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AUTHOR	Bartolini Bussi, Maria G.
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ABSTRACT

This paper discusses some findings of research studies developed in Italy on the approach to theoretical knowledge. Two examples are discussed in detail: (1) a classroom discussion concerning the shift from an empirical to a theoretical compass in primary school (fifth grade); and (2) the scheme of a teaching experiment on overcoming conceptual mistakes by an original recourse to Socratic dialogue (grades 5-7). (Contains 19 references.) (KHR)

The Theoretical Dimension of Mathematics: A Challenge for Didacticians

Maria G. Bartolini Bussi
University of Modena and Reggio Emilia - Italy

Introduction

The aim of this presentation is to discuss some findings of a few research studies developed in Italy about the approach to theoretical knowledge. Four different teams have worked in a coordinated way for years, namely the teams directed by F. Arzarello (at the University of Turin), M. Bartolini Bussi (at the University of Modena), P. Boero (at the University of Genoa) and M. A. Mariotti (at the University of Pisa).[1]

The set of studies is based on the strict interlacement of several kinds of analysis, for designing purposes and for modeling the processes as well:

- first, the historic-epistemological analysis of ways of mathematical reasoning (with a special focus on proofs);
- second, the didactical analysis of the interaction processes developed in the classroom within suitable teaching experiments;
- third, the cognitive analysis of the processes underlying the production of reasoning and of arguments in proofs.

Whilst the first kind of studies might be considered as an heritage of the Italian tradition of studies on the foundations of mathematics and on the didactical implications, the second and the third are more related to the development of the international literature on the field.

To keep under control the complexity of the system, some theoretical constructs were assumed and/or produced in the research development. The early theoretical constructs concerned the setting of students' activity (*field of experience*[2]) and the quality of classroom interaction (*mathematical discussion*[3]). Exploratory studies were produced at very different school levels.

The teaching experiments of the coordinated project were aimed at creating suitable settings for most of learners (from primary to secondary and tertiary education) being able to develop a theoretical attitude and to produce proofs. The experiments shared some common features from the design phase to the implementation in the classroom:

- the selection, on the basis of historic-epistemological analysis, of fields of experience rich in concrete and semantically pregnant referents (e.g., perspective drawing; sun shadows; Cabri-constructions; gears; linkages and drawing instruments);
- the design of tasks within each field of experience, which require the students to take part in the whole process of production of conjectures, of construction of proofs and of generation of theoretical organization;
- the use of a variety of classroom organization (e.g., individual problem solving, small group work, classroom discussion orchestrated by the teacher, lectures);
- the explicit introduction of primary sources from the history of mathematics into the classroom at any school level.

Teacher participation helped to determine activity in each phase (design, implementation, collection of data and analysis). Their sensitivity and competence proved to be essential not only in the careful management of classroom activity but also in the elaboration of analytical tools and of the theoretical framework: in a word, they have become *teacher-researchers*. Last but not least, while taking part in the design of experiments, the teachers were put in the condition of deepening some issues concerning the theoretical dimension of mathematics and its relationship with experiential reality. In other words, the theoretical dimension of mathematics became part of the intellectual life of teachers, an essential condition, as they were expected to be able to foster the development of similar attitudes in their pupils.

The outcomes of the teaching experiments were astonishing, if compared with the general plea about the difficulty (or the impossibility) of coping with the theoretical dimension of mathematics. Just to quote a case, most of students even in compulsory education (e.g., Grades 5-8) succeeded in producing conjectures and constructing proofs (Bartolini Bussi et al., 1999) in the setting of the modeling of gears. Were these studies action research based, the process could have stopped here with the production and documentation of facts, i.e., paradigmatic examples of improvement in mathematics teaching. But an additional aim of 'research for innovation' concerned the study of the conditions for realization as well as the possible factors underlying effectiveness; in other words, this success had to be treated as a didactic phenomenon (Arzarello & Bartolini Bussi, 1998). This created the need of framing in an explicit way the existing studies within a theoretical framework, that might have interpreted them in a unitary way and might have suggested issues for a research agenda. The general framework was enriched by two specific additional theoretical constructs, elaborated on the basis of epistemological and cognitive analysis: the idea of *mathematical theorem*[4] and the idea of *cognitive unity*.[5]

In this presentation two examples will be discussed in more details:

- a classroom discussion concerning the shift from 'empirical' to 'theoretical' compass in primary school (5th grade);
- the scheme of a teaching experiment about overcoming conceptual mistakes by an original recourse to Socratic dialogue (5th - 7th grades).

