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ABSTRACT

The proceedings of the 24th annual meeting of the Psychology of Mathematics Education North American Chapter (PME-NA) contains research reports, plenaries, and poster presentations. Papers include: (1) "What Do We Believe about Teacher Learning and How Can We Learn With and From Our Beliefs?" (Deborah Loewenberg Ball); (2) "Toward Evidence for Instructional Design Principles: Examples from Cognitive Tutor Math 6" (Kenneth R. Koedinger); (3) "Achieving Equity and Improving Teaching in Mathematics Education through Teacher Education and Professional Development" (Laurie E. Hart, and Martha Allexsaht-Snider); (4) "Exploration of Patterns and Recursive Functions" (Lourdes Guerrero and Antonio Rivera); (5) "Exploring the Role of Metonymy in Mathematical Understanding and Reasoning: The Concept of Derivative as an Example" (Michelle J. Zandieh); (6) "A Content Analysis of Exit Level Mathematics on the Texas Assessment of Academic Skills: Addressing the Issue of Instructional Decision-Making in Texas" (Jere Confrey and David Carrejo); (7) "Entailments of the Professed-Attributed Dichotomy for Research on Teachers' Beliefs and Practices" (Natasha M. Speer); (8) "Writing Conjectures in Geometrical Activities with Cabri-Geometre" (Ernesto A. Sanchez Sanchez and Miguel Mercado Martinez); (9) "Student-to-Student Questioning in the Development of Mathematical Understanding: Six High School Students Mathematizing a Shell" (Janet G. Walter and Carolyn A. Maher); (10) "Developing Students' Understanding of Exponents and Logarithms" (Keith Weber); (11) "Understanding Change: Epistemologies of Three Future Teachers

of Elementary School Mathematics" (Jennifer Chauvot); (12) "A Framework for Posing Technology-Rich Mathematics Problems" (Nicholas Oppong); (13) "Theory and Practice of Mathematics Education Reform: Including the Teachers' Voice" (Azita Manouchehri); (14) "Learning Trajectories as Tools for Supporting Teacher Change: A Case from Statistical Data Analysis" (Kay McClain); (15) "The Impact of Teachers' Content Knowledge and Attitudes on Instructional Beliefs and Practices" (Jesse L.M. Wilkins); (16) "A Length Model of Fractions Puts Multiplication of Fractions in the Learning Zone of Fifth Graders" (Karen C. Fuson and Mindy Kalchman); (17) "Listening as a Vital Characteristic of Synergistic Argumentation for Enhanced Mathematical Learning in a Problem-Centered Environment" (Darlinda Cassell and Anne Reynolds); (18) "Teacher Identity and Knowledge in Elementary Mathematics" (Susan B. Empson, Corey Drake, and Debra L. Junk); and (19) "Viewing a Reform-Based Mathematics Curriculum through the Eyes of Two Veteran Teachers" (Ann Wallace and Miriam Gamoran Sherin). (NB)

Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education

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Psychology of Mathematics Education

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PME-NA XXIV

October 26-29, 2002
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History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implementation thereof.

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2002	PME 26	England	Not yet available

Preface

It is with great pleasure that we at the University of Georgia host the 24th annual meeting of PME-NA as we also hosted the 4th annual PME-NA meeting exactly 20 years ago in 1982. A quick glance at the differences between then and now shows how the field has grown and changed: The proceedings that year were about 250 pages long, and there were 34 papers presented in 7 topic areas. This year the proceedings will be nearly 2000 pages long with over 200 presentations in 15 topic areas. Despite the growth of the field and the organization, the hallmarks of collegiality and open intellectual exchange remain.

The theme of this year's conference is *Linking Research and Practice*. The theme is intended to highlight the interplay between the ways that research is used in practice and the ways that research grows out of practice. The invited plenary speakers were asked to address the theme in their areas of expertise by challenging the audience to think critically about the research we do–the questions we ask, the methods we use, the contexts in which we do research, the people with whom we do research, how we communicate the results of our research, etc. We hope that those in attendance as well as those who will read these papers in years to come will be stimulated to think deeply about our roles as researchers and consumers of research.

We received over 250 proposals for sessions at PME-NA and are grateful for the work of the many reviewers who helped shape the program. We undertook only structural editing (format and references) on the final papers so as to leave intact the integrity of the authors' work.

We wish to express our appreciation to the many people at the University of Georgia who have made these volumes and this conference a possibility, including, but not limited to, Patricia S. Wilson, Margaret Caufield, Elizabeth Platt, Salli Park, Bernice Peters, Teresa Banker, Nancy Williams, Joseph Allen, Brian Wynne, and all of the faculty and graduate students in the Department of Mathematics Education.

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WHAT DO WE BELIEVE ABOUT TEACHER LEARNING AND HOW CAN WE LEARN WITH AND FROM OUR BELIEFS?

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Everyone seems to have strong convictions about teacher education—about what teachers need to learn, about how they can be helped to learn those things, about the factors that affect professional growth and the improvement of teaching and learning. There is widespread consensus about some claims, but significant dissent about others. Reasonable as they may seem, do we really *know* these things? In many cases, the evidence on which particular claims rely is inadequate. The claims may be true. But they may also require qualification. Some are probably dead wrong. Some compete with others, equally popular. In this session, we will consider a set of claims about teacher learning and what it would take to deploy them as resources for developing a stronger knowledge base about professional education for teaching. This paper is not intended as a review of the literature, and is also not meant to suggest that we know nothing about teacher learning. Instead, it is meant to provoke a collective discussion of how inquiry and practice in teacher education might support its development and improvement.

The Value of Skepticism to the Development of Knowledge

One hot summer Sunday morning, my attention was captured by a provocative headline in the *New York Times*: "Science needs a healthy negative outlook" (Kolata, 2002). When scientific experiments are successful, noted the writer Gina Kolata, they are published, and that this is how we think knowledge grows. "But the sad truth about science is that most experiments fail and the hypotheses that seduced researchers turn out not to be true, or at least, the studies provide no evidence that they are true." Are such studies are any less important than those that we call "successful?" asks Kolata. Isn't it success to show that something we thought was true may not be, or that what we believed may need some revision?

I often make myself unpopular among my colleagues by asking, "Do we really know that?" about some popular idea or another. My intention is neither to suggest that we know nothing, nor to be unproductively cynical. But I am as prone as anyone to accord unwarranted weight to my assumptions and preferred ideas. The *New York Times* headline gives me courage to remind myself that being wrong is useful to learning. I know this already from my experience as a teacher. Children's errors can be exploited to make productive mathematical progress. Indeed, in mathematics, the failed proof may yield as much insight as the successful one (see, for example, Lakatos, 1976). Why should the progress of knowledge in teacher education be any different?

Perhaps it is the pressure—from funders, from the public, from our administrators and colleagues—to show that what we do works, or perhaps it is the natural internal need of teachers everywhere to believe that we are making a difference, and to think we are "right." We take firsthand, memorable experiences and generalize them; we develop beliefs and see them at play around us; we use our operating assumptions to explain events and outcomes. It is no easy task to push oneself to consider alternative explanations, to question the reliability or validity of evidence, to see the shadows as well as the foreground, to wonder and question. Yet the capacity for surprise is at the heart of learning, the possibility of refuting hypotheses the core of good science. Scheffler (1977/1991) captures this when he writes, "Receptive to surprise, we are capable of learning from experience—capable, that is, of acknowledging the inadequacies of our prior beliefs and recognizing the need for their improvement (p. 12). And much contemporary philosophy of science reveals the importance of the deliberate effort to refute operating assumptions, of the careful pursuit of counterexample and counterevidence (Kuhn, 1962; Popper, 1963).

Disciplined inquiry depends fundamentally on both faith—the formation of strong convictions and initiatives that seek to build on those—and doubt—the deliberate effort to be skeptical about those convictions. If hypotheses are not falsifiable, for example, then learning is hampered, because experience can do no more than confirm the premises. If the data gathered cannot offer evidence that calls those premises into question, then it does not support learning. If ideas are held as principles, and never unpacked and developed, their use for practice is likely to be negligible.

The Challenge of Combining Action and Inquiry

We find ourselves in a period where developments in professional education have perhaps never been more important. New professional standards, increased pressure for testing and results, new curriculum materials, and an ever-diversifying student population all imply the need for high levels of professional skill. But ours is not a highly selective profession like law or medicine. On a massive scale, we must prepare elementary and secondary teachers to begin teaching mathematics with reasonable proficiency. And, equally daunting, we need to build a culture, including structures and resources, for the ongoing professional learning that developing the complex practice of teaching mathematics requires. Rarely has the demand for teacher education and learning been as widely articulated, and rarely have as many opportunities and as much funding existed. It sometimes seems a bit frightening, for with all humility, we know that we do not know all that we need to know. Yet we must act. As teacher educators, leaders, and professional developers, we must design courses, programs, materials, and workshops. As teachers, we must choose opportunities for our own learning. This must all build on the best current knowledge, the best wisdom of practice, and on experience.

These conditions also present the field with unprecedented opportunities to develop knowledge about teacher learning and teacher education. Making clear our assumptions, beliefs, and current ideas to frame strategic working hypotheses can enable us to use practice itself as a medium for the development of such knowledge. Doing this means identifying the best of what we currently know, considering the basis for that knowledge, and actively cultivating skepticism. It means seeking disconfirming evidence, valuing negative results, and questioning the basis for our claims. Without this stance, teacher education runs the risk of being based on ideology more than wisdom. And without evidence for what we know, and the willingness to leave open that which we do not know, we have no better claim to what it would mean to create good professional development than anyone else. Our beliefs will compete, often ineffectively, with others'. And the quality of teachers' opportunities to learn will be based on caprice and personal bias rather than evidence.

Commonly Held—and Sometimes Competing—Beliefs About Mathematics Teacher Learning

I turn now to a small set of ideas that are often heard as claims about teacher learning. My purpose is to consider what does—or would—make these "knowledge," and what it would take to use these ideas as levers for developing reliable, valid, and useful knowledge about teacher education, and to avoid reinforcing ideology that limits the development of such knowledge. Grouped into a few important categories—what mathematics teachers need to know, for example, or the structure of effective professional development—some common beliefs compete with other, also commonly espoused, beliefs. The empirical bases for any of these claims vary widely. Some have been produced from studies of teacher learning and teacher education while some are part of a contemporary ideology in this country. Worth considering is how our ideas are both helpfully shaped and also limited by culture and experience. These ideas, then, have different origins, and possess uneven warrants. But, ubiquitous, they are widely held. And each makes sense, and is in some important ways, likely valid. But each also deserves to be questioned, for its basis, and likely validity. Put too simply: Where did each of these ideas come from? What about them do we know to be true? What is likely right about them? What are you suspicious about with each one? What would it take to unpack each one, to sort out its constituent ideas and assumptions, and use the results of this analysis deliberately to make progress in what we know about teacher learning?

1. The pedagogy and curriculum of mathematics courses for teachers
   Mathematics courses for teachers should be designed to model the ways that teachers should teach their own students.
2. The structure of professional learning opportunities for teachers
   (a) One-shot workshops are not effective.

(b) Teachers learn when they can work closely with colleagues.

(c) Lesson study is an effective form of professional development.

(d) Lesson study can work in a country like Japan, but would not work here.

3. The nature and scope of the mathematics that teachers need to know

(a) Teachers need to know mathematics of the school curriculum in depth; courses that treat advanced mathematics are useless to teaching.

