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AUTHOR	Star, Jon R.
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ABSTRACT

This paper explores the development of students' knowledge of mathematical procedures. Students' tendency to develop rote knowledge of procedures has been widely commented on. I postulate an alternative, more "intelligent" endpoint for the development of procedural knowledge, where students choose to deviate from solving patterns on particular problems for greater efficiency. The purpose of this study was to explore the instructional conditions that facilitate the emergence of this outcome. Students with no prior knowledge of formal linear equation-solving techniques were taught the basic transformations of this domain. After instruction, students engaged in problem-solving sessions in two conditions. In the treatment group, students completed "alternative ordering tasks," where they were asked to re-solve previously completed problems but using a different ordering of steps. Completing alternative ordering tasks was found to lead to more intelligent solving. (Author)

RE-CONCEPTUALIZING PROCEDURAL KNOWLEDGE: THE EMERGENCE OF "INTELLIGENT" PERFORMANCES AMONG EQUATION SOLVERS

Jon R. Star
Michigan State University
jonstar@msu.edu

This paper explores the development of students' knowledge of mathematical procedures. Students' tendency to develop rote knowledge of procedures has been widely commented on. I postulate an alternative, more "intelligent" endpoint for the development of procedural knowledge, where students choose to deviate from established solving patterns on particular problems for greater efficiency. The purpose of this study was to explore the instructional conditions that facilitate the emergence of this outcome. Students with no prior knowledge of formal linear equation-solving techniques were taught the basic transformations of this domain. After instruction, students engaged in problem-solving sessions in two conditions. In the treatment group, students completed "alternative ordering tasks," where they were asked to re-solve previously completed problems but using a different ordering of steps. Completing alternative ordering tasks was found to lead to more intelligent solving.

Introduction

For much of this century, mathematics educators have sought to address students' tendency to view school mathematics as a series of procedures to be memorized. Researchers in mathematics education concur that (a) procedures learned by rote are easily forgotten and error-prone; and (b) the learning of procedures must be connected with conceptual knowledge in order to foster the development of understanding (e.g., Hiebert & Carpenter, 1992). The National Council of Teachers of Mathematics (NCTM) has articulated this emphasis on conceptual learning by calling for decreased attention to "memorizing rules and algorithms; practicing tedious paper-and-pencil computations; memorizing procedures ... without understanding" (NCTM, 1989, p. 71); and "rote memorization of facts and procedures" (NCTM, 1989, p. 129).

There is little doubt that the rote execution of memorized procedures does *not* constitute "mathematical understanding." However, there are other ways in which a procedure can be executed other than by rote, some of which could be characterized as "intelligent" or even as indicative of "procedural understanding" (Greeno, 1978). But few prior studies have considered procedural outcomes other than rote knowledge, much less explored its development. This paper attempts to map out this terrain: I examine the development of students' knowledge of mathematical procedures, with particular emphasis on examining learning outcomes other than rote execution.

Perspective

Fundamentally, executing a procedural skill requires that one have knowledge of its component steps and the order in which these steps should be applied. But not all performances of a skill are the same. In particular, skillful execution in mathematics can mean two very different things. On the one hand, skillful execution involves being able to use procedures rapidly, efficiently, with minimal error, and with minimal conscious attention; in other words, to execute a procedure automatically or by rote (Anderson & Lebiere, 1998). On the other hand, being "skilled" means being able to select appropriate procedures for particular problems, modify procedures when conditions warrant, and explain or justify one's steps to others; that is, to execute a procedure thoughtfully or deliberately (Ericsson & Charness, 1994; Karmiloff-Smith, 1992), "relationally" (Skemp, 1976), "mindfully" (Brown & Langer, 1990; Langer, 1993), or "intelligently" (Ryle, 1949).

Although acknowledging that both notions of "skilled" are important and necessary (National Research Council, 2001), mathematics educators have had difficulty integrating these two competing visions of mathematical proficiency. The tension between these two visions is a foundational issue in mathematics education: it not only pertains to our educational goals for students, but also speaks directly to what it means to know and to do mathematics.

While the first outcome for successful skill execution (automaticity) has been frequently examined by cognitive scientists (Anderson, 1982; Anderson & Fincham, 1994; Anderson & Lebiere, 1998), "intelligent" execution of procedures has been less widely studied and is thus the focus of this paper. I begin by articulating what I mean by intelligent execution of procedures. I then describe a study that explored the development of this capacity.

