whereas teachers spent time coming to understand the content themselves.

The eye-tracking technology used in our study was developed by Marshall (1998). The information on eye movement is coupled with documentation of effortful cognitive activity that accompanies enlargement or dilation of the pupil. A large body of research has shown that "the pupil dilates on presentation of cognitive or affective stimuli; pupil dilation occurs with effortful information processing in many areas;...(and) the degree of dilation varies by individual and task (p. 2)".

METHOD

Four prospective elementary teachers, four experienced elementary teachers, and two mathematics educators (with doctorates) participated in the study. Each viewed a videotape of a fifth-grader being interviewed by one of our project researchers (R). While watching, each was fitted with equipment used to track point of gaze and level of cognitive activity. Each individual then was taken to another room where she viewed the video again, this time with the white dot tracking her eye movements. During this second viewing, a researcher from our project, pausing frequently, interviewed the participant regarding her reactions to the videotape.

The videotape selected for this experiment was a 22-minute interview of a fifth grade girl who was said to be of "average mathematical ability." The girl was poised, pleasant, and cooperative, and appeared to be quite comfortable with the videotaping process. The interview was taped in a studio that allowed a double screen to be shown on the final version of the video. The top of the screen showed the interviewer on the left, the girl on the right; the bottom half of the screen, called the "work area", allowed simultaneous viewing of all paper work and manipulative use. The interview had three distinct parts. In the first part, Terry's fraction knowledge was assessed. She was asked to circle the larger of pairs of fractions or to place an equal sign between two fractions if they were equal. Her knowledge of fractions was very weak. In the second part of the interview, R gave an explicit, procedurally oriented lesson on changing mixed numbers to improper fractions and back again. Terry learned the procedure but could not apply it later in the interview. The third part of the interview was a conceptually based lesson during which R used Pattern Blocks[1] to begin to develop Terry's understanding of fractions.

Data collected. In addition to a videotape showing point-of-gaze for each participant, the eye-tracking staff provided us with (a) gaze traces of each individual during a 2-minute interval; (b) graphs indicating the percentage of time spent by each of the three groups looking at the interviewer, at the child being interviewed, and at the work area; and (c) estimates of the cognitive activity over 30-second intervals of each individual as she watched the video. To make sense of this data, we then divided the entire video into 30 second intervals, documented what was happening in the interview during each interval, and charted the level of cognitive functioning for each individual during that 30 second interval. We of course also had the transcribed interviews from the second viewing of the video.

RESULTS OF INTERVIEWS

During interviews we asked the same set of questions of each participant and they were encouraged to say anything about the video at any time.

Initial Reactions to the Videotape. The two mathematics educators both made initial comments on the structure of the interview, on expectations, and on technological aspects of the videotape. The teachers were very aware of the conceptual problems children have with fractions. They tried to understand Terry's thinking, but sometimes found it difficult to do so. All teachers commented on Terry's confidence and poise. T2 thought it was frightening that she was so confident in her misconceptions, "She spoke like a very confident child and she's obviously had much success in her schooling. You could just tell by the way she articulated her responses to R's requests." The prospective teachers had few comments to make before reviewing the video tape.

Reactions to Assessment Section. The mathematics educators were not surprised with Terry's poor performance, but they realized that her knowledge of fractions was very weak for a fifth grader. From their own experience, they thought that their own students (preservice teachers) would find some of the questions unfair. "(My preservice teachers) often pick up on the nuances of the interviewer's questions, and say, "That was a trick question" or "She didn't understand" and "It's not the student's fault". For the most part, teachers were not surprised by Terry's responses to the assessment questions. They thought Terry should know more about fractions than she did, but thought the errors she made were typical. The prospective teachers appeared to be at least somewhat surprised at how weak Terry's knowledge of fractions was, but they were careful not to fault her. One said that "It looked like she had not learned it yet, but considering if she had not learned that, she was doing pretty good" (although she was not doing well at all).

Reactions to Procedural Lesson. ME2 thought that the interview demonstrated the chasm between what teachers think they are teaching and what is actually being learned. ME1 found value in the fact that the interview showed how easily Terry picked up on the procedures without a clue as to what they meant, and that she held on to her misconceptions. Teachers reflected on the kind of teaching Terry had probably experienced. All commented on her lack of sense-making. One said "That it's probably the kind of teaching that she's had, what she's comfortable with." The teachers realized that Terry needed something concrete to help her understand fractions as quantities. Interviews of the prospective teachers indicated that they had particular ways of doing things that made it harder for them to understand Terry's thinking. For example, PT3 said, "I'm trying to figure out why she would choose one-half because I'm so used to, well, I was converting it to decimals, like three-tenths was point three and one-half was point five." There was also a tendency on the part of these four to excuse Terry for her weak knowledge base.

Reactions to Conceptual Lesson. ME2 recognized that Terry's knowledge was still very fragile at the end of the lesson, and thought Terry was beginning to feel some conflict. "In fact, she called it *I'm pretending whether I know fractions or whether I don't.* That is how she handles the conflict. Obviously she's recognizing that her old way failed her." ME1 did not think this lesson helped Terry see any connection between the blocks here and the procedures from the earlier lesson. Both educators also made comments about the use of manipulatives. "When I think about using the Pattern Blocks like that, I'm also aware of the ... I'm not convinced about what she really understands about fractions in terms of relating them to a whole other than a hexagon."

Teachers were dismayed that even after her work with pattern blocks, Terry still responded that 1/2 plus 1/2 was less than one. "She wasn't convinced even though they had done all that work (with blocks). Then when prompted to push forward, to visualize the pieces, she's got something to grab onto but that one experience wasn't enough to undo the damage." They could see that she was struggling to accommodate the new and contradictory information she was being exposed to. T2 said "She has all her misconceptions now and she's really struggling because R's really shifted her paradigm a little bit but she's still not there yet. She's really in disequilibrium." The teachers picked up on small things as telling. For example, T4 noticed that Terry never used the word 'equal' even though R did, and even though equality was a key concept in understanding fractions.

When asked for comments on the second lesson, the prospective teachers had little to say. One said "I think the blocks helped her to visualize and really understand." She later said that "I think it was good that you allowed her to see things first and you didn't contradict her and tell her 'No that was wrong' and let her see it for herself."

Would it be helpful to show this video to preservice teachers? Both mathematics educators thought it would be valuable for preservice teachers, but they were cautious in talking about how it could be used. ME1 said "It is long. It's something that might be done over two weeks. . . For me, it raises some issues. It would allow me to talk about the whole variety of issues as well as the questioning." The four teachers also felt that the videotape should be shared with prospective elementary teachers of mathematics. From T1 we heard, "Absolutely. I think it would be a great way to help them come to believe that there is a reason to explore concepts deeply before procedures," and from T3: "They can have an idea as to how concept development happens and why that concrete part is so important." Interestingly, T2 would show it to prospective teachers, but not to teachers in her own district because they would simply blame the teachers Terry had had. All four thought that the most important segment to show was the teaching with manipulatives. Prospective teachers who had been exposed to the use of manipulatives in their content class for teachers thought that the video should be shown to their peers. PT4's statement was similar to those of PT2 and PT3: "I think it is a different approach to fractions and I happen to like it. I think it's very cool. I would show the last part definitely.

Comparing the two lessons. Both mathematics educators found value in both the procedural and the conceptually oriented lesson. ME1 found it interesting that "She doesn't seem uncomfortable or threatened by either lesson. That was a pretty mature kid, really. It was pretty evident that she was punting, but it seemed to me that she was able to keep a presence of mind, like 'Am I going to be able to pull this one over on him? Maybe he won't notice.'" Teachers were well aware of the difference in the manner in which Terry responded in the two lessons. "In the second segment (the procedural lesson), she didn't seem to connect the numbers to any conceptual knowledge. She was able to see R's procedure and use it. But I don't think that she connected it to anything at all. In the third segment, again, she's a bright girl; she caught on quickly. But she would need a lot of time with the manipulatives to really cement her knowledge and to realize that some of the things she had thought before weren't true." When the prospective teachers were asked about the differences in the two teaching sequences, their answers showed that they saw differences, but in a very vague ways. PT3 said "(The last lesson) was more conceptual, I guess, visual. The other one was more math, as long as you knew how to do your math." They were asked which parts should be shown to prospective teachers. "I would show the last part definitely. I think the last part took a different approach because most everyone knows how to find the improper fractions."

WHAT WE LEARNED FROM THE EYETRACKING DATA

The point-of-gaze information we have on a selected 2 minute segment confirms that the prospective teachers spent up to 75% of their time looking at the work area where the work was being shown. All four PTs looked at the work area (and sometimes at R or Terry) during almost every 5-second interval of this time period. These prospective teachers appeared to spend a good deal of their time mentally figuring out the answers to the problems given to Terry so that they would know if she was doing them correctly. "I think during this time I was probably trying to answer the questions in my head . . . and trying to see if it would match hers" (PT3). We conjecture that part of the reason for this focus on the work area was that these individuals needed to first work out each problem given to Terry for themselves, and that they were not always able to do this quickly. The mathematics educators, on the other hand, spent about half the time looking at the work area, and practically no time looking at R. They were most interested in Terry and what she was doing. Teachers spent about 60% of their time looking at the work area and less time looking at R than did the prospective teachers.

The three groups had significantly different levels of cognitive functioning ($F(2,43) = 106.05$), and each group was significantly different from the other two. We were surprised to find that the mathematics educators had the lowest cognitive workload, followed by the prospective teachers, with the teachers registering the highest of the three groups. Upon reflection, and after analysing interview protocols, we speculate that the mathematics educators had the lowest cognitive workload because they found very little of surprise in the interview. Both had undertaken many such interviews themselves and viewed many interviews similar to this one. The teachers, on the other hand, have little opportunity to see one-on-one interviews with children. Also, they had developed expectations built on their own experiences. Each of the teachers has a strong interest in mathematics learning, and thus found this video interview interesting, causing them to reflect on what they were seeing at the same time they were viewing the interview. The prospective teachers lacked knowledge needed to interpret what was going on in the interview. They did, however, appear to attend to the problems being given to Terry, working them out, sometimes too late, so that they could try to make sense of her responses.

SUMMARY AND DISCUSSION

The university mathematics educators acted in ways one would expect experts to act. They responded effortlessly to the interview situation, as was evident in the cognitive workload graphs of each individual and in the composite graphs of the two individuals. Their level of cognitive functioning remained in the low range throughout. They agreed that the procedural lesson should confirm, for any viewer, the ineffectiveness of this type of instruction. Both were aware of the limitations of the Pattern Blocks as an instructional aid. Both commented on the interviewer's skill at questioning. Both were cautious about using the videotape with prospective teachers, commenting on the need to show pieces over a period of time and plan for appropriate discussion. In our discussions of this data, we realized that having the interviewer on the screen may have been an unnecessary distraction to others, who spent considerably more time looking at R.

The cognitive workload for the teachers was in the moderate to high range for the most part; higher than that of the professional mathematics educators. The teachers knew what an average fifth-grader should know about fractions, and this knowledge played out in the interviews. They recognized Terry's answers as typical of a student with poor understanding of fractions, and tried to understand the underlying misconceptions Terry had, knowing that those misconceptions would need to be overcome for Terry to progress. But they expected the work with the Pattern Blocks to be more effective than it was, not realizing that the lesson was too brief to have lasting effects. We considered the teachers to be experts, but at a less advanced level than the professional mathematics educators. The teachers' access to their own knowledge of the mathematics involved in R's questions, their interest in Terry's responses (as compared to the greater interest of the prospective teachers in the problems posed rather than in responses), and their ability to understand and interpret the various parts of the interview were qualitatively different from those of the prospective teachers.

The prospective teachers did express some surprise at Terry's lack of understanding of fractions, but thought she had probably not learned about fractions yet. They had their own ways of solving the problems posed to Terry, and consequently found her answers hard to follow. The reactions of the prospective teachers were, in most respects, quite different from those of the teachers. They could accurately be classified as novices. As in the Livingstone and Borko study (1990), the prospective teachers had different knowledge structures and processed information differently. They spent a high percentage of the time looking at the work area, trying to understand the mathematical content of the questions rather than focusing on the verbal and nonverbal behavior of Terry.

What lessons did we learn from this study? We recognize that the teachers reactions to the video interview were heavily influenced by their many years of experience in the classroom, working with students like Terry. They had knowledge of the mathematics appropriate for this grade level, and they had expectations that helped them evaluate Terry's mathematical knowledge. However, the data from this study led us to believe that there are experiences we can provide for prospective teachers that can help to refine their knowledge of what children know, change their expectations of what children should be able to do, and offer them new ways to observe and make sense of the children's responses to problems posed to them. This belief has guided our design of experiences that can have these effects. For example, we have prospective teachers who have just completed a highly-structured early field experience during which they, in pairs, interview individual elementary school students, then discuss, as a class, what they have learned. With another set of prospective teachers, we offered a seminar during which the prospective teachers viewed and discussed a carefully selected and sequenced set of videos of interviews with students. We believe that such experiences can change the knowledge structures of prospective teachers, the manner in which they process and interpret student answers, and make decisions about what next steps should be taken. They can move, in the words of Saber et al. (1991), from being novices to being advanced beginners, and thus will be more likely to succeed in teaching, and to take less time becoming experts. The major implication of what we learned from this study for our work with prospective elementary school teachers was that careful attention must be paid to how videos are used with prospective teachers. For them to benefit from viewing a video in the manner intended, particular care must be given to the selection of the video and to the preparation for the viewing. But simply solving the mathematics problems beforehand may not be sufficient preparation for viewing a videotape if the children's solutions in the video do not model their thinking; prospective teachers need to anticipate the kinds of solutions children may provide. We also must pay attention to the tools used in a video, so that, for example, before showing a video of a child using a hundreds chart, it is necessary to acquaint them with the hundreds chart and even help them consider ways the chart might be used.

Observing a video twice is often helpful; the first time watching all the way through without considering any guiding questions and the second time watching with particular questions in mind. A purpose for watching videos needs to be understood, especially when if videos are assigned for homework. For example, in preparation for conducting an interview with a child, we often gave as an assignment watching a video of a child solving the same problem that they was posed to the child during the interview. In this case, we ask the prospective teachers to pay attention to particular aspects of the video such as the interviewer's prompts or wait-time or the children's use of language. Another advantage of observing video for homework is that the prospective teachers can rewatch a video as they need, which some do. Thus, the eye-tracking study has guided us in further project activities aimed at helping prospective teachers gain more expertise about children's thinking.

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A CASE STUDY OF A UNIVERSITY STUDENT'S WORK ANALYSED AT THREE DIFFERENT LEVELS

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An on-going study of a long-term one-to-one work with a future mathematics teacher will be presented. The framework for a threefold analysis of data is suggested. The first level of the analysis of data yielded several settings which will be described by some variables (e.g. didactic contract), the second level of analysis concentrates on different situations within the settings and the third level of analysis investigates communication patterns. The first results of our analyses are briefly given. Finally, further research is proposed.

In 1998, we started a research project focused on the building of internal mathematical structures [1]. A series of semi-structured interviews was conducted with several future mathematics teachers who volunteered to be part of the research. One of them, Molly, who was in her first year of study then, was very excited about the mathematical topic she was working on – so called restricted arithmetic RA [2]. She studied it at home and often came back to the experimenter (the author of this paper) with new suggestions. Very soon the previously formal interviews transformed into a qualitatively different setting in which the research purposes grew less and less important, while the teaching-learning purposes became prominent. Molly continued investigating RA with growing autonomy, formulated hypotheses, definitions, tentative theorems, meeting regularly with the experimenter to discuss her work. Her 'co-operation' with the experimenter spanned all five years of her study and culminated in her writing a two-hundred-page diploma thesis on the topic.

What contributed to Molly's enthusiasm and endurance? Why did she do so much additional work in her free time? Why did she feel such a need to explore RA as much as possible? Why did she go on with the work while the other students were content with several interviews and stopped? What teaching-learning settings did she go through during her interaction with the experimenter and how can they be characterised? We do not know the answers to these questions yet but we have tried to find a methodology of how they might be found. This will be the focus of this paper.

THEORETICAL BACKGROUND

At the university level, many types of teaching-learning situations can be determined, some similar to those in elementary or secondary schools, others specific to university teaching: tutorials, lectures, seminars, individual instruction, demonstration, class discussion, home study, office hours, etc., among non-standard teaching-learning situations we have, for instance, scientific debate (Alibert, Thomas, 1991), and using constructive, interactive methods involving computers and co-operative learning (Leron, Dubinsky, 1995). Some attention has been paid to individual types of situations, e.g. tutorials (Nardi, 1996, Jaworski, 2001), lectures (Boero, Dapueto, Parenti, 1996), however, the author does not know of any research which focuses specifically on situations at the university level and determine their invariants and specifics.

We will use a qualitative research paradigm and our considerations will be grounded in the theory of constructivism and its basic tenets that knowledge cannot be transmitted but must be constructed by the learner (von Glasersfeld, ed., 1996) and that it is necessary to create contexts which stimulate creativity (Hejný, Kurina, 2001).

METHODOLOGY

As stated above, the object of our research will be our work with Molly. For the present purposes, we will focus mainly on interactional and emotional matters, neglecting the cognitive point of view (i.e. what knowledge is being learned) which will be the topic of subsequent studies.

The data consist of tape recordings of interviews, protocols, Molly's work from the interviews, the experimenter's field notes, Molly's solutions to mathematical problems, subsequent versions of Molly's 'mathematical' text (mathematical description of RA), her concept map of RA and the protocol of her description of it, the protocol and recording of Molly's teaching experiment and subsequent versions of her analysis of the experiment, and finally subsequent versions of her diploma thesis. The framework for the interpretation of data we want to propose here is based on a threefold analysis.

First level of analysis

First, seven settings S1-S7 [3] will be distinguished in the course of our work with Molly and characterised by variables. For our considerations at this level of analysis, we will borrow a general term of the *definition of situation* from interactionism and a more specific term of *didactic contract* (Brousseau, 1997).

Every situation can be viewed as a communication situation in which its participants play some role. Each participant defines the situation on the basis of his/her prior experience with similar situations. Their definitions of situation come to the fore when they are not compatible and when, for instance, some participants' expectations are violated. When defining a situation, a person assesses (McHugh, 1968) the theme and its elaboration during the interaction, how individual parts of the interaction fit the theme, if the instance in the interaction is typical or likely, if it is given by some previous event, etc.

It is not easy to describe briefly the term didactic contract introduced by Brousseau. Bodin and Capponi (1996) give a concise explanation: Didactic contract "refers to the system of reciprocal expectations held by teachers and students, in other words all the rules, mostly *implicit*, which determine the part for which each of them, teacher and students respectively, is responsible for handling in the teacher-student relationship".

Similarly to the *definition of situation*, the *didactic contract* is being constantly redefined by the participants on the basis of their interaction.

Second level of analysis

Next, within these settings different situations will be identified. Here, we will draw on the theory of didactic situations (Brousseau, 1997): situations of action (students first attempt to solve a problem), situations of communication (students communicate the results of their work), situations of validation (the results are justified), and situations of institutionalisation (results are summarised, the 'official' terminology is used).

Third level of analysis

Last, the analysis of interaction will be made with the aim to identify communication patterns in individual situations. For the analysis of interactions between a student and the experimenter, we will use the framework presented in Dreyfus, Hershkowitz, Schwarz (in press) and used for the analysis of pair interactions. They distinguish six types of statements which establish and maintain the flow of the conversation: control statements (proposals, plans), elaborations (what is done to continue or develop an idea), explanations (comments on the results of actions), queries (which put in question previous utterance(s)), agreement (or concession) and attention. The last type of utterances only show that the person pays attention to the actions of others.

In the interactionist research, several patterns of interaction in the classroom have been identified, e.g. recitation pattern, funnel pattern and focusing pattern (Wood, 1998), repetition pattern (Cestari, 1998), various thematic patterns of interaction (Voigt, 1995, cited in Sierpinska, Lerman, 1996, p. 854). Dreyfus, Hershkowitz and Schwarz, (in press) identified four patterns of interaction which can lead to abstraction: guidance/self-explanation, symmetric argumentation, asynchronic collaboration and collaboration in parallel. Brousseau's Topaze effect and Jourdain effect (Sierpinska, Lerman, 1996) can also be seen as interactional patterns. It is important to note that the various forms of interaction "are not always consciously recognised by students, or even by teachers" (Ellerton, Clarkson, 1996).

ANALYSES AND THEIR FIRST RESULTS

The paper presents an on-going study. In this section, the process of analysing the data will be briefly presented and some preliminary results discussed.

The settings presented below can be defined by the didactical variables, and also by the psychological and sociological variables (Sierpinska, 2000), i.e. the personal characteristic of the teacher/experimenter, and those of the student. Molly has always been a diligent student. Her mathematical ability is above average (in comparison with her fellow students), her attitude towards mathematics (and German language which is the second subject she will be teaching) has always been positive. She is a conscientious worker. She often underestimates her abilities. She is communicative and willing to speak about her thoughts and feelings.

First level of analysis – identification and description of settings

So far seven settings S1–S7 which we feel are different in at least some characteristics have been identified in the course of our work with Molly. Using the comparative analysis of the settings, the following variables have been determined so far for the description of the seven settings: MR – Molly's role, MG – Molly's overall goal, ER – experimenter's role, EG – experimenter's goal, MDS – Molly's definition of situation (part of didactic contract), EDS – experimenter's definition of situation, E – emotions.

One setting will be described in more detail, the others only listed and briefly characterised. The names of the settings are only tentative. In S1-S3 and S7 there are two participants – Molly and experimenter, in the others Molly alone.

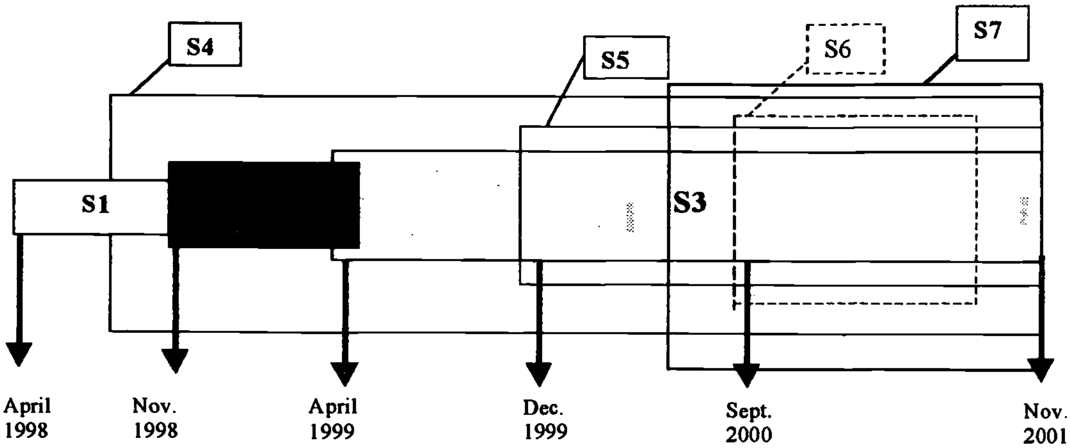
<u>S1 – semi-structured interview</u>

<u>MR:</u> pupil
<u>MG:</u> to try her best to solve the tasks
<u>ER:</u> expert
<u>EG:</u> to get data for the research on structuring mathematical knowledge; to motivate Molly for independent work; not to teach anything
<u>MDS:</u> expects to be told what to do; to be presented with some mathematical problems; maybe to be taught something
<u>EDS:</u> expects Molly to be a problem solver; expects Molly to communicate about her ideas and thinking; expects to be an observer
<u>E:</u> Molly is a bit apprehensive about the non-standard situation (being recorded), she thinks of the situation as similar to an oral examination; gradually she is more and more relaxed. Her worries whether she will be able to solve the problems diminish when the experimenter keeps encouraging her and saying that the situation is non-standard and nobody is expected to understand it straight away. Molly is more and more attracted by the mathematical tasks.

Notes: The expectations of both participants are to a certain extent violated. Molly expects more help from the experimenter. During the first session, Molly redefines the situation and starts to behave in a more independent way.

The other settings are: <u>S2 – teaching interview</u> (unlike in S1, the experimenter has didactic intentions and behave more like a teacher, rather than an observer), <u>S3 – discussion</u> (Molly continues investigating RA, the experimenter is a discussion partner, sometimes a teacher), <u>S4 – home study</u> (Molly is engaged in studying on her own), <u>S5 – writing a mathematical text</u> (Molly acts as a 'mathematician', she was asked to write in a concise way everything she had learnt so far about RA for a journal), <u>S6 – Molly's teaching experiment</u> (Molly acts in the role of a teacher and researcher, she prepares, carries out and analyses her own teaching experiment), <u>S7 – spontaneous discussions</u> (while S3 are sessions planned in advance and Molly knows that she will be expected to speak and explain work she has done at home, S7 is more like impromptu 'office hours' with Molly coming to ask a question or make an observation).

Setting S4 does not exist by itself. It is combined with S1, S2, S3, similarly S5 is combined with S3 and S7. The complex situation is best illuminated by the following figure which shows the time sequence of our work with Molly.



Molly's role has changed during the course of our co-operation. From the role of a pupil (expected to be taught) she got into the role of an independent problem-solver, autonomous learner, 'mathematician' at times, teacher and teacher researcher. We consider the process of change between the individual roles an important aspect of the whole process and will analyse it in more detail in the future.

Second level of analysis - situations

Every effort was made to suggest the "natural order" of the growth of scientific knowledge (Sierpinska, 2000) from action, formulation and validation to institutionalisation during our work with Molly. Moreover, whenever Molly validated anything it was her own conjecture or the conjecture conceived in co-operation with the experimenter, not given as a ready-made product.

Sierpinska (2000) also claims that at the university level, the last type of situation is the most frequent while the others do not appear at all or only in a degenerate form. She gives an example of proofs of theorems which are not students' own conjectures as a degenerate situation of validation; an example of a degenerate form of a situation of formulation when a teacher punishes an incorrect formulation of a definition; a degenerate form of a situation of action when a teacher gives students hints and suggestions before they tackle the problem. The first analyses of our data showed that despite the experimenter's effort on the contrary, some degenerate actions can be identified especially during the first interviews with Molly. They will be the subject of further analyses.

Third level of analysis – communication patterns

In this section, we will restrict ourselves on S1 – semi-structured interview [4]. The communication patterns can be briefly characterised as follows (E stands for the experimenter, M for Molly):

- a lot of *encouragement* from E ("Well done");
- virtually no *rhetorical questions* (i.e. utterances which only pretend to be questions);
- E's queries (the result of her need for clarification) have often the form of *repetition* of M's utterances;
- a good deal of M's *thinking aloud* (we do not classify it as explanation because it does not explain the results of action but rather M speaks as she thinks);
- there were some instances of E's utterances which we classified as *pointers* which were specific to the experimental situation (e.g. E repeats what M is writing in order to record it on the tape recording and thus make the analysis of the data later easier; or utterances like "Write the solution to a new problem on a different sheet", "Write in a different pen", "Say aloud what you are doing on the calculator"; some of them can be classified as control statements);
- E's *prompting* (e.g. "So...?"); a lot of E's *control statements* (E sets the agenda most of the time) – often it has the form of: E's control statement, E's question ("Do you understand?"), M's consent or M's *reformulation* of the task in her own words;
- E's *admittance* that she herself does not know the answer or that M found something which E did not see before (e.g. "I do not know if there is a pattern.", "Well done! I did not see this before.");
- demonstrations of *rapport* between E and M (e.g. jokes, laughs);
- M's *questions* are of a technical nature ("Shall I do it here or at home?", "Shall I finish the problem?"), she never asks for strategies; towards S2, the typical pattern is: E's question, M's explanation, E's brief consent (often non-verbal), M's explanation, E's consent, etc.

PEDAGOGICAL IMPLICATIONS

On the one hand there are indisputable advantages of the long term individual work with a student (it has a very strong and long-term influence on the individual, a student <u>experiences</u> 'doing' mathematics and constructivist learning and thus it is one of the ways how to get constructivist teaching to schools [5]). On the other hand, we cannot omit the fact that there was no co-operation with other students and that it is time consuming for both participants. Moreover, it is very difficult, if not impossible to reproduce a didactic situation (see obsolescence of didactic situations, Brousseau, 1997), "the practices observed are the products of multiple interactions whose elements are not always the same, even for any one teacher " (Bodin, Capponi, 1996).

PROPOSED FURTHER RESEARCH

Within the given framework of analysis, we have so far completely omitted at least three aspects which strongly contributed to the success of our work with Molly. First, the <u>mathematics</u> Molly was working on – RA, which is a rich context suitable for a student's independent investigation. Second, the <u>personality of the experimenter</u> and her motivation. Mathematics, no matter how interesting, does not work by itself in the above way. And third, the experimenter previously investigated RA herself and was herself very excited about it. We hypothesise that if a teacher works on the same topic as his/her students, it is an important source of a positive motivational climate. All three aspects will be the focus of our analysis in the future.

It was stated at the beginning that the present study is an on-going one. Next, the analysis outlined above will be consolidated and applied to data, the cognitive aspect will be put under scrutiny and last but not least, Molly's long-term progress as a practising teacher will be followed. We hypothesise that Molly's experience with her long-term work will influence her teaching strategies towards the constructivist approaches.

In order to answer the questions stated at the beginning of the paper, a comparative analysis of interviews with other students must be made, too. Why did Molly go on with her work? We are sure that the reason must be a combination of several factors.

NOTES

1. First results were given in Stehlíková, Jirotková, 2001.

2. For its description see Stehlíková, Jirotková, 2001.

3. We will need a more convenient term 'situation' for other purposes.

4. We presume that further analysis will show that the communication patterns differ in individual settings. However, only S1 and S2 were tape recorded, in S3 and S7 our considerations can only be based on written materials, the experimenter's memory and Molly's self-reflection.

5. In view with Becker and Selter (1996), we strongly believe that simply hearing or reading about new teaching approaches is not enough. The student – future mathematics teacher – has to experience the new approaches in a way we want him/her to use at school.

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The research was supported by grant GACR 406/02/0829.

TEACHERS DIFFER IN THEIR EFFECTIVENESS

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This article reports an unanticipated result from a large scale project that sought to examine ways of improving numeracy learning. We found that there were marked differences in achievement between classes, irrespective of geographic or socioeconomic variables, and we argue here that these differences are attributable to the teachers. In other words, teachers influence the learning of their students and there are marked differences between teachers in their effectiveness.

INTRODUCTION

One sometimes gets a sense that the community at large, and even some education bureaucrats, believe that anyone can teach, or that teachers are born, or that new teachers can learn to teach just by watching others. The assumptions behind those beliefs seem to be that effective learning is the result of curriculum policy statements, resource provision, school leadership, or other factors amenable to central policy decisions. We argue the opposite: children from similar backgrounds have markedly different experiences at school, and these differences are attributable to their teachers.

Of course teachers have responsibility for more than the learning of the students, but this learning is clearly fundamental to the teacher's role, and the following argument is based on an assumption that a measure of the effectiveness of teachers is the growth in learning of their students.

There is a tradition that connects learning to teaching effectiveness. For example, Darling-Hammond (1997), in proposing a case for teacher education, summarised research on data from 900 school districts in Texas that found that 40% of the measured variance in student achievement across grades 1 to 11 was due to teacher expertise. She argued that, even after controlling for socio-economic status, the large differences in achievement between "black and white were almost entirely accounted for by differences in the qualifications of their teachers" p 8.

Such research can be unconvincing because of the way that achievement is sometimes measured using multiple choice items on tests of low level skills. The following argument draws on results from a large scale numeracy project to examine this issue using carefully collected, ample and representative data, based on a research informed view of a particular domain of mathematics, measurement of length. We conclude that teachers make a difference to the learning of the children in this domain, and we suggest that such differences may be evident in other domains as well.

THE ROLE OF THE TEACHER IN IMPROVING LEARNING

The object of educational reform is improved learning of the children. Clearly the instruments of any improvement in learning are teachers. Yet we suspect that there is an assumption that teachers are more or less equally skilled, or that reforms are to some extent independent of the teacher. Certainly the structure of terms and conditions of employment, and of professional development programs themselves, do not seem to make assumptions about differences between teachers.

One possible explanation for this lack of acknowledgement of the importance of the teacher is connected to the way that learning is conceptualised. It is common for reforms to be informed by a social constructivist perspective that was summarised by Ernest (1994) as recognising that knowing is active, "individual and personal, and that it is based on previously constructed knowledge" (p. 2), and that the knowledge is not fixed, rather it is socially negotiated, and is sought and expressed through language. It is possible to infer that learning is independent of the teaching. Yet Ernest saw the teacher as central listing among the pedagogical implications that teachers need to be sensitive to learners' previous constructions, to seek to identify errors and misconceptions, to foster metacognitive techniques, and to acknowledge social contexts of learners and content. Likewise, Cobb and McClain (1999) argued that teachers should have a clear impression of the direction that the learning of the individuals and the class will take. They proposed that the teacher should form for the class an "instructional sequence (that) takes the form of a conjectured learning trajectory that culminates with the mathematical ideas that constitute our overall instructional intent" (p. 24).

Another possible explanation for the underemphasis of the teachers' role might be a belief that the primary determinant of students' learning is their family or cultural background. Tschannen-Moran, Hoy and Hoy (1998) reviewed research on teacher efficacy that contradicted this. Teacher efficacy refers to the extent that teachers believe that they can influence how well students learn, independent of their motivation, their background, their prior learning or other factors. Tschannen-Moran et al. cite a range of studies that connect high scores of efficacy by the teachers with higher achievement of their students.

In other words, teachers have an active role in promoting learning, and this is connected to their beliefs about the nature of learning and the nature of learners, but this active role is not necessarily acknowledged in educational policy and practice.

SEEKING A MEASURE OF THE EFFECTIVENESS OF TEACHERS: SOME RESULTS FROM THE ENRP

To explore the impact of teachers on student learning, we draw on results from the Early Numeracy Research Project[1] (ENRP) that investigated mathematics teaching and learning in the first three years of schooling, involving teachers and children in

35 project ("trial") schools and 35 control ("reference") schools (for details see Clarke, 2000; Sullivan, Clarke, Cheeseman, & Mulligan, 2001). One source of data was a one-to-one interview over a 30 to 40 minute period with every student at the beginning and end of the school year (Feb/March and November respectively). Interviews were conducted by the classroom teachers, who were trained in all aspects of interviewing and recording. The processes for assuring reliability of scoring and coding are outlined in Rowley and Horne (2000). The data reported in this paper were collected in the year 2000, the second year of the project.

The data from this project arise from intensive interviews with large numbers of children, with trained interviewers, and experienced coders, with double data entry, and using a framework for learning based on interpretation of research. We argue that these data provide a reliable measure of learning, and a further perspective on previous research on the ways that teachers influence student learning.

Learning of Length Measurement

The data, on which the argument proposed below relies, is based on growth in students' learning of aspects of length, one of nine domains on which data were sought within the ENRP. Growth points for the key aspects of learning length were proposed, and assessment tasks administered individually were developed, after consideration of some common findings in length research.

Much research on learning of measurement is influenced by work of Piaget and his colleagues, who identified stages of development in coming to understand measurement concepts such as conservation, the idea of a unit, transitivity, and iteration, with focus on development of cognitive abilities within individuals. Nunes, Light, and Mason (1993) present an alternative Vygotskian perspective in which competence is not fixed within individuals but cultural representations play a mediating role in development of understandings. In measurement, such cultural representations are often conventional rather than conceptual. In developing the assessment framework, we were concerned that Piagetian stages did not lead to clear teaching and assessment guidelines (see Carpenter, 1976; Kamii & Clarke, 1997).

Through the framework and interview for the domains of Length, the ENRP gives emphasis first to whether children show awareness of length through tasks that facilitate use of language. Measurement by direct comparison is then considered. We intended that both the growth point and the task prompting direct comparison be inclusive and suggestive of conservation as well.

The next growth point requires the use of a non standard but consistent unit to quantify a length measurement, including some of the requirements for iteration. The final two points relate to the use of standard units and their application. We did not include transitivity directly in the framework.

The six proposed growth points for length are as follows:

No apparent awareness of the attribute of length and its descriptive language. (*Not apparent*)
Awareness of the attribute of length and use of descriptive language (*Awareness of the attribute*)
Compares, orders, & matches objects by length (*Comparing lengths*)
Uses uniform units appropriately, assigning number and unit to the measure (*Quantifying lengths*)
Uses standard units for estimating and measuring length, with accuracy (*Using standard units*)
Can solve a range of problems involving key concepts of length (*Applying*).

These were developed as a conjectured sequence. It was assumed that students will follow different pathways in their learning, but nevertheless the intention was to describe the learning trajectory of the majority of students. To illustrate the style of the assessment, the following were the first two tasks posed (with italics indicating what to do, and the normal text indicating what to say):

The string and the stick
Drop the string and the skewer onto the table.
A) by just looking (without touching), which is longer: the string or the stick?
B) how could you check? *(touching is fine now)*
C) so, . . . , which is longer?
The straw and the paper clips
Get the straw and show the child the eight (5 cm) paper clips.
Here are some paper clips. Here is a straw.
A) measure how long the straw is with the paper clips. . . . *(if child hesitates)* use the paper clips to measure the straw.
B) what did you find? *(no prompting)*
If correct number is given (e.g., 4), but no units, ask "4 what?"

The interviewers proceeded through the interview in order, but moved directly to the next domain (e.g., Mass) if the student answered a question incorrectly. A coding rubric was used to score the students' responses.

Entry Level Students and the Length Growth Points

The issues of interest here are how the students responded to the tasks, and how they improved over the year. Table 1 presents the percentage of entry level students (commencing at age 5) in project schools rated at each growth point in both March and November, near the start and end of the school year respectively.

Table 1: Entry Level Students (%) on Length Growth Points (March and November)

	March (n=1488)	November (n=1484)
Not apparent	18	3
Awareness of attribute	6	1
Comparing lengths	62	50
Quantifying lengths	13	43
Using standard units	0	3
Applying	0	0

The students are spread over the growth points, even at the start of their schooling. It is suspected that these differences are due to home or other specific experiences prior to school, recognising that language would be a contributing factor in some cases.

Given the low number of students who have reached the growth point, *Awareness of the Attribute,* the first two points are combined for the subsequent discussion, and termed *Not yet comparing*. To examine further the improvement over the year, individual responses in March are compared with those in November. Table 2 presents these as numbers of students.

Table 2: Comparisons of Student Growth from Respective Growth Points (n=1369)

	November		
March	Not yet comparing	Comparing lengths	Quantifying lengths or beyond
Not yet comparing	37	187	92
Comparing lengths	11	453	400
Quantifying lengths		40	149

Under 4% of the students went backwards, 47% stayed at the same growth point and 50% improved. The argument here is about growth, and to explore this further, we examine the largest set of students whose rating did not improve over the year.

The 453 students who were at *Comparing lengths* in March who were still at that point in November represent 33% of this group. These "*Comparing length*" students have not had whatever experiences may have been necessary to grow to *Quantifying length.* To explore further the nature of the development and potential of such students, some other aspects of their learning of mathematics were explored. Table 3, for example, compares their responses on the *Counting* domain in November to those of the group overall (n=1369).

Table 3: The *Comparing length* students (%) on *Counting* in November

Growth point descriptor	The *Comparing Length* students (n=453)	The whole group (n=1369)
Not yet able to count to 20	7	6
Can say number sequence to 20	4	4
Can count a collection of 20 objects	62	57
Counts forward & back by 1s from x	13	15
Can count from 0 by 2, 5, 10	13	17
Can count from x by 2, 5, 10	1	1

The *Comparing length* students have a similar profile on the *Counting* domain to the group overall. This suggests that there is little direct connection between *Counting* and *Comparing length*, at least at this level. For example, over one quarter of these *Comparing length* students were able to count forward and backwards from various starting points. Only 11% were not able to count a collection of 20 teddies. That these students could not progress to *Quantifying length* over the year is not due to inability to count.

Similarly, in the other domains of *Place Value*, *Addition and Subtraction*, *Multiplication and Division*, and *Time*, the profile of these *Comparing length* students was similar to the group overall. This suggests that to develop to the next growth point the students need particular experiences associated with the learning of length, rather than general mathematical development. In other words, the mathematics experiences that these students have had has not resulted in across the board changes but rather that improvements are a result of domain specific experiences.

To explore this further, we examined the improvement by classroom on the *Length* domain for Entry level students. As Table 4 shows, there are marked differences between the classrooms. Note that the table presents results for Entry level students who are in the 54 classrooms with only entry level students, and the results of students who are in multi-age classes are not included in the analysis.

Table 4: Percentage of students per class in Entry level classes who improved (n=54)

Percentages of students per classroom improving	Number of Entry level classes
Up to 20% of the students per classroom improving on Length	5
More than 20 up to 40% of the students per classroom improving on Length	22
More than 40 up to 60% of the students per classroom improving on Length	19
More than 60 up to 80% of the students per classroom improving on Length	6
More than 80 up to 100% of the students per classroom improving on Length	2

To be more specific, there was a class in which 24 out of 27 children improved, another where 21 out of 24 improved, whereas there was a class where 2 out of 25 improved, and others where 3 out of 24, 5 out of 28, and 2 out of 21 improved respectively. These are very noticeable differences and are unlikely to have occurred due to chance factors. An examination of the schools and other factors indicated that neither being effective nor being less effective teachers in terms of promoting improvement was dependent on school size, socio-economic community, student language background, or years of experience of the teacher.

This suggests it is possible for students at this level to move through the growth points but it is teacher dependent. A similar analysis of results of Grade 2 teachers on Length showed similar although less striking differences. This result is also similar to that reported by Sullivan, Clarke, Cheeseman and Mulligan (2001) with respect to differences in effectiveness in the teaching of multiplication and division at Grade 2.

Characteristics of teachers who made a difference

To explore the nature of these differences between the results of the classrooms, six teachers who had higher proportions of students improving on length in each of the first two years of the project were interviewed. Among other items, each of the teachers was asked to describe an example of an activity they used in their teaching of Length. A particular feature of the teachers was that they seemed able to describe rich experiences for the students and the purpose of those experiences. For example, the following is an extract from the response of one of the effective teachers:

> Well my favourite one … "The Long Red Scarf" and I based the series of lessons on that covering the different growth points … I had a whole lot of teddies that the children made scarves for and we compared lengths and then we actually taught them how to measure

using blocks and bears and things and we measured our scarves and … language because a lot of our children do not have the language so even simple things like longer and shorter …

I started with reading the book and we talked about scarves, then I brought in scarves and we put them on the floor in the middle of the big circle and I spread them out haphazardly and I said well "which scarf here is the longest?" and the children said "have a guess at anything sort of thing" and I have got a very bright boy who said "no you can't do it like that, you have to line them up" and he lined them up and then someone else said "no but you've got to match them at the end" so they matched them at the end … so we got lots of language.

The second lesson … we actually said we were going to make scarves so I gave them paper. … they had to make a scarf long enough to go around their teddy …they proceeded to make their scarves and some of them even decorated them and then we actually compared lengths again so that was all one lesson, they came back and they put their scarves down and we talked about who had the longest and some of them had very long scarves because they had bigger teddies and some had short scarves.

It seems that this teacher had a clear vision of the experiences that were needed, was able to engage the students, and was not deterred from such a rich experience by the unfamiliarity of some of her students with the language demands. Other effective teachers gave similarly rich examples. Another common theme, was that these teachers were prepared to probe the thinking and understanding of the children. For example, in response to the same prompt another teacher said:

I always try and make sure that there's a sharing of findings at the end of each session …and I always ask the kids "how did you obtain such a result?" or "how did you get your answer?". So there's that constant reflection … "if you measured your foot and you found out that it was 22" … also I try and challenge the kids by asking them "if we've all measured our feet and we've all measured the length of a basketball court and we've all got a different response, why is this?" so I'm actually getting them to think a little bit beyond just obtaining a result.

In other words, the teachers seemed to be aware of characteristics of rich experiences and how to use those experiences to extend the students' thinking.

CONCLUSION

In a detailed study of a small aspect of a large project, an unanticipated result emerged. Teachers who were given extensive professional development including teaching advice some of which focussed on teaching of measurement, participated in structured planning teams, and released from teaching to interview all of their students, differed substantially in the extent to which their students improved in defined growth points in Length. That teachers make a difference is supported by other studies. The data presented here suggested that the differences between the most effective and least effective teachers are substantial. Effective teachers seemed able to articulate focused, developmentally appropriate and engaging activities for their students, and engage them actively in interrogating those experiences.

Before substantial policy decisions are taken more research is required. If further research confirms these results then there may be some important and somewhat challenging implications. If there are teachers who are substantially more effective than others, presumably steps could be taken to find out who those teachers are and acknowledge their effectiveness in some way. If there are teachers who are substantially less effective than others, we could seek to find out who those teachers are, and find ways to assist them to understand the impact of their teaching, and to examine strategies that might assist them to become more effective. We fully recognise the dangers of simplistic solutions to such issues, but suspect that it is possible to devise strategies that avoid undesirable side effects.

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[1] The *Early Numeracy Research Project* (ENRP) is a collaborative venture between Australian Catholic University, Monash University, the Victorian Department of Employment, Education and Training, the Catholic Education Office (Melbourne), and the Association of Independent Schools Victoria, and is directed by Doug Clarke, and the team includes Barbara Clarke, Jill Cheeseman, Ann Gervasoni, Donna Gronn, Marj Horne, Andrea McDonough, Pam Montgomery, Anne Roche, Glenn Rowley and Peter Sullivan (who was at Australian Catholic University for most of the project).

CHANGES IN MATHEMATICS TEACHING: LEARNING WITH AN EXPERIENCED ELEMENTARY TEACHER

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Despite a history of teachers' resistance to change, researchers in mathematics education have learned that mathematics teachers can change. The process of change, however, is usually long-term and difficult. This paper discusses the issue of change from the teachers' perspective. How do teachers' perceive their own, professional changes? What are factors that teachers consider as catalysts for their processes of change? Through the stories of a very experienced elementary school teacher, one learns that teaching is a dynamic job, impregnated with short and long-term changes. Contradicting the image of teaching as repetitious, this teacher sees the development of her mathematics teaching as a continuous learning process that has gone through many phases and incorporated many changes.

If a group of elementary teachers from the early 20th century could travel in time and visit a 21st century classroom, there would be few elements they would not recognize. For the most part, however, these educators from the last century would be able to understand what the classroom teacher was trying to do and could take over the lesson. This parable, used by Papert (1993) to claim that education has not changed as much as it should over the past century, brings to mind an image of teaching as a repetitive, monotonous job in which changes seldom occur. Although in his book Papert presents innovative teachers and separates issues about teachers as individuals from those about schools as institutions, images of teaching as a static job and of teachers as professionals who perpetuate certain practices are pervasive in the educational literature. Teachers, it appears, do the same things one day after the other, one year after the other. Teachers also do the same things generation after generation. After all, teachers tend to teach in the way they were taught (Ball, 1988).

In recent change- and reform-related studies in mathematics education, we have learned that despite a history of resistance to change (Cuban, 1993) teachers can successfully implement modifications in their practices (e.g., Fennema & Nelson, 1997; Ferrini-Mundy & Schram, 1997; Schifter & Fosnot, 1993; Webb & Romberger, 1994; Wood, Cobb, & Yackel, 1991). Changes in mathematics teaching, however, are usually the result of sustained partnership and leadership, on-going collaboration, and long-term support for teachers to embrace new recommendations for school mathematics. Thus, although researchers in mathematics education have come to accept that mathematics teachers can change, the process of change is perceived as a long-term, difficult, and slow effort.

This paper discusses the issue of change from the teachers' perspective. How do teachers' perceive their own, professional changes? Do teachers believe they modify their teaching practices over the paths of their careers? If so, what are factors that teachers consider as catalysts for their processes of change? These are questions addressed in this report.

Analyzing data from Helen[1]—an elementary school teacher with 31 years of teaching experience—one learns that teaching is a dynamic job, impregnated with short and long-term changes. Contradicting the image of teaching as repetitious and static, Helen sees the development of her mathematics teaching as a continuous search process that has gone through many phases and incorporated many changes.

The Research Project

My research aims at better understanding elementary school mathematics teaching. Toward such a broad goal, it is necessary to identify the knowledge elementary teachers construct, and understand how such knowledge changes throughout the teachers' careers. It is also important to relate changes in practices to changes in knowledge, connecting specific practices to knowledge that supports it. One of the specific objectives of this study was to trace the history of teachers' personal knowledge and teaching practices, identifying turning points and factors that, from the teachers' perspective served as catalyst for change. This report focuses on the teachers' perceptions about changes in practices.

During the past decades, researchers' understanding of teachers has greatly expanded. Beginning in the 80s, there has been an ongoing movement to strengthen teaching as a profession (Carnegie Forum, 1986; Holmes Group, 1986), and the development of teachers' knowledge-base for teaching has become of utmost relevance. Various theoretical positions have led to different typologies that categorize teachers' knowledge and its sources, as well as research programs that target the topic (Fenstermacher, 1994; Gauthier, 1998; Schulman, 1986, 1987; Tardif, Lessard, & Lahaye, 1991). Despite differences, it is accepted that teachers construct professional knowledge through their teaching practice.

To understand teachers and their knowledge, an "epistemology of professional practice" is needed. Tardif (2000, p.13) defines this epistemology as the study of the set of knowledge used by professionals in their everyday working space to accomplish all their tasks. From this perspective, teachers' knowledge is personal and time dependent. It depends on teachers' life histories and develops throughout their careers. Tardif and Raymond (2000) propose that to learn about a teacher's

[1] Names are pseudonyms.

knowledge one should ask her/him the history of her/his personal knowledge-base for teaching. Connected to such history is the history of the teacher's mathematics teaching practices.

This report is part of a larger project in which I worked with 5 elementary school teachers, exploring their stories about mathematics teaching. All teachers have over 20 years of teaching experience, and they have been teaching at the same school for the past 10 years. I identified factors that, from the teachers' point of view, were important in the establishment of their ways of teaching mathematics to young children, and factors that served as catalysts for change their mathematics instruction.

To discuss her career and consider factors that were relevant in the establishment of her mathematics teaching, I interviewed each teacher three times. The instruments used in these interviews were a timeline based on the teachers' responses to an inventory of their teaching experiences, a logic-spider with knowledge for teaching mathematics as the central theme, and a life-line (Connelly & Clandinin, 1994). The goal of these instruments was to offer teachers different approaches and opportunities to recall their experiences in different schools and classrooms. The research instruments also helped teachers organize their accounts, looking for continuity and change. In a final group interview, teachers discussed issues that were repeatedly mentioned in individual interviews.

All interviews were audiotaped and transcribed. Teachers received copies of the transcriptions. Data was analyzed through constant comparison method using initial descriptive categories such as knowledge, early teaching, later years, types of change, catalysts for change, and developing own teaching. Although the stories of all 5 teachers were interesting and had many similarities, the case of Helen was selected for presentation in this paper because she is the participant with the most teaching experience. Many ideas presented by Helen, however, were true for other teachers as well. In particular, the notion that becoming a teacher is an on-going change process was addressed by these experienced teachers.

Helen, Mathematics Teaching, and Perceptions on Change

Helen earned a B. S. in Elementary Education in 1969 and a master's degree in Elementary Education in 1970. She began teaching 3rd grade in 1970. After that, she taught 2nd grade, 5th grade, and remedial middle school mathematics. Since 1984 she has been teaching 1st grade, and in 1990 she began teaching at the school where she is now. During her many years of teaching, Helen has completed graduate level courses in education and has attended staff development courses in mathematics education.

Helen thinks her mathematics teaching has greatly improved during her career, particularly during the past decade. She considers herself a good teacher, especially because she thinks her students are excited about mathematics; they enjoy studying it, they think it is fun, and they appreciate when Helen tells them "it's math time."

> I think I am a good teacher. I have taught a long time and I have changed some things, some things that didn't work. You know, you have to change. ... I would say the last third of my teaching career has been my best years of teaching math, because of the changes that I have made in the way that I have taught math. ... I am more satisfied with what I have done. Because I can feel a sense of pride whereas before, you know, there was not that much. ... I think that my teaching style developed, it sure matured. Then you become so much better at it. And I think any time you have children involved in what they are doing and have a say in it, then it is better teaching.

For Helen, teaching mathematics is an ongoing learning and change process. In her path to becoming a better mathematics teacher, Helen believes she formed her own way of teaching mathematics. She feels great ownership of her current mathematics teaching, which she calls eclectic. Her teaching approach represents a mixture of teaching resources she acquired throughout her career. Helen claims teachers need to try a variety of ideas in their classrooms and keep the ones that work, that is, that help students learn. In this way, after many years of teaching experience, teachers build a rich web of ideas for the classroom.

> You can't just say, okay, open up your math book and do page so and so. You can't do that. I mean, that is not teaching. So I think that you have to just pull from every source, and do everything that you can, and try new things. And then, if they don't work, then okay, I have tried it, it didn't work, so I will move on to something else that will work. ... There is a time when you have to do some paper and pencil things. I mean, there are times when you have to do that. But then there are some times when you can teach it another way that is not paper and pencil, there are some manipulatives that you can use and games that you can use. ... [Because] there is no one best teaching method. You know, you have to pull from everything. You just can't do it one way. ... If you use a variety of teaching methods you will have better math students.