Alternatively:

(b) Teachers need to know more than they are responsible for teaching—advanced mathematics study is important to being able to have perspective and make good judgments.

4. The role of curriculum materials in mathematics teachers' practice

(a) Highly detailed teachers' guides and curriculum materials limit teachers' professional autonomy. Teacher education should prepare teachers to work as professionals, using curriculum materials as a resource as needed.

Alternatively:

(b) Highly detailed teachers' guides and curriculum materials can make up for lacks in teacher knowledge and experience. Teacher education should help teachers learn to interpret, adapt, and use curriculum materials.

Before reading further, it might be useful to consider your own appraisal of each of these commonly heard beliefs about the professional education and learning of teachers. For which do you think there is substantial evidence? For which do you think the evidence is sparse? Which are you inclined to agree with, and why? And with which do you take issue? On what basis?

The Pedagogy and Curriculum of Mathematics Courses for Teachers

Frequently it is claimed that teachers need to learn mathematics in the same ways that their students should experience that mathematics. An alternative formulation of this idea is that mathematics courses for teachers should "model" the pedagogy and curriculum that teachers would learn to enact with students. An under-elaborated idea, it is not clear what is included in the notion of "modeling. " Somehow, however, it is thought that instructors or professional developers would work with teachers as they learn mathematics in ways similar to what those teachers should do with their own students.

The idea of "modeling" good practice is intuitively appealing. In fact, it would seem difficult to argue that the teaching should not exemplify good practice. It seems obvious that the better the teaching, the more effective the course or workshop will be. And, most important, this "good teaching" can not only help teachers learn mathematics, but it will also help them learn about teaching mathematics. However, what would it mean for this to be something we could say we *know*, as opposed to an idea to which we are attracted?

First, it would help to unpack the idea of "modeling," and decide what the elements are that might be modeled. A second question would be what it means for an approach to be "similar." Without this sort of clarity in the core ideas of the claim, it is difficult to investigate it.

Second, it would be useful to consider what might make this a valid claim, and what might militate against it. On the face of it, we suspect that how ideas are taught and experienced shapes what is learned. If the point is for teachers to develop the sort of knowledge they will need when they work with students, then they may need to encounter it in similar ways. Moreover, teachers may learn about teaching from the ways in which they are taught mathematics, and so "modeling" teaching may deliberately take advantage of this to make it possible for teachers to learn about teaching mathematics even while they are learning the mathematics itself.

This leads to a sub-claim about what is learned from "models." What do we mean when we speak of "modeling"? What do teachers actually learn from watching their instructors teach mathematics to them? How is their learning from modeling affected by whether or not the instructor narrates what he or she is doing? How is their learning affected by their opportunities to analyze the instruction?

Stepping back from the question about what constitutes modeling and what is learned from models, some observations suggest that, for all the intuitive appeal of the idea, it might not always make sense to teach mathematics to teachers as one might teach mathematics to students. After all, teachers are not eight-year-olds, and they have already been taught the mathematics that comprises much of what they encounter in teacher education. They have been taught to divide fractions, multiply decimals, to identify polygons and quadrilaterals. They have solved countless puzzles and problems. The challenge for some teachers, documented in many studies of teachers' mathematical knowledge, is that they have learned these ideas with little meaning or connective tissue. They know the forms, but not the ideas. Sometimes they have not even really learned the forms. All of this suggests however, that as teachers approach learning mathematics, they bring different resources and foundations than do their students. So, for example, problems that are fruitful for children may not be so for their teachers. And vice versa. Take, for example, the well-known problem of constructing a story problem for $1\ 3/4 \div 1/2$ (Ball, 1988; 1990; Ma, 1999):

(a) Calculate the answer to $1\ 3/4 \div 1/2$. What do you get? What method did you use to get the answer?

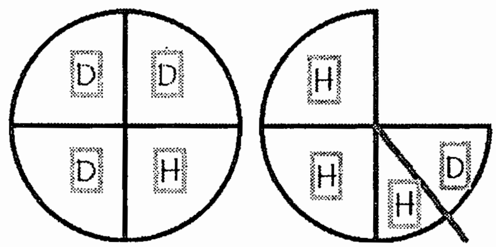
(b) Write a story problem, or describe a situation, for which $1\ 3/4 \div 1/2$ is the mathematical formulation. What do you get? How does it fit with the answer you got when you calculated?

While most preservice and practicing teachers remember the procedure of "invert and multiply" and can get 3 1/2 for the calculated answer, most cannot produce a story problem that correctly maps to this expression. Instead, most stories involve dividing by 2 rather than 1/2. A typical story is:

I had two pizzas. My roommate ate 1/4 of one of them, and then I had 1 3/4 pizzas left.

I shared the remaining pizza with one of my friends. How much pizza did we each get?

Automatically assuming the pizzas to be divided into fourths, many imagine the following:



This representation results in each of two people getting 3 1/2 pieces. What people do not often notice is that this solution changes the unit from halves to quarters. Each person in the pizza story is getting 3 1/2 *quarter-pizzas*, not 3 1/2 *half-pizzas*. (Worth noting is that this common "solution" to the problem of generating an appropriate story problem is produced by most adults—not just teachers—to whom the question is posed.) What is important for the present discussion is that this problem, posed to a group of teacher education students or practicing teachers, can be fruitfully used to develop a number of significant mathematical ideas, including interpretations of division, the importance of the unit, and what is involved in building a mathematical correspondence between a model and a mathematical expression. That this is a fruitful workspace seems related to the fact that the arithmetic territory is not new to them, that the work is engaging them in a disequilibrating encounter with something they already know. This would *not* make a good problem for sixth graders first learning to divide fractions. They would not be able to calculate with such ease, nor use the algorithm of "invert and multiply" to produce the initial answer. The discrepancy between the calculated answer and the story problem would not emerge, and there would not arise the surprise factor that animates adults' learning in this problem.

Developing the concept of division and extending it to fractions, as one would do with sixth graders is a different pedagogical undertaking than challenging complacent procedural knowledge of adults. That teachers already know a great deal of mathematics, albeit often without deep knowledge of the fundamental ideas, makes productive mathematics learning experiences often different for the two groups. Hence, the simple adage that teachers should be taught as they would teach students, is likely *too* simple. The mathematics *problems* that afford appropriate intellectual space for students' learning are often different from those that afford learning spaces for teachers. Teachers would not necessarily profit from problems designed for student learning. And problems that are designed to be fruitful for teachers may not set up students' learning as well.

Treating as a working hypothesis this reasonable idea that professional development should "model" the mathematics teaching that teachers should use with students would involve examining the sorts of mathematical tasks that seem to afford productive learning for teachers, and for students. It might also involve close analyses of the interactive dynamics of mathematics discussions among teachers, compared with those of children. Just as there may be some distinctive features of problems that are fruitful for teachers' learning, there may also be particular possibilities or challenges in helping teachers work—independently and with others—on mathematics. Their histories, identities, and understanding are all likely different in some important ways from those of students, and good pedagogy is about building bridges between learners and the mathematics to be learned. Engaging this principle as a hypothesis rather than mandate would likely uncover interesting areas of overlap between the teaching and learning of children and the teaching and learning of teachers. Doing so would require careful unpacking of the constituent elements of the claim, and scrutiny of their validity in a variety of contexts and under different conditions. Are the issues different for beginning versus experienced teachers? For elementary versus middle or high school teachers? Does it matter what the mathematical content is? For example, does it make a difference if the mathematics to be learned is *new* to teachers—that is, not content that they are revisiting, but content they have never explored before? Does it matter if the purpose of their work is to prepare them to work on it with students versus to develop some broader sense of the territory, some sensibilities or peripheral vision? And throughout, what we need is a thoughtful development of what "modeling" might involve, and what it would take for it to be educative.

There are also important teacher learning questions involved in this set of ideas. What might it mean to approach an opportunity to learn mathematics as a teacher rather than as a student? And are there versions of such a stance that would make more mathematics courses and workshops into useful sites for developing mathematical knowledge, with less reliance on the instructor's ability to model a particular pedagogy? For example, a limiting version of such a stance might mean that a teacher would see as irrelevant forays into mathematical ideas and issues not directly related to the curriculum. But generative versions of such a stance might mean that a teacher would actively take note of others' mathematical ideas in the session as a window into how others might solve a particular problem or understand a specific idea. Or, a teacher, aware of the sort of depth and flexibility of understanding needed in teaching, might not settle for getting right answers, and be more demanding of himself or herself, and of the instructor to reach for a stronger sense of the ideas. While the notion of "modeling" may include some important claims worth unpacking about what instructors can do, there may be some parallel ideas about what it means to learn mathematics as a teacher that may be worth uncovering and exploring as well.

The Structure of Professional Learning Opportunities for Teachers

Many hold strong beliefs about what constitute productive structures for teachers' learning. Widely believed is that "one-shot workshops"—that is, single session professional learning opportunities, without follow-up—are not useful. And many believe that when teachers work with colleagues, the time spent is worthwhile. Reactions to the newer phenomenon that is referred to as "lesson study" are similarly impassioned, often also related to structure of the work, and the professional and social features of the learning opportunities. Although likely important in combination with other more "curricular" aspects of professional development, *what* teachers work on—the curriculum of the professional learning—seems likely to be as big a factor as the *structure* of the opportunity. Notably relevant is that organizational features of students' learning of mathematics have not been shown to have effects on their learning of mathematics on their own. This should give pause to strong claims about the effects of structure on teachers' learning.

Why might we believe such claims? Let's take the "one-shot workshop" first. One reason to disparage such formats is that most of us, whether teachers or teacher educators, have seen or participated in shoddy one-shot teacher "inservice" sessions. We have had strong reactions to the waste of time, to the lack of engagement or useful knowledge, to the often-poor pedagogy or dramatic style of such sessions. Yet many school districts continue to provide such opportunities, and many of us have ourselves offered such sessions. We may even think that some of what we do in such sessions has value, and may also hear from participants that they found them useful. Reflecting further, many of us may have also had important insights in the context of a single session—a lecture, a workshop, a meeting—that turned out to be significantly generative for our learning. Taken together, there may be more to ponder than at first meets the critical eye. If in fact, districts are likely to continue sponsoring such sessions, there are good reasons to investigate the sorts of experiences, content, and ways of working that *can* be productively packaged into single sessions.

Moreover, and equally important are the teacher learning questions involved. Teachers, like most professionals, have many opportunities to learn and to develop their knowledge, skill, and practice. What does it take to be a good user of professional learning opportunities—to combine different kinds of learning, to use sessions and people as resources in an ongoing learning program of one's own? In other words, what would it mean to be a discriminating and constructive user of multiple forms of professional development, across sessions, workshops, courses, meetings—with one's own needs for learning and one's own practice as the drivers? Viewed from the perspective of the teacher as a learner, a series of one-shot workshops that the teacher coordinates in his or her own learning may have more coherence and value than can be seen by the outside observer.

The current enthusiasm for and concomitant skepticism about "lesson study" is no more than a specific case of the claims often made about forms of teacher education. Made visible through the popular book, *The Teaching Gap* (Stigler & Hiebert, 1999), and actively developed in research and development programs such as those led by Catherine Lewis (Lewis, 2000) and Clea Fernandez (Fernandez, Chokshi, Cannon, & Yoshida, in press) and their colleagues, "lesson study" is a term given to describe a kind of professional work in which Japanese teachers engage to develop their teaching. Probing the surfaces of the practices involved in lesson study suggests that there are important aspects of this work that offer promise for teachers to learn in and from practice, in the company of other professionals. Teachers work closely on a particular mathematical idea, examining its elements and connections, probing ways it might be represented to students, and investigating difficulties children might have in learning it. They collaborate to design and teach the topic to students, testing their design ideas in practice, analyzing how they work, and revising and reteaching the lesson.

