DOCUMENT RESUME

ED 471 763 SE 067 022

AUTHOR	Weber, Keith
TITLE	Developing Students' Understanding of Exponents and Logarithms.
PUB DATE	2002-00-00
NOTE	10p.; In: Proceedings of the Annual Meeting [of the] North American Chapter of the International Group for the Psychology of Mathematics Education (24th, Athens, GA, October 26-29, 2002). Volumes 1-4; see SE 066 887.
AVAILABLE FROM	ERIC/CSMEE Publications, 1929 Kenny Road, Columbus, OH 43210-1080. Tel: 800-276-0462 (Toll Free).
PUB TYPE	Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE	EDRS Price MF01/PC01 Plus Postage.
DESCRIPTORS	*Mathematics Instruction; Algebra; Concept Formation; Elementary Education; *Innovation; Logarithms
IDENTIFIERS	Exponents (Mathematics)

ABSTRACT

In this paper, we describe instruction designed to teach students about exponents and logarithms and report a pilot study to test the effectiveness of this instruction. Based on the theoretical work of Dubinsky and Sfard, we postulate a set of mental constructions that a student could make to understand the concepts of exponents and logarithms. We then describe computer and paper-and-pencil exercises to induce students to make these constructions. We report a pilot study assessing the efficacy of these exercises. Students receiving our instruction outperformed students receiving traditional instruction across a variety of measures, including performing basic computations, recalling formulae, explaining why rules of exponents and logarithms are true, and answering conceptual questions. (Author)

DEVELOPING STUDENTS' UNDERSTANDING OF EXPONENTS AND LOGARITHMS

Keith Weber
Murray State University
keith.weber@murraystate.edu

In this paper, we describe instruction designed to teach students about exponents and logarithms and report a pilot study to test the effectiveness of this instruction. Based on the theoretical work of Dubinsky and Sfard, we postulate a set of mental constructions that a student could make to understand the concepts of exponents and logarithms. We then describe computer and paper-and-pencil exercises designed to induce students to make these constructions. We report a pilot study assessing the efficacy of these exercises. Students receiving our instruction outperformed students receiving traditional instruction across a variety of measures, including performing basic computations, recalling formulae, explaining why rules of exponents and logarithms are true, and answering conceptual questions.

Introduction

Exponents and logarithms are important mathematical concepts that are useful for modeling and understanding population growth, radioactive decay, and compound interest. Further, exponential and logarithmic functions are central concepts for many college mathematics courses, including calculus, differential equations, and complex analysis. Unfortunately, research indicates that students' understanding of these concepts is quite limited (e.g., Confrey & Smith, 1995). In particular, students often forget many properties of exponents and logarithms shortly after they learn them and can seldom explain why these properties are true (Weber, in press).

While mathematics educators have proposed instructional techniques to supplement or replace traditional pedagogy of exponents and logarithms (e.g., Confrey & Smith, 1995; Rahn & Berndes, 1994), to our knowledge, the efficacy of these techniques has not been assessed. The purpose of this paper is to describe instruction designed to teach students the concepts of exponents and logarithms and to report a pilot study in which we tested the effectiveness of this instruction.

Theoretical Framework

Students are often told that the exponential operation represents "repeated multiplication" (e.g., $2^3 = 2\times2\times2$). However, as many researchers have pointed out (e.g., Confrey & Smith, 1995; Lakoff & Nunez, 2000), this conception is inadequate to perform much of the reasoning that we associate with exponents and logarithms. For instance, to a student who can only view exponents as repeated multiplication, expressions such as 2^{-1} and $2^{1/2}$ will be nonsensical, as you cannot multiply a number by itself negative one or one half times. In this paper, we attempt to teach students about exponents and logarithms by first having students understand exponentiation as a process, then having them view exponential and logarithmic expressions as results of applying this process (i.e., b^x represents the number that is the product of x factors of b), and finally generalizing this understanding for cases where the power in an exponent is not a natural number. We discuss the theoretical underpinnings behind these constructions below.