Both examples concern young pupils, to emphasize also the need of starting quite early to nurture the theoretical approach to mathematics.

First Example: From 'Empirical' To 'Theoretical' Compass (5th Grade)

The Protocols

The pupils have been given the following individual problem on an A4 sheet: *Draw a circle, with radius 4 cm, tangent to both circles. Explain carefully your method and justify it.*

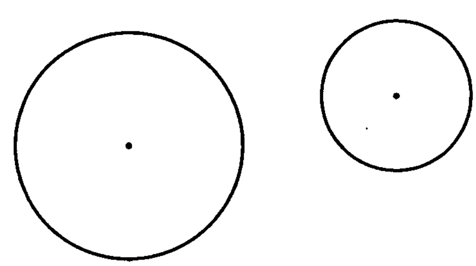


FIGURE 1 [The circles have radii 3 cm and 2 cm and the distance of their centres is 7 cm.]

All the pupils of the classroom have produced a solution by trial and errors, adjusting a compass to produce a circle that looks tangent to both. Some of them have found two solutions (symmetrical). The teacher (Mara Boni) collects all the individual solutions, analyses them and, a week later, and gives all the pupils a copy of Veronica's solution. Then she introduces the theme of discussion.

Veronica's solution:

The first thing I have done was to find the centre of the wheel C;

I have made by trial and error, in fact I have immediately found the distance between the wheel B and C. Then I have found the distance between A and C and I have given the right 'inclination' to the two segments, so that the radius of C measured 4 cm in all the cases. Then I have traced the circle.

Veronica's justification:

I am sure that my method works because it agrees with the three theories we have found:
i) the points of tangency H and G are aligned with ST and TR;
ii) the segments ST and TR meet the points of tangency H and G;
iii) the segments ST and TR are equal to the sum of the radii SG and GT, TH and HR.

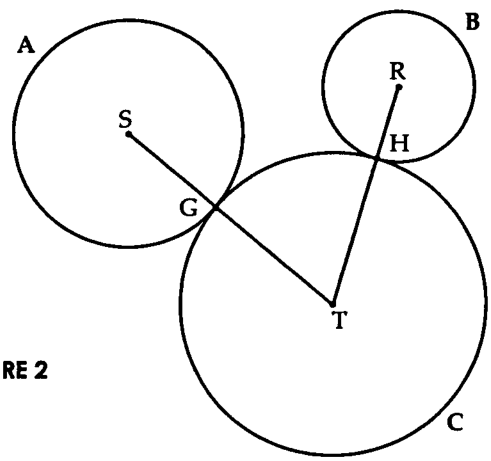


FIGURE 2

The classroom discussion:

Teacher: *Veronica has tried to give the right inclination. Which segments is she speaking of? Many of you open the compass 4cm. Does Veronica use the segment of 4cm? What does she say she is using?* [Veronica's text is read again.]
Jessica: *She uses the two segments ...*
Maddalena: *.. given by the sum of radii*
Teacher: *How did she make?*
Giuseppe: *She has rotated a segment.*
Veronica: *Had I used one segment, I could have used the compass.*
[Some pupils point with thumb-index at the segments on Veronica's drawing and try to 'move' them. They pick up an ideal segment as if it were a stick and try to move it.]
Francesca B.: *From the circle B have you thought or drawn the sum?*
Veronica: *I have drawn it.*
Giuseppe: *Where?*
Veronica: *I have planned to make RT perpendicular [to the base side of the sheet] and then I have moved ST and RT until they touched each other and the radius of C was 4cm.*
Alessio: *I had planned to take two compasses, to open them 7 and 6 and to look whether they found the centre. But I could not use two compasses.*
Stefania P.: *Like me ; I too had two compasses in the mind.*
Veronica: *I remember now: I too have worked with the two segments in this way, but I could not on the sheet.*
[All the pupils 'pick up' the segments on Veronica's drawing with thumb-index of the two hands and start to rotate them. The shared experience is strong enough to capture all the pupils.]

