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ABSTRACT

In 2001, at Snowbird, Utah, members of the µ-group from the University of British Columbia, under the guidance of Dr. Susan Pirie, engaged in an innovative, continuous session of linked papers which consider the notion and nature of mathematical understanding and reviewed a number of different theories and approaches to this phenomenon. We presented, discussed and contrasted a variety of key ideas in this area and considered their usefulness and applicability in analyzing the mathematical actions of pupils working in school classrooms. We also presented in detail how we undertake our video analysis and for this, the reader is referred to the paper by Pirie in last year's proceedings entitled "Analysis, Lies and Video-tape." This year we intend to use a slightly differently structured session to enable even greater audience participation, especially by those who are unable to attend the entire session. We will still be show-casing the collaborative work of the teachers and mathematics educators who are members of the µ-group, but we will offer an interactive symposium space in which to discuss three papers and three posters, all presented together in one long session. Participants will be invited to examine the presented research, and to enter into dialogue with the authors both as a group and individually. As part of a SSHRC funded project into the effects of Problem Posing on ways of understanding mathematics, µ-group members have taken a common mathematical prompt: "If the answer is 72, what is the question?" to a wide variety of students and explored the mathematical understanding revealed as the students grappled with posing problems from their own, individual, mathematical perspectives that fit the provided situation. Each researcher presented the problem as they saw fit to their students, given the students' level of mathematical knowledge and sophistication Our session as a whole will encompass papers and posters presented in interactive ways by: Susan Pirie, with a review of the group's theoretical position and methodological stance; Katharine Borgen, whose work is with Grade 9 students tackling mathematics in English which is their second language; Lionel LaCroix, who has taken the common prompt out of the classroom to an industrial- trades worker of 25 years experience of using mathematics in the workplace; Donna Jenner, who has taught and studied Grade 2/3 students working individually and in small groups. Shelley Saltzman, who has worked with a small group of Grade 4 students with learning difficulties in mathematics; Shevy Levy, who approached pairs of students of a wide variety of ages; Amy Bamford, who challenged her Grade 12, enriched curriculum class. Lyndon Martin will act as a discussant for the session. (Author)

PROBLEM POSING: WHAT CAN IT TELL US ABOUT STUDENTS' MATHEMATICAL UNDERSTANDING?

Susan E.B. Pirie
The "μ-Group"
University of British Columbia

Setting the Scene

In 2001, at Snowbird, Utah, members of the μ-group[i] from the University of British Columbia, under the guidance of Dr. Susan Pirie, engaged in an innovative, continuous session of linked papers which consider the notion and nature of mathematical understanding and reviewed a number of different theories and approaches to this phenomenon. We presented, discussed and contrasted a variety of key ideas in this area and considered their usefulness and applicability in analysing the mathematical actions of pupils working in school classrooms. We also presented in detail how we undertake our video analysis and for this, the reader is referred to the paper by Pirie in last year's proceedings entitled "Analysis, Lies and Video-tape." This year we intend to use a slightly differently structured session to enable even greater audience participation, especially by those who are unable to attend the entire session. We will still be show-casing the collaborative work of the teachers and mathematics educators who are members of the μ-group, but we will offer an interactive symposium space in which to discuss three papers and three posters, all presented together in one long session. Participants will be invited to examine the presented research, and to enter into dialogue with the authors both as a group and individually.

As part of a SSHRC[ii] funded project into the effects of Problem Posing on ways of understanding mathematics, μ-group members have taken a common mathematical prompt—"If the answer is 72, what is the question?—to a wide variety of students and explored the mathematical understanding revealed as the students grappled with posing problems from their own, individual, mathematical perspectives that fit the provided situation. Each researcher presented the problem as they saw fit to their students, given the students' level of mathematical knowledge and sophistication

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Theoretical Framework

Since 1987 Thomas Kieren and Susan Pirie have been involved in a collaborative research project to develop, test and refine a theory for the growth of mathematical understanding. Rather than think of understanding in terms of singular (or even multiple) acquisitions, or test for it merely by noticing differences in pre and post instructional behaviours, our theory allows one to observe growth of understanding as a dynamical, non-linear process. It offers a way to examine the cognitive activities of students as they 'come to know' mathematical concepts and therefore also offers the possibility for a fine-grained analysis of the problem-posing process, not so far undertaken.

The theory (Pirie & Kieren 1991, 1994a, 1994b, 1996) posits eight layers of understanding together with the cognitive activity of 'folding back' as crucial to growth of understanding. As illustrated in the figure, these layers each contain all previous understanding and are each contained by all outer more sophisticated understanding. A student's growth of understanding is traced out by a 'back and forth' pathway across the layers. The core of all understanding is one's Primitive Knowing, defined as everything one knows *except* knowledge already created on the particular topic under consideration, which is, of course, understanding at one of the other levels *for that topic*. Three of the inner layers are defined as Image Having, Property Noticing and Formalising. At the Image Having level, for any particular topic, students are observed to hold one or more mental image (often very specific, limited and context dependent), which they

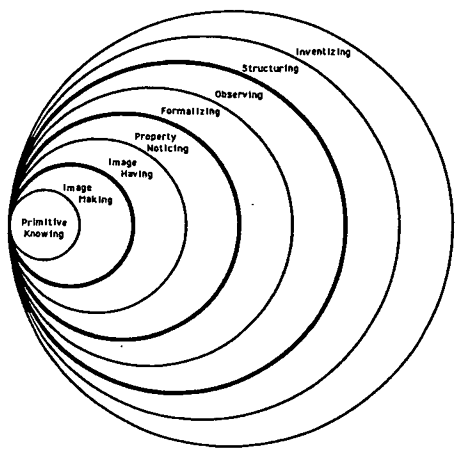

Figure 1. The Pirie-Kieren model for the dynamical growth of mathematical understanding.

are able to employ when working on mathematical tasks. At the Formalising level, students are seen to be working with generalisations of concepts, and they no longer need to relate back to the specific mathematical contexts that gave rise to their understanding. Between these two layers lies that of Property Noticing. Here students are engaged in the reflective activity of examining their images: making distinctions and connections between them. It is this level of cognitive activity that enables students to formalise their mathematical knowledge with comprehension, and it is this level that is so often ignored by advocates of discovery-based learning.

Problem Posing

In mathematics, "problem posing" refers to both the creation of questions in a mathematical context and to the reformulation, for solution, of ill-structured existing problems. Over the past two decades, problem posing has received only sporadic interest within the field of mathematics education, and this focused mainly on students writing story problems in a given context or with given data. Cudmore (2000) tends to show that the value of this work is limited in terms of growth of mathematical understanding. Research has also been largely focused at the elementary grades, and therefore involved comparatively unsophisticated mathematics and questioning. (See a review of the literature by Silver (1994) and more recently work published by Brody and Rosenfield (1996) and English (1997)). To pose mathematical questions at any level, however, involves more than being able to do the mathematics. It requires some understanding of the mathematical concepts involved - at the very least a feel for when a concept can be appropriately invoked. As a part of the SSHRC funded project, Pirie has developed an additional, somewhat more specific meaning for problem posing. We have asked students to pose, or create, mathematical questions in a specific topic area that they have studied, "like the ones you did in the text book" when all they are given is "the answer from the back of the book." We have hypothesised that learning to problem pose might also enhance students' learning for understanding. That is to say that, in coming to ask questions on mathematical concepts, students might come to understand those concepts in a more generalised, less context-dependent way.

Methodological Stance and Methods of Analysis

Since the aim of our research is to tease out the "specific structure of occurrences" (Erickson 1986) of problem posing, the methodological approach is an interpretative one, based in case studies and ethnographies, using participant observations, video-stimulated recall techniques, clinical interviews and analysis of students' written work. Since so little is known about the nature of problem posing as a cognitive activity, such a methodological approach allows us to view the process of data collection as "progressive problem solving, in which issues of sampling, hypothesis generation and hypothesis testing go hand in hand" (Erickson 1986). This approach enables decisions of which cases to focus on to be taken as the research progresses and we are building a "portfolio" of examples of growth of mathematical understanding that illustrate learners working at the Property Noticing level. Pirie's specific form of problem posing is proving very productive in providing evidence of work at this level.