This cycle of study and development in practice offers some important opportunities for teachers' learning. It also shares family resemblances with other forms of professional learning being explored in the United States and in other countries—the use of cases, for example, or the study of videotapes or other records of practice. Learning from experience is difficult. It is also both essential and inescapable. What these different forms of professional development share is the attempt to "harness" practice to make it a productive site for professional learning. Closer analysis of the underlying ideas and the ways in which they are enacted would afford important possibilities to gain knowledge about how teachers can learn in, from, and for practice (Ball & Cohen, 1999).

Yet some of the current attention to lesson study does not yet probe its fundamental conceptual structure as an opportunity for teacher learning. Ironically, both its proponents and its skeptics can at times remain focused on its external structure and form. People ask questions such as: How much time do teachers spend? How often do they meet? How many lessons do they work on in a year? While its forms matter, what is significant about lesson study is probably not merely those forms. Most likely what is significant is not mainly that teachers work with one another. It is more likely that what matters is the unusual ways in which it engages teachers in learning mathematics in ways connected to practice. Also important may be the work on lesson design that organically connects attention to student thinking and to the integrity of the mathematical ideas. If the structures used in Japan do not fit the common organization of teachers' work in the U.S., this provides fodder for discussion. But if these structures are taken as the dominant feature, then the possibilities of learning about the interplay of structure and what teachers work on, and in what ways, are limited.

Lesson study is no more than an instance of a larger challenge we face in developing knowledge about teacher learning and teacher education. Surely the structures of teachers' opportunities to learn do matter. But to develop what we know beyond claims about surface features of professional education, we will need to unpack the dynamics of form and content—of what teachers do, in what forms, and what sorts of professional learning by teachers occur across different permutations of these.

The Nature and Scope of the Mathematics that Teachers Need to Know

Many strong views exist about what teachers need to know. Few, however, are based on more than anecdotal or small-scale evidence about how teachers' knowledge of mathematics affects their effectiveness with students. Yet because mathematical knowledge for teaching is a domain that also lends itself to logical analysis, countervailing claims permeate the discussion, each with reasonable bases. Professional opinion provides the foundation for many claims in this arena. However, few disciplined means exist for sorting out these competing claims. It makes sense to claim, for example, that what teachers most need is to understand the mathematics for which they are responsible. In order to teach multiplication of decimals well, for instance, there is much to understand about number and operations, about models for each, and about algorithms and place value, about the distributive and commutative properties. Yet closer scrutiny may also reveal that mathematical issues can arise as teachers work with their students, issues for which more advanced mathematical knowledge might be useful (see, for example, Lampert, 2001, for an up-close view of the mathematical complexity of elementary teachers' daily work). When students propose alternative approaches, teachers need ways to size up their mathematical validity and value in order to make sound decisions about what to take up and what to deal with individually (Ball & Bass, 2000). Seeing connections between these elementary arithmetic procedures, such as multi-digit multiplication, and ideas and work with polynomials can also offer teachers a sense of the mathematical horizons, of the trajectories along which their students are traveling. Ma's (1999) notion of "knowledge packages" offers one way to conceptualize the structure of usable mathematical knowledge, emphasizing the importance of core ideas and their connections and development over time. A close focus on the curriculum, narrowly interpreted as what teachers teach, can limit teachers' peripheral vision and lead to a kind of myopia in teaching decisions. However, it is also easy to overlook the complexity and richness important to unpack and learn right in the immediate mathematical territory. A commitment to extending teachers' mathematical reach can inadvertently shortchange what there is to learn about the ideas within the student curriculum.

So far, we see that competing claims about the extent of teachers' knowledge are viable, based on reasoned argument. However, significant problems exist that limit our progress on this important set of issues. First, we lack specificity about what it might mean to learn "the mathematics of the curriculum" or to engage with "advanced ideas." We need ways to make these notions more concrete and be able to develop indicators or measures of each, and of other conceptions of teacher knowledge. And we need to be able to track and analyze how different kinds of mathematical knowledge bear on practice. How, for example, do teachers with different kinds of mathematical understanding appraise and use curriculum materials? How do they size up the mathematical quality of alternative materials, perceive and compensate for distortions, transform weak presentations, learn from unfamiliar but promising representations or approaches? Evidence on these and other such questions about practice would help to mediate these different claims. It is even possible that the more closely tied to practice the discussion becomes, the closer, too, will grow the different claims. And remaining to study, most important, is how teachers' mathematical knowledge affects what their students learn. Progress on this question depends on developing the sort of more nuanced conceptions of what might be meant by "knowing mathematics for teaching," for the past efforts to use proxies for teachers' mathematical knowledge (e.g., courses taken, degrees held) have generally been too imprecise to detect effects.

It may also be too imprecise to view all mathematical knowledge similarly. Perhaps the nature of knowledge for teaching depends on the ideas themselves. For example, maybe knowing geometry for teaching is qualitatively different from the knowledge needed to teach number and operations. What about elements of mathematical knowledge that are less topical—knowledge about the nature and role of mathematical definitions, for instance, or mathematical reasoning? It may also be that some mathematical ideas and skills have high leverage for teachers' mathematical proficiency in teaching. Possibly certain elements of mathematics not easily considered "knowledge" have significant power in teaching. Examples might include sensibilities about what makes a mathematical solution elegant, fascination with symmetries and correspondences, appreciation of particular representations, sense of a good problem. Might it be that these mathematical qualities and orientations bear in important ways on teachers' decisions and capacities? Investigating these and other questions about the mathematical resources that matter for teaching requires moving beyond the sort of blunt (e.g., numbers of courses) or vague (e.g., descriptors such as "deep" or "flexible") claims about what teachers need to know. It requires a closer probing of mathematics itself and what there is to know and appreciate about the domain. It requires also an equally closer probing of the mathematical demands of teaching, and improving our working hypotheses in ways that will allow us to test and develop our claims.

The Role of Curriculum Materials in Mathematics Teachers' Practice

Do curriculum materials hamper or enable teachers? Strong views run in different directions on these questions, complicated by equally strong views about the quality of any particular textbook. Some are sure that textbooks determine the curriculum; others are convinced that teachers are a bigger determinant than the materials they use. Perhaps it is because teaching is a mass profession, or perhaps because of the importance of design and development in learning, curriculum materials do play a visible and important role in instruction. Much evidence suggests that despite an ideology that suggests that teachers who "make" their own curriculum are more "professional," commercially published curriculum materials dominate teaching practice in the U.S. (Goodlad, 1984). This role may be greater in mathematics teaching even than in other subjects. Textbooks provide a structure and organization for the ideas and skills, and order the development of content over time. With both assignments for students and guidance for teachers, curriculum materials are a relatively elaborated resource for teaching. Unlike frameworks, objectives, assessments and other mechanisms that seek to guide curriculum, instructional materials are concrete and daily. They are the stuff of lessons and units, of what teachers and students do. That centrality affords curricular materials a uniquely intimate connection to teaching. An important question, then, is how teachers use them, and how much and what aspects of curriculum developers' visions are enacted in classroom lessons.

Because of this close connection to the daily work of teachers, the design and spread of curriculum material is one of the oldest strategies for attempting to influence classroom instruction. Reformers have often used instructional materials as a means to shape what students learn (Bruner, 1960; Dow, 1991) and some developers have operated as though curriculum materials could operate nearly independently on students (Dow, 1991). Recent strong efforts in states such as California and Texas reflect policymakers' assumption that controlling textbook adoption will determine the curriculum. Critics argue that this strategy "de-skills" the professional work of teaching and severely limits local discretion over curriculum (Apple, 1990). Many of the recent debates in mathematics education have centered on this set of issues: on arguments about curriculum materials, their endorsement and adoption, their evaluation and effects.

However, too little is known about how teachers use textbooks, or more, what they learn from them. Long-term research on teachers' content decisions by Porter and his colleagues (e.g., Schwille, Porter, et al., 1983) suggest that teachers' mediate curriculum developers' designs and intentions. And many researchers claim that teachers include and omit lessons, follow and modify teachers' guides, as a function of their own knowledge and beliefs about mathematics, learning, students, and goals (Fennema & Franke, 1992). Still, while some argue for highly elaborated materials with detailed teachers' guides, others claim that such curriculum materials deny teachers professional discretion. Here, as in other areas, ideology and evidence co-mingle. Imagine another profession where the essential tools of the trade were as contested as textbooks are in teaching. The notion of a "professional" creating his or her own tools de novo is not only impractical; it is both risky and foolish. None of us would prefer surgeons who departed from detailed protocols for particular procedures or who defined professionalism as the right to be creative and find one's own style. A recent essay in the *New Yorker* illustrates just how complex is the learning to use and carry out such procedures effectively, and how long is the "learning curve" in doing so (Gawande, 2002). To complicate matters further, as in teaching, professional practice changes across a surgeon's career, making it necessary to continue learning new procedures, approaches, and ideas.

Yet even to agree that curriculum materials provide important tools for instruction does not answer major questions about what makes curriculum materials usable, what makes them resources with and from which teachers can learn, and what might constitute the needed guidance for the necessarily particular nature of helping children learn mathematics. Turning our beliefs and worries about the role of curriculum materials in teachers' learning and in their practice demands a closer examination of what might be meant by "guidance" and ways to distinguish among different kinds of detail in materials. Are some forms of detail prescriptive and inflexible, while others provide specifics that are usable and adaptable? How do teachers read and use different kinds of material, and what factors seem to influence differences in teachers' uses? Do some materials make it possible for even relatively inexperienced teachers, or teachers with weaker mathematics backgrounds, to teach well, while others are most useful to very skilled teachers?

Learning more about how curriculum materials might function in practice requires setting aside worries and ideologies, and turns assumptions about teachers and teaching into questions that can be investigated in relation to teacher education and learning over time. Although this set of ideas pertains to instruction rather than to teacher education, it is important because of the centrality of curriculum materials in mathematics teaching. A recent study (Cohen & Hill, 2001) suggests that professional development may be more effective when teachers' opportunities to learn mathematics, and to learn about how to help students learn that mathematics, are connected with the materials they use. What we understand about the interplay of curriculum and teachers in teaching can contribute significantly to our ideas about teacher learning and teacher education. What do teachers need to learn to appraise, interpret, and use curriculum materials wisely? How might professional education and curriculum be designed to work together more effectively in improving the quality of mathematics instruction and in helping teachers develop their practice?

On Knowing What We Know

The domains briefly explored above are riddled with beliefs. Firsthand experience, anecdotes, and commitments can lead to strong views about good professional development, what mathematics teachers need to know, or how textbooks shape instruction. But in each of these areas—as in any others we might explore—opposing views compete without grounding to mediate their claims. Our arguments are based as often on firsthand perspectives and personal experience or anecdote as on more rigorous evidence. Even where we do have better evidence, that evidence is often currently limited in a variety of ways that matter for our claims to "knowledge." Our evidence in some cases does help to support our ideas, but there is more to ask, and to investigate. For example, we have not put in the foreground the "who" of teacher learning as often as we might. We tend to know only of teachers who stay *in* professional development, or who *do* participate in programs, workshops, or who pilot materials. What might be learned by following those who leave professional development, who work more on their own, who do not work closely with university educators? What about preservice teachers who are less visible or less enthused about their education classes or their mathematics courses, who are more or less engaged? There are also important questions about generalizability. What of teachers in contexts where less research has been done on teacher learning, where fewer interventions are tried, where resource conditions and the contexts of practice are different? What about beginning teachers versus teachers with many years of experience? Or teachers with substantial depth in mathematics for teaching compared to those without? Does who the teacher is, or where he or she works, shape the usefulness of different ideas, programs, or approaches?