Developing this eclectic approach, that combines different resources, ideas and teaching methods, symbolizes for Helen her growth in mathematics teaching. This development is a learning experience that occurred over time and required many changes from Helen. Her current teaching was constructed and no one could have given it to the teacher. Helen's eclectic combination of ideas is the result of her many experiences in mathematics and the different trends she faced in the teaching of mathematics. It was built through many trials, revisions, reflection, and changes.

> [In my personal life] I don't like change very much. I mean, I stress over it. Whereas in teaching, you have to change because there are so many things going on, you know, so many cycles that you are going through. ... I think I was taught in college right out of the book. And so when I started my teaching that was what I did. It was right out of the book, there was no hands-on.... I put a lot of board work up, you know, lots of problems on the board and all. And for a long time, that was the way we did it. ... And then I found out, you know, that really isn't the way to teach math. And probably some of the greatest experiences would have been with, when I came here and they were doing Math Their Way which I was not familiar with. And it was so hands-on. It was just such a different type of math that, really, you could understand the why of math.

Helen attributes the changes she has made in her mathematics teaching style to a few different factors—for example, the courses she took on Math Their Way, a late 1970s activity-centered mathematics program for early childhood education. Among the factors that mostly influenced her changes, Helen mentions moving to a different school, sharing teaching ideas with colleagues, and attending workshops.

> I think probably one of the things that have had a great impact was the change in schools and the different colleagues. I think that probably has had one of the greatest impacts on my teaching of mathematics. ... There have been a lot of opportunities here [at this school] where you can grow and do a lot of things, and then the teachers in my hall worked so well together, so if somebody is doing something that you like, you know, you can say, hey, you know, let me do that. ... [In other schools] there was no sharing. ... I think probably some of the influence of the Math Their Way when I came here which I had not had in other systems. And when I saw how it was being taught and that whole philosophy, how to teach math was so different from these other things. And using so many different ways to teach one concept. That had a big impact on me. ... And then going on to math workshops and finding out what is going on.

Schools, colleagues, and participation in workshops are factors one can consider as directly related to Helen's own career and personal choices. Helen also attributes some of her changes to another set of factors more related to changes in education in general, and mathematics education in particular. Over her many years of teaching experience, Helen has lived various recommendations for school mathematics. She has experienced different trends and the availability of different resources for mathematics teachers.

> And then also, the new ideas that came from, that came into being. Like when I was telling you earlier. When I first, those first years when I went

into teaching, individualized structure, now, that was the catch phrase. And then after that it was open classrooms, where we had centers. Well, then that was kind of passé and they went into using a lot of manipulatives and that kind of thing. And then we came into Math Their Way. So, I mean, it is just the different, all of these different things have been floating around ... A lot of times we do have a math book, then all of the things that are taught in there are taught according to whatever is the "hot" way to do it at the time. So, that influences, too. And also, then all of the monies that are available in the school. And they will say, "You can spend the money on something, what do you want to buy?" Well, then we are looking through [lists of resources] and seeing all of the things that seem to be working in classrooms ... [You are also] influenced by the materials that are bought.

Discussion

It is interesting to observe that, in Helen's stories, recommendations for change in mathematics teaching that are "floating around" make their way in to the classroom. These recommendations are mediated by school administrators, mathematics consultants for school districts, books and other resources available for teachers. As education goes on different pendulums, so do teachers. And as teachers swing, they learn new ideas, incorporate new suggestions, and develop their eclectic teaching. Teachers learn from different sources. They use all they can to do what they believe is best for their students. Thus, as new educational ideas come around, teachers change; as new recommendations for mathematics teaching are proposed and shared with the greater educational community, teachers change teaching of mathematics.

This study is based on teachers' careers, their perspective on their development in mathematics teaching, and their history in teaching mathematics. It is not an evaluation of whether teachers are teaching in accordance with this or that recommendation for the classroom. It is also not a judgment of the teachers' implementation of reform rhetoric. With this approach to the study of teachers' practices and change, it is quite striking to notice to the many changes teachers are making in their practices.

The history of Helen's professional career in no way resembles the image of teaching as a job in which changes seldom occur. Quite the contrary, listening to Helen's stories, one becomes immersed in change, new experiences and searches for professional growth. Helen's ongoing quest for new ideas, together with changes in working milieu and new educational trends, have led her onto a path of many revisions and alterations of her teaching practice. From Helen's point of view, change is not a slow, difficult process; change is a necessary component of good mathematics teaching, it is part of teachers' everyday experiences.

The difference between the two perspectives on mathematics teachers' practice —one more static and one more dynamic—deserves attention. One way to approach this difference is to take into account the idea that researchers and teachers may have different perspectives on classroom practices (Simon & Tzur, 1999) and different meaning for what counts as change. Many researchers have an ideal image of what should constitute mathematics teaching. With this image as a starting point, it becomes hard to observe change—probably because the changes required to become like the envisioned, ideal teacher are too big. Also, researchers are typically looking for profound changes that affect teachers' practice in all of its dimensions. These changes are probably less often, and represent turning points in teachers careers. Finally, researchers focus their studies on particular points in time, looking at teachers during specific moments of their careers. Loosing a career perspective might make it harder to detect changes.

Teachers, on the other hand, understand change in a way that is more in tune with their step by step, everyday attempts to improve teaching. From this perspective, changes become a long-term and a short-term process at the same time. It is about everyday life in the classroom as well as about opportunities that appear throughout one's career. Change, from teachers' perspectives can be about trying new ideas, changing the way a certain lesson is structured, changing the books used for the introduction of a certain topic, adding a new assignment to their syllabus. In the case of very experienced teachers, the vision of change is strengthened by the privilege of exploring it from a 30-year perspective.

When researchers approach teachers with a static perspective of mathematics teaching, they fail to capture the many changes teachers implement in their daily practices and in the paths of their careers. They also fail to notice the many ways in which research recommendations for change might be reaching the classrooms, and the ways in which the mathematics education reform rhetoric circulate within schools. Not understanding teachers from their own perspectives, one might not recognize small steps toward change as important components in the development of mathematics teachers. In this case, teachers on-going development and growth goes unnoticed.

It is time we better understand teachers and their mathematics teaching. There are many questions to be answered, some of which related to the issue of change in mathematics teaching practice. It is worthwhile to notice that, as Helen puts it, teachers learn from each other. In schools where many teachers are searching for new ideas for improving elementary school mathematics teaching, transformation and growth might gain bigger momentum and happen in an easier, quicker fashion. Studying teachers who work such schools might allow us to improve our understanding of mathematics teaching and change.

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CONSTRUCTION OF KNOWLEDGE AND ITS CONSOLIDATION: A CASE STUDY FROM THE EARLY – ALGEBRA CLASSROOM

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Abstract

The process of abstraction is central to construction of knowledge. It has been discussed intensively, but only seldom studied experimentally. The following study exemplifies a way for tracing processes of knowledge construction and its consolidation. In doing so, we extend the nested model of abstraction elaborated by Hershkowitz, Schwarz, & Dreyfus (2001), to study two 7-graders, collaborating to investigate algebra problem situations in successive activities, along the year, in technological learning environment. The analysis demonstrates the construction of knowledge in an ongoing dialectical process, between construction and consolidation, which took place along three activities, when pieces of knowledge incrementally accumulate from one activity to the other.

This study is part of a longitudinal study, which examined the collaborative work done by two 7-grade students, in five activities of investigating problem situations along the academic year. One activity was analyzed in Tabach, Hershkowitz & Schwartz (2001). Here we will analyze selected episodes from other 3 activities, where all of them deal implicitly with the exponential growth phenomenon.

THEORETICAL FRAMEWORK

Mathematical activity, like any other human activity, is embedded in a socio-cultural environment (e.g., Voigt, 1995). This view is increasingly accounted for by the mathematics education community, which sees mathematics learning as a culture of mathematisation in practice. Such approach gains from combining collective with individual activities, analytic with reflective stages, and integrating intra with inter-processes that are at the root of mathematical development (Hershkowitz & Schwarz, 1999a, see related ideas of scientific enculturation in classrooms in Woodruff & Meyer, 1997).

Abstraction is at the heart of mathematisation (Freudenthal, 1991; Gravemeijer, 1995). To study abstraction experimentally, Hershkowitz, Schwarz, and Dreyfus (2001) gave an operational definition of abstraction: *an activity of vertically reorganizing previously constructed mathematical knowledge into a new structure.* They suggested a model, which is based on three observable epistemic actions, nested one in each other: Constructing (C) is the central action of abstraction. It consists of assembling knowledge artifacts to produce a new structure with which the participants become acquainted. The action of Recognizing (R) a familiar mathematical structure, occurs when a student realizes that the structure is relevant to the problem situation in which participants are engaged. The Building-With (B) action consists of combining existing artifacts in order to comply with a goal such as exploiting a strategy or justifying a statement. The term Consolidation denotes (according to Hershkowitz, Schwarz, and Dreyfus, 2001; Dreyfus & Tsamir, 2001) the progressive familiarization through observable recognizing and building-with actions, in four types of situations: Reconstruct the new structure or actualizing it by recognizing it in various contexts. Use it with increasing facility for building-with in various contexts. Use it in the construction of higher structures for which it is a necessary prerequisite. Verbalize about it – possibly during or after an activity of reflection, such as reporting or summary discussion in class. The RBC model of abstraction will be used in this article to trace the construction and consolidation of new mathematical knowledge along various activities.

THE STUDY

We focus here on the work of two Grade 7 students who participated in a one-year algebra course. The basis for choosing of these two students was their high verbal ability. Three activities, which deal with exponential change phenomena, were chosen out of the algebra course of the CompuMath project. The approach adopted by the development team of the CompuMath project is a function approach to algebra (Hershkowitz et al, in press). The activities goals are the construction of some generalizations of growth phenomena patterns within problem situations, and the use of these generalizations to build the phenomena numerically and graphically, using spreadsheet program (Excel). The problem situations, which are the mile stones of the algebra course, were designed to give opportunities to students' construction of new knowledge structures concerning mathematical concepts (algebraic variables and models) and of various mathematical processes (hypothesizing, making generalizations, testing hypotheses, interpreting representational information, solving and justifying). In the present study, we examined three aspects, in which these constructions take place. That is, we observed (a) the types of interactions while collaborative work is taking place (due to space shortage it will not discussed here). (b) The construction of a shared knowledge of the pair and, (c) The contribution of each participant, as well as what is left of it in the individual. All these aspects are examined within and between the three activities as one continuum along the academic year.

All activities were open - no guidance for solution was provided to students and no instruction if and how to make use of Excel was given.

The class work of the pair was videotaped and written works were collected. The videotapes were transcribed. Following Chi (1997), the protocols were divided into "cognitive segments".

In this presentation we will analyze selected parts from these 3 activities (They were the first, third and forth activities of the research sequence of activities). In all of them creation of new structures of knowledge, concerning exponential change phenomenon takes place, while students collaborate together. The designers of the Algebra Course intention was that students will be involved in investigating such a phenomenon, but not via the explicit algebraic formulae of exponential growth.

In the next part we will describe shortly each problem situation, and the work done by the pair of students Avi&Ben on it. In our analysis here we will present selected utterances from the full transcript of their work in these activities, as evidences to our claims.

THE PROBLEM SITUATIONS

Efrat's Savings activity

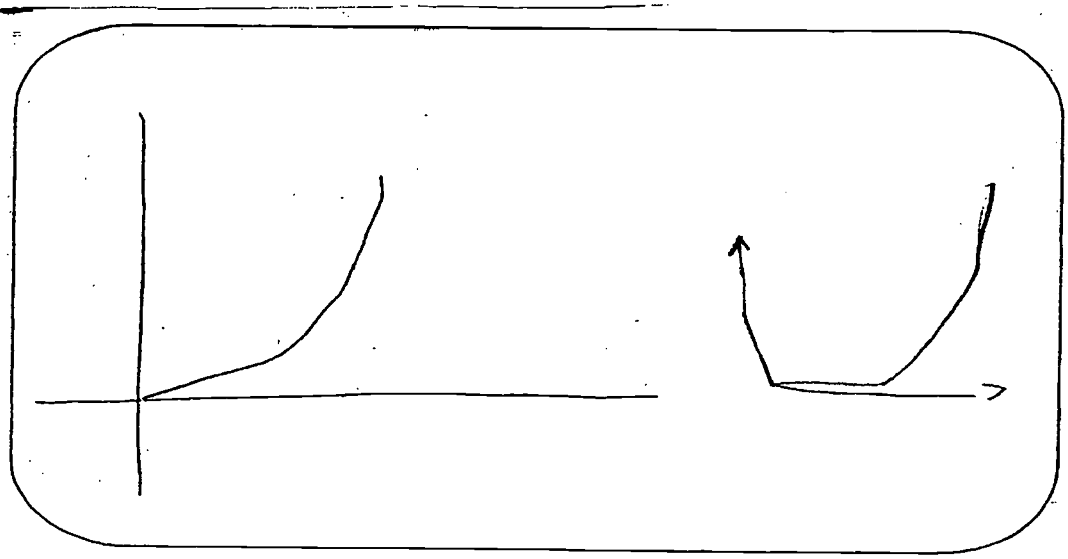
Efrat received her weekly savings as follows: at the end of the first week two cents, (which are 0.02 $), and in every weekend she received the same amount that she had in her saving box in the last week. Efrat saved all the money.

The students were asked to hypothesize how much money will be in Efrats' saving box by the end of the year, in comparison with other four linear ways of savings, which they explore during the week before (for more details, see Freidlander & Tabach, 2001). Then the students were asked to investigate the phenomenon in the computer laboratory with the help of Excel, and to check if their hypothesis was correct. It is important to note that it was their first meeting with the exponential growth.

The students' initial hypothesis is that the amount of money in Efrat's saving box will be the lowest among all other linear ways of saving. In the computer laboratory Avi&Ben try to find an algebraic generalization that will help them to construct the numerical representation of the phenomenon in the computer. Avi says: *Because she started from two. What, one times one, one times one, one times one, what, we should start from two* (A31). Avi is trying to develop an algebraic generalization for that change. Avi feels that this phenomenon has some thing to do with repeating multiplication, and maybe he is starting to see the generalization as an expression, which includes powers. Ben doesn't understand what Avi is saying, but both are aware to the need for an Excel formula. Avi says: *We should write a formula* (A36), and Ben reacts: *Exactly, so I will do A2* (B37). And then Ben continues: *Oh, yes, =B2+B2* (B39). Ben suggests a 'translation' of the verbal representation of the situation to a local connection, $\boxed{=B2+B2}$. Ben ignores Avi's suggestion to consider powers: *No, no, just a second, if we have here powers it will be good* [pointing to the screen] (A40), and drags "his Excel" formulae until B20, then the whole phenomenon appears numerically. They are both quite surprised from the large numbers they receive, and react: *Until 20. Yooo!* (A45).

The explicit algebraic generalization of Efrat saving, $\boxed{0.02*2^x}$, were x stands for the week number, is beyond 7 graders knowledge. And yet, Avi tries to reach such generalization. However, as we can see, Ben easily creates the whole phenomenon by recursion relation between the savings of two successive weeks, and the "dragging" operation in Excel. That is, by writing in cell B2 $\boxed{0.02}$, and in cell B3 $\boxed{=B2+B2}$, and by 'dragging' this formula to the next cells in the same column. In that way Avi&Ben get the whole phenomenon in its numerical representation.

Next, they are asked to sketch a graphical hypothesis of the same phenomenon, and then to check their sketch on the computer screen. They accomplish it quite quickly, with no evidential difficulties. Yet, if we compare their sketches, we can see that Avi's is much more accurate in his graphical hypothesis then Ben's.



Ben's sketch Avi's sketch

In this activity Avi&Ben constructed some new knowledge regarding the exponential growth: at the beginning they underestimated the exponential growth, but when they received its numerical representation, they were surprised by its rapid growth. Their graphical representation sketches were quite close to graphical representations of the exponential change. And yet, we can't be sure what kind of knowledge concerning exponential growth was emerged.

***Aunt Berta* activity.**

Five months later, the students are asked to solve the Aunt Berta activity. (It is important to mention that during this five month, they didn't meet any situation regarding exponential change).

Yosi received a letter from his rich aunt Berta.
Dear Yosi!
I have reached the age of 65, and my life is comfortable. I would like to give you some of my money. You can choose one of the options:
One. I will give you this year 1,100 \$, next year 1,200 \$, and so on. Each year 100 \$ more then the last year.
Two. I will give you this year 2,000 \$, next year 1,900 \$, and so on. Each year 100 \$ less then the last year.
Three. I will give you this year 100\$, next year 150\$, and so on. Each year 1.5 more then last year.
Four. I will give you this year 8 \$, next year 16 \$, in two years 32\$, and so on. Each year 2 times more then last year.
The agreement will go on while I am still alive. Let me know your decision soon, yours, Berta.

The students were asked to help Yosi to make the best choice, meaning to make hypotheses, taking into consideration that aunt Berta will live at least until the age of 80. Then they are asked to investigate the phenomenon in the computer laboratory with the help of Excel, and to check whether their advises to Yosi (their hypotheses) are correct.

Avi&Ben's advice to Yosi is to take the forth option. Ben estimates that the amount of money will grow quickly after it will reach 100\$, and Avi estimates that the amount of money will grow in several millions.

They try to construct the generalization with the help of Excel. For the third option Ben starts to build a recursion generalization: *100, equals D2, ahh...* (B40). Avi helps by saying: *times 1.5* (A41). They drag the generalization $\boxed{\text{=D2*1.5}}$, were in D2 they put $\boxed{100}$. In investigating the forth option, Ben takes the lead again by starting to build a recursion formula. Avi interferes by saying: *Make Power* (A43). Ben objects: *No, No, I will just double it, don't you think so?* (B44). Avi hesitates: *no,...try it on* (A45). So Ben writes $\boxed{\text{= E2*2}}$, drag it, and they both agree on the numerical results. Avi's reaction to the numerical data is: *I was wrong here* (B49). He explains that his estimation was for 80 years, and he tries to drag the generalization more, up to 80 years. But, since the column width is limited, he does not get a proper reaction from the computer, and he remains frustrated: *irritating, O.K.* (A69).

In Aunt Berta Activity, Avi&Ben had no doubt that the forth option is the best. How can we explain the over estimation they gave, especially Avi? It seems that the source of this over-estimation is the surprise they had in the first activity, concerning the large numbers, which were received. This over-estimating may serve as evidence that during the first activity they indeed construct the knowledge that exponential growth is very fast. This knowledge is consolidated here, as evidenced by the correct choice they made, and their over-estimation. Moreover, this time Avi has no doubts that he can use powers for the forth option. The way Avi tries to check his estimation, shows that he understands intuitively that the exponent is responsible for that rapid growth.

The Crazy Candy **activity.**

Lets assume that in the next years the rate of inflation will be 10% per year.
Write the name and price of your favorite candy.
What will be the price of that candy when you will reach the age of 120?
Suppose we could do the following changes:
Cut down by half the price of your candy, and increased the inflation rate to 11% (and not 10%).
Is this deal profitable?

The students are asked to make hypotheses concerning the prices of their candy, and then they are asked to investigate the phenomenon in the computer laboratory with the help of Excel, and to check their hypotheses.

The current price of Avi&Ben's candy is 3.5$. Ben's estimation is that in 100 years it will be ~100,000. Avi's estimation is about 11,000,000. They try to build the right expression, and Ben's suggestion is: *oh, B2/10+B2. Is it?* (B52). They are both surprised by the "low price" of the candy after 107 years. Avi tries again to "drag" the expression for some more years, and is disappointed.

The generalization of the price in an explicit algebraic formula, $\boxed{A1*1.1^x}$, were x stands for the year number and A1 is its present price, is again far beyond the knowledge of 7 graders. However, finding a local recursion connection between the prices of two consecutive years is quite easy. If we write in cell B2 the current price, and in cell B3 we write $\boxed{=B2*1.1}$, and "drag" the formulae to the next cells in the same column, we will receive the whole phenomenon numerically.

Here it is interesting to note that: (a) both students knew that the phenomenon has a rapid change. As they dealt with 1.1^x and not with 2^x, they over-estimated this change. (b) The generalization Avi&Ben used is of additive nature $\boxed{=B2/10+B2}$, which might be explained by the fact that this time the verbal representation of the situation involved percentages, a topic in which their knowledge is quite poor.

Reading the second part of the problem, Ben immediately said that this deal is not profitable. However, when trying to generalize it, Ben offered the expression =B2/11+B2, which is obviously a wrong one, but it supports the assumption about their poor knowledge concerning percentages.

Their right hypothesis shows us, the researchers, that another piece of knowledge was constructed and consolidated here: intuitively, they understand that the exponent basis, weighs more in the rate of change of the exponential phenomenon.

CONCLUDING REMARKS

Tracing the action of abstraction (including the construction of new structures of knowledge and their consolidation) is a complicated task. The evidence for the construction of knowledge is sometimes by actions that took place in the same activity, as was shown in Tabach, Hershkowitz & Schwarz (2001). In other cases, the experimental evidences for the construction of knowledge took place in later activities, while the constructed knowledge is consolidated, in the sense it was described in the theoretical framework section. Therefore, longitudinal studies like the one that was described in this paper are needed, in order to trace such evidences for construction and consolidation of knowledge.

The knowledge of Avi&Ben regarding the exponential change is not yet fully formalized – for example, they do not know the explicit expression of it. And yet, we can see from one activity to the other how the knowledge incrementally accumulated: At first they approached the exponential growth with an under estimation, and they were surprised.. As a second step in this activity we observed their graphical hypotheses which were quite close to the right exponential graph. Whether they constructed some knowledge concerning the exponential growth or whether they succeeded to "translate" from table to graph is not clear yet. In the second activity they over estimated the same phenomenon, but this time Avi was sure that this change has some thing to do with powers. Avi than reflected critically on his own hypothesis in the light of the findings he got, and dragged the generalization down to the eighty row. This action evidences a kind of consolidation leading to further construction (reconstruction). In the third activity they again over estimate the growth, but this time is due to the small basis of the exponent (1.1^x and not 2^x). In the second part of the third activity they have to choose between 1.1^x and 1.11^x, and surprisingly they made the right choice. This is an additional evidence for the dialectic process of construction and consolidation of incrementally accumulative knowledge between and within activities.

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UNIVERSITY STUDENTS' EMBODIMENT OF QUANTIFIER

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This paper investigates the novice university students' understanding of the formal definition of "equivalence relations", especially their understanding of the quantifiers in the definition. Even though the definition is relatively simple and only involves the universal quantifier, we find that half of a class of highly qualified university students are unable to test whether an explicit relation on a set with three elements is an equivalence relation. Analysis of the data, from a questionnaire answered by 277 students and interviews with 36, reveals subtle influences of language and of conceptual embodiments. In particular, the transitive law, which is shared with the notion of order relation, may evoke an embodied image of order that is highly misleading.

INTRODUCTION

Chin & Tall (2000) proposed a theory of development of formal thinking, moving from informal concepts, to the introduction of definitions, to 'definition-based' deduction and on to 'theorem-based' deduction, in which we hypothesised that successful students would compress formal concepts into cognitive units appropriate for powerful formal thinking. In particular, we focused on 'equivalence relations & partitions' which students claimed to be the most difficult topic in the course. Our purpose was to determine *why* the students found the topic to be problematic.

In our first study above, there was evidence of responses in the categories of informal responses, definition-based deduction, theorem-based deduction, with some evidence of a few students having cognitive units that linked the notions of equivalence relation and partition. Our second study (Chin & Tall, 2001) followed the development over a period of time and revealed a general shift from 'definition-based' deduction using the formal definition to 'theorem-based' deduction referring to already proven theorems. However, it also revealed quite different developments for 'equivalence relation' and 'partition'. An equivalence relation has an apparently simple definition that almost all students are able to reproduce but it does not seem to have a natural embodiment. A partition has a subtler definition but has a simple prototypical embodiment as a set broken into subsets. Even though equivalence relations and partitions are mathematically equivalent, they develop as qualitatively different cognitive units linked by a formal theorem.

Our focus here is on the way that students employ the definition of 'equivalence relation'. Is the definition *operable* in the sense that it can be used as the basis for formal deduction (Bills & Tall, 1998)? In particular, do they have a full understanding of the use of the quantifiers, as, for instance, in '$a \sim a$ *for all* $a \in S$', or are there other aspects of the definition that the students use intuitively or implicitly?

THEORETICAL FRAMEWORK

Formal definitions and deductions are the essential ingredient in advanced mathematical thinking (Tall, 1995), but they are known to cause great difficulties (Vinner, 1991). Some formal thinkers attempt to use logical deductions, others use a natural approach that relies more on their concept image (Pinto, 1998). The latter often involves thought experiments using embodied images in the sense of Lakoff and Johnson (1999, pp16-44). Our earlier papers showed that the notion of equivalence relation did not have a strong embodied image in the same way as the notion of partition. So, precisely how do students reason when they use the definition? For instance, how do they use the universal quantifier which occurs in the reflexive law 'for all $a \in S$, $a \sim a$'? The use of quantifiers in definitions has been considered "one of the least often acquired and most rarely understood concepts at all levels, from secondary school on up — even, in many cases, into graduate school" (Dubinsky, et al., 1988, p.44).

The definition of equivalence relation, uses only the universal quantifier:

> An *equivalence relation* on a set S is a binary relation $\sim$ on S that is
> *reflexive*: $a \sim a$ for all $a \in S$
> *symmetric*: if $a \sim b$ then $b \sim a$ for all $a, b \in S$
> and *transitive*: if $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in S$. (Stewart & Tall, 1977)

However, the quantifier here plays a very subtle role, as is shown by the following question, (which the first author well remembers being unable to answer for several days whilst studying Birkhoff & Maclane (1953) as an undergraduate):

> Given an equivalence relation "$\sim$" on A. Let $a \sim b$, then $b \sim a$ (by symmetry) and $a \sim b$, $b \sim a$, implies $a \sim a$ (transitivity). So symmetry and transitivity imply reflexivity.

The subtlety is two-fold. The first axiom asserts that the relation $a \sim a$ must hold *for all* $a \in S$. The second and third axioms use implication in a more subtle way, that *if* the premises holds, *then* the consequence follows. We decided to use a specific instance of this idea to test if the students' definitions are truly operable.

EMPIRICAL STUDY

The study was performed on 277 (first year mathematics) students taking a course on 'foundations' in one of the top five ranked mathematics departments in the UK, 151 from a class of pure mathematics majors and 126 in a class consisting of those following courses such as statistics, economics, or physics. Both classes covered the same material over a ten-week period with three-hour lectures per week supported by weekly examples classes in groups of up to four students. The second author acted as supervisor for 12 pure mathematics students selected to cover a range of abilities. The topic of 'equivalence relations' and 'partitions' was formulated in the third week and developed in subsequent weeks. The students conceptions were studied by a questionnaire given to all students in the ninth week, (six weeks after the definition of equivalence relation was given and subsequently developed) and the responses were triangulated with interviews with the other 24 selected students, fourteen in pure mathematics and ten in other mathematically-linked subjects, together with field notes made in tutorials with the 12 pure mathematics students.

In this study we focus on the following question to investigate how the students understand the definition of "equivalence relations":

> Let $X=\{a, b, c\}$ and the relation ~ be defined where $a\sim b$, $b\sim a$, $a\sim a$, $b\sim b$, but no other relations hold. Is this an **equivalence relation**? If not, say why?

Student Responses & Analysis

The student responses to the main question were categorised as follows:

- Correct deduction & answer,
- Incorrect deduction with correct answer,
- Incorrect deduction & answer,
- Don't know/ no response.

The following examples illustrate the categories.

To be classified as 'Correct', an answer must either have a general statement that the reflexive property does not hold for all elements, or specifically note that $c\sim c$ does not hold. For instance, ANNWAN (pure mathematics) wrote:

No because $c \nsim c \Rightarrow$ not $\forall\, a,b,c \in X \Rightarrow$ not reflexive. $\Rightarrow$ not equiv. relation.

GILWIN (other mathematics) was classified 'Incorrect deduction with correct answer.' Not only did he miss the universal quantifier in the reflexive property, he tested the symmetric property only with the distinct elements a, b, then denied transitivity because "$b\sim c$ doesn't hold" for making $a\sim b$, $b\sim c \Rightarrow a\sim c$. However, he even did not notice that $a\sim c$ does not hold either when making the deduction. In the interview, he explained what he means is it needs three *different* elements to make transitivity hold.

does it follow rules? $a\sim a$ ✓ $b\sim b$ ✓
$a\sim b \Rightarrow b\sim a$ ✓ symmetry
$a\sim b,\ b\sim c \Rightarrow a\sim c$, but $b\sim c$ doesn't hold, so this is not an equivalence relation.

JOAITE (pure mathematics) was classified as 'Incorrect deduction and answer'.

1) $a\sim b$, $b\sim a$
2) $a\sim a$, $b\sim b$
3) No occurrence Yes it is an equivalence relation.

He used only the four related pairs and assigned each of them to his own versions of the three rules, (presumably in the order 1: symmetry, 2: reflexivity, 3: transitivity) without exhibiting any conception of the subtlety of the universal quantifier.

	Correct deduction & answer	Incorrect deduction		Don't know/ No response	Total
		correct answer	incorrect answer		
Pure mathematics	94	24	30	3	151
Other mathematics	45	44	18	19	126
Total	139	68	48	22	277

Table 1: Responses to the "use of quantifiers" question

The distribution of the categorisation is shown in table 1. Visibly, the pure mathematics students give more correct responses (in terms of both deduction and answer) than the other mathematics students (χ^2=19.34, p<0.0001). These 'correct' responses show a satisfactory use of the universal quantifier in the reflexive property at either a general or specific level. (This is not to say that a student who gives a correct response to the reflexive property necessarily understands the whole definition, for there are seven students who correctly assert the falsehood of the reflexive property who are categorised as having an 'incorrect deduction' because of difficulties with the symmetric or transitive properties (usually the latter)).

We investigated whether the nature of these responses correlated with the quality of the definition that the students offered for an equivalence relation. To determine the latter we combined the results of the following question:

> Say what "***equivalence relation***" means to you,

and a later question focusing on their concept definition:

> Look back at what you wrote about the meaning of "***equivalence relation***", do you consider it to be a *formal definition*? If you consider it is not a proper formal definition, please write down the *formal definition*.

The combined response to these questions were placed into four categories:

- formal-detailed (including full use of quantifiers),
- formal-partial (giving the three properties in symbolic form without quantifiers),
- informal outline (mentioning 'reflexive, symmetric, transitive' only)
- other, or no response

The student responses are cross-tabulated in tables 2, 3. There is no statistical correlation between the responses of the pure mathematicians and the quality of their definitions (χ^2=10.94, p=0.28). But for the other mathematicians, the correlation is highly significant (χ^2=29.39, p<0.001). However, closer examination shows that the significance arises from the correlation between those students responding 'other/no response' in both categories. If these are removed, then the correlation on the remaining data is χ^2=3.11, p=0.54 for the pure mathematicians and χ^2=5.91, p=0.21 for the others. Thus the statistical difference is due solely to the fact that a significant minority of the other students cannot handle the definition at all.

Our attention focuses on those students who gave unsatisfactory responses.

Deduction / Definition	Correct deduction & answer	Incorrect deduction		Other/ No response	Total
		correct answer	incorrect answer		
Formal-detailed	31	7	5	1	44
Formal-partial	25	6	7	1	39
Informal-outline	35	10	15	0	60
Other/no response	3	1	3	1	8
Total	94	24	30	3	151

Table 2. Pure mathematics students: giving definitions versus making deductions

Deduction / Definition	Correct deduction & answer	Incorrect deduction		Other/ No response	Total
		correct answer	incorrect answer		
Formal-detailed	9	5	1	0	15
Formal-partial	21	23	6	4	53
Informal-outline	12	12	9	11	41
Other/no response	3	4	2	8	17
Total	45	44	18	19	126

Table 3. Other mathematics students: giving definitions versus making deduction

Unsatisfactory responses declaring the relation to be an equivalence relation

Forty eight students asserted incorrectly that the relation is an equivalence relation. Thirty gave no explanation. Of these, three were interviewed and in each case the response indicated that the students concerned did not think deeply about the problem. For instance, one was capable of giving a 'formal/detailed' response with full quantifiers in the interview, but was unable to explain any further.

Of the 18 students offering a written explanation, there was a tendency to simply overlook the role of the quantifier in the reflexive property. SANSON (pure mathematics), for example examined the reflexive rule for all four given relational pairs, then indicated one clear example of the transitive law, but did not consider symmetry in detail.

This is reflexive because if a~b then b~a
if a~a then a~a
& if b~b then b~b
it is transitive because a~b b~b & a~b etc.
& I think it will also be symmetric ∴ it's an equivalence relation

MAROOD (pure mathematics) wrote out the definition in full, but then did not apply the definition to the specific example.

Yes. $d \sim d \ \forall d \in X$. (Reflexive).
If $d \sim e$ then $e \sim d$ (Symmetric)
If $d \sim e$ & $e \sim f$ then $d \sim f$ (ie $a \sim b$ & $b \sim a \Rightarrow a \sim a$) (Transitive)

In general, even though many of these students could give the definition in detail, they failed to implement it in the given case.

Saying the relation is not an equivalence relation

The case of students who asserted the relation was *not* an equivalence relation, but made errors in explanation, is the most revealing category of all. *Eighty two percent of them (19 of 24 pure mathematicians and 37 of 44 other mathematicians) offered the same reason: "the relation is not transitive so it's not an equivalence relation".*

Their reasons varied. SUSDLE (pure mathematics), who could give a formal/ detailed definition, correctly dealt with the reflexive property, but noted incorrectly, that the symmetry law needed to hold for more elements:

No, because if it was, for it to be symmetric
$a \sim c$ and $c \sim a$, and $b \sim c$ and $c \sim b$.
For it to be reflexive, $c \sim c$.

In the interview, he was asked how about transitive property. He replied that it must include all the relations between each two out of any three *different* successive elements. He also explained that he thought in this way because the universal quantifier is used in all three axioms.

LUCCCO (other mathematics), classified as 'formal/detailed', misunderstood transitive property because he did not think $a \sim b$, $b \sim a \Rightarrow a \sim a$. He clearly pointed out there should be three elements for transitivity to hold.

No because $a \sim b$, $b \sim a \not\Rightarrow a \sim a$.
need 3 elements for transitivity to hold.

SUSURT (pure mathematics), classified as 'formal/ partial' because he omitted quantifiers in his definition, missed the universal quantifier in reflexivity, only examined symmetry with the distinct elements *a* and *b*, then disagreed with transitivity "as *c* is not involved".

$a \sim a$ hence reflexive
$a \sim b$ & $b \sim a$ hence symmetric
However not transitive as c is not involved. $\therefore$ not

In the interview SUSURT was asked why he thought *c* should be involved for transitivity to hold. He echoed the thinking of LUCCCO and GILWIN (quoted before) by replying "because it needs *three* elements to make transitivity hold". In the interviews, we found that six of the seven interviewees who asserted transitivity failed also shared this view. Furthermore, when asked, 'If the two relations $a \sim b$, $b \sim a$

are removed from this question, what will happen?', five out of these seven replied, 'symmetry will not hold either'. Three students said explicitly that they used the concept of an order relation as an embodiment of transitivity and two of these explicitly said they used '$a<b$, $b<c$ imply $a<c$' as a special example for transitivity in their concept image.

DISCUSSION

In our data, approximately half the students are unable to handle the definition in a simple example using only three elements. The reasons are diverse, but 82% of those giving an incorrect reason for the example not being an equivalence relation focus on the transitivity law where there is a sense that 'the transitivity law must involve three elements' and even that the transitive law is interpreted using an embodiment that is the same as the axiom in an order relation. This has been a fundamental underlying conception in mathematics even amongst those who insist on formal thinking. In his famous address to the International Congress of Mathematicians in 1900, David Hilbert said:

> Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea "between"? (Hilbert, 1900)

The students involved in this study have been given an exposition of the theory in which the concept of relation is defined first, then, in quick succession, the special cases of function, order relation and equivalence relations. We therefore hypothesise that what is happening is that the students fail to get a workable mental image of each of these three conceptually different kinds of relation. In the case of an order relation, it is a natural thought process to imagine the elements ordered in a line, and, in the absence of an embodied image of the notion of equivalence relation, in using the transitive law, it is natural to link to the self-same image.

This phenomenon is an essential element in the transition from elementary to advanced mathematical thinking. In moving from a way of thought that uses related imagery at will to a formal way of thinking that is intended to be formal, the student can not have conscious control over all the mental connections that are made. Lakoff and his colleagues (Lakoff & Johnson, 1999; Lakoff & Nunez, 2000) argue powerfully that all thought is embodied. However, it is tautological to claim that human thought occurs in a physical brain. A more useful distinction needs to be made. Mathematicians are acutely aware that their formal thought occurs in an embodied mind, but they struggle to make their deductions as free from embodied influences as is humanly possible. Tall (2002) theorises how embodied thought experiments can suggest formal theorems and how formal deduction can prove them, sometimes giving structure theorems that have new, more sophisticated, embodiments.

What is happening with these students is that their introduction to formal thought concerning equivalence relations, order relations, and functions as examples of relations occur in a brain in which the concepts are intimately linked together. Some make connections in a manner familiar to the mathematical community, but most, if not all, have a variety of other mental linkages which need to be addressed for serious progress to be made.

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Connecting Children's In-School With Out-school Mathematics By Using Mathematical Writings

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Abstract

The purpose of this study was to explore the process of mathematical enculturation in children's cultural activities through the use of mathematical writings. There were ninety-nine second graders participating in the study. Three stages were developed in the use of mathematical journal writings. The results indicated that (1) the daily activities mathematics embedded into that were perceived by second-graders included drastic disasters, causal accidents, and surrounding environments. (2) The quality of children's mathematical writing was improved by evaluating each other on their writing journals. (3) Five commentary levels of mathematical writing were characterized. They were moved toward advanced commentary level. (4) Children became flexibly and efficiently in resolving the mathematical problems they posed.

Introduction

School mathematics experiences at all levels should include opportunities to learn about mathematics by working on problems arising in contexts out of school of mathematics (NCTM, 2000). Therefore, the opportunity for students to experience mathematics in variety of contexts is important. In recent years, many studies have focused on mathematical cognition relating to individual competence in daily life context (Bishop & Abreu, 1991; Carraher, 1988; Lave, 1988; Saxe, 1991; Tsai & Post, 1999). A review of children's out-of-school mathematics raises critical questions about how children come to understand mathematics and how they connect informal knowledge out of school with formal knowledge in school (Hibert & Carpenter, 1992; Millory, 1994; Resnick, 1987). In keeping with the connection between in- and out-school mathematics for children, the researcher has developed a teaching model called the Cultural Conceptual Learning-Teaching Model (CCLT) (Tsai, 1996) that attempts to combine individuals, activities, concepts, and culture together. The previous studies focusing on CCLT suggest that the linkage from children's cultural activities to school mathematics contributes to children's better performance in school mathematics and their abilities in solving problems in daily life (Tsai & Post, 1999; Tsai, 2000; Tsai 2001). However, An important question is how children applied school mathematics they learned in the CCLT model into real life situation. Thus, this study intended to provide children an opportunity with the mathematical journal writing to help them make sense school mathematics implicitly embedded in everyday activities.

Mathematical writing is considered as a tool of assisting children in making sense of, formulating, and solving mathematics problems in daily activities and since it provides an opportunity to construct ones own knowledge (Countryman, 1992). This process is part of mathematical enculturation (Bishop, 1991).

The Cultural Conceptual Learning Teaching Model (CCLT)

The CCLT (see Figure 1) contains three learning environments and six learning stages. Play Stage provides children with cultural activities. In this stage, children share, negotiate, and construct their immediate experiences to achieve the emergent goals of arithmetic problems with peers and more advanced children (the expert children). In the Construction Stage, the teacher designs a worksheet that has structural objectives those need to be accomplished by students. In the Connection Stage, based on children's experiences or strategies, the teacher tries to help children construct a connection between their experiences and concrete materials like ten-based blocks or mathematical symbols and procedures. In the Reapplication Stage, the teacher provides another similar or same cultural-conceptual activity for children to reapply to the learned mathematical concept. In the Practice Stage, children try to practice school mathematics in everyday situations by using opportunities provided for them. In the Reflection Stage, children are trained to monitor their thinking and to be aware of where and how they can apply school mathematics in everyday activities. The CCLT model contains two parts: play stage, construction stage, and connection stage are the first part of the area of cultural mathematization; Reapplication stage, practice stage, and reflection stage are the second part of the area of mathematical enculturation.

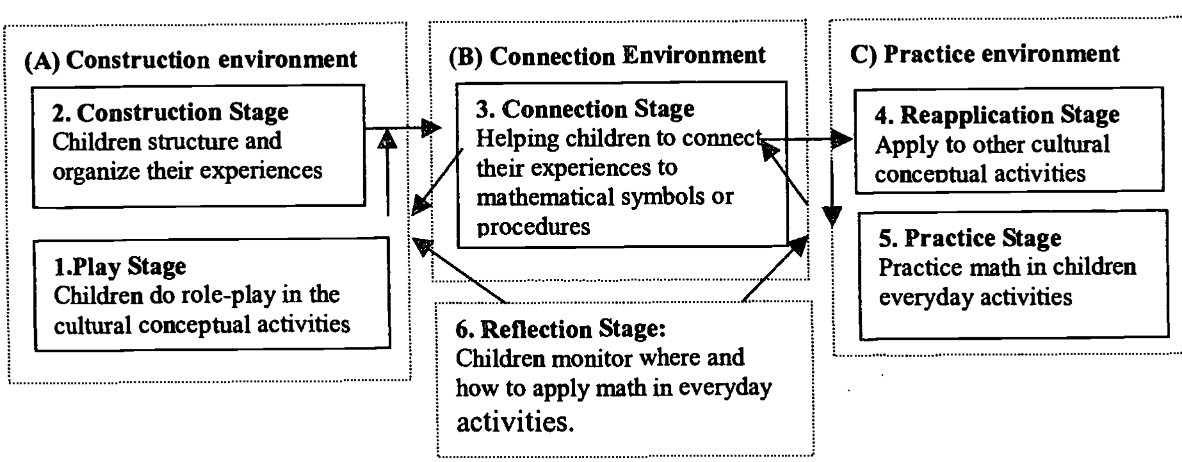


Figure 1: The Cultural Conceptual Learning Teaching Model

In the first part, several studies show that learning arithmetic through children's cultural activities based on the CCLT model not only affect children learning of school mathematics but also improve their ability to solve task problems (Tsai &

Post, 1999; Tsai, 2000; Tsai 2001). This study focused on the second part that created a learning environment by using mathematical journal writing for children to perceive school mathematics in which of daily life cultural activities.

Methodology

The ninety-nine participants were second grade students from three classes in one school. The class size includes thirty-three students. Limited space prevents to report how the students make sense of mathematics in daily settings from each class. An illustration of the effect of journal writing on children perceiving mathematics in daily settings is only taken from one class.

To improve second-grade students' abilities to formulate and solve problems, three stages were considered in the use of mathematical journal writing. First, children were expected to be aware of mathematics in which of activities or events in daily cultural activities. They were asked to answer the question: What are activities or events you encountered in daily situation involved in mathematics? Second, the students were asked to write down or draw a picture to represent the mathematical problem embedded in daily activities or events. Finally, the students were asked to solve the mathematical problems that students created in the first two steps. Between the second and the third stage, students were administrated with a critical task that mutually makes comments and criticizes on the problems students created. The first two stages were undergone at the beginning of the study. At the period of the second stages including critical task, each student was asked to write mathematics journal entries twice for every month and class discussion about students' commentary once for every month. One circulation from first stage to the third stage took two months to accomplish. Four circulations of using mathematical writings have been carried out throughout the entire school year.

In terms of the critical task, the classroom teacher attached all students' journal writings on blackboard and asked them to chose one or two journal entries in which they valued good or bad, followed by making comments on the 76.2cm×76.2cm Post-it-Notes and attaching the Post-it-Notes on the journal writing. Likewise, the student who wrote the journal entries was asked to give follow-up responses whether s/he agreed with the comments from others. The responses are written on different color Post-it-Notes. The critical task provided students the opportunity of mutual support with learning from other students how mathematics is relevant with daily cultural activities. Besides, the critical task was designed to improve the quality of making comments for second-grade students.

Results

The results indicated that not only students improved their writing journal but also progressed on the critical thinking on the writing comments. Several narratives delineate how students perceived, formulated, and solved mathematics problems in everyday settings. The comments and follow-up responses children made are described in the second part. The development of quality of comments will be described in the third part.

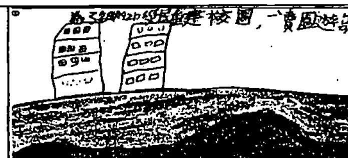
Cases analysis in writing journal

The results showed that the students used more efficient and flexible methods to solve the problems they formulated. The use of writing journals contributed to deepening and connecting the concepts what they learned in classrooms.

(1) Mathematics embedded in an accident event

The data of Table 1 portrays that S1 drew a picture representing the situation of the 921 earthquake happened in Taiwan. To assisting the people ruined in the disaster in rebuilding their houses, S1's school had an activity for collecting money. He posed two mathematics problems about buying the tickets. When he solved the first problem described in Table 1, he counted the money one by one. But in the second problem, he used a unit of five-tickets to calculate the answer. Therefore, he used a flexible strategy to solve a complicated problem.

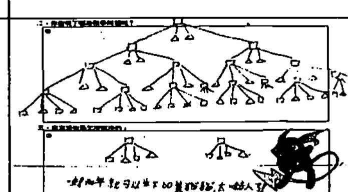
Table 1: S1 used mathematics in an accident event

Setting: To assisting the people ruined in the disaster in rebuilding their houses, the school held an activity of selling tickets for collecting money.	
Problems: (1) One carnival ticket is 200 dollars. Class A bought 5 tickets. How much are there altogether? (2) Class B bought 15 tickets. How much are there altogether?	園遊券每張200元，2年12班買5張，共要幾元？ 2年3班買15張，共要幾元？
Solutions: (1) 200+200=400 400+200=600 600+200=800 800+200=1000 Answer: 1000 元 (2) 1000+1000=2000 2000+1000=3000 Answer: 3000 元	① 200+200=400 400+200=600 600+200=800 800+200=1000 答：1000元 1000+1000=2000 2000+1000=3000 答：3000元

(2) Mathematics embedded in a casual accident

The data of the Table 2 shows that S2 and his mother caught a cat with 4 kittens. S2 and his mother go to find Doctor for help.

Table 2: S2 used mathematics in a casual accident

Setting and problems: Mother and I caught a cat with 4 kittens in our community. Mother and I bring the cat to zoo hospital for treatment. The doctor said that a cat gave birth to 4 kittens in a year, how many kittens would be after two years? If half of cats are female.	社區的流浪貓生了四隻小貓咪，我幫著媽咪捉住母貓，帶去給獸醫叔叔，叔叔說，一隻母貓一年可以生兩次，一次平均生三到四隻小貓，那麼，一隻母貓，兩年可以生下幾隻貓貓呢？ 如果：生下的貓貓，一半是公的一半是母的。□代表母的 △代表公的
Solutions: S3 used the tree diagram to solve the problem, and then she felt surprise and said " Whoa! After two year, the cat would give birth to 60 kittens. It is amazing!"	

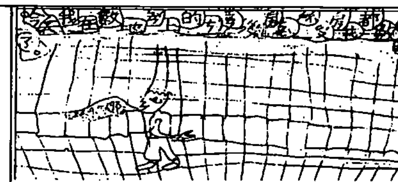
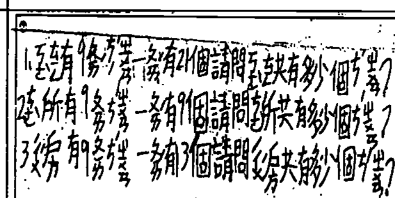
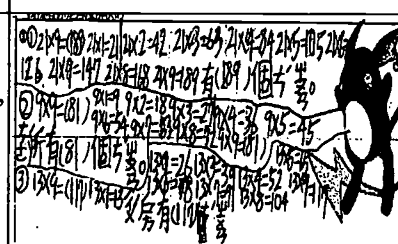
Doctor gave him a mathematical problem to solve: A cat gave birth to 4 kittens in a year, how many kittens would be after two years? If half of cats are female.

S2's solution was represented with a tree diagram in which square represents female cats and triangle represents male cats. S2 posed a complex problem but used an elegant method to represent his way of thinking. The second-graders have not been taught the tree diagram. He used an efficient way to solve mathematical problems in his familiar situations.

(3) Mathematics embedded in surrounding environments

The data of Table 3 shows that S3 was aware of the amounts of bricks on the floor equivalent to the area of the floor in each living room, kitchen, and restroom. The second-graders did not learn the formula of an area until the fourth grade. However, S3 has extended what he learned the fundamental meaning of area into a familiar situation to her. Moreover, she did not learn the multiplication with two-digit numbers until she entered into third grade. In accordance with her solutions, she was able to extend the operation of multiplication with one-digit into two-digit numbers. The result indicates that mathematical journal writing not only connected the mathematics students learned in school with daily situations but also made connections within mathematics.

Table3: S3 used mathematic in her house

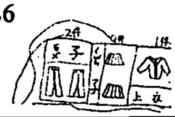
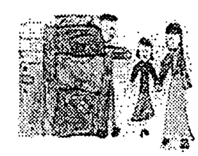
Setting: S3 described how to count the bricks in living room, kitchen, and restroom. He tried to understand the amount of bricks in each row multiplying by the amounts of rows in each room.	
Problems: 1. In the living room, there are 9 rows and each row has 21 bricks, how many bricks have totally? 2. In the restroom, there are 9 rows and each row has 9 bricks, how many bricks have totally? 3. In the kitchen, there are 9 rows and each row has 13 bricks, how many bricks have totally?	
Solutions: 1、 21x9=(189) 21x1=21,21x2=42,21x3=63,21x4=84,21x5=105,21x6=126,21x7=147, 21x8=168,21x9=189.There are(189)bricks. 2、 9x9=(81) 9x1=9,9x2=18,9x3=27,9x4=36,9x5=45,9x6=54,9x7=63,9x8=72,9x9= 81.Restroom has (81) bricks. 3、 13x9= (117) 13x1=13,13x2=26,13x3=39,13x4=52,13x5=65,13x6=78,13x7=91,13x 8=104,13x9=117, kitchen has (117) bricks.	

Improving children's journal writing by evaluating each other

The exchange of points of view was an essential part in using journal writings. The result indicates that it was a powerful way to evaluate journal entries each other for improving the quality of journal writing.

The data of table 4 shows that the writer of the journal and the commentator interacted frequently when they inspected the journals. S4 would teach the new idea to the commentator. S5 accepted the commentator's suggestion and applied the suggestion to the next journal. S6 explained the process of solution to commentator. S7 and S8 asked the commentator to indicate which part was good or clear. In the increasing use of journal writings, the students were required to write the journal or comment as clear as they could; otherwise, other students would criticize on unclear parts.

Table 4: Cases of children's comments and responses.

Setting	Posing problem	Solution	Comment	Response
S4 	Mother buys 6 hot dogs, 20 dollars per hot dog. Mother brings 150 dollars, how much left does mother have?	20×6= (120) 150 - 120 30	Wu! We did not learn multiplying with 20 yet, so I do not understand your calculation, ok!	My mother taught me, if you like, I can teach you, ok!
S5 	One egg is 5 dollars, how much does it cost for 10 eggs?	5+5+5+5+5+5+5+5+5+5= (50)	Your problem is too easy!	Thanks! I will change at next time!
S6 	There are two skirts, two trousers, and one T-shirt, how many clothes do I have?	2+2+1=5 2+2=4 4+1=5	Your problem is clear, but I do not understand your solution.	2+2+1=5 2+2=4+1=5 include skirts, trousers, and T-shirt.
S7 	There are 22 apples on the tree and 19 apples drop, how many apples are still on the tree?	22-19=(3) 20-10=10 10+2=12 12-9=(3)	Tang! Your solution is very good!	Thanks your comment, but can you tell me where is good?
S8 	Father gives me 60 dollars. Mother gives me 50 dollars. How much money do I have in total?	60+50=110 60 + 50 110	Ya-tang! You need to work hard and I will give you 100 scores.	Chung! Can you tell me what the problem is? I do not know what you mean? Next time, please tell me clearly, OK?

Five levels of comments were identified in this study

Five levels of comments were identified in this study, when analyzing students' comments on Post-it Notes: unrelated comment, general comment, question comment, suggestion comment, and evaluation comment. (1) **Unrelated comment** is that students gave comments irrelevant with the problem. For instance, "Your paper is very clean!" "You need to write the date". "If you change your temper you will write better". (2) **General comment** is that students gave general comments but did not indicate what is or where is clear. For example, "I do not understand what you say" "Your problem has a contradiction" "Your problem cannot be solved". Those comments do not specify the clear part of the problem. (3) **Question comment** is that students gave comments that pointed out the problematic parts but did not give further suggestions. For example, "You do not write the price of apple so how you can calculate the answer" "39+52=81 is wrong". (4) **Suggestion comment** is that students gave some suggestions but did not illustrate the reasons. For example, "You can use multiplication to do it". When a student calculates with 30+50=80, 100-80=20, A suggestion from others is that "You can calculate it by 100-30=70, 70-50=20". (5) **Evaluation comment** is that students identified the problematic parts and also gave some suggestions. For example, "Your answer 71 is wrong, you should write 43+26=69".