Each of the researchers videoed their students "problem posing" and used multimedia techniques, mainly based around VPrism software, to assist in the analysis of the data from the perspective of the mathematical understanding revealed by the students through use of the "answer is 72" starting situation. As can be imagined, some very different and interesting features have emerged from the research and these will all be open for discussion at the session.

The Portfolio

There is clear evidence from our data (and that of many others) of a disconnected mental leap that many students are required to make as they move from having constructed some mental image(s) of a concept, to its formal representation, without having grasped the relationships and features of their images that enable this abstract generalisation of their concrete experiences. This transition between levels of understanding is not well understood by the mathematics education community. Action and discovery learning, while enabling students to form richer mental images for mathematical concepts, have failed to provide the link, that many students lack, between action-tied images and generalised formalism. Asking students to pose mathematical problems within a given topic, rather than to answer well constructed teacher/text questions, appears to provide a much needed, connective bridge to formal generalisation with understanding.

The specific form of problem posing that we are using currently was developed out of an incident where students were working from a standard text book, with questions in the topic of quadratic equations such as: Solve $5x^2 + 3x + 0.25 = 0$, which simply needs students to remember the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A question posed by one of the students in the class, however, - "Why does 3.5 'solve' that equation?" - led to a class discussion around the meaning of "solve" in this context and what the formula was actually 'doing'. From this Pirie constructed the question:

"If the page giving the exercise had been torn from your text book, but you still had the answers at the back of the book could you reconstruct the quadratic equations? For example, if for question (1) the solution was 3.5, what was the quadratic equation?"

This notion of giving students an answer in a specific mathematical context, and expecting them to create a "standard" question to fit it was then expanded to other topics. This is proving to be a very effective method of provoking learners to think at the Property Noticing level in order to connect the images for the topic that they have built up, with the formalisations that they have been using to answer the questions in the text book exercises.

The rest of the contributors to this μ-group presentation will illustrate, using a few of the examples in the portfolio, the widely differing ways in which a single prompt can provoke or reveal growth of understanding at a whole range of ability and topic levels.

PROBLEM POSING IN A SECOND LANGUAGE

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Mathematics has often been considered "language free." At the same time, problem solving and problem posing are considered appropriate means of determining students' conceptual understanding of mathematics. This article considers the working of four relatively capable mathematics students in Sweden, whose *primitive knowing* was developed in Swedish, who were asked to pose a problem in English. Results indicate that for these students, an obstacle to their ability to express their mathematical understanding was their need to concentrate on the correct grammatical presentation of their work. This distracted from their concentration on the mathematical presentation, possibly making them look less capable.

An individual's first language is the language through which he/she initially perceives the environment, through which first concepts are developed, and through which identity as a person is achieved (Pattanayak, 1986). While many consider that mathematics is an absolute with universal symbols, making language less important in the understanding of it than in other subjects, mathematics has a very precise language with subject specific meanings that are often combined with symbolic notation (Scholnick, 1988; Mestre, 1988; Lass, 1988; Kimball, 1990). The ability to express oneself mathematically requires logical and abstract thinking (van Heile, 1986).

Due to the considerable amount of immigration around the world, the language of communication in the classroom is often not the first language of the student, or even the language spoken in the home. While students who come from a home where English is the language of communication will be aware of, although probably unconsciously, the linguistic structures that affect meaning, this is not necessarily true for second language learners. Thus, while many second language students are placed in a regular mathematics class long before they are placed in a regular class where language is thought to be more significant, they may lack knowledge of the linguistic structure that gives meaning to the expressions encountered in the mathematics classroom (Cuevas, 1991), possibly placing them at a cognitive disadvantage.

Standards based mathematics promote sense making by being comprehensive and by developing ideas in depth in a coherent and motivating manner (Taftan, Ryes & Wasman, 2001). The 2000 National Council of Teachers of Mathematics (NCTM) Standards have stressed the importance of conceptual understanding in mathematics as a component of proficiency. This implies an understanding that is beyond the simple manipulation of symbols, allowing, and even requiring, students to express their understanding verbally. However, teachers often find that, while students may be able to perform the mathematical algorithms appropriately, they often struggle in their attempt to describe the mathematics they have done or to explain why they used a particular approach. This inability to verbalize their procedures may not necessarily indicate a lack of conceptual understanding, but rather, may be due to the abstractness and theoretical nature of many mathematical concepts (Miller, 1993).

When students work at mathematics in a language that is not their first language and which is not the language spoken at home, the ability to express their understanding is even more of a problem than for first language students because they often "think" the mathematics in their first language (Borgen, 1998). Thus, when second language students seems to perform less well than first language students, it may be that they simply lack the linguistic ability to express their cognitive understanding (MacGregor, 1993; Adler, 1995). In order to put the mathematics into context, they must, to borrow an expression from the Pirie and Kieren (1994), *fold back* to terminology and images formed and described in their first language, and may attempt to transpose and translate these to the new situation. Attempts at translations may not help as direct translation seldom, if ever, provides the same meaning as the original expression. Also, the students might not have encountered the relevant mathematical terminology in their first language. Translation may thus create disjoint connections between cognition and verbal expression, a discontinuity of ideas from one language to the other. So, when second language students are asked to express their mathematical understanding, even though they may have learned the requisite terminology in the second language, they may simply be unable to verbalize their thoughts because the foundational concepts are understood in their first language. Discussion of the new material cannot therefore take place in a coherent manner.

The emphasis on understanding being placed on mathematics by NCTM Standards has lead to an increased emphasis on problem solving. This, of necessity, requires understanding of the language and of the context of the question. The area of problem solving, therefore, is one in which second language learners often have difficulty. They thus often rely on patterning solutions based on problems previously done in class (Borgen, 1998).

Problem posing is often seen as a means by which students can display their level of understanding of a mathematical concept. In forming a question for which the answer is known, it is often assumed that students can display their understanding more clearly than if they are simply asked to solve a question.

This paper will consider a problem posing situation involving students for whom English is not their first language. I will consider the question they created and their solution to it, although the two do not agree, and discuss the effect that using a second language might have had in the problem-posing situation.

Answer is 72: What is the Question?

The Swedish Setting

The setting for this study was Umea, a medium sized city in the northern part of Sweden. Four grade nine students, Klara, Stine, Nathaniel and Maria, agreed to take part in a videotaped session. It is important to note that in Sweden, the use of English as a language for communication is promoted in many ways. In most schools, in the first grade, students learn songs, colours and some useful words. By grade four they start a more formal approach by learning to read and write in English (S.-M. Kuoppa, personal communication, 2002). Most television programs imported from English speaking countries are not dubbed, as is the norm in many European countries, but rather are presented in English with Swedish subtitles. And, although Sweden has a strong rock and pop culture of its own, not only are North American pop songs widely circulated and listened to, as far back as the early 1970's the well known Swedish rock group, ABBA, recorded in English as have, more recently, Ace of Base and Roxette, to name a few. Swedish Metal bands that record in English include Inflames, Children of Bottom, Opeth and many more. Thus, most Swedish people have been exposed to the English language on many levels, and, by the time students are in grade nine, they appear fluent in oral English.

The Problem Posing Session

When I met the four students involved in this session, I spoke to them briefly to ascertain their level of English and to explain what was planned during the session. As expected, their understanding of English, and their ability to express themselves in it created no major obstacle. I explained to them that I was from the University of British Columbia and that I was part of the MU Group, a research team of teachers, university students and professors who were interested in mathematical understanding and that we were presently working on a particular aspect of that understanding. Specifically, we were interested in something called "problem posing." Therefore, in this session, I would not give them a question, but rather, I would give them a solution and they would be asked to create a question that suited it. The catch was, however, that my particular interest was in mathematical understanding in a second language. Therefore, I was asking them to create and discuss the question in English as much as possible. However, if at any point they could not find the expression they needed in English, they should naturally speak in Swedish. I had made the assumption that since they had learned mathematics in Swedish their primitive knowing (Pirie & Kieren, 1994) would be in Swedish, that they might not know all the appropriate English terms, and that they might need to use each other as "sounding boards", knowing what they wanted to say (in Swedish) but not knowing how to express it (in English).