Framework

What does it mean to intelligently execute procedures?). I have proposed elsewhere that two central features of intelligent execution of procedures are (a) flexibility, and (b) innovation (Star, 2001a, 2001b). Flexibility refers to the ability to use a wide range of mathematical procedures in order to generate the best solution for particular problems (Beishuizen, van Putten, & van Mulken, 1997; Feltovich, Spiro, & Coulson, 1997). Flexible solvers have knowledge of standard solution procedures, but they also choose to use alternative or non-standard procedures on certain problems, when doing so results in a better or more efficient solution. Metaphorically, flexible solvers have more tools in their procedural toolbox.

Another feature of intelligent execution is innovation (Gick, 1986; Ryle, 1949; Simon & Reed, 1976). Innovation refers to the ability to use steps within a procedure in atypical ways in order to produce a more efficient solution. An innovative solver is able to use the individual steps of a procedure in ways other than that suggested by a standard solution. Metaphorically, innovation refers to the ability to use the tools in one's toolbox in non-standard ways that do a better job of performing certain kinds of tasks.

Both innovation and flexibility can be seen in the example solutions shown in Table 1. Note that the three problems in Table 1 are almost identical, but they have been solved using three different solution strategies. Problem A is solved using a standard solution method, one that is commonly and explicitly taught as *the* way to solve linear equations in US schools. Problems B and C could have been solved using this exact same method, but the solver has recognized and capitalized on the opportunity to use different strategies for these two problems -- strategies that are at least as good as the standard method but that could not be used in problem A. A solver who chooses to use the three solution strategies shown in Table 1 on this set of three problems is demonstrating flexibility; she has knowledge of multiple solution procedures and can select the most appropriate one for a particular problem.

Table 1. Example Solutions to Three Linear Equations

Problem A:

$$
\begin{aligned}
4(x+1)+2(x+2) &= 3(x+4)\\
4x+4+2x+4 &= 3x+12\\
6x+8 &= 3x+12\\
3x &= 4\\
x &= 4/3
\end{aligned}
$$

Problem B:

$$
\begin{aligned}
4(x+1)+2(x+1) &= 3(x+4)\\
6(x+1) &= 3(x+4)\\
6x+6 &= 3x+12\\
3x &= 6\\
x &= 2
\end{aligned}
$$

Problem C:

$$
\begin{aligned}
4(x+1)+2(x+1) &= 3(x+1)\\
6(x+1) &= 3(x+1)\\
3(x+1) &= 0\\
x+1 &= 0\\
x &= -1
\end{aligned}
$$

Innovation can be seen by looking closely at the solution steps used in problem B. In her first step, the solver has combined the terms $4(x + 1)$ and $2(x + 1)$ to yield $6(x + 1)$. The way that this solving step, "combining like terms," is more typically used is to combine variable terms (such as $4x$ and $2x$) or constant terms (such as 4 and 2), as was done in the standard solution method seen in problem A. In problem B (and also in problem C), the solver has used the same "combine like term" step, but in an atypical way that results in a solution that is arguably better. This atypical or non-standard use of an equation-solving step is what is meant by innovation.

Framing intelligent equation-solving in this manner raises the question of how innovation and flexibility develop. This question is largely unexplored. Basic skill practice has been linked to the development of rote knowledge (Anderson, 1982; Fitts, 1964), but the development of more flexible knowledge appears to require a different kind of practice, which has been referred to as "deliberate" (Ericsson, Krampe, & Tesch-Romer, 1993). One hypothesis for what such deliberate practice looks like comes from studies where participants were asked to solve a problem repeatedly in order to observe changes in their solutions that emerged with practice. There is ample evidence that solving a problem multiple times can lead to more automatic execution (e.g., Simon & Reed, 1976; Anzai & Simon, 1979; Blessing & Anderson, 1996; Koedinger & Anderson, 1990). However, there is also reason to hypothesize that, under certain conditions, re-solving previously completed problems can lead to more intelligent solving (Blöte, Klein, & Beishuizen, 2000; Krutetskii, 1976).

In the present study, I test this hypothesis by utilizing a task I refer to as the "alternative-ordering task." Participants are asked to re-solve previously completed problems but using a different ordering of steps. In this task, students are not merely practicing the same solution over and over again, but instead are generating, comparing, and evaluating the effectiveness and efficiency of different solution procedures. There is reason to speculate that such a task may lead to more intelligent solving, in the form of greater innovation and flexibility.

Goals

In my prior work (Star, 2001a, 2001b) I demonstrated that intelligent execution of procedures, as described above, exists in school-aged learners, and I explored the development of this capacity among solvers working individually on a paper-and-pencil task. In the present work, I build upon these initial findings by examining the development of intelligent execution among groups of students in a simulated classroom setting.