Those five levels of comments are characterized as from the lowest level to highest level. Suggestion level is higher than question level. The students who were able to identify the problematic parts moved toward the advanced level that gave further suggestions without illustration. Evaluation comment contains the questions addressed and suggestions.

Developing the levels of comments in this study

From Figure 2, initially, students didn't know how to write comments when they inspected others' journals. The number of students' comments stayed at unrelated comment, then decreased from general comment to question comment. In the second month and the third month, students' comments decreased the unrelated level and increased to question level and suggestion level. In the fourth month, students' unrelated comments disappeared and moved toward the evaluation level.

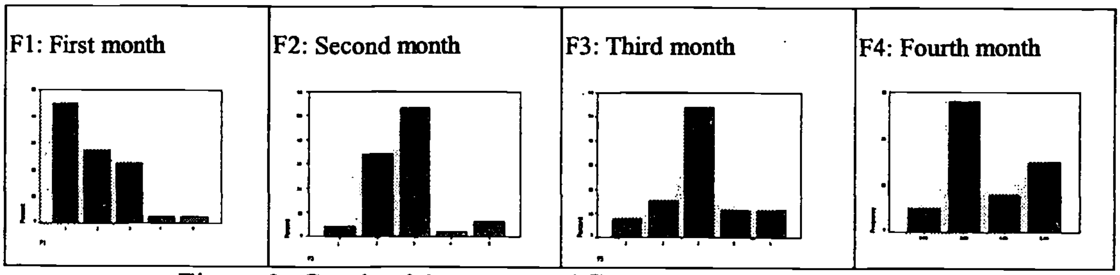


Figure 3: Graph of the Levels of Commends Within Four Months

Conclusion and Discussions

The result indicated that most students performed well in solving the problems they created from daily activities by using mathematical journal writings. The daily activities mathematics embedded into that were perceived by the students included drastic disasters, causal accidents, and surrounding environments. It appears that children could use flexible and efficient strategies to solve the problems relevant with familiar situations. Nevertheless, the strategies were been not taught in school for the second graders. One possible interpretation was that the mathematical problems they formulated were based on their familiar situations; as a result the solutions were the ways of sense-making to them. The other interpretation was that there was a close relationship among the three steps so that students understood the relationships among setting, problem, and its solution. A significant finding was that the mathematics concepts students learned were easily extended and connected by the use of mathematical journal writings.

The critical task involving the study provided the opportunities for children to take-and-share perspectives and sequentially resulted in the improvement of writings and comments. The students evaluated each other and discussed in class about the way of writing journals and what comments are valuable for journal writers. The higher quality of comments brought about the higher quality of mathematical writings. Finally, children were aware of the mathematics they learned in school embedded in various cultural settings and daily activities through the use of journal writings. As a result, their abilities in formulating and solving the mathematics problems were improved.

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ALGORITHMIC MODELS: ITALIAN AND ISRAELI STUDENTS' SOLUTIONS TO ALGEBRAIC INEQUALITIES

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This paper describes a study regarding Israeli and Italian students' solutions to algebraic inequalities. Fischbein's notions of intuitive and algorithmic knowledge are used to analyze the data. The findings presented here show similarities in students' correct and incorrect solutions, in both countries. The findings indicate that students intuitively considered the balance model, saying that it is always permitted to "do the same thing on both sides" of a given inequality. Most students were intuitively drawing analogies to the solutions of related equations, either by excluding only zero values when dividing both sides of an inequality by a not-necessarily positive value, or when dividing by an expression without the exclusion of zero values as well.

There is a wide call for using students' correct and incorrect ways of thinking in teaching (e.g., NCTM, 2000). While this recommendation seems to be "speaking for itself", any attempt to take it from theory to practice, shows how complex this is. Among the prerequisites for such teaching, are familiarities with students' correct and incorrect reactions to various related tasks, with possible reasons for students' errors, and with available teaching approaches to be considered under specific circumstances. All of the above are needed, but do not guarantee that the student whom we teach will gain mathematical understanding. In this paper we use Fischbein's notions of intuitive and algorithmic knowledge as a theoretical framework to analyze students' solutions to inequalities, and consequently, we suggest possible implications for teaching.

Fischbein's theory deals with the formal, the intuitive and the algorithmic components in one's mathematical performance (e.g., Fischbein, 1993). According to Fischbein, formal knowledge is based on propositional thinking. It relates to rigor and consistency in deductive construction, being free of the constraints imposed by concrete or practical characteristics. Intuitive knowledge is a kind of cognition, which is accepted directly and confidently as being obvious, imparting the feeling that no justification is required. Algorithmic knowledge is the ability to use theoretically justified procedures. Each of the three components plays a vital part in students' mathematical performance, but since they are usually inseparable, the relations between them are not less significant. Fischbein explained that "sometimes, the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures" (ibid. p. 14). Consequently, he identified and investigated with his colleagues a number of algorithmic models related to various mathematical operations, such as subtraction of natural numbers, and methods of reduction in processes of simplifying algebraic expressions (e.g., Fischbein & Barash, 1993).

They explained, for instance, that the distributive law serves as a prototype in the algorithmic models of simplifying algebraic expressions, triggering students to present $3(a+b)^2$ as $3a^2+3b^2$. Here, the algorithmic model is expressed in the shift from $(a+b)^2$ to a^2+b^2 to if these are equivalent expression, like $2\cdot(a+b)$ and $2\cdot a+2\cdot b$. This paper discusses Italian and Israel secondary-school students' intuitive ideas and algorithmic models when solving algebraic inequalities.

Usually, publications in mathematics education journals present instructional suggestions with no research support. They recommend, for instance, the sign-chart method (e.g., Dobbs & Peterson, 1991), the number-line method (e.g., Parish, 1992), and various versions of the graphic method (e.g., Dreyfus & Eisenberg, 1985; Parish, 1992). Only little attention has been paid to students' conceptions of inequalities (e.g., Bazzini, 2000; Linchevski & Sfard, 1991; Tsamir, Almog & Tirosh, 1998; Tsamir & Bazzini, 2001). These studies pointed, for instance, to students' difficulties in grasping the role of the sign, and in using logical connectives.

The present study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities (see also Tsamir & Bazzini, 2001). In this paper we focus on the question: What intuitive ideas and what algorithmic models can be identified in Israeli and Italian secondary school students' solutions to algebraic inequalities?

METHODOLOGY

Participants

One-hundred-and-ninety two Italian and 210 Israeli high school students participated in this study. All participants were 16-17 year old who planned to take final mathematics examinations in high school. Success in these examinations is a condition for acceptance to academic institutions, such as universities.

In both Italy and Israel, algebraic inequalities usually receive relatively little attention and are commonly presented in an algorithmic way by discussing various algebraic manipulations. That year, the participants in this study had studied the topic of algebraic inequalities, including linear, quadratic, rational and absolute value inequalities. In both countries, the participants were taught different methods for solving the different types of inequalities, such as, graphic methods to solve quadratic inequalities, and "multiplying by the square of the denominator" for the solutions of rational inequalities.

Tools

Italian and Hebrew versions of a 15-task questionnaire were administered to the students. Here we focus on three tasks dealing with "dividing an inequality by a not-necessarily-positive factor". Two tasks ask to judge statements regarding parametric inequalities, and the third task, poses a parametric "solve" inequality.

Research findings indicate that when solving rational inequalities, students frequently multiplied both sides of given inequalities by a negative number without changing the direction of the inequality (e.g. Tsamir, Almog & Tirosh, 1998). It was also reported that students encounter difficulties when solving mathematical tasks, presented in a way different from the way they are used to. For example, when having to deal with parametric equations and inequalities that are commonly not discussed in class (e.g., Furinghetti & Paola, 1994; Ilani, 1998). Taking into account these data, we constructed tasks I, II, and III.

Task I: Examine the following claim: for any a in R, $a \cdot x < 5 \Longrightarrow x < 5/a$

Task II: Examine the following statement: for any $a \neq 0$ in R, $a \cdot x < 5 \Longrightarrow x < 5/a$

Task III: Solve the inequality: $(a-5) \cdot x > 2a-1$ x being the variable and 'a' a parameter.

The tasks were designed to provide information regarding students' distinction between the sufficiency of mentioning $a=0$ as a counterexample in Task I, and the insufficiency of the '$a \neq 0$' condition given in Task II, a condition, which is sufficient in the case of equations. Our aim was to see whether students regarded this limitation as sufficient for the inequality as well. While Tasks I and II asked the students to examine the claim regarding the equivalence of parametric inequalities, Task III dealt with the same issue in a different manner, asking the students to solve a similar given parametric inequality.

Procedure

The students were given approximately one hour, during mathematics lessons, to complete their written solutions. In order to get a better insight into the students' ways of thinking, forty-five students were individually interviewed, usually being asked to elaborate on their written solutions. Each interview lasted 30 to 45 minutes.

RESULTS

The results will be presented in the following order. First, we present an analysis of students' responses to Task I, Task II and Task III. Then, we present the intuitive ideas and the algorithmic models we identified in the students' reactions to the three tasks. It should be noted that no significant differences were found between the Israeli and Italian students' solutions.

Students' Reactions to Task I

Most of the participating students correctly responded to this task (see Table 1). About three- quarters of the students accompanied their correct judgement with an acceptable justification – 18% of them elaborated on the role of the sign of the value substituted for a in determining the direction of the ">", and 54% mentioned only the "zero case" as a counterexample to the given statement.

In their explanations, students who *related to the sign of a* wrote, for instance, "for $a>0$ this is a correct statement, but for $a \leq 0$ it is not"; or "there are three cases: when $a>0$ the statement is correct, for $a = 0$ it is impossible to divide by zero, and when a is negative the conclusion is that $x<5/a$". Others explained more briefly that, "this statement is correct only for positive 'a's"; or "when a is negative the direction of the sign changes".

A number of students provided specific *counterexamples*, such as, "if a=(-1) the statement is not correct"; or "if –5x<5 the conclusion is that x>(-1), instead of x<(-1)". A few added "I gave one example, but a single counterexample is sufficient for proving that the statement is false."

Table 1: Frequencies of students' solutions and explanations to Task I (in %)

Judgement Justification	**ISRAEL** N=210	**ITALY** N=192	**TOTAL** N=402
False*			
Relating to the sign of a	20	16	18
Relating only to a $\neq 0$	59	48	54
True	6	2	4
Other	15	34	24

* Correct solution

Most prevalent was the students' tendency to use *only the 'a=0' case,* as a *counterexample* to refute the statement. They wrote, for instance, "this statement is false when a equals zero"; or "the statement is false, because of the case of a=0". Many added "division by zero is undefined, therefore the statement is not *always* correct."

In their oral interviews these students' typically commented,

> Sophia: the statement here refers to any number. BUT, since it is false for a=0, the statement is not true for ANY number. It is, therefore, false.
>
> Gabby: It is enough to show that the statement is false in one case, and zero is such a case.
>
> Dana: One counterexample is sufficient for proving that the statement is incorrect. Zero, that is a=0 is such a counterexample.

A number of them added,

> Gabby: It is similar to equations.
>
> Dana: I am sure, because I know it from equations.

The few students, who regarded this statement as correct, explained, "we divided both sides by the same thing". One student based his incorrect "true" judgment on an example: "this is correct. For example, if 2x<5 then x<2.5".

In their oral interviews these students added,

> Naomi: I divided both sides by a [pause]. It's OK. I operated in the same way on both sides.
>
> Jonathan: It's OK to do the same thing on both sides. When doing the same operation on both sides, the equivalency is preserved.
> Interviewer: What do you mean?

Jonathan: In equations, it is allowed to add, subtract, and multiply [pause] to do any operation with the same number on both sides.
Interviewer: What do you mean by "allowed"?
Jonathan: [thinking] it does not change the solution...
Interviewer: You related to equations, but here we have an inequality...
Jonathan: It's the same...

Students' Reactions to Task II

Only 30% of the participants correctly responded that the claim is false and accompanied their response with a valid justification (Table 2). All of them related to the role of the sign in their decision. They explained, for instance, "if a is negative then $x>5/a$"; or "the claim is correct only for positive a". Some students added specific examples, "It is false, because it holds only when a is positive. For example, if $a=(-2)$, then $-2\cdot x<5 \Longrightarrow x>(-2.5)$". Students were usually satisfied with a single counterexample, occasionally explaining, "one counterexample is sufficient in order to show that the proposition is false".

Table 2: Frequencies of students' solutions and explanations to Task II (in %)

Judgement *Justification*	**ISRAEL** N=210	**ITALY** N=192	**TOTAL** N=402
False*			
Valid explanation	37	23	30
True			
Relating to $a \neq 0$	50	57	53
Other	13	20	17

* Correct solution

Most prevalent in both countries (over 50%) was the incorrect response that the statement is true, which was explicitly based on the given that $a \neq 0$. Students wrote, for instance, "It is correct because of the given condition that $a \neq 0$." In the interviews, these students' elaborated explanations pointed to connections they made between equations and inequalities. In the interview Daniel explained,

Daniel: I can show how this works in an easier case [pause]...
Interviewer: What do you mean?
Daniel: I am looking for an example of a similar, but easier, task... like $2x=6$. Yes, here, x simply equals 6/2 [pause], then, $ax=6$ means that $x=6/a$, but this is true *only* if a is not zero. If $a=0$ than it is impossible to divide... by zero.
Interviewer: Your example is an equation, and here we have an inequality.
Daniel: It's the same [pause]. In a way inequalities are a certain type of equations. Just that equations are easier, so I use simple examples of equations when I have a difficult inequality.

Several students gave explanations, similar to the following one given by John,

John: In equations and inequalities, dividing by zero is problematic. But if we deal with this problem, operating with identical numbers and by means of the same operation on both sides is not only permitted, it is actually *the way* to solve the given tasks.

Students' Reactions to Task III

Fewer than 15% of the participants provided a comprehensive analysis of the various (positive, zero and negative) options for 'a' (Table 3). About 50% of the participants did not solve this task, and more than a quarter used an "equation-like" approach, writing that $x>(2a-1)/(a-5)$ for $a\neq 5$. In their interviews, a substantial number of them clearly mentioned drawing analogies to equations. Bettina, for instance, explained,

Bettina: I divided both sides by the same expression, but I had to make sure that it is a non-zero expression. So, I wrote that a cannot be 5, because then a-5 equals zero...

Interviewer: Are you sure that five is the only problematic value here?

Bettina: [confidently] sure. I have done that a million times when solving equations.

Table 3: Frequencies of students' solutions to Task III (in %)

	ISRAEL N=210	ITALY N=192	TOTAL N=402
Full Solution*	13	12	12
$x>(2a-1)/(a-5)$	18	15	16
$x>(2a-1)/(a-5)$, $a \neq 5$	29	26	27
Other	41	48	46

* Correct solution

About 15% of the students gave the $x>(2a-1)/(a-5)$ solution (with no mentioning of the $a\neq 5$ limitation), and in the oral interviews they frequently mentioned the use of equation-ideas. All of them explained that it is allowed to "do the same thing on both sides of an inequality". Many, like Anna, related to equations and to inequalities,

Anna: I divided both sides by a-5?

Interviewer: Is it OK to divide both sides by a-5?

Anna: Yes. I have done the same thing on both sides. If you do the same thing on both sides of an equation [pause], I mean an inequality [pause], actually both, you reach an equation or an inequality that has the same solution as the given one.

Interviewer: Always?

Anna: It is not only allowed, it is necessary to do that in order to solve the problem.

DISCUSSION

We opened this article by referring to Fischbein's notion of algorithmic models, rooted in the application of intuitive ideas and/or formal overgeneralizations to rigid algorithms (e.g., Fischbein, 1993). These models are usually coercive, used with confidence and grasped as being self-evident, even though they frequently lead to erroneous solutions. According to Fischbein, algorithmic models evolve when students' intuitive ideas manipulate their formal reasoning and/or their use of algorithmic procedures. In this spirit we posed the question: What intuitive ideas and what algorithmic models can be identified in Israeli and Italian secondary school students' solutions to algebraic inequalities?

An initial examination of the Italian and Israeli participants' written solutions to Task I revealed almost no intuitive, erroneous ideas. Most of the students correctly rejected the statement: $\forall a \in R\ a \cdot x < 5 \implies x < 5/a$, and based this rejection either on a comprehensive analysis of the sign of a, or on a single counterexample. In the latter case, many students used zero as their counterexample, and most of them, correctly stressed that one counterexample is sufficient for the refutation of a proposition.

Had we stopped here, we might have assumed that most students have a good formal understanding of such parametric inequalities. However, an examination of their responses to tasks II and III revealed that this was not the case. The students' solutions to the latter two tasks, and their explanations in the interviews, clearly pointed to their intuitive grasp of inequalities as being "similar to", "the same as", "a certain type of" or analogous to equations. That is to say, equations were found to serve as a prototype in the algorithmic models of solving inequalities. This algorithmic model had two appearances: (1) "do the same operation with the same numbers on both sides"; or (2) "exclude the possibility of zero in the denominator, and then, do the same operation with the same numbers on both sides".

Those who applied the concise version of the algorithmic model (1), could easily be identified, since they incorrectly solved Tasks I and II by judging the statements to be true, or erroneously wrote in Task III that $x>(2a-1)/(a-5)$ is the solution to the inequality $(a-5)\cdot x>(2a-1)$. These students used the balance model when incorrectly solving equations, assuming that "doing the same thing" on both sides of an equation *always* leads to an equivalent equation, and consequently to the solution. When solving inequalities, they based their solutions on the equation-model, and thus, made no restriction when dividing (same operation) both sides of the inequality by the same expression. They were so sure that equations and inequalities follow the same set of rules, that they occasionally confused the two terms when explaining their solutions (See interviews with Anna, Dana and Gabby).

Most participants applied the second version of this algorithmic model. That is to say, they knew that when solving equations they should be careful not to divide by zero, and since they held the equation-model for solving inequalities, they imposed the same condition. These students usually answered Task I correctly, providing the '$a=0$' case as a counterexample to refute the given proposition. However, they frequently, incorrectly regarded the proposition of Task II as valid or suggested $x>(2a-1)/(a-5)\ a\neq5$ as the solution for Task III. The use of the equation-model by these students could not be identified by their written solutions to Task I only.

We have seen three, parametric inequalities that may be helpful in triggering students who hold different equation-based models to answer differently, and occasionally also incorrectly. We would like to suggest such tasks to be presented and discussed in class, in order to promote students' awareness of their intuitive ideas and the resulting algorithmic model they intuitively use. How to implement such instruction and their impact on students' performance should further be investigated.

In the presentation we will show, how the equation-model was identified in students' solutions to additional inequalities in our study.

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FROM THEORY TO PRACTICE: EXPLAINING SUCCESSFUL AND UNSUCCESSFUL TEACHING ACTIVITIES (CASE OF FRACTIONS)

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Abstract: In a teaching experiment I examined a theoretical model of mathematics teaching and learning in practice. In this paper I focus on how the model can guide the teacher's thinking about students' understandings and the generation of activities that foster intended transformations in those understandings. As a researcher-teacher I taught, twice a week for four months, basic ideas of fractions to 28 third graders in a public school in Israel. The analysis of both classroom data and researcher's documented reflections indicates how the model can empower the generation and explanation of successful teaching activities, as well as thinking about and adjusting unsuccessful ones. It also highlights the importance of research-based, content-specific models of students' understandings for successfully implementing the model in practice.

Recent reform documents in the USA (NCTM, 1991) stressed that a primary goal of mathematics teaching is to promote students' conceptual learning of key mathematical ideas and procedures. In line with a constructivist principle of active learning, teachers are encouraged to use valuable methods such as problem solving, manipulatives, and whole class discussions. Sometimes the techniques bring about conceptual understandings of the intended ideas but often times they don't. In both cases, what seems to be missing is a systematic, developmentally appropriate guidance for teachers how to generate and adjust activities that advance students to intended understandings. To provide such guidance, my colleagues and I (Simon et al., 2000; Tzur & Simon, 1999) have recently developed a model that relates learning with teaching of new mathematical conceptions. The purpose of the present study was to examine in practice how this model can guide the teacher in (a) generating and/or selecting activities that successfully bring about the intended learning and (b) thinking about and adjusting unsuccessful teaching activities.

CONCEPTUAL FRAMEWORK

The study was part of a four-month teaching experiment that drew on a social-constructivist approach (Cobb & Bauersfeld, 1995). This approach coordinates social and psychological perspectives of teaching-learning processes. The model at issue is an elaboration of the cognitive perspective and, itself, served as a framework for the study. Since the study focused on relationship between teaching activities and students' learning, I briefly present only components of the model that served in the data analysis of successful and unsuccessful teaching activities (for details see Simon et al., 2000; Tzur & Simon, 1999). I also present key constructs of a content-specific model of children's fraction learning that served me, as the researcher-teacher, in making sense of students' thinking.

Knowing, Learning, and Teaching. The model regards knowing and understanding as using a conception--a dynamic mental compound that relates between an activity and its effect(s) (abbreviated as A-E relationship). The compound is not mainly the activity nor the effect(s) but the dynamic relationship consisting of both. In this paper I will use the term A-E relationship and conception interchangeably. Learning is regarded as the process of transforming current conceptions into new ones and/or coordinating anew established conceptions (Piaget, 1985). The mental mechanism that enables learning is reflection on A-E relationship (abbreviated as Ref*A-E relationship). The term reflection refers to the continual mental comparison between one's goals for and effects of his or her activities. Note that from the learner's point of view, Ref*A-E relationship does not need to be directed toward making specific conceptual advances (awareness is not implied). Teaching is regarded as a cycle of four principal activities. It begins with inferring learners' current conceptions. It proceeds through hypothesizing a learning trajectory (Simon, 1995) from those current conceptions to the intended ones. It moves to designing and engaging learners in activities they can assimilate and use as a means to form the intended conceptions. Finally, it implies orienting learners' reflections on A-E relationships via probing questions and tasks which also serve to infer learners' conceptions, and so on.

Conceptualizing Addition of Fractions. The content-specific model that I used in the study does not regard unit fractions (1/2, 1/3, etc.) merely as parts of equally partitioned wholes. Instead, it regards unit fractions as particular relational quantities, that is, entities that stand in a multiplicative relation to a given whole (cf. Behr et al, 1992). For example, a given quantity called "a third" is not just one-out-of-three equal parts; rather it is a third because another quantity, which was designated as "a whole," is **three times as much** as the given quantity. (Children often say, "The given piece fits-in-the-whole 3 times.") Thus, learning to add unit fractions requires that learners establish those units as relational and symbolically meaningful entities in and of themselves. Only then addition, which learners have established for whole numbers, can be coordinated with and applied to those entities.

METHODOLOGY

For four months I conducted a classroom teaching experiment (Cobb, 2000), while teaching 28 third-graders in a public school in Israel twice a week. Prior to the period reported herein the students formed their conception of unit fractions (1/n) via the Repeat Strategy that they used in tasks of equally sharing a given whole among a number people. To partition the whole (24 cm long paper strips), they estimated the size (share) of one person, repeated it the number of times implied by the number of people, and compared the iterated whole with the given whole. Then came the critical step of the repeat strategy: adjusting the size of the initial piece and reusing the previous steps (if the iterated whole was longer than the one to be shared--they shorten the initial piece; If the iterated whole was shorter--they made the initial piece longer). When the present study began, 26 out of 28 students had established unit fractions as unique quantities. This was indicated by their ability to order unit fractions while explaining the inverse relationship between size and number of parts (e.g., 1/5 is larger than1/7 because to fit-in-the-whole 7 pieces one must make each piece smaller).

Data for the present study consist of:

1. Videotapes of 7 lessons conducted between December 10, 2000, and January 7, 2001. I taught the lessons on Friday and Sunday to create a sense of continuation while allotting the time needed between lessons for ongoing analysis and planning.
2. Tapes and transcripts of my reflections, recorded (audio) after every lesson.
3. Tapes and transcripts of audio-recorded conversations with the homeroom teacher (who observed and assisted my teaching) that took place 3 hours after every Sunday lesson.
4. Students' written works from classroom.

A retrospective analysis of the data focused on relating teaching activities and students' learning. The term 'successful teaching activity' is used to imply that it brought about students' learning of the intended idea ('unsuccessful' if it did not). The 7 lessons chosen represent both unsuccessful and successful activities.

ANALYSIS

This 3-part section tells the story of successfully teaching the students addition of like-denominator fractions. In the first part I portray my inference of the students' powerful understandings which prompted my decision to begin teaching them addition of unit fractions. In the second part I analyze my first, unsuccessful attempt to teach that topic. In the third part I analyze my second and very successful attempt to teach it.

<u>Part 1: Students' Thinking</u>. Before teaching addition of unit fractions, I evaluated the students' thinking to make sure that for them - these units were meaningful entities. On December 12, I engaged the students in a simple task. I gave each small group one long paper strip (the whole) and an envelope with nine, exact-size paper strips that I cut in advance (1/2 through 1/10). They were to randomly pick a piece from the envelope and find what fraction it was. After each group had picked at least three pieces we shifted to a whole class discussion. ('R' stands for researcher-teacher.)

R: (Holds an envelope like theirs) I pick a piece and I want to know what fraction is it of the whole on the board? Who can tell me how to figure it out? (Only a few students raise hands.)
R: Please go ahead and talk about this question again in your groups. (Group work, one minute.)
R: Okay, so what would I have to do?
Sally: Measure how many times the piece fits-in-the-whole.
R: How should I do that? Nora?
Nora: Put the piece exactly under the whole; mark its right end; then move it to the right and…
R: (Stops Nora before she explains the entire process) Okay, before I do this, what am I trying to find? When I'm done, what would I find? (A little later) Why am I doing what I'm doing?
Ruth: To find what fraction is it.
R: How would I know what fraction is it by doing what Nora said?
Ruth: You'll count how many such pieces fit-in-the-whole.

Reflecting on their activity of picking pieces from the envelope and stating what one <u>would have</u> to do was not a trivial task for the students. Thus, I asked them to re-discuss it over in the small groups so that their reflection would be based on what they found key to their activity. Then, by stopping Nora's articulate way of explaining the process, I attempted to orient students' reflections on the multiplicative relationship between their goal and activity. Key to my intervention was to let them carry out the activity in their mind, not in actuality. As a result of both interventions, Ruth was able to clearly state the intended relationship.

Following Ruth's reply we executed Nora's suggestion and all said that the piece was one-tenth because it fit-in-the-whole 10 times. So I asked, "What if I drew a piece that fits 7 times?" and Bob replied, "It would be one-seventh." To the same question only with '3 times' and '6 times' Yoel and Ellen replied 'one-third' and 'one-sixth,' respectively. The lesson on December 15 began with short individual work on a worksheet. Then, I presented a new situation. I put a long, red paper strip and said it was a whole, then put another red piece under it and the students easily detected that it was half of the longer strip. Then I put a green strip, shorter than the red whole but longer than its half, and under it a green strip that the students easily detected as one half of the green whole. Before I could say anything, Aaron brought up my intended issue.

R: Aaron raised a point that I would like you to discuss as a question in your groups. (Holds up the green and the red halves) Both the red half and green half are halves, but the red half is longer than the green half. How do you explain this? (Group work, 1.5 minutes.)
R: (To the whole class) Okay, what if I would put a red third and a green third on the overhead. Would the red third be longer or shorter than the green third?
Jack: The red third will be longer.
R: Why?
Jack: Because the red whole is longer than the green whole. Therefore, [all] parts of the red should be longer than parts of the green whole. So the red third must also be longer.
R: (A little later) Why is the red third longer than the green third?
Noel: Because to fit-in-the-red [longer] whole three times the third has to be longer.
Judah: And to fit-in-the-green [shorter] whole three times the third has to be shorter.
(A little later, without any prompt, Lee says he has a different idea and comes to show it.)
Lee: If you take this (puts the green third below the red half) it also comes as a half.

The students' work on different-size halves and thirds indicated a significant understanding of unit fractions: they saw them as multiplicative relations with respect to a given whole. They could anticipate the effects of the activity to determine a fraction and hence could shift their attention from one whole to another as a means to compare parts of those wholes. Moreover, Lee's initiative indicated a powerful understanding—he was able to re-designate a part as a whole and compare it to a part of a different whole. I thought they were ready for addition.

Part 2: Unsuccessful Attempt. On December 15, toward the lesson's end, I distributed a worksheet that introduced a coloring activity with which I planned to initiate adding of unit fractions. They worksheet showed a whole partitioned into 6 equal parts. I asked the students six questions that involved coloring certain parts and they completed the first two questions. The first asked them to color a part and say what fraction it was (answer - '1/6'); the second asked to color another part on that whole and say what fraction was that part only (answer - '1/6'). To my genuine surprise, some replied '1/5' to the second question whereas others simply did not know. In my written notes on the matter I wrote:

Journal entry, Saturday, December 16: I thought again on tomorrow's lesson, Dec. 17 [my initial plan was to go on with coloring]. Already by the end of class I thought, 'What might have been the reason that the [coloring] activity caused some students difficulties to disembed [dissociate] the pieces of the partitioned whole?' In class I thought that this is only a difficulty of unclear convention, hence I'll have to tell them to count the colored parts. Then I realized - I should focus on the A-E relationships involved in the task, which re-focused my attention on 'What activity could be focusing the students' attention when coloring?' The coloring activity creates a background-foreground problem and masks the counting activity to which I wanted to orient them. Coloring is an activity of pointing at the specific unit [to count] but it creates problems that I must also think about in terms of conceptualizing the situation, not merely in terms of conventions. The specific conception is the understanding that each part is 1/6 no matter what its color.... Because the students completed only the first two tasks I had the time to reflect on their thinking and on my initial tendency (in class) to dismiss the problem [convention]. My focused reflection about the activity as a plausible source for the students' difficulties led to a change in my plan for the next class. First, I'll ask every child to point to each part, colored or not, and ask what fraction was it of the whole [anticipated – '1/6' for each]. Only when I'll be clear how students 'see' the colored parts [separately] shall we move to adding fractions.

The entry above indicates both the power and limitation of the model. It provided powerful guidance to focus my reflection on plausible conceptual sources for students' difficulties. In particular, it guided me to infer the particular activity that the students might have been using to assimilate the task. My knowledge of content-specific models of students' fraction learning, brought forth the 'disembedding activity' notion. I used this notion to bridge (so I thought) the gap between my expectation that coloring parts will simply serve to point at individual parts to be counted and the students' difficulty to see the parts individually. The limitation is that a teacher's inference can be inappropriate. Here, I did not infer that a problem to isolate parts could result from the coloring activity itself. In that case, just asking students to identify each part as 1/6 could not help because it only dealt with effects of the coloring activity.

On December 22 we returned to the coloring activity via the 6-question worksheet. This was a daunting, teacher-led lesson. I read out loud the first question. The students replied that the first colored part was 1/6 of the whole. I read the second question. A few students replied that the second colored part was ... 1/5 of the whole. On the spot I thought that we must establish some convention, so I told the students that we also have to count the colored parts.

I then asked the third question, 'What fraction were both colored parts combined?' which fostered a very interesting learning experience. During a 3-minute small-group work, four (out of six) groups had figured out that the two parts combined would fit-in-the-whole three times, hence it is one-third. I devoted 8 more minutes for a whole class discussion about this exciting, student-generated idea. Then, 10 minutes to the end of the lesson, frustrated for not achieving my goal, I decided to present the conception of adding non-unit fractions as if it was a matter of learning a new convention. I told them that just like when one adds apple plus apple to get two apples or a flower plus a flower to get two flowers, one adds one-sixth plus one-sixth and gets two-sixths. Never before or after this repetitive teaching activity had I used such a method and, not surprisingly, it was unsuccessful. For example, when asked to color and add the third part students responded, 'I don't know,' '1/3,' '1/2,' and only a few wrote '3/6.' Students' difficulties on December 22 perturbed me greatly and led me to back to the model as a guide for resolution.

Journal entry, Sunday, December 24: Immediately after Friday's class, I thought that the coloring activity failed to foster the intended learning because it might not be appropriate. The homeroom teacher said that the students would need to work on something concrete, to 'feel the units in their hands.' I thought, 'True, but coloring is a concrete activity.' Reflecting on that thought, I realized the need to have an activity that focuses students' attention on counting, where they separately point to each part while accumulating the parts. This need to figure out such a counting activity focused me back on the problematic issue of disembedding. I immediately asked myself why didn't I think about that issue earlier. This self-criticism pushed me to re-examine my reasoning about the failure and I realized that a wrong inference about the students' current understanding was also a possible. Thus, I decided to re-evaluate students' thinking before using a different activity for coordinating their established counting operations with the newly established unit fractions.

The entry above indicates the practicality of a model that combines general constructs of learning and teaching with articulated, content-specific constructs. The model implies to re-examine not only the activity but also students' thinking and enables the teacher to analyze specific component understandings needed for successfully learning to add unit fractions. Thus, consciously using the model to guide my reflection on students' work resulted in executing a specific plan that, here, rectified my previous inferences. On the basis of my re-examined inferences I generated a teaching activity that was more likely to foster the intended coordination between students' counting operations and unit fractions.

Part 3: Successful Attempt. On December 31 the students worked on a 2-page worksheet that contained 4 problems designed for the re-examination. I read every student's worksheet as soon as he or she finished it and this re-confirmed my inference. Thus, I began teaching them the game that I designed in line with my December 24 reflections. Each pair received a paper strip - the whole. Each student received 6 pieces marked 1/10. First, they checked that a part was, in fact, 1/10. Then, taking turns, they rolled a die and laid down as many 10ths as it indicated. Each student was supposed to say the names of the fractions as they accrued, write the fraction that each of them laid down on a chart, add their two fractions, and figure out game points (0, 1, 2, or 3, governed by chance) that the pair received in that turn. Before the lesson ended we had enough time to let two students demonstrate one turn of the game.

On January 5 the students quickly got engaged in the joyful game, which quickly brought forth their counting activity as a means to add fractions. They realized that what matters in adding and writing the results is the two non-unit fractions each player had and hence applied addition of whole numbers. The pairs quickly completed 10 turns of the game (10ths) and I gave them another worksheet for a game with 8ths. I moved about pairs to watch their work. Although successful, I was aware of two issues that may be hidden. First, several sums were improper fractions (e.g., 5/10+6/10, 6/8+4/8) and I needed to know what sense they made of them. Second, I wanted to make sure that as they were coordinating counting with unit fractions they were not losing the meaning of the units added.

R: (About 15 minutes to the end of lesson) Here's a question that we would talk about later. Some of you saw that sometimes we go beyond the whole. Please pay attention to the number of parts in such situations.

R: (A whole class discussion about 8 minutes later) There were times when you both rolled the die and got a number of pieces that exceeded the whole. Let's talk only about the game with 10ths. We have a whole and we have 10ths. When does it go beyond the whole? Please raise your hands if you can explain it. You may like to recall what you've just done in the game. (Several students raise hands.) Let's hear from Joe.

Joe: With twelve- and with eleven [pause], 10ths.

(R. takes a minute to write on the board Joe's examples, 6/10+5/10=11/10, 6/10+6/10=12/10.)

R: Joe says that in these cases we have more than the whole. If you think that Joe is right please just raise your hand quietly. (A majority of the students raise their hands.)

R: Who thinks that Joe is wrong? (Nobody raises a hand.) Aaron, would you explain Joe's idea?

Michael: (Bursts out) Because the whole is ten 10ths.

R: (Hisses them) Only Aaron.

Aaron: Because, because ten 10ths is one whole.
R: And so if we have 11/10, or 12/10 (points accordingly on the board), then it is more than the whole?
Many students: This happened to us, too.
Tom: This did not happen to us with 10ths but it did with 8ths.... We got ten 8ths!

The model-based question I asked in the first line of the protocol above meant to orient the students' reflection on relations between their activity and an effect of mathematical interest (to me). Consequently, students related between their activity—adding unit fractions beyond the whole—and the numerosity of the partitioned whole. Thus, I fostered an understanding beyond simply being able to add unit fractions. This was particularly indicated in Tom's voluntary contribution, he learned to conceive the partitioned whole as an entity and could then make the abstract application to his personal experience of 10/8.

R: (On Jan. 7, after returning the all-correct worksheets to the students) Today, I want us to make a rule that will help you from now on. (Writes on the board and says) 'How do we add fractions?' We want to come up with a rule that always works. And just as usual, we want to see that everybody understands why this is the rule. (A little later) Okay, now I'll write the question the board. (Writes on the board while saying) 2/7+4/7= . We want to know how much is it, this time with no dice and no pieces. And I wrote these two only as an example. Because the question is not simply if you know the answer but if you can state a rule how will we always know. (About half of the class raises hands.) Hmm, okay, Jan?
Jan: Six-sevenths.
R: And how should I write it [on the board]?
Jan: (Directs R.) Six; fraction-bar; Seven.
R: Jan, how did you know this was 6/7 and not 9/10 or 85/12?
Jan: I did it in my head.
R: What did you do in your head?
Jan: I added 2 and 4 and got 6.
R: (Asks who agrees. About half agree, no one disagrees. R. calls on Tom who said he was not sure.)
Tom: Ah, 2+4 is, in fact, 6. But I was not sure that she has to write: 'fraction-bar, 7.'
R: Good observation. Tom says that Jan added the 2 & 4, but why do we have to write 7 here (points to the sum's denominator)? Why don't we add 7+7 and write 14? Many think it's obvious but Tom asks us why.
Aaron: I think she cannot explain if it's supposed to be 7 or 14, but her answer is fine.
R: Why can't we explain this?
Aaron: Because...
R: 'Because' is not an answer that helps us to understand. (A little later, Noel raises his hand) Noel?
Noel: Ah, this is what we learned, six is with sevenths (his hand movement indicates, 'this is obvious').
R: Why is it with 7ths and not 10ths, 14ths, 3rds?
Noel: (Voluntarily gets up and goes to the board) Look. These (the 2/7) are 7ths. And if you add 7ths to it (points to the 4/7), it must come out with 7, not with 14.
R: (A little later) Noel, why must it be 7 and nothing else?
Noel: Because if you have 2 and 4, and they **are** 7 [he seems to mean 7ths], and you add them together, the '7' does not go away [i.e., the parts, 7ths, do not change].
R: (To the entire class) Look, we added the 2 and 4 but we did not change the 7. Why?
Nora: (A little later) We knew we had two-sevenths and four-sevenths, so we say it's six-**sevenths**.
(A little later R. engages them in small-group work to write their rules. Five out of six groups write rules that clearly indicate their understanding. A student from each group reads their rule.)
Judah: When adding fractions of the same size [denominator], the number on the bottom always remains the same and you have to add the numbers on the top.
R: (Two minutes before the end of the class.) We would now use your rule to solve some problems. (R. distributes a worksheet of 28 problems of adding like-denominator fractions, e.g., 3/9+4/9, 2/5+4/5. All 27 students in attendance completed the entire worksheet, all correct, in less than 5 minutes.)

The last two protocols indicate that the model-based game successfully fostered students' coordination between counting activities and unit fractions. It was a powerful activity that promoted students' thinking about the unit fractions as invariants. Thus, it succeeded where the coloring activity failed. Key to the creation of a successful activity was my model-based reflection on the previous failure: analyzing students' understandings and articulating ways in which their understandings support counting activities.

DISCUSSION

This study examined how a general model of teaching-learning a new mathematical conception, combined with content-specific models of students' thinking, can guide a teacher in practice. It demonstrated how useful models are in devising successful teaching activities and in analyzing and adjusting unsuccessful ones. It highlighted the critical role that well articulated, content-specific models play in making sense of students' evolving understandings and in creating tasks and activities that both 'meet' the students' current conceptual state and foster an intended advance. That is, it indicated that the model can help the teacher to understand the activity from the student's point of view and thus to consider the pedagogical potential of an activity. A particular, content-specific implication of the study is that coloring activities, which are frequently used in schools, may be very difficult and sometimes misleading even for students with robust understandings.

The study also supported the key role that the postulated mechanism of learning a new conception—reflection on A-E relationship—plays in advancing students' conceptions. This implies a critical role for teachers. Engaging students in thoughtful activities must be followed with thoughtful probing of reflective processes in directions that can foster the intended learning. In this sense, the study implies that the model of teaching and learning is a plausible goal for teacher education. That is, in order for teachers to be articulate about their teaching activities, especially unsuccessful ones, they need to develop understandings of both general constructs of learning and teaching and content-specific models regarding students' thinking so they can intentionally focus students' reflections.

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SECONDARY SCHOOL STUDENTS' ILLUSION OF LINEARITY: EXPANDING THE EVIDENCE TOWARDS PROBABILISTIC REASONING

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Many secondary school students have a strong tendency towards improper linear reasoning in the domain of geometry, e.g. by believing that if the sides of a figure are doubled, the area is also doubled. In this paper, the evidence for this "illusion of linearity" is expanded to a new application domain: probabilistic reasoning. The paper reports an empirical investigation on the ability of 10th and 12th grade students to compare the probabilities of different situations. It is shown that most students have a good capability of comparing two events qualitatively, but at the same time incorrectly quantify this qualitative understanding into linear relationships between the varying quantities. It is shown how the research findings can shed a new light on some well-known probabilistic misconceptions.

THEORETICAL AND EMPIRICAL BACKGROUND

Because of its wide applicability for understanding problems in mathematics and sciences, the linear (or proportional) relationship is a key concept in primary and secondary education. However, together with its intrinsic simplicity and self-evidence (see, e.g., Rouche, 1989) the reinforcement of the linear model may lead students to "the seduction to deal with each numerical relation as though it were linear" (Freudenthal, 1983, p. 267), a tendency which is sometimes referred to as the "illusion of linearity". This phenomenon can appear at different levels and in many domains of mathematics and science education, such as elementary arithmetic, geometry, algebra, probability and physics (see, e.g., De Bock, Verschaffel, & Janssens, 1999). The best-known case of the overreliance on the linear model is situated in the domain of geometry: many students of different educational levels believe, for example, that when the sides of a figure are doubled, the area and volume will be doubled too (National Council of Teachers of Mathematics, 1989). In the past years, we performed a series of empirical studies to evidence this irresistible tendency in secondary school students, to identify influencing task variables (see, e.g., De Bock et al., 1999) and to unravel the underlying problem-solving processes (De Bock, Van Dooren, Verschaffel, & Janssens, 2001).

Besides continuing our studies in the domain of geometry, we set up a new line of research which aims at searching for the illusion of linearity in other mathematical domains. The first new domain that we chose for exploration of the overgeneralization of proportionality, is probabilistic reasoning. As explained in Van Dooren, De Bock and Verschaffel (in press), this domain is particularly interesting since the learning of probability is often hindered by students' primitive conceptions, wrong intuitions, fallacies, etc. (see, e.g., Shaughnessy, 1992). Moreover, the notion of chance itself shows some very strong similarities to the notion of proportion (Fischbein, 1975; Truran, 1994), suggesting that the overreliance on proportions might very well occur when students approach probabilistic situations. Recently, we have performed a review of the literature on probabilistic misconceptions, and made an inventory of those specific misconceptions which are conceptually related to the (unwarranted) application of proportions. This inventory contains a wide variety of erroneous reasonings (both famous and intensively studied misconceptions and anecdotal phenomena) for which the illusion of linearity yields a proper explanation (see Van Dooren et al., in press; Van Dooren, De Bock, Verschaffel, & Janssens, 2001). The current paper reports the next phase in this new research line: after the conceptual analysis of linearity-related probabilistic misconceptions, we empirically tested whether the overreliance on the linear model is actually present in students' probabilistic thinking.

FOCUS OF THE CURRENT PAPER

The empirical study reported in the current paper focuses on one particular class of misconceptions that was distinguished in the theoretical inventory (Van Dooren et al., 2001, in press): the scope is on those misconceptions that could possibly be explained by erroneously assuming a linear relationship between the variables (n, k and p) of a binomial chance situation, on the one hand, and the final chance for success (P), on the other hand, and that can be illustrated by the following example:

> The participants in a television game can roll 12 times with a fair die. If they obtain at least 4 times a six, they win a car. At Christmas day, the game leader is in a generous mood and tells the participants that they get 24 instead of 12 trials, so that their chance for winning the car is doubled.

The probability P of winning the car in the regular game is about 12.5 %, and it is determined by three variables: n is the number of allowed trials (12), k is the required number of successes (4) and p is the probability for success in a single trial (the chance to obtain a six with a fair die is 1/6). The game leader is mistaken, however, when he claims that the chance P for winning the Christmas game is doubled at the moment when the number of trials is doubled ($n = 24$). He wrongly assumes a linear relationship between n and P, and he would be surprised that in fact, the probability of winning the Christmas game is not $2 \times 12.5 = 25.0$ % but 58.4 %!

In the above example, the mistake was a wrongly assumed proportional relationship between n and P. Analogously, we can think of situations where a variation of k or p is expected to have a proportional effect on P: the game leader might think that the chance for winning the car (P) is doubled when k is halved (e.g. only 2 instead of 4 sixes are needed to win the car), or that P is tripled when p is tripled (e.g. the goal is to obtain even numbers instead of sixes). Moreover, also the combination of two variables can lead to erroneous reasonings. For example, one could reason that the regular game is equally favourable as a game in which you get 24 trials, but have to obtain at least 8 sixes (n and k are doubled), or a game in which you have to obtain 4 even numbers, but at the same time get only 4 trials (p is tripled, n is divided by three).

RESEARCH QUESTIONS AND HYPOTHESIS

The goal of the current study is to test to what extent the above-mentioned linear misconceptions in binomial chance settings are actually present among secondary school students with and without formal instruction in probability in general and in the binomial probability model in particular. More specifically, the study aims at answering the following research questions:

- Do secondary school students have a good *qualitative* insight in the effect of a variation of the different variables (n, k and p) that determine a binomial chance setting?
- To what extent do these students have a tendency to *quantify* these qualitative insights *as proportional relationships* between n, k and/or p, on the one hand, and P, on the other hand?

Our hypothesis is the following. Since several authors (e.g., Fischbein, 1975) have claimed that even very young children have an elementary understanding of probability, we expect secondary school students to have a good qualitative insight in probabilistic situations. But because of the intrinsic simplicity and self-evidence of the linear model and students' well-established tendency to overrely on the proportional model in other mathematical subdomains, we expect that most of them will erroneously translate these correct qualitative insights into linear relationships between the available variables.

More concretely, we make the following predictions: First, we expect that the students will be able to make appropriate judgements when they have to qualitatively compare the probability of two events that differ with respect to one of the variables n, k or p. We expect that this capacity is present in students without formal probability instruction, and that the formal learning of probability will have an additional positive effect on it. Second, we predict that the large majority of the students will quantify these correct qualitative insights in terms of a linear function between n, k and/or p on the one hand and P on the other hand. We expect this tendency to be persistent: it will be present in students with and without formal instruction in the binomial probability model.

METHOD

A paper-and-pencil test was taken from 225 secondary school students divided in two age groups: 107 10th graders and 118 12th graders[1]. Participants had one hour to solve a test consisting of 7 experimental items and 3 buffer items, which were offered in

	n	k	p	$n \times k$
Qualitative items	1	1	1	
Quantitative items	1	1	1	1

Table 1 : Design of the experimental items

randomised order. The 7 experimental items were multiple-choice problems in which the students had to compare the probability of two specific events. All 10 problems were situated in the context of rolling fair dice. The design of the test is shown in Table 1. It can be seen that for each of the variables n, k and p there were two sorts of items: students had to make either a qualitative or a quantitative comparison between two situations. Additionally, there was one item in which n and k were varied simultaneously[2]. Table 2 gives an example of a qualitative item, a quantitative item (in which the variable n is varied), and the item where n and k are varied simultaneously.

Variation of n		**Variation of $n \times k$**
Qualitative	**Quantitative**	**Quantitative**
I roll a fair die several times. The chance to have at least two times a three if I can roll four times is □ *larger than* □ *smaller than* □ *equal to* the chance to have at least two times a three if I can roll five times.	I roll a fair die several times. The chance to have at least two times a six if I can roll twelve times is *three times as large as* the chance to have at least two times a six if I can roll four times. □ *This is true* □ *This is not true*	I roll a fair die several times. The chance to have at least two times a five if I can roll six times is *equal to* the chance to have at least once a five if I can roll three times. □ *This is true* □ *This is not true*

Table 2 : Examples of experimental items

For the *qualitative* items, the student had to indicate whether the first event had a higher, lower or equal probability as the second event. The correct answer always was either "larger than" or "smaller than". The *quantitative* items were necessarily formulated differently: they contained an explicit quantified comparison of the probabilities of the two events, and the students had to judge the correctness of this statement. The quantification was always done proportionally (e.g., in the example in Table 2: n is tripled, thus P is tripled). As a consequence, the correct answer always was "This is not true" [3]. For each item, the students were asked to indicate the correct alternative and, moreover, to write down an explanation for their answer.

RESULTS

Tables 3 and 4 give an overview of the answers of the 10th and 12th grade students respectively on the seven experimental items.

As predicted, the large majority of the students in both age groups performed very well on the qualitative items. In about 90 % of the cases, the correct alternative was chosen, indicating that even before formal instruction in probability, students have a good qualitative understanding of how the probability in a situation evolves when an aspect (n, k or p) of this situation changes. Contrary to our prediction, the 12th

Variable	Qualitative items			Quantitative items		
	Correct	Incorrect	No answer	Correct	Incorrect	No answer
n	87.9	10.3	1.9	15.9	84.1	0.0
k	83.2	15.9	0.9	17.8	81.3	0.9
p	93.5	6.5	0.0	16.8	82.2	0.9
n × *k*				22.4	77.6	0.0
Total	88.2	10.9	0.9	18.2	81.3	0.5

Table 3 : Frequency (in %) of correct and incorrect answers of the 10th graders on the experimental items

Variable	Qualitative items			Quantitative items		
	Correct	Incorrect	No answer	Correct	Incorrect	No answer
n	92.4	6.8	0.8	18.7	76.3	5.1
k	78.8	19.5	1.7	38.1	61.9	0.0
p	97.5	2.5	0.0	16.1	83.1	0.8
n × *k*				30.5	67.8	1.7
Total	89.5	9.6	0.8	25.8	72.4	1.9

Table 4 : Frequency (in %) of correct and incorrect answers of the 12th graders on the experimental items

graders, who had already met the binomial probability distribution in their curriculum, performed only slightly better on these qualitative problems than the 10th graders (89.5% versus 88.2% correct answers), but considering the already high performance of 10th grade students, there might have been a ceiling effect. This good qualitative understanding of the chance situation was present for all of the three items. For the k-problem, there was a higher error rate which is probably due to the inversed effect of k on P: if *more* successes are needed, there is *less* chance of succeeding.

As expected, the high performance on the qualitative items is in sharp contrast with the low score on the quantitative problems. For this last category, the students most frequently chose for the incorrect alternative, which expressed a proportional relationship between n, k or p and P. Apparently, the vast majority of the students agreed with a proportional quantification of their correct quantitative insights. Also for the problem in which n and k were varied simultaneously (see Table 2), most of the students believed in the linear effect. The 12th graders performed slightly better on the quantitative problems (on the average, 25.8 % correct answers) than 10th graders (18.2 % correct answers) but as expected, the tendency towards linear modelling is still strongly present in these students. In particular, 12th grade students had a better score on the k-problem than 10th graders. Apparently, after instruction in probability, more students were aware that when the number of required successes is doubled, the probability is not necessarily halved.

We will also perform a qualitative analysis of the written notes and explanations of the students to investigate what specific strategy led the students towards the correct or incorrect answering alternative. A first round of qualitative analysis of the written notes and explanations accompanying the incorrect answers on the quantitative problems already revealed that more than 80 % of the incorrect answers on the quantitative problems can be clearly identified as resulting from students' overreliance on the linear model. Examples of such statements – referring to a linear relationship between the variables in the n-problem in Table 2 – are:

> In the first case, you can try three times more to obtain the same result (two sixes), so it is evident that you have three times more chance of winning.
>
> The chance of getting 2 sixes in 12 trials is a lot bigger than getting 2 sixes in 4 trials. And 12 is three times larger than 4, so the statement is true.

A more fine-grained analysis of students' written notes and explanations will provide richer data on the mechanisms and origins of students' improper proportional reasoning in binomial probability situations.

CONCLUSIONS AND DISCUSSION

The results of this study have confirmed our global hypothesis. Secondary school students have a good qualitative understanding of probabilistic situations, and are able to compare two such situations that differ in one variable. The understanding is even present in students without formal instruction in probability. At the same time, however, most students have a strong tendency to incorrectly quantify their correct qualitative insights as linear relationships between the variables in a binomial chance situation. In our multiple-choice items, the large majority of students chose for the alternative that stated a linear increase (or decrease) of the probability of the described event if one or two variables in the situation were increased (or decreased). This tendency towards linear reasoning is strongly present in all students in the research group, even those who met the binomial probability distribution in their mathematics curriculum.

Further qualitative analyses of the answers, as well as in-depth interviews with selected students will have to clarify how different components in the students' mathematical knowledge base lead them towards unwarranted linear reasoning.

Another remaining question, which we will address in our future research, is to what extent our findings are significantly affected by the way in which the test items were administered to the students. It could be argued that so many students fell into the linearity trap because they were seduced to do so, by confronting them with proportional statements with which they either had to agree or disagree. The tendency to reason linearly might considerably decrease when an open-answer format is used.