According to Dubinsky, a mathematical operation can be understood as an action or as a process (Dubinsky, 1991; Asiala et al., 1996). An *action* is a repeatable physical or mental transformation of mathematical objects to obtain other objects. Students limited to an action understanding of an operation can apply this operation only in response to an external cue explicitly detailing what steps to make. In the case of exponents, students with only an action understanding can do little besides calculating an exponent when they are given a specific base and power, and only if the power is a positive integer. After repeating an action and reflecting upon it, the student may interiorize the action as a *process*. Individuals with a process understanding of an operation can imagine the result of the transformation without actually performing the corresponding operation and can construct a new process by reversing the steps of the original transformation. A student with a process understanding of exponents can imagine b^x as a number that is the result of applying the operation of exponentiation, an ability that we believe is necessary to understand the rules of exponents, and can imagine reversing the process of exponentiation to obtain the process of taking a logarithm.

Many researchers have noted that expressions such as b^x have multiple meanings: this expression can be viewed as an operation- multiply b by itself x times- or it can be viewed as a mathematical structure- the number that is the result of applying the process of exponentiation. It can also be viewed as a function, a family of functions, or a string of symbols, depending on the context in which it is used (e.g., Sfard & Linchevski 1994). In particular, Sfard (1991, Sfard & Linchevski, 1994) distinguishes between an operational understanding of a concept- which focuses on its algorithmic nature- and the structural understanding of a concept- which treats the result of a process as an object in its own right. While students are generally capable of acquiring an operational understanding of a concept, acquiring a structural understanding appears to be quite difficult. In our view, an operational understanding of exponents would be tantamount to understanding exponential expressions as calls for repeated multiplication, while a structural understanding would be interpreting b^x as *the number that is the product of x factors of b* and $\log_b m$ as *the number of factors of b that are in the number m*.

At this point, a critical reader may question why a student needs to understand b^x as a mathematical object. According to Sfard, possessing a structural understanding of mathematical objects is necessary to reason about mathematical concepts because it makes one's knowledge of the concept more compact and intuitive (Sfard, 1991, p. 23). We believe that this is the case with exponents and logarithms. Consider the following rule about exponents: $b^x b^y = b^{x+y}$. A student with a structural understanding of exponential expressions could interpret this equation as "The product of x factors of b and y factors of b is (x+y) factors of b". Perhaps a student could also explain this equation as a statement about processes, at least in theory. (e.g., You obtain the same result if you multiply b multiplied by itself x times with b multiplied by itself y times or if you multiply b by itself (x+y) times). However, such an explanation would be longer and less intuitive. Hence, we believe that possessing a structural understanding of exponential and logarithmic expressions greatly aids students in understanding their properties.

After a student acquires an operational understanding of b^x as "the number that is the product of x factors of b", that student must generalize that understanding to account for cases where x is not a natural number. How students may do this is an interesting and difficult question, but it is beyond the scope of this paper. Discussion of this topic can be found in Confrey and Smith (1995), Lakoff and Nunez (2000), and Weber (in press).

Instruction Used in Our Study

In this section, we describe instructional activities designed to lead students to make the mental constructions we describe above. After receiving this instruction, we would like students to be able to complete routine tasks that are traditionally associated with exponents and logarithms. That is, students should be able to perform basic computations with exponents and logarithms and be able to recite and apply the "rules" of exponents and logarithms. We would also like students to acquire a deeper conceptual understanding of these topics. Students should understand why the rules of exponents and logarithms are true and they should be able to use their conceptual understanding to answer traditional and non-traditional questions.

The first goal of our instruction is to have students understand the act of "taking an exponent" as a process (as described in section 2 of this paper or in Asiala et. al. (1996)). Research indicates that a particularly effective way for leading students to construct this understanding is to have students write a computer program to apply this operation. (Tall & Dubinsky, 1991). Researchers argue that in writing a computer program, students are forced to reason about and explicitly describe the steps of an operation. In doing so, students are likely to reflect on the steps of an operation and interiorize it into a process.

In our instruction, students were first taught the structure of a basic for loop and were given a for loop program in MAPLE that performed multiplication of integers as repeated addition (or repeated subtraction, in the case where a factor is negative). Students were asked to write a similar program that performed exponentiation as repeated multiplication (or repeated division). The students worked in pairs to write these programs. The instructor would answer questions and would help the students with MAPLE syntax, but would not help the students in other aspects of the program writing. All students completed this task in a one hour class period.