There are questions based on *how* we work, as well as with whom. For example, many opportunities for teachers' learning are designed based on one view or another, without the possibility of scrutinizing the validity of any particular claim. For example, mathematics courses are offered for teachers that engage them in advanced study, while others aim to probe the intricacies of the mathematics of the school curriculum. In some cases, deliberate effort is made to model the pedagogy teachers should use with students, in others not. Some professional development is strategically tied to specific curriculum materials while other programs offer supplementary material and ideas, unconnected to any particular textbook series. In general, opportunities for teachers' learning are designed based on the views held by those who develop and run any particular experience. Only rarely are these assumptions articulated as such, and even less often are they systematically tested. If more such deliberate articulation, design, and delivery could be designed, and then followed, we might be able to engage in various forms of comparative analysis. Doing this would engage us in a useful discussion about evidence.

Wilson and Berne (1999), surveying the practice of professional development, and our beliefs about it, note that the qualities of high-quality teacher professional development can be found in lists sprinkled through our literature. These lists are highly consistent, and, they note, perfectly reasonable. "Yet we know as little about what teachers learn in these kinds of forums (conforming to the principles which we embrace) as we do about what teachers learn in traditional staff development and inservice. Our readiness to embrace these new principles may, in fact, be rooted in a desire to escape collective bad memories of drab professional development workshops rather than sound empirical work. Replacing our old conceptions of professional development with new only makes sense if the new ideas are held up for rigorous discussion and evaluation. New isn't always right." (Wilson & Berne, 1999, p.176) In their review of four thoughtful intensive professional development programs, each focused on subject matter instruction in a particular area, Wilson and Berne (1999) acknowledge the complexity of the work, but prod: Too often, examination of program effects was based on teachers' reflections on what was useful and helpful, comments about what they used, and reports of what they learned. While participants' reactions are obviously important in seeking to understand how teachers experience particular learning opportunities, imagine if we relied on children's reports of what was helpful for our examination of the effectiveness of particular instructional approaches. Wilson and Berne push for more developed methods of studying what teachers actually learn, and how, and for how what they learn affects their effectiveness with students. They note that the progress of the field depends on more systematic ways of studying what teachers learn and how this affects what they do, and more careful examination of how teachers learn inside of particular approaches and experiences.

Our charge, then, is to take our fond beliefs and current ideas, and turn them into explicit working hypotheses that can be tested, compared, and falsified in a variety of settings, and with different kinds of teachers. If all we do is create programs for professional learning based on what we currently believe and then generate illustrations of how these ideas play out in practice, we do not push our ideas nor test them. We do not make progress as a field, and we condemn ourselves to a morass of endless irreconcilable arguments about the quality and effects of professional development.

So, for instance, we need to articulate more clearly what it means to "model" good teaching in professional development, and what, specifically, it would mean to teach teachers as they should teach students. We need to track closely how this might be done, and examine critically what teachers attend to, how they interpret what they experience, see, and discuss, and what impact this has on what they know and believe and what they do. And we need to compare different ways of enacting the principle of "modeling" good instruction. Only if we collectively and individually develop ways to isolate our most compelling current ideas, and subject them to testing, questioning, and study, will we be able to make the sort of progress that the field needs.

Our beliefs are resources for such progress if we wield them as hypotheses rather than mantras. It is to that endeavor that this session intends to turn us. Instead of arguing about the mathematics requirements needed for certification, without grounding or evidence, ask: what difference *does* advanced mathematical study make to teachers' mathematical resources, or to any aspect of the practice of teaching, and how could we know? How *do* teachers use textbooks and what do they learn as they use them? How *does* professional development interact with the materials that teachers are using? Or take the one-shot workshop: Are there *no* things that lend themselves to this format, and how could we find out? Are there things one might try to do within such workshops that would intervene in their assumed limitations?

Converting assumptions to empirical questions, beliefs to testable hypotheses, and ideology to theory, the field of teacher learning and teacher education will begin to develop more reliable, warranted knowledge—and new and more subtle and sophisticated ideas that can, in turn, be tested. Not only do we need to deploy our ideas as questions, but we need also to think more systematically about where and with whom we investigate these questions. How can we design work such that we test ideas across settings and with teachers with different backgrounds, levels and kinds of experience, attitudes and needs? Developing a self-critical and constructive enterprise in which we seek to use our well-honed beliefs as resources for the development and improvement of our knowledge about teacher learning and teacher education is an endeavor well worth our collective commitment and engagement.

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TOWARD EVIDENCE FOR INSTRUCTIONAL DESIGN PRINCIPLES: EXAMPLES FROM COGNITIVE TUTOR MATH 6

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There is a significant gap between theories of general psychological functions on one hand (e.g., memory) and theories of mathematical content knowledge on the other (e.g., content of algebra). To better guide the design of ground breaking and demonstrably better mathematics instruction, we need instructional principles and associated design methods to fill this gap in a way that is not only consistent with psychological and content theories but prompts and guides us beyond what those theories can do. Toward this goal, I reflect on lessons from past and current Cognitive Tutor mathematics projects. From this experience, I have abstracted four instructional bridging principles, Situation-Abstraction, Action-Generalization, Visual-Verbal, and Conceptual-Procedural, and associated methods for applying them. I illustrate these in the context of the design of the successful Cognitive Tutor Algebra course (now in more than 800 schools) and the on-going research and development of a Cognitive Tutor course for 6th grade mathematics.

Introduction

My first goal in this paper is to summarize a successful educational innovation grounded in psychological research on mathematical learners and learning. Cognitive Tutor Algebra is a full year course that combines a problem-based consumable textbook, used 3 days a week, and intelligent tutoring software, used 2 days a week. Cognitive Tutor Algebra has been successful commercially, sold to over 800 schools, and pedagogically, shown to yield higher student achievement than alternatives. My second goal is to summarize aspects of the psychological theory, ACT-R, that supported the design of this course and helps explain its apparent success. My third goal is to motivate the need for theory at an intermediate level between general cognitive theories, like ACT-R, and models specific to narrow content areas, like proportional reasoning. Such theory is needed to fill the gap that exists between general theory and instructional innovations, so that fewer unguided, purely intuitive decisions need to be made in instructional design. I will discuss four principles to guide such decisions and describe domain-specific methods to implement these principles. My fourth and final goal is to introduce a new Cognitive Tutor mathematics course for 6th grade and to illustrate the four principles using text and tutor lessons from the on-going course development.

Success of Cognitive Tutors and Humility Message

Before discussing the Cognitive Tutor Algebra project, let me start with some background. Following an experience creating and field testing an intelligent tutoring system for geometry proof (Koedinger & Anderson, 1993), in which I was one of four teachers involved, it became particularly clear to me how difficult and important it is to attend to the social context of an educational innovation (cf., Lehrer, Randle, & Sancillo, 1989; Schofield, Evans-Rhodes, & Huber, 1990). Two key features of the social context are integrating the innovation with other aspects of the curriculum and innovation-specific teacher professional development. When we started the algebra project in 1992, we took what we called a "client-centered approach" and sought guidance from mathematics educators toward meeting national standards (Koedinger, Anderson, Hadley, & Mark, 1997). Perhaps most importantly, we decided that to address the curriculum integration challenge, we would create a full course including both the text materials (used 3 days a week) and the technology (used 2 days a week). The mathematics curriculum supervisor for the Pittsburgh Public Schools, Diane Briars, pointed us to teacher Bill Hadley who had already been writing new algebra text particularly focused on helping students make sense of algebra. We worked together, combining classroom intuition and cognitive science, to produce a problem-based course that connects multiple representations of functions and employs advanced computational tools. Problems and projects in both the text and software connect to real-world uses of algebraic reasoning like estimating the cost of a rental car, choosing between long-distance phone services, predicting the decline of the condor population, planning profits for shoveling snow, comparing the current quantity and growth rate of old growth forest in the US to the harvest rate.

In three full-year multi-school field studies, we demonstrated that students in a Cognitive Tutor Algebra course learn more than students in comparison classes both on assessments of problem solving and representation use and on standardized assessments of basic skills (Koedinger et al., 1997; Koedinger, Corbett, Ritter, & Shapiro, 2000). As the dissemination of the algebra course was ramping up, in 1995 we obtained funding from local foundations (Heinz, Buhl, Grable, Mellon, and Pittsburgh Foundations) to create full courses for geometry (led by myself) and algebra 2 (led by Albert Corbett). With the help of Carnegie Mellon's technology transfer office, in 1998 we created Carnegie Learning, Inc to further develop and market all three courses. The number of schools using Cognitive Tutor Algebra has roughly doubled each year and is now over 800.

Such success is encouraging, but the job is not done. Education is a very hard problem. Besides tough issues of values, culture, race, politics, respect for teachers, testing, etc., we certainly have not cracked how the mind works or how learning is best achieved. We need to *remain humble*. We need to recognize that our instructional beliefs, practices, theories may be wrong. We need to put them to the test and be proud of failure. In the context of hypotheses and principles, failure advances theory. Given the difficulty of the education problem, we also need to collaborate. No one can successfully tackle a significant educational problem alone. We need to combine expertise from multiple domains, particularly education researchers, cognitive psychologists and learning scientists, but also anthropologists, computer scientists, measurement, policy experts, etc.

In the remainder of this paper, I reflect upon the lessons learned in our Cognitive Tutor projects particularly at the level of principles and design methods. I illustrate these primarily with examples from the on-going research and development of the Cognitive Tutor course for 6th grade mathematics. In 1999, with support from Carnegie Learning, we began a project to create full Cognitive Tutor courses for middle school mathematics with Albert Corbett leading the 7th and 8th grade course development and myself leading the 6th grade course development.

Theory, Principles, and Methods behind the Design of Cognitive Tutors

ACT-R and General Principles for Instructional Design

Cognitive Tutors are based on the ACT-R theory of learning and performance (Anderson & Lebiere, 1998). The theory distinguishes between tacit performance knowledge, so-called "procedural knowledge" and static verbalizable knowledge, so-called "declarative knowledge". According to ACT-R, performance knowledge can only be learned *by doing*, not by listening or watching. In other words, it is induced from constructive experiences—it cannot be directly placed in our heads. Such performance knowledge is represented in the notation of if-then production rules that associate internal goals and/or external perceptual cues with new internal goals and/or external actions. Three examples of English versions of production rules are shown in Table 1, which is discussed below.