Finally, we want to show in an exemplary way how our research can shed some new light on a number of other well-documented misconceptions in the domain of probability, by looking at them from the perspective of the improper assumption of linear relationships between quantities. A first example is a problem used by Tirosh and Stavy (1999, p. 190):

> The Carmel family has two children, and the Levin family has four children. Is the probability that the Carmels have one son and one daughter larger than/equal to/smaller than the probability that the Levins have two sons and two daughters?

More than half of the 7^{th} to 12^{th} grade students in their study answered that the probabilities are the same. According to Tirosh and Stavy (1999, p. 190), students here applied the intuitive rule Same *A*–Same *B*: "Because the target boys:girls ratio in the two families was the same (1:2), the probability would therefore be the same." A second example comes from Fischbein (1999), who found that the majority of 5^{th} to 11^{th} grade students answered erroneously on the following problem:

> The likelihood of getting heads at least twice when tossing three coins is smaller than/equal to/greater than the likelihood of getting heads at least 200 times out of 300 times.

For a similar problem, Fischbein and Schnarch (1997, p. 103) argue that "The principle of equivalence of ratio imposes itself as relevant to the problem and thus dictates the answer."

Taking into account our own research findings, we believe that these misconceptions can be explained in a related, but somewhat different way, with a stronger emphasis on linear functions instead of ratios. In the study reported above, we observed that many students believe that (1) P is doubled when n is doubled (2) P is halved when k is doubled and (3) these two effects can also play simultaneously. The improper assumptions of linearity might also be a valid explanation for Tirosh and Stavy's and Fischbein's findings. We are even able to provide a useful framework for understanding a famous historical problem: Chevalier de Méré knew that it was advantageous to bet on at least one six in 4 rolls of a single die. He deduced that it should be equally advantageous to bet on at least one double-six in 24 rolls of a pair of dice. This did not yield the financial gain he had hoped for... Probably, most of our students would also agree that multiplying n by 6 and dividing p by 6 is a neutral operation for P.

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[1] According to the Flemish curriculum, probability (and the binomial probability model in particular) is a part of the mathematics curriculum in the 11th grade.

[2] As can be seen in the $n \times k$-problem, the simultaneous variation of two variables (in a proportional way) necessarily implies a quantified statement (i.e. the *equality* of the probabilities). Therefore it was impossible to formulate a qualitative item for this category.

[3] The buffer items were manipulated so that they were similar to the qualitative items (but 'equal to' was the *correct* alternative) or to the quantitative items (but this time, 'This is true' was the *correct* alternative).

MECHANICAL LINKAGES AS BRIDGES TO DEDUCTIVE REASONING: A COMPARISON OF TWO ENVIRONMENTS

Jill Vincent, Helen Chick and Barry McCrae

The University of Melbourne

This paper reports part of a study investigating the use of mechanical linkages as contexts for establishing a classroom culture of conjecturing and proof in geometry at Year 8 level. The focus is on a comparison of students' use of a physical model and a Cabri Geometry model of a linkage, examining in particular how the students exploited the features of Cabri to assist them in producing conjectures and proofs.

The lack of success with traditional methods of teaching geometric proof has prompted researchers to seek alternative approaches, involving new and old technology. Recent research on proof has focused on the role of dynamic geometry software, for example Cabri Geometry II™, as an environment for geometry learning, with debate as to whether these environments assist or, rather, hinder the development of deductive reasoning. Hoyles (1998), for example, expresses concern that unless we can develop a 'need for proof', dynamic geometry software may merely contribute to a 'data' gathering, empirical approach to geometry.

Bartolini Bussi (1998) reports on the use of historic drawing instruments to create an environment where students can 'relive' the work of mathematicians in theorem production. She describes the conjecturing and proof construction processes of a group of five year 11 students who investigated the geometry of Sylvester's pantograph, working with a physical model.

Bartolini Bussi notes that "producing the conjecture was difficult and slow" (p. 742), and that the students' written proof was incomplete and not in a logical sequence, but when they refined it with the teacher's help, it remained meaningful for them:

> The order of the steps recalls the sequence of production of statements, as observed during the small group work, rather than the logical chain that could have been used by an expert. Nevertheless it was easily transformed later with the teacher's help into the accepted format ... yet, what is important, the time given to laboriously produce their own proof ensured that the final product in the mathematician's style, where the genesis of the proof was eventually hidden, retained meaning for the students. (p. 743)

What is it, then, about an environment that assists students' conjecturing and proof?

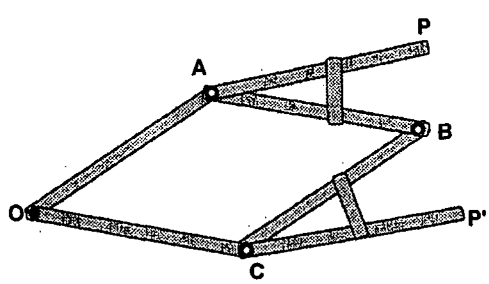
THE STUDY

The research presented here forms part of a study on mechanical linkages, using both physical and computer models, as an environment for Year 8 students to conjecture, argue, and construct proofs (see Vincent & McCrae, 2001). The linkages were carefully selected on the basis of appropriate geometry for Year 8. This report focuses on a comparison of students' use of a physical model and a Cabri Geometry focuses on a comparison of students' use of a physical model and a Cabri Geometry model of a linkage, examining in particular how the students exploited the features of Cabri to assist them in producing relevant conjectures.

The participants, who had no previous formal exposure to deductive reasoning, were from an extension Year 8 Mathematics class at a private girls' school in Melbourne. Prior to the research the following geometry was taught/revised: angles in parallel lines, triangles and quadrilaterals; Pythagoras' theorem; and similar and congruent triangles. Tchebycheff's linkage for approximate linear motion (Vincent & McCrae, 2001) was used to introduce the students to the need for proof by showing them that, although the midpoint of one of the links appeared to move on a linear path, closer investigation showed that the path was in fact not exactly linear. For each of the linkages studied, the students worked in pairs, constructing their linkages from plastic geostrips and paper fasteners, as well as having access to a Cabri model of the linkage which had been previously prepared by the teacher-researcher. The students were video-taped during their conjecturing, argumentation, and proof construction. At the conclusion of each task, the students were asked which model they considered most useful in helping them in their conjecturing and proof construction, and which one they most enjoyed working with.

Sylvester's pantograph: Case studies of two pairs of students

Before investigating Sylvester's pantograph, the students had completed pencil and paper proofs for the angle sum of triangles and parallelogram properties, and had worked with at least three other linkages. The students constructed the rhombus version of the pantograph, using the diagram shown in Figure 1, where OA = OC = AB = BC = AP = CP' and ∠PAB = ∠P'CB. They explored the linkage (Figure 2) for approximately 10 minutes before being given access to the Cabri model. They were then free to choose the model with which they preferred to work. Each pair of students worked for two 50-minute lessons, conjecturing, arguing, and constructing their proofs, with the teacher-researcher (TR) occasionally intervening.



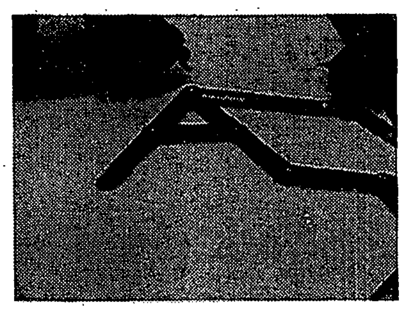


Figure 1. **Figure 2.**

Students Ce and Ch

Ce and Ch drew a shape on paper and traced over the shape with point P of their geostrip pantograph so that P' traced an image of their shape (Figure 3).

Ce: It's the same but it's not in the same direction. It's been turned.

Ch: [Spreads her thumb and fingers to compare the size] It's the same size. It's turned 45 degrees.

TR: How do you know it's 45 degrees?

Ch: Just guessing, 'cause it was about half 90.

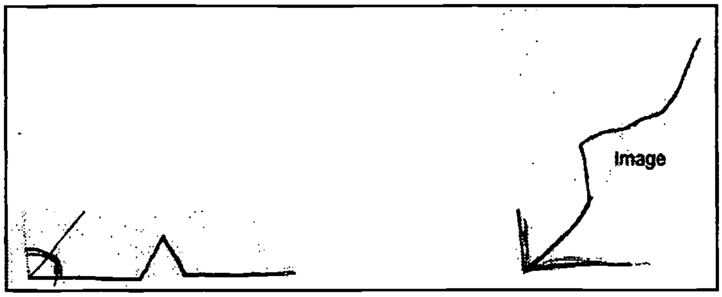


Figure 3.

Ce and Ch were confident that the image was the same size as the original, but they were less confident of their conjecture about the angle of rotation, particularly as there seemed no obvious reason why it should be 45 degrees. When given the Cabri model, their initial reaction was to use the same approach as for the geostrip model, but they were not sure how to proceed. With *Trace* on P', Ch carefully dragged P in the shape of a square so that P' traced the same shape. She then switched on *Trace* for P and tried to drag P so that P' moved around its previous path. However, she had trouble controlling the mouse to follow the small shape and began to move P haphazardly, before dragging it horizontally across the screen (Figure 4), when she suddenly noticed that the paths of P and P' were converging:

Ch: [Excitedly] Goody goody goody … we've got 'em to meet! And then look …

Ce: Oh, we've got the angle there …

Ch: We'll make a segment. And from there down to about there. [Draws segments over traces of P and P' — see Figure 5] Measure the angle … 30.1. [Measures angles PAB and P'CB] 30! 30 and 30. So it rotates the angle of that [PAB or P'CB]!

Ce: Yes!

Ch: They're the same distance [indicates from P to O and from P' to O] … maybe if we draw segments. [Ch draws segments OP and OP' — see Figure 6]

Ch: Let's measure that angle … [POP'] … it looks like the same angle again.

Ce: 30! So that [OP'] is always turned around 30 degrees from that [OP].

Ch and Ce recognised that their method of drawing segments over the converging traces (see Figure 5) involved some error, so they seemed unconcerned by the discrepancy between the measured angles, 30° and 30.1°; the size of angle POP' seemed to carry greater significance. Ch and Ce were now confident about the rotation of the image: "The image rotates [through] the angle of BAP".

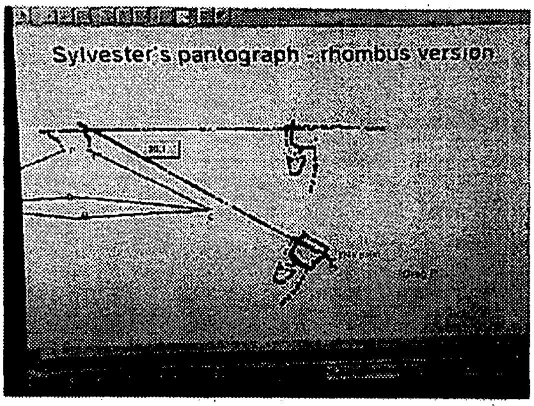


Figure 4.

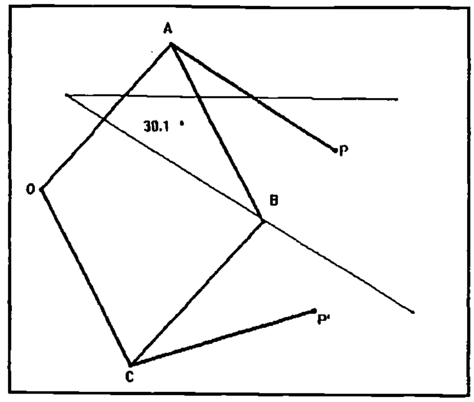


Figure 5.

Ce and Ch were now able to explain why their drawing and its image were the same size, and were able to write the geometric proof (Figure 7):

Ch: They're congruent triangles [OAP and OCP'].

Ce: Yeah, 'cause the sides are the same [OA, OC; AP, CP'] and so are those angles [OAP and OAP'].

TR: Why are those angles equal?

Ch: The [opposite] angles in the rhombus are equal and then they both have the same fixed angle bit added on.

Ce: So therefore OP and OP' are equal.

TR: And what does that tell you?

Ch: That's why the copy is the same size.

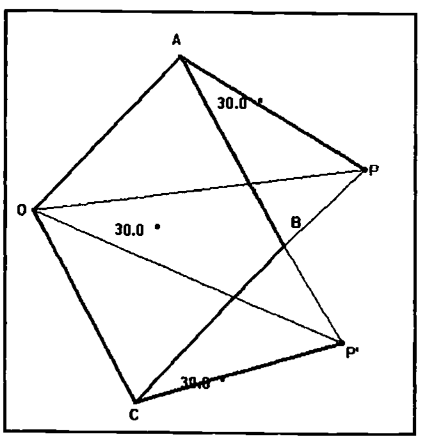


Figure 6.

Prove: OP = OP'

Proof: ∠OAB = ∠OCB (rhombus)

∠PAB = ∠P'CB (Given)

∴ ∠PAO = ∠P'CO

∴ ΔP'OC ≅ ΔPOA (SAS)

∴ OP' = OP

Figure 7.

Their next task was to prove their conjecture that the angle of rotation POP' was equal to the fixed angles PAB and P'CB. Ch drew segments from P to B, B to P' and P to P' on the worksheet diagram (see Figure 8).

Ce: Well ... could we make a triangle there? [points to O and from P to P']

Ch: But B's not in line ... see ...why don't we use the angles in the rhombus?

Ce: OK, let's put some letters in. We'll call this [PAB] *a* and these [BCP' and POP'] *a*.

TR: We don't know yet ...that's what you're trying to prove.

Ce: Oh, yeah, we'll call it [POP'] *b*, and we'll call this [OAB] *c* and this [OCB] is *c* too.

Ch: And call those *d* [points to AOP and COP'].

Ce: And those [APO and CP'O] are *d* as well.

Ch: Oh, yeah ... isosceles triangles.

Ce: And this [ABC] is *e*. OK ... *e* equals 2*d* plus *b*.

Ch: *d* plus *d* plus *a* plus *c* equals 180 'cause that's a triangle

Ce: And 2*d* plus *b* plus *c* equals 180 'cause angles in the rhombus.

Ch: So *b* must equal *a*!

Ce: Yeah, that's it!

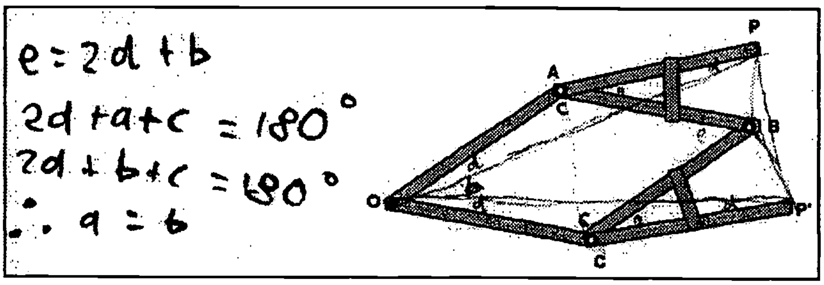


Figure 8.

Both students then wrote correct proofs; Figure 9 shows the proof written by Ch.

Prove: ∠POP' = ∠BCP' = ∠PAB
Proof: ∠COP' = ∠CP'O (isosceles triangle)
= ∠AOP = ∠OPA (congruent triangles) = d
let angle OCB = c
d+d+c+a = 180° (triangle)
∠AOC + ∠OCB = 180° (rhombus), let angle POP' = b
∴ d+d+c+b = 180°
∴ b = a
∴ ∠POP' = ∠BCP' = ∠PAB

Figure 9.

Students Lu and Li

Like students Ch and Ce, Lu and Li were reasonably confident that the geostrip pantograph was producing a congruent image but they too were less certain about the angle of rotation and unsure how to determine the angle with the geostrip model:

Lu: Well, it creates a mirror image ... well, not exactly a mirror image but ...

Li: ... if it was more accurate.

Lu: A congruent shape of the shape and it goes in a circle. If you kept on following the trace it would go round and round [points to rotation of the linkage about O]

Li: Maybe that angle [points to the fixed angle, PAB] ... I'm not sure ... maybe not ...

Lu: When you move it [points to P] up at the beginning it [points to P'] moves around and then it just makes the rest of the shape.

Li: I was thinking that many degrees [points to angle PAB] ...but I don't know why.

At this point the girls were given the Cabri pantograph. They used the same approach in Cabri as they had done with the geostrip model, drawing a triangle and moving P around it so that P' traced the image (Figure 10). Because of the transitory nature of Cabri traces, Li then drew a triangle over the image trace.

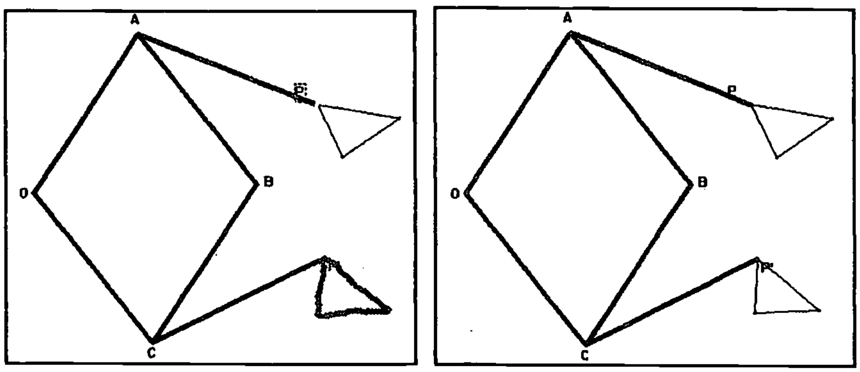


Figure 10.

Lu measured angles PAB and P'CB as 30 degrees (Figure 11), then carefully dragged the original triangle, placing it over the image in order to measure the angle between the two triangles. Seeing the linkage as an accurate geometric diagram seemed to encourage the girls to add construction lines and to notice congruent angles.

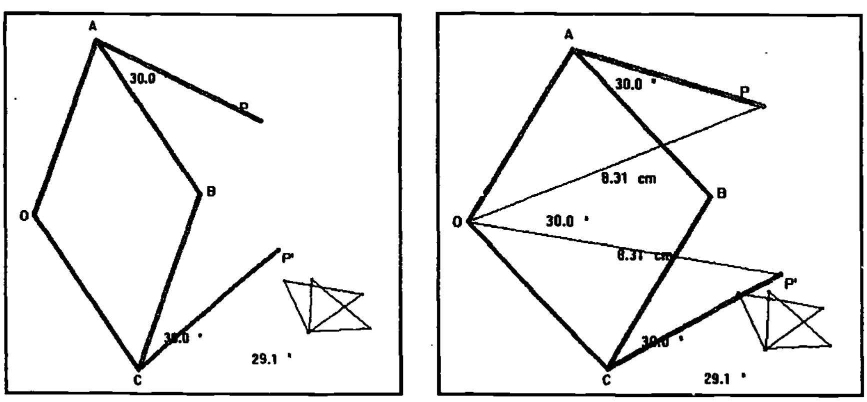


Figure 11.

Having drawn and measured segments OP and OP', and measured angle POP', the girls now understood how the pantograph was working:

Lu: 29.1 … it's point 9 off.

TR: So what is your conjecture?

Lu: The angle which the copy of the shape rotates is that angle of the pantograph [PAB].

Lu and Li then began writing their proofs, with Li demonstrating that she had a clear understanding of the side-angle-side condition for triangle congruency. The girls discussed and sorted their statements into a logical sequence as they wrote, for example:

Li: Let's do the sides first. OA equals AP equals OC equals CP' … then angle OCP' equals … OAP because they both have 30 degrees … they share 30 degrees … we shouldn't do that yet [erases] … angle OA … angle OAB equals …

Lu: Angle OCB.

Li: And angle BCP' equals BAP because given … OCB plus BCP' …

Lu: Those two added together, that whole angle … that means …

Li: Once we've proved that angle, then the whole thing's easy 'cause side angle side … see if you have two sides and how big it's going to be in between … when you join them up the triangles will be the same …

DISCUSSION

In their responses to the linkage questionnaire which they completed after working with the pantograph, all the students agreed that operating models of the linkage made the geometric properties more obvious. Even though three of the four girls had enjoyed working with the geostrip model more than with the Cabri model, they all believed the Cabri model to be more useful than the geostrip model for finding out why the linkage worked. Significantly, perhaps, Ch — who had become quite excited by the converging traces of P and P' (see Figure 4) — indicated that she had enjoyed the Cabri model more than the geostrip model.

The students' use of Cabri was by no means restricted to dragging. The function of the pantograph led naturally to the use of the Cabri *Trace* facility, and transformation of the screen construction indicated where additional construction lines, for example, OP and OP', might be useful. In other linkages explored by the students, tabulation of angle measurements was also used. The students' prior familiarity with the features of Cabri was therefore an important aspect of their successful use of the Cabri linkage in helping them to produce their conjectures and construct proofs.

The ease with which they could trace the paths of points, add construction lines, and accurately measure angles and distances meant that the students made little attempt to return to the geostrip model after they had been given the Cabri model. Once the students were confident of their conjectures, they tended to work with paper and pencil diagrams during the proof construction. This was, however, not always the case; the students worked with a number of linkages during the classroom research and with some of these they would sometimes go back to the physical linkage to check an observation they had made with the Cabri model, or return to the Cabri model while they were constructing their proof.

The students in the study described by Bartolini Bussi (1998) were required to produce their own geometric drawing of the linkage from the model and to provide a description from which another person could construct it. This resulted in the need to determine the structure of the linkage, for example, by measuring the lengths of the linkage bars. This was, of course, an important aspect of Bartolini Bussi's project, which aimed to make these historic drawing instruments 'transparent' to students. The Cabri model, on the other hand, provided a ready-made geometric figure, which allowed the students to focus more immediately on the functional relationships within the linkage, rather than its basic structural properties. This may have contributed to the ease with which these Year 8 students were able to present their statements in a logical order, compared with the initial difficulty experienced by the Year 11 students in the study described by Bartolini Bussi (1998).

It would seem, then, that the imagery, both static and dynamic, of the Cabri environment made a substantial contribution to the conjecturing process, but also challenged these students to produce an explanation. The features of dynamic geometry software — constructions based on Euclidean geometry, accurate measurements, the tracing of loci and the drag facility — make the software highly suitable for exploring the geometry of linkages. Rather than eliminating the need for proof, the convincing evidence and the unique opportunities for exploration and discovery which the software provided gave the students the confidence and desire to go ahead and prove their conjectures. However, the tactile experience and satisfaction of working with actual physical linkages may also represent a significant motivational aspect, at least for some students, and should not be over-looked.

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ABOUT THE FLEXIBILITY OF THE MINUS SIGN IN SOLVING EQUATIONS

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ABSTRACT

Among the difficulties the students have in solving equations with negative numbers, the 'detachment from the minus sign' (Herscovics & Linchevski, 1991) and solving an equations such as -x = 7 are often pointed out in the research literature. However, the study of their origin was not yet debated a lot. This article intends to propose an analysis of these difficulties. It reports results from a clinical study carried out with 8th grade students. The analyses stress that the presence of negative numbers in equations needs some level of flexibility in conceiving the minus sign that most students have not yet reached. They show that, beyond the equations and negative numbers, this lack of flexibility leads to an erroneous or superficial understanding of algebraic operations and symbolism.

INTRODUCTION

Many researchers showed that the presence of negative numbers in first degree equations with one unknown leads to several errors (Cortès, 1993, Gallardo & Rojano, 1994 ; Gallardo & Rojano, 1990; Herscovics & Linchevski, 1991; Vlassis 2001). Three obstacles are regularly pointed out by these authors and they thus seem to indicate major difficulties for students: (1) the avoidance of a negative solution (Gallardo & Rojano, 1990 ; Gallardo & Rojano, 1994 ; Glaeser, 1981; Vergnaud, 1989), (2) the 'detachment from the minus sign' (Herscovics & Linchevski, 1991 ; Linchevski & Herscovics, 1996; Vlassis 2001) and (3) solving an equation such as -x = a (Cortès,1993 ; Gallardo & Rojano, 1994 ; Vergnaud, 1989 ; Vlassis, 2001). The difficulties raised by the negative solution of an equation have already been analysed (Gallardo & Rojano,1990 ; Gallardo & Rojano, 1993 ; Vergnaud, 1989). However, whereas errors linked to 'the detachment from the minus sign' and those linked to solving '-x = a' have been often mentioned, their origin has not yet been clearly established. Historical epistemological arguments are usually pointed out when negative numbers are involved. We think that this perspective has to be enlarged to other analyses in order to define more completely the origin of these obstacles. In this article, we intend to propose an original approach of the difficulties stemmed from the 'detachment from the minus sign' and the solving process of -x = a. Problems associated with a negative solution to an equation will not be developed in this proposal. Our analyses are based on interviews of 18 students of the 8th grade. They show that beyond equations and negative numbers, those errors are due to a lack of flexibility in the conception of the operation signs, leading to a wrong or superficial understanding of the concepts involved in algebraic operations and symbolism.

METHODOLOGY

We interviewed 18 students of the 8^{th} grade level at the end of the school year. These students learnt elementary algebra (reduction of expressions, distributive law, ...), the solving of first degree equations with one unknown and the negative numbers during their 7^{th} and 8^{th} grades. These students came from two different schools, one of them recruiting in a favoured population (6 students), the other one in a non-favoured population (12 students). The student selection was based on test results. Here is the 18 students distribution: 4 students of good level (results in the 80% range); 6 students of middle level (in the 70% and 60% ranges); 4 students of low level (in the 50% and 40% ranges); 4 students of very low level (less than 40%). The students that obtained 90% and more were not interviewed.

The aim of the interview was to list and analyse students' reasoning when they are confronted with negative numbers in solving 'arithmetical' and 'non-arithmetical' equations (Filloy and Rojano, 1989) and in the reduction of polynomial expressions. The interview also aimed at considering the students' conceptions of mathematical objects and operation signs in an expression such as '-18a - 2y + 5a – y'. The students were invited to tick the negative numbers of the expression. They also had to say what the letter represented and to give some examples of numbers that could replace 'y' in -2y and -y, and 'a' in -18a and + 5a.

THE DETACHMENT FROM THE MINUS SIGN

Results

In 1991, Herscovics & Linchevski already stressed the following result: 'The detachment from the minus sign was somewhat of a surprise to us. The high incidence of this mistake indicates that the problem is not idiosyncratic but may well reflect unsuspected cognitive obstacles.' (p. 179). These authors defined that issue as 'a tendency to ignore the minus sign preceding a number'. In their experiment, that error concerned the 7^{th} grade pupils that had not yet received any algebra teaching As for our 18 interviewed students, they had been learning algebra for two years. However, 'the detachment from the minus sign' still remains often observed and raised the following errors :

1) In solving an equation

error (a) : $2 - 3x + 6 = 2x + 18$

becomes

$3x + 8 = 2x + 18$

error (b) : $2 - 3x + 6 = 2x + 18$

$-2x\downarrow \qquad \downarrow -2x$

$2 - 1x + 6 = 18$

Eight students on the 18 are concerned by one or the other of these errors.

- In case (a), 4 students simplify the first member and copy it, forgetting the minus sign before 3x. Cortès (1993) also stressed the high incidence of that error. This one is mainly made by good-level students (3 of good level and 1 of middle level). Here is how Coralie explains her error :

1 Interviewer: Which error did you made here?
2 Coralie : h'm, ... it's -3x.
3 Interviewer : Why did you forget your minus sign before 3x?
4 Coralie : Because there was a number before it.
5 Interviewer : Yes, and what did you think so?
6 Coralie : I deleted the 2 and the 6. Then I looked at the 3x and I forgot the minus sign before.

- In case (b), 4 students want to cancel -2x in each member and get 2 - 1x + 6 in the first member. This error is more typical of lower level students (2 of middle level, 2 of low level). The lowest students are not concerned since they did not begin any solving process. During the interviews, all the students who made that error explained that they removed 2x from 3x and that they obtained 1x. Then, three of them reintroduced the minus sign before 1x.

In both cases, the minus sign before a literal term is either completely forgotten (error a), or forgotten and then reintroduced (error b).

2) In the reduction of a polynomial expression

2.1. The following expression was presented to the students : $-18a - 2y + 5a - y =$

In order to simplify this expression, they had to group like terms. Four of them made a mistake when simplifying 'y terms'. They made $-(2y - y) = -y$ instead of $-2y - y = -3y$. The reasoning is related to error (b) in the equation solving : again, the students took in consideration 2y and not -2y. In this case, the operation sign was forgotten and then reintroduced since they obtained -y. The error consisting in forgetting the minus sign before the 18a did not appear. No student operated $-18a + 5a = 23a$ nor -23a. One of my hypotheses about the error on the 'y term' was that moving terms, in order to group them, 'made easier' the omission of the minus sign before 2y. Obviously, in order to solve the equation as well as to reduce the expression, it was needed to 'build' a new expression with the like terms (-3x - 2x in the equation and -2y - y in the expression).

2.2 In order to check that hypothesis, the following expression was proposed to the students : '$-24b + 4b - 6a - 2a =$'

I thought that, since it was not needed to change the place of the like terms, the error consisting in 'forgetting' the minus sign would appear less regularly. In such a case, in my opinion, the error consisting in operating 6a - 2a rather than -6a -2a should have appeared less regularly. I was disappointed. Five of the students simplified the terms -6a and -2a in -4a. The explanations were the same for the 5 students: 'I made 6a - 2a, that is 4a, and then, as there was a minus sign, I put it back before'. Again, the students concerned by this error are not the lowest ones. Here is their distribution: 1 is good level, 3 are middle-level and only 1 is low level. The error consisting in forgetting the minus sign before b was made by only 1 student, who obtained -28b for -24b + 4b.

Comments : from arithmetic operations to a combination of signed entities.

I was quite surprised by those results because, on the one hand, the 'detachment from the minus sign' concerns students of various competence levels, and not only low-level students. On the other hand, it is important to note that the students do not understand easily their error. Some of them, even among those of good level, did not see why I underlined an error in the first member of their equation when they wrote : 2 - 1x + 6. I had thus to re-formulate the operation like this : '-3x - 2x is -1x ? ', and to insist on the minus signs to make them finally become aware of their error. And even in this case, it was really very difficult for some of them to admit that -3x -2x produced - 5x and not -1x. Those results made emerge three main analyses :

1) The students do not always forget the minus sign

The 'detachment from the minus sign' could be assimilated to 'an avoidance syntome'. Gallardo and Rojano (1990) already stressed on that difficulty in solving equations. A link could be established between that students' error and the historical development of the negative number where, at a given moment, scientists used some processes to avoid the obstacle of negative numbers (Glaeser, 1981). The 'avoidance syntome' actually exists by students (in the interviews, I could observe that equations such as 4 - x = 5 were modified into x = 5 - 4) and could explain errors such as the omission of the minus sign when copying a simplified equation (error (a) in equation solving). The results nevertheless show that not all the students forget the minus sign. On the contrary, most of them put it aside to make their operations, and then reintroduce it. Some students even tell us they put the expression between brackets and copy the minus sign before the answer they found. I would thus slightly qualify the Linchevski and Herscovics's definition (1991), adding that this error consists in ignoring or *temporarily omitting* the minus sign preceding a number or a literal term.

2) Moreover, the omitted minus sign is not any minus sign.

The omitted minus sign is located *in* the expression, before a number or a literal term. Coralie explains (line 4 of the interview here above), that what disturbs her is the number before the sign. The minus appearing in the beginning of the expression, such as in the expression -18a -2y + 5a - y is not or almost not omitted by the students. Another example of that difficulty can be found in the question about the mathematical objects of the expression -18a - 2y + 5a - y. When the students have to tick the negative numbers of the expression, almost all the students tick -18a, but -2y and -y are ticked more randomly. For the students I interviewed, the term -18a is the 'prototype' of a negative number and cannot be considered the same as -2y and -y. The minus signs in the middle of the expressions have an double status : they have to be considered like operation signs ('subtraction' level of Gallardo, 1994) and like signs attached to the number ('signed number' level of Gallardo, 1994). The difficulty does not depend on reaching one or the other level of conceptualisation, but rather on being able to go to one or to the other, according to the situation. The students that omit the minus sign do not consider that duality. On the contrary, they seem to keep an arithmetical and inflexible conception and do only consider the minus sign inside the expressions as the symbol of subtraction. This can explain why some of them put it aside in order to operate (it is the symbol of subtraction and it is thus not attached to the literal term), and then re-introduce it, once the operations are made.

3) The meaning of the operations changed

The introduction of negative numbers in algebraic expressions undoubtedly leads to another conception of the operations. In primary school, the students are used to operate on concrete entities, which means without signs, they have to subtract or add up. In expressions involving positive numbers, the students can keep most of that operation conception. For instance, in the expression 5b + 3c + 2b + 4c =, the plus signs can be seen only as operation signs. The students can then add up the 'b terms' and the 'c terms' which are seen as objects without sign. With negative numbers, that perspective isn't possible anymore. The students cannot consider anymore the operations as a set of numbers without signs (about numbers with and without signs, see Vergnaud, 1989) that have to be subtracted or added up. The duality of the operation signs implies to consider the expressions as a combination of signed entities, *i.e.* as '-18a followed by -2y, by +5a and by -y'. That notion requires a deep understanding of these mathematical objects. Signs need to be considered as attached to every term while acting also as operator. But, when it is needed to make the operations, the meaning of addition and subtraction is modified and becomes abstract. The first concrete arithmetical representations becomes then an obstacle to that evolution because, when it is about simplifying -3x - 2x, for example, the result (-5x) seems to be that of an addition (that of 3x and that of 2x). This perhaps explains one of the difficulties faced by the students. During the interviews, Julie (a middle-level student) helped me to think in that sense. When she understood her error (-3x - 2x is not -1x) and gave the correct answer (-5x), I was trying to make her say she had forgotten the minus sign before 3x.

Interviewer: What did you forget while making 3x - 2x? (expected answer : the "minus" before 3x)

Julie : to add them up.

I did not understand immediately what she meant. As to me, I did not see where it was about adding up. In fact, each of us referred to two distinct models. Julie was reasoning in very concrete terms, linked to her first arithmetic learning (if -3x - 2x = -5x, it is just like an addition); as to me, I referred to the model of the number line. It is possible that, for some students, the answer –5x is counter-intuitive, even if preceded by the minus sign. The meaning of the subtraction is still related to elementary arithmetic (the operation must produce a smaller result). The reference to the number line does not seem to be a conceptual instrument for these students, even though they have experimented that model during their learning of algebra.

SOLVING -x = a OR A MINUS SIGN CAN HIDE ANOTHER ONE.

Another example of the students' lack of flexibility about the minus sign was pointed out when solving the equation $-x = 7$: many students made a confusion between the minus sign before the x and that of the solution -7. The difficulties in solving that kind of equation have already been underlined by Cortès (1993), Gallardo and Rojano (1994), Vergnaud (1989) and Vlassis (2001). Vergnaud (1989) thinks that difficulty comes from the negative solution. I do not think the explanation is mostly based on the negative value of the solution. Without denying the obstacle related to a negative solution, it is shown that other equations produced the same type of solution but didn't present so much problems to the students. According to Gallardo and Rojano (1994), the problem is that the students don't decode $-x = 7$ as $-1x = 7$. Our analyses also show that students do not interpret the equation in that way. We did not find out any explanation of that kind in our interviews. It seems that this formulation is difficult to be understood by students at that school level (Vlassis, 2001). It does not represent, however, the only possible way of solving.

Results

A major part of the issue seems to come from the ambiguity of the minus sign before the x. Among the 15 students who solved the equation, five of them gave 7 as the answer of the equation and 10 gave the correct answer. The students who gave 7 as the solution were not able to consider –7 as the solution since the minus sign was already noted. But the most interesting analyses come from the students who gave the correct answer. I first asked them to explain how they had found their solution. Here are the types of answers I obtained : *(a)* the value of x is 7 and, since there is a minus sign before, I added it up (2 students). *(b)* I transferred the minus sign to the other member (5 students). *(c)* The value of x is -7 since, when I put -7 instead of, I get $-(-7) = 7$ (2 students). *(d)* I took the opposite (1 student). Seven of the 10 students justify their answer only on basis of algebraic rules - true or invented- (answers *a* and *b*). When I asked to these students to explain me more about the method they follow, they told me it was just like that, that it was a trick in algebra. Thus I invited the 10 students to check their solution. All the students, but the two who had done it spontaneously, were perturbed by my question. Most of them did not understand what I wanted. When I reminded them that the solution of an equation is checked by replacing the letter by its numerical value, some of them thought their answer was wrong : 'If I replace x by -7, I get $-7 = 7$, which is not correct, so my answer is wrong'. It turned out to be quite difficult to make them understand that the minus sign before the x was not the same as that which was attached to the 7 of the solution and to make them copy the equation again, replacing the x by -7.

Obstacles to solving $-x = 7$

The difficulty in solving this equation is due to the multiplicity of concepts it covers :

- First, the students must be able to detach the minus sign from the letter x. They must consider the sign attached to the numerical value of the letter, *independently* from the sign attached to the letter. The students' answers show it is an obstacle quite difficult to overcome. It seems that -x is considered as a 'prototype' of a negative number (as well as -18a) and that they see -x in an inflexible way, as a 'signed number'. Another example of that difficulty appeared during the discussion about the expression -18a - 2y + 5a - y. When the students had to decide whether the letter y of -2y could replace a positive number, a negative number, or both, most of them answered without hesitation a negative number. Only one student told me it was not possible to answer the question because he did not know whether the letters replaced negative or positive numbers. Finally, when the students were asked to give examples of numbers that could be replaced by y, they gave examples of positive numbers, without looking disturbed by the apparent contradiction of their reasoning. I then asked whether 'y' could replace, for instance, -6. The answer given by a large half of them was 'No, since the minus sign is already wrote'.

- The students must be able to enlarge their notion of 'signed number', but also to juggle with high levels of conceptualisation of negative numbers (Gallardo, 1994) : that of 'isolated number' (the solution is a negative number), that of the formal conception of a 'negative number' (a letter replaces a number: the negative numbers are part of the numbers that could be replaced by a letter). The 'relative number' level (the opposite) is not compulsory but it rather represents a way to give sense to the equation.
- Decoding -x = 7 in -1. x = 7 seems to be a problem because it needs not only to detach the minus sign from x, but also to attach it to a "hidden" number (number 1) and to consider the presence of an operation (that of a multiplication), whereas a lot of students still just see a number in -x. It seems that such decoding of the implicit imbedded in the algebraic symbolism is out of sight for numerous students of that school level.

CONCLUSION AND FINAL DISCUSSION

The analyses presented in this article show that the presence of negative numbers in the equations causes some errors with various origins : they are related, not only to negative numbers and equations, but also to essential algebraic concepts. These difficulties appear indeed as an expression of a lack of understanding of the concepts involved in the algebraic operations and symbolism. The 'detachment from the minus sign' is performed by students that are not aware of the duality of the minus sign in algebraic expressions: they always consider these expressions as operations performed on numbers without signs. To give some meaning to the reduction of these expressions, they must be able to consider them as a combination of signed entities. This ability needs not only some flexibility in the meaning given to operation signs that are both attached to every term and operation sign, but also an abstract vision of subtraction and addition. That flexibility of the minus sign is also required to solve significantly an equation such as -x = 7. The minus sign in the initial equation is the sign attached to the letter x, but it is not attached to the value of x, which can be itself positive or negative. Students meet some significant difficulties at that level. Even those who find the correct answer explain it with algebraic rules and are not able to re-write the equation, replacing x by its value -7. The minus sign before the solution -7 is very often mistaken with the minus sign attached to x. We can conclude that the students' expertise doesn't consist only in reaching such or such level of conceptualisation of the negative numbers, but also in being able to adapt this level to the situation. It also seems that a part of those difficulties is linked to a pseudo-structural notion of algebraic expressions, as think Sfard and Linchevski (1994, p. 222). « Students acted as if they were handling some kind of objects, but their thinking was completely inflexible and the appropriate kind of structural interpretation was unavailable... The majority of pupils could not provide any sensible justification for the permissible operations and it was obvious that for them these were no more than arbitrary « rules of the game »

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EXPLORING THE POTENTIALS OF HANDS-ON INVESTIGATIVE TASKS FOR CURRICULUM EVALUATIONS

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A performance assessment for grade 8 students was administered in the Netherlands in 1995. The test consisted of practical, investigative tasks. Dutch curriculum experts considered this test to fit well with the Dutch mathematics curriculum, which is based on Realistic Mathematics Education (RME). However, Dutch students scored lower than expected on this practical test. This result demanded follow-up research. Therefore, the performance assessment was repeated in the Netherlands, in 2000. Trend data (1995-2000) show that the achievements of Dutch students show little gain. Also, the research gave new evidence on issues pertaining to reliability and comparability in practical tests.

INNOVATING ASSESSMENT IN MATHEMATICS

When measuring student achievement in mathematics for a large population, in many cases, paper-and pencil tests have been developed and issued to students. Especially, multiple-choice items have been popular because of automated scoring and their presumed high reliability. Yet, this method for measuring student achievement has come under debate, especially with respect to multiple-choice questions. These are associated with low-valued factual knowledge, asking for limited thinking processes. Assessment methods have been altered, although it proves difficult to break with traditions and find valid and reliable alternatives. The labels used for innovated assessment are for example: performance assessment, practical assessment, alternative assessment, and authentic assessment (see e.g. Burton, 1996; Clarke, 1996; Niss, 1993; Wiggins, 1989). The descriptions show considerable overlap and the terminology applies if some of the following criteria are met:

- testing through open questions and for higher order skills,
- being open to a range of methods or approaches,
- making students disclose their own understanding,
- allowing students to undertake practical work,
- asking for performances and products,
- being as an activity worthwhile for students' learning, and
- integrating real-life situations and several subjects.

To assess student achievement, the used formats can be portfolio, observation, interview, and so forth. But in large-scale testing, these formats can be too labour-intensive. Baxter & Shavelson (1994) compared the exchangeability of different assessment methods. These were observation, notebook reports, computer simulation, short answer questions and multiple-choice questions. They found that observation yielded most detailed information on the achievement, with notebook reports providing a reasonable "surrogate". All other tests failed to approximate the same information.

An example of a large-scale study attempting to realise an innovation in assessment is the TIMSS Performance Assessment. In 1995, this test was optional within TIMSS (Third International Mathematics and Science Study), the international, comparative study. In TIMSS, a test is developed and translated into different languages and then issued to students of different nations, in order to compare their achievements internationally. Within TIMSS, in 1995, two different tests were available. Besides a standard written test, a practical test with hands-on items was developed. The practical investigative tasks of this Performance Assessment were considered to complement the written test with a higher focus on practical skills and a lower focus on knowledge reproduction. This practical test was developed from the educational vision that seeks coherence between procedural, declarational and conditional cognition. Students are expected to investigate systematically, contrary to cookbook-demonstrations. Being provided with manipulatives and instruments, they are tested through open tasks like: designing and executing an experiment, observing and describing their observations, using calculators, looking for regularities, finding notations and interpretations of their measurements, etc. The TIMSS Performance Assessment can be associated with Gal'perin's view of *learning by doing* in which mental acts (manipulating objects in the mind) develop from material acts (manipulating tangible objects) (Van Dormolen, 1993). Though, in the assessment, manipulatives and instruments are not seen as mere demonstrators of taught concepts. They are integrated into the assessment to trigger investigative activities (Harmon et al., 1997; Garden, 1999).

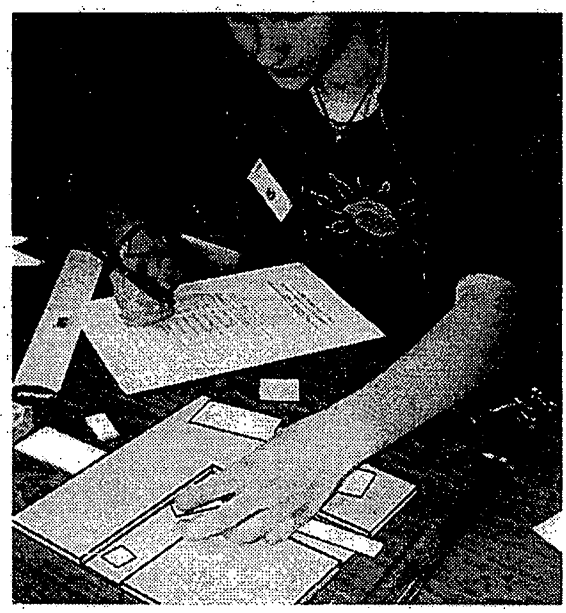

Figure 1: The task *Around the bend.*

TASKS OF THE TIMSS PERFORMANCE ASSESSMENT

The TIMSS Performance Assessment is administered in a circuit format in which students take turns in visiting stations. At each station they find a task, which guides them to carry out a small investigation. They write their answers on a worksheet. There are mathematics tasks, science tasks or combined tasks (overlapping between science and mathematics). There are five tasks with a clear mathematical focus:

- The task *Dice* is related to probability: students are given a die and a transformation rule for each throw (even: plus 2, odd: minus 1). They are asked to throw 30 times, record their findings and explain why one result (the "4") has a higher frequency.
- The task *Calculator* is related to number sense: students are given a simple calculator and are asked to discover a pattern in the multiplications of 34x34, 334x334 and 3334x3334. As the calculator holds only eight positions in the display, this is not an obvious task. The second part of the task consists of factorising 455 into two integers between 10 and 50.
- The task *Folding* is related to symmetry and spatial abilities: students have to make certain required figures by cutting, using a pair of scissors. Because only one cut is allowed for each figure, the paper has to be folded.
- The task *Around the bend* (see Figure 1) is related to scale drawing and finding rules: students are given a cardboard model of a corridor and have to cut rectangles (modelling furniture). By testing which rectangle fits through the corridor, they have to find a rule for the critical lengths.
- The task *Packaging* is related to measuring and the design of nets: students are given four table tennis balls and have to design different boxes for these.

Besides tasks with a mathematical focus, the test also contains tasks from biology, chemistry and physics. In these tasks, science investigations meet with mathematical activities, as students have to measure using instruments (using stopwatches, rulers, thermometers, and scales). But other mathematical activities are also required. For example, the task *Rubber Band* covers the topic of extrapolation. In this task, a number of washers are attached to a rubber band. Students have to measure the stretching of the band, related to the number of washers. With only ten washers given, students are asked to predict the length of the rubber band, if twelve washers were attached. Another task, named *Shadows*, is related to geometrical transformations. Students are given a torch, a card and a white screen. They have to project a shadow, which is twice as wide as the object, and find a rule for the distances between torch, card and screen. Finally, the task *Plasticine* asks for problem solving heuristics. Students are provided with a two-sided (uncalibrated) balance, two weights (20g and 50g) and a lump of plasticine. They are asked to make smart combinations in order to produce pieces of plasticine of 10g, 15g and 35g. Details of all tasks can be looked up in Harmon et al. (1997).

THE TIMSS PERFORMANCE ASSESSMENT IN THE NETHERLANDS

In 1995, the Netherlands participated in TIMSS for grade 8, both with the standard written test and the practical test. This was not without debate, as the mathematics curriculum in the Netherlands differs from the mathematics curriculum in many other countries. The Dutch mathematics curriculum is based on the principles of Realistic Mathematics Education (Freudenthal, 1975; De Lange, 1983). Dutch students learn mathematics starting from real-life contexts. Also in assessment, each test item has a theme like transport, retail prices, or sports. Each mathematics test item has narrative texts explaining the context. The mathematical tasks are embedded into the theme.

Objections to participating in an international comparative study pertained to the validity of the comparison: if Dutch students had learnt mathematics through a different approach, they would not be able to show their particular competencies. From the RME-point of view, the written TIMSS test was too traditional and therefore, results based on it were not authoritative (Bos & Vos, 2000; Kuiper, Bos & Plomp, 1997). On the other hand, Dutch mathematics curriculum experts valued the additional test, the TIMSS Performance Assessment. It was considered to match well with the intentions of the Dutch curriculum. Instead of narrated contexts, students would now apply their mathematical skills in tangible contexts.

DUTCH STUDENTS' RESULTS IN TIMSS-1995

In 1995, 18 countries participated in both the written TIMSS test and in the TIMSS Performance Assessment for grade 8. Most countries reached a position in the international comparison on the practical Performance Assessment, which had a comparable ranking to the position in the standard written test. If a country ranked high on the league-table of one test, it would rank high on the other test. But the Netherlands was a marked exception here. Despite the fit of the TIMSS Performance Assessment with the Dutch intended mathematics curriculum, Dutch grade 8 students did not score as expected. Unlike on the written test, their achievement was at the level of the international average and not significantly above.

Therefore, an understanding of Dutch students' achievements was needed. Maybe, TIMSS in 1995 had come too early. The new RME-based curriculum was only introduced in 1993. At some schools the textbooks had not yet been replaced. And maybe teachers had not yet had time enough to adopt their instruction to the new curriculum. Thus, if the TIMSS Performance Assessment could be replicated at a later stage, trend data could establish whether the new curriculum was starting to settle. The repeat of the TIMSS Performance Assessment was planned for 2000. Unfortunately no other countries were interested in participating.

DESIGN OF THE STUDY AND RELIABILITY ISSUES

The TIMSS Performance Assessment was repeated by copying all international TIMSS protocols of 1995 (see Harmon et al., 1997; Garden, 1998). A two-stage stratified sample of 50 schools was drawn. The instruments of 1995 were re-used. The science tasks were maintained because of their mathematical aspects and to maintain the task-interaction effects during testing. For 2000, a sample of n=234 students at 27 schools was realised. This response of 54% is good according to Dutch standards. In 1995, with more funds being available, withdrawing schools had been replaced, resulting in n=437 students at 48 schools being tested.

Also for coding of students' answers, the 1995 procedures were followed. But this proved to be an ambiguous task. As already pointed out in the international 1995 TIMSS Performance Assessment report (Harmon et al., 1997), inter-scorer agreement can vary considerably. As a check on coding, two independent coders coded 10% of students' work. In the extreme case of one sub-item in the task *Shadows*, the Dutch coders only agreed in 52% of the cases on the correctness of students' work. As Zukovsky (1999) has pointed out, the coding of answers is conditional to the coders' background (e.g. coding experience, subject matter knowledge, teaching experience, etc). Some tasks showed such a low inter-scorer agreement that the results had to be doubted.

In another case, with the task *Rubber Band*, the protocol did not cover a strategy that was used by a considerable number of Dutch students. In this task the students had to measure and record the stretching of the rubber band with each washer. The result would yield an irregularly increasing graph (growing with 2-5 mm per washer) with slightly diminishing growth. But approximately 10% of Dutch students did not measure. Instead, they drew a graph that grew consistently with exactly 5 mm per washer. Their graph was a perfect straight line. There was no appropriate code for this styling strategy (a heuristics which reduces realism from the onset). Depending on the interpretation of the scheme, a coder could either give full or no credit.

Comparability of testing circumstances of some tasks in the Performance Assessment proved problematic. Although test instruments of 1995 were copied in 2000, there were minimal mutations in the laboratory equipment. These mutations were within the narrow range that the international protocol allowed. One example will illustrate the effect. In the task *Shadows* a torch is used. The torch used in 1995 gave a vaguer shadow, while the torch of 2000 gave a sharper edge to the shadow. The latter made student's measurements easier giving them more time for remaining items in the task.

To eliminate unreliable and incomparable results, two tests were carried out. First, for each task, Cronbach's alpha was calculated for 1995 and 2000 separately. Results higher than 0.6 were considered acceptable. Second, a chi-squared test was carried out, revealing that answer patterns differed because of altered testing circumstances. As a result, four science tasks (*Magnets*, *Rubber band*, *Shadows* and *Plasticine*) had to be eliminated from analysis. Fortunately, the five mathematics tasks passed these tests.

Based on these five mathematics tasks, the initial Dutch agitation about the disappointing test results dwindled. In the international reports, the results of the task *Plasticine* had been included into the Mathematics league table. This task had been pulling down the overall results of Dutch students. Omission of the unreliable results raised the Dutch position above the international average. Table 1 shows the results in TIMSS 1995 of the 19 countries that participated both in the standard written test and in the TIMSS performance Assessment. The first column shows the mathematics results of the TIMSS written test. The second column shows the results on the

TIMSS Performance Assessment as presented in the international TIMSS report. This average score is based on five mathematics tasks *Dice*, *Calculator*, *Folding*, *Around the bend* and *Packaging* plus the combined task *Plasticine* (Harmon et al., 1997). The third column shows the results based on the five mathematics tasks only. The position of the Netherlands in each league-table is indicated in grey. As can be seen in the two columns for the Performance Assessment, the average scores of each country do not change much after deletion of the *Plasticine* task. The difference between the average scores of a country is at most 2% (with the exception of the Netherlands: +3, and Iran: -6). In fact, the correlation of the two scores at country level is r=0.97 (n=19). Yet, the position in the league-table can vary considerably, because of the large number of countries with only slight differences in their scores. The positions of countries with ranking 2 up to 13 are based on scores that are very close. Therefore, it can be concluded, that the test in itself is robust (the average scores do not change much after deleting of a task). Yet, the presentation in a league-table is misleading because differences between almost equal scores are enlarged as average scores (on a continuous scale) are being transformed into a ranking (on a discrete scale).