Method and Data Sources

Thirty-six 6th grade students volunteered to participate in this study. Students attended one-hour experimental sessions in groups of six for five consecutive days. The mathematical domain that I chose to use in this study was linear equation solving. A pre-test on linear equation solving was administered on Day 1. Students were then given a brief 30-minute scripted lesson on the steps used to solve linear equations (e.g., adding a constant to both sides of an equation, adding like terms, etc.). Following instruction, students were given a post-instruction test on these steps.

At the conclusion of instruction, all six students were randomly assigned to a treatment or a control group. Both groups devoted the sessions on Days 2, 3, and 4 to equation-solving practice. During these three problem-solving sessions, students solved a great variety of equations, some of which were very straightforward (e.g., $2x + 4 = 10$), while others were much more complex (e.g., $4(x + 2) + 6x + 10 = 2(x + 2) + 8(x + 2) + 6x + 4x + 8$).

In the problem-solving sessions, students engaged in alternating cycles of individual work followed by group discussion. The treatment and control groups differed only in the content of the group discussion. In the treatment groups, the discussion centered on students comparing their solution methods, discussing the differences between solution methods, and generating alternative solution methods. In the control groups, students participated in a discussion of identical length and concerning the same problems. However, the discussion focused on the correctness of numeric solutions and methods of checking numeric solutions. On Day 5, students in both conditions were given a common post-test.

Results

The most interesting result was that students in the treatment and control groups differed in the ways that they chose to approach equations by the end of the study. Significant differences emerged in both students' flexibility and innovation.

With respect to flexibility, treatment students were significantly more likely to become flexible solvers than control group students: In particular, treatment students were more likely to use several different solution methods on the post-test problems, while control students were more likely to rely upon a single solution for all problems. The repeated equation-solving practice that control group students received resulted in the discovery, for each individual, of a dependable, favorite solution method that was used on many subsequent problems. Sometimes a student's favorite solution method was an efficient one; however, in other cases, students reliably and consistently used solution methods that were quite inefficient. For example, Table 2 shows the solution strategy that Billy (a student in the control group) used on several problems toward the end of the study. Billy repeatedly moved variable and constant terms back and forth, from one side of the equation to the other. Despite the inefficiency of this approach, it is one that Billy used consistently. In contrast, treatment students, ostensibly because they were repeatedly asked to consider alternative ways that equations could be solved, did not settle into a favorite, consistently used solution strategy. Rather, treatment students varied the ways that equations were solved in the post-test depending on the specifics of individual problems, demonstrating knowledge of multiple solution strategies. Treatment students were not content merely to solve an equation using an algorithm that was known to always work for them; they tried multiple approaches in order to arrive at the best solution for a particular problem.

Table 2. Billy's Solutions to Problem 4.2

Billy:	$3(x + 1) + 6(x + 1) + 6x + 9$	=	$6x + 9$
	$3x + 3 + 6x + 6 + 6x + 9$	=	$6x + 9$
	$3 + 6x + 6 + 6x + 9$	=	$3x + 9$
	$3 + 6x + 6 + 6x$	=	$3x$
	$3 + 3x + 6 + 6x$	=	0
	$3 + 6 + 6x$	=	$3x$[a]
	$3 + 6 + 3x$	=	0
	$6 + 3x$	=	-3
	$3 + 3x$	=	0[*]
	$3x$	=	-3
	x	=	-1

[a]Indicates an error in how a transformation has been applied.

This increased flexibility appears to be related to innovation: there is evidence that treatment students in this study were more likely to innovate than control group students. Recall that innovation is the use of an equation-solving step in an atypical way that results in a better solution for a particular problem. 81% of treatment students showed signs of emerging innovation on at least one problem attempted during the 3 problem-solving sessions, while only 15% of control group students did so. Innovation is illustrated in Table 3, which shows Anna's (a student in the treatment group) solutions to equations she encountered in the second and third problem-solving sessions. Note the difference between how Anna solved problems 2.2 and 3.2, two problems that are structurally identical. Anna's solution on problem 3.2 is more efficient than the one used on problem 2.2, in that she chose to divide to both sides as a *first* step (on 3.2) rather than a last step (on 2.2). (Dividing to both sides as a last step is the typical way that students solved equations such as this one.) Anna's use of the "divide to both sides" step" atypically is an example of an innovation. Anna came to this knowledge as a result of the treatment: the generation, comparison, and reflection on multiple solution strategies. Control group students were significantly less likely to produce innovative solution strategies such as this one illustrated in Table 3.