Developing Cognitive Tutor software involves the use of the ACT-R theory and empirical studies of learners to create a "cognitive model". A cognitive model uses a production system to represent the multiple strategies students might employ as well as their typical student misconceptions. The following provides a simplified example from algebra equation where these three production rules are alternative ways to respond to the same goal (a, b, c, and d are any numbers and x is any variable to be solved for):

Strategy 1:	IF the goal is to solve $a(bx+c) = d$ THEN rewrite this as $bx + c = d/a$
Strategy 2:	IF the goal is to solve $a(bx+c) = d$ THEN rewrite this as $abx + ac = d$
Misconception:	IF the goal is to solve $a(bx+c) = d$ THEN rewrite this as $abx + c = d$

The cognitive model is used with an algorithm called model tracing to follow students through their individual approach a problem. In so doing, the tutor is able to provide context-sensitive instruction. The cognitive model is also used by an algorithm called knowledge tracing that assesses students' knowledge growth as they succeed or fail on actions associated with the production rules in the cognitive model. The results of knowledge tracing are displayed with "skill bars" in the computer tutor interface (see the top right in Figure 5) and are used to select activities and adapt pacing to individual student needs. The cognitive model is not only the key workhouse within the automated Cognitive Tutors, but it is also a theoretical tool used to guide the design of other aspects of instruction including problem and interface design, text materials, and classroom activities.

In 1995, we published a report on the status of the lessons learned to date from Cognitive Tutor development (Anderson, Corbett, Koedinger, & Pelletier, 1995). We described some general Cognitive Tutor design principles consistent with ACT-R and our research and development experience to that date. A key principle recommended to "represent student competence as a production set". In other words, to base instruction on an analysis not of mathematical content per se, but of the way in which students think about the content. The notion of "theorems in action" (Nunes, Schliemann, & Carraher, 1993; Vergnaud, 1982) is very similar to production rules.

The idea here is to characterize how students may think about mathematics differently ("informally" or "intuitively", Resnick, 1987) than is normatively taught or present in textbooks (e.g., see production #1 in Table 1). In addition, production rules represent not just *how* students think about mathematics operations, but *when* they retrieve relevant knowledge. This feature of production rules is critical to representing the fact that the knowledge students acquire is sometimes overly specific, so it does not transfer well, and is sometimes overly general, so it leads to errors. In other words, students' "theorems in action" are often more specific than actual theorems or rules. For instance, students can combine like terms in an equation when coefficients are present (e.g., $2x + 3x \Rightarrow 5x$) but not when a coefficient is missing (e.g., $x - 0.2x$). See production #2 in Table 1. Alternatively, students' theorems in action are often more general than actual theorems or rules. For instance, they may learn to combine numbers by the operator between them (e.g., $2 * 3 + 4 = x \Rightarrow 6 + 4 = x$) without acquiring knowledge that prevents order of operations errors (e.g., $x * 3 + 4 = 10 \Rightarrow x * 7 = 10$). See production #3 in Table 1. It should be clear that the "rules of mathematics" and the "rules of mathematical thinking" are not the same.

A fundamental assumption of ACT-R is that people learn by doing as the brain generalizes from one's explicit and implicit interpretations or "encodings" of one's experiences. This assumption is, in my opinion, quite consistent with constructivism. It is not the information or even the instructional activities students are given per se that matters, but how students experience and engage in such information and activities that determines what knowledge they construct from them. Thus, another principle in Anderson et al. (1995) was "provide instruction in the problem-solving context".

The strength of ACT-R and the principles summarized in Anderson et al. (1995) are that they are general and can apply in multiple domains. However, this strength of generality comes with a limitation. An instructional designer can easily apply the

Table 1. Example Production Rules

Production Rules in English	Example of its application
1. Production acquired implicitly (not explicitly taught) IF you want to find Unknown and the inal result is Known-Result and the last step was to apply Last-Operator to Last-Num, THEN work backwards by inverting Last-Operator and applying it to Known-Result and Last-Num	Part of informal "unwind" strategy that solves problems of the form "3x + 48 = 63" without by working backwards: 63 – 48 -> 15 / 3 = 5
2. Overly specific production IF "ax + bx" appears in an expression and c = a + b THEN replace it with "cx"	Works for "2x + 3x" but not for "x + 3x"
3. Overly general production IF "Num1 + Num2" appears in an expression THEN replace it with the sum	Leads to order of operations error: "x * 3 + 4" is rewritten as "x * 7"

theory and principles in multiple ways, some good, some OK, and some bad. For instance, the principle "minimize working memory load" can be implemented in many ways including having the technology perform trivial operations for students. However, this approach will only be effective if the designer is right about what is trivial. Getting the application of the theory and principles right depends on getting domain-specific details right. *Thus, a general theme of this paper is that we not only need to make progress in better articulating theory and principles, but also in specifying associated methods that better ensure these principles will be appropriately applied.* This point applies as well to other proposals for instructional principles.

Instructional Design Principles require Empirical Methods to Successfully Implement

The National Research Council's volume How People Learn (Bransford, Brown, & Cocking, 1999) proposes three general instructional principles: 1) build on prior knowledge, 2) connect learning of facts and procedures with conceptual learning, and 3) support meta-cognition. These are common to many other approaches. To take the first, for instance, the NCTM states: "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge," (NCTM, 2000, p. 16). Romberg & de Lange (2002) describe the Realistic Mathematics Education approach as recommending "making use of students informal mathematical activity to support their development of more formal strategies".

But, what prior, informal knowledge do students have? Sometimes theoretical analysis is used to predict prior knowledge under the assumption that smaller component tasks are more likely to tap prior knowledge than larger whole tasks (cf., Van Merrienboer, 1997). Such analysis focuses on mathematical content and its external forms. However, smaller tasks are not always simpler. It is not the surface form of tasks or external representations that determine how accessible they are to students. Instead, it is the internal mental representations that students acquire and use in task performance that determines what will be simple or not. Thus, to identify what prior knowledge students have and whether task X is going to be more likely than task Y to tap prior knowledge, it is not sufficient to analyze the content domain. Instead it is critical to study how students actually perform on tasks – to see student thinking as it really is, not as a content analysis might assume it to be.

Results of a Difficulty Factors Assessment indicating that "smaller is not always simpler"

Consider the three problems shown below, a story problem, word problem, and equation, all with the same underlying quantitative structure and the same solution.

Story Problem: As a waiter, Ted gets $6 per hour. One night he made $66 in tips and earned a total of $81.90. How many hours did Ted work?

Word Problem: Starting with some number, if I multiply it by 6 and then add 66, I get 81.90. What number did I start with?

Equation: $x * 6 + 66 = 81.90$

Which would be most difficult for high school students in a first year algebra course? Nathan & Koedinger (2000) discussed results of surveys of teachers and mathematics education researchers on a variation of this question. The survey respondents tended to predict that such story problems would be most difficult and such equations would be easiest. Typical justifications for this prediction include that the story problem requires more reading or that the way the story problem is solved is by translating to the equation.

In contrast, what we found in the two original studies (Koedinger & Nathan, in press) and in subsequent replications is that students perform best at the story and word problems (70% and 61% respectively) like these and worst at the analogous equations (42%). Clearly many students were not solving the story and word problems using equation solving. Instead they used alternative informal strategies like guess-and-test and "unwinding", working backwards from the result, inverting operations to find the unknown starting quantity (cf., Bednarz, Kieran, & Lee, 1996). Students had difficulty in comprehending equations and, when they did succeed in comprehending, they often had difficulty in reliably executing the equation solving strategy.

This result is important within the algebra domain. It indicates that if we want to create instruction that builds on prior knowledge, we should make use of the fact that beginning algebra students have quantitative reasoning skills that can be tapped through verbal or situational contexts. Unlike many textbooks that teach equation solving prior to story problem solving, it may be better to use story problem situations and verbal descriptions first to help students understand quantitative relationships with them before moving to more abstract processing.

However, there is also a more general message. In order to apply principles like "build on prior knowledge" we cannot assume that it is obvious what prior knowledge students' posses. Instead, we need to do empirical studies to assess what prior knowledge students have and what forms of presentation best elicit this knowledge. To remind us of this message, we need to repeat the mantra: "The student is not like me".

Cognitive Tutors are Developed Using the Design Experiment Methodology

Assessment experiments, like the one above, that compare the effect of various difficulty factors on student performance are a common component in my instructional design approach. Such "Difficulty Factors Assessments" are one part of my version of the "design experiment" (Brown, 1992) approach. Design experiments involve theoretically motivated cycles of design, feedback, and redesign. While use of theory is critical to the approach, design experiments do not involve a linear process of applying theory to create innovations that are then tested. The approach is neither purely basic nor purely applied, but combines the goals of both. It is between "the hare of intuitive design and the tortoise of cumulative science". In addition to theory, we rely on intuitions of multiple team members, particularly practicing teachers to help generate design ideas. We test and prune ideas in embedded cycles at different time scales (see Table 2). Some cycles are short, like a 5-minute assessment to test a prior knowledge hypothesis or feedback from a project teacher on how a lesson went today and how might it be improved for tomorrow. Some cycles are at the curriculum unit (or chapter) level, like investigating how much a unit improved learning from pre- to post-test relative to an alternative approach. A typical "parametric" study compares different versions of a Cognitive Tutor that vary on a key dimension of interest. The longest cycles are at the full course level, comparing student end-of-course performance with that resulting from alternative courses.

Instructional Bridging Principles

I now describe four "instructional bridging" principles that are more specific, particularly to mathematics, than the more general principles from ACT-R and others

Table 2. Empirical Methods at Different Temporal Grain Sizes

1. Single point assessment of performance. Paper quiz or informal observation (minutes).
2. Lab study or class pull-out. Pre-Post lesson length comparison (day)
3. Replacement unit "parametric" study. Pre-Post unit length comparison (week/ month)
4. Full course field study. Pre-Post course length comparison (semester/year)

3. Instructional Bridging Principles

described above. I have abstracted these principles from our experience developing Cognitive Tutor courses, from associated observations and experiments, and from the mathematics education and cognitive and developmental psychology literatures. In this section, I introduce the four principles and the underlying motivation for them. In the next section, I use examples from the 6th grade Cognitive Tutor course my team is developing to illustrate an on-going application of these principles.

The four "instructional bridging" principles are:

1. *Situation-Abstraction*: Bridge from concrete situational to abstract symbolic representations.
2. *Action-Generalization*: Bridge from doing with instances to explaining with generalizations.
3. *Visual-Verbal*: Integrate pictorial and verbal/symbolic representations.
4. *Conceptual-Procedural*: Integrate conceptual and procedural instruction.

These are specific variations on more general principles like "build on prior knowledge" and "connect and integrate multiple representations of knowledge".

Different kinds of evidence for these principles can be associated with the different kinds of studies in Table 2. One heuristic form of evidence (#1 in Table 2) assesses students' performance to see whether one form of activity/representation is closer to students' prior knowledge and thus easier. Another stronger form of evidence is instructional studies (#2 or #3) where an experimental comparison is made between instruction based on the principle and some other plausible form of instruction. A third less strong form of evidence is full course field studies (#4) where the experimental curriculum is consistent with the principle in question, but the comparison curriculum is not. The limitation of full course studies is that experimental and comparison curricula inevitably differ in many more ways than whether they are consistent or not with the principle in question.

Situation-Abstraction Principle: Bridge from Concrete Situational to Abstract Symbolic

The "Situation-Abstraction" principle is a consistent with one of the key features of the Cognitive Tutor Algebra course. It recommends instruction that uses more familiar concrete problem situations and descriptions as a bridge to powerful abstract symbolic forms. A key premise is that for many ideas and concepts, we can find concrete situations or verbal descriptions that use or communicate these ideas in ways that are more comprehensible and accessible to students than more abstract symbolic and conventionalized forms of mathematical expression. Of course, these conventional symbolic forms are crucial to powerful thinking and communication, particularly in math and science and so are critical goals of instruction. The key hypothesis is that robust competence with abstract symbols can be reached more effectively and efficiently by employing this Situation-Abstraction principle.