Elisabetta [excited]: *She has taken the two segments of 6 and 7, has kept the centre still and has rotated: ah I have understood !*
Stefania P.: *... to find the centre of the wheel ...*
Elisabetta: *... after having found the two segments ...*
Stefania P.: *... she has moved the two segments.*
Teacher: *Moved? Is moved a right word?*
Voices: *Rotated .. as if she had the compass.*
Alessio: *Had she translated them, she had moved the centre.*
Andrea: *I have understood, teacher, I have understood really, look at me ...*
[The pupils continue to rotate the segments picked up with hands.]
Voices: *Yes, the centre comes out there, it's true.*
Alessio: *It's true but you cannot use two compasses*
Veronica: *you can use first on one side and then on the other.*
Teacher: *Good pupils. Now draw the two circles on your sheet.*
[All the pupils draw the two circles on their sheet and correctly identify the two possible solutions for the centres.]

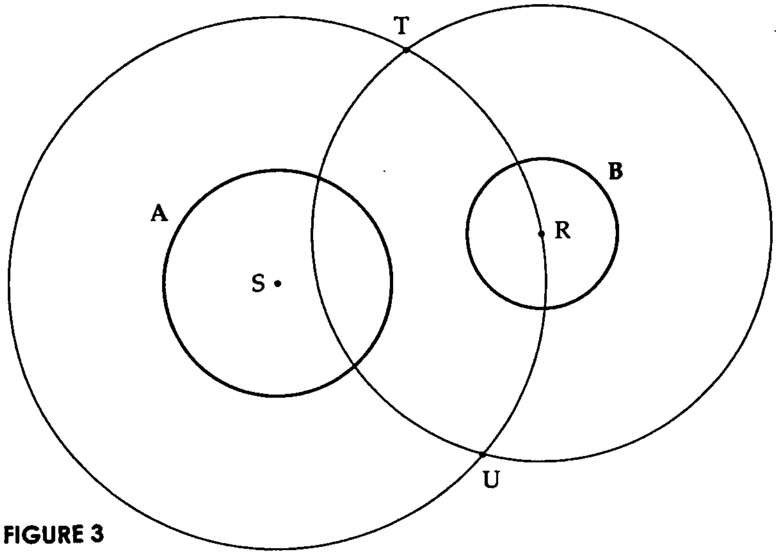


FIGURE 3

Discussion of the Protocols

The reader might be astonished to think that this problem is given in primary school; s/he might be even more astonished if s/he knew that this classroom is in a district with a very low socio-cultural level. We might offer some elements to understand better, before starting the discussion of the protocols.

First, the classroom is taking part in a long term teaching experiment from the 1st grade, concerning the modeling of gears. The project has two aspects: the (algebraic) modeling of functioning (Bartolini Bussi et al., 1999) and the (geometrical) modeling of shapes (Bartolini Bussi et al., to appear). For the pupils, (toothed) wheels have been naturally modeled by circles and wheels in gear with each other have been modeled as tangent circles (the evocation of wheels appears also in Veronica's protocol). They have discovered by experiments and transformed into fundamental statements some elementary properties of tangent circles, such as the alignment of the two centres and the point of tangency and the related relation between the distance of centres and the sum of radii.