Our next goal was to have students understand b^x as the number that is the product of x factors of b and $\log_b m$ as the number of factors of b that are in the number m, a construction that Sfard coins *reification*. Sfard (1991, 2000) emphasizes the role that names and symbols play in acquiring a structural understanding of a mathematical concept. We designed paper-and-pencil worksheets in which the student was asked to describe exponential and logarithmic expressions as mathematical objects. They were also given exercises in which they were required to use a structural understanding of the exponential expressions to complete. With the judicious use of worked out examples, the students had several prompts to help guide their work. Some examples of these exercises are given in the Appendix. Students worked in groups of two or three to complete these activities. After the activities were completed, they were discussed, handed in, corrected, and returned to the students. This continued until all the activities were complete.

A skeptical reader may question whether something as trivial as paper-and-pencil exercises and classroom discussion can trigger a mental construction as sophisticated as reification. The only response we can offer at this time is the results of our study indicate that it can. Students who received this instruction appeared to be more capable of treating b^x as a mathematical object than students who received traditional instruction. We will describe these results in the next section.

Evaluation of Our Instruction

To evaluate the effectiveness of our pedagogy, we conducted a pilot study in which we implemented this instruction in a college algebra and trigonometry course and then compared the performance of our students with students who received traditional instruction on a set of interview questions.

Methods

Participants

Two groups of students from a regional university in the southern United States participated in this study. The experimental group of students was enrolled in the first author's college algebra and trigonometry course. The control group of students was enrolled in a separate section of the college algebra and trigonometry course taught by a different instructor. The instructors of the control and experimental groups spent roughly equal time reviewing exponents and logarithms. 15 students in each course volunteered to participate in this study.

Procedure and Materials

Three weeks after receiving instruction, the students were interviewed individually. During the interview, the students were asked the following questions:

Basic computations

B1. What is 2^3?

B2. What is $\log_2 64$?

B3. What is $\log_x x$?

B4. $\log_9 729=3$. Use this information to find $\log_3 729$.

Rules

R1. $b^x b^y$ can be simplified to what? Why?

R2. $\log_a x^r$ can be simplified to what? Why?

R3. How can you express the square root of x as a power? Why?

Conceptual questions

C1. Is $(1/2)^x$ an increasing function or a decreasing function? Why?

C2. Is $(-3)^{10}$ a positive or negative number? Why?

C3. Is 5^{14} an even number or an odd number?

C4. How would you find $\log_5 78125$?

Results of the Pilot Study

The number of correct responses for each of the Basic Computation and Rules questions is presented in Figure 1.

Basic computation questions- Every student was able to compute 2^3, indicating that all participants had some basic notion of exponent. As can be seen from Figure 1, the participants in the experimental group performed much better than their counterparts in the control group on the remainder of the Basic computation questions. No student in the control group was able to answer questions B3 or B4.

Rules questions- The data in Figure 1 indicate that students in the experimental group were able to recall more rules of exponents and logarithms than students in the control group. The difference between the groups becomes more pronounced when one examines the students' responses when they were asked why the rules of exponents and logarithms were true. Not a single student in the control group was able to explain why any of the rules of exponents were true. However, students in the experimental group were often able to give an explanation of why the rules were true. For instance, eight students were able to explain why $b^x b^y = b^{x+y}$. One typical response from a student was "Because we're having b x amount of time and y x amount of time, so when you set it up...it's basically like your adding up all the repititions of b". Six students in the experimental group were able to explain why $\log_b x^r = r \log_b x$ and why $x = x^{1/2}$. The latter result was particularly impressive, as why $x = x^{1/2}$ was never explicitly discussed in our activities.