Table 1: Ranking of 19 countries in 1995 on two TIMSS mathematics tests.

TIMSS Written Test			Performance Assessment					
			(six mathematics tasks)			(five mathematics tasks)		
	Country	Score points		Country	Avg % correct		Country	Avg % correct
1	Singapore	643	1	Singapore	70	1	Singapore	70
2	Czech Rep	564	2	Switzerland	66	2	Romania	67
3	Switzerland	545	3	Australia	66	3	England	65
4	Netherlands	541	4	Romania	66	4	Netherlands	65
5	Slovenia	541	5	Sweden	65	5	Norway	65
6	Australia	530	6	Norway	65	6	Australia	65
7	Canada	527	7	England	64	7	Switzerland	64
8	Sweden	519	8	Slovenia	64	8	Slovenia	64
	Intl average	*509*	9	Czech Rep	62	9	Sweden	63
9	Nw Zealand	508	10	Canada	62	10	Nw Zealand	61
10	England	506	11	Nw Zealand	62	11	Canada	61
11	Norway	503	12	Netherlands	62	12	Scotland	61
12	USA	502	13	Scotland	61	13	Czech Rep	60
13	Scotland	498		*Intl average*	*59*		*Intl average*	*59*
14	Spain	487	14	Iran	54	14	USA	54
15	Romania	482	15	USA	54	15	Spain	54
16	Cyprus	474	16	Spain	52	16	Portugal	49
17	Portugal	454	17	Portugal	48	17	Iran	48
18	Iran	428	18	Cyprus	44	18	Cyprus	42
19	Colombia	385	19	Colombia	37	19	Colombia	35

RESEARCH RESULTS FOR THE REPEAT STUDY

By repeating the study, a trend in the achievement of Dutch students could be established. The resulting scores are given in Table 2. For each task the average percentage of correct scores on the items was calculated.

Table 2: Mathematics tasks from the TIMSS Performance Assessment 1995-2000, average percentage correct score of Dutch students.

Task	1995 (n=437)	2000 (n=234)
Dice	77	74
Calculator	62	59
Folding	73	77
Around the bend	68	69
Packaging	52	54
Average score on mathematics tasks	*66*	*67*

Compared to 1995, the scores do not show any significant changes on the five mathematics tasks. The average score correct of 66 on these tasks in 1995 does not differ significantly from the average score of 67 in 2000. Also, on each task separately, the shifts were statistically insignificant.

These results show that Dutch students have not gained in practical, investigational skills. This could be caused by the classroom practice, in which they never encounter hands-on tasks like the ones in the TIMSS Performance Assessment. Students told us, during testing, that they had never done this before. Still, the tasks match well with the intended Dutch curriculum. But the testing practice has stuck with a paper-and-pencil format in which students have to read the texts in which real-life contexts are described. Tangible real-life contexts with manipulatives are habitually not utilised in Dutch testing. However, as an additional result, the TIMSS Performance Assessment proved to be an eye-opener to many Dutch mathematics teachers. During the testing sessions, they observed the tasks and how their students coped with these. Some teachers admitted that they had never thought mathematics could be tested in this way, through a mathematical practical test. As such, the TIMSS Performance Assessment could prove to be part of the exemplary material that is needed to support curriculum reform (Fullan, 1991).

CONCLUSION

The TIMSS Performance Assessment clearly has potentials in monitoring students' mathematical investigation skills. It is a valid addition to standard, paper-and-pencil tests. It also shows, that manipulatives are useful to organise time-restricted hands-on tasks in mathematics, linking mathematics to other areas. Still, much more experience is needed when it comes to reliability and comparability issues.

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TEACHER HEARING STUDENTS

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Ruhama Even — Weizmann Institute of Science

This study examines elementary school teachers' "hearing" students, i.e., understanding what they are saying, showing, feeling and doing. In the larger study there are 25 elementary school teachers who participated in a weekly four-month long inservice workshop. In this paper we analyze the case of Ruth. We identify four types of teacher interpretations of students' talk and action: describing, explaining, criticism and justifying. We illustrate the nature of hearing characteristics, and examine two possible sources for Ruth's overhearing and under-hearing.

INTRODUCTION

It is widely accepted today that teachers should be aware of and knowledgeable about their students' thinking and understanding in order to make appropriate decisions for helping and guiding students in their knowledge construction (e.g. Ball, 1997; Even & Tirosh, in press; NCTM, 1991; Shulman, 1986). Teachers who understand where students are can challenge and extend their thinking; respond to their questions, remarks and ideas in meaningful ways; conduct discussions; promote discourse; and modify or develop appropriate activities for them.

However, assessing students' knowledge, understandings and ways of knowing, learning and thinking is not an easy task. Understanding what students are saying, showing, feeling and doing—what Ball (1997) calls "hearing students"—is complex. Teacher interpretations of students' talk and action are closely connected to cognitive, social and affective aspects. For example, to teachers' own ways of understanding the subject matter and their commitment to students (Ball, 1997).

Currently, research that explores teacher "hearing" students is scarce. The study reported in this article is an initial investigation of this issue. It is part of a larger study that examines elementary school teachers' interpretations of students' talk and action when students are engaged in mathematics problem solving. In this paper we focus on Ruth's (pseudonym) hearing two of her students. We analyze her interpretations of the students' work and their ways of thinking, both with regard to the nature of the interpretations and to their possible sources.

DESCRIPTION OF THE STUDY

Participants

There are 25 elementary school teachers from several schools in a southern town in Israel in the large study. The teachers participated in a weekly four-month long inservice workshop, led by the first author at a regional teacher center. The group was heterogeneous with regard to the grade level taught, teaching experience and age. Most of the teachers were teacher-college graduates, and so was Ruth, the teacher who is the focus of this paper. Ruth had 11 years of teaching experience in elementary upper grades. In the year of the study she taught in fourth grade. Like most elementary school teachers In Israel, she taught all subjects, among them mathematics.

The workshop

The workshop's main component centered on work on several mathematics problems taken from Wallach and Regev (1999). The main characteristics of these problems, as indicated by a field-test conducted with elementary school students, are: challenging; require long time to solve; facilitate different ways and different strategies of solving; do not have an immediate solution; encourage mathematical thinking such as: reasoning, conjecturing, explaining, justifying, etc; encourage cooperative work and discussions; integrate different mathematical topics and concepts from the school curriculum and encourage reflective thinking.

At the beginning of the workshop, the participating teachers were introduced to these problems. They solved several problems and then discussed their solutions in small groups. Later, each teacher chose one of the problems and presented it to a pair of students from her own class. The teachers observed and videotaped the students n working on the solution of the problem. The teachers were instructed not to intervene by suggesting comments, hints or advice to the students. Then, each teacher summarized and reflected on her observation in writing and met with the workshop leader to discuss her interpretations of the students' work and ways of thinking while focusing on one episode from the students' video-tape chosen by the teacher.

Data collection

Data collection includes:

- Observations of all workshop sessions. The sessions were video-recorded. The first author kept a journal where she documented her ideas and reflections after each session.
- Written work prepared by the teachers, such as reflections on their students' problem solving.
- Videotapes of the pairs of students' problem solving sessions.
- Individual interviews conducted with each teacher for about 45 minutes. These interviews centered on one episode that the teacher chose from the videotape of her students.
- Focus-group interviews were conducted twice. Once at the beginning of the workshop to discuss the teachers' own solutions of the mathematics problems and how they expect their students to solve them (about one hour). Another focus-group interview was conducted toward the end of the workshop to discuss the work of an unfamiliar pair of students who worked on the same problem that the teachers chose for their students (about one hour).

Data analysis

The analysis is based on the Grounded Theory method (Glaser & Strauss, 1967). We code the data using utterances as the unit of analysis and generate initial categories. We constantly compare new data with current categories, refine them and identify core categories, looking for integration and hierarchy among the categories, and using them as source for theoretical constructs.

RUTH'S HEARING AND INTERPRETATIONS

The "Shirts and Numbers" problem

After solving several problems at the workshop, Ruth chose to present two of her fourth grade students—Sigal and Ore—with the "Shirts and Numbers" problem (Figure 1). This problem deals with sport-team players who wear numbered shirts. The students are asked to divide the team into two groups, according to different criteria that are indicated in the problem. For example, in one part of the problem, they are asked to divide a team of 15 players so that in one group there are 5 players less than in the other group. In another part, they are asked to divide a team of 12 players so that in one group there is one half of the number of players in the second group. The part of the problem, on which Ruth chose to focus at her individual interview, deals with a task that has no solution, as is described below:

****

The task in front of you does not have a solution.

Divide a team of 15 players, so that in one group there are 4 players less than in the other group.

a) Explain why such a division is impossible:_______

____

b) Change the number of players, so that there will be a solution.

Demonstrate the solution:

****

Figure1. The "Shirts and Numbers" problem

Sigal and Ore's solution

The video recording of the students' work on the problem shows that immediately after reading the instructions Sigal says: "Because the number 15 is odd and the number 4 is even, then it's impossible". Then Sigal and Ore start trying to change the number of the team's players. During about three minutes they try to find a suitable number by thinking, reading again the problem and examining different divisions of the shirts that are drawn on their worksheet. Sigal and Ore suggest the numbers 16 and 12, but do not pursue these possibilities. After three more minutes of trying to find two groups that would satisfy the conditions, Sigal says: "Here are seven children [Ore confirms by saying "yes"], and here are three [pointing to the appropriate places on the worksheet drawing, see Figure 1]...Up to here it's seven. And three, so it's ten [players]. Four, seven minus four is three. Then here it is. Here are four children [players] less. Do you understand? [Sigal asks Ore] so ten children [players]". They both agree on ten players divided into one group of seven and one of three players. They make appropriate drawings on their worksheets. Below (Figure 2) is Sigal's drawing:

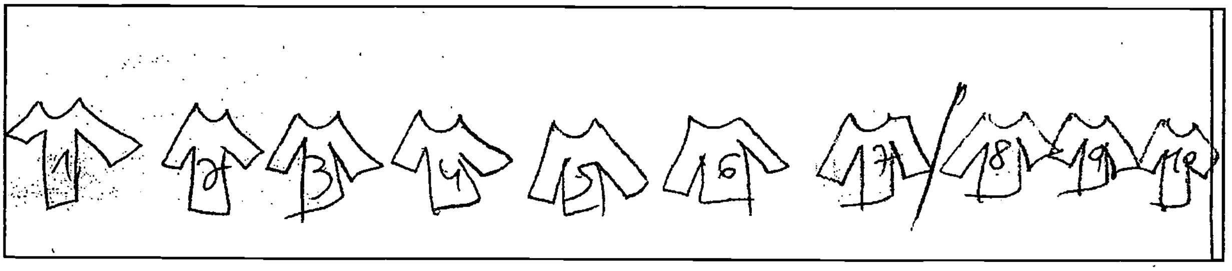

Figure 2. Sigal's drawing

Ruth's types of interpretation

We identified four types in Ruth's interpretations of Sigal and Ore's talk and action: Describing, Explaining, Criticism and Justifying. Following is an elaboration of each.

- Describing—the teacher describes students' talk and action by direct (or almost direct) "quotation" or portrayal.
- Explaining—the teacher explains students' talk or action. This includes ideas about students' thoughts, reasoning, knowledge, and assumptions. The explanation is the <u>meaning</u> the teacher attributes to students' talks and actions.
- Criticism—the teacher criticizes students' talk and action, based on the "Meaning" she assigns to them.
- Justifying—the teacher justifies her criticism of, or the meaning she attributes to, students' talk and action. This type of interpretation is different from the other three types, because here the teacher reflects on her own interpretation and not on the students' talk and action.

The following two excerpts from Ruth's interview illustrate these different types of interpretations. The first excerpt demonstrates the first three, and the second excerpt demonstrates the last two types.

First excerpt:

Teacher: She says, let's take 4, she marked 4. Now she wants to divide it by 2 (Describing). Because she forgot that it is impossible with an odd number (Explaining). She forgot again. She tried to counterbalance the 4 (Explaining). This is not a method at all (Criticism).

Second excerpt:

Teacher: They got mixed-up when working on this (Criticism).

Interviewer: What would you expect them to do?

Teacher: Frankly, I expected them to work more, a lot faster and a lot more easily (Justifying).

Ruth's hearing her students

Ruth's interpretations of Sigal and Ore's talk and action are reflections of her hearing Sigal and Ore, of her understanding what they are saying, showing, feeling and doing. Below we illustrate the complexity of hearing students, what might be entailed in interpreting students' talk and action, and suggest possible sources for Ruth's hearing.

Overhearing. We start with an example that portrays how even the relatively simple type of interpretation—describing—is not as simple as might first meet the eye. When Sigal and Ore attempted to solve the "Shirts and Players" problem, Sigal said, "Because the number 15 is odd and the number 4 is even, then it's impossible". Before we look at Ruth's interpretation, let us examine this statement. What does Sigal mean?

One possible interpretation is that Sigal knows that to solve the problem she can remove 4 players from the entire team and then divide the reminder players into two equal size groups. Finally, adding the four players to one of the groups would give the solution of the problem. However, Sigal understands that in this specific case the partition is impossible because 15 is odd, and when she subtracts 4 from 15 the difference is 11, which is an odd number, and an odd number cannot be divided by two. Consequently, Sigal claims: "it's impossible". It might even be that Sigal understands the generalization of this method, that the partition is possible only when the number of the players is even. Because only in this case the removal of 4 players would leave an even number of players, which can then be divided into two equal size groups.

Another possible interpretation is that Sigal knows that the partition is impossible because this is stated in the text "The task in front of you does not have a solution". She also knows and identifies that 15 is odd and 4 is even. So she connects these two pieces of information, and without any logical argument states that "15 is odd, 4 is even, then it's impossible". Of course, there may be other possible interpretations.

Let us examine now Ruth's interpretation of Sigal's statement. During the interview, on two different occasions, Ruth "quotes" Sigal's explanation (Describing type) as "...If you take away an even number from an odd number, you are left with an odd number, which you cannot divide by 2". In another occasion she says " They said: Oh, for sure it's impossible. Odd minus even, it's impossible to divide it by two". However, Sigal did not say all that. She refers only to the two numbers, 15 and 4 and indicates that they are odd and even respectively. She does not mention any operation on these numbers or on the difference between them. Still, what Ruth hears is different. Ruth hears Sigal talking in generalizations, without mentioning any specific number. She hears her discussing odd and even numbers, the difference between an even number and an odd number, and of its divisibility by 2. Even when the interviewer tries to shake Ruth's confidence about her interpretation and asks: "Perhaps here Sigal did not understand what you think that she understood" Ruth remains confident and responds without hesitation "Then what, she just said it? Like a parrot?" and " It seemed to me real understanding".

This short episode illustrates that Ruth "hears" things that were not said by the students. We define this characteristic *overhearing*. Interestingly, such overhearing is reflected even in the relatively simple type of interpretation—describing—which naturally would be expected to be an accurate duplicate of what the students said. The discrepancy between what students say or mean and what the teacher hears could be significant to teacher decision-making.

<u>Under-hearing</u>. When later Sigal suggests the solution of 10 players, Ruth does not seem to believe that there is any substantial reasoning underlying this solution. Several times throughout the interview Ruth is asked to explain how the students reached this solution. Her responses (Explaining type) did not focus on the students' reasoning. For example, "The solution just came out of the blue", or " She just said 10 off the top of her head". On another occasion Ruth says: "Suddenly, she had a flash of the 3 and the 7. OK, 10 shirts [players]... Fine, she got it, she reached an even number" or " It seems to me that she just got it". These explanations do not seem to get into the details of the students' reasoning. Ruth does not refer to the process in which Sigal builds the two groups, by circling first 4 shirts followed by creating two groups accompanied with her explanation: "Here are 7 players and here are 3 [pause] so [pause] 10 players. 7 and 3 makes 10. And 7 minus 3 is 4, so 10 players". Ruth "ignores" some of the things said or done by the students. We define this characteristic as *under-hearing*.

SOURCES OF RUTH'S HEARING

Why does Ruth "hear" what was not said or done by the students? Why does she "mishear" what was said or done by them? Possible sources of teacher hearing their students are teacher own knowledge of mathematics, conception of the solution of the problem at stake, beliefs about the nature of mathematics learning and knowing, understanding of the nature of mathematics teaching, disposition towards the teacher role, and so on. In the following we analyze Ruth's concern for her students' success and Ruth's own conception of the solution to the "Shirts and Numbers" problem. Then we examine the possibility of explaining Ruth's hearing Sigal and Ore on the basis of these two sources.

Ruth's concern for her students' success

Throughout the interview Ruth expresses clearly her desire to see Sigal and Ore succeed in solving the problem. While watching the video-recording of them working, she looks happy and satisfied when they say what she considered as good and upset and disappointed when it seems to her that they are not on the right track. She says, for example, "I really enjoyed it", referring to Sigal's statement that 15 is odd and therefore it is impossible, or "I enjoyed it very much, I was astonished by the way she came up with it immediately", etc. This concern to see her students succeed may emerge simply from her liking them. It may also be related to her role as their teacher, where Sigal and Ore's success may be regarded as her own success.

Ruth's conception of the "Shirts and Numbers" problem's solution

When solving the problem by herself, to show that there is no solution to the "Shirts and Numbers" problem when the group-size is 15 (part (a) of the problem), Ruth subtracts 4 from 15 and receives 11. Then she points that the division of the result by 2 does not give a whole number, which means that the problem has no solution. Throughout the workshop, when discussing hers and her colleagues' solutions, Ruth emphasizes that it is enough to state a "short version" of this solution, that "15 is odd and 4 is even. It is not possible to divide 11 by 2". This short version includes a claim about the impossibility of solving a more general problem when the group size is odd and the difference between the sizes of the two teams is even. When discussing the solution with her colleagues, Ruth is willing to accept even a shorter version, where only two components are mentioned: "15 is odd and 4 is even". For her, such a statement represents the more elaborated solution.

When solving part (b) of the problem, Ruth changes the number of players to 14. Analysis of the data suggests that Ruth's solution strategy is to *remove* a *minimal* number of players to reach an even number of players. For example, in her interview Ruth says: "…It bothered me that they didn't remove a shirt immediately, [pause] and to reach an even number. She [Sigal] knows that it [the number order] goes even, odd, even, odd. So, remove a shirt and then get an even number". On another occasion Ruth says: "I was really surprised that they [Sigal and Ore] changed to 10, [that they] removed 5 shirts. Remove 1[pause]… I don't know, it seemed to me that you need to remove 1 and try".

Hearing through

Ruth's concern for her students' success and her own conception of the problem's solution seem to contribute to Ruth's overhearing and under-hearing Sigal and Ore.

Wanting to see them succeed, Ruth is receptive to interpret their talk and action in this way. When combined with her own conception of the problem solution, Ruth overhears in Sigal and Ore's statement that "15 is odd and 4 is even" the answer she expects. However, when Sigal and Ore propose the solution of 10 players that they found by actually *constructing* two teams in which one has 4 players more than the other (3 and 7), Ruth's under-hears them. She is unable to figure out how they reached this solution because their *constructing* strategy is different from her *removing the minimum possible to reach an even number* strategy. Even after watching this part of the video recording several times, Ruth under-hears her students and claims that " The solution [10] just came out of the blue", or that " She just said 10, from the top of her head". Ruth knows that 10 is correct but because both the strategy of solution and the resulting number are different, Ruth mishears her students.

FINAL REMARKS

In our investigation of Ruth's hearing her students, of her understanding what Sigal and Ore were saying, showing, feeling and doing, we identified four types of teacher interpretations of students' talk and action: describing, explaining, criticism and justifying. We illustrated the complexity of hearing students, even in the relatively simple case of describing, and examined two possible sources for Ruth's overhearing and under-hearing: Ruth's concern for her students' success and Ruth's conception of the problem's solution. Ruth's case is an initial investigation of teacher hearing students that raises interesting questions, such as, Are there additional interpretation types? Are they interrelated? How? These and other questions are the focus of the next stage of our research.

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TELLING OTHER PEOPLE'S STORIES: KNOWLEDGE PRODUCTION AND REPRESENTATION LESSONS

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This paper raises epistemological questions which have to do with how we know one another. These questions are directed towards inquiry, and ask how knowledge of other people is constructed and circulated. I begin by looking at how the scientific model stakes out certain rhetorical spaces for establishing credibility and for gaining acknowledgment, noting how our current research models limit and regulate epistemological legitimacy. In the second section I introduce poststructuralist ideas of knowledge as a useful way of dealing with intersubjective arrangements in which cognitive resources and positions of authority and expertise are unevenly distributed. The claim is that inquiry informed by these ideas necessarily invokes political deliberation.

INTRODUCTION

Posing one of the most provocative questions in the field of educational inquiry, Patti Lather asks: "How does a researcher work to not see so easily in telling stories that belong to others?" (1994, p44). Framed within the context of a contemporary interrogation into modernist descriptions of reality and truth, the question presents to those of us working in mathematics education an abrupt challenge to the way we have always done things. It invites an investigation of conceptual issues concerning what it means to know others and tell their mathematical stories. In doing so, it moves us from a preoccupation with what ideal researchers ought to do, opening up a space less comfortable than the certainties and absolutes of our research practices and all the relationships within those practices.

Certain principles have always guided our practices. The science from which those principles are derived has played an important role in cultivating the means by which those of us doing research might be empowered to make the classroom and the wider communities of knowers, a better place. *That* science claims to produce "paradigmatic instances of the best knowledge possible, for everyone, in all circumstances" (Code, 1995, pxi). I question those claims. Within the terms of the disintegration of the classical *episteme* of representation, I investigate the implications of that crisis for the gathering and analysis of qualitative data. My purpose is to encourage a move away from the desire to establish true and accurate accounts about others and suggest how we might rethink our practices to accommodate an awareness of the limits of knowing. This question is not to anticipate the end to representation *per se* and abandon wholesale engagement, but to draw on the Derridean (1976) realisation about the end of 'pure presence', in order to suggest a transformed discursive practice.

The discussion is in two parts. First I revisit the research traditions which occasion most of the investigative work in the field. I look at how we have constructed mathematical knowledge of other people, noting how our current research models limit and regulate epistemological legitimacy. What obtains from this critique is the opening of a rhetorical space for more sophisticated analytical tools. In the second section I present a theoretical justification for a form of research which demands attention to negotiations among people who are intersubjectively constituted. In this practice the researcher assumes a self-reflexive stance, asking questions about the methodological implications for the way reality is understood and represented. This is the point where the interest moves from establishing truth onto an understanding of how meaning is produced and created and in how these productions factor into larger decisions concerning power and privilege.

HISTORIES OF RESEARCH PRACTICE

Research is about making a difference through science. Central to this taken-for-granted understanding, is the modernist belief in the perfectibility of society. If it is the modernist gesture of progressive change by which science is circumscribed, then it is this same science which came to be responsible for the administration of that progress. Foucault (1972) has provided a full conceptual articulation of how science, invested with multiple interests of regulation and redemption, became an important social force. Although twenty-first century science has long since located itself beyond the modernist project, the ideas of truth to which that project subscribes still play a major role in the structure and process of education. By creating a regime of order which organises and regulates our practices of inquiry, this particularly powerful discourse occupies a standard-setting position, determining what counts as valuable knowledge and who has access to the production, the distribution, and the legitimation of that knowledge. Research in mathematics education, as one of these practices, is complicit within this regime of truth.

Fundamental to this regime is a grand design of systematising and tidy partitioning, which lends coherence of the advancing of a universal explanation of the world. From this perspective, patterns of an individual's life reveal observable facts, and hence are able to be classified, categorised, and ordered systematically and linearly as, using Foucault's (1972) term, positivities. These facts can then be interpreted. Indeed it is a principal concern with an interpretation of gathered data which enables one to discover the truth in human terms, and by this means to normalise experiences (for example, stages of mathematical development), understandings and even our desires, with a view towards a definitive order upon which judgments can be substantiated.

To engage the idea that it is possible to make a difference through systematic order is to make an assumption of a coherent and transparent reality which awaits expression. This scheme of things is the site *par excellence* of a metaphysics of presence: reality is attributed evidential status as if it were existing 'out there' waiting to be captured (Walshaw, 2001a). Since seeing is the origin of knowing, reality assumes a fixed character, exhibiting certain qualities regardless of who is observing. Knowers are interchangeable spectators, abstracted from the particularities of their circumstances; objects of knowledge are separate from knowers, functioning as inert items in the observational knowledge-gathering process. This conceptualisation of the autonomous knower prompts and sustains a belief in the existence of a universal, homogenous and 'essential' human nature which allows those doing research to put themselves in another's place and know his or her circumstances and interests in exactly the same way as she or he would know them. The researcher's role then is that of disembodied arbiter of knowledge – one who can bestow authority and credibility onto the experiences of others and provide access to truth and certainty.

These ideas are still current, informing conceptions in both quantitative and qualitative approaches of what it means to know and what it means to know others. Both subscribe to a set of assumptions to the effect that knowledge is made by the abstract, interchangeable individual whose stories have been spoken from nowhere and everywhere. What emerges from this is a staging of truth, in which the concepts of objectivity and abstraction play a central role.

This is not in any way to suggest that all research is premised on notions of abstraction and universality. Some theorists, working to escape from the logic of the abstract, 'generalised', disengaged individual of the Western tradition, have argued for the concept of difference (for example, Bishop, 1988; Carraher, 1988; D'Ambrosio, 1985; Lave, 1997; Nickson & Lerman, 1992; Restivo, 1992). In their discourses, categories which have traditionally been regarded as commonplace in mathematics education are reordered and in the process common arguments about knowledge and its transference are undermined. In this way these theorists are able to demonstrate how the concepts and categories fundamental to an epistemology of mathematics derive their authority and value from what is repressed and excluded.

Difference carries infinitely rich connotations and continues to be harnessed to a wide variety of political and cultural projects. But aren't those who articulate the construction of difference and the reification of voice situated both within and against established traditions for doing research? Popkewitz (1997) sets forth a similar criticism of the construction of difference in contemporary social and educational theory. He argues that "the concept of voice maintains the very rules of 'sameness/difference" that it seeks to violate" (p25).

Recently the same kinds of arguments about the complicity that goes hand in hand with the concept of difference have been levelled at the reconstructionist feminist project (Walshaw, 2001b). Isn't the theorising of women's mathematical experience not, in a very real sense, entrapped in the very logic which it seeks to subvert? Doesn't the method of feminist resistance (for example, Becker, 1995; Burton, 1995; Damarin, 1995) work within established frames by reinscribing that which it is resisting? By reconstructing the language and reversing the binaries standpoint feminism remains imprisoned within modernity's parameters - parameters which have the effect of masking a complicity in structures of power and privilege.

Talk about the celebration and elevation of female difference bypasses the problematic of representation, and those questions which relate to social reality, to institutions and to power remain fully imperceptible from these standpoints. Clearly, the epistemic project needs to be conceptualised quite differently from one contained within a master discourse which obscures intersubjective negotiations of mathematical learners, teachers, and researchers of differential power and privilege. It needs to be superseded with a new set of perspectives, more in keeping with the unpredictability of our contemporary way of life.

RETHINKING RESEARCH

Whereas qualitative research moves forward positivist conventions of how we go about reality construction, poststructuralism offers an interrogation about those very practices, exposing the inadequacy of objectivist epistemologies. In claiming redundant a theory of knowing organised around uncontaminated situational exigencies, poststructuralism registers a realisation that modernist values, assumptions and explanations are no longer adequate nor desirable when we try to make sense of our contemporary world. Poststructuralist ideas then become key resources for showing how the conventions of research practice are mapped in such a way as to preclude it from even and equitable possibilities of establishing credibility. It offers those of us committed to understanding the epistemic implications of knowing others well, a potential vantage point from which to rethink the way we do research. At the same time, however, this theorising elicits highly charged reactions precisely because there is so much more at stake derived from an interrogation about the limits of knowing than the existence or non-existence of a 'correct' research procedure.

Poststructuralism is an intellectual movement informing a constellation of theoretical positions ranging from phenomenology to deconstruction. Each of these discourses takes as its founding principle the disintegration of the classical episteme of representation. Read within those terms, poststructuralists maintain that, irrespective of our efforts to contain it, knowledge will always escape our grasp. Couching their work in a language which destabilises and challenges, poststructuralists historicise our 'safe' and 'true' understandings, offering critical interrogations of familiar ideas of knowing, description, and the rational subject. Through these interrogations they expose the limits of knowing, encouraging a scepticism about the possibility of true and accurate research findings, and, moreover, about the very possibility of knowing others and telling their stories.

Questioning the operative logic in contemporary research - an assumed homology between observation and knowledge – poststructuralists argue that representation can no longer be considered a politically neutral and theoretically innocent activity. Their work is not an argument for relativism, but a claim that representation is necessarily always partial, historically specific and interested. What this invites is a new understanding of how we go about reality construction.

A poststructural inquiry redeploys the meaning of research to offer "less false stories" (Harding, 1991) and to enable concrete social changes. Premised on the essential indeterminacy of human experiencing, these inquiries explore how knowledge production and its legitimation are historically situated and strategically practised. Mindful of the problem of unmediated access to a *transparent* mathematical reality, poststructuralists consider the problem of access to a mathematical reality from the perspective of local and marginalised practice. However, unlike in our research traditions, the intent is not to seek common denominators and homogeneous networks of causality and analogy in specific mathematical practices. Nor is it to promote a list of determinations and categories such as those of gender, race, or ethnicity. Rather a poststructural approach proposes an understanding of the categories by which mathematics education is organised as historically emergent rather than naturally given; as multivalent rather than unified in meaning; and as the frequent result and possible present instrument in struggles of power. Within this proposition reality emerges as fluid in nature, forever in process, continually being reshaped by the changing categories individuals use to understand themselves, others, and the spaces they share.

A recognition of unstable competing realities points towards a different research practice. Relativising the status of all truth claims requires "changing the subject" (Henriques et al., 1984), shifting the emphasis from the learner as the site of original presence, to a decentred, relational complex process. Actualising the site of this different research practice demands attention to how the researcher is also implicated (Britzman, 1995). If there is no "view from nowhere" (Haraway, 1988), and if representation must pass through the filter of the researcher's discipline, biography, and social determinations, such as race, class, gender, ethnicity, and so on, then the researcher's knowledge of mathematics education always privileges particular interests. If reality is understandable only through the use of the abstract categories which the researcher employs, then those very categories researchers are productive of shifts and movements themselves. Following from this, the researcher becomes with the learner and the teacher, a key player in the production of educational knowledge.

In Foucauldian work the claim is that *all* categories and concepts of practice are the effects of specific relations of power, all producing some dissonance between and within the individual. Inquiry derived from Foucauldian ideas (for example, Klein, 2000; Walshaw, 1999) investigates the power relations which make a focus on the production of mathematical knowledge both possible and an effective tool of subversion. This form of inquiry draws attention to lived moments of practice where cognitive resources and positions of authority and expertise are unevenly distributed to inform, constrain, and implicate mathematical work. By attending to these concerns and to the broader historical contexts of mathematical experiences, Foucauldian ideas then become a productive means to account for different degrees of coherence between subject positions and mathematical practice. They become a key resource for unmasking the ways in which the teaching and learning of mathematics is intimately tied to the social organisation of power.

CONCLUSION

Opting out of the impossible, yet nevertheless seductive, desire for coherency, and the impulse to access the 'truth' in mathematics education, the poststructural research project pushes the traditional boundaries in terms of how we know others and how we tell their stories. Concerning itself less with establishing researcher authority, and more with questioning the very construction of that authority, the investigation advances claims of multiple and contradictory positionings. It questions conventional constructions of objectification in order to take into account competing stories working through and against the stability of meanings, identities, experiences, the treacheries of language, and the conceptual order constructed by all those involved in the research. But far from dismissing observation *per se* the intent is to query the uncritical appropriation of our conceptual categories and the logic we deem necessary to access 'reality'. Such an appropriation has everything to do with the power of the science, on which research is based, to disengage itself from contradiction, disunity, and multiplicity.

All research, even that named as qualitative and couched in the language and rhetoric of postpositivist discourse, operates within certain codes and conventions. The question raised by poststructuralism is not the existence of such commodified complicity, but the conditions of its deployment, and its effectivity. In drawing attention to the categories we construct and their derivative conceptual order necessary to access truth, we need to think about the way in which the political impinges upon and infuses all of our thinking and acting about research. The question we should be asking is not 'is this research objective enough', but rather 'if this research is authenticated and validated, what motivates its deployment? What are the political effects?'

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FLEXIBLE MATHEMATICAL THOUGHT

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We consider different aspects of flexible thinking in mathematical contexts, and illustrate these by examples from recent mathematics classrooms. We argue that aspects of flexible mathematical thought can be encapsulated as an ability to use long-term declarative knowledge in novel situations. Both Karmiloff-Smith's representational re-description and recent work in the psychology of memory are related to flexible thinking.

INTRODUCTION

Cognitive flexibility is a well studied, but not especially well-defined, notion in the psychology of mathematics education. Flexibility of thought is often treated in the psychological literature as a given which is approached operationally through measurement (Berg, 1948; Edwards, 1966; Busse, 1968; Jausovec, 1994), or via specific problems that are thought to require flexibility for their solution (Dover & Shore, 1991; Kaizer & Shore, 1995). With a few significant exceptions, studies in mathematics education treat flexibility generally as the capacity to exhibit a variety of novel strategies for solving problems (Dunn, 1975; Carey, 1991; Klein & Beishuizen, 1994; Vakali, 1994; Beishuizen et al, 1997; Heirdsfield, 1998; Imai, 2000; see also Shore, Pelletier & Kaizer, 1990). The significant exceptions are Krutetskii (1969b), Shapiro (1992) and Gray and Tall (1994). Shapiro was a PhD student of Krutetskii in Russia in the late 1960's. Krutetskii and Shapiro characterize flexible thinking as reversibility: the establishment of two-way relationships indicated by an ability to "make the transition from a 'direct' association to its corresponding 'reverse' association" (Krutetskii, 1969b, p. 50). Gray and Tall (1994) characterize flexible thinking in terms of an ability to move between interpreting notation as a process to do something (procedural) and as an object to think with and about (conceptual), depending upon the context:

> "We characterize proceptual thinking as the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object. It is proceptual thinking that gives great power through the flexible, ambiguous use of symbolism that represents the duality of process and concept using the same notation." (Gray & Tall, 1994, p.7)

In this paper we consider in detail what aspects of "flexibility" might be considered reasonable and desirable in a mathematical context. Although this is a theoretical analysis, aimed at clarifying the notion of flexibility, we base our discussion largely upon empirical work carried out in a grade 6/7 problem solving class.

THEORETICAL BACKGROUND

Flexibility of thought we will argue, is basically an ability to use long-term declarative knowledge in novel situations (Squire & Kandell, 2000). As simple as this statement seems it encapsulates a lot of background discussion and thought from mathematics education and psychology.

Part of the reason that the psychological literature has had so much difficulty with flexibility is that it may not be, and probably isn't, a trait. Like bodily flexibility, some people may naturally incline to a greater degree of mental flexibility, but all people can improve their flexibility. This is an important issue to bear in mind in tests of flexibility: if flexibility is not a trait, as we suspect it is not, then tests of flexible thinking will generally have low reliability (Edwards, 1966), especially if practice on the tests substantially enhances one's flexibility of thought.

Flexibility of thought as conceived by psychologists working on the neuropsychology of memory, is essentially long-term declarative memory – memory that has been processed in its formation by the hippocampal system (Squire & Kandell, 2000). The significance of the hippocampus is that, according to Eichenbaum (1994, 1887), it establishes a "relational space" in which prospective long-term knowledge becomes embedded relationally with other long-term knowledge. The well-known properties of the hippocampus in relation to spatial orientation are just one aspect of a more general capacity of the hippocampus to represent knowledge and memory in ways that allow flexible use in novel settings. The basic idea is that declarative memory - long-term memory that, in students, is expressed through words or diagrams - is inherently flexible in nature. Neuropsychologists, particularly those engaged in memory research, tend to speak of "flexible memory representations" rather than "flexibility of thought". This aspect of flexible thinking, involving a transition from implicit to declarative memories, is also a prominent feature of Karmiloff-Smith's theory of representational re-description (Karmiloff-Smith,1992). One interpretation of her theory of representational re-description is that it involves a transition, for an individual student, from short-term working memory to long-term flexible declarative memory, and thereby achieves a greater capacity for storage and retrieval of relevant memories, and also a wider applicability of those memories.

ASPECTS OF FLEXIBLE THINKING IN MATHEMATICS

Apart from a capacity for using novel strategies to solve problems, which is the most commonly used definition of flexibility in the mathematics education literature, we illustrate, below, a number of aspects of flexible thinking that are consistent with the neuropsychology of declarative memory. These all involve students in a "re-orientaton": of place, situation, of person, direction, or interpretation of notation. This re-orientation is consistent with the nature of long-term declarative memory as essentially relational (Eichenbaum, 1994; Squire & Kandell, 2000). From this perspective cognitive neuropsychology may be catching up with the ideas of Skemp (1976).

Capacity to interpret someone else's thinking

A student might explain another student's thinking and use it, build off it, try to prove that it is invalid, or ask questions about it. Consider the following example. A grade 6 student, Amanda, presented to her class a connection between three problems: (1) building all possible towers 4 high, using black or purple cubes; (2) finding all ways 2 teams can win the World Series playoff in 4 games; and (3) finding all possibilities for walking stairs 4 high, taking double or single steps. The following day a student, Jessica, gave a presentation to the class trying to explain what Amanda was saying, and convince the class that Amanda's connection was not valid. Jessica showed a key that equated a black cube, *b*, with *Y* (representing the Yankees) and a *2* (representing a double step). She did the same for the other possibilities: *purple* ↔ *M* ↔ 1. She showed a four tall tower (*bpbp*) that should, under Amanda's connection, be equivalent to an example of a way to play the World Series in four games (*YMYM*), and equivalent to a possibility for walking stairs four high (*2121*). She stated the connection would not work "because it would go overboard": 2121 would make six stairs high, instead of four.

This is an example, on Jessica's part, of interpreting another student's thinking (in this case to show a connection was invalid). The situation that is novel for Jessica is Amanda's schema of connections. Jessica brings to this novel situation her recollections of working on all three problems and is able to argue that Amanda's connections do not do what Amanda intended.

Using an idea or strategy across different contexts or changing an existing strategy to fit a new context

Consider a grade 6 student, David, who mapped out an equivalence between the number of handshakes at a 4 person party (where each person shakes hands once with every other person at the party) to the "two purple case" of building towers four-tall, choosing from two colors, black and purple. He labeled the position of each block beginning at the top of a tower, working down (1, 2, 3, 4), focusing on the position of the two purple blocks. He also labeled the people at the party 1, 2, 3 and 4 and color-coded them. He began with the numerical pair "1, 1" and claimed that two purple blocks cannot both be placed in position 1, and, corresponding to his labelling, person 1 cannot shake hands with themselves. He then listed "1, 2" and said the two purple blocks would be in the top two positions and, correspondingly, person 1 would shake hands with person 2. He listed "1, 3" and "1, 4" showing how the towers would look, and represented this with two people shaking hands. He did not count "2, 1" because, according to him, person 1 already shook hands with person 2 and the purple block was already in position 1 and 2. He continued to show each possibility with towers and people and why he would or would not list each. Here he used the same strategy for two different problems, creating an exact mapping of possibilities from one problem to the other. In essence David extracted the scheme of working systematically through the number pairs "a, b" with a < b, and interpreted both the tower building problem and the handshake problem from this point of view. This involved a double re-orientation: first interpreting towers as handshakes and vice versa, and then seeing both problems as instances of ordered pairs "a, b" with a < b. David utilized his knowledge of building towers, and of calculating handshakes, in a novel setting in which number pairs were his focus of attention.

~~1, 1~~	1, 2	1, 3	1, 4
~~2, 1~~	~~2, 2~~	2, 3	2, 4
~~3, 1~~	~~3, 2~~	~~3, 3~~	3, 4
~~4, 1~~	~~4, 2~~	~~4, 3~~	~~4, 4~~

Table 1. David's explanation that C(4,2) = 6, relating towers of height 4, with 2 black and 2 purple blocks, to handshakes between 4 people.

Using multiple representations for the same mathematical problem or using multiple representations to express the same idea

Use of multiple representations is an essential aspect of mathematical flexibility. One can see it operating superbly at an advanced level in Thurston (1997), where, for example, different representations of manifolds and knot groups are utilized to enhance understanding.

We give an example of multiple representations by considering Jessica as she built towers two blocks tall from those one block tall by placing a black cube and purple cube under each possibility. She showed this by drawing an arrow from the original tower to the two new towers that she created and numbered each pair with the tower it came from. She also did this to create all towers four blocks tall from those towers three blocks tall. She later changed her recording to a tree diagram to show how to recursively build all towers of any height. Jessica re-oriented herself through encapsulation of the recursive process of building towers via a tree diagram. The tower building problem as conceived as a branching process (from our perspective, not Jessica's) and the tree diagram was used as a representation of this branching process.

Raising hypothetical problem situations based on existing problems

These are essentially "what if" questions, and involve imagined scenarios, regarded as similar to an existing problem. For example, a grade 6 student Michelina asked in relation to building towers: "What if we have three colors to choose from, instead of two?" This question indicates that Michelina was not stuck in the constraints of the problem but imagined, what was to her, a similar problem in which she could utilize three, rather than just two, colors. Again, this involved a re-orientation into scenarios that are similar to those already encountered, but not yet acted out.

Using notation ambiguously

This is Gray & Tall's (1994) definition of flexible mathematical thinking. They give an example of moving comfortably from using the division sign to a fraction bar representing division.

Consider the example of a grade 6 student, Jessica, who explained that the number of possibilities doubled when she moved from towers built from black and purple blocks, of a given height, to towers of height one greater. She described doubling by multiplying by two. She then to switched to doubling by adding the number plus itself. This way of describing doubling apparently allowed her to see things more clearly. After she began describing doubling this way, she saw that her possibilities were doubling recursively. (Initially she had doubled the *height* to get the number of possibilities). This allowed her later to make a connection with doubling of the number of possible towers and her strategy of placing a black and purple cube under each tower to get all possibilities for the next height. The ability to see the process and results of calculation in different lights is a form of mental re-orientation: one obtains alternative and complementary points of view.

Transition from a 'direct' association to a corresponding 'reverse' association

This is Krutetskii (1969b) and Shapiro's (1992) definition of flexibility of mathematical thought. A clear example of his aspect of flexibility occurred when sixth graders found a formula for the handshake problem for different numbers of people at a party. Michelina asked if they could find a formula when they know the number of handshakes and don't know the number of people. Two further examples, one from a college level remedial algebra course, the other from a pre-service elementary methods course, also illustrate this aspect of flexibility. For the first example, consider the remedial college algebra student who wrote, as part of his final self evaluation:

> I now have the skills to interpret and use mathematical notations appropriately, reflected in my work and ability to interpret a function machine; convert from one type of mathematical notation to another, convert a function machine to an equation/equation to function machine.... I have identified inputs and outputs, which was the key to answering the questions, appropriately and accurately in my tests on questions involving: recognizing the given slope to make a table, showing input/output; identification of a given notation and breaking it down by input/output; naming input/ output from a given function machine or equation; finding output if given input/finding input if given output.... I am knowledgeable in understanding the difference between evaluating and solving: solving for X; evaluating for Y.

This student's work indicated that he did indeed know the difference between evaluating an expression and solving an equation, a distinction he did not make at the beginning of his course.

Second, consider the example of an elementary teacher who, in a mathematics methods course, turned around a problem, utilizing a reverse association. The problem was to begin with a sheet of paper in one person's hands, to tear it into two and pass one piece to some one else. That procedure was repeated until everyone had a piece of paper. The problem was to figure how many pieces of paper there were after the last tear. The reverse problem, raised by a teacher on the course, was to count the number of people in the room and figure how many tears were needed so that everyone had a piece of paper.

CONCLUSIONS

Building on the work of Krutetskii (169b), Gray & Tall (1994) and other research in mathematics education we identify several major aspects of flexibility of mathematical thought:

1. Use of novel strategies to solve problems.
2. Ability to establish both direct and reverse associations.
3. Ambiguous use of notation.
4. Capacity to interpret someone else's thinking.
5. Using an idea or strategy across different contexts or changing an existing strategy to fit a new context.
6. Using multiple representations for the same mathematical problem or using multiple representations to express the same idea.
7. Raising hypothetical problem situations based on existing problems.

These aspects of flexibility are, from our point of view, all manifestations of a fundamental ability to utilize what is stored in long-term memory in novel settings: of place, time, direction, representation, or person. This ability we hypothesize is related to the capacity of the hippocampus to establish long-term relational memories. A model of Davis, Hill & Smith (2000) suggests that a teacher can act as an external agent in taking students' procedures, helping to make them explicit and emotionally colourful, and so assist in transferring those procedures into explicit long-term memory. The classroom practices of the first author are entirely in accord with that model. Whether, and to what extent, all students can increase their flexibility of mathematical thought is a question we are currently investigating.

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UNKNOWNS, ARITHMETIC TO ALGEBRA: TWO EXEMPLARS

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This paper examines young children's ability to solve sentences with unknowns. A semi-structured interview was conducted with 87 children who had just completed their first three years of formal schooling. The purpose of this interview was to ascertain their thinking when solving problems with unknowns and draw implications for early years algebra. The results of the interviews indicated that many children are experiencing difficulties in solving for unknowns.

INTRODUCTION

The transition from arithmetic to algebra has been a major focus of recent research into the learning and teaching of algebra (Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 2000). A focus of this research is the gap between the knowledge required to solve arithmetical equations by arithmetical methods and the knowledge required to solve algebraic equations by operating on or with the unknown (Booth, 1988; Herscovics & Linchevski, 1994; Filloy & Rojano, 1989). Student's inability to operate with or on the unknown has been central to delineating arithmetic from algebra, where algebra traditionally follows arithmetic. Early algebra research is moving towards the integration of arithmetic and algebraic reasoning in the elementary grades. Thus the importance of how young children deal with unknowns is growing.

ARITHMETIC IN THE EARLY YEARS

A variety of addition and subtraction problems exist in the real world. Fuson (1992) identified four broad categories that addition and subtraction problems seem to fall into. For each, the use of the unknown varies. The first is referred to as *Change-Add-To* and *Change-Take-Away*. This involves beginning with a single collection and changing the initial collection either by adding or removing something from it. This category covers the majority of problems children solve in the elementary school (Baroody & Standifer, 1993). In this instance, the unknown is the outcome of the change. Its manifestation in algebra (e.g., $4+3=x$) is trivial and represents limited challenges in developing meaning for unknowns.

The second category focuses on considering arithmetic situations as comprising three components, two parts and a whole and arithmetic operation as joining the parts or removing one part. In this understanding, in order to find the unknown it is simply a matter of working out which component is missing and applying an appropriate solution strategy. For example, if we have 3 + ? = 7 or 7 - ? = 5 then we have the whole and one of the parts. To reach a solution we need to work out the other part. For problems such as 3 + 4 = ? or ? - 4 = 7 we have the two parts and need to work out the whole. This thinking not only assists in classifying problems but also suggests ways of reaching appropriate solutions for number sentences that fit the *Part-Part-Whole* framework. It also represents many of the unknown situations covered on the way to algebra. However, its limitations lie in its inability to model equivalent situations with two or more terms on both sides (e.g., 2+5=1+?). Thus, while it serves an understanding of arithmetic processes, its usefulness for algebra is uncertain.

The third and fourth categories are similar. The third, *Equalize*, involves removing the difference between two collections, for example, Jill has 5 cars and John has 8 cars, how many more does Jill have to buy in order to have the same amount of cars as John? In this instance, the unknown is the difference. The last category, *Compare*, considers the difference between two numbers, and is similar to *Equalize*, but does not entail any action and the difference between the two numbers persists. For example, John has 8 cars, he has three more than Jill, how many cars does Jill have? In both these instances the unknown represents change or difference.

EARLY ALGEBRA AND UNKNOWNS

Early algebra is not about introducing formal algebra in the early years but is about developing arithmetic reasoning in conjunction with algebraic reasoning. Thus exploring problems with unknowns in the early years is important. Slavitt (1999) suggested that a key to early algebraic competence is the ability to abstract computation to more structural realms, commonly referred to as generalised arithmetic. In this generalising process it is the numbers that map to variables and the operations remain the same, although the meaning of the operations could change.

Recent research has begun to focus on the development of young children's algebraic thinking (Falkner, Levi, & Carpenter, 1999), with a focus on children's understanding of equals as equality. Most students do not have this understanding; rather they have a persistent idea that the equals sign is either a *syntactic indicator* (i.e., a symbol indicating where the answer should be written) or an *operator sign* (i.e., a stimulus to action or "to do something") (Behr, Erlwanger & Nichols, 1980; Filloy & Rojano, 1989). Warren (2001) indicated that (i) young children are capable of reaching generalisations, (ii) classroom experiences can interfere with them reaching valid generalisations, (iii) in some instances teaching materials act as cognitive obstacles to abstracting the underlying mathematical structure, and (iv) new learning in mathematics can be incorrectly used to solve old problems. Incorporating 'real' world language can further complicate many situations. Pririe and Martin (1997) claimed that language only serves problems where the equal sign appears just before the answer, thus reinforcing the equation as an action and privileging equations with the unknown occurring directly after the equal sign.

Several studies have investigated student's equation solving strategies. The various methods used by algebra students have been classified as follows; using number facts, using counting techniques, cover-up, undoing (or working backwards), trial-and-error substitution, transposing from one side of the equation to the other, and performing the same operation on both sides (Kieran, 1992). Commonly, it is assumed that students bring with them from their experiences in arithmetic in the elementary school some understanding of the first two methods, number facts and counting techniques. Both these methods assist young children to find the 'missing addend' in number sentences. Counting techniques consist of counting on, counting up, or counting back (Fuson & Fuson, 1992). The questions are: do these techniques assist children in solving missing addend problems in a range of situations or are there problems that are beyond these techniques, how are the methods aligned with preparing young children for algebra, and are young children capable of using other strategies in meaningful ways in the early years?

The specific aims of this paper were to investigate young children's strategies for finding the unknown in first degree equations with one unknown, and to delineate the thinking that supports the development of algebraic understanding.

METHODS

The sample comprised of 84 children from four elementary schools in low to medium socio-economic areas. The children are all participants in a three year longitudinal study investigating early literacy and numeracy development. The average age of the sample was 8 years and 6 months and all had completed the first three years of formal education. In Queensland children spend 7 years in the elementary school. Five tasks were developed for the semi-structured interview. Two tasks were developed to probe young children's ability to ascertain the unknown in equations (see Figure 1).

Task 1 $\boxed{16 + \square = 49}$ **Task 2** $\boxed{54 = \square - 12}$

Figure 1 Tasks presented for the interview

All children had completed their formal introduction to the concepts of addition and subtraction and could add and subtract numbers involving tens and ones. The two tasks chosen for this segment of the interview were believed to probe children's ability to operate with equations in differing formats and to understand the inverse relationship between addition and subtraction. The format reflected formatting commonly used in algebra, that is, the equations were represented horizontally, the unknown was on different sides of the equation, and more than one term followed the equal sign. Both tasks were presented in a symbolic format. Task 1 represented format commonly presented in most classrooms with the unknown on the left hand side and one number of the right hand side. Task 2 was atypical with the unknown

on the right hand side of the equation and the equal sign followed by two terms. Two digit numbers were deliberately chosen for the two tasks. It was believed that this would force children to use equation solving strategies other than number facts and counting on. Throughout the interview students had access to a calculator and were encouraged to use it if they so wished.

The script for this segment of the interview was: *What is the card asking you to do? How can you find the missing numbers? What is the missing number? How did you find it?* The interviews were audio-taped and the scripts transcribed for analysis.

RESULTS

Task 1

An examination of the transcripts indicated that the responses to the first task **(16 + □ = 49)** fell into four broad categories, namely, counting up, counting back, subtraction, and no response, each reflecting the general approach taken to reach a solution. Each category consisted of a number of differing solution paths. The next section describes each of the categories and includes some typical responses.

Category 1 **Counting up**

Counting up consisted of three strategies, counting up in ones, counting up in tens then ones, and using a trial and error method on a calculator. A typical response for counting in 1's was:

What is the card asking you to do? Well it doesn't have a number in there and it is not a turn around and it equals 49. I have to try and figure out the answer in there. *What are you thinking about?* I was just counting in my head 17,18, 16, 17, 18, 19, 20 ... ...33, 34, ... 41, ...49. [While counting up the student kept track of the number of tens with her fingers.] That is 34.

A typical response for the trial and error strategy was:

What is the card asking you to do? 16 plus something equals 49. *How do you find the something?* You find it by well is should be something else so you can figure it out. *So can you work it out for me?* [Long silence] 16 + 16 equals [the child enters the two numbers into a calculator] No. 16+17. No. 16+29. No 16 +34. No. 16+33. It is 33.

Category 2 **Counting back**

A typical response for this category was:

What is the card asking you to do? Put the number in there. *How can you find the missing number?* Well you go from 49 and count back 16 and whatever you land on is your answer. 49, 48, 47, 46, 45, ... .33 So you put 33 in the box.