Second, the pupils are capable to discuss effectively with each other, as their teacher is a member of the research team on *Mathematical Discussion*, which analyses the limits and advantages of different types of discussion orchestrated by the teacher (Bartolini Bussi, 1996; 1998) and has introduced in that classroom mathematical discussion since the 1st grade. The pupils are also accustomed to produce detailed individual written protocols, with a clear explanations of their processes: this custom too is the outcome of a very special classroom culture, where individual tasks and discussions on the individual strategies are systematically interlaced with each other.

Third, the pupils are acquainted with the use of the compass to draw circles more precisely than freehand. Between the 2nd and the 3rd grade they have worked for some sessions on the compass, first trying to invent their own instruments to draw circles and then appropriating the existing instrument 'compass' and the manual procedure (not so easy for young pupils) to use it effectively. They have used often the compass to produce round shapes, also in art lessons with the same teacher (to imitate some drawings by Kandinsky).

In this background, we may now discuss the protocols. In this episode we are real time observing the emergence of an enriched use of the compass, that is entering the classroom culture. The compass is going to be used not to produce a fair round shape, but to find a point (or better, two points) that are at a given distance from two given points. The way of using the compass (i. e. the gesture of handling and tracing the curve) is the same when a pupil wishes to produce a round shape and when s/he wishes to find a point at a given distance, but the senses given by the pupils to the processes (gestures) and to the products (drawings) are very different. When the compass is used to produce round shape, its main goal is communication; when the compass is used to find the points which satisfy a given relationship, it becomes an instrument of semiotic mediation (Vygotsky, 1978), that can control - from the outside - the pupil process of solution of a problem, by producing a strategy that (i) can be used in any situation, (ii) can produce and justify the conditions of possibility in the general case and (iii) can be defended by argumentation referring to the accepted theory. We shall reconsider this point later.

The geometric compass, embodied by the metal tool stored in every school-case, is no more a material object only: it becomes a mental object, whose use may be substituted or evoked by a body gesture (rotating hands or arms) or even by the product of the gesture, i. e. the drawn curve. Even if the link with the body experience is not cut (it is rather emphasized), the loss of materiality allows to take a distance from the empirical facts, transforming the empirical evidence of the drawing that represents a solution (whichever is the early way of producing it) into the external representation of a mental process. The (geometrical) circle is not an abstraction from the perception of round shapes, but the reconstruction, by memory, of a variety of acts of spatial experiences (a 'library' of trajectories and gestures, see Longo 1997).

A Short Interlude: Towards Semiotic Mediation

In the above episode we observed the integration of two ways of thinking of circles. Recalling the history of geometry, it is the integration of the mechanical/dynamic/procedural approach of Hero ('a circle is the figure described when a straight line, always remaining in one plane, moves about one extremity as a fixed point until it returns to its first position') with the geometrical/static/relational approach of Euclid ('a circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another'). In this case the standard compass is only the prototype of a larger class of instruments (drawing instruments) which were used for centuries to prove the existence of and to construct the solutions of geometrical problems and of algebraic equations as well (Lebesgue, 1950). The experience of the continuity of motion was in the place of the still lacking theoretical foundation of mathematical continuity.

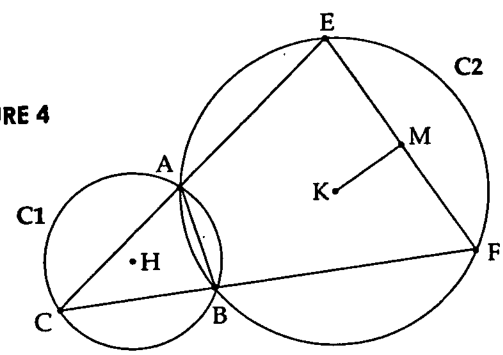
Hence, exploring the relationships between these two ways of thinking of circles and the role played by the compass is an epistemologically correct way to approach very early the problem of continuum. Even if the flow of the discussion is natural and fluent, the process is not spontaneous at all: it is evident the care of the teacher in choosing an effective protocol and in encouraging the use of gestures and the interaction between pupils with a deep exploration of the mental processes.