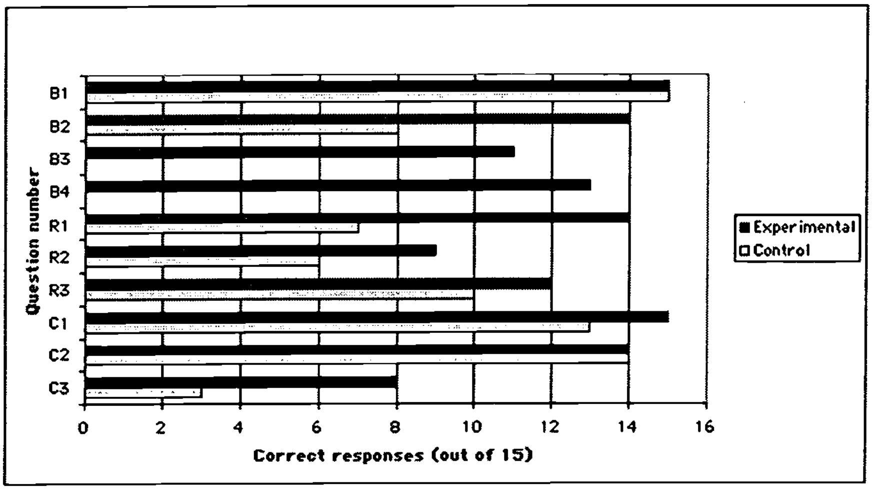

Figure 1. Performance on interview questions.

Conceptual questions- Both groups generally performed well on questions C1 and C2; they both were aware that $(1/2)^x$ was a decreasing function and $(-3)^{10}$ was a positive number. However, students in the experimental group were better at stating why these statements were true. When asked why the first property was true, no student in the control group could give an adequate response. Most students relied on looking at specific examples, usually only looking at the cases where x is one or two. However, five students in the experimental group were able to explain why this rule was true by using their understanding of exponentiation as a process. For instance, one student said, "If you keep multiplying [by one half], the number is going to keep getting smaller and smaller". Likewise, many students in the experimental group noted that the number $(-3)^{10}$ could be decomposed into the product of positive numbers, while such responses from the control group were uncommon.

12 students in the control group believed that 5^{14} would be an even number, generally because they believed an odd number raised to an even power would be even. The three students who correctly stated that 5^{14} would be odd conjectured this by looking at simple cases such as 5^1, 5^2, and 5^3. 8 students in the experimental group also answered this question correctly. While four also did so by looking at simple cases, four other students offered deeper explanations such as, "An odd to any power is always going to be odd… 'cause you keep on multiplying by an odd number, so it can never turn even" and "It'll be odd. If you multiply two numbers ending in five, it's going to end in five. So 5 to the anything will end in five".

When asked how to compute $\log_5 78125$, eight students in the control group knew this was tantamount to solving the equation $5^x = 78125$, but none could offer anything more than this. In contrast, several students in the experimental group offered responses that demonstrated an understanding of logarithms as reversing the process of exponentiation. For example, four students suggested repeatedly multiplying by five until the result reached or exceeded 78,125 and another suggested dividing 78,125 repeatedly until he reached one. The number of repetitions required would be the answer.

Reconstruction of forgotten knowledge- Perhaps the most promising result from this pilot study was that students in the experimental group often could not recall properties of exponents and logarithms, but were able to use their conceptual knowledge of these topics to reconstruct these rules. For instance, three students in the experimental group initially believed that $b^x b^y$ was equal to b^{xy}. (This was also a common mistake in the control group). When these students attempted to explain why this rule was true, they wrote out b^x as x factors of b and b^y as y factors of b. At this point, the students realized that there were (x+y) factors of b in $b^x b^y$, so the correct answer must be b^{x+y}. In contrast, this phenomenon did not occur with any of the students in the control group. In fact, not one student could explain why a single rule of exponents was true. There were several other instances of students in the experimental group using their conceptual knowledge to correct an initially erroneous response. For instance, one student initially believed that $(1/2)^x$ would be an increasing function, but then realized that as x grows, "we are going to be taking half of it more often, so it will be getting smaller." This illustrates an important point. As time passes, one's knowledge of symbolic rules will generally decay. If one has a deep understanding of the concepts involved, these rules can be reconstructed. If not, the rules cannot be recovered.

Discussion

Summary of Our Data

Students who received our instruction performed better than students who received traditional instruction at performing basic computations, recalling rules, and explaining why the rules of exponents and logarithms are true. They were also better able to answer questions that required them to use their conceptual knowledge of these topics.

Limitations of this Study

We consider the results of this pilot study to be encouraging. However, we should note that any conclusions drawn should be tentative, due to shortcomings in the design of this study. First, the author of this paper also served as the instructor of the experimental students and the investigator in the experiment. This raises a host of methodological concerns. Perhaps the instruction to the experimental students was superior to that of the control students, but only because the instructor of the former group of students was more motivated or more able. Perhaps the students in the experimental group performed so well in an effort to please or impress the investigator, who also happened to be the individual who assigned their grades at the end of the semester. Students in the control group would feel no such obligation. Also, although there was no *a priori* reason to suspect that students in the experimental group had more knowledge of exponents and logarithms prior to instruction, this possibility cannot be dismissed since no pre-test was given to these students. Clearly these concerns need to be addressed before definite conclusions can be drawn.