This principle is similar or even identical to recommendations of other approaches (e.g., Nunes, Schliemann, & Carraher, 1993; Collins, Brown, & Newman, 1989; Cognition and Technology Group (CTG), 1997). For instance, the Realistic Mathematics Education approach recommends "developing instruction based in experientially real contexts" (Romberg & de Lange, 2002, p. 9). This approach has been nicely implemented in the Math in Context curriculum (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1997–1998).

Evidence for this principle includes of a number of full-year field studies (CTG, 1997; Koedinger et al, 1997) and some unit-level studies (Brenner, Mayer, Moseley, Brar, Duran, Reed, & Webb, 1997; Nathan, Stephens, Masarik, Alibali, & Koedinger 2002). However, all these studies involve substantial curricula that vary in other ways from the comparison curricula (e.g., use of multiple representations). The theoretical rationales for the benefits of early use of concrete or authentic situations include better connection with student prior knowledge, facilitating student motivation, facilitating memory, and facilitating transfer to authentic real world goals. We need more studies to test these different underlying rationales. My colleagues and I have emphasized using situations to connect with prior knowledge and demonstrated, as described above, that certain problem situations are easier for students to understand than corresponding symbolic problems. However, we have not "closed the loop" in a scientifically rigorous way, for instance, by showing that instruction that introduces situations before abstractions leads to better learning than instruction that introduces abstractions before situations but is otherwise the same.

While this principle is well known, it is not necessarily easy to implement effectively in the classroom as Deborah Ball (1993) noted:

> How do I (as a mathematics teacher) create experiences for my students that connect with what they now know and care about but that also transcend the present? How do I value their interests and also connect them to ideas

and traditions growing out of centuries of mathematical exploration and invention? (p. 375)

We should not expect that all kinds of situational support are valuable in all domains. Whether and what kinds of situational supports make sense or are interesting to students depends on the domain – a point I will illustrate below with data from the 6th grade Cognitive Tutor project.

Action-Generalization Principle: Bridge from Doing with Instances to Explaining with Generalizations

The "Action-Generalization" principle is also consistent with the Cognitive Tutor Algebra course. It recommends instruction that has students engaging in problem solving (doing) with specific instances of a mathematical relationship as a bridge to explaining that relationship in general terms. For instance, we might have students solve a problem for specific numerical instances (e.g., $2 * 4 = 8$ and $3 * 4 = 12$) before having them explain what they are doing in a general form (e.g., $x * 4$; cf., Bednarz, Kieran, & Lee, 1996). Or, we might have students measure and add the angles in specific triangles before discovering the generalization and explaining it in their own words.

In the process of designing the Algebra Cognitive Tutor, we performed a parametric study in which we compared tutor versions with different orderings of questions about the problem situation (Koedinger & Anderson, 1998). One tutor version was modeled after a popular algebra textbook (Forester, 1984). Here students were asked first to define an independent variable and write an expression for the dependent quantity then asked to solve for multiple instances of the variable. Another tutor version was consistent with the Action-Generalization principle. In this version, students were first asked to solve the problem situation for a couple specific instances and then asked to write an expression for the dependent quantity. Students showed significantly greater pre-to-post learning gain in the version in which instances preceded the general expression, consistent with the Action-Generalization principle.

The Action-Generalization principle is a variation on the Realistic Mathematics Education notion of progressive formalization. Hans Freudenthal (1983) believed that "students are entitled to recapitulate in a fashion the learning process of mankind" (p. ix). This principle is also related to other recommendations for promoting student meta-cognition (e.g., Bransford et al., 1999), self-reflection, and particularly self-explanation (e.g., Chi, de Leeuw, Chiu, & Lavancher, 1994). We performed a series of parametric studies with the Geometry Cognitive Tutor (Aleven & Koedinger, 2002) comparing a textbook-like version in which students solved for measures in given figures with a self-explanation version in which students explained their problem-solving steps by indicating the relevant geometry rule. These experiments demonstrated a "less-is-more" benefit for self-explanation whereby students learn deeper, more transferable knowledge from self-explaining even though the extra time needed for explanation means they solve fewer problems in the same instructional period. These experiments also demonstrated that such benefits can be achieved with a simple approach to explanation (referencing rules in a glossary) that is easily implemented in computer software. We are currently making progress on the more difficult task of having the software give feedback on student explanation written in their own words (Aleven, Popescu, Koedinger, 2002).

Both the algebra and geometry studies were motivated, at least in part, by results of Difficulty Factors Assessments. We had found that students were more successful at problem solving with specific instances than at explaining in general terms both in algebra (Heffernan & Koedinger, 1998) and geometry (Aleven & Koedinger, 2002). Unlike the evidence for the Situation-Abstraction principle, the evidence for the Action-Generalization principle fairly nicely "closes the loop" in the following sense. First, the two parametric studies show how a result from a single point assessment can be used to guide the application of an instructional principle to create a novel instructional design. Second, they show that the students learn more from an instructional design consistent with the Action-Generalization principle than from one that is otherwise the same, but is not consistent with the principle.

Visual-Verbal Principle: Integrate Pictorial and Verbal Representations

The Visual-Verbal principle recommends instruction that helps students integrate visual, spatial, or analog representations of an idea with verbal, sequential, or digital representations of that idea. In Cognitive Tutor Algebra, the use of multiple representations of functions, the more visual graph and table representations and the verbal-symbolic situation and equation representations, is consistent with this principle. However, it was not generally or explicitly applied. This principle is an expression of a major theme of the work of Robbie Case and colleagues (Griffin, Case, & Siegler, 1994; Kalchman, Moss, & Case, 2001). Case observed that students typically come to a new idea both with relevant visual intuitions (e.g., sense of size of quantities) and relevant verbal knowledge (e.g., knowing the counting sequence: one, two, three...). He suggested that deeper understanding and competence comes from a careful integration of this visual and verbal knowledge into a "central conceptual structure". Case and colleagues have successfully employed this principle in at least three areas, early number, rational number, and functions (Kalchman, Moss, & Case, 2001). Typical results show students in experimental conditions with poorer backgrounds (e.g., low SES 4 year olds) learning more than students in control conditions with similar backgrounds and catching up or exceeding students in control conditions with better backgrounds (e.g., high SES 4 year olds).

Another influence on this principle is the Singapore Primary Mathematics textbook series (Singapore Ministry of Education, 1999), the preface of which states:

> The main feature of the package is the use Concrete => Pictorial => Abstract approach. The pupils are provided with the necessary learning experiences beginning with the concrete and pictorial stages, followed by the abstract stage to enable them to learn mathematics meaningfully. (p. 3)

This statement and the associated materials combine (and confound) aspects of the Situation-Abstraction, Action-Explanation, and Visual-Verbal principles. However, visual "pictorial" representations are clearly emphasized.

Conceptual-Procedural Principle: Integrate Conceptual and Procedural Instruction

The Conceptual-Procedural principle recommends instruction that supports learning of both conceptual and procedural knowledge and facilitates students in integrating the two. Rittle-Johnson & Siegler (1998) define conceptual knowledge as the understanding of principles and of relations between pieces of knowledge that are needed to solve novel tasks. They define procedural knowledge as the step-by-step actions (algorithms) efficient solving of routine problem-solving tasks. Many other researchers have addressed this distinction (e.g., Hiebert, 1986; Lesh & Landau, 1983; Ma, 1999; Starr, 2000) and there has been considerable debate about which kind of knowledge comes first developmentally and which kind should be instructed first. Some advocate that students learn concepts first and should use that knowledge to generate and select procedures (e.g., Gelman & Williams, 1998; Hiebert & Wearne, 1996). Others advocate that students learn procedures first and then should extract domain concepts from that experience (e.g., Hiebert & Wearne, 1996; Siegler, 1991). It may well be that different orderings are appropriate for different mathematical ideas. More importantly, it may be better to focus on how instruction can integrate the two in an iterative process. In such an approach, increases in one type of knowledge may lead to increases in the other type, which leads to further increases in the first (Rittle-Johnson & Alibali, 1999).

While many will acknowledge the effectiveness of Cognitive Tutors for procedural knowledge acquisition, the question arises about whether they are effective in supporting student acquisition of conceptual knowledge. Our work on Cognitive Tutor Math 6 has attempted to address this question (Rittle-Johnson & Koedinger, 2001; 2002).

Using the Bridging Principles in the Design of Cognitive Tutor Math 6

Course Overview

We started work on the middle school text and software in the fall of 1999. In 2001-2002 school year the 6th grade course included 9 units of text and 36 units of software, summarized in Table 3. To illustrate the use of the principles, I will focus on the algebra and rational number strands running through the curriculum.

Table 3. Cognitive Tutor Math 6 Course Content—2001-02 School Year

Textbook Units	Tutor Units	Strands
1. Finding Patterns & Writing Rules	Algebraic Expression Generation, Expression Evaluation, Picture Algebra	Algebra
2. Grounding Place Value in Measurement	Decimal Number Line, Decimal Place Value, Decimal Addition & Subtraction	Number
3. Grounding Decimals in Area & Perimeter	Area & Perimeter with Decimals, Picture Algebra with Multiplication	Number Geometry
4. Understanding Division	Expr Gen with Div, Compatible Numbers, Simplifying Div, Decimal Div with Area	Number Algebra
5. Understanding Fractions, Equivalence, Addition & Subtraction	Fraction Concept, Proportional Reasoning, Probability, Fraction Arithmetic, Picture Algebra with Fractions	Number Data & Probability
6. Factors & Multiples	Common Factors, Common Multiples, Scaling, Proportionality with Factors & Multiples	Number
7. Fraction Multiplication & Division	Fraction Mult with Area, Picture Division, Fraction Mult & Div with Area	Number Geometry
8. Data Analysis	Simple Histogram & Central Tendency	Data
9. Properties of Angles & Triangles	Sorting & Defining, Drawing, Discovering Properties, Angle Solving, Similar Triangles	Geometry

Emphasixing the need for content-relevant data.

Instructional principles are true or useful only to the extent that the application of them is guided by content-relevant data. Our research related to the Situation-Abstraction principle provides an example. Table 4 illustrates different content areas in middle school math where we compared concrete story problem situations with abstract context-free problems. The table shows 6th graders' average percent correct on multiple pre-test items in each category. In three of the areas, the problem situation consistently facilitates performance significantly above the abstract problem. These are decimal place value and decimal arithmetic (Rittle-Johnson & Koedinger, 2002) and fraction addition (Rittle-Johnson & Koedinger, 2001a). In data analysis (cf., Baker, Corbett, & Koedinger, 2001), the situation facilitated performance on a global interpretation task, but not on a local interpretation task. In the area of factors and multiples, the situation reduced performance.

Application of the Situation-Abstraction principle may not be effective for concepts and procedures related to factors and multiples unless situations can be found that are easier to understand than abstract problems. While one might still want to use such a problem situation as motivation for learning, given this data, it does not appear that such a situation will provide a student with easier or more direct access to understanding.

Examples from the Algebra Strand of Cognitive Tutor Math 6

Unit 1 of the 6th grade curriculum is "Finding Patterns and Writing Rules". The curriculum starts with a pre-algebra unit because it provides a good introduction to pattern finding, induction, and explaining patterns in general terms. This unit thus sets a tone for the rest of the school year where students will engage in pattern finding and rule discovery to gain deeper access into new concepts.

Pattern induction and rule writing lessons illustrating Situation-Abstraction and Action-Generalization.