These are young pupils, yet the difficult relationship between the two ways of thinking of circles emerges also with elder students. We may quote a very interesting clinical interview (Mariotti et al. 2000), carried out with an 12th grade student, Giulia, familiar with the Cabri environment.

> ***Problem***
> *Two intersecting circles C1 and C2 have a chord AB in common. Let C be a variable point on the circle C1. Extend segments CA and CB to intersect the circle C2 at E and F, respectively. What can you say about the chord EF as C varies on the circle? Justify the answer you provide.*

Giulia has produced a right conjecture (if K is the centre of C2 and M is the midpoint of EF the segment KM does not vary in length) and justified it. In particular, she has proved, in standard way that, while C is dragged on the circle C1, the triangle EKF does not change in size. A very brief excerpt of the last part of the interview follows. The complete protocol with a detailed epistemological and cognitive analysis is in the quoted paper.

FIGURE 4



Giulia: *What is the geometric locus of the midpoint of EF as C moves on the circle? The geometric locus of the midpoint of EF .. the midpoint of EF.... if it doesn't vary in length... can I draw? [referring to the drawing with Cabri]*

Int.: *Sure...*

G.: *This is the midpoint ... when C ... well.. what does it mean? When C moves on the circle the midpoint [M] draws a circle around the centre, another circle with the same centre as the bigger one... shall I say why? If I need to say that it is a circle I must prove that it is always equidistant from the centre ... because.. well... I must prove that it is always equidistant from the centre ... well... at this point this distance here... why? ....*

Int.: *Try and use what you already know ... what you have just proved.*

G.: *Yes, if this segment has to be always.... it is always equal to ... it has always the same distance from the centre, because if you change the segment ... I mean... equal segments are those for which... I don't remember exactly the theorem, but equal segments are.... have the same distance from the centre ... equal segments on the same circle have the same distance from the centre... therefore if I prove ..I mean.. no I have already proved that that segment is always constant. [...] I haven't proved it because I haven't proved that this one rotates ...or something like that....[...] Now I must also say why the geometric locus is a circle, mustn't I? Shall I prove it?*

Int.: *Haven't you done it? you said that one always stays constant...*

G.: *stays constant....*
Int.: *How do you define a circle?*
G.: *I define it as locus.. you are right... locus of points equidistant from the centre... it crossed my mind that I had to prove also.... no.. maybe it is stupid ... that I had to prove that it was rotating around the centre....*
Int.: *Oh, right... that it was rotating....*
G.: *I mean, wasn't it something ...? if I say ... it is fine that it forms a ... but the circle... I can see that it forms a circle when I drag the point C and drag it around the circle [...]*
G.: *Mustn't I prove that it [M] rotates?*
Int.: *How can you prove that it rotates?*
G.: *Well.. I don't know.. actually that is the problem....*
Int.: *The fact is that once you proved that as C varies the triangle which has the chord as one side is always a constant triangle, you have proved that that one is a circle because it is the locus of points equidistant....*
G.: *Is that enough? Is that enough to show that it is a circle?*

Giulia feels that the dynamic features of the locus cannot be taken on board with the standard proof. Also the rotation of the chord EF is perceived as something to be proved. Giulia is not happy with the simple visual perception of the movement and she seems to feel the need to 'translate' it into a mathematical statement to be justified in some way. There is a sort of gap between Euclid's definition of circle Giulia knows and the perceived rotation of the line KM. The definition of circle as a locus of points having the same distance from a given point is perceived as detached from what happens in the Cabri screen: there is a point C whose movement directs the movement of the point M. The situation is similar to the one experienced by using a drawing instrument, where the directing point control the motion of the tracing point (Bartolini Bussi, 2000). The relationship of pointwise generation of curves by means of geometrical constructions to continuous generation of curves by means of instruments was debated for centuries (in ancient Greece and in the 17th century Europe as well) and enlightened when the meaning of continuity was clarified, in geometry and in calculus. It is worthwhile to remind that, until then, the continuous generation of a curve by a drawing instrument was used to prove the existence of and to construct the solutions of geometrical problems and of algebraic equations as well (Lebesgue, 1950).