How do we know that the experimental students' performance was due to our instruction? First, many of these students' responses were analogous to the way that they completed our worksheet. For instance, to answer question B4, one could simply combine several rules of exponents. (e.g., $\log_9 729 = 3$. So $9^3 = 729$. $(3^2)^3 = 729$. $3^6 = 729$. $\log_3 729 = 6$). However, not one student solved the problem this way. All 13 students who answered this question correctly wrote 729 as 9×9×9, and then noticed that each 9 "split" into a pair of 3's so 729 can be written as 3×3×3×3×3×3. Thus there were six 3's in 729. The language that the experimental students used was also indicative of their thinking. For instance, many students spoke of there "being six threes in 729", indicating that students were thinking in a way that was consistent with our worksheets. Students' explanations of why the rules of exponents and logarithms were true were also consistent with their work on our worksheets.

Second, students' responses to our conceptual questions were consistent with our theoretical analysis. We predicted the conceptual questions could best be answered if the student had a process understanding of exponents. Many of the students' successful responses explicitly drew on this understanding.

Conclusion

In the past decades, our understanding of how students acquire mathematical concepts has increased immeasurably. This study marks our first attempt to apply the influential learning theories of Dubinsky and Sfard into the classroom. Using their theories, we postulated a set of mental constructions that a student could make to understand exponents and logarithms. We then designed instructional activities to lead students to make these constructions. We described the results of a pilot study in which we assessed the effectiveness of this instruction. The results are encouraging; students who completed our instructional activities outperformed students who received traditional instruction in a variety of measures, including recall of formulas, simple computation, and specification of why the formulas are true. Perhaps most significant is that students can use their deep understanding of these topics to reconstruct forgotten symbolic knowledge. However, due to limitations in the design of our study, any conclusions drawn from this study should be tentative. We are attempting to replicate the results of our pilot study in a more controlled setting with more students. If our attempts are successful, this study will be the focus of a future report.

References

Asiala, M., Brown, A., Devries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. *Research in Collegiate Mathematics Education, 1,* 1-32.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions, *Journal of Research in Mathematics Education, 26*, 66-86.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking*. Dordrecht: Kluwer.

Lakoff G., & Nunez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Rahn, J. R., & Berndes, B. A. (1994). Using logarithms to explore power and exponential functions, *Mathematics Teacher, 87*, 161-170.

Sfard, A. (1991). On the dual nature of mathematical conceptions. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K. E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating: Perspectives on mathematical discourse, tools, and instructional design*. Mahwah, NJ: Erlbaum.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification -- the case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Tall, D., & Dubinsky, E. (1991). Advanced mathematical thinking and the computer. In D. O. Tall (Ed.), *Advanced mathematical thinking*. Dordrecht: Kluwer.

Weber, K. H. (in press). Students understanding of exponential and logarithmic functions. *Proceedings from the 2nd international conference on the teaching of mathematics*.

Appendix

Sample exercises from our worksheet (Desired student responses given in **bold**).

Describe each of the exponential expressions in terms of a product and in terms of words.

4^3 = **4×4×4 = the number that is the product of 3 factors of 4**
b^x = **b×b×b×... (x times) = the number that is the product of x factors of b**

Simplify each of the expressions below by writing each exponential term as a product. Summarize each simplification in words.

b^2b^4 = **(b×b)×(b×b×b×b) = b^6**
The product of 2 factors of b and 4 factors of b is 6 factors of b.
bb^x = **b×(b×b×b×... (x times)) = b×b×b×b×... (x+1 times) = b^{x+1}**
The product of b and x factors of b is (x+1) factors of b.

NOTICE

Reproduction Basis

X	This document is covered by a signed "Reproduction Release (Blanket)" form (on file within the ERIC system), encompassing all or classes of documents from its source organization and, therefore, does not require a "Specific Document" Release form.
	This document is Federally-funded, or carries its own permission to reproduce, or is otherwise in the public domain and, therefore, may be reproduced by ERIC without a signed Reproduction Release form (either "Specific Document" or "Blanket").