I had had a chance to peruse the textbook that the students had been using to determine what work they had covered during the year. Although I could not read Swedish, I was able to ascertain the gist of what they had done by the mathematical symbolism. I had also talked with the teacher who indicated that the most recently covered unit was a review of fractions. I therefore suggested to the students that they could use this as a start, or they could choose any area of mathematics where they thought they could come up with a suitable question. Thus, the students began their solution, or question making, by discussing a setting in which Ana and Alex buy some apples. They briefly considered a scenario in which one person buys more apples than another. Klara then suggested that they should come up with "the mathematic (sic)" first, and then make the "problem." The students briefly discussed the format the "equation" should take. (In transcriptions, **bolding** is used to indicate that words were stressed orally, --- for a pause, and ... to indicate missing sections.)

Klara: *Well, if we should come up with like the mathematic, the mathematic problem first then it, then put it in the words.*

Klara: *Maybe we can, so if the answer is seventy-two. We can have like --- seventy-two something times something divided by something and the answer is something and then you have to count out what, what x is and x will be seventy-two.*

Maria: *Uh hu*

Klara: *Do you get it?*

Nathaniel: *Yeah*

Klara: *So, uh, x is seventy-two. OK, seventy-two, then we can have seventy-two times something that works. --- Seventy-two times --- Four maybe? Or four --- OK seventy-two then its x times four and that is two hundred and eighty-eight. Two hundred eighty-eight. That's divided by ---*

Stine: *Two hundred eighty eight*

Maria: *It could be like Ana buys*

Nathaniel: *Its divided in four you get seven --- You get --- no*

Klara: *No, but if you divide in four you get seventy-two. And that's the answer, but*

Nathaniel: *Yeah, I know, I know*

Klara: *Uhm if you divide it by --- maybe --- six you get forty-eight, so if it is x times four divided by six equals forty-eight, what is x?*

Maria: *Uh hu.*

Nathaniel: *Seventy-two*

The students thus formed their "equation" and went on to form a problem using this form as their solution. The students' decision regarding the format of this "equation", it seems, dictated their thinking about the problem, and they considered it as the solution. The actual problem they created did not agree with the solution they had predetermined. Specifically, the problem the students posed and their solution are presented in Figure 2.

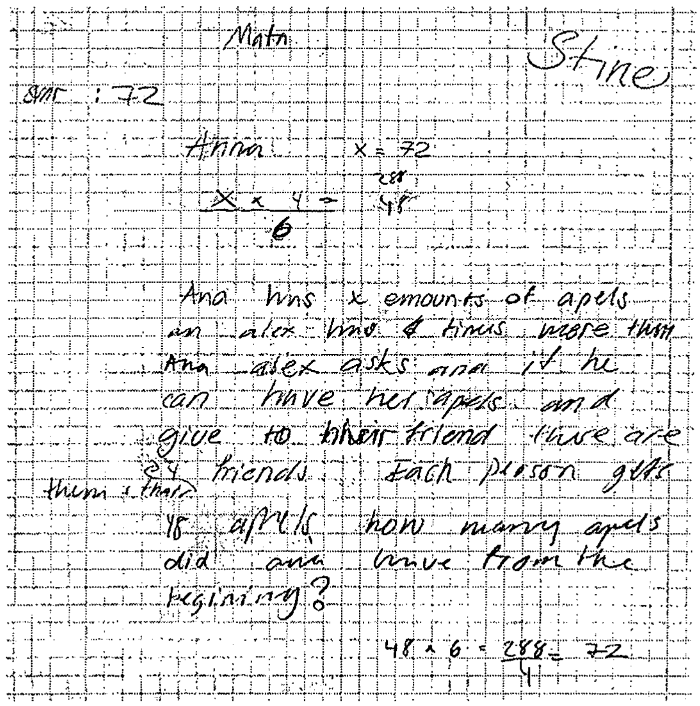

Figure 2. Problem posed by the students.

Rather than transcribe the entire discussion, in this next section I will state some of the inferences I made with respect to the students' performance of mathematics using English as their language of communication and will give the transcription of only those statements that prompted me to arrive at these inferences.

Inferences

The problem posed was a mere "copy" of problems from previous mathematics work. Little creativity went into formatting the question. The students had decided that they would form a "word" problem and discussed how to form it:

Nathaniel: *We gotta write a problem and solve.*

Klara: *Mmmm. Well should we like --- apples and that's x and that times something is seventy-two or shall we do like another --- "*

. . .

Stine: *We ca --- They always --- Ana and Steve always bla bla bla and umm umm umm are going to give us some apples. (At this, all the students laughed.)*

Later, they specifically indicated that they had simply formulated a question similar to those that they were accustomed to doing.

Nathaniel: We make like example from the book. They are all ---

Klara: They are always taking Ana. They have to divide apples or potatoes or whatever.

It is unclear whether they would have been more creative if they had been asked to form a question in Swedish. However, what is apparent that they simply used an existing model to format their ideas – that they felt that this was a suitable type of question to form in a mathematics class.

The students, it seems, needed a mathematical model and only then could they make a problem (albeit incorrect) around this "equation." As noted earlier, Klara had indicated that they come up with the mathematics first, and then created a situation in which the equation *4x/6=48* applied. Following was a set of disjoint statements in which the students seemed to be trying to formulate their thoughts as to how the question could be worded.

Klara: ... But then you have to have like Ana has x amount of apples --- and uh

Nathaniel: Times five by four times more

Klara: No, like Alex has four times more than Ana

Nathaniel: How much?

Klara: Ans --- so maybe

Nathaniel: And then they

Klara: And then they ---

Stine: So they put them together and [She swore in English](laughter)

Nathaniel: Well, if we do like and they put them together

Klara: Ana. Put them together and divided by divided by six

Nathaniel by six persons

Klara: Yes, by six, by six persons

Nathaniel: By? And how, yeah, and how how many apples

Klara: Does Ana have, yeah

Few, if any, theoretical mathematics concepts were expressed during the problem posing session. The students simply tried to state a problem that suited their "equation."

Correct grammar seemed more important than correct mathematics. Once the format of both the equation and the problem were outlined, the students continued to discuss the actual situation, but concentrated more on the grammatical structure of the problem they were creating than on the mathematics involved, being sure that numbers and tenses agreed.

Klara: So that's Alex asks Ana, asks Ana if he can have hers --- hers and divide them --- to her, and give --- give to his friends.

Nathaniel: And give them to his friends

Klara: And give to his friends

*Nathaniel: **Six** friends*

*Klara: Or, **to** give to his friends. For hers to*

Nathaniel: Hers apples

*Three girls: **Her** apples*

Nathaniel: Her, I mean

*Klara: Give it to her his um ... And together Alex and his friends are --- Alex's **five** friends*

Maria: Yeah. Are divided them

Klara: along if if five friends, then he gets some, too. Or maybe there are four friends and they divided by

Maria: by friends.

*Klara: And by **their** friends, not her friends, not his friends, his friends, umm her friends, **their** friends.*

Nathaniel: Their friends are four

*Klara: **Their four** friends. Or maybe just like them and their friends --- them and their friends --- and their **four** friends*

She again corrected Nathaniel.

Nathaniel: Alex asked Ana if he can have her apples and give to his four friends."

*Klara: And give to **their**.*

In these latter two incidents, Klara seemed to be making sure that the grammar was appropriate to the numbers. Later she also corrected herself from *"did Ana had"* to *"did Ana have,"* showing a concentration on tense, thus the importance of grammar.

Once the "equation" was formed, the students considered it the solution to the problem they had posed without considering how they would actually do the problem had it been presented to them. No thought was given to the relationship between the problem (words) and the mathematics (equation). When asked to find the solution to the question they had created, Klara read the question (not verbatim).