Hence, the activity of the 5th graders who were building the link between pointwise and continuous generations of circles was very relevant from an epistemological perspective. After revisiting the standard compass, we might wonder whether other artifacts may be used effectively by teachers to nurture a theoretical attitude towards mathematics. The analysis of other artifacts is surely beyond the scope of this presentation. If the compass (and other drawing instruments) may be oriented towards the problem of continuum, the abacus may be oriented towards the polynomial representation of natural number and the perspectograph towards the projective extension of the euclidean plane and the roots of projective geometry. These examples concerns very intrusive old physical artifacts which explicitly require a physical (gesturing/handling) activity of the pupils. Other examples are offered by the recent development of 'virtual' microworlds: we may quote the approaches developed by Mariotti (and coworkers) in the Cabri setting (Mariotti et al., 2000) and in L'Algebrista, a symbolic manipulator created within Mathematica to introduce pupils to algebra theory (Cerulli & Mariotti, 2000).

All these examples are relevant cases of psychological tools or tools of semiotic mediation, as meant by Vygotsky[6]. It is worthwhile to remind that Vygotsky himself quotes, as examples, language, various systems for counting, mnemonic techniques, algebraic symbol systems, works of art, writing, schemes, diagrams, maps and *mechanical drawing* (including the use of drawing instruments). All these tools are, coherently with the cultural - historical approach of Vygotsky, taken from the history of mankind and are to be introduced into the life of young learners by an interactive practice guided by adults. This observation hints at the possible differences between these psychological tools and those teaching aids that are, very often, produced and sold by commercial agencies (think for instance of the multibase material for the representation of numbers): first the cultural - historical relevance; second the not obvious transparency of concepts to be conveyed, rather dependent of the practices realized by adults and young learners together (see Meira, 1994).

The power of the theoretical construct of semiotic mediation, borrowed from Vygotsky's work, is evident if we apply it to analyze briefly another teaching experiment, aimed at developing metacognition as the consciousness of the ways of overcoming conceptual mistakes.

Second Example:
Overcoming Conceptual Mistakes by Imitating a Socratic Dialogue (5th - 7th Grades).

This study has been carried out by a group of teachers-researchers coming from two different teams (Genoa and Modena), under the direction of P. Boero (see Garuti et al., 1999). It concerned the capacity to detect conceptual mistakes and overcome them by general explanation. The object of the experiment was a well known piece of the Plato's dialogue *Meno*, that concerning the problem of doubling the area of a given square by constructing a suitable square (this means overcoming the mistake which consists of doubling the side length). The crucial mediational tool is the Socratic dialogue, i.e., a dialogue intended to provoke crisis and then allow it to be overcome. In this framework, the excerpt concerning doubling the area of a given square is crucial as a practical demonstration. The dialogue consists of three different phases:

A. Socrates asks Meno's slave to solve the problem of doubling the area of the square by constructing a suitable square; the slave's answer (side of double length) is opposed by Socrates through direct, visual evidence (based on the drawing of the situation).
B. Then the slave is encouraged to find a solution by himself—but he only manages to understand that the correct side length must be smaller than three halves of the original length.
C. Socrates interactively guides the slave towards the right solution (achieved through a construction based on the diagonal of the original square).

Drawing on the text of the dialogue, a teaching sequence was organized as follows:

1. students were briefly informed about the whole activity to be performed; then they individually tried to solve the same problem posed by Socrates to the slave.
2. students, under the teacher's guidance, read and tried to understand the three phases of the dialogue; then they read the whole dialogue aloud (some students playing the different characters); finally, they discussed the content and the aim of the whole dialogue, trying to understand (under the teacher's guidance) the function of the three phases. After negotiation with students, a wall poster was put up summarizing the three phases in concise terms.
3. the teacher presented the students with some, possible mistakes that could become the object of a dialogue similar to Plato's, and they were invited to propose other mistakes.[7]
4. students discussed about the chosen mistake, trying to detect (under the teacher's guidance) good reasons explaining why it is a mistake, then trying to find partial solutions, and finally arriving to a general explanation.
5. students individually tried to produce a "Socratic dialogue" about the chosen mistake;
6. students compared and discussed (under the teachers' guidance) some individual productions.