Category 3 **Subtraction**

Subtraction consisted of two strategies, using a pen and paper algorithm (either 49-16 or 16-49), or reaching the answer by mental computation processes.A typical response for the pen and paper algorithm was:

What is the card asking you to do? Something plus 16 equals 49. *How would you do it?* Take 16 away from 49. [The child proceeded to write the sum 49 - 16 in the vertical format and obtained the answer of 33]

Each response was coded according to the four categories. Table 1 summarises the frequency of responses for each category.

Table 1 Frequency of response for each category to 16 + □ = 49

Category	Strategy	Frequency	Correct
1.Counting up	Counting up in 1's	26	8
	Counting up in 10's and 1's	25	22
	Trial and error	6	3
2.Counting back		2	1
3.Take away	Correct algorithm (49-16)	7	7
(inverse operation)	Incorrect algorithm (16-49)	2	0
	Mental computation	6	6
4.No response		10	1

The most commonly chosen strategy was counting up. Not surprisingly many of those who chose counting up in ones experienced difficulty in reaching the correct solution. They either included the starting number in their solution or simply lost track of where they where up to. Only 15 children (18%) used the inverse relationship between addition and subtraction to ascertain the unknown with all of those who chose the correct algorithm (49-16) reaching the correct solution.

Task 2

An examination of the transcripts indicated that the responses to the second (**54 = □ - 12**) fell into four broad categories, namely reformatting the original problem, interpreting = as 'the answer must follow' and thus the unknown must be 54, reaching the solution by addition, and no response

Category 1 **Reformatting the problem**

Three different strategies were used for reformatting the original problem. Each resulted in a format where there was only one 'thing' after the equal sign, either the unknown or one of the numbers in the problem. The first simply entailed 'flipping the format over' (**12 - □ = 54**). The second involved swapping '-' and '=' (**54 - □ = 12**). and the last strategy placed the unknown after the equal sign (**12 - 54 = □ or 54 - 12= □**). Some typical responses were

What is the card asking you to do? 54 take something = 12 no 54 = something take 12. You have to put the missing number in the box ... but it is going backwards. *How would* *you find the missing number?* Count on. *What way would you like to see the problem?* 12 take something = 54. *Can you work out the number?* No you can't do it

What is the card asking you to do? 54 minus something = 12. 54 equals something minus 12... I don't know what that means. *So what is wrong with it?* Because the = is before the 12 and it shouldn't be. *What should it look like?* Well 54 - 12 = something. The box is after the equals it should be at the end. *How would you work that one out?* I don't know how to .... I will try 26 [The child entered 54 -26 into the calculator]. I'll try 35 Oh I am doing it wrong it is 42

Category 2 **Interpreting = as 'the answer"**

What is the card asking you to do? 54 = something take away 12. *What is that asking you to do?* 54 equals 54. *Why? Can you explain?* The answer is 54 because it must be 0 plus 54 equals 54.

Category 3 **Solving by addition**

What is the card asking you to do? 12 take away.. is it going that way?... 66 goes in there.*How did you work that out?* Because the 12 needs a higher number... the 54 is the answer the 12 has to have a big even number and it was a 6 and then the other one needed a 6. *Why did the 12 need a 6?* To make that number and it was 66 and that takes away a 10 then it has to be 6 and then the 12 take away the two and it equals 54

Each response was coded according to these categories. Table 2 summarises the frequency of response for each category.

Table 2 Frequency of response for each category of responses to 54 = ☐ - 12

Category	Strategy	Frequency	Correct
1. Reformatting the problem	**(12 - ☐ = 54)**	18	
	(54 - ☐ = 12)	3	
	(12 - 54 = ☐)	7	
	(54 - 12= ☐)	20	
2. '=' as the answer		3	
3. Addition (inverse)		21	20
4. No response		12	

Over half the sample (58%) needed to reformat the problem before they could even attempt to reach a solution, with most insisting on placing the unknown after the = sign. As compared with Task 1, the number of children who reached a solution by using the inverse relationship increased from 15 to 21 children and all but one of these indicated that the unknown must be 66. Only 8 of these 21 children used the inverse relationship to solve both tasks. The rest either solved Task 1 by counting on in ones (4 children) or counting on in tens then ones (9 children).

DISCUSSION AND CONCLUSIONS

As expected, most children found Task 1 easier than Task 2. Some conjectured reasons are that Task 1 meets student's understanding of equals and it can be easily represented in everyday language in all four categories (Change-Add-to, Part/Part/Whole, Compare & Equalise). By contrast, Task 2 seemed not to meet student's understanding of equals, and cannot easily be represented in everyday language in any of the categories. Arazello (1998) believed that natural language is crucial for developing an algebraic way of thinking. So how do young children deal with problems that cannot be easily translated into real world contexts? Does continually situating problems in real world contexts privilege some types of problems over others, as suggested by Pririe and Martin (1997)?

Early algebra involves a reconceptualisation of arithmetic in the elementary school. One expectation is that children will be able to solve for the unknown/unknowns in a wide variety of contexts and formats, including number sentences such as 12+?-1=23+5. As we move towards asking young children to be able to solve an array of problems with unknowns that are not commonly presented in our elementary schools, the role of natural language in developing an algebraic way of thinking takes on new meaning. Is it necessary condition to represents problems in real world contexts or is it sufficient for children to simply to be able to translate problems into spoken language?

The format of the tasks seemed to evoke differing solution strategies. For Task 1, the inclusion of addition seemed to suggest counting up rather than using the inverse relationship between addition and subtraction, whereas Task 2 did not suggest counting back but rather either reformatting the problem to an easily recognised form, or using addition, the inverse operation. This conjecture is supported by the increase in the number of children who chose to use the inverse relationship to solve Task 2. The use of the inverse algorithm for Task 2could also reflect young children's unease with counting back (Fuson, 1992). The need to reformat the problem supports previous findings with regard to interpretation of equal as a *syntactic indicator*, that is, a symbol indicating where the answer should be written.

Young children are using a variety of strategies for finding the unknown. While the majority used counting strategies, some used an undoing strategy (recognising addition as undoing subtraction and vice versa). In fact, all the children who were successful in Task 2 used this strategy. The tasks did not seem to elicit an understanding of balance to reach the solution. This is reflected in their lack of success in rewriting Task 2, with all maintaining the '-' sign in their transformations.

Trial and error was also a strategy used by some. These children already have equation solving strategies beyond number facts and counting techniques.

As we broaden the exemplars involving unknowns in the early years, the nature of categories, the role of natural language and equation solving strategies need further investigation, as all seem to play some part in successfully reaching solutions.

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EMBODIED ACTION, EFFECT AND SYMBOL IN MATHEMATICAL GROWTH

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This study of the learning of vectors is situated in the intersection of embodied theory relating to physical phenomena, and process-object encapsulation of actions as mathematical concepts. We consider the subtle effects of different contexts such as vector as displacement or force, and focus on the need to create a concept of vector that has greater flexibility. Our approach refocuses the development from 'action to process' as a shift of attention from 'action to effect' in a way that we hypothesise is more meaningful to students. At a general level we embed this development from enactive action to mental concept within a broad theoretical perspective and at the specific level of vector we report initial experimental data.

INTRODUCTION

This study is part of an ongoing enterprise to build a practical cognitive theory embracing human learning and the powerful use of symbolism in mathematics (Gray & Tall, 2001). As such, it stands at the conjunction of two major theories of cognitive development: the embodied cognition of Lakoff and others (Lakoff & Johnson, 1999, Lakoff & Nunez, 2000) and the development of symbolic mathematics through process-object encapsulation (Dubinsky, 1999, Tall *et al*, 2001). The current three-year study focuses on the concept of vector. This is particularly appropriate for the wider development of the theory as it encompasses embodied aspects in Physics—such displacement, force, velocity, acceleration—and a complementary approach in Mathematics that is based on a text with an implicit process-object approach (Pledger *et al*, 1996).

In the first year, (September 2000 – July 2001), the first author followed a strategy using full class plenaries to encourage students to construct their own coherent conceptions. As we shall see, this had positive effects in improving the flexibility of the students' concept of vector. In the second year, which is now in progress, a new initial emphasis focuses on embodied activities in which the new students enact vectors as transformations moving a shape on a flat surface. The movement of any point on the shape can be represented by an arrow from start to finish, and all the arrows have the 'same effect'. This idea (formulated by a student, Joshua) has proved to be a helpful bridge in translating the sophisticated theory of process-object construction to a practical idea of action-effect. In this article we discuss the findings of the exploratory study and data analysis of the first stages of implementation.

This paper was written jointly with David Tall, but is presented as a single-authored paper to comply with PME policy on the maximum number of papers per author.

THE COMPLEXITY OF THE LEARNING SITUATION

Students meet the notion of vector in different contexts with subtle differences in embodiment. For instance vectors may be encountered as displacements sensed as physical journeys from one place to another, or as forces acting at particular points. In the addition of displacements, one journey followed by another is naturally interpreted using the triangle law, but the addition of forces operating at a point is more naturally represented by the parallelogram rule. In the curriculum we are considering, the notion of vector is first introduced as a translation in the plane and dealt with as a column matrix in mathematics, or as the separate horizontal and vertical components in physics. Both versions are linked to a picture of the vector as the hypotenuse of a right-angled triangle with components as horizontal and vertical sides. In turn this links more easily to the triangle law than to the parallelogram law. Given a problem solvable by horizontal and vertical components such as figure 1a, 25 out of 26 students were able to solve it. However, given a more complex physical problem such as that in figure 1b, asking the student to mark the forces involved with an object on a rough sloping plane, only 4 out of 26 students were successful. In interviews, it transpired that several students, who used the triangle law to draw a picture as in figure 1c, used the triangle of forces to mark the components; because the force parallel to the plane is drawn well below the object, it did not seem to be acting *on* it and was ignored.

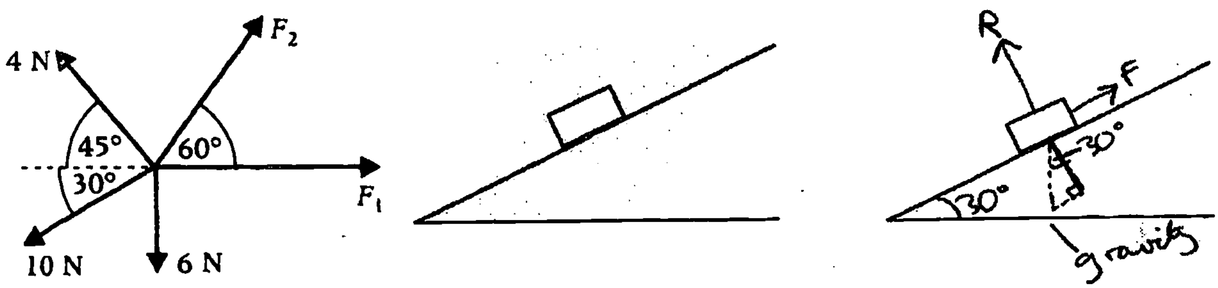


Fig 1a: find F_1, F_2 **Fig 1b: describe & mark forces** **Fig 1c: forces as marked**

The research project therefore focuses on two main objectives, the first is to analyse the cognitive development of the notion of vector in the curriculum, the second is to help students develop the notion of vector as a flexible cognitive unit that can be applied transparently in its various incarnations.

METHODOLOGY

The research method draws upon qualitative and quantitative data and includes lesson observations, standard class tests to assess progress, and a specially designed conceptual questionnaire coordinated with clinical interviews with students and Mathematics and Science teachers (Ginsburg, 1981; Swanson *et al.*, 1981). The data was triangulated, by analysing the books used by students and teachers, by videoing and observing classes, by interviewing teachers on their preferences and their expectations of the students' knowledge about vectors, with particular emphasis on the questions used in the conceptual questionnaire.

The research is conducted at a Comprehensive School with a good academic reputation (for example, in 2001, 63% attained a grade C or above in mathematics in the GCSE examination taken at age 16, as compared with a national average of 54%). The research involved 23 Lower Sixth students (aged 16-17), 26 Upper Sixth students, (aged 17-18), 2 teachers of Physics, and 4 teachers of Mathematics (two covering the preliminary work on vectors at GCSE level, and two teaching the two-year 'A' level course in the Sixth Form).

THEORETICAL FRAMEWORK

The topic of vectors spans both mathematics education and science education. In science the ideas often begin from what we would now call a real-world embodied viewpoint. For instance, in dealing with vectors, Aguirre and Erikson write:

> Teachers could [...] build upon students intuitions (developed through experience in everyday settings) by relating these intuitions to the more formal problem settings in the scientific domain. (Aguirre & Erikson, 1984, p 440.)

They proceed by detailing networks of vector concepts to support this approach, however, their network shows no indication of the mathematical concept of vector. After many years of such developments, Rowlands, Graham and Berry observe:

> ... various attempts at classifying student conceptions has been by and large unsuccessful [...]. A taxonomy of students conceptions may be impossible because the considerations of 'misconceptions' require a specific regard for the framework from which the 'misconception' occur [...] and how misconception is linked to the other forms of reasoning. (Rowlands, Graham & Berry, 1999, p 247.)

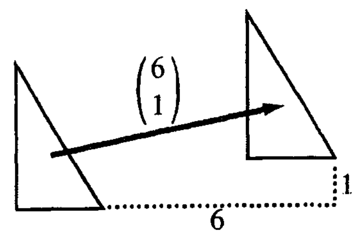
The recent development of embodied cognition, particularly in the formulation of Lakoff (eg. Lakoff & Johnson, 1999, Lakoff & Nunez, 2000) even formulate the idea that *all* thought is embodied. Here we are faced with a genuine quandary. If the attempts to classify the development of vector through intuitive student concepts is, by and large, unsuccessful, and the evidence is that the notion of vector has subtly different meanings in different contexts, how are we to progress in teaching the subject? One strategy that seems evident is to encourage students to reflect on the different aspects of different embodiments and to help them rationalise the various contexts to give an entry into the mathematical notion of vector wherein the ideas of the triangle law and parallelogram law are different aspects of the same theoretical idea, not distinct rules that each have their separate domain of meaning. To gain insight into this possibility, the first author began a programme of plenary discussions with the Upper Sixth based on the underlying unity of the idea of vector.

In linking various embodied ideas to the mathematical notion of vector, we were conscious of the large number of theories relating embodied experiences with mathematical symbolism. These include Piagetian stage theory developing through sensori-motor via concrete operational and formal operational, the Bruner (1966) theory of enactive-iconic-symbolic representations, the work of Krutetskii (1976) on geometric, harmonic and symbolic styles of thought, the van Hiele (1986) development in geometry, and the SOLO taxonomy (Structure of Observed Learning Outcomes) of Biggs & Collis (1991), not to mention the wide range of work in using visual and symbolic interfaces with computers. Amongst all of these we found most empathetic was the notion of how successive modes of thought arise in the SOLO taxonomy whereby each broadens to be included in the next and how physical action (in the sensori-motor mode) is broadened to the ikonic mode, then, through the introduction of symbols, to the symbolic mode and on to successive formal modes of operation. As each of these becomes available and is added to previous modes, we found ourselves dealing with students in a situation where the two main modes of operation are a combined sensori-motor/visual embodied mode of thought and a fundamentally concrete-symbolic mode.

ANALYSIS OF THE SCHOOL APPROACH TO VECTORS

The text-book (Pledger *et al*, 1996) used in the school for introducing vectors to the students in the previous year followed a pattern that is reminiscent of the encapsulation of processes as objects. In this approach, the processes are translations of objects in the plane and these lead to vector concepts, as follows:

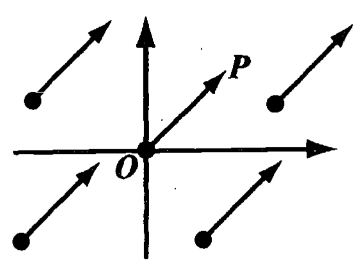
1. translations are described using column vectors, $\begin{pmatrix}x\\y\end{pmatrix}$ [...] with the column vector $\begin{pmatrix}6\\1\end{pmatrix}$ meaning 6 units in the positive x direction and 1 unit in the positive y direction.



2. an alternative notation which can be used to describe the translation is $\overrightarrow{AB}$ representing where A is the starting point and B is the finishing point. [...] The lines with arrows are called **directed line segments** and show a unique **length** and **direction**.

3. a third way to way to describe a translation is to use single letters such as **a** .... Translations are referred to simply as **vectors**. [...] [Each vector] has a unique **length** and **direction** ...

4. **Position Vectors**. The column vector $\begin{pmatrix}x\\y\end{pmatrix}$ denotes a translation. There are an infinite number of points which are related by such a translation. ... The diagram shows several pairs of points linked by the same vector. The vector which translates O to P, $\overrightarrow{OP}$, is a special vector, the *position vector of P.*



This pragmatic approach has some of the aspects of process-object encapsulation. Stage 1 sees a vector as an action on a physical object. The translation is already represented as an arrow. In stage 2 the object is omitted and the focus of attention is on the line segment as a journey from a point A to a point B. Stage 3 shifts the focus to the vector as a single entity drawn as an arrow and labelled with the single symbol **a**. This entity has both an enactive aspect (the movement from tail to nose of the arrow) and an embodied aspect (as the arrow itself). In stage 4 the column vector is used to denote an infinite family of arrows with the same length and direction, with one specific vector starting at the origin singled out as a position vector as a special representative of the whole family.

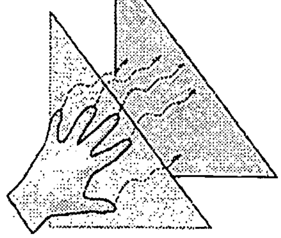
In practice, the Physics teachers preferred to 'simplify' the ideas by referring separately to horizontal and vertical components of vectors. For example, to add two vectors, they would consider each vector separately, calculate its horizontal and vertical components and add them together to get the components of the sum. In parallel, the students would often use the equivalent matrix method to add vectors in pure mathematics. Thus, although they had been taken through the spectrum of development in stages 1 to 4, to make any computations, they were encouraged to fall back to level 1.

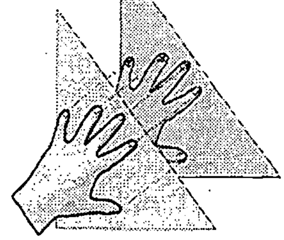
ACTIONS AND EFFECTS – THE INSIGHT OF A SPECIAL STUDENT

In attempting to build a more flexible conception of the notion of vector that encapsulates the whole structure of embodiment and process-object encapsulation, we were struck by the interpretation formulated by one particular student whom we will call Joshua. He explained that different actions can have the same '*effect*'. For example, he saw the combination of one translation followed by another as having the same effect as the single translation corresponding to the sum of the two vectors.

By focusing on the effect, rather than the specific actions involved, we realised that it proves possible to get to the heart of several highly sophisticated concepts. For instance, in fractions, 'divide into three equal parts and take two' is a different action from 'divide into six equal parts and take four' but they have the same *effect,* giving rise to the central idea of equivalent fractions. The same idea occurs in algebra where $2(x + 4)$ and $2x + 8$ involve different sequences of actions with the same effect, leading to the notion of equivalent expressions. We hypothesize that the notion of action-effect is a more approachable way of describing the theory of action-process (Dubinsky, 1991) or procedure-process (Gray & Tall, 1994).

In dealing with this approach, we encouraged students to participate actively by shifting a triangle placed on the table. Figures 2a and 2b have the same start and endpoint (and therefore the same effect, even though the journeys they take in between are different.) The arrows in figure 2b represent equivalent vectors (free vectors having the same effect) and figure 2c represents Joshua's idea that the sum of two vectors has the s





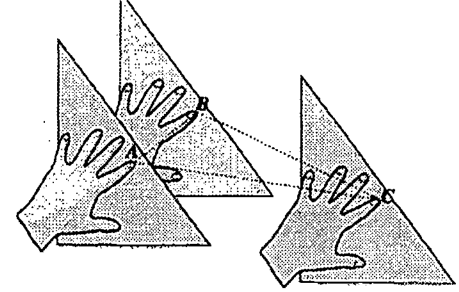


Fig 2a: a translation **Fig 2b: a translation** **Fig 2c: The sum of two vectors**

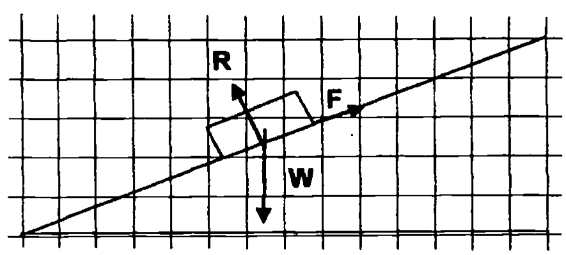
Figures 2a, 2b 'have the same effect' **Two translations and their total effect**

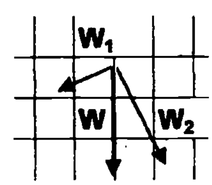
ame effect as the two vectors applied one after the other.

This approach implicitly encapsulates the notion of equivalent free vectors 'having the same effect' and encourages students to feel able to shift free vectors around in any appropriate manner. We would therefore expect students following this approach to be more flexible in handling free vectors.

EMPIRICAL DATA ANALYSIS

To investigate the reasons underlying the original problem in figure 1c, a question was given showing a body on an inclined plane, as in figure 3. Figures 3b and 3c represent the ways in which two students James and Chris split the weight W into components W_1 and W_2. Are either or both of James and Chris right?





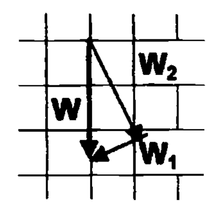
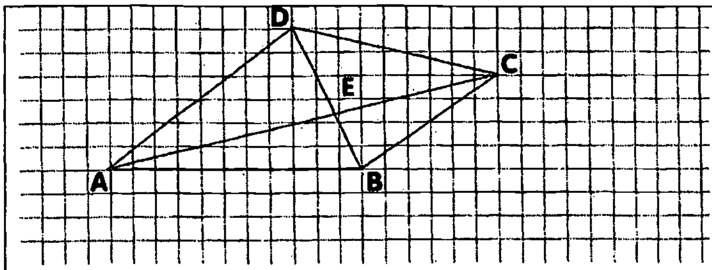


Fig 3a: a body on an inclined plane **Fig 3b: Jame's picture** **Fig 3c: Chris's picture**

The 23 students beginning the course in the lower 6th gave a variety of responses, 11 said both were right, 4 chose fig 3b, 1 chose fig 3c and 6 said neither. All of the students in the sixth form who had taken part in the reflective plenaries found the question trivial and saw the triangle and parallelogram as equivalent.

To test the student's ability to deal with vectors graphically, we gave the question in figure 4. The first part is a natural triangle problem with the vector *AB* followed by *BC*, the other two benefit from being able to see the vectors as free vectors to be able to move them so that they follow on end to end.

In the test, *all* the students were easily able to cope with the sum $\overrightarrow{AB}+\overrightarrow{BC}$. However, parts (ii) and (iii) were more problematic. When we consider those students who were able to solve all three problems, we get the data in tables 1a and 1b.



Show clearly a vector equivalent to:

(*i*) $\overrightarrow{AB}+\overrightarrow{BC}$
(*ii*) $\overrightarrow{AB}+\overrightarrow{AD}$
(*iii*) $\overrightarrow{AB}+\overrightarrow{CA}$

Fig. 4: Testing the visual sum of two vectors

In table 1a those upper sixth students who participated in reflective plenaries were more successful than those following standard class lessons and in table 1b, those following an embodied approach also had more success.

Upper 6th	Reflective	Standard
All 3 correct	7	1
Other	8	10

Table 1a: effect of reflective plenaries

Lower 6th	Embodied	Standard
All 3 correct	5	1
Other	2	15

Table 1b: effect of embodied approach in reflective plenaries

Interviews with six selected students in the lower sixth confirmed that students following a standard course had problems adding two vectors that did not follow on one after the other, especially in cases where they were joined head to head. In the latter case, two out of three students thought that two vectors pointing to the same point would have resultant zero, because they would cancel out.

REFLECTION

The study so far has revealed the complexity of the meaning of vectors as forces and as displacements and the subtle meanings that are inferred in differing contexts. Studies in science education have attempted to build a classification of misconceptions without clearly identifying the underlying problems. Our approach is to develop a pragmatic method that will work in the classroom. One aspect is the use of conceptual plenaries, which are already becoming part of the formally defined curriculum in England. The other is to continue to develop a theory that links physical embodiments to mathematical concepts via a strategy that focuses on the effects of actions. Our experience shows that such an approach can be beneficial in the short-term and we are continuing our practical and theoretical developments over the longer term.

Acknowledgement

In the development of these ideas we have been privileged to have the inspiration and support of Professor Panayotis Spyrou of the Department of Mathematics, University of Athens.

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Extending example spaces as a learning/teaching strategy in mathematics

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Abstract *This paper gives an illustrated theoretical account of some roles for student-generated examples in the teaching and learning of mathematics. We give instances which show that the development of example spaces contributes to conceptual engagement, and that such spaces are personal and situated.*

Introduction

In this paper, learning is seen as growth and adaptation (*extension*) of personal, situated, example spaces; teaching involves providing situations in which this can take place. We find this metaphor useful in characterising many teaching strategies which promote active engagement with concepts, and also in characterising experiences of conceptual learning. This theoretical approach has been developed over a number of years through examination of our own mathematical work, through the responses of groups to mathematical tasks we have offered them, through our observations of teachers and their students, and through some of the literature on exemplification.

We recognise the importance of interaction, discourse, ethos and enculturation in the learning of mathematics, but we find ourselves unable to adopt a solely social constructivist or socio-culturalist position when thinking about learning and doing of mathematics. When we do mathematics for ourselves, we do not recognise our experience as anything other than internal struggle with the unknown to achieve what feels like a self-contained sense of understanding. As institutional learners, personal struggle for sense was supplemented by a need to satisfy an external authority (the teacher). We acknowledge that these descriptions, our behaviour and our judgements are subtly guided by social practices, of which we are a constituent part, but that has little bearing on our *sense* of relationship with the mathematical problem or concept once we have engaged with it. The questions which guide us are:

How can engagement in mathematics be encouraged?
How can construction of mathematics be supported?

We see mathematics as a structure of agreed quantitative and spatial conventions which can be extended in agreed ways into abstract worlds, about which we communicate in ways which imply agreements. Learning mathematics involves making personal sense of experiences of quantitative, spatial and abstract objects and relationships, and matching these senses to further experiences and conventions. The interactive aspects of these experiences are crucial, because language (verbal, non-verbal, symbolic) is the medium through which a teacher can construct situations in which learners meet abstract concepts and compare the implications of their constructions with the implications of conventions. Thus we are interested in 'good-fit' relationships between language and active construal, that is, relationships which are effective in leading learners to construe mathematics in useful ways.

Put more simply, we ask what tasks teachers can set to help students learn conventional mathematics, given that learning involves constructing meaning in the environment created predominantly by the teacher.

The role of examples in learning mathematics

It is has long been acknowledged that people learn mathematics principally through engagement with examples, rather than through formal definitions and techniques. Indeed, it is only through examples that definitions have any meaning, since the technical words of mathematics describe classes of objects or relations with which the learner has to become familiar.

We use 'example' to cover a broad range of mathematical genres, including examples of classes, examples illustrating concepts, worked examples demonstrating techniques, examples of problems and questions which can be resolved, examples of appropriate objects which satisfy certain conditions, examples of ways of answering questions, constructing proofs, and so on. Thus learning mathematics can be seen as a process of generalising from specific examples: learning to add involves generalising a process which works for given examples so that it can be applied to examples one has not met before; learning about quadrilaterals involves understanding what types are possible, and what their properties are. The broader the range of examples, the richer the possibilities for generalisations and connections to be made, so the extent of the set of familiar examples is influential in construction of conceptual understanding.

Most examples come to students from authorities (by whom we mean teachers, textbook and examination writers), but there is a contrast between a single example seen as paradigmatic, or generic, by the expert but as something merely to learn by a student (Mason and Pimm, 1984). For instance, a teacher might multiply some numbers by 0.3 to show, generically, that multipliers between 0 and 1 make 'smaller' products, but are students aware that this is an example of a range of numbers which act similarly, or do they perceive merely a set of practice exercises?

It is useful, then, for students to have several examples from which to get a general sense of what is being taught. Firstly, it is important to compare examples to see what features are common, and hence to appreciate generality. This helps avoid students' fixations with figural concepts which have unhelpful features. Choice of examples is important in helping students develop generalisations of structures rather than surface features (Bills and Rowland, 1999). The process of comparing examples can also highlight the conceptual, semantic and structural similarities between problems rather than superficially seeking cues in syntax, habit or context (Reimann &Schult,1996). Learners may however only use worked examples as templates to be followed using different inputs, rather than as demonstrations of how to manipulate representations, to transform relationships, or to synthesise facts to achieve solution (Anthony 1994). For example, the processes of elimination and substitution essential to solving simultaneous equations may be lost in the detail of "multiplying and subtracting".

There has been much debate about the usefulness of including counter-examples in students' experience. In some studies, counter-examples appear to be helpful in focusing students on what is relevant and what is irrelevant; in other studies, the role of counter-examples appears to confuse students who do not understand how or what it refutes (Zaslavsky and Ron, 1998).

The extent and extension of example spaces

There is a special kind of counter-example, referred to by Askew and Wiliam (1995): 'The ideal examples to use in teaching are those that are only just examples, and the ideal non-examples are those that are very nearly examples' (p. iii). Our searches for such examples have led us to develop the more general notion of boundary examples. For example: if a general straight line is given as $y = mx + b$, then lines of the form $x = a$ may be excluded from students' experience because they are not expressed in the general form; consequently teachers have to introduce these deliberately in some way. We call '$x = a$' a *boundary* example for straight lines expressed as $y = ax + b$.

We use the word 'boundary' because we see students' experiences of examples in terms of spaces: families of related objects which collectively satisfy a particular situation, or answer a particular mathematics question, or deserve the same label. Such spaces appear to cluster around dominant central images.

For example, some mathematics postgraduates all offered $|x|$ as a function which is continuous but not everywhere differentiable but they needed considerable prompting to extend their images and provide other examples. When asked for a second example most translated $|x|$; on being asked for a third example some commented (excitedly!) that a vast number could be generated from $|x|$; others searched for very different examples and reported looking for other central images from which to work, such as an image of a curve becoming a straight line. None of them tried to construct anything solely from definitions. A group of mathematics graduate teacher trainees nearly all drew segments of circles when asked to give an example of an image for a quarter; a few gave sections of rectangles; one wrote (1/4); none gave a point on a number line. Only after coaxing, critique and discussion did other images arise.

Student-generated examples

Waywood (1992) encouraged a metacognitive shift by asking students to write journals which included collections of examples. He detected a range of responses from the inclusion of examples copied from other sources, through the use of examples to demonstrate an application or use or illustration, to the construction of annotated examples to summarise aspects of a topic or idea. Over time, his students progressed towards the personal construction of examples. In the two incidents described just before this section, students worked on extending their example spaces by generating more examples for themselves. In the process, they had to think about the meaning of concepts and draw on experiences which had not readily come to mind.

There is a growing body of work in which the use of student-generated examples is advocated. Mainly these are for assessment purposes, to motivate interest in a topic, or to give students the opportunity to pose their own problems in the hope that this will make them better able to solve problems posed by authorities.

Contextual questions created by students are motivating for the producer and also likely to relate more closely to the experience of other students. Shifting the responsibility for test construction and problem posing from teachers to students can have clear affective gains, and necessarily involves some review of material used. Ellerton (1988) comments that the questions and examples produced by students generally reflect what they are used to from their teachers' styles. Further, experience of problem-posing enabled some students to 'reason by analogy' when presented with similar questions (English, 1999).

But the ability of students to create and answer questions which are *dis*similar from those previously experienced would be a more powerful indicator of mathematical learning. Silver and Cai (1996) working in urban middle schools with ethnically and linguistically diverse populations in economically disadvantaged communities, found that their students could indeed create complex problems involving several relationships or challenging given constraints if they had appropriate support. In their study, active extension of the example space of possible questions took place, moving students away from the centrality of the teacher's example as dominant template.

Use of SGEs for concept development

A few writers use SGEs to promote conceptual development of new mathematics rather than primarily for motivation or assessment. For example, Sadovsky (1999) challenges her students to exemplify division operations which give a dividend of 32 and a remainder of 27. She asks 'How many are there? If you think there are less than three write them all down, and explain why there are no other ones. If you think there are more than three write down at least four of them and explain how other solutions can be found' (p. 4-147). One outcome of her study, whose main aim was to explore algebraic shifts, was her conclusion that '… these problems are simultaneously a chance to find the limits of arithmetic practices and to enrich the conception of Euclidean division'. But concept development through exemplification need not be an incidental effect; it can be an explicit pedagogic aim. Sowder (1980) reports using the prompt 'Give me an example, if possible, of …….', with the teacher taking responsibility for guiding students towards peculiar examples as an integral component of concept construal. Dyrszlag (1984) suggests asking learners to give examples as a way of expressing their own understanding of a concept, and thus affecting future learning[1]. Zaslavsky (1995) asked students to 'find an equation of a straight line that has two intersection points with the parabola $y = x^2 + 4x + 5$.' Attention was thus directed to features of a parabola and a straight line, rather than to algebraic or trial-and-error techniques for finding intersections. Each strategy led to further questions such as 'find an equation of a straight line which does not intersect twice with the parabola....'. Students encountered most of the analytical geometric syllabus, engaging with the structures and equations of straight lines and parabolae through their own examples.

A particularly interesting account of the use of exemplification for conceptual understanding is given by Dahlberg and Housman (1997). They asked their tertiary students to give examples and counter-examples of analytical concepts as well as their own explanations, having previously been given only formal definitions. Students who consistently employed example generation as an integral part of their learning strategy, even when not specifically prompted to do so, underwent more shifts of concept image, could give better explanations, developed broader example-spaces and hence had a more complete understanding of the taught concept. "Example usage, particularly example generation and verification, is crucial for understanding a new concept." (p. 284). These reports fall far short of the practices described by Brown and Coles in several accounts (e.g. 2000) in which students raise mathematical questions and examples as a natural part of their classroom discussions.

Is it realistic to expect *all* students to be able to generate examples of newly-met concepts? Students in a low-attaining year 9 class were nearly all able to produce examples of a variety of mathematical ideas when asked, having become used to being asked for these by a teacher during only three lessons. Sometimes, unexpected examples would be produced which had been constructed rather than recalled from previous lessons. On one occasion, a student responded to the question 'do you know what a prime number is?' by offering a non-example with its factors and saying 'you can't do that with prime numbers'.

For a few years we have been working on ways to characterise teaching actions which engage students in active reorganisation of knowledge structures or creation of mathematical objects. The purpose of such characterisation would be to give teachers access to a range of strategies which they may adapt and develop, while recognising that every individual teacher's practice is particular to them (Watson and Mason, 1998). Many of the teaching approaches we have observed can be described as offering opportunities for extending personal example spaces.

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[1] A translation of Dyrszlag's suggestions, provided for us by Anna Sierpinska, can be found in Watson and Mason (1998 p31)

The individuality and situatedness of example spaces

When we consider a concept, a paradigmatic image might come to mind and, as we have already suggested, this image might have special features which make it rather less general than we would hope. For instance, a rectangle with sides parallel to the edges of the paper appears when someone says "quadrilateral", a unit fraction appears when someone says "fraction", a decimal number including lots of 9s appears when someone asks for a number very close to 1. Some of these dominate because they have been presented frequently to learners through texts, others because of recent exposure. For some learners, central images have an affective dimension: they might be associated with a favourite teacher, or a particular memory of a page of a textbook. Hence, although a significant number of a group of teachers drew 'page-parallel' rectangles with sides roughly 2:1 when we asked them to draw a quadrilateral, because it is a dominant image in (and outside of) mathematics texts, not everyone did. For some, other influences were brought to bear. Their experience of mathematics, and of us, led them to be more creative or devious in what they drew. Some drew a square, because their interpretation of the task was that something very special is required; some felt too insecure to draw anything because they did not know the success criteria. We then asked them to draw further quadrilaterals which satisfied more ands more constraints, structuring our request so that eventually they had to abandon their rectangles and construct shapes from properties, rather than just searching through their existing categorisations and figural examples.

What comes to mind for individual learners when working on known concepts, or when asked for examples of mathematical objects, can relate to dominant or figural images in the topic, but can also be influenced by past experience, preferences, interpretations of what is required and what is valued. The teacher trainees who drew segments of circles as fractions may have been interpreting the task as about offering images to their students; or believing that an image had to be a picture. The post-graduates offering $|x|$ were giving it as a starting point for whatever was to come, not as an illustration of their total knowledge.

Similarly, how dominant images provide access to other examples and images varies from person to person. We asked a group to find families of quadratics which have the same 'inter-rootal' distance[2] One, a secondary mathematics educator, imagined a family of quadratic curves, all the same 'shape', related to each other by lateral translation and then paused, hoping to find something else as well. Another, who worked much of the time with dynamic images, saw these as essentially the same curve. For him, the family of translations was one example, not infinitely many. His focus was on generating curves which were different shapes but were all 'pinned' at two fixed points on a notional x-axis. Both of these knowledgeable mathematicians knew about all the curves that would have this property, but their senses of what

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[2] We define 'inter-rootal distance' as the distance between the roots along the x-axis

could be varied and how to vary it were different because of their different ways of working in that moment. Their examples were the same, but they were in differently structured spaces in terms of relations between examples.

These stories from our experiences of working with others illustrate the potential for variety in the examples, and their structuring, which are evoked in response to teachers' prompts. From these and many similar experiences we deduce that example spaces come into being in response to prompts, and are differently structured, with different contents, for different learners, at different times and in different contexts.

Extending example spaces

Individual spaces, situated by time, place, people and prompt, provide the starting point for students' work. Even with a new topic, there are connections with previous knowledge which will bring images and examples to mind. Whatever the teacher says, students will work with their own example spaces, and they will try to fit what is said into them. Sometimes learning involves restructuring the contents of such a space; sometimes learning involves extending the space to include new items. Consider this prompt: *Find two numbers such that the square of the smaller is larger than the square of the larger.*

For some students this may well be a mystery, because their example spaces of numbers only include positive integers; in order to answer this they are going to have to begin extending it to include some other numbers – but what other numbers? Responding to this prompt might lead to positive numbers between 0 and 1, but it might also raise the question of what is meant by 'smaller' and 'larger' when negative numbers are considered. The search for two appropriate numbers, probably aided with a calculator and number line or graph, encourages an extension of the example space which may be lasting for some students, or ephemeral for others.

The prompt to find functions with the same inter-rootal distance can lead people to try to restructure their example spaces, some wishing to remove the limitations of fixed axes from their spaces, others wishing to see curves as representing an infinite number of translated versions of themselves from now on.

Conclusion

We have illustrated how seeing teaching/learning mathematics as creation and extension of personal example spaces can inform the construction of tasks in which students can work directly on their own mathematical structures and relationships. Achieving competence in mathematics can be seen as the development of complex, interconnected, accessible example spaces.

Acknowledgements: As well as the references we have used in this short paper, this theory builds on the work of: Bruner, de Morgan, Fischbein, Halmos, Lakoff and Johnson, Michener, Piaget, Skemp, van den Heuvel-Panhuizen, van Hiele, and von Glasersfeld, among others. A full bibliography is available on request.

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GRADE 5 STUDENTS' APPRECIATION OF VARIATION

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This report focuses on one aspect of a larger study of school students' understanding of statistical variation. Although the study included students in grades 3, 5, 7, and 9, this paper will focus on grade 5 students only. Students experienced a unit of 10 lessons on the chance and data part of the mathematics curriculum conducted over an eight-week period. Lessons included a particular emphasis on variation and its role in statistical understanding. Pre- and post-tests were administered and improvements were found in overall performance and for variables reflecting appreciation of variation in chance, variation in data, and variation in sampling. Some comparisons are made with grade 3 students' performance.

INTRODUCTION

Variation is an important element in the basic understanding of statistics (Moore, 1990), yet many middle school students have little understanding of what constitutes appropriate variation. Partly because of the complex formula traditionally used to measure variation, educators and curriculum developers tend to avoid it at this level. As a result, students usually do not encounter the concept of variation or its applications until senior level mathematics in school or university statistics courses. Until this time there is a tendency to focus on centers with little regard for spread or variation within data sets. It is possible, however, to help primary school students reach a broader appreciation of variation without the computation of complex formulas. In a related investigation by the same researchers (Watson & Kelly, 2001) it was shown that students as young as grade 3 could develop a foundation for the understanding of some aspects of statistical variation. Although this understanding was limited, it was hypothesized that slightly older students, with the same instruction and materials, could gain a firmer grasp of the concepts involved.

The data used in this study were part of a larger project designed to model development of, and monitor change in, students' understanding of statistical variation in grades 3, 5, 7, and 9. This report will focus on the change in understanding following instruction for grade 5 students, comparing improvements with grade 3 students in the same study (Watson & Kelly, 2001). Similar patterns of improvement and an understanding at a slightly higher level were expected.

SAMPLE AND TEACHING UNIT

The sample of grade 5 students in the present study consisted of 82 children from three primary schools in the Australian state of Tasmania (n = 35, 31, and 16). Students were taught 10 lessons covering aspects of the chance and data curriculum with a special emphasis on variation. All lessons were taught by the same experienced teacher, provided by the research project.

The first lesson was an investigation of a small packet of coloured chocolates called Smarties™. Students were asked to investigate the contents of the packets in pairs or groups and develop column graphs to show the number of different colours in each box. Whole-class stacked dot plots were created from the individual graphs and these were compared against the data collected in individual pairs. Lesson 2 was about "People in a Family" (Russell & Corwin, 1989), and students recorded the number of people in their families. Once again, different types of graphs were created; people graphs, linked blocks, and paper-and-pencil graphs were used to display the information. The types of information that can be obtained from such representations, the general shape of the graphs, and the existence of outliers were discussed during this session. The third and fourth lessons dealt with equiprobable events using spinners and dice, and non-equiprobable events using two dice. The activities allowed students to compare class graphs and discuss the shapes of graphs generated from one die with graphs generated from two dice (Edwards & Hensein, 2000). Variation was a strong theme during these sessions, linked with discussions of the theoretically expected outcomes. Lessons 5 and 6 were related to sampling and the first was introduced with a discussion of what constitutes a sample. Activities included selecting randomly drawn samples from the classroom and predicting to the wider classroom population. The issue of fairness and the idea of predicting from a sample were the most difficult concepts for students to grasp.

The final four lessons were designed to introduce students to experimental methods. In lessons 7 and 8 students investigated how long they could stand on each foot with their eyes closed (Rubin & Mokros, 1990). The two data sets (for left and right feet) were then represented on a whole-class-generated stacked dot plot for discussion and analysis. Students focused on clumps, outliers, common scores, range, and the differences between the two sets of data. The final lessons followed on from this and gave an opportunity for students to plan and conduct their own experiments. All students were told they would be given a pencil, measuring equipment, and a recording sheet, and would be asked how far the pencil could be blown across a flat surface. Each class then had to decide on a question and develop a hypothesis. Most classes focused on possible differences between boys and girls, sports players and non-sports players, and so on. Even though the analysis of the data required a great deal of teacher direction, the students conducted their experiments efficiently. Once again, discussions focusing on differences, common scores, outliers, and the shapes of the graphs reinforced the notion of variability in the data sets.

SURVEY INSTRUMENTS AND ANALYSIS

The pre- and post-tests consisted of the same 29 items, covering four aspects of the curriculum as emphasised in the teaching unit: Basic Chance and Data, Chance Variation, Data Variation, and Sampling Variation. The items on Basic Chance and Data were adapted from the studies of Watson, Collis, and Moritz (1997) involving dice and coloured marbles, of Torok (2000) involving spinners, of Watson (1998) involving basic table reading, and of Watson and Pereira-Mendoza (1996) involving reading pictographs. For Chance Variation there were items involving spinners over many trials developed by Torok (2000), as well as a question concerning the prediction of outcomes for 60 tosses of a die. Data Variation was assessed through pictograph items adapted from Watson and Pereira-Mendoza (1996), as well as three items asking for comparison of two differently scaled stacked dot plots representing the same data on how long students in a class had lived in town (Konold & Higgins, in press; see Figure 1). The items asked students what they could tell from each of the plots and then asked for a decision as to which stacked dot plot told the story better. Sampling Variation items included giving a definition of "sample"; suggesting a method of sampling, and evaluating the bias in four sampling methods, for the "raffle scenario" of Jacobs (1999); and suggesting a fair way to select students to lead a parade in the context of the table reading items of Watson (1998). For a more detailed description of the items and their contribution to the four variables (Basic Chance and Data, Chance Variation, Data Variation, and Sampling Variation), refer to Watson and Kelly (2001).

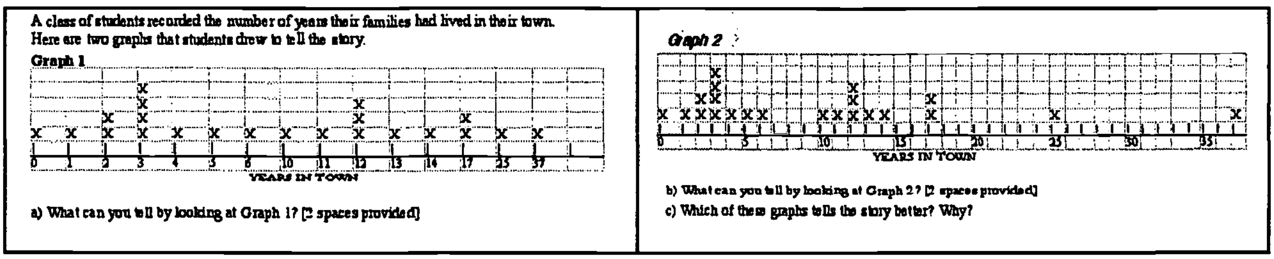

Figure 1. Stacked dot plot problem from Konold and Higgins (in press).

The post-test was administered approximately seven weeks after the completion of the teaching units. Five variables were defined, representing the four aspects of the curriculum covered in the subscales described above, and the total score for the test. The scoring for each item ranged from 0-1 to 0-5 depending on the potential sophistication or complexity of the response. Of interest in this study is the change, potentially the improvement, in scores from the pre- to the post-test and the improvement of performance as compared to the grade 3 students. Paired t-tests were performed for the five variables (n = 82). The tests for grade 3 were the same as for grade 5 except that they did not include the three items on stacked dot plots (Konold & Higgins, in press) or one of the biased sampling items (Jacobs, 1999).

RESULTS

Pre- and post-test means and standard deviations for the five variables defined for the grade 5 students are given in Table 1. As can be seen, there is a statistically significant improvement for the three themes of variation and on the overall total. The paired t-test for pre- and post-tests for the Basic Chance and Data variable was not significant. The maximum possible score on this variable was 16 and as can be seen, the grade 5 mean pre-test understanding of Basic Chance and Data was relatively high. The non-significant change may have been due to a ceiling effect.

Table 1. **Paired t tests for the grade 5 survey**

Variable (number of items)	Pre Mean (SD)	Post Mean (SD)	*t*	*P*
Basic Chance and Data (9 items)	12.1 (1.9)	12.4 (2.1)	-1.3	NS
Chance Variation (5 items)	5.6 (2.6)	6.7 (2.1)	-4.3	<.001
Data Variation (7 items)	8.4 (2.9)	9.3 (3.0)	-3.1	<.01
Sampling Variation (8 items)	8.7 (4.0)	11.2 (4.8)	-5.2	<.001
Total (29 items)	34.8 (8.5)	39.6 (9.6)	-6.2	<.001

For the Chance Variation variable students improved on 3 out of the 4 spinner items. Two items showing significant differences required a prediction of how many times the spinner would land on the shaded part out of 10 spins and asked how many spins landing on the shaded part out of 10 would be surprising. In response to the first, a student in the pre-test wrote, "6 times, [because] you spin and hope it will come up", whereas in the post-test she used a more theoretical prediction, "5 times, because they have an equal chance." A few grade 5 students on the post-test were able to expand on this and acknowledge the role of variation. An optimal response was "4, 5, or 6, because it would land on the shaded part approximately half the time." On the spinner item that asked for a prediction of a shaded outcome for a 50-50 spinner spun 10 times on six different occasions, one initial response indicated a strict probabilistic view ("5,5,5,5,5,5"), whereas the corresponding response on the post-test acknowledged the potential for variation over the six trials ("5,3,6,4,7,5").

For the Data Variation variable, all three items comparing the two differently scaled stacked dot plots in Figure 1 revealed a significant improvement from pre- to post-test. One student, for example, in the pre-test responded by reading the data on the first stacked dot plot incorrectly; she wrote, "Most people have lived here one year, and the person who has been here the most has been here for four years." This student mistakenly read the X's on the stacked dot plot to be the years (instead of people) and the numbers on the horizontal axes to be individual people (instead of years) even though it was labelled. The same student responded in the post-test by stating "3 years has the most; and, 0, 1, 4, 5, 6, 10, … all have one X on them." Although this student did not put the data into the full context, she read and interpreted the horizontal axis correctly, an improvement over the earlier response.

The pictograph items, part of the Data Variation variable, showed no improvement on the post-test. Of interest was the item about the gender of a new student if the new student arrived by car, which required students to make an inference based on the data presented in a pictograph. Statistical reasoning based on the observed majority (girls), accompanied with an element of uncertainty, for example, "Girl, I don't know [why], I'm having a guess … because there is more chance of it being a girl," was shown by only 1.2% of grade 5 students on the post-test. None answered like this on the pre-test and the majority of students both times responded by focussing on patterns in the data, with no emphasis on uncertainty or observed majorities. The possible reasons for this are discussed later.

For Sampling Variation, there was a significant improvement in performance on five of the eight items: judging two of the survey methods presented in the "raffle scenario", choosing the best survey method out of the four presented, defining a sample, and suggesting fair selection methods for students to lead a parade. Overall, students demonstrated a better post-test understanding of what a sample is and applied this knowledge to examine critically the survey methods presented and suggest methods of selection on their own.

Table 2 shows the means and standard deviations for grades 3 and 5 for the variables defined from the subset of the 25 common items completed by both grades. As can be seen, the grade 3 students improved significantly on the Basic Chance and Data variable, whereas the grade 5 students improved very little. The grade 5 pre-test mean, however, was significantly higher than the grade 3 post-test mean ($p<0.001$) and as noted there may have been a ceiling effect for grade 5. Similarly, for the common items in the Data Variation variable, grade 3 students improved significantly whereas the grade 5 students showed no improvement. Again the grade 5 pre-test mean was higher than the grade 3 post-test mean ($p<0.05$), but this difference was not considered meaningful given the small number of common items.

***Table 2.* Comparison of paired t tests of survey results for the grades 3 and 5**

	Grade 3*				Grade 5			
Variable (number of common items)	Pre Mean (SD)	Post Mean (SD)	*t*	*P*	Pre Mean (SD)	Post Mean (SD)	*t*	*P*
Basic Chance / Data (9)	9.7 (2.8)	10.8 (2.4)	-3.7	<.001	12.1 (1.9)	12.4 (2.1)	-1.3	NS
Chance Variation (5)	4.1 (2.8)	5.5 (2.6)	-5.3	<.001	5.6 (2.6)	6.7 (2.1)	-4.3	<.001
Data Variation (4)	3.9 (1.8)	4.4 (1.9)	-2.6	<.01	5.1 (1.5)	5.0 (1.6)	0.9	NS
Sampling Variation (7)	5.3 (3.6)	6.5 (4.8)	-2.7	<.01	7.8 (3.6)	10.1 (4.1)	-5.6	<.001
Total (25)	23.0 (8.6)	27.2 (9.8)	-6.4	<.001	30.6 (6.9)	34.1 (7.8)	-5.6	<.001

*Data from Watson & Kelly (2001)

The Chance Variation variable showed a significant improvement for both grades 3 and 5. The grade 5s performed at a higher level than the grade 3s in both the pre- and post-tests but the post-test mean for the grade 3 students was equivalent to the pre-test mean of the grade 5 students. The analysis of the Sampling Variation common items showed a significant improvement from the pre-test to the post-test for both grades 3 and 5, with greater improvement for the grade 5 students.

DISCUSSION

Two aspects of the outcomes of the research with grade 5 students will be covered in this section: the overall improvement in their performance following instruction and their performance relative to that of grade 3 students. Finally some suggestions will be made for future research.

The improvements observed for the variables reflecting the three themes of variation are almost certainly a result of exposure to experiences reflecting variation in the chance and data lessons taught. The Data Variation variable showed a significant improvement only when the three items from Konold and Higgins (in press) were part of the variable. The improvement on all three items provides evidence of the emphasis on collecting and displaying data in stacked dot plots within the teaching unit presented. Lessons 7 to 10 dealt explicitly with gathering data and generating stacked dot plots from the data. In relation to this, Konold and Higgins state that it is important for students not only to be able to read and interpret data from graphs but also to be able to refer the data back to the context in which they are set. An integrated and optimal response to this item would thus include the context of the question (i.e., how long students have lived in town). In response to the second stacked dot plot, one student, after incorrectly reading the plot in the pre-test, responded in the post-test by saying, "A lot of people have lived in the town between 0 and 5 years." This post-test response not only summarizes the data correctly, but also refers the data back to the context within which the task is embedded, as is recommended by Konold and Higgins.