Klara: *Ana has x amounts of apples and Alex has four times more than*

Ana. *And Alex asks Ana if he can have her apples and so they can divide them to* ***their*** *four friends also. And so then each person, when they have divided them, each um gets forty-eight apples. How many apples did Ana have from the beginning?--- So then you take forty-eight that everyone gets times six 'cause there are six persons. Then you get* [uses calculator] *two hundred and eighty-eight. Yeah. And two hundred and eighty-eight divided by four, 'cause you got Ana's ... four that's around*

Nathaniel: *Divided by four is. Divided by four is seventy-two. So we have the answer that is seventy-two*

Klara: *Ana has seventy-two apples from the beginning. Ta da*

During this conversation, all four wrote down the solution as they had discussed it in their posing situation. They did not realize that their working and solution did not match the problem that they had posed. Notice that Klara had to clarify again in her mind that there were six people, and stated the four "*'cause you got Ana's.*" However, she did not identify the fact that Ana had x apples, and it was Alex who had four times more, or that, when they were put together, there would be $5x$ apples. Thus, the students did not really rework the problem, but simply used the numbers that they knew.

Again, it is unclear whether this was due to the use of a second language, forcing them to concentrate on the linguistic structure of the problem created, or because they already "knew" the process or equation through which they got the solution. This area of the research needs more study in other situations and should be compared to similar posing situations for first language students.

Discussion

The students involved in this discussion were described by their teacher as being mathematically able and seemed to have a good command of the second language, English, in which they were asked to pose a question. However, their *primitive knowing* (Pirie & Kieren, 1994) was constructed through learning in Swedish. Although they had been advised that they could use Swedish in their discussion if it helped them clarify their meaning, they did not feel the necessity to do so. It seems that there were no situations in which they felt that they could not express themselves adequately in English. They concentrated more on the language of presentation than on the mathematical presentation, seeming to want to be sure the problem was grammatically correct. The result was a solution that did not agree with the actual problem posed. It agreed with their original "equation" and did not consider the details they inserted into the problem, nor was the problem at all creative. It was simply a copy of a typical textbook problem. It may be that, since the students had been taught mathematics in Swedish, they had not encountered more complex mathematical terms in English, so that their ability to express themselves was limited to the terminology and concepts that they understood in English. It could also be that their concentration on the correct grammatical presentation deterred them from concentrating on the mathematics.

While one must be careful not to take the situation out of context or to generalize from one case, one must still consider the similarities between this situation and the situation of students new to a country where the language of the classroom is not the student's first language. Many of the students placed in an English speaking mathematics classroom in North America have a much poorer command of the English language than do these students. Learning to converse in a second language requires approximately two years of extensive exposure, while learning to communicate technically requires closer to seven years (Cummins, 1979). Thus, for second language learners, concentration on the correct use of language (conversation) may detract from their ability to concentrate on the mathematics involved (technical). Understanding mathematics involves an interplay of conceptualizations and verbalizations that must be meshed together to create meaning. If the languages of the two (conceptualization and verbalization) are different, the verbalization, the need to understand and be understood, and the need to express oneself as appropriately as possible, may take precedence over the conceptualization. In concentrating on the language, students may not be able to attend to the mathematics.

In a problem-posing situation, the linguistic structure of the problem being posed, as they are trying to make meaning of the language used as well as the mathematics involved, may be of greater significance to second language students than the mathematical structure of the problem and its solution. Therefore, problem posing may not be a suitable method for determining their level of mathematical understanding.

A TRADESMAN'S MATHEMATICS UNDERSTANDING IN A MACHINE SHOP: INSIGHTS PROVIDED BY THE PIRIE-KIEREN THEORY

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This case study examines the understanding of ratio used on the job by a successful tradesman as he designed and built a milling machine gear mechanism. The tradesman had limited formal mathematics and technical training and in problem solving he made extensive use of hands-on materials, trial and error and the freedom to try out an answer and modify it as needed. Using the Pirie-Kieren theory of mathematical understanding, it was determined that the tradesman worked at the levels of image making, image having and property noticing. The power and utility of these problem

solving methods evident in this study and, from the perspective of the Pirie-Kieren theory, the necessity to work through these levels of understanding in the process of developing the higher levels of understanding associated with the school mathematics challenges the authenticity and validity of much of what is done in schools in the name of developing students' understanding of mathematics and teaching students to become problem solvers.

This case study examines some of the mathematics understandings used by a tradesman in solving problems encountered in his work. This individual is of particular interest in that, while highly regarded for his expertise within his industry, his formal mathematics and formal industrial training have been minimal--he developed the vast majority of his technical knowledge on the job in a variety of industries over his career, and he has been self employed and working independently in his own machine shop for most of the past 25 years. (The participant, now 81 years of age, began working full time at age 14 in a garage after receiving three years of elementary school education.) Thus, many of his mathematical understandings and problem solving methods have been constructed in response to particular problems encountered on the job, making use of the tools and equipment at hand, and reflect a limited degree of prescribed school taught methods. It should be noted that a minimum of research has been published which focuses on the mathematics and problem solving performed on the job in an industrial setting. (e.g., Scribner, 1984a, 1984b, 1984c) It is also the purpose of this study to examine the usefulness of a type of novel and open-ended question to initiate dialogue with an individual for the purpose of revealing mathematics understanding. The particular question used in this study was, "The answer is 72, what is the question?"

Mathematical understanding was examined in this interpretive case study through the lens of the Pirie-Kieren dynamical theory of mathematical understanding (Pirie & Kieren, 1994). In comparison with situated cognition, for example, which focuses on identifying and comparing mathematics practices in different settings, the Pirie-Kieren theory facilitates the analysis of even brief mathematics problem solving episodes in terms of a developmental perspective of individual mathematical understandings.

The volunteer subject was interviewed over a number of days in his workplace and asked to talk about his use of mathematics and problems that he solved in his work. These exploratory discussions were initiated, in part, using the question indicated above. On the basis of what was said, further questions were created *in situ* and posed by the researcher to encourage the participant to elaborate on the types of problems that he solved and the use of mathematics in his work, his ideas and his methods for performing these and his criteria for assessing the appropriateness of his methods and results.

Audio recordings of the interviews, all written work produced by the participant, photocopies of printed reference materials cited and photographs of artefacts referred to during the discussions were retained for analysis. These data sources were reviewed by the researcher, significant episodes were identified and coded in the interview dialogue, and verbatim transcripts were constructed of many of these episodes. All of these information sources were then used in concert to construct the case description. Finally, the conclusions constructed by the researcher were shared with and validated in further discussion with the tradesman himself.

A number of features of this man's use of mathematics and mathematical problem solving figured prominently in the discussions. These included the use of charts for useful numerical data and unit of measure conversions, the use of various specialized measuring tools, the use of tools and machinery in the problem solving process, the use of standardized and prefabricated components when building things, and the use of conventions and "rules of thumb" specific to machining. In all cases the appropriateness of his work was tested empirically—did it do the job?

For the first interview, the tradesman brought and presented a number of charts that he used on a regular basis. He explained that he didn't really do much math on the job, instead he relied on these charts and others like them in his machinist's handbook. A number of these charts were posted prominently on the walls in both the machine shop and office work area. The charts that he brought included: industry standard drill sizes in inches to three decimal places and the drill sizes needed for tapping holes of different sizes (i.e., making a thread within a hole for a bolt), measurements for standard male threads of different sizes, the dimensions for various shapes and sizes of steel bars available for the manufacturing process, the dimensions for locating between five and 12 holes equidistant around a unit circle, recommended torque values for fastening bolts of various sizes and various metal strengths, the required lathe settings to create standard sized threads, and a standard trig table which included the formulas for determining all possible unknown values within a right triangle given one side length and one other piece of information. It was the tradesman's view that if you didn't have the required information at hand, you could always look it up in a machinist's handbook or equipment manual. In a subsequent discussion, the tradesman agreed with the researcher's conclusion that using these charts minimized the need to perform multi-step calculations, use formulas and perform any form of algebraic manipulation.

Throughout the interviews the tradesman also explained how he used specialized tools and/or equipment for: measuring the thickness or diameter of an object to the nearest ten thousandth of in inch, measuring the depth of machine treads (used in conjunction with a special chart), setting material to be machined at a predetermined and precise angle of inclination in a vice, precisely rotating and securing a work piece to be machined using specialized chuck, as well as how he used the digital readouts on a number of his machines to locate the center of a piece of work, to the nearest thousandth of an inch, by taking an average of the co-ordinates of the outer edges, and how he used these same digital readouts for converting between imperial and metric measures of length. Like the charts described in the paragraph above, the tradesman used specialized tools and his shop equipment to perform routine mathematics tasks in ways which simplified his work and minimized his effort.