It is beyond the scope of this paper to detail the findings, that are, by the way, framed by an original 'voices and echoes game', that cannot be described here. It is enough to say that 86 pupils (out of 114) showed to be aware that appropriate counter-examples can reveal a conceptual mistake. More than 50 pupils tried to give a general explanation of the mistake or to find a general rule, showing to be aware of the necessity of doing it. Last but not least, 19 pupils showed to be conscious of the way to overcome the chosen mistake and were able to guide the slave towards a general solution.

The results were astonishing for the exceptional ability in keeping the roles of Socrates and of the slave in the dialogue and for the good choice of counterexamples.

Comparison of the Two Examples

Beyond the young age of the pupils, the originality of the tasks and the astonishing results, is there any relationships between the two examples we have considered?

The relationship emerges if we compare them as tools of semiotic mediation.

The aims of the two activities concern higher psychological processes: what is into play is, for the compass, the theoretical nature of actions with a physical instruments and, for the Socratic dialogue, the overcoming of conceptual mistakes. For these aims original tasks are chosen: the task is, for the compass, the production of a method to find points according to the standard of geometric construction, and, for the dialogue, the production of a dialogue according to Plato's model. The tools introduced are rather standard: the compass is included in every school bag from primary school, and this excerpt of *Meno* is often used to introduce the problem of doubling the cube. However, what is original is the intentional and carefully designed kind of social practice, where the use of the tools is introduced. What is asked is not an instrumental use of them, rather it is the internalization of the activity with the concrete compass and of the model of the Socratic dialogue (that has the potentiality to transform them into psychological tools). The semiotic mediation is started in the collective phases (discussions orchestrated by the teacher, after or before the individual task), with a strong emphasis on imitation of gestures, words and, in the case of the dialogue, of the structure of the text itself. The teacher's role in guiding the whole process is essential. Hence, in spite of the big difference between the two artifacts, strong and deep similarities can be envisaged in the management of school activity. They can be summed up by three keywords: social interaction; teacher's guide and imitation. The resulting processes draw on a careful a priori analysis of the potentialities of the activity towards the development of the higher psychological process, required by the approach to theoretical knowledge.

Concluding Remarks

In this presentation, we have discussed two examples which are provocative for both teachers and researchers. The 'good' functioning of the above experiments might be analyzed by means of different tools:

- epistemological (to enlighten the nature of mathematics itself);
- historical (to clarify the process of constitution of a piece of knowledge to be taught);
- anthropological (to uncover hidden conceptions);
- cognitive (to interpret the nature of individual processes and the weight of students and teachers conceptions in shaping them);
- didactical (to elucidate the function of contexts and tasks and on the role of the teacher in classroom interaction).

The above list might be enlarged to enclose other issues. It is trivially clear that, on the one hand, every factor offers only a particular perspective and, on the other hand, a global analysis is needed for educational purposes.

We believe that the process of building a theoretical attitude towards mathematics is quite long and can last for years. In our framework, this process is developed under the guide of a cultured adult (the teacher), who, on the one hand, selects the tasks and, on the other hand, orchestrates the social interaction before or after the individual solution. For the learners, gaining a theoretical attitude does not cut the link with concrete (and bodily) experience, but rather gives a new sense to 'the same' concrete experience.

From a research perspective, this set of studies opens a lot of interesting questions. For instance, they concern the analysis of distinctive features of theoretical knowledge, at least when the didactical purpose is in the foreground; the listing of a larger and larger set of artifacts analysed as tools of semiotic mediation and the study of the effective introduction of these artifacts in the classroom; the study of the relationships between individual and collective processes in selected cases.