Figure 1 shows a text sample from the first lesson in Unit 1, which is intended as a fun opening activity. Students engage in both a simple pattern discovery, find the minimum number of sections (pieces of cake) for N cuts (N+1), and a difficult pattern discovery, find the maximum number of sections for N cuts (left to reader!). Students achieve various levels of success from finding correct section values for small N's, through finding how the section values increase going down in the table (i.e., the "y differences"), to stating a rule for computing the sections given the cuts (i.e., the function). In our pilot classes, most 6th graders find the pattern of y differences for both minimum and maximum, and find and state the function for the minimum but not the maximum. When students do state a rule, it is usually in words ("just add 1 to the cuts") and possibly including an instance ("say the cuts is 12, then one more is the sections"). In later lessons, students work through of a series of scaffolded situations (starting with single operator addition and going to combinations of two operators) to improve their ability to find patterns and to read and write verbal rules and algebraic expressions (see Figure 2).

Rule writing is also practiced in the tutor. The Algebraic Expression Generation tutor unit (see Figure 3) is similar to early lessons in the Algebra Cognitive Tutor. Following the Action-Generalization principle and the results of the parametric study

Table 4. Comparisons of Situational and Abstract Problems in Five Content Areas

	Decimal place value	Decimal arith	Fraction addition	Data interp-global
Situation	Show 5 different ways that you can give Ben $4.07. [A place value table was provided to scaffold answers, but is not shown here]	You had $8.72. Your grandmother gave you $25 for your birthday. How much money do you have now?	Mrs. Jules bought each of her children a chocolate bar. Jarren ate 1/4 of a chocolate bar and Alicia ate 1/5 of a chocolate bar. How much of a chocolate bar did they eat altogether?	[2 scatterplots given] Do students sell more boxes of Candy Bars or Cookies as the months pass?
% Correct	61%	65%	32%	62%
Abstract	List 5 different ways to show the amount 4.07. [Place value table given.]	Add: 8.72 + 25	Add: 1/4 + 1/5	[Scatterplots given] Are there more Moops per Zog in the Left graph or the Right graph?
% Correct	20%	35%	22%	48%

	Data interpretation-local	Factors & Multiples
Situation	[Scatterplot given] When 9 players from Anchoville were present, how many pizzas got eaten?	You work at a candy store. Your boss has asked you to figure out the different ways she could package the jelly beans and chocolate eggs, and she wants to know all the possible ways. If there are 64 jelly beans and 40 chocolate eggs and she wants each package to be the same, what are the different numbers of packages you could make?
% Correct	64%	20%
Abstract	[Scatterplot given] In the Left graph, when there were 9 Pooks, how many Feeps were there?	The common factors of 64 & 40 are:
% Correct	65%	37%

To celebrate the start of sixth grade, your class is going to have a party. They are going to buy a rectangular sheet cake. The cake needs to be sliced into pieces.

Situation 1

Your job is to get the minimum number of pieces from each cut. All the cuts must be a straight line and must start and end on a side.

Complete the chart as you finish each cake. Don't be timid; draw extra rectangles if you need them.

Number of cuts	Minimum number of sections
1	2
2	
3	

Below are two examples of single cut cakes that are iced. Notice that the sections do not need to be even, just be certain that the lines are straight.

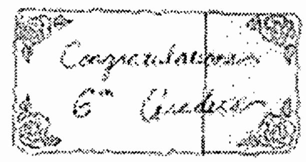


Figure 1. Text excerpt from lesson 1.1 in the 2001-02 version of Cognitive Tutor Math 6.

described above, students are first asked to solve for instances (rows 1 and 2 in Figure 3) before writing a general expression (final row). A major difference in the 6th grade tutor is the addition of the "Show Your Work" column. Having students move from instances to the algebraic expression is consistent with the notion of algebra as a generalization of arithmetic (e.g., Bednarz, Kieran, & Lee, 1996). Much to our surprise, however, we discovered that this generalization step is not the difficult one for students. In two studies reported on and referenced in Heffernan & Koedinger (1998), we found that students were not much worse at writing a general expression for story problems (49% correct) like "$X * 2.0 - 1.25$" for the problem in Figure 3 than writing an instance-based expression (53%) like "$5 * 2.0 - 1.25$". However, writing either kind of expression was significantly more difficult than finding a numeric solution like 8.75 for the same story problems. In other words, it is the explanation step, going from "doing with an instance" to "explaining with an instance", that is particularly difficult for students. Speaking in terms of students' learning difficulties, one might say that algebra is the "explanation of arithmetic" rather than the "generalization of arithmetic".

As indicated above, the Situation-Abstraction principle is at work in multiple places in the curriculum. Some problem situations are more authentic to real world concerns, while others, like the Cake and Marble Machine problems, are more playful. How well individual situations work either in terms of motivating students or making ideas more accessible are open questions. The process of generating problem

Situation 3 Carol, feeling confident that she could break any code, went to the Marvelous Marble Machine. Unlike the computer code maker, the Marvelous Marble Machine uses two operators to compute its secret number.

The Marble Machine will take ordinary marbles and replace them with even more beautifu colored marbles. Of course there is a catch! You must break the code in order to keep ar marbles. If you cannot break the code, you will lose your marbles.

The following table shows how many colorful marbles you will get when you insert ordinar marbles. Help Carol break the code.

Ordinary Marbles	Colorful Marbles
2	10
3	14
6	26
9	38
125	502

1 4

~~2*5~~ 2*4+2

3*4+2

6*4+2

9*4+2

125*4+2

a) How do you get from 2 ordinary to 10 colorful marbles?

b) Does this same procedure work from 3 ordinary marbles to 14 colorful marbles? (If not try another possibility for the previous question)

Figure 2. Text excerpt showing an example pattern finding activity with student work.

situations is primarily an art at this point, but the process of evaluating problem situations (what situations are more accessible or motivating and why) could be more of a science than it is now.

A key premise of the algebra strand is that it will be pedagogically more effective if students first learn the syntax and semantics of the algebra language by writing and reading it before learning to use this language in problem solving. While writing and reading algebraic expressions is a regular activity throughout the 6th grade curriculum, little emphasis is placed on using algebraic expressions in problem solving. Students frequently engage in algebraic problem solving, but it is supported using strategies other than equation solving. One such strategy involves the use of pictures as I describe in the next section.

Picture Algebra lessons illustrating Visual-Verbal and Conceptual-Procedural.

Figure 4 shows our initial instruction of Picture Algebra from Lesson 1.9 of the 2001-02 version of the text and Figure 5 shows an image from the associated tutor unit. This strategy is a variation of strategies used in Asian curricula (e.g., Singapore

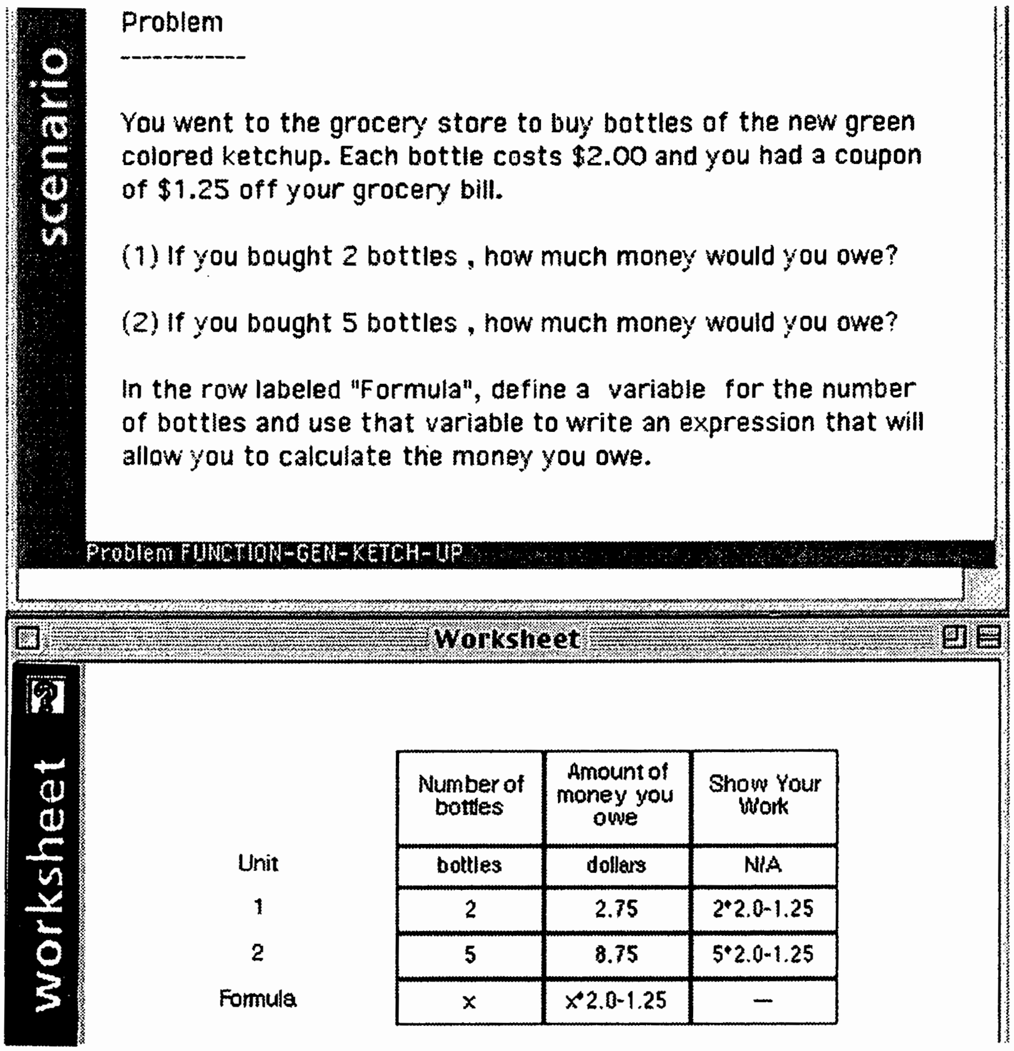


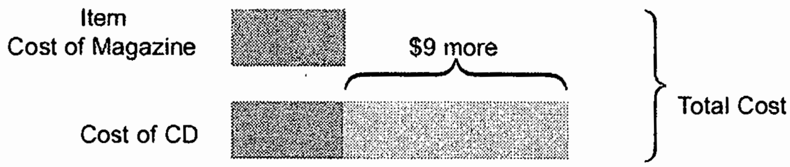
Figure 3. Image of Algebraic Expression Generation tutor unit with "Show your work" column.

Ministry of Education, 1999). It also bears similarity with picture strategies used in Math in Context. As one point of evidence for the power of this strategy, we have observed 6th graders using it to successfully solve problems that many older students (who usually attempt an equation solving strategy) fail to solve (Koedinger & Terao, 2002). Whereas 7th graders in a sample from Bednarz & Janvier (1996, p. 120) were only 5% correct on the problem below, 6th graders in a Cognitive Tutor class were 26% correct on this problem after Picture Algebra instruction.

The use of Picture Algebra is consistent with the Visual-Verbal principle. This strategy integrates students' visual sense of quantity size with their prior verbal and symbolic skills for performing arithmetic operations. By drawing boxes of appropriate sizes, labeling, and performing associating arithmetic operations, students construct

Situation 1
Suppose you want to buy a CD and magazine. The two cost $17 together. If the CD costs $9 more than the magazine, how much does the magazine cost?

One way to view this problem is to draw a picture to represent the objects. Be sure to label the drawings



If you remove the extra $9 from the total cost of $17, what will be left?