Another prominent part of the discussion related to conventions or "rules of thumb" and the use of available standard or prefabricated tools and components in his work. On technical drawings for example, dimensions given in decimal form were understood to be accurate to a thousandth of an inch with tolerances of plus/minus " a thou"—the standard unit of measure for amounts less than an inch. Decimal values of tenths, hundredths and ten thousandths of an inch were all expressed in terms of thousandths of an inch. Dimensions given on technical drawings in fraction form were understood to have tolerances of plus/minus 1/64 of an inch or plus/minus 15 thou. (Author's note, 1/64 = 0.015625) When cutting steel on a band saw a rule of thumb for selecting the right type of saw blade was that ratio of teeth on the blade to thickness of the material be no less than one and a half to one. This was to prevent the work from vibrating as it was being cut. And, when cutting a keyway slot in a hole (for securing a gear or wheel onto an axle), the width of the slot depended upon the standard sized cutters that were available to do the job and it was a rule of thumb that the depth of the keyway slot was half of it's width. Threads were always cut using standard metric or imperial thread sizes because of the tools and equipment used to perform these tasks, pre-existing and standard material sizes and prefabricated components were routinely used or modified for use in the design of objects to simplify the manufacturing process. And lastly, all of the cutting, milling, drilling and grinding operations could only be performed as permitted by the machinery available. The effect of these practices was to set a highly codified context in which problem solutions were conceptualized. For example, when approached about building a specialized latch for a garden gate, the tradesman's first response was to reframe the proposed design in terms of the material and components that he currently had in stock, and in terms of standard sizes and standard machine operations.

Unlike school mathematics or school problem solving where the answer is at the back of the book or the teacher is the arbiter of solution correctness, the tradesman was able to test the appropriateness of his efforts by trying them out and checking to see if they functioned as they were supposed to, and in the end they always did! Whenever a design did not meet the intended criteria, this indicated simply that the process was not yet finished. In fact, it was not uncommon for a client to return after a job had been completed requesting that further modifications be made to the device that had been made because new ideas arose from seeing it and using it.

If the answer is 72, what's the question?

When asked what question could be answered in his work with the answer 72 the tradesman, after an extremely long pause, explained that three of the gears, in a complex gear mechanism that he designed and built some years earlier for manufacturing hand reamers on his milling machine, each had 72 teeth. A reamer is a tool used for shaving off a minute amount of metal inside of a hole to make it a precise size after it has been drilled, or to resize a hole precisely after it has become worn. To make reamers, pieces of round bar are first cut to length and small center holes are drilled in each end to a depth of approximately a quarter of an inch. These pieces are then held by their center holes and passed lengthways through a rotating cutter on a milling machine to make between six and twelve evenly spaced indentations or "flutes" in the reamer between four and a half and five inches in length at one end. The reamers are then hardened and the metal which remains between the flutes is then ground to an exact size to serve as cutting blades. A large portion of the interview time was then spent discussing the design criteria and construction of the gear mechanism used on the milling machine in this manufacturing process.

The gear mechanism was designed in order to perform three operations for milling reamers of different sizes: 1) to allow for the reamers to be rotated or "indexed" on their center axes after each pass through the cutter so that a predetermined number of flutes could be cut, 2) to rotate the reamers slightly as the they passed through the cutters to produce a helix or spiral cut pattern in the flutes and to adjust the degree of rotation for different sizes of reamers, and 3) to hold, index and rotate up to four reamers at a time for simultaneous machining. The power to the gear mechanism was fed from a shaft on the table of the milling machine.

The tradesman explained that when he designed the gear mechanism he drew upon his experience using various combinations of gears--which he changed manually on an old lathe in order to cut helical threads. He had often used a gear with 72 teeth with various smaller gears to reduce the speeds on the lathe. He chose to use three 72 tooth gears with three smaller gears in the gear mechanism for his milling machine to obtain the slow rotation needed for turning the reamers as they passed through the cutter. The choice of 72 tooth gears in particular allowed for simple gear ratios with many different sizes of small gears, these particular gears met the size and strength requirements of the job and by choosing three gears of the same size he would need only to design and build one kind of bracket to hold each of them in place. The 72 tooth gears were paired with of a 16 tooth, a 36 tooth and an 18 tooth gear which the tradesman explained " all divided 72 evenly." He explained further,

> My math wasn't that great, that I could figure things out in advance, to come to the exact ratio.... I wasn't interested in accuracy, where one or two teeth would make a difference, but I had to be in a certain range. I assembled a couple of them and then checked the results and then assembled one more and then I made a note of it, it was a matter of trial and error. This was pretty much the process.

Two other gears were needed to complete the mechanism, one at the beginning of the gear sequence and one near the end. Initially two specific gears were selected to provide the appropriate rate of rotation of the reamer to produce the helical pattern

Size	Flutes	Power Gear	1st Stud	2nd Stud	3rd Stud	Spindle Gear	Index turns
1/4" up	6	27	72/18	72	72/16	48	1/2
7/16"	6	24	72/18	72	72/16	48	1/2
1/2"	8	24	72/18	72	72/16	64	1/2
5/8"	8	36	72/18	72/36	72/16	64	1/2
11/16"	8	27	72/18	72/36	72/16	64	1/2
1 1/8"	8	24	72/18	72/36	72/16	64	1/2
1 5/16"	10	27	72/18	72/36	72/16	80	1/2
1 3/4"	12	18	72/18	72/36	72/16	64	3/4

Figure 3. The gear configurations for machining various sizes of reamers.

needed for a one inch reamer and to allow for the indexing required to make eight flutes. Then over a period of many months, other gears were purchased, one or two at a time, for use in the beginning and end positions of the gear sequence to obtain the spiral patterns required for reamers ranging in diameter from one quarter of an inch to one and three quarter inches for reamers with six to twelve flutes. The specific combinations of gears that worked best were then recorded on a chart for future reference. A recreation of this chart is included here as Figure 3. When asked if, at the start of the design process, he was sure that the speed reduction and spiral cut features of the gear mechanism would work as needed using the initial three pairs of gears, the tradesman answered "No. Probably 80 percent sure."

When asked about how well his gear mechanism worked for making reamers of different sizes, the tradesman explained that, "I ran into some problems with the one and one eighth inch reamer. I still haven't figured out how to give it enough spiral on the flute. It's not quite enough. There's enough to get by. It's not that critical."

The tradesman was asked if he could relate the numbers on the gear combination chart for the gear mechanism. To get at this the researcher posed the question, "If part of this chart was missing, could you use the existing numbers on the chart to figure out the missing ones?" He responded by explaining that he would set the machine up with a finished reamer in place and observe how it moved using different combinations of the gears until he found the combination which gave the required indexing and spiral.

Assessment of Mathematical Understandings

The focus of this analysis is the tradesman's understanding of ratio as evidenced in his description of designing and building the gear mechanism for his milling machine and, as mentioned earlier, mathematical understandings evident during the interviews were assessed using the Pirie-Kieren theory. In order to make the case that the tradesman was able to work at least to the level of property noticing, evidence for the levels leading up to and including this level will be presented in turn.

The level of image making was evident when the tradesman tried different combinations of gears to adjust the speeds of both his lathe and milling machine, and when he worked with different combinations of gears by hand, counting the resultant number of turns of the small gears for each turn of a 72 tooth gear. The level of image having was evident when the tradesman commented that linking 16, 18 and 36 tooth gears with 72 tooth gears all resulted in simple gear ratios because these numbers "all divided 72 evenly." Finally, the level of property noticing (specifically the property of equivalent ratio) was evident when the tradesman responded that a 90-30 gear combination would act the same way as a 72-24 combination when he was asked why he chose to work with 72 tooth gears instead of, for example a 90 tooth gear attached to a 30 tooth gear. When asked about the possibility of using other gear combinations, he explained that a 36-18 gear combination would work in the same way as a 72-36 combination. Elsewhere the tradesman explained that linking a large gear with a small gear would reduce the speed in a machine and that the larger the difference in size, the greater the change in speed.