Table 3. *Anna's Solutions to Problems 2.2 and 3.2*

Problem 2.2:	$3(x+1) = 15$
	$3x+3 = 15$
	$3x = 12$
	$x = 4$
Problem 3.2:	$3(x+2) = 21$
	$x+2 = 7$
	$x = 5$

Discussion

This study provided evidence that engaging in alternative ordering tasks, which involved re-solving previously solved equations using a different ordering of steps, led students to believe that equations could be solved in more than one way and that some strategies were better than others. Treatment students' cognizance of multiple ways that equations can be solved led to an increase in their ability to innovate, where innovation refers to the use of a step in an atypical way that results in a better solution. The ability to innovate was also related to increased flexibility in treatment students' solutions, where flexibility refers to a reluctance to rigidly adhere to the exact same solution sequence when solving similar problems. Students who did not experience this treatment were more likely to develop one solution method that was rigidly adhered to on all problems.

This study adds to the literature on equation solving by shifting the focus from students' errors to the capacities that successful performers exhibit. A review of the literature on the use of mathematical procedures (with its emphasis on cataloging the multitude of errors that students make) suggests that the most important feature of success in this domain is the ability to rapidly execute error-free procedures. The present study suggests that another important feature of a successful solver is the ability to intelligently use procedures; that is, to selectively choose to deviate from standard and practiced methods in order to produce even more efficient solutions. Students who have capacities for innovation and flexibility have more sophisticated knowledge of equation solving transformations that only emerges in their application. This outcome of learning procedures has not previously been considered in the mathematics education literature.

Implications. The fact that the alternative ordering task was effective in this study suggests that these results could be used to inform classroom practice in several ways. First, during instruction on equation solving (and other symbolic mathematical procedures), teachers should frequently and regularly ask students to re-solve previously completed problems using a different ordering of steps. The multiple solutions that are generated in such a task can then be compared and contrasted. The study described in this paper suggests that the incorporation of such tasks will result in substantial gains in students' ability to innovate and be flexible.

Implicit within this recommendation is a caution *against* direct and explicit instruction on the "standard solution method." Especially for novice learners, teachers should avoid labeling any one solution method as being the best way, the right way, or the only way. Benefits that arise from engaging in the alternative ordering tasks come when students think carefully about how to generate additional solution strategies and how to compare multiple solution strategies. Students come to their own conclusions about the features that identify one solution as different from another (e.g., efficiency), and direct instruction on a standard, efficient procedure would appear to subvert this process.

A challenge that necessarily accompanies these recommendations concerns student motivation. Many teachers find students to be uninterested in learning how to solve equations, and so it might appear that asking students to re-solve previously completed equations would further reduce already low levels of motivation. This is certainly a valid concern. However, there is a great deal of evidence from the elementary grades that such concerns can be addressed. Many examples exist of classrooms where a climate has been created that incorporates the features that are integral to the recommendations detailed above: student collaboration, the sharing of multiple solution strategies, and the group comparison and evaluation of mathematical procedures and reasoning (Ball, 1993; Chazan & Ball, 1999; Lampert, 1990). There are fewer examples of this kind of classroom environment at the high school level, particularly related to the instruction of mathematical procedures. The study described here suggests that there is much to be gained from efforts to make such changes at the secondary level.

Conclusion

Procedures are an integral component of mathematics. While fluency is certainly one educational outcome, this paper has identified another in the ability to vary the ways that one uses procedures on particular problems in order to arrive at maximally efficient solutions. Krutetskii captured this distinction as follows:

> Incapable students are marked by inertness, sluggishness, and constraint in their thinking in the realm of mathematical relations and operations. ... Mathematically able students are distinguished by a greater flexibility, by mobility of their mental process in solving mathematical problems. It is expressed in a free and easy switching from one mental operation to another qualitatively different one, in a diversity of aspects in the approach to the problem to problem-solving, in a freedom from the binding influence of stereotyped, conventional methods of solution, and in the ease in reconstructing established thought patterns and systems of operations. ... Very typical of capable students is a striving for the most rational solution to a problem, a search

for the clearest, simplest, shortest, and thus most 'elegant' path to the goal. (Krutetskii, 1976, p. 282-3)

While incapable students -- those with rote knowledge of procedures -- are relatively easy to find, intelligent solvers present a much more significant challenge. The study described in this paper represents a first attempt to re-conceptualize procedural knowledge so as to include such a relational outcome. If flexibility and innovation in the use of procedures are integral to our educational goals for students, further investigation of the development of this kind of procedural knowledge is vital.

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