The greatest improvement occurred in the variable measuring Sampling Variation. Lessons 5 and 6 of the teaching unit focused specifically on sampling. In the first session the students demonstrated pre-instructional knowledge of sample through giving examples encountered in everyday life. Anecdotal evidence from videos of the lessons indicates most students had a good intuitive basis on which to build an understanding of sample (e.g., "a lady giving out food in a supermarket"; "a doctor takes a sample of your blood"). Jacobs (1999) in her study of students in grades 4 and 5 found a similar occurrence, stating that her students had a sound knowledge of sampling before instruction began and were using examples containing elements of statistical samples when asked for a definition. The objective of the formal work with students in this study was to improve on the developing understanding of sample by putting it into different contexts and highlighting its potential for description and prediction. The results in Table 1 highlight the significant improvement in the understanding of sampling after the teaching unit.

In comparison to the grade 3 students, the grade 5 students performed at a higher level on the Basic Chance and Data variable and on the common items in the Data Variation variable, even though they did not improve significantly after instruction (Table 2). Anecdotal evidence from preliminary analysis of grade 7 and 9 data shows that the pre and post means for these grades were equivalent to or slightly lower than those reported for the grade 5s, supporting the idea of a ceiling effect for these items. The lack of improvement on the common items for Data Variation for grade 5, however, may be related to mathematical experiences in the classroom outside of this study. On the tests the items used were based on interpreting a pictograph, whereas during teaching stacked dot plots were used more frequently. Patterns of various types (e.g., girl, girl, boy, girl, girl, boy) were present in the pictograph data and recognising such patterns as significant is part of most primary school mathematics programs. Responses to the pictograph items often reflected pattern recognition and whereas for grade 5s this was consistent across the pre- and post-tests, for grade 3 it occurred more in the post-test indicating an "improvement" from not being able to interpret the graph at all. The lack of specific discussion of pattern and pictographs during the lessons meant that students were not specifically encouraged to move beyond this level in either grade. For the grade 3 students—who were just starting graph reading and interpretation—any experience during the lessons was helpful in improving their performance. Grade 5 students had previous experience and began at a slightly higher level on the pictograph interpretation tasks. The lack of discussion of pictographs meant their performance did not change on these items (Table 2) but they reacted well to stacked dot plots, and hence improved on the Data Variation variable with more items (Table 1).

In relation to the Sampling Variation subscale, grade 5 students made a larger absolute gain in this area than the grade 3 students. Students in grade 3 had good out-of-school experiences (e.g., were able to give reasonable examples of a sample) but struggled with making inferences from samples to populations (Watson & Kelly, 2001). This made it particularly difficult for the grade 3 students to choose which of the survey methods was the best in the "raffle scenario". Perhaps the greater initial understanding demonstrated by the grade 5 students led to adopting the concepts and applying them to an unfamiliar context with greater ease and to an improvement in the ability to judge the more appropriate survey methods.

Overall, the difference in average levels of performance favouring grade 5 students over grade 3s would be expected due to their greater time in school and experience in everyday life, as well as their potentially greater levels of cognition and better ability to comprehend the objectives of the lessons. Although the teacher made every effort to conduct each lesson on the same topic in a similar fashion, student input sometimes influenced the direction of the lessons. In grade 3 this sometimes resulted in the reinforcement of ideas already presented, and for the grade 5 students it sometimes led to an extension of the original ideas. All in all, what was encouraging was that within the expectations for the grade levels, both groups improved in their understanding of aspects of variation with chance and data.

Three directions for future research are suggested by the outcomes of this study. First is an analysis of how much transfer is to be expected between the content of the lessons taught and the content of the pre- and post-tests. Obviously, using identical content in the tests and lessons is likely to lead to statistically significant but educationally less useful results. On the other hand if the differences are too great, as perhaps was the case with pictographs and stacked dot plots, then some alterations need to be made after clarifying objectives. Second, research into the performance of other groups after instruction would increase the depth of understanding of how elementary and middle school children understand variation. Are there differences in the impact of instruction between private and public schools in pre and post understanding? Do samples from other countries that perceive chance and data to be important behave similarly to this Australian sample? Is there a universal developmental pattern emerging for a concrete understanding of variation in chance and data concepts? Third, there is the issue of using classroom teachers as the conveyors of the lessons in future research. In this study, classroom teachers were reluctant to organise and teach lessons themselves. Since teachers' own competencies and understanding of the topic are important determinants of the successful teaching of that topic, professional development is strongly advised for those taking part. Teacher attitude, enthusiasm, and topic knowledge, may affect pre-post results.

Acknowledgments: Research funded by an Australian Research Council grant (No. A00000716). The authors thank Patricia Jeffery, the project's classroom teacher.

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THE INSTABILITY OF YOUNG STUDENTS PROBABILITY NOTIONS

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This paper draws together the findings of a series of probability studies with students aged 11 to 14 years and incorporates new data not previously reported. The collective data emphasises the fragility of probability notions and the strong effect of confirmation or refutation of outcome 'predictions' on probabilistic reasoning. Although some evidence of development of probabilistic reasoning with age was found, instruction appears to have a positive effect. The studies confirm the usefulness of videos of seemingly random experiments as research tools.

INTRODUCTION

Probability remains a pedagogically problematic topic in mathematics education at both primary and secondary levels. While the topic is recognised as being important in some countries (for example; USA, National Council of Teachers of Mathematics, 2000), there is enough concern about the efficacy of teaching the topic that it has been downgraded or even omitted in other curricula (UK, Department for Education and Employment, 1999; New South Wales Department of Education, 1998 respectively).

Much work in several avenues of inquiry is still needed:

1. Information about specific probability concepts and decision-making strategies and their relationship to other mathematical concepts and processes.
2. A coherent framework of the development of probabilistic thinking, which will support assessment procedures.
3. Development and evaluation of instruction programmes for the classroom

In recent years, work in each of these three areas has begun to overlap productively (e.g. Fischbein and Schnarch, 1997; Jones, Langrall, Thornton & Mogill, 1996, 1997, 1999; Watson, Collis & Moritz, 1997). The ultimate goal of most educational research is to inform classroom practice, and research in this third area is long overdue. However, we are far from achieving a complete model for the development of probabilistic thinking, and researchers are still discovering new information about children's perceptions of probability situations and their patterns of thinking.

Understanding of probability involves the integration of several concepts, including randomness, ratio and proportion (the structure of the sample space) and likelihood of particular events. In children, the connections between these concepts, when made, remain tenuous and can be adversely effected by occurrences that challenge their thinking (Jones, Langrall, Thornton & Mogill, 1997, 1999). For example, a child might appropriately state the most likely outcome of drawing a specified object from a box, based on either knowledge of the sample space or on a known set of previous outcomes, only to have their 'prediction' refuted by the next draw producing the least likely outcome. While the expectation that the long-term distribution of outcomes would be representative of the sample space is appropriate, believing that a small set of outcomes should be representative is inappropriate. This notion of *representativeness* has received much attention (Amir, Linchevski & Shefet, 1999; Fischbein & Schnarch, 1997; Shaughnessy, 1981), yet the quite commonly occurring scenario described above has not. The dramatic effect it can have on children's fragile probabilistic thinking is just being realised (Proposed by Truran, 1996) and confirmed by Ayres & Way, 1999, 2000).

A major hindrance to probability research with children is, ironically, randomness itself. In other areas of mathematics the concrete experiences needed to support children's learning can be prepared and predetermined. However, when engaging children in probability experiments and games, the outcomes naturally remain unpredictable. With adults and adolescents this difficulty is overcome by providing hypothetical situations and lists of outcomes, but a less abstract approach is generally desirable for children. A surprisingly little used solution is to create videos of seemingly random experiments. This enables researchers and educators to present predetermined and consistent experiences to different groups of children at different times.

This paper reports on a number of linked studies that have utilised this technology. A series of studies has been conducted by Ayres and Way (1999, 2000, 2001) with Australian students aged 11 to 14 years to investigate particular aspects of their understandings of probability. Although each of these three studies contained a new element of investigation, all of them included a common experience in the form of a pair of pre-recorded videos depicting a series of apparently random draws from a box of balls. From 1999-2001 we reported the results of these studies at various stages of completion. However, in each case, additional data was collected which was not previously reported. Consequently, the new data has enabled us to conduct further analysis, as well as more consistent statistical testing, on a larger sample size. This paper focuses on the new and consolidated findings.

Video recordings as a research instrument

The major research tools used in the studies were video-recordings. Two videos were made which featured a presenter making thirty selections of coloured balls from a box, with replacement. In each video, 19 whites (63% of the total selections), 7 blues (23%) and 4 yellows (13%) were drawn from the box. For each group of five selections, the total number of whites, blues and yellows balls that occurred were identical for both videos. Consequently after each set of five selections the progressive experimental probabilities of each colour occurring were identical. However, for the 6^{th}, 11^{th}, 16^{th}, 21^{st} and 26^{th} selections, the colours varied according to the video. For one video, which we called the *typical sequence video*, a white was selected in four of these five positions. For the second video, called the *non-typical sequence video*, one of the less likely colours (blue or yellow) occurred in place of a white four times out of five. As students were required to predict the most-likely colour for the next draw after observing the five previous selections, these differences were highly significant. Students who consistently chose the most frequently occurring colour (white) would be successful in their predictions if they viewed the *typical sequence*, but unsuccessful if they viewed the *non-typical sequence.* As a result of this design we were able to investigate how confirmation or refutation of success affected student choice.

Although the outcomes of these videos were manipulated and not random, it was expected that students observing these videos would believe that they were genuine random experiments. This was tested (see Ayres & Way, 1998a, 1998b) and found to be the case. Furthermore, over the whole series of experiments completed so far, no evidence has ever emerged that students doubted the validity of the videos. By controlling the colour sequences we were able to not only compare how students reacted differently to the two videos, but also see how different groups of students reacted to the same videos. Consequently, it was possible to make meaningful comparisons across studies, as the instruments are identical. A major empirical difficulty in conducting probability experiments of this nature is that random generators produce random outcomes. Video-recording techniques, such as the one described here, can overcome this problem.

RESULTS

Prediction analysis

For each study, students observed one of the videos and were asked to make six predictions. Students were told that the prediction tasks were a game and they should try to predict as many correct colours as possible. Because the selection of white (the most frequent colour) was considered an important statistic, as it indicated probabilistic reasoning, the number of whites chosen was reported in each study. Furthermore, as students often changed their strategies over the course of the trial, prediction trends were noted by recording the number of whites chosen in the first and last three predictions. It should be noted that none of the participants in these studies had any formal classes in understanding likelihood, as probability was not part of the syllabus for these students

1999 Study

The main goal of the 1999 study was to investigate how students were influenced in their probability judgements by confirmation or refutation of their predictions. Students from the same school participated from two grades (Grade 6, aged 12 and Grade 7, aged 13). Half the student from each grade observed one of the two videos. The mean number of white balls by group and location selected are reported in Table 1. The inclusion of the Grade 7 data, previously not reported, doubles the sample size and allows a 2x2 (grade x typicality) ANOVA to be conducted on the total number of white balls selected. A significant main effect for typicality was found; $F(1,116) = 4.3$, $p < 0.05$; indicating that the students who viewed the *typical* video selected white more times than those who viewed the *non-typical* video. No main effect was found for grade: $F(1,116) = 2.5$, $p = 0.12$.

Table 1: Mean number of white balls selected by group and location (1999 study)

	Typical sequence			Non-typical sequence		
	1st 3	2nd 3	Tot	1st 3	2nd 3	Tot.
Grade 6 N= 60	1.1	1.7	**2.8**	1.2	1.0	**2.2**
Grade 7* N=60	1.3	1.7	**3.0**	1.5	1.2	**2.7**
Combined	1.2	1.7	**2.9**	1.4	1.0	**2.5**

*Data not previously reported

A notable feature of this study was how prediction strategies changed over the course of the task. For the *typical* group (Grades 6 and 7 combined), students chose significantly more whites in the second half of the task (final three predictions) than on the first (first three predictions) under a paired t-test: $t(61) = 3.6$, $p < 0.01$. In contrast, the *non-typical* group chose significantly less white balls for the second half compared with the first half under the same test: $t(57) = 3.3$, $p < 0.01$. These comparisons indicated that students who were being rewarded for choosing the most likely colour (*typical* group) continued more frequently with this strategy, whereas students who were not being rewarded (*non-typical* group), reduced their selections of the most likely colour.

2000 Study

In the 1999 study, students were not told about the sample space (ratio of the coloured balls in the box) prior to starting the trial. However, in 2000, the effect of knowing or not-knowing the sample space was investigated. The experimental design of the previous study was extended by subdividing further each of two groups who received either a *typical* or *non-typical* video into two subgroups, one of which knew the sample space and the other did not. In addition to the previously reported data on Grade 8 students, a cohort of Grade 7 students participated from the same school. This data is reported in Table 2. A 2 x 2 x 2 ANOVA (grade x typicality x sample space) was conducted. A significant main effect was found for grade: $F(1, 100) = 15.5$, $p < 0.01$; indicating that the older group of students selected more whites than the younger group. There was also a significant main effect for typicality: $F(1, 100) = 12.6$, $p < 0.01$; indicating that students who observed the *typical* video predicted more whites than those who viewed the *non-typical* video. However, there was no main effect for sample space: $F(1, 100) = 0.2$, $p = 0.64$; indicating that knowing or not-knowing the sample space made no difference to student predictions. No significant interactions were found.

Analysis of the prediction trends revealed that students who viewed the *typical* video had no significant difference in their choice of whites between the first and second three predictions under a paired t-test: $t(53) = 0.2$, $p = 0.82$. However, students who viewed the *non-typical* video predicted significantly less whites during the second half of their predictions than the first half: t (53) = 3.5, $p < 0.01$.

Table 2: Mean number of white balls selected (2000 study)

	Typical sequence						Non-typical sequence					
	Unknown Sample Space			Known Sample Space			Unknown Sample Space			Known Sample Space		
	1st 3	*2nd 3*	*Tot*	*1st 3*	*2nd 3*	*Tot*	*1st 3*	*2nd 3*	*Tot*	*1st 3*	*2nd 3*	*Tot*
Grade 7* (n=48)	2.0	1.6	**3.6**	1.9	1.5	**3.4**	1.5	1.1	**2.6**	1.3	0.8	**2.1**
Grade 8 (n=60)	1.8	2.2	**4.0**	1.8	2.2	**4.0**	1.8	1.1	**2.9**	2.2	1.7	**3.9**
Comb.	1.9	1.9	**3.8**	1.8	1.9	**3.7**	1.7	1.1	**2.8**	1.8	1.3	**3.1**

*Data not previously reported

Interview data

Twenty-four (three chosen at random from each cell in the study) of the 108 participating students were asked to give reasons for each prediction made. The four most frequent reasons given were: a) more white balls in the box (42%), b) the colours formed a pattern (22%), c) guessed (15%), and d) a particular colours turn to come up (15%). Students who stated that they chose white because there were more whites in the box were demonstrating probabilistic reasoning. However, the frequency of this response varied according to which video was viewed. By recording group means for the number of times this "more whites" reason was given a 2 x 2 (typicality x sample space) ANOVA could be conducted (grades combined). There was a significant main effect for typicality: F (1, 20) = 6.5, $p < 0.05$; indicating that students who viewed the *typical* video gave more reasons consistent with probabilistic reasoning than those who viewed the *non-typical* video. No significant main effect was found on knowing the sample space or not. By further analysing the "more whites" reason over the first and second three predictions made, a further difference was found. Students who viewed the *typical* video, gave the "more whites" reason significantly more times during the second three predictions (2.1) than the first three (1.1): t (11) = 3.4, $p < 0.1$; indicating that they became more convinced as the trial progressed. However, for those who viewed the *non-typical* video, no significant difference was found between the two phases: t (11) = 1.1, p =0.31. Although, "more whites" was offered as the reason more on the first 3 predictions (mean = 1.1) than the second 3 (mean = 0.8). Overall, the trends for the reasons given, consistently matched the predictions made.

2001 Study

The main aim of this study was to investigate the effect of instruction (small-group practical activities) on predictions. As reported previously (Ayres & Way, 2001), the different instructional experiences and group dynamics affected student predictions when individuals responded to the *non-typical* video. In addition to this data, students were asked to give reasons why they made each prediction. The mean number of times the "more whites" response was given is recorded in Table 3. Although, knowing or not-knowing the sample space failed to reach significance, knowing the sample space produced higher means for both the number of whites chosen and the "more whites" reason, and suggested there may be a real effect for a larger sample. Furthermore, consistent with the other studies, there was a significant decrease in the number of whites chosen; $t(21) = 2.7$, $p < 0.05$, and the "more whites" reason; $t(21) = 3.5$, $p < 0.01$; from the first 3 predictions to the last 3.

Table 3: Mean group responses to the untypical video (2001)

Grade 6 students (n = 24)	Unknown Sample Space			Known Sample Space		
	$1^{st}3$	$2^{nd}3$	**Tot**	$1^{st}3$	$2^{nd}3$	**Tot**
Number of whites chosen	2.3	1.6	**3.9**	2.6	1.9	**4.5**
Number of times "more whites" given as a reason*	2.2	1.3	**3.5**	2.6	1.6	**4.2**

*Data not previously reported

CONCLUSIONS

The inclusion of new data and its analysis has strengthened and extended the findings of the 1999-2001 studies. Overall, the findings fall into three main categories. Firstly, it is clear that students in this age range appear to develop an understanding of likelihood with age, even without instruction. By choosing a high percentage of the most-likely colour throughout these studies, students have demonstrated some understanding of basic probability. Interview data, collected on students' reasons for particular predictions, have reinforced this conclusion. However, many misconceptions have also been observed. Notably, many students believe that colours that have not occurred for a while are then more likely to occur- "it's that colour's turn to come up". This *negative recency* effect has been observed in other studies (see Fischbein and Schnarch, 1997). Of particular interest has been the discovery that students often look for patterns. This finding, which is easily explained considering the emphasis placed on patterns in modern curricula, may make understanding randomness for students of this age, more difficult, especially if outcomes are displayed in pattern-like sequences found in these studies.

Secondly, students in this age group have a fragile commitment to using probabilistic reasoning consistently. The differing responses to the two videos have demonstrated this effect conclusively. Students, who viewed the *non-typical* video, switched from choosing the most-likely colour into strategies based around various misconceptions. In contrast, students who viewed the *typical* video tended to choose the more-likely colour more often, as the trial progressed. Refutation or confirmation of 'prediction' is an important factor in this type of task.

Thirdly, instruction appears to have a positive effect. Students presented with the *non-typical* video after instruction, extensively chose the most frequently colour and argued that they did so because it was more likely. Nevertheless, this strategy still decreased for many students as a result of their predictions being refuted. However, of particular note, was the high number of whites chosen by the Grade 6 students compared with all other students in these studies, many of whom were older. In particular, students who were aware of the sample space had a mean score of 4.5 out of 6 - the highest score for any group in all three studies. Although, knowing the sample space or not, did not produce a significant difference when directly tested, it was done so only on students who received no instruction. It is feasible that following instruction, knowledge of the sample space may become more meaningful.

Further research directions follow directly from these studies. The previous studies were deliberately limited to one type of random generator, a particular sample space structure and two specified sequences of outcomes. However, future research could expand to include a wider variety of random generators, such as dice and spinners; a range of sample space ratios; and various sets of frequency data. The value of using video-recorded probability trials in such research has been established. Therefore, to enable the use of videos as instructional aids and/or assessment tools a range of videos could to be developed. Using such videos, the effectiveness of instruction can be directly tested, as well as the types of instruction, and the influence of knowledge of the sample space.

Finally, in reference to the debate about inclusion of *chance* into school curriculum, it should be acknowledged that it is a difficult concept for young children to understand; nevertheless, our research suggests that students as young as eleven can benefit from instruction.

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ASSOCIATIONS BETWEEN MATHEMATICALLY INSIGHTFUL[1] COLLABORATIVE BEHAVIOUR AND POSITIVE AFFECT.

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Associations between mathematically insightful behaviour and student affect are studied. A tool is developed to identify this behaviour and study its association with positive affect. Mathematically insightful behaviour displayed by a collaborative pair of senior secondary calculus students is used to illustrate this tool and discuss a situation in which student autonomy, spontaneity and creativity accompanied positive affect. The tool builds upon the ideas of Dreyfus, Hershkowitz, and Schwarz (2001), the concept of 'flow' (Csikszentmihalyi & Csikszentmihalyi, 1992), and the identification of student engagement as they discover complexities (Williams, 2000).

INTRODUCTION

This paper illustrates associations that can exist between positive affect in students and the creative development of a new cognitive structure (a mathematically insightful process of abstraction). An analytical tool that simultaneously displays student cognition, social interaction patterns (Dreyfus, Hershkowitz et al., 2001) and affective indicators (Csikszentmihalyi & Csikszentmihalyi, 1992; Williams, 2000) was used. Attention to whether the dialectic interaction occurred in response to earlier dialogue within the interaction under study facilitated identification and analysis of student spontaneity and student autonomy. An interaction was seen to be more autonomous and spontaneous where the students within the pair responded to interaction from within rather than outside the pair. Where autonomous and spontaneous behaviours were exhibited, further analysis was undertaken to find evidence of creativity. The simultaneous display of affective indicators 'made visible' associations between engagement with the task and students' autonomy and spontaneity in the process of 'abstraction' (Dreyfus, Hershkowitz et al., 2001). It is recognised that the cognitive artefacts students assemble during the creative process could include strategies, ideas and concepts that relied upon contributions from individuals external to the group prior to this interaction. In relation to the interaction under study, these previous influences contribute to the cognitive artefacts students assemble (Dreyfus, Hershkowitz, et al., 2001). The extent to which an interaction is seen as creative in this study depends partly on the spontaneity with which students assemble these artefacts and the students' prior knowledge of their relevance.

[1] Acknowledgement: Support for this research by the Spencer Foundation through the Learners' Perspective Study directed by David Clarke and support for conference attendance by the Mathematical Association of Victoria. David Clarke's comments on an earlier version of this paper led to my further clarification of some ideas.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Student alienation with school is an issue of current concern (Marks, 2000) and one solution is to create enhanced learning situations where positive affect is linked with new learning (Csikszentmihalyi & Csikszentmihalyi, 1992). The presence of positive affect associated with mathematics learning has been identified in Nicholls' (1983) task-centred rather than ego-centred learning. The characteristics of students undertaking task-centred learning can be seen in instances of learning through 'discovered complexity' as identified by Williams (2000). Discovered complexity occurs during task completion where a group of problem solvers perceive intellectual and conceptual complexities not evident at the commencement of the task. When a complexity is discovered, the group spontaneously formulate a question that leads to higher level thinking (analysis, analytical-synthesis, synthesis, and evaluative-synthesis) in the domain of mathematics. The resolution of the question leads to an abstraction associated with this complexity. Both Nicholls (1983) and Williams (2000) linked student learning accompanied by positive affect to 'flow' (Csikszentmihalyi & Csikszentmihalyi, 1992). Flow is an optimal learning condition that may occur when a person works just above their present skill level on a challenge almost out of reach. Individuals or groups in flow become so engrossed with the task at hand that they lose awareness of self, time and the world.

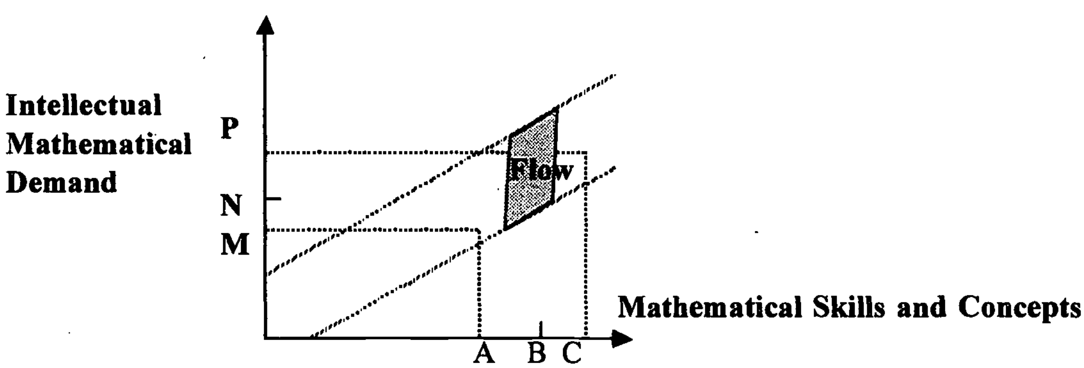

Figure 1. Representation of associations between discovered complexity and flow

Williams (2000) illustrated the fit between discovered complexity and the conditions for flow by modifying the schematic representation (Figure 1) developed by (Csikszentmihalyi & Csikszentmihalyi, 1992). A student's perceived level of skills and concepts and perceived challenge comfortably overcome are represented by **A** and **M** respectively. Students in flow are seen as working to achieve a goal represented by a point within the shaded region of flow **(B, N)**. This goal is just above their perceived skills and concepts level and involves a challenge perceived to be almost out of reach (flow). Once this goal is achieved, each student's perceived skills and concepts level is **B** and they can comfortably achieve a challenge of **N**. The shaded region representing flow is now located to the right of **B** above **N** between the parallel lines. To sustain flow would require a discovered complexity that led to a goal represented by a point like **(C, P)**.

By definition (Williams, 2000), the mathematical ideas in a discovered complexity are new to all students in the group and the teacher does not contribute new mathematical ideas during the interaction. Before the present interaction, other class members, the teacher, or another source may have contributed to the cognitive artefacts the students chose to assemble. In such situations where the student group develops new concepts through discovered complexity, the type of student response described is seen to be partially attributable to the task and the implicit pedagogical approach (Williams, 2000). The process students enact when working with discovered complexity is an insightful process of autonomous, spontaneous, and creative abstraction similar to the activity described by a research mathematician (Chick, 1998). Dreyfus, Hershkowitz and Schwarz (2001) are interested in 'abstraction'—an activity of vertical reorganisation of 'previously constructed mathematical knowledge into a new structure' (p. 377). (Vertical refers to a new mathematical structure as opposed to strengthened connection between a mathematical structure and a context ('horizontal')). They have diagrammatically represented a simultaneous display of cognitive activity and social interaction present during the abstraction process. Six categories of dialogue were used in the analysis of social interaction—control, elaboration, explanation, query, agreement and attention. Through analysis of the dialogue of participants as they undertake the social process of critical inquiry, the cognitive elements of the process of abstraction were made more visible. These nested elements of the process of abstraction as identified were: (a) 'recognising'—seeing a previously known mathematical structure within a new context or realising a previously known mathematical structure fits a new context; (b) 'building-with'—using a combination of previously generated abstracted entities in a new context; and (c) 'constructing'—use of assembled resources to vertically re-organise a mathematical structure.

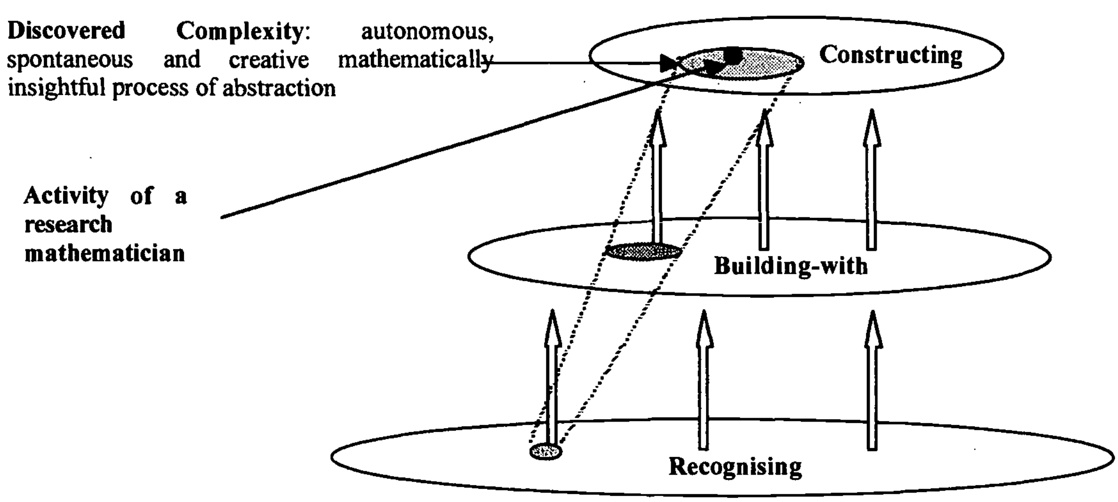

Figure 2: Relationship between nested elements of abstraction and discovered complexity

Figure 2 illustrates the relationship between discovered complexity, the activities of a research mathematician (Chick, 1998), and the epistemic elements in the abstraction process. In examining the process of creation of mathematical concepts, Chick (1998) reported the presence of positive affect as strategies, ideas and concepts were synthesised to produce a novel mathematical insight about the complexities in a mathematical structure. This is consistent with Krutetskii's (1976) description of the mental activity of combining concepts to create a new idea.

Engagement in a task was monitored by Williams (2000) through observation of video data to identify indicators of positive affect: (a) eyes on the task; (b) pens on the task page or bodies leaning in towards the task; (c) unaware of the world around; (d) participating in the interaction; (e) students building on each others' ideas (latching comments); (f) exclamations of pleasure.

Analysis of what triggered each part of the dialectic interaction and the nature and source of the conceptual artefacts upon which students draw will facilitate the identification of insightful mathematical behaviour.

BACKGROUND ORIENTATION FOR THE ILLUSTRATIVE TRANSCRIPT

Three students, William, Talei and Gerard were videotaped working as a collaborative group to solve the unfamiliar challenging problem 'Understanding the Double Derivative'. The teacher selected the groups, provided the task sheet and stated that $f''(x)$ meant the derivative of $f'(x)$. Students followed a familiar classroom-working pattern; they worked in groups for twelve minutes then made their first brief report to the class as a whole. The interaction reported occurred eleven and a half minutes into group-work and continued for one and a half minutes.

In the previous twelve months, all three students had worked briefly with sign diagrams and extensively with gradients of curves in individual and group settings through investigation and exercises. The students had not previously encountered second derivatives. Gerard had not been exposed to the term 'curvature' in a mathematical context. Talei and William were working to mathematically formulate 'curvature' in another mathematics subject studied simultaneously (with another teacher). At the time of this research, they had not achieved their goal. The teacher made no reference to the idea of curvature or to the other subject.

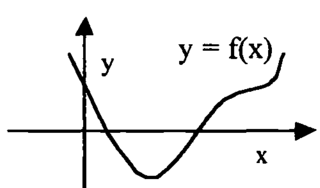

Figure 3: The shape of y = f(x) in the 'Understanding the Double Derivative Task'

The task 'Understanding the Double Derivative' required students to use the graph (Figure 3) as a starting point. By sketching the graphs of $f'(x)$ and $f''(x)$ and the sign diagrams to the three graphs and generating other information if they chose to do so, students were required to investigate links between $f''(x)$ and $f(x)$. The task sheet contained general suggestions and questions about possible ideas to explore but these were not prescriptive and did not provide hints or directions about specific ideas to explore, strategies to use or pathways to follow. They included questions and statements like: 'What happens to key features on one graph in another graph?' 'Search for patterns and reasons why these patterns exist'. 'Predict'. 'Check'. 'Is the $f''(x)$ graph sufficient information to be able to generate the f(x) graph? Explain'.

Key to transcript: '}' one individual completed or extended the previous statement (latch); '/' one individual cuts across the statement of another (cut).

1	William:	The shape.	[Time 11 min. 30 secs.]
2	Talei:	Uh hu.	
3	William:	To there.	
4	Buzzer:	[Denotes the end of collaboration time].	
5	Teacher:	[Begins to organise the reporting phase. No mathematical talk, just organisational. Continues simultaneously with interaction to Line 24]	
6	William:	Well if you look between there and there it is curving}	
7	Talei:	{Yes.	
8	William:	The curve is getting less and less}	
9	Talei:	{Yes ... it's the gradient yes it's}	
10	William:	{'till there which means it's negative ... it's steep}	
11	Talei:	{Yes}	
12	William:	{'till there when it starts to get more}	
13	Talei:	{So when the curvature is	
14	William:	and more}	
15	Talei:	{smaller but that's	
16	William:	but what about between there and there ... what happens? Why is that?/	
17	Gerard:	/What are we going to talk about? Do you want/	
18	William:	/Why is that ... a positive?/	
19	Gerard:	/to talk?	
20	Talei:	It's sort of ... that is ... that's true .../	
21	Gerard:	/Do you want to talk?/	
22	William:	/It turns there but it doesn't turn and go the other way ... it always turns the same way- THAT'S IT! [Moves his hand in the shape of a minimum turning point and watches his hand move]	
23	Talei:	Yes} [Smaller hand movement in the shape of a minimum turning point without appearing to see William's hand movement]	
24	William:	{It's positively turning that way.	[Time 13 mins.]

By the eleventh minute, the group had found several patterns through the discovery of six successive complexities (Williams, 2001). William then commented 'there must be something more'. William's focus of attention on a level beyond searching for patterns and towards finding an over-arching reason for why these patterns occurred produced the implicit spontaneous focus question to another discovered complexity. It appears the pair were responding to an implicit question like 'why do the patterns we have found exist? Gerard did not participate in this final collaboration because his attention was drawn back to the class task (and away from the spontaneous question explored by William and Talei) by the buzzer (signalling the end of group-work) and the organisational comments of the teacher preparing students for the reporting session. For this reason, in this paper Talei and William have been described as a collaborating pair and Gerard as an outsider for this final one and a half minute interaction where 'constructing' occurred between Talei and William. William and Talei gave no indication that they were aware of the activity around them; they did not respond to the external attempts to focus their attention on the reporting process (buzzer, teacher, or Gerard).

Key: Cognitive **C B R** (constructing, building-with, recognising). Social **O W T** External // Internal. Affective: **Ey D U P L Ex** (eyes, body, unaware, participate, latch, exclaim)

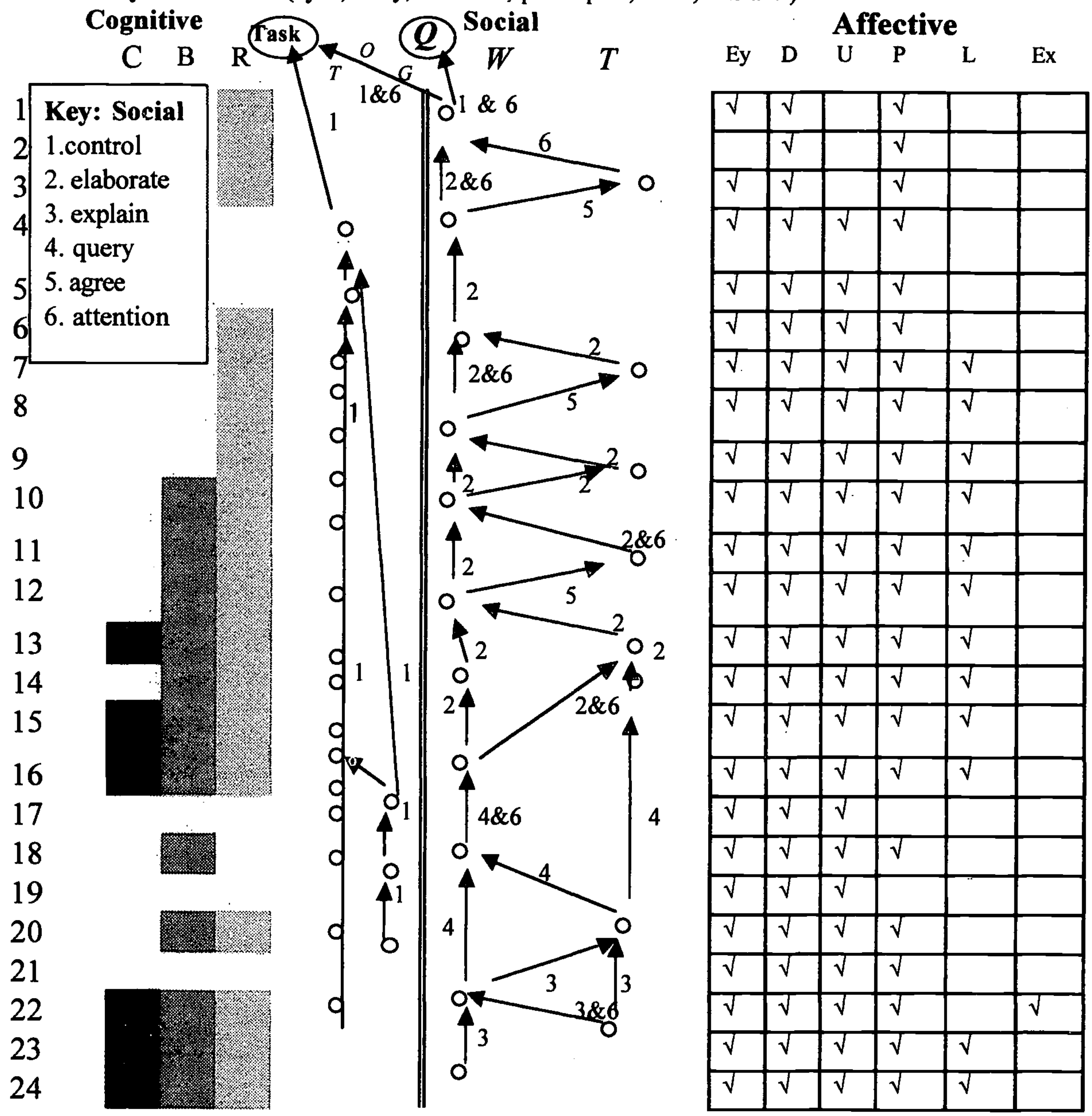

Figure 4. Tool to aid identification of student autonomy, spontaneity, and creativity in the abstraction process (from Dreyfus, Hershkowitz & Schwarz; 2001; Williams, 2000)

DISCUSSION AND CONCLUSIONS

The analysis tool (Figure 4) used to explore associations between student affect and the creative process of abstraction is a diagrammatic representation displaying the cognitive, social, and affective features of the interaction. Numbers down the left hand side of the diagram represent the line of transcript. The shaded squares in the first column indicate the inferred presence of epistemic cognitive elements in the process of abstraction for the individual speaking in that line of transcript. The centre column contains a vertical line that separates the small circles representing those within the collaborating pair from those outside the collaborating pair. The numbered arrows indicate the individuals to whom the speaker appears to attend and the social interaction category to which those prior statements belong. The number of ticks in the grids on the right hand side provides an indication of task engagement. Arrows that cross the separating line in the centre column suggest less spontaneity and autonomy than when the arrows from members of the collaborating pair are directed to dialogue within the pair. By considering cognitive activity in conjunction with social interactions, inferences can be drawn about student creative behaviour. Where students are constructing and the social interaction pattern indicates students are responding only to sources internal to the group, the students display autonomous, spontaneous behaviour that may be creative. The pattern shown by the directions of the arrows from William and Talei (Figure 4, centre column) indicate the pair responded only to each during the interaction. William's first statement is in response to the overall task and the pair's discovered complexity within it (Q in Figure 4). They displayed numerous indicators of positive affect (Figure 4, right column) as this autonomous, spontaneous pair employed mathematically insightful behaviour whilst constructing a new mathematical insight. Evidence of creative behaviour exists and requires further exploration.

William explored the complexity by directing attention to a feature of a segment of the graph using common language to elaborate ideas (Transcript Lines 1,3,6 and 8 categorised as attention in Figure 4). The use of common language rather than precise mathematical language indicated the presence of an amorphous idea rather than a well-structured mathematical idea. Talei recognised the mathematical significance of William's comment (Transcript Line 9); she synthesised the amorphous structure of 'curvature' she recognised as applicable and built-with this as she elaborated further (Transcript Line 13 & 15). Talei and William now moved their pen along the curve undertaking evaluative-synthesis between the f(x) graph (Figure 3) and $f''(x)$ graph. Almost simultaneously they found the minimum point in f(x) (Figure 3) appeared to contradict their theory (the 'but' in Line 15 and 16). Their subsequent behaviour suggested they explored 'Why doesn't the minimum point on f(x) show as a key feature on $f''(x)$?' William's exclamation and comment framed partially in common language (Line 22) and Talei and William's hand movements (Line 23, 24) provide evidence of vertical cognitive reorganisation.

A more detailed characterisation of the factors that contributed this student interaction would require exploration of factors such as classroom culture, task characteristics, and the teacher's introduction to the task. This brief illustrative example shows how this tool can be applied to distinguish between the processes of 'abstraction' in general and a subset of that process where student behaviour is spontaneous and creative. Further work is required to extend the tool to examine the nature of dialectic intervention from an 'outsider' who stimulates rather than inhibits spontaneity and creativity. The tool presents possibilities for examining such incidents and has been designed for use with the video and interview data in the Learners' Perspective Study of Year 8 Mathematics classrooms being conducted in ten countries.

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SEMIOTIC ANALYSIS OF DREYFUS' POTENTIAL IN FIRST-YEAR CALCULUS

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This paper presents a semiotic analysis of the (often claimed) potential of computer algebra systems (CAS) for enabling mathematical activity on a higher conceptual level than usual. Theoretical points are illustrated by an example from a development project in the context of a first year university course on calculus. We also discuss how they may be used in 'a posteriori' didactical analysis.

DREYFUS' POTENTIAL.

Broadly speaking, there are two main types of issues relating to the use of standard CAS, such as *Maple* or *Mathcad*, in undergraduate mathematics teaching:

- *Pragmatic issues*, concerning the competencies that students need in present and future mathematics-related practice, where CAS is or could be a relevant tool,
- *Didactical issues*, concerning the actual or potential impact of CAS-use on the students' learning of mathematics.

In the first case, we are talking about *needs for actual practice*, such as solving concrete mathematical problems, while in the second case, CAS is viewed as a vehicle for *learning*.

We shall discuss, in this paper, the didactical aspects of *Dreyfus' potential*, defined as the possibility that CAS may serve as follows:

> The idea is for students to operate at a high conceptual level; in other words, they can concentrate on the operations that are intended to be the focus of attention and leave the lower-level operations to the computer. [Dreyfus (1994) p. 205]

The didactical interest of Dreyfus' potential is rather obvious, in particular it is likely to reflect the ideal of most CAS-using university teachers. However, as it stands, it is exactly 'an idea' that may or may not 'appeal' to their colleagues, depending on their experience and personal preferences. In order to rationally discuss, plan, implement and document realisations of this idea, we need to formulate it in terms of a clear theoretical framework. In particular we need to clarify the notion of a 'high conceptual level' in its (potential) relation to CAS use in university education. This is where the analytic tools of *semiotics* come in; specifically, we draw mainly on Duval's semiotic analysis of mathematical cognition (cf. the next section). Of course, we must also relate the theoretical model with actual practice; at the end of this paper, we give examples of how it may be used in didactical analysis and design based on semiotic variables.

SEMIOTICS AS AN ANALYTIC TOOL FOR DIDACTICS.

Duval (2000) points out three major cognitive functions related to the semiotic registers used in mathematics: *representation* (e.g. a graph representing a function), *processing* (e.g. computation with number symbols) and *conversion* (change of register of representation). Even though students do not acquire mastery of these functions separately, their separate analysis proves useful for our purposes.

Semiotic *representations* in mathematics do not 'represent' in the naïve sense: *the entire discourse refers to...nothing other than its own signs* (Rotman, 1988). Mathematical objects are created *and* invoked through semiotic representations (Sfard, 1999). Yet in many (if not all) mathematical contexts it is crucial not to confuse an object with a particular representation (Duval, 2000), e.g. not to identify a 'triangle' with a concrete drawing. In fact, particular representations are less important than the changes (*processing*, *conversion*) of such representations that might be effected, while retaining a representation of the object. A simple semiotic description of an individuals' conceptual image of an object can thus be said to be a class of representations invariant under some class of transformations (processing, conversion). These classes will typically be implicit for the learner, and will develop gradually through actual instances of representing, processing and converting, briefly, through *semiotic activity*. This provides a theoretical basis for the need that students engage in such activity using a wide range of relevant registers and transformations. It may, in particular, challenge the modern contempt of 'training exercises' to the extent these exhibit such variation – even if variation pertains only to processing. However, as pointed out by Duval (1995, pp. 45-59), many of the most interesting and persistent learning difficulties arise in the context of conversion. Hence the 'degree of freedom' of the individual with respect to conversion (coordination of registers pertaining to the same conceptual object) is essential for the individual's conceptual development (*opus cit.*, p. 69).

In university level mathematics, patterns of *relations* among *different* conceptual objects are at the heart of the learning enterprise. To establish and develop such relations coherently, very specific forms of discourse – involving more or less formal uses of natural language in coordination with 'simple' semiotic registers – are needed. These forms may be classified as *apophantic*, *expansive* and *reflective* functions of discourse (*opus cit.*, chap. II). While a 'high conceptual level' may be indicated in semantic terms, it is usually accompanied by (at least implicit) semiotic and discursive complexity, and so its realisation in undergraduate mathematics education is intrinsically linked to developing the students' degree of semiotic freedom within this larger discursive framework. This implies the need for a careful design of learning situations centred on specific discursive and semiotic functions, where the latter includes pertinent forms of processing and conversion. It points to semiotic activity and discursive functions as crucial variables in didactical engineering (Artigue, 1989 and Brousseau, 1997, pp. 24f) at the undergraduate level.

The semiotic activity of students serves *pragmatic* as well as *didactic* purposes.

Pragmatic, because mathematical competencies are articulated – and can only be observed and evaluated – through discursive and semiotic performance (e.g. Sfard, 1999). Didactic, because an individual's conceptual development depends on his effective range of semiotic activity, as explained above. Notice that this is by no means a denial of the social nature of the learning enterprise, as semiotic activity to a large extent derives its meaning from a social context (interchange with other agents). Moreover, semiotic activity is influenced by the social context in the forms of agents, media and codes (the latter being a matter of *consensus* but also a *condition* for semiotic activity). Only codes may be independent of the context, that is, they belong to a large extent to a *global* context of mathematical discourse (Duval, 1995, p.225). The shared perception – largely implicit – of codes among participating *agents* is, on other hand, decisive for their potential for engaging in shared semiotic activity.

SEMIOTIC VIEW OF DREYFUS' POTENTIAL.

In a semiotic interpretation, the basis of Dreyfus' potential is that a CAS may provide a higher degree of semiotic freedom, primarily by facilitating *processing*, and – in a few cases, such as plotting – *conversion*. To 'operate on a high conceptual level' implies, in functional terms: to engage in discourse involving complex semiotic activity. The 'lower level operations' – carried out by the CAS – consist mainly of the *processing* parts of this activity. Typically, the explicit complexity of semiotic activity is reduced by the use of CAS, as the CAS tends to leave out several intermediate steps that may, or may not, be made explicit at wish. In fact, the 'black box issue' (Dreyfus, 1994) arises to the extent this is the case.

It follows from the above interpretation that the use of a CAS – at least, a priori – facilitates neither *coordination* of registers nor the main *discursive* functions. The simple *representation* of objects and transformations is not simplified, either. On the contrary, we have an extra medium (the computer), an additional special code (depending on the CAS) for semiotic activity, and a kind of 'automatic semiotic agent' with a potential influence on discourse (Winsløw, 2000). These additions may be particularly disturbing for novice users of CAS, but their influence remains important even in a context with experienced CAS-users.

An example.

Our key example is an event observed during a development project (Solovej et al., 2001) in the context of a freshman calculus unit. It occurred towards the end of a two-hour class session, in which students had presented and discussed a number of exercises on linear differential equations. In this unit, *Maple* was generally used by students in their homework, and it was demonstrated and used as illustrating device during lectures. In class, the computer was mainly used as a medium for presenting homework (students bring floppy disks to the class) and, occasionally – but with increasing frequency – as a semiotic agent in 'whole-class' exploring. There was one PC in the classroom, connected to a screen projector; both students and teacher used this PC to present their work and ideas. At the point we shall consider, all exercises set for that day have been treated. Then a student – let us call him Peter – raises a question. He has been working on an exercise from the book (in the same section, but not among those given as homework), concerning the initial value problem

$$(*) \qquad u''(t) + u(t) = \sin \nu t,\ \ u(0) = 1,\ \ u'(0) = 0$$

where ν is an undefined parameter. Using *Maple*'s routine (*dsolve*) for solving ordinary differential equations, he had obtained the output [copied from *Maple*]:

$$(1) \qquad \mathrm{u}(t) = \cos(t) + \frac{\nu \sin(t)}{-1+\nu^2} - \frac{\sin(\nu\, t)}{-1+\nu^2}$$

and then wondered how this should be interpreted in the case $\nu = 1$. The teacher asks Peter to show his solution, as outlined above, to the class (which he does). The teacher then proposes to let *Maple* solve the problem (*) with ν substituted by 1. Peter copies the first input to a separate input line, and makes the proposed change. This results in the following new output:

$$(2) \qquad \mathrm{u}(t) = \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)\, t + \cos(t)$$

which, on the face of it, seems quite different from the first answer. The students are confused. Peter comments: 'That looks strange'. The teacher says: 'Clearly, one cannot simply substitute ν by 1 here [points to the first output, then hesitates for a while]. But how about taking a *limit* of it as ν tends to 1?' The student at the PC copies (1) to a new command line and ads the said limit. This gives the second output (2). The teacher says a few words about the possibility of studying the 'continuity of solutions' with respect to a parameter before the lesson is over.

This appears to be a fairly clear-cut example of how the *processing* powers of CAS may be used to realise Dreyfus' potential. The teacher suggests the *coordination* of the (for the students at this level) separate registers of 'ODE solving operations' and 'limit operations', in order to enrich the concept of 'solution' with a relation to 'con-tinuity'; the *didactic intention* of the teacher is served by CAS as 'processing agent'. While the students had some hands-on experience with using the standard procedure for solving inhomogeneous ODE's, the discussion – that occurred during the last 5 minutes of a lesson – can be 'lifted' to a higher conceptual level (discussion of solu-tions in terms of a parameter) only because semiotic *processing* is left to the CAS.

THE ROLE OF TEACHER CONTENT KNOWLEDGE.

In the classroom example outlined above, the discussion (discourse beyond simple semiotic activity) is entirely governed by the teacher. However, Dreyfus' potential talks about *students* operating on a higher conceptual level, while in the situation just described, these operations are only formally performed by the student (assisted by

CAS). The students do not actively participate in the 'higher level' *discourse* about 'continuity of solutions', and it is not clear to what extent they are informed by attending it. In a pessimistic evaluation, we may thus have a CAS version of the *Jourdain effect* (Brousseau, 1997, p. 26): students are led to perform certain CAS-assisted semiotic actions, and are then told 'what they have done' in terms of a higher-level discourse that is essentially beyond their reach.

In the example, this does not quite seem to be a fair evaluation of the situation. The discourse was indeed initiated by the student's question on the special case $\nu = 1$ of the solution. The students understand the solution of the special case, but Peter's comment ('That looks strange') certainly calls for an explanation; one input (ODE) is a special case of the other, yet the two outputs (solutions) look different. Given the impending end of the class, the teachers' choice of a quick explanation cannot surprise, except perhaps that it is both improvised and correct. However, in retrospect, the question could be made useful in many different ways, and this might be a starting point for CAS-based didactical engineering.

In another class, *Maple* was *only* used by the teacher as an alternative way of processing (within one register, except for standard plots of functions), usually demonstrated just after 'blackboard presentation' of the same problem. Discussions of an abstract nature never involved use of CAS. Students in this class primarily saw CAS as a *quicker* and *easier* means of reaching a pragmatic goal (processing), and their expressed motivation for trying to use it was almost exclusively the advantage they supposed to achieve for the upcoming written exam. Their conceptual understanding was most likely not enriched by the repetitive demonstration that a CAS can do in seconds what they had struggled to do on paper. One may say that, in this case, CAS was allowed as a semiotic agent only in cases where it 'echoed' other agents. The teacher repetitively used expressions like 'finally, it's nice to see these calculations in reality' (meaning, on the blackboard rather than on the screen).

While the teacher of the first class happens to be an eminent researcher of mathematics with a deep and flexible knowledge of mathematics, the teacher of this second class was a TA with his main occupation in high school teaching, and a strong focus on formal aspects (procedures, 'official' definitions etc.) of the topics taught. Throughout the project, he maintained a pessimistic view of the effect of CAS use on students' performance and understanding. The importance of *teacher content knowledge* – and its flexibility – is often found to be a main determinant for the development of CAS-based pedagogy (see Kendal et al., 2001). It may, in our framework, be interpreted as the importance of the teachers' own freedom of semiotic action within the discourse that he should help students to engage in.

TOWARDS CAS-BASED DIDACTICAL ENGINEERING.

We now return to the key example from the first class in order to sketch how this didactical situation could be improved by didactical engineering based on semiotic and discursive variables, in order to more fully realise Dreyfus' potential. This discussion is partially based on a conversation with the teacher, a few days after. It is important that in this *a posteriori* analysis, the situation must be conceived in its total discursive context, which we can only describe here to the extent it enters our discussion. It must also be stressed that this kind of analysis and design depends crucially on collaboration with teachers who, based on their own agency in mathematical discourse, are aware of the importance of the variables mentioned. This awareness is, initially, likely to be mainly non-explicit, but it is an important part of collaboration to change that.