There was no direct evidence of the level of formalizing. At this level, a research subject is able to abstract a method or common quality from his previous image dependent know how and would be ready for, capable of enunciating and capable of appreciating a formal mathematical definition or algorithm. It is unknown whether or not the tradesman was ready for this as it did not arise as a part of the conversation. It is this researcher's conjecture that the tradesman would be able to follow and appreciate a formal mathematical explanation of his gear mechanism, but this remains only strong hunch.

It is asserted that the tradesman did not work at level six, that of observing at which a research subject is able to reflect upon and co-ordinate formal activity and express patterns as theorems. When asked if he had a way to relate the numbers on the gear chart, he could not without deferring or "folding back" to actually using the gear mechanism with a reamer which is the level of image making. The process of folding back, which is vital to the growth of understanding, is a crucial feature of the Pirie-Kieren theory.

Using the Question, "If the answer is 72, what is the question?

It appears that the researcher had luck on his side when he posed this question to the tradesman. The number 72 had come up in the interview a few minutes earlier when the tradesman showed the gear configuration chart for the gear mechanism to the researcher. When asked if he could think of another question from his work with 72 as the answer, the tradesman could not. A review of all of the printed material from the interviews did not reveal any other meaningful instances of the number 72 and the tradesman reported that measurements of 72 thousandths of an inch and 72 inches were very unusual in the kind of work that he did. And, when asked what other number might make this question more meaningful for him, he picked things that were constant terms in his calculations, specifically pi and diameter--a value typically given or known in this man's work, not something to be determined. His response to this latter type of open ended question for revealing mathematics understanding appears to relate to the context specific way that this tradesman used mathematics and solved problems.

Conclusion

The mathematical understandings revealed in this real-world and authentic problem solving context are significant for a number of reasons. First, little detail is known about the kind of mathematics and mathematical problem solving that is done in industrial settings such as a machine shop. These findings provide evidence that that a tradesman can work effectively, at least some of the time, with little formal mathematics. Second, the context in which this tradesman solved problems, specifically with outcomes that he valued highly (i.e., the desire to get the job done) and a need to be self reliant stands in marked contrast to the context in which mathematics and mathematical problem solving are done in a mathematics classroom. It is reasonable to conclude that these non-mathematical dimensions of problem solving contributed greatly to the tradesman's persistence as a reflective problem solver, which ultimately has led to his effectiveness without the need for more sophisticated understandings of mathematics. And lastly, the viability of the methods that were developed and used successfully in this setting, namely the use of hands-on materials, trial and error and the freedom to try out an answer and modify it as needed reveal the power and utility of these methods for problem solving. These methods are seldom allowed in school classroom, and yet an individual can use them to solve complex problems effectively. Furthermore, the Pirie-Kieren theory indicates that these types of activities, which were associated with the image making, image having and property noticing levels of mathematical understanding, are essential in the development of higher levels of mathematical understanding customarily associated with school mathematics. This finding challenges the authenticity and validity of much of what is done in schools in the name of developing students' understanding of mathematics and teaching students to become problem solvers.

It is clear that further research on the use of mathematics in industry would further our understanding of the development of mathematics understanding beyond the classroom; the connections, or lack thereof, between classroom mathematics and the workplace; and the design and implementation of mathematics curriculum, particularly for students who may enter a technical field when they finish school.

STORIES OF "72"

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This study illuminates the rich, flexible ways in which primary children search for and access current knowledge of number or further develop their mathematical understanding of number while working on a novel problem task. A group of four grade Two and Three was videotaped as they developed a question or task where the answer is 72. The seventy minute videotape was then analyzed using the Pirie-Kieren theory and model for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994a). Two aspects of the theory, folding back and primitive knowing were useful in describing and analyzing the children's actions. The notion of 'interventions' provided a powerful vehicle for considering how the actions and interactions of the teacher occasion the emergence of mathematical understanding. The notion of story as articulated by Borasi, Sheedy & Seigel (1990) was also useful in describing and analyzing how the idea of story impacted how the children approached and develop their problem solutions of stories.

By focusing the lens on a group of four children, this study illuminates the ways in which some primary children search for and access current knowledge of number or further develop their mathematical understanding of number while working on a novel problem task. This study is part of a larger research project headed by Dr. Susan Pirie that is focused on the individual's growth of mathematical understanding in diverse settings. In developing an analysis of the students' activity, the paper draws on elements of the Pirie-Kieren theory for the growth of mathematical understanding and on the power of story in developing children's mathematical understanding as elaborated by Borasi et al. (1990).

The class and the small group were videotaped as the children worked on a task that asked them to create a question for a given answer of 72. The children were taped and observed during the seventy-minute session and the author took some field notes. The notes are brief however, as the author is also the classroom teacher who is seen on the tape interacting with the featured group of four students. This is a mixed ability, combined Grade Two and Three class consisting of 11 Grade Three and 10 Grade Two students. All the children were asked to record their thinking on paper about the question related to an answer of 72. The videotape is focused on four students, three Grade Three and one Grade Two children as they discuss their understanding of the number 72 and how to create a question where the answer is 72. Like the rest of the class, this group recorded their personal approach on paper. The class has engaged in a variety of tasks in the previous months through calendar math, specific activities and explorations designed to develop their knowledge and understanding of number in flexible ways. In other classrooms in previous years, the children became used to seeing and solving story problems where all the information is included in the story and solving means to merely use the given information. These experiences with the typical story problem form an obstacle in this atypical problem situation. Now they are asked to invent the information for the story. 72 is not part of the story but is its resolution. On the previous day, the children and myself informally discussed how many different questions we could ask someone where the answer is 25. During this discussion it became apparent that the concept of question was very difficult for them to use in a practical way. Many of their responses were equations that "equaled" 25. I began using the phrases 'story question' and 'task question' as a way to focus their thinking on how to create a question that did not include the answer of 25. This same language was used to pose "the answer is 72, what is the question?" task on the following day.

In order to analyze the students' mathematical understanding of number, use was made of the Pirie-Kieren theory and model for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994a). This theory has been presented and discussed at a number of previous PME and PME-NA meetings (see for example Pirie & Kieren, 1992; Martin, Towers, & Pirie, 2000). The Pirie-Kieren theory provides a way to analyze, describe and account for the growing understandings of individual learners of mathematics and, through the notion of 'interventions', provides a powerful vehicle for considering how the actions and interactions of the teacher occasions the emergence of mathematical understanding. Two aspects of the theory, folding back and primitive knowing were useful in describing and analyzing the children's actions. Primitive knowing refers to the mental images a child "holds in her head" when approaching a new mathematical task. Folding back describes how the learner, when faced with a novel question or task that is not immediately resolvable, will return to an inner level of understanding (like primitive knowing) for concepts such number. This mental action involves reviewing and reshaping earlier understanding in the context of the present problem situation (Pirie & Kieren, 1992). As well, the notion of story as a powerful vehicle in helping children examine and represent their mathematical understanding, as articulated by Borasi et al. (1990) was useful in describing and analyzing how the idea of story impacted how the children approached and developed their problem solutions or stories.

What follows is a detailed analysis of the videotaped session featuring the activity of the group of four students. The analysis includes a description of the children's activity to provide the reader with a flavour of what was observed.

The notion of using story as a context for this task is strong for all the children and is evident in the way many of the students frame the task for themselves. In our mathematics' classroom, books are frequently shared and discussed and the concept of story is used extensively when learning concepts. When reading books together, the children are invited to talk about the mathematical ideas they see embedded or emerging from the story. At other times, the children are asked to "tell the story behind numbers" and to write about what they are learning and have understood. As with other researchers such as Borassi et al. (1990) and Whitin and Whitin (2001), stories in books, personal stories and a variety of writing is valued in our classroom as a powerful way to learn mathematics and to represent one's mathematical thinking and understanding. As well, our pre-discussion shifted from using the word "question" to what "story task or story question" you could ask someone that would result in an answer of 25. The word "question" is confusing for most of the students. As indicated earlier, their experiences with problems has developed an expectation that problems are stories where all the information one needs to find a solution is present in the given story. Now they are asked to write such a story themselves where the only information is the solution. In this initial discussion, it is also clear that many of the children do not understand what is meant by a question. This is not surprising when I recalled all the instances when various children would launch into personal stories in response to a presenter's invitation to ask her questions.