The discussion in Montreal has shown that linking material activity with theoretical reasoning and emphasising the teacher's guide and the role of imitation may clash against the deep beliefs of some mathematics educators. This is what happens when different classroom cultures meet each other.

Notes

1. The project was directed by F. Arzarello and funded by the MURST (Ministry of University and Research in Science and Technology).
2. *field of experience:* a system of three evolutive components (external context, the student's internal context and the teacher's internal context) referred to a sector of human culture which the teacher and students can recognize and consider as unitary and homogeneous (Boero et al., 1995)
3. *mathematical discussion:* a polyphony of articulated voices on a mathematical object, which is one of the motives of the teaching - learning activity (Bartolini Bussi, 1996)
4. *mathematical theorem:* the system of three interrelated elements: a statement (i.e., the conjecture produced through experiments and argumentation), a proof (i.e. the special case of a discourse that is accepted by the mathematical community) and a reference theory (including postulates and deduction rules); this conception emphasizes the importance that students are confronted with this complexity rather than with the mechanical repetition of given proofs (Mariotti et al., 1997).
5. *cognitive unity:* the continuity between the processes of conjecture production and proof construction. During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the previously produced arguments according to a logical chain' (Garuti et al., 1996). The cognitive unity is recognisable in the close correspondence between the nature and the objects of the mental activities involved. This theoretical construct was the base of further developments, playing different roles: a formidable tool for designing activities within the reach of students (Bartolini Bussi et al., 1999); a pointer of the difficulty, for analyzing activities and understanding some reasons for success and failure (Bartolini Bussi & Mariotti, 1999); a key for understanding the deep nature of proving tasks, as far as the immediacy of the solution is concerned (Mariotti et al., 2000). In the ongoing research, the issue of mental processes underlying the production of proofs and more generally the genesis of (abstract) mathematical objects is at the very core (see Bartolini Bussi et al., 1999, Arzarello, 2000).
6. A long quotation from Vygotsky may be useful here:

   Every elementary form of behavior presupposes direct reaction to the task set before the organism (which can be expressed with the simple S - R formula). But the structure of sign operations requires an intermediate link between the stimulus and the response. This intermediate link is a second order stimulus (sign) that is drawn into the operation where it fulfils a special function: it creates a new relation between S and R. The term 'drawn into' indicates that an individual must be actively engaged in establishing such a link. The sign also possesses the important characteristic of reverse action (that is, it operates on the individual, not the environment).

   Consequently, the simple stimulus-response process is replaced by a complex, mediated act, which we picture as:

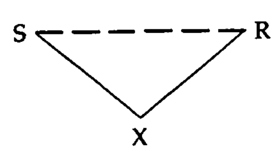


FIGURE 5

   In this new process the direct impulse to react is inhibited, and an auxiliary stimulus that facilitates the completion of the operation by indirect means is incorporated.

   Careful studies demonstrate that this type of organization is basic to all higher psychological processes, although in much more sophisticated forms than that shown above. The intermediate link in this formula is not simply a method of improving the previously existing operation, nor is a mere additional link in an S - R chain. Because this auxiliary stimulus possesses the specific function of reverse action, it transfers the psychological operation to higher and qualitatively new forms and permits humans, by the aid of extrinsic stimuli, to control their behavior from the outside. The use of signs leads humans to a specific structure of behavior that breaks away from biological development and creates new forms of a culturally-based psychological process. (Vygotsky, 1978)

7. Here is a sample of the three mistakes that were chosen in the six classes (5th or 7th grades):
   a. "By dividing an integer number by another number, one always gets a number smaller than the dividend" (7th grade)
   b. "By multiplying an integer number by another number, one always gets a number bigger than the first number" (5th grade)
   c. "By multiplying tenths by tenths, one gets tenths" (5th grade) (Garuti et al., 1999).

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