Processing and the 'black box' issue

In order to achieve an inclusive mode of discourse, where students participate actively, it is clearly desirable to maintain *coherence* with previous (not too distant) elements of discourse that included the students. In the class situation considered, the previous discourse focused on the problem of *solving* linear differential equations with constant coefficients (in the sequel, abbreviated LODE). In particular, the focus had been on: the solution of *homogeneous* LODE, partial solutions to *inhomogeneous* LODE (obtained by 'informed guessing'), and the principle for finding the complete solution to inhomogeneous LODE (sum of two previous). The black box issue was partially mitigated in the case of homogeneous LODE's, by reading off the characteristic polynomial from the coefficients of the equation, and then use the *solve* routine to find its roots. This way, the form of the solution of the LODE could be related to intermediate steps of the solution process. After identifying the 'homogenous part' of the general solution of an inhomogeneous LODE, the partial solution could be motivated as the result of an appropriate 'guess' based on the form of the right hand side. A strategy for dealing with Peter's question might thus be to suggest a comparison between the two *problems* that result from (*) in the special cases $\nu = 1$ and (for example) $\nu = 2$, rather than between the two solutions (1) and (2), directly. Using the same patterns of analysis as previously, the students would find the crucial difference between the problems: in case $\nu = 1$, the right hand side, $\sin \nu t$, is a homogenous solution, while it is not in case $\nu = 2$. Then the different forms of (1) and (2) may be explained by differences in the 'good guess' for a partial solution. And this might bring forward *the point of knowing (in principle, and perhaps a choice of) the intermediate steps of processing* involved in the solution process – *and* their function in discourse – for the task of evaluating and comparing solutions, and otherwise reflect on their status. Notice that while this does not prevent usage of *Maple* as a semiotic agent, it requires a more flexible and informed usage than simply asking for final solutions.

One might also proceed with the original suggestion of taking a limit, and then use the reaction of Peter ('That looks strange') to bring out the point above, by asking the question: *how could one proceed to justify this limit?* Incidentally, the students learned about l'Hospital's rule 6 weeks before this event, and about partial derivatives just 3 weeks ago. This might, for instance, lead to a discussion involving the following two events of *Maple* processing:

```
> diff(nu*sin(t)-sin(nu*t),nu)/diff(-1+nu^2,nu);
```

$$\frac{1}{2}\,\frac{\sin(t)-\cos(\nu\, t)\, t}{\nu}$$

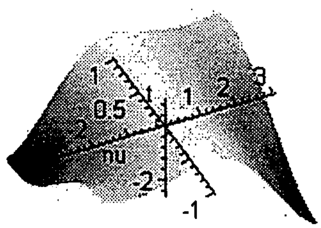
```
> limit(%, nu=1);
```

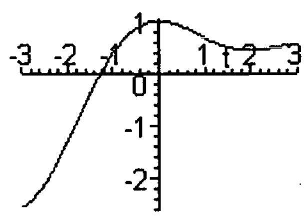
$$\frac{1}{2}\sin(t)-\frac{1}{2}\cos(t)\, t$$

(Here, the first line is *Maple* input code for: $\dfrac{\partial(\nu\sin t-\sin\nu t)/\partial\nu}{\partial(-1+\nu^2)/\partial\nu}$.)

Conversion and coordination

Continuing with the idea of simply exploring the relation between (1) and (2), one might profit from the fact that the students had studied functions of several variables 4 weeks before, in particular they were familiar with the conversion of *algebraic* to *graphical* representations of functions of two variables. In order to connect with this idea, one might suggest that the right hand side of (1) be regarded as the algebraic representation of a function of the two variables t and ν, and see if one might use the corresponding graph to relate it with (2). After a little experimentation with the settings, one might then produce the following two graphs of (1) and (2), respectively, understood as above, and with the domains [-3, 3] for t and [-1,1] for ν:





In *Maple* (not here!) the left image may be rotated in space, but even in the above 'static' form, it is not difficult to see the similarity between the 'upper edge' (corresponding to fixing $\nu = 1$) and the graph to the left. The discussion that could evolve from this would have, as its main point, to provide an exercise in the (difficult) coordination of algebraic and graphical registers that is motivated by a concrete problem. It is facilitated by CAS as an agent of *conversion*.

Expansive and reflective discursive functions

In order to avoid the Jourdain effect described in the previous section, the students must be acquainted with – and ideally control – the discursive contexts of the proposed semiotic activities. Notice that the actual *devolutions* have yet to be devised (for an example, see Brousseau, 1997, 33-35). Typically, students must be in the presence of (or have access to) a number of discursive units (apophantic sequences of phrases, including semiotic elements) on which they will, in the adidactical situation, have to *expand discursively*, using their observation of *similarity* (semantic or semiotic) among these elements, or to produce *reflective* discourse concerning the status (logical, semantic) of these elements. In the case of the example discussed above, the units present *a priori* may not suffice for the students to expand on (1) and (2) in order to relate them, and we have only outlined possible extensions of this fragment of the 'given' discursive inventory. The talk will present results (in terms of realised discursive expansion) from planned classroom experiments with more concrete extensions of this discursive basis of evolution.

But, in terms of discursive functions, the above expansions do not quite cover what the teacher, in the example, had in mind. Namely, once it has been established that (2) may be seen as the limit of (1) – and one has done similar investigations with other LODEs as well – one might try to help students proceed to a new kind of expansion: to develop (perhaps also prove) hypotheses in terms of more general forms of the *problem* (*) and its solutions. We then approach the professional mathematicians' (pragmatic!) version of Dreyfus' potential: studying 'concrete' examples using CAS as a 'semiotic slave' in order to generate hypotheses at a 'high conceptual level'.

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MAPPING OVERLAPPING WAVES OF STRATEGIES USED WITH ARITHMETICAL PROBLEMS

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This study is based on a systemic, school-focused initiative and uses data from a very large number of children (23 121) who ages range from 4.5 to 9.9 years. Each child was assessed by their class teacher using an interview-based approach and a schedule of tasks. The study examines the use of Siegler's overlapping wave theory to map the range of levels of strategies used by children to solve addition and subtraction tasks. The levels of strategies are derived from psychological models developed by Steffe, and links are made to Gray's notion of preferential hierarchies. The results indicate that the overlapping wave theory is useful to demonstrate progression with age, of levels of strategy use.

A child provided with a pile of counters and asked to determine 6 + 3 could count out 6, then count out 3, and then count all nine items from one. This strategy which we call "counting from one three times" could be applied to a range of problems. Alternatively, the child might count-on from 7 to 9 and keep track of 3 counts. Strategies differ in the amounts of time their execution requires, in their processing demands, and in the range of problems to which they apply. On a problem-by-problem basis, children of the same age often use a wide variety of strategies (Geary & Burlingham-Dubree, 1989; Goldman et al., 1988; Gould, 2000; Siegler, 1988; Steffe et al., 1983). In a similar vein Carpenter et al. (1999) suggest that there is "a great deal of variability in the ages at which children use different strategies" (p. 26).

BACKGROUND

The Cognitively Guided Instruction (CGI) professional development program (eg Carpenter, Fennema & Franke, 1996) draws on word-problem research (eg Carpenter, 1985), and focuses on the ways teachers use knowledge of student's thinking in making instructional decisions. In New South Wales, a systemic professional development program, Count Me In Too (CMIT), has been developed to improve students' learning outcomes through a school-focused method of teacher learning (Bobis, 1997; Bobis & Gould, 1998), drawing on a research-based learning framework (see below). The learning framework used in the CMIT program is a synthesis of multiple research studies (Wright & Gould, 2000). CMIT emphasises the advancement of children's arithmetical solution strategies — a recognised need in teacher development:

> The research on addition and subtraction has identified a progression of concepts and skills that is generally not reflected in instruction. Most instruction jumps directly from the characterization of addition and subtraction using simple physical models to the

memorisation of number facts, not acknowledging that there is an extended period during which children count on and count back to solve addition and subtraction problems (Romberg & Carpenter 1986, p. 856).

Beyond recognising that children can use multiple strategies to solve arithmetic problems, the question arises: How do they construct such strategies in the first place? This question has been investigated from a range of theoretical viewpoints (Carpenter & Moser, 1984; Steffe & Cobb, 1988; Steffe, 1992). The research methods employed include longitudinal studies, constructivist teaching experiments and microgenetic studies — small-scale studies of the development of a concept (Siegler & Crowley, 1991; Kuhn, 1995). This paper reports a study involving a large population of children and uses the theory of overlapping waves of strategy use arising from the microgenetic approach (Siegler, 1996; Siegler & Jenkins, 1989). The overlapping wave theory is based on three assumptions: (a) children typically use a variety of strategies and ways of thinking to solve a given problem; (b) the diverse strategies and ways of thinking coexist over prolonged periods of time; and (c) experience brings changes in relative reliance on existing strategies and ways of thinking, as well as introduction of more advanced approaches.

Stages in number development. According to Fuson, "children in the United States display a progression of successively more complex, abstract, efficient, and general conceptual structures for addition and subtraction. Each successive level demonstrates cognitive advances and requires new conceptual understandings" (1992, p. 250). The CMIT program uses a research-based Learning Framework in Number (LFIN) (Wright, 1998; Wright & Gould, 2000; Wright, Martland & Stafford, 2000) which has as one of its key components, a progression of conceptual structures which we refer to as the Stages of Early Arithmetical Strategies, and which is based on psychological models developed by Steffe et al. (eg Steffe, 1992):

Emergent: A child who is an emergent counter may have some number knowledge but it is generally made up of discrete pieces of information. For example, a child may know some of the sequence of number words and be able to identify some numerals while still being an emergent counter.

Perceptual: A child at the perceptual stage can count perceived items, matching the number word sequence to the items.

Figurative: A child at the figurative stage can determine the total in two concealed collections of items but typically counts from one to do so.

Counting-on-and-back: A child at the counting on stage uses advanced count-by-one strategies to solve a range of addition and subtraction tasks. A number takes the place of a completed count and a child can count on or back to solve problems.

Facile: A child at the facile number sequence stage can use a range of strategies other than counting by one. This includes a part-whole knowledge of numbers that enables children to draw on doubles or known combinations to five or ten.

Emphasised in the LFIN is that children frequently use strategies that are less sophisticated than those of which they are capable. This may happen for one or more reasons. A child may use a basic strategy because it is easier and although it may take more time this may not be of concern to the child. Alternatively, some feature of the child's thinking immediately prior to solving the current task may focus the child's attention on a less sophisticated strategy than the child is capable of. The emphasis in LFIN just described accords with the first two listed assumptions of Seigler's overlapping wave theory (see above).

Preferential hierarchies. Gray (1991) investigated the strategies used by 72 children to solve addition and subtraction tasks. The children were equally spread across the six age ranges of 7+, 8+, ... 12+ at each of three teacher-defined ability levels - below average, average and above average. Addition strategies were classified as known fact, derived fact, count-on, or count-all; and subtraction strategies were classified as known fact, derived fact, count-up or count-back, or take-away. Gray describes these four-level classifications of addition and subtraction strategies as preferential hierarchies. If unable to solve a task by immediate recall (ie known fact) the child reverted to what might be regarded as a preferred level. Gray identifies "two distinct approaches to the regression" (p. 569-70):

> The first makes use of other known knowledge, the deductive approach [used mostly by above average and average children]. The second is dominated by the use of counting, the procedural approach [used mostly by below average children]'....What has become fairly clear ... is that the below average ability child is neither successful at learning the number bonds nor in making use of the ones that they do know .... [For younger below average children] memory is abandoned for a procedure that involves the use of physical or quasi-physical objects. The bits they do know do not appear to be held together, with the result that this change in strategy may involve the child in long sequences of counting....In contrast, condensing the long sequences appears to be almost intuitive to the above average child. This eventually becomes the cornerstone to their higher level of attainment; they can take short cuts and operate with increasing levels of abstraction.

Gray's preferential hierarchies suggest that students might use strategies that are less sophisticated than those of which they are capable.

Mapping strategy use. Carpenter and Moser (1984) identified inconsistency of strategy use as an issue. "When children have several strategies available, they often use them interchangeably rather than exclusively using the most efficient one. Even when a more efficient strategy like counting-on from larger has been acquired, children often revert to a less efficient strategy like counting all" (p. 189). In the CMIT program, strategy use is documented using an interview-based assessment which we call the Schedule for Early Number Assessment (SENA) (NSW Department of Education and Training, 2000) and the subsequent analysis of children's responses. Each child's performance is recorded as the highest level of strategy use demonstrated during the interview. Thus the child's highest level of strategy is taken to indicate the child's level of conceptual development in number in terms of the model of stages of early arithmetical strategies.

Research question. This report addresses the following research question: Is the overlapping wave theory useful to demonstrate progression with age, of levels of strategy use?

METHOD

The data reported here is derived from analyses of individual interviews (using SENA) of 23 121 children, across 327 schools in New South Wales. The children were in classes from Kindergarten to Year 4 and ranged in age from 4 years 6 months to 9 years 11 months. Each child was interviewed by their classroom teacher at the start and end of the CMIT classsroom project. In each school the team of participating teachers was assisted by a district-based mathematics consultant trained in CMIT. In each class, the assessment interviews of at least three students were videotaped. These tapes were used in consultant-led, professional development meetings focusing on children's solution strategies. The meetings constituted a forum for collaborative planning of instruction aimed at advancing children's strategies. As well, the consultants assisted the classroom teachers with assessing, analysing and planning for teaching. Analyses of the assessment interviews enabled determination of each child's stage of early arithmetical strategy, and levels of facility with number words and numerals (eg Wright, Martland, Stafford, 2000). This report focuses only on the children's stages of early arithmetical strategy, as determined in the first assessment interview, that is, the initial interview. Any child whose data set was incomplete was excluded from the analysis.

RESULTS AND DISCUSSION

Table 1 shows the numbers of students in each age group, determined to be at each stage of early arithmetical strategy use, at the time of their initial interview. These numbers are consistent with the notion that children's highest level of strategy use tends to increase with age.

Table 1.

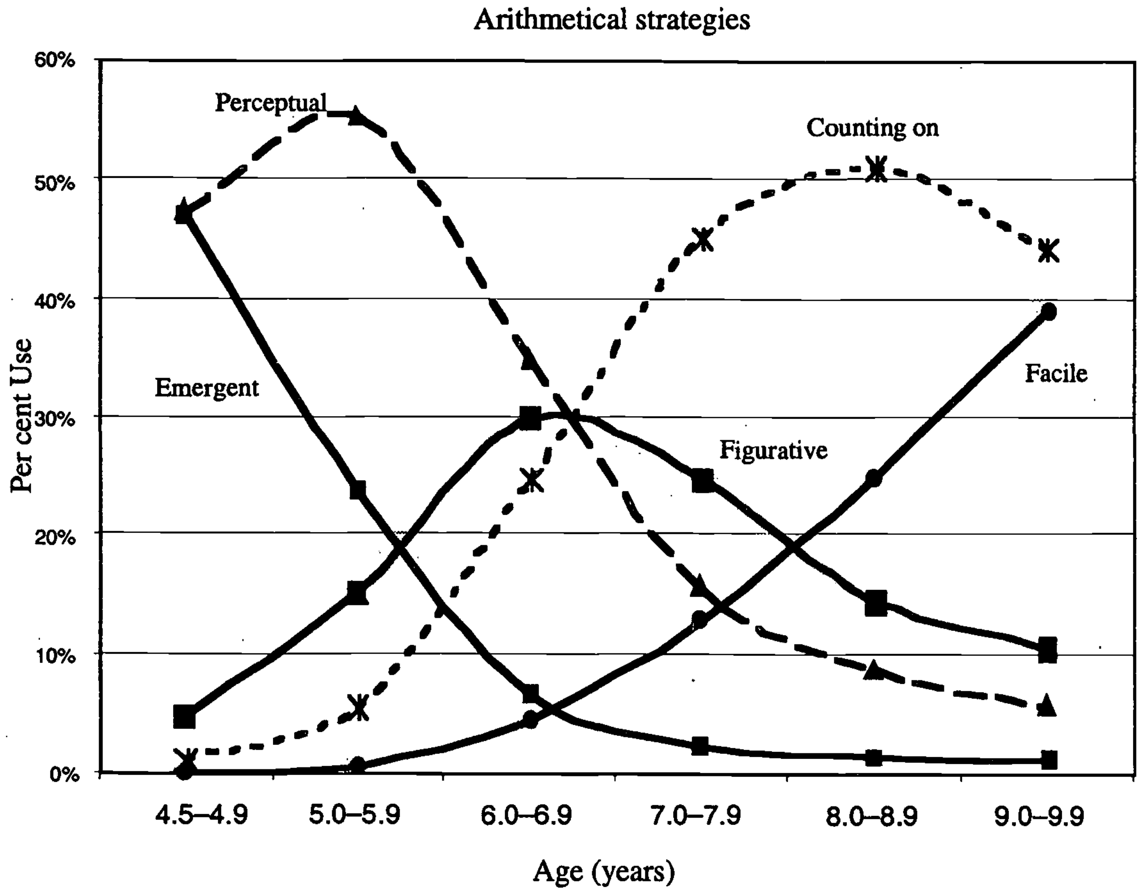
Number of children at a each stage, for each age group (n = 23 121)

	4.5 – 4.9 yrs	5.0 – 5.9 yrs	6.0 – 6.9 yrs	7.0 – 7.9 yrs	8.0 – 8.9 yrs	9.0 – 9.9 yrs
Emergent	552	1406	388	113	39	18
Perceptual	555	3254	2032	825	262	97
Figurative	56	887	1749	1321	441	182
Counting on	13	322	1445	2413	1563	772
Facile	0	29	259	682	765	681
Totals	1176	5898	5873	5354	3070	1750

Figure 1 is obtained from the grouped frequency distribution data in Table 1 as follows. For each stage of early arithmetical strategy a line graph is plotted across the six age ranges using the cell numbers from Table 1 expressed as percentages. Each cell number is expressed as a percentage of the total number in the corresponding age range. Curve smoothing is then applied to the resulting line graphs. The overlapping wave theory suggests that children in a given age range have access to more than one level of strategy. Figure 1 gives an indication for each age range, of the proportion of children for whom a given strategy level is their most advanced.

Figure 1.

Strategy use by age



The data in Table 1 and the resultant overlapping waves in Figure 1 indicate that the overlapping waves theory is useful to demonstrate progression with age, of levels of strategy use. All of the levels of strategy use are present in the population at any given age range with the exception of the facile level at the lowest age range (4.5-4.9), but there is significant variation in the degree to which levels of strategy use occur. Perceptual counting appears to peak between 5 to 6 years of age and counting-on appears as the dominant strategy used between 8 to 9 years of age. A significant range and change of strategy use appears to occur between 6 to 7 years of age. A positive feature of this approach to mapping children's levels of early arithmetical strategy use against age is that it incorporates both a sense of directionality in children's learning and a sense of children's movement among the levels. At the same time, applying overlapping wave theory in this way does not capture the insidious nature of inefficient strategies. Inefficient arithmetical strategies continue to be used by some students long after they need to, simply because they work. Thus inefficient strategies can be very persistent. A child asked to find 8 + 3 could count out 8, then count out 3 and finally count all the objects to obtain an answer. If this strategy persists in later years, the amount of mental effort needed to obtain the answer can make it difficult to achieve a necessary cognitive reorganisation.

Overlapping waves theory specifies four dimensions along which learning occurs: (a) acquisition of novel ways of thinking; (b) more frequent use of the more effective ways of thinking from among the existing possibilities; (c) increasingly adaptive choices among alternative ways of thinking; and (d) increasingly efficient execution of the alternative approaches (Siegler, 2000). Clearly, Gray's (1991) notion of preferential hierarchy has implications for the second and third mentioned dimensions. In reviewing video-taped assessment interviews we have seen that occasionally, children become quicker at using inefficient strategies. As well, children do not necessarily spontaneously progress to higher levels of strategy use. Thus it seems that instruction should focus directly on the development of higher levels of strategy use.

CONCLUSION

One of the features of assessment in the CMIT project is explicit acknowledgement that children are not always consistent in their choice of strategy. Consequently, the assessment tasks have been designed to elicit the most advanced strategy that children can demonstrate. Although a child might be able to count on or use non-count-by-one strategies, it does not mean that he or she will always do this. Competent adults might revert to perceptual counting for comfort when placed under stress or if materials are present. In our view, any approach to mapping children's solution strategies needs to take account of changes of strategy use. This study shows that an overlapping wave model of strategy use is can be applied to a large cross-sectional sample of children. Mapping individual strategy use as a function of age is a useful graphic organiser for capturing students' arithmetical strategies. The overlapping wave model constitutes a broad response map for displaying children's solution processes, describes the flow of development as observed in the population and suggests movement between the various strategies is possible at a range of ages. Thus a model arising from a microgenetic approach can be applied in a macrogenetic cross-sectional study of arithmetical solution strategies. This is consistent with the view that "the most important aspect of a research study is the constructs and theories used to interpret the data" (Kilpatrick, 1981, p. 27). Changing the theoretical lens need not distort the view.

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SEVEN YEAR OLDS NEGOTIATING SPATIAL CONCEPTS AND REPRESENTATIONS TO CONSTRUCT A MAP

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We report on-going research on the spatial – geometrical meanings generated by two groups of seven-year olds collaborating to construct a map of the area surrounding their school. The learning environment involved the use of a G.P.S. device dynamically linked to cartography software and was designed so that each group worked in a different external representations environment and was reliant only on verbal communication with the other in order to co-construct meanings which would enable them to make the map. Results showed student negotiations about concepts of motion, location and orientation to include instances where they dissociated from the software representations and made choices from a variety of geometrical reference systems.

FRAMEWORK

We report on-going research on the spatial – geometrical meanings generated by two groups of seven year olds collaborating to construct a map of the area surrounding their school. The learning environment was designed so that each group worked in a different external representations environment and was only able to communicate with the other group verbally in order to co-construct meanings which would enable them to make the map. One group physically roamed the area carrying a Global Positioning System device. The other worked indoors using cartography software[1] where they could see an icon of the G.P.S. carrier on the screen dynamically moving and turning in the corresponding direction in synchrony to the carriers' movement. Both groups could communicate verbally with walkie-talkies. We studied their actions and talk while they made discriminations between their different environments (screen representation and actual space) and attempted to develop some joint understanding about position and orientation in order to construct the map in the context of making a treasure hunt game.

Related research has highlighted the dual nature of the map as representational medium and as embodiment of spatial concepts i.e. scale and absolute (Cartesian, polar) and relative (Euclidean) orientation, position and motion (Liben &Yekel 1996). In that context, map *construction* may provide an environment for creating and understanding *representations* of such spatial concepts in order to use them for negotiating about the concepts themselves. As Kaput et al (2001) put it:

> 'People need to represent for themselves how things work, what makes systems fail and what would be needed to correct them. This kind of knowledge is increasingly important;

it is knowledge that potentially unlocks the mathematics that is wrapped invisibly into the systems we now use and yet we understand so little of' (p. 16).

This research draws from two perspectives in analyzing the problematic in students' understandings of spatial concepts and representations. One has to do with the symbol–referent relationship and specifically what we could refer to as the *distance* between the two, i.e. the level of abstraction needed to relate surface structure (signifier) and deep structure (signified). In discussing algebraic notation, Kaput (1987) stressed the importance of students' abilities in choosing, building and interpreting mathematical representations and their difficulty in relating these to the represented concepts, rather than merely focusing on the surface rules of the representations themselves. Research with younger children involved attempts to narrow the distance between symbol and referent by means of using tangible objects to represent real objects in space (e.g. a model of a classroom and the objects inside it, Lieben and Yekel, 1996). DeLoache et al, (1999), however, found that the children have difficulty dissociating from the tangible properties of the representational objects in order to relate them to the represented ones and displayed better understanding of iconic representations which were more distant to the referent space. DiSessa studied primary students' bottom-up methods of building abstract representations (graphs) in order to negotiate and explain experiences with physical phenomena (diSessa, 2000). On the other hand, however, Nemirovsky et al (1998) suggested that fusion between signified and signifier can be a source of intuition for understanding representations, in environments designed to encourage linking kinesthetic experiences with their immediate iconic representation on a computer screen. In our study we had two groups of students negotiating to understand and build on representations of space by providing one group with tools to interpret and create iconic representations and by placing the other in an environment constituting the referent space itself.

The other research perspective relevant to our study has to do with the spatial – geometrical concepts embedded in map construction and students' related intuitions. Even though Papert coined the term body-syntonicity (1980) in describing students' use of their intuitions about their own movement in order to understand motion and orientation on the plane, subsequent studies revealed different sets of intuitions employed in settings involving a different system of reference to the plane, such as the Cartesian (Lawler, 1985). Students find it particularly difficult to relate their intuitions about Topological, Cartesian and Intrinsic systems of reference (Kynigos, 1992). Most research, however, is based on learning environments focusing on the use of one system at a time and on its use for the understanding of geometrical figures and properties, rather than studying the ways in and the reasons for which students choose between systems to understand concepts related to location, and orientation. In this research, we focused on the students' respective choices in a setting where these reference systems were simultaneously available since they could choose to refer to external objects, to bodily movements and to relative positioning of external objects.

SETTING

The activities

Two pairs of seven year-old pupils collaborated for the joint construction of a map. One pair was the "base team", situated indoors in front of the computer screen. The other was the "roamers", following the teacher who carried the bulky G.P.S. Whole group discussions took place before and after each research session, involving setting out goals and experience exchanges. The two groups exchanged roles when a part of the activity was completed. The activity involved creating a map of the area around the school and then using it to play a treasure hunt game. That area had a number of identifiable objects, such as trees, water taps, waste bins, lamp - posts, the flag. The two groups had to communicate about the objects in the area so that the base team could place ready made or child - created icons on the corresponding locations on the screen. The rationale underlying the deliberate differentiation of pupils' perspectives while working on the same task was to pivot the activities around language exchanges (Nemirovsky et al, 1998). Before the map construction activity, the students engaged in two dot-to-dot type tasks. The first involved rubber cones placed on the playground and numbered icons of these on corresponding positions on the screen. The base team had to drive the roamer team from cone to cone in the right sequence (the roamers could not see the numbers) to form a star shape. The second involved the base team instructing the roamer team to place the cones in the corresponding positions to form the shape of a fish.

The computer environment

From a technical point of view, the environment was based on synergy between high accuracy G.P.S. (around 1 meter) and corresponding cartography software. The G.P.S. continually fed data to the computer via mobile phone so that the effect would be a dynamic synchronous correspondence between G.P.S. motion and orientation change and the respective changes of the roamer icon on the screen. The software involved a combination of a G.P.S. receiver component with a map and an icon editor component ('E-slate' software, see endnote). It was thus designed to embody a scalable representation of physical motion on the plane (the roamer icon moved leaving a linear trace and turned in the direction of the motion) in combination with the students placing pictures or self – constructed icons of objects in positions and orientations decided after negotiation with the roamers (see picture next to excerpt 1). The teachers or authors of the software configured the scale and orientation of the simulated plane so that it matched the base teams' experience of the physical space outside the classroom.

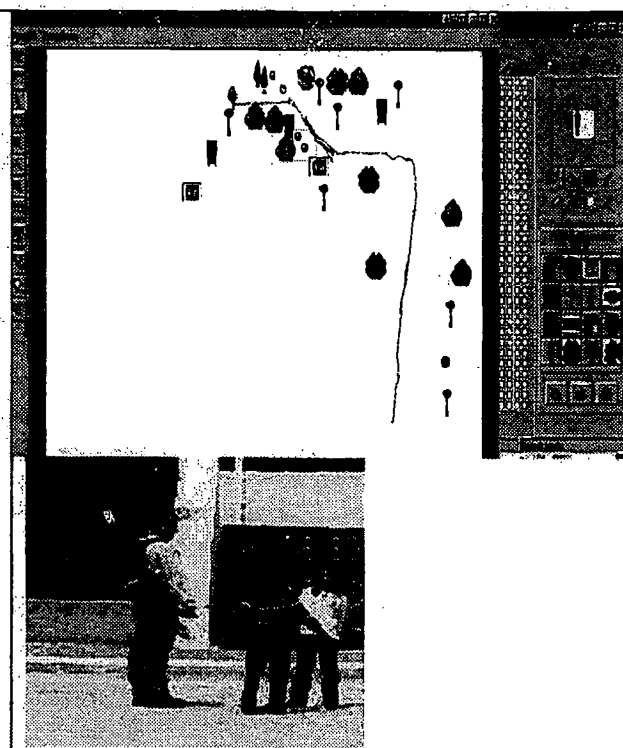
RESEARCH METHOD

The study lasted for 16 hourly sessions with the same seven year-old students, 3 girls and 1 boy. They were selected in collaboration with their teachers so that they ranked at the middle of their class with respect to their classroom general abilities, knowledge and willingness to collaborate and communicate. We adopted a generative methodological stance, attempting to gain some detailed insight into student - generated meanings. Both groups were videotaped simultaneously throughout all the sessions of the research. For the data collection we engaged in participant observation intending to stimulate students to make their thinking explicit as well as to challenge their actions and explanations. During the observations we focused on a) the verbal exchanges between groups and among the members of each group separately b) students' motions and gestures and c) the data captured on the screen. In our analysis we consider the exchanges between the groups in relation to the exchanges within each group. The discourse analysis focused on our subjective interpretation of the students' communicational intent and meanings rather than on some pre-defined 'objective' methodological construct.

USING THE REPRESENTATION AND ACTUAL SPACE AS RESOURCE

One of the key features of the learning environment requiring rigour in the children's exchanges was that one group used representation on the computer screen as resource for these exchanges while the other used their direct experience of the area in question. During the activity, neither team had access to the others' experience apart from being able to talk about it. The base team could also "see" the roamers' position and heading represented on the software. In accordance with our theoretical framework, its not surprising that at the beginning, the base team seemed bound to the representations on the screen, not being able to discriminate which of these were dependent on its orientation and which could be directly associated with the roamers' physical experience. This was manifested in the language they used in their attempts to communicate as shown in the following excerpt. S1, S2 are the base team, S3, S4 are the roamers, and wt stands for talk into the walkie-talkie. S2 used the word "downwards" signifying the literal orientation of the screen. S1 asks S2 to clarify some end point as direction finder, in effect requesting for an external reference point (line 2). The roamers' also take S2's comment literally thinking she's asking them to sit on the ground. However, since this request did not seem reasonable, the need to re-negotiate words and meanings arose. S1 tries a different idea, linking the requested heading change to the roamers' present one, in effect, using an intrinsic frame of reference by taking the roamers' point of view (line 7). From a mathematical point of view, this is correct. S2 cannot see that, bound to the upright orientation of the screen. S1 tries to mediate between the two parties by eliminating the word "downwards" which was explicitly causing confusion. However, she uses the word "diagonally" indicating that she too is bound to the screen representation, even though in her previous comment she seemed to have disentangled from the screen addressing the roamers' heading alone.

1. S2wt: As you are [means keep the heading you have] move downwards a bit.
2. S1: Where to?
3. S4: Should we sit on the ground?
4. S3: What did she say?
5. S4: As you are move down a bit.
6. S4wt: What do you mean? How? Towards the ground? [waiting]
7. S1: Move a bit towards the direction you are looking at.
8. S2wt: Move left, diagonally down.
9. S4wt: What do you mean move down?
10. S1wt: Move left diagonally.



Excerpt 1

At another session, this issue arose again, provoking more articulate language in the children's attempt to negotiate a request for a roamer move (excerpt 2). S3 indicated the roamers' position in space in terms of her perspective (line 11). S2 appears to try to relate the referent space with the next target point on the screen and to make an inference on what the change of heading should be (line 13). S1, on the other hand seemed to prefer to use external points of reference (using the word "towards" and pointing, line 12) but with no indication that she was thinking about the external space and not just the screen. However, her adamant comment that "down" was not communicable (line 16), brought S2's comment on the walkie-talkie, requesting a change in direction and position using intrinsic terms of reference associated with the roamers' perspective (line 19).

11. S3wt: We are looking towards the flag
12. S1 They should go towards there [indicates the point on the screen with her finger]
13. S2: So if they are heading to the flag they should move with their left hand [he means towards their left hand side] down
14. S1: Yes
15. S2: What?
16. S1: Down, they won't understand
17. S2: So, turn on your left
18. S1: And move a bit

19. S2wt: Turn left and walk four steps

Excerpt 2

The specific episodes we presented here along with the body of our data indicate that there were instances where students attempted to relate symbols and referents and vice versa even though they initially referred to spatial concepts with expressions bound to the specific resources (line 4). Even though the students had to grapple with a set of representations with varying distances between symbol and referent, the dynamic link between physical motion and its representation and the continual negotiations to relate symbol and referent provided them with rich opportunity to develop rigorous communication and generate corresponding meanings.

USING MULTIPLE REFERENCE SYSTEMS

As mentioned before, the environment was set up so that the children would have opportunities to use concepts which we may assign to three different representational systems, the projective or intrinsic, the Euclidean and the topological. In line with previous research, we were expecting the children to use concepts from all three, but were interested to investigate whether they might show preference in using one system over the others and in the case where a child would use concepts from more than one, what was it that triggered their choice. Furthermore, we were interested to study the ways in which they made relations between concepts belonging to different systems, having in mind that the previously mentioned research has shown this to be particularly difficult. Our data indicates that children consistently used concepts from all three systems. The excerpts we present here illustrate this use. In the following excerpt for example, S2 seems to employ a combination of intrinsic and Euclidean concepts, since she uses the term "left" and the term "diagonally". With respect to the latter, however, the dialogue indicates that it was the tangible size of the screen representation which enabled some logical connection between the two concepts. It may have well been that her intention was to find words for the concept of a 45 degree angle turn, and since the children did not know about degrees or angle size yet, it was the next best way to convey that meaning. S3 seems to consider the terms "left" and "diagonally" as contradictory, having a notion of left as a right angle turn and not being able to see himself walking on the edge of an area so that its diagonal would have meaning. However, that specific moment would have been an opportunity to help the children focus on the meaning of turn size and of the scale of the area they were mapping. It important to consider this, since the whole activity was replete with such events.

20. S2: Go left diagonally. Over
21. R2: Look at the agent. Ok? Look also at the agent.
22. S3: Left. Diagonally. I didn't understand a thing.

Excerpt 3

So, this use of concepts from different systems in an ad hoc fashion brought about the need for rigour in the children's communication, since the result of what they jointly understood was immediately visible on the screen. In amongst the occasions where a combination of concepts were used, there were instances of expression of the respective mathematical meaning. The following excerpt is such an example. In excerpt 4, S3 used a combination of an intrinsic ("move left") and a Topological ("not the first but the second cone") notion. He does not use some screen – bound representation and addresses the cones (which a mathematician would perceive as standing for points on the Euclidean plane) as a means to specify the direction he wanted S2 to turn towards. In doing that he makes a relative use of two cones.

23. S3wt: Move left towards not the first but the second cone
24. S2: Left, which means this way [shows left hand]

Excerpt 4

Finally, during the map construction activities, there were many instances where topological relations between objects came into play (Excerpt 5).

25. S3 wt What is next?
26. S1 What did we tell them before?
27. S2 About the water taps...
28. S1 They have to put the trash can and the lamp post
29. S2wt: ... In front of the water-taps there is a trash-can and further ahead there is a lamp –post.

Excerpt 5

In fact, S2's comment is quite difficult to disentangle, since she uses the water –taps as a central reference point to place the other two objects taking for granted that her own position and heading are clear and that the base group can associate her comment with that. This may mean that it was hard for S2 to conceptualise the topological relation between the objects she could see in conjunction to her own intrinsic position and heading. The excerpts in this section, however, indicate that the environment provided rich opportunity for the use of concepts from all three reference systems and in this way invited appropriate didactical engineering for the children to create relations amongst these.

CONCLUSION

The learning environment seemed to provide rich opportunity for the children to negotiate spatial concepts during the co-construction of the map. With respect to the kinds of intuitions employed for referring to position and orientation changes and to objects and the relative positions, we did not detect some preference with respect to system of reference. It would be interesting, however, to study over more time perhaps what kinds of strategies and connections children might make amongst concepts associated with the three systems. The interdependency between symbols and referents inherent in the learning environment played an important role in students' understandings of relationships between the two. In this sense, relating symbol and referent in meaningful settings with carefully designed dynamic representational tools may play an equally if not more important role in understanding and creating representations to the distance between these and the corresponding concepts. This may provide some explanation of the seemingly disparate views of di Sessa and Nemirovsky et al described earlier.

NOTES

C Cube: Children in Choros and Chronos, European Commission, Esprit LTR, Experimental School Environments, #29346, 1999-2000. The cartography software was co-developed by the authors with the E-slate software (http://e-slate.cti.gr). In a version for high school students, the object icons can be linked to information on a student created database and scaling of building perimeters can be implemented through variable Logo procedures dynamically manipulated with a variation tool.

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LEARNING FROM LEARNERS: ROBUST COUNTERARGUMENTS IN FIFTH GRADERS' TALK ABOUT REASONING AND PROVING

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I, a teacher-researcher, presented my fifth graders with an interesting but incorrect student-constructed proposal first seen in 1996 (Zack, 1997). The students used the patterns they had detected while solving the chessboard task to formulate counterarguments. The five types of counterarguments which emerged offer insights into the children's understanding of the mathematics. The children's perspectives, in turn, changed my understanding in substantive ways.

For the past ten years, as a teacher and researcher in a fifth grade classroom, I have been studying the problem-solving work and talk of ten to eleven year-old learners, with a particular interest in recent years in the children's notions of proving. This study is a follow-up to a study reported previously at PME (Zack, 1997). In that paper I showed how, in solving a variant of the chessboard problem, one team is convinced that their patterns work, and they use what they know of their patterns to refute an interesting but incorrect proposal put forth by the other two group members, Ted and Ross. I subsequently decided to offer the Ted-Ross idea to all of my fifth grade students for consideration (1999-2000, 2000-2001). My aim was to see whether, and if yes, when and how the fifth graders would refute this idea. It has been noted that "'wrong' ideas can be opportunities for important mathematical discussions and discoveries" (NCTM, 2000, p. 191). The authors of the recent NCTM (2000) *Principles and standards for school mathematics* envision children constructing valid arguments and evaluating arguments of others, detecting fallacies and critiquing others' thinking, and reasoning about mathematical relationships, such as the structure of a pattern (p. 188). I will describe the diverse counterarguments which emerged, and dwell on one in particular, the "array strategy" counterargument, which surprised and intrigued me. I have shown elsewhere (Zack, 1996, 1997, in press) and will also point out here in regard to the children's "array strategy," that the children's ways of seeing have changed my understanding in fundamental ways.

THE SCHOOL COMMUNITY AND CLASSROOM SETTING, AND ASSIGNED TASKS

St. George's is a private, non-denominational school, with a middle class population of mixed ethnic, religious, and linguistic backgrounds; the population is predominantly English-speaking. The homeroom class size in the 1999-2000 year was 27, in the 2000-2001 year was 26; the work, however, is always done in half-groups (13/14 children in each group) of heterogeneous ability. Problem-solving is at the core of the mathematics curriculum in this classroom. The school and classroom is one in which the children are expected to publicly express their thinking, and engage in conjecture, argument, and justification. The groundwork laid during the year by the teacher included an expectation that the children would be looking for patterns, and that they could be nudged to think about the mathematical structure underlying the pattern (Zack, 1997).

Mathematics class periods are 45 minutes, and twice a week are extended to 90 minutes. In addition to the in-class problem-solving sessions, each week the children also work on one challenging problem for which they are expected to record their work and reflect on their strategies. They write in a Math Log which serves as the initial basis of their group discussions in class. In class much of the session is conducted by the children as they discuss the problem first with a partner, then in a group of four or five, and finally with the entire group of thirteen students.

The children are videotaped throughout the school year as they work in their groups. In addition to the videotape records, data sources include focused observations, student artifacts (Math Logs), teacher-composed questions eliciting opinions (written responses), and retrospective interviews.

The mathematical context of the problem/discussion

The task is a variant of the 'Chessboard' problem (see Mason, Burton & Stacey, 1982). The work was assigned as follows:

> **Task 1:** Find all the squares [in a four by four grid given as a figure]. Can you prove you have found them all?
>
> **Task 2:** What if . . . this was a 5 by 5 square? How many squares would you have? Extensions were subsequently posed.
>
> **Task 3:** What if this was a 10 by 10 square? What if this was a 60 by 60 square? How many squares would there be?

Interesting talk about proving arose with the 'What if it was a 60 by 60 square?' task. The proposal that 2310 would be the answer for the sum of squares for a 60 by 60, what I have called the Ted-Ross strategy, was first seen in 1996 (Zack, 1997), and then came up again in subsequent years. In the 1999-2000 and 2000-2001 school years, I distributed to all the students in the class the proposal which Ted and Ross made in 1996, and asked the students to respond. It was presented to each pair in late April after they had completed their discussion about their work done independently in their Math Log on the task, 'How many squares would there be if it were a 60 by 60 square?' The 'Ted-Ross question' was posed as follows:

> Imagine that two of your classmates, Ted and Ross, came up with the following solution for the 60by60: The answer for the 10by10 grid is 385 squares. So take the answer for the 10by10 square (385) and 10 x 6 = 60 so multiply 385 x 6 = 2310 and you have the answer for the 60by60. What would you say?

I will speak first about the different kinds of counterarguments, and what the ten to eleven year-old students draw upon to shape their counterarguments.

FIVE COUNTERARGUMENTS

In the first study (Zack, 1997) Will, Lew, and Gord, in 1996, constructed three counterarguments (CA), namely CA #2A, #2B, and #1, in that order, in their quest to refute their partners', Ted's and Ross's, idea. A number of other counterarguments emerged in subsequent years, and all are described below using categories outlined in late 2001.

CA #1: 60 by 60 gives you 3600 squares of the smallest size, the 1by1 squares (not counting the others) and 3600 is already larger than 2310.

CA #2: The growth pattern is one in which the pattern of differences between the numbers grows as the numbers get bigger. There were a number of counterarguments which dealt in some way with the idea that the numbers increase in size, and are numbered 2A through 2D below.

(2A) The answer for the [number of squares in the] 10by10 (which is 385) is not double the answer for the [number of squares in the] 5by5 (which is 55). If you could just double the answer for the 5by5 to get the answer for the 10by10, the answer for the 10by10 would be 110, and it is not. (There were variants on this as the children chose different pairs to present their arguments, e.g. the 4by4 and the 8by8, the 3by3 and 6by6, but the reasoning was the same.) This counterargument proved hard to express. A few of the students (5, and 2 in the respective years) could not understand Ted and Ross's method. The majority however did understand Ted and Ross's method, and understood as well the connection between Ted and Ross multiplying by 6, and the presenters of the counterargument choosing to show that when one multiplied by two it did not work.

(2B) The pattern does not stop at 385 and then 'restart' itself. It keeps growing.

(2C) Look at the string of numbers (1x1, + 2x2, + 3x3 . . .). By the time one reaches a certain point, for example, the 20by20 square, the answer for the total number of squares in the 20by20 was a number already greater than 2310.

(2D) Refer back to a task previously done in class, one which had a similar growth pattern. For example, in the task 'What is the number of diagonals in an n-sided polygon?' one could not take the answer for the number of diagonals in a 10-sided polygon, multiply that number by 2 and get the answer for the number of diagonals in a 20-sided polygon.

CA #3: If one takes a 10by10 grid and lays six of them side by side, representing what Ted and Ross are suggesting, one omits counting all the different-size squares within each grid, and the squares which overlap the (10by10) grids. (This counterargument was offered only twice, by Jake in 2000, and by Dora, in 2001)

CA #4: There are 51 times 51 10by10's in a 60by60. This statement was offered once only as a counterargument, by Jake in May 2000, immediately after he presented CA #3. This counter argument will be elaborated below.

CA #5 If theirs (Ted and Ross) works, you should be able to take the answer for the 6by6 (91) and multiply it by 10 and it should give you 2310, but it does not. One student, Theo, presented this argument, in 2001.

The distribution of the different kinds of arguments is presented in Table 1. At times a student constructed more than one counterargument. In most cases it was 2 counterarguments, and in one case 4 (May, 2000), in the years 1999-2001. There is therefore a discrepancy between the total of counterarguments, that is, 20 and 23, and the total number of students who offered them, that is, 13 and 18, in 2000, and 2001. The total class size each year was 27 and 26 children respectively.

	CA #1		CA	#2		CA #3	CA #4	CA #5	Total
		#2A	#2B	#2C	#2D				
1999-2000	6	4	3	5		1	1		20
2000-2001	5	8	4	2	2	1		1	23

Table 1: Distribution of the five counterarguments

There was potential seen for a sixth counterargument, but the presentation of that counterargument did not materialise. In this case, the child, Kate, used inductive reasoning, appeal to empirical evidence, and reference to authority to support her way of arriving at her answer for the 60by60. She stated that "Leo's way", that of adding 1x1 plus 2x2 plus 3x3 and so on, as used by her classmate Leo, had been tried and tested for many examples by many of the students and it had proven true. She rejected the Ted-Ross strategy, giving the following reasons: "They didn't prove it and they didn't check it. They can't prove it's right." However, she did not then proceed to offer a counterargument to disprove their method. Had she made the move, she might have pointed out that using Leo's way and with the help of the calculator, she had arrived at a sum far larger than 2310. (In fact, Kate had arrived at the correct answer, 73 810, one of only three students to have the correct sum.) Although she could not be certain that her answer was correct, she might have insisted that the discrepancy in size deserved attention and deliberation. The point she might have made was both that the Ted-Ross method does not work, as the answer must be a far larger number, and that the Ted-Ross method is not needed because they are convinced that Leo's way works and will result in the correct answer (provided that the calculator work is done without error).

The benefit of having diverse counter-arguments

Some might say that one counter-example is all one needs. Others might suggest that one should consider in the main the most elegant one. However, having a diverse assortment of counterarguments is beneficial. The counter-arguments tell us different things about the mathematics in the task. A look at the counterargument helps me as a teacher to come to a better understanding both of the mathematics, and of the children's understanding of the mathematics. CA #1, that 3600 is already bigger than 2310, is the 'simplest' and some of the children voice their appreciation of it as the most powerful. Mark, who had used the Ted-Ross strategy in his independent work, states that he was quickly convinced by his partner Chris's refutation: "I did what the person did on the sheet and he [Chris] proved me wrong. . . . [Chris] ended up showing me in like one rather small sentence that it was wrong, that sixty times sixty is three thousand six hundred." CA #2 and the talk about growth rates reveals an understanding of the mathematics pattern which is at work here. The children at times themselves refer to the fact that CA #2A is difficult to articulate. However, those who propose and those can understand CA #2A -- and try to show for example that the answer for the 10by10 is not 'double of 'the answer for the 5by5 -- reveal an understanding of the method proposed by Ted and Ross, and its underlying structure.

The students "learn to describe relationships that hold across many instances and to develop and defend arguments about why those relationships can be generalised and to what cases they apply" (NCTM, 2000, p. 190). Lew, for example, builds upon what he knows of the number of little squares in the 5by5 square, and applies it to the 60by 60. This constitutes the foundation of CA #1. Lew explains the connection and justifies how he and his partners arrived at the number 3600 (of the littlest size square); while gesturing to both Gord's and Will's Math Logs, he points out: "Because here you would do five times five to see how many little squares there are, so we did sixty times sixty . . ." The students also push for reasons. When Adele keeps saying that the Ted-Ross strategy sounds good, stating: "I think it's right but I don't know how it works," Maggie (who has just finished presenting CA #1 to her group of five) insists: "Then you have no reason to think it's right." Soon after she repeats again: "But Adele, really, I think you should have a reason to think it's right."

The students who have generalised a rule after testing it, use what they know and trust it. Reid (in press) has shown for example how Will (from the Zack 1997 study) formulates a conjecture about one of the patterns he has seen, tests it, and once he feels sure that the pattern works, he is prepared to generalise it as a rule. Tom Kieren spoke recently about a "truth box" in relation to cases in which the children are secure with the knowledge, and do not feel they have to continue to test (Gordon Calvert, Zack, & Mura, in press). I hear them predict with confidence, and state that 'it will always work like this'. There are a number of patterns which are pivotal, all of which the children introduced to me. One, the 1, 4, 9, 16 'criss-cross' pattern (see Zack, 1997); two, adding 1 + 4 + 9 + 16 without having to worry about the number of squares (see Reid, in press); three, the pattern of differences between the above numbers a difference which increases by 2, i.e., 3, 5, 7 (Zack, 1997); and the "array strategy," a means whereby the children see that they can determine the number of squares there are of each particular size, which will be described below,.

The "array" strategy

The counterarguments are robust in that they are deeply rooted in the children's grasp of various aspects of the mathematics of the task. I have seen how the students formulate generalisations about observed regularities in regard to diverse patterns which they have detected (NCTM, 2000, p. 262). I will use as an example a counterargument which startled me, namely CA #4, when Jake said that there are 51 times 51 10by10s in a 60by60.

Jake's second counter-argument came swiftly upon the heels of his presentation of CA #3 without any overt deliberation or elaboration with his partner or small group members. This took me by surprise, and I had to sit down and figure out by working down from 60 times 60 1by1's, to 59 times 59 2by2's, etc., to 51 times 51 10by10's that it was indeed so. In the follow-up interview (May 23, 2000), when asked how he had come to that answer, Jake said at first that he could not recall, and two of his peers were asked what they thought, and how was it that they knew he was correct. Dexter said: "I visualised (the) others would be off the grid." Another clue to Jake's approach might be found in his partner Ari's explanation of his/their way. In answer to the question "Why does it go down by one?" (i. e., from 60 x 60, to 59 x 59 etc.), Ari said: "The size goes up by one row of squares so there's one fewer." Ari's reasoning introduced to me a new perspective: First there are 60 x 60 little 1by1 squares with none going off the grid. The next size square, the 2by2, is one square larger than the 1by1 going up and across, and therefore only 59 x 59 2by2's fit on the grid. Jake quickly dropped 9 squares down to get 51 x 51 10by10's, while I had to work it out one step at a time.

Jake may possibly have worked out yet another generalisation. He says the following during the interview as part of his response to how he arrived at the 51 x 51 10by10's:

> I remember Jennine showed there are four 2by2's [in a 5by5]. The one itself plus 3. There are 50 rows plus the one at the bottom that's 51.

Jake may have made the following connections, building upon Jennine's idea: In the 5by5, 5 minus 2 (the size of the square) gives you three more 2by2 squares which you can fit; three plus the first one gives you four 2by2's (along one dimension) in a 5by5. Hence there would be 4 times 4, or 16, 2by2's. In a 60by60, it would be 60 minus 10 (the size of the square) which gives you fifty more squares which you can fit on the grid; fifty plus the first one gives you fifty-one 10by10's in a 60by 60. Hence there would be 51 times 51 10by10's in a 60by60 (David Reid, personal communication, January 5, 2002). Jake found it difficult to articulate how he had arrived at his counter-argument of 51times51 10by10's. Doubtless, articulation of the idea is challenging because the procedure itself is complex

For the past three years a number of students have 'seen' and have attempted to explain in writing and in talk their strategy of how to figure out how many squares of different sizes are contained within a large square, the "array strategy." These students are seen to develop and test conjectures about mathematical relationships. They work with generalisations they have made while doing their first tasks; they apply to the larger square, the 10by10, the generalisations about how many squares one can fit along two dimensions, that is both vertically and horizontally. For example, Clare in discussing the 10by10 showed how she arrived at 64, or 8 times 8, squares of the 3by3 size, saying: "It would only be 8 [3by3's], cause you would go off the grid otherwise." I saw this "array strategy" for the first time in 1999 in one child, Walt's, work. In the past two years, seven and two students respectively have independently or with partners constructed and used this 'rule' about the array to arrive at their answer for how many there are of each size square. However, Jake was the only student seen to use this 'rule' to counter and refute.

As a result of the children's work I am constantly developing my own knowledge of children's thinking and my understanding of the mathematics in the task. My learning trajectory of the "array strategy" had its genesis in Lew's sliding of each size square across each row, from left to right (proof by cases) in 1996; was extended by students seeing the array as a way to arrive at the answer for the number of each-size square (Walt, 1999); and was further extended by Ari and Jake's knowing why it works that way (2000). This informs my work with future cohorts of children, in a continuing cycle of generative growth (Franke, Carpenter, Levi & Fennema, 2001).

The teacher's role

The children who invented the "array" procedure have a tacit but not conscious awareness, and work ahead of formal instruction; indeed, in this case they were teaching me. One of my enduring questions is in regard to the role the teacher should play in making more explicit what some of the children are doing naturally. Richard Pallascio suggested recently that the teacher is instrumental in making certain aspects visible (Gordon Calvert, et al., in press). The students use sophisticated reasoning, but may well not see the power in the reasoning they are doing. The teacher can bring to the surface some of the implicit items. She can point out the mathematical concepts the students have used perhaps without being conscious of them, and the types of arguments they have used. The teacher can come back to look at what the students have said, and to connect their talk with the ways in which a mathematician would express those ideas. My personal goal as teacher lies in encouraging the children to keep in touch with their personally meaningful ways of seeing, while valuing and learning the conventions of their culture. As I have elaborated elsewhere, the challenges inherent in that endeavor are substantial (Zack, 1999). Most important is to encourage the students to stay connected to the meaning they make personally.

Acknowledgements: I am indebted to David Reid whose provocative and insightful involvement have helped me both with the project itself, and with this paper. This research was supported by a Social Studies and Humanities Research Council grant from the Government of Canada #410-98-0427.

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21 - 26 July 2002
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