To set the stage, the children are asked to recall our discussion the previous day about the various ways that we could ask someone "to do something", a story question, a story task or activity, that would result in an answer of 25. The students then are asked to consider 72 as an answer and are invited to select from a variety of manipulatives to help them create their question, task, story or activity either with their group members or on their own. They are also given paper to record their thinking so that all the children's work could be examined later.

After carefully examining all the children's recordings and sorting them into categories, it was clear that the small group's activity could be used as a way to discuss how various children understand the number 72. What follows is a discussion of how each child (Noel, Elizabeth, Cole and James) understands 72 and use what he or she knows to create a story or a story game situation that represents 72 or has a goal of 72. The discussion will also suggest instances when individual children grow in their understanding through interactions with each other or with myself.

James and Cole both create similar stories about a store that sells items like cupcakes, muffins or cookies. They both use their selected manipulatives to initiate a story idea. At first, it looks like they are merely "playing." The arrows that are constructed are unrelated to the stories they eventually develop on paper. As described by James, the arrows are merely "thinking arrows, they help me think." But as they progress through the task, the materials seem to represent story elements rather than specific mathematical concepts and processes. It appears that James and Cole are engaging in a meaning making process where their understanding of number is being developed and represented as specific story elements (Borasi et al., 1990). The unifix cubes, tiles and dual coloured chips help them shape the story and the question that is posed at the end of their stories. As the two boys describe their stories, it is evident that their understanding of number is very different. In James' story problem, he begins by describing a store that has 72 cupcakes to sell. James appears unable to conceive of 72 as being an answer. Perhaps due to previous experiences with typical story problems, he sees the need to include 72 in his story as part of the story question. In contrast, Cole's story about a store includes concepts of money, multiplication, addition and subtraction as he describes how the shopkeeper sells various amounts of muffins and cookies (ten cents each). Cole's story also describes how the shopkeeper suffers the theft of differing amounts of money at two different points in the story before he poses the question of "how much does the person have?" Cole has flexible, multifaceted images for the number 72 that he accesses mentally and represents as elements of his story. Through mapping (Pirie & Kieren, 1992) it is possible to draw a "map" of Cole's activity and to identify, track and analyze the growth of Cole's understanding with the aid of the Pirie-Kieren model (see Figure 4 for the pathway mapping Cole's growing understanding).

The videotaped recording gave me a way to analyze Cole's actions with the manipulatives, his conversation with group members and our discussion together, providing me with a way to trace his movement between the layers of the model. Cole is at a level in his mathematical understanding that the Pirie-Kieren model describes as image having. He no longer needs to perform certain actions with materials to demonstrate concepts such as addition, subtraction, multiplication or division. Rather, Cole can manipulate his mental images as if they were objects to compose his story. The previously described events are all image having activities. Cole has also crossed what the Pirie-Kieren model describes as the "don't need boundary" between the image making level and the image having level. Cole's understanding of "the number 72" is not tied to a specific image. Rather he is able to flexibly and mentally create story details that require him to mentally calculate the total earnings from the sale of various amounts of muffins and cookies and to track the total earnings after losing various amounts of money through theft. Cole recognizes that 72 can be represented by groups of muffins and cookies, as differing amounts of money and as mathematical processes where 72 is the result. The Pirie-Kieren theory describes this level of understanding as property-noticing, a level where learners recognize properties of a concept such as "that 72 can be represented by multiplication or by the processes of adding and subtracting amounts of money." Unlike Cole, property noticing is absent in James' understanding. James prides himself on being able to quickly calculate equations. Numbers are for calculating. He does not seem to make the connection between calculating and the reason for calculating or in other words, the story behind the numbers.

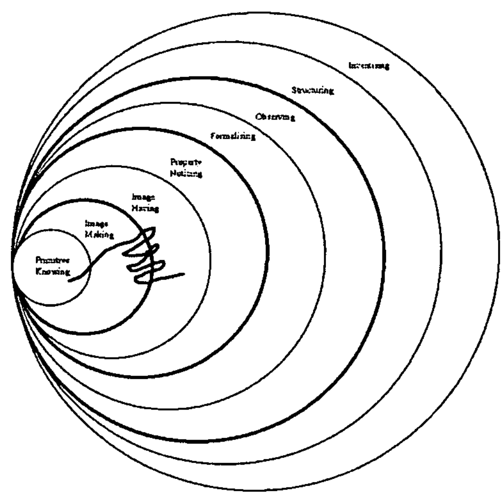

Figure 4. A mapping of Cole's growing mathematical understanding.

James and Cole worked independently, there was virtually no discussion of their stories either between the boys or with myself. In contrast, manipulatives and prolonged discussions were central in how Elizabeth and Noel created their games. The game idea was a radical departure from the approach used by other children to weave a story around the number 72. This is a highly atypical context for exploring number for these children. There is no guiding information or story leading them to a correct answer. They have to imagine and write the story themselves. Creating games was the way Elizabeth and Noel respond to this challenge. These children used their manipulatives extensively to make and explain their individual games. Ongoing discussions with each other and with myself prompts both Elizabeth and Noel to engage in specific actions that results in the emergence of richer, personal understandings for number. Both children recognize that 72 can be broken down in a variety of ways but their progress is slowed by the need to "fold back" to examine, strengthen or create images for specific concepts such as multiplication and division. The number of players in Noel's game changes progressively as his images for multiplication and division become more developed. The number 72 is initially represented in his game as the number of stops a player's bus can make and the total number of passengers that get off the bus at the various stops. Each bus holds four people (the bus is eight multilinks snapped together with four bingo chips on top) and players race each other to see how many people they can "put off at the most stops." These story elements represent Noel's primitive knowing about the structure of games which he uses to build images for the number 72. After beginning to make rows of pennies (a penny represents one stop), Noel asks "what is half of 72?" He appears to be trying to create a game where two players try to race each other to see which one can get to their half of the stops first. Noel needs to fold back from image making activities to his primitive knowing that an even number can be broken into two groups. He then uses this for the number 72. My intervention is intended to help him make an image of how 72 can be divided into two groups, an intervention described by Pirie & Kieren as invocative. After placing dual coloured chips into 'partners' while counting by twos to 72, two rows are created with 36 chips in each row which I believe "shows" Noel that half of 72 is 36. However, it is not apparent to Noel as he continues to count all the chips then one row of chips and eventually the number of pennies in his rows to establish and re-establish the number he seeks. He folds back to primitive knowing about how to count objects to 72, one by one. Noel seems to be working at the image making level, trying to create an image that will help him to build the game but he returns to an earlier understanding of number to help him create the images he needs to create the game. As he continues to develop the game for four players then six players, our discussions center on the total number of stops in the game and the number of stops in each of the four then six "alleys." I am trying to grasp his current level of understanding. While this is the intention of my validating intervention, I am not sure that Noel responds in the manner the intervention intends. At times, he would begin another image making action such as counting or placing more pennies on rows or would simply wander off! By the time Noel scribes his directions to me for how the game is played, it is apparent that he has worked out how 72 can be developed into a game for two, four or six players.

The goal of Elizabeth's game is also 72 people being dropped off at stores but three people disembark at each store. Each player has a "train" of 10 connected multilink cubes with 10 coffee beans on top of each train. Coffee beans represent the people and dual coloured chips represent the stores, one row of red chips or "stores" and one row of yellow chips or "stores. Elizabeth attempts different ways to set up the game with the various materials using her primitive knowing about numbers. When we begin to talk about her game, she tells me she has "too much" because 40 and 40 is 80. She is referring to the two rows of 13 chips with 3 beans on each one and a single bean at the end of the row or 13 x 3 + 1 = 40 plus 13 x 3 + 1 = 80 for the two rows. Our conversation centers on having her explain what she is doing and thinking. Elizabeth has folded back to her image for the quantity 80 but this is not a useful image. Elizabeth very quickly says she needs to figure out "how many pairs of chips with three coffee beans on top would be 72." She needs a new richer image that will help her to continue designing the game. Without further conversation, Elizabeth continues to add chips to the rows until she has counted by three's to 72. She needs an opportunity or space to consider what she has done and is thinking about and how to proceed. In folding back, Elizabeth may have accessed images she has for number and developed richer images (image making) to guide her in completing the game.

This study suggests that primary children approach problem tasks with rich, diverse and flexible levels of mathematical understanding. In this study, the children were asked to use what they understand about number in a situation they had never encountered before. In previous experiences, problems had been solved by using the information in the story problem. Now they are asked to create the information or story where the answer would be 72. For some children like James the novel problem situation remained an obstacle preventing him from seeing the answer of 72 as separate from the story information. For other children like Cole, stories provide the means to represent their thinking and understanding. Composing a story provides children with a way to "generate and elaborate mathematical ideas" and to engage in problem posing in a manner that honours current thinking while encouraging a deeper understanding of mathematics (Borasi et al., 1990). The context of story helped children like Elizabeth and Noel to grow in their mathematical understanding of number. While weaving details of the game, these two children used current number knowledge and evolving knowledge to develop the games. At the same time, the story game context created a need for new images to guide the creating process. Teacher interventions play a key role in supporting the growth of the children's understanding by providing thinking space. The intention of teacher interventions is to provide children with opportunities for them to independently construct or modify their personal images. Often teacher interventions are meant to verbally prompt children to return to their conceptual images, to create new richer images that will guide their outer actions. However such interventions can also create a space where the child is given a quiet place for reflection unhindered by time or conversation. An unusual task occasioned the creation of oral and written stories as the primary children grappled with how to create a problem task or question where the answer is 72. While writing their stories, some of these children also grew in their understanding. The stories of 72 authored by these young children reveal an understanding of number that is both diverse and complex.

...AND THE ANSWER IS 72: MATHEMATICAL UNDERSTANDING THROUGH OPEN ENDED TASKS

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This poster focuses on children's mathematical understanding of basic operations in mathematics, such as addition, subtraction, multiplication as well as estimation and place value, when they are given an open-ended task.

The research involved the videotaping of an interview with a group of grade 4 students. This particular group is comprised of children, about whom their teacher has some concerns regarding their understanding and performance of mathematical skills. It is for this reason that these particular children became members of the focus group. The data was analysed through the use of the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding. (Pirie & Kieren, 1994).

The task posed was to make up a problem for which the answer is 72. The students were divided into two groups of three and were initially given about 15 minutes to come up with as many problems as they could. They were each given paper and a pencil, and had access to their textbooks.

It is worth noting that in the three months that I worked with, and observed this class, the students had no exposure to any sort of open-ended activities or problems. In addition, I never observed any group activity during math class. The students' sole direction in learning and understanding has been by way of a single dimensional approach to the learning of a particular algorithm. In the time that I was in the classroom with these students they did a short unit on estimation. The textbook presented the idea of estimation, for the most part, by a series of addition and subtraction exercises, whereby the students were asked to give, not an exact number, but rather an estimated one. The exercises were generally not situated within an appropriate everyday context.

It is clear that the mathematics used by the students in the new problem situation reflected their prior learning experiences. In general, the students did not come up with situational problems, but rather exercises in addition and subtraction and, in some cases, multiplication. When asked to come up with their own problem, not one of the students entertained the idea of coming up with a problem involving the concept of estimation. When it was suggested that the students think about using their estimation skills in the creation of their problems, one girl said, "I can't do estimating." One of the boys said, "Estimation is just guessing." In this school year the students spent several weeks practising algorithms for addition and subtraction, and it is these algorithms and rules which form the pupils images for number operations. The poster will elaborate on the nature of images and explore how they were able, and unable, to use them in creating questions which led to an answer of 72.

IS IT AN ALGEBRA QUESTION? – ANALYSIS OF STUDENTS' STRATEGIES WHEN SOLVING A NUMBER THEORY PROBLEM

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The theoretical underpinnings of this study came from the Theory for the Dynamical Growth of Mathematical Understanding (Pirie, 1988; Pirie & Kieren, 1989, 1994). The Pirie-Kieren theory provides a way to visually trace the thinking actions of individuals working mathematically, and the growth of their mathematical understanding in a non-linear process related to *particular mathematical topics*. The theory posits *Primitive Knowing* as the inner layer of the model, which contains all a student's prior understandings that may influence the growth of the *new topic* being considered. In addition, it has been shown that when faced with a problem, students need to *fold back* to an inner layer of understanding, often their primitive knowing, to enable them to extend their current inadequate and incomplete understanding.

Several questions have arisen which form a focus for the paper: 1) How do students solve a non-routine mathematical problem? 2) What connections, if any, do students make to their primitive knowing when introduced to a new problem category? and 3) to what extent do students' existing understandings assist or hamper their growth of understanding of a new problem category. It is the purpose of the poster to discuss possible answers within the context of observing how students solve a particular number theory problem, a topic not included in school curricula. The problem was:

Alfie: Can you guess the ages of my three children? The product of their ages is 72.

Beatrice: That's not enough information. Give me another hint.

Alfie: My youngest child is more cheerful than my twins.

Beatrice: Now I can figure it out!

Six pairs of middle school to university level students were engaged for one-hour sessions in informal settings. The participants were asked to think aloud as they worked and responded to a word problem. All sessions were videotaped and transcribed. The students were encouraged to use pencils and papers to record their work. The analytical model for examining the videotape data emerged from the research activity of Davis, Maher & Martino, (1992), Mercer (1995), Pirie (1996, 2001) and Powell (2001). Analysis of the transcripts focused on characterizing students' individual behaviour as well as identifying trends or similarities across their actions. Analysis of students' actions, inter-student discussions, and their written work were used to produce a description of the students' growth of understanding as they worked through the mathematical task. With the investigation of each individual, clearer patterns of students' mathematical growth of understandings emerged. Most students considered the problem "algebraic", and tried to solve the problem through a two-equation system. It was evident that existing understandings of algebra did not assist their growth of understanding of a new problem category. In my poster I will illustrate some of the students' errors and discuss possible answers to the research questions.

THE ANSWER IS THE QUESTION: THINKING BACKWARDS WITH EXPONENTS AND LOGS

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This poster investigates students' growth of understanding during an open-ended problem solving session. Through focusing on this topic review session involving senior level high school students, I investigated the different images held among students for exponents and logarithms. I also investigated the different ways in which students accessed these existing understandings as they were required by the problem solving situation. More specifically I considered the ways in which students recalled and used their existing understandings when faced with a novel problem solving opportunity.

The theoretical framework I employ is the Pirie-Kieren Theory for the Growth of Mathematical Understanding. (Pirie & Kieren, 1994) This theory provides a means of analysing the growth of understanding as students revisit their existing images of exponents and logarithms and try to apply these understandings in new situations.

The students involved in this research are grade 12 level mathematics students in an enriched class at a private Catholic high school in North Vancouver, BC. The students are capable and motivated mathematics students. The students are working on an open-ended problem solving opportunity as a review of a recently completed unit on exponents and logarithms. The unit included topics such as exponential functions, the laws of logarithms, mathematical modelling, geometric sequences, and logarithmic functions. Students are placed in small groups and are posed with the problem "The answer is 72. What is the question?" The students are further challenged to create "challenging" questions to puzzle their classmates and problems that review each of the concepts in the unit. The mode of inquiry is video recording of classroom small group work in problem solving. Two digital video cameras were placed around the classroom, each one focusing on the discussion of one small group for the duration of the session. The video data was analysed to produce a meaningful picture of the growth of understanding of each small group of students as they worked on the problem.

The poster offers an analysis of this video footage by mapping the growth of understanding of each of the three groups through the levels of the Pirie-Kieren model. It also highlights the common features and the differences in the growth of student understanding for the different small groups. Particular attention is paid to the place of 'folding back' to prior understanding and to the working of students at the Property Noticing level of the theory.

Notes

[i]This is a group of teachers, faculty members and graduate students from British Columbia, Canada who work regularly together and are undertaking and interested in research into the growth of mathematical understanding at all stages and levels of learning, from early childhood to the work place.

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