Figure 4. Text excerpt showing initial instruction of Picture Algebra from Lesson 1.9 of the 2001-02 version.

an integrated representation that may assist them in developing a "central conceptual structure" for algebraic reasoning.

Picture Algebra is also consistent with the Conceptual-Procedural principle. It is meant to help students acquire a conceptual understanding of key algebraic ideas. One key idea regards transformations that maintain equality, for instance, subtracting 9 from the bars on the left and the total (cf., Rittle-Johnson & Alibali, 1999). A second regards methods for recombining constraints on unknowns, for instance, seeing that the equal parts of the unknowns can be combined (analogous to transforming "$x + (x + 9)$" into "$2x + 9$"). Currently, instruction that emphasizes integration of this conceptual approach with equation solving procedures is being left for 7th and 8th grade in our curricular program. It is a good question whether or not saving the integration for later is a violation of the Conceptual-Procedural principle and, separately, whether or not earlier integration would help learning.

Examples from the Rational Number Strand of Cognitive Tutor Math 6

Rational numbers are a cornerstone of advanced mathematics and a major stumbling block for many children. Multiple units in the curriculum, especially Units 2-7, address number knowledge and rational numbers in particular.

Decimal place value and arithmetic lessons illustrating Situation-Abstraction and Conceptual-Procedural.

In text unit 2 and associated Tutor units, we targeted decimal number understanding with a particular emphasis on combining the Situation-Abstraction and the Conceptual-Procedural principles. The Conceptual-Procedural principle was employed

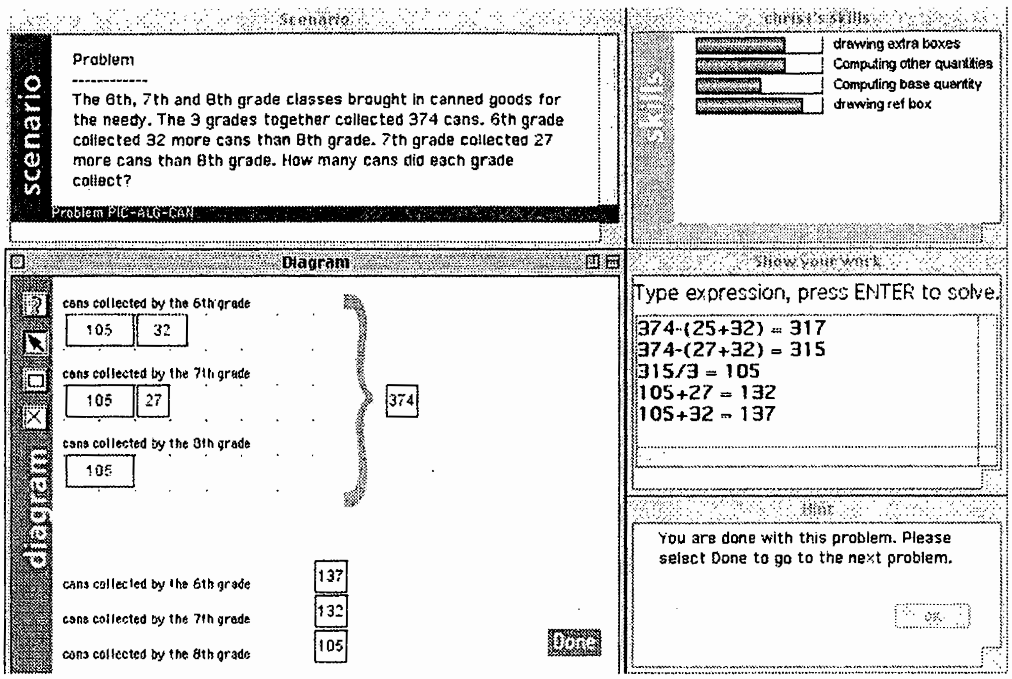


Figure 5. Image of problem from Picture Algebra tutor unit. In the Diagram window, students draw boxes of appropriate size and label them (the text is given but the boxes are not). In the "Show your work" window, students must justify their answers. The tutor monitors each step and provides context-sensitive hints only as needed (by clicking on the ? icon).

through the integration of place value concepts and decimal addition and subtraction procedures. The curriculum emphasizes the alignment and adding of equivalent place values (add ones to ones, tenths to tenths, etc.) as a rationale for the procedure of aligning the decimal point. The Situation-Abstraction principle was employed through the initial use of situations involving money quantities before other non-money and abstract situations. Money contexts appear to dramatically reduce decimal alignment errors in arithmetic, as we saw in Koedinger & Nathan (in press) and in the difficulty factors comparison illustrated above. The two principles combine well here as the key abstract conceptual-procedural link, do not add different place values, has an easier to understand analog in the concrete money situation, do not add dollars and cents. In other words, add ones to ones and tenths to tenths is analogous to add dollars to dollars and dimes to dimes. Rittle-Johnson & Koedinger (2002) report on an experiment demonstrating the value of iterating between conceptual and procedural lessons, first within the familiar money situations and then more abstractly.

Fraction concept and addition lessons illustrating Visual-Verbal and Action-Generalization.

Learning to add and subtract fractions is particularly difficult for students as indicated by the fact that incorrect procedures persist into high school and reflect basic misunderstandings of fractions (e.g., Kouba, Carpenter, & Swafford, 1989). The most common error children make is to add the numerators and denominators (e.g., 1/4 + 1/5 = 2/9), which violates basic part-whole concepts and can lead an answer that is smaller than either addend. In Rittle-Johnson & Koedinger (2001) we explored whether presenting problems with pictures or within a situational context would enhance performance and, in particular, provide a source for sense-making that would reduce this common error. Indeed, on a pre-test given to 6th graders, students performed better when given a problems with pictures (37%) or within a situation (32%; see Table 4) than when given an abstract number sentence (22%). In particular, the combine numerator and denominator error, while still common prior to instruction, was significantly lower for both pictures (38%) and situations (39%) than for the number sentence (48%). Relevant prior knowledge appears to be elicited by pictures and situations and it appears that, even without instruction, some students use such knowledge for sense making.

In addition to indicating room for improvement among 6th graders, these results suggest promise for applying the Situation-Abstraction and Visual-Verbal principles to fraction concepts and fraction arithmetic. Lesson 5.10 addressed fraction addition, but it is important to first review what ideas were addressed earlier in the unit along the developmental corridor leading to fraction addition. Students are encouraged to discover multiple ways of expressing fractions of equivalent value. Following the Visual-Verbal principle, the concept of fraction equivalence (and comparison) is supported through

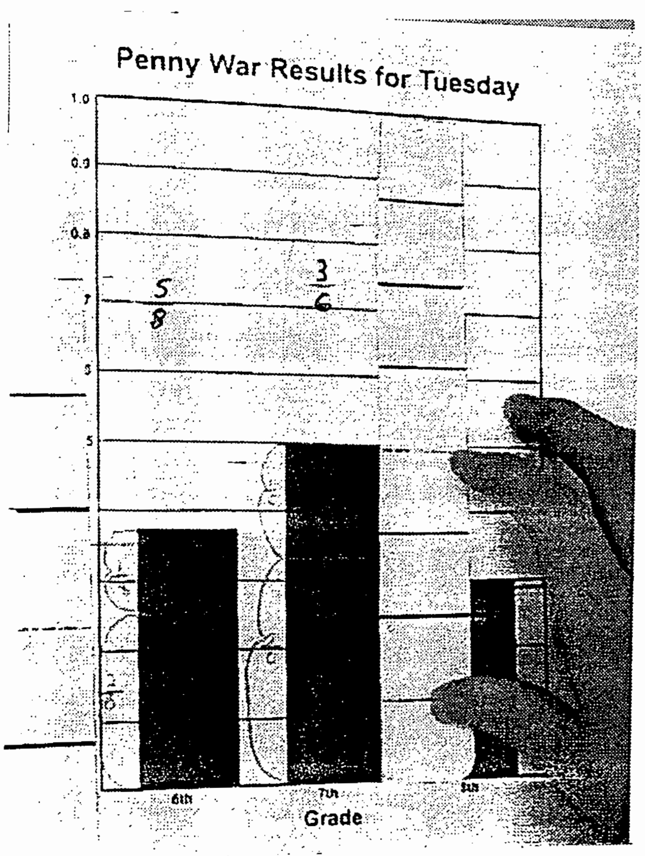


Figure 6. Student using her fraction strip to measure the result of combining the 1/3 of the weekly goal collected on Monday and the 1/6 of the weekly goal on Tuesday. She measures the result to be 1/2.

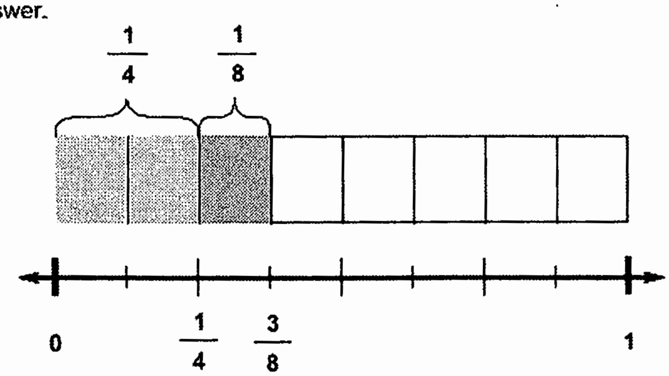
measurement activities and pictorial representations—fraction bars, number lines, and analogous representations in the Cognitive Tutor software. As part of the measurement activities, students create a fraction ruler that is a whole unit in length and labeled with multiple fractions. This ruler is both a frequently used tool and a concrete reminder of fraction equivalence as many positions on the ruler are labeled with multiple fractions (e.g., 1/4 and 2/8).

In lesson 5.1, students begin an activity involving a fund raising effort at the middle school (see Figure 6). (Note: "Penny Wars" is the name of an annual fund-raiser at a local school, but we are changing the name in the text to "Penny Race".) The fund-raiser has a goal for the week and students record in a bar chart the fractional progress toward that goal for each grade on each day. As illustrated in Figure 6, students use their fraction ruler to measure the total of the 7th graders' penny collection on Monday, 1/3 of the goal, and on Tuesday, 1/6 of the goal. The measurement shows that the 7th graders are now 1/2 of the way to their goal.

At this point, the Action-Generalization principle is employed. Students record such measurements (actions) and instances of associated arithmetic facts (e.g., 1/3 + 1/6 = 1/2). As shown in Figure 7, students revisit these facts in lesson 5.10 and have a discussion about how to add fractions. Why is it that 1/3 + 1/6 is 1/2? The prior emphasis on fraction equivalence and its reification on the fraction ruler has the consequence that it is not a difficult leap for students to think of 1/3 as 2/6. Thus, they are in position to recognize that they already know how to add 2/6 + 1/6 (like fraction addition is a topic prior to 6th grade). Having begun to recognize a pattern, students are asked to experiment with their procedure idea on other arithmetic facts. Once their

Situation 2 – Adding Fractions with Unlike Denominators

Look back at the penny war results for Tuesday (page 7). The sixth grade collected 1/4 of its weekly goal on Monday, and 1/8 of its goal on Tuesday. The total for the week was measured at 3/8 of the total goal. The diagram and number line display the problem and the answer.



$$\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$$

Figure 7. Revisiting the fraction addition facts determined by physical arithmetic to use as a basis for finding patterns and writing a fraction addition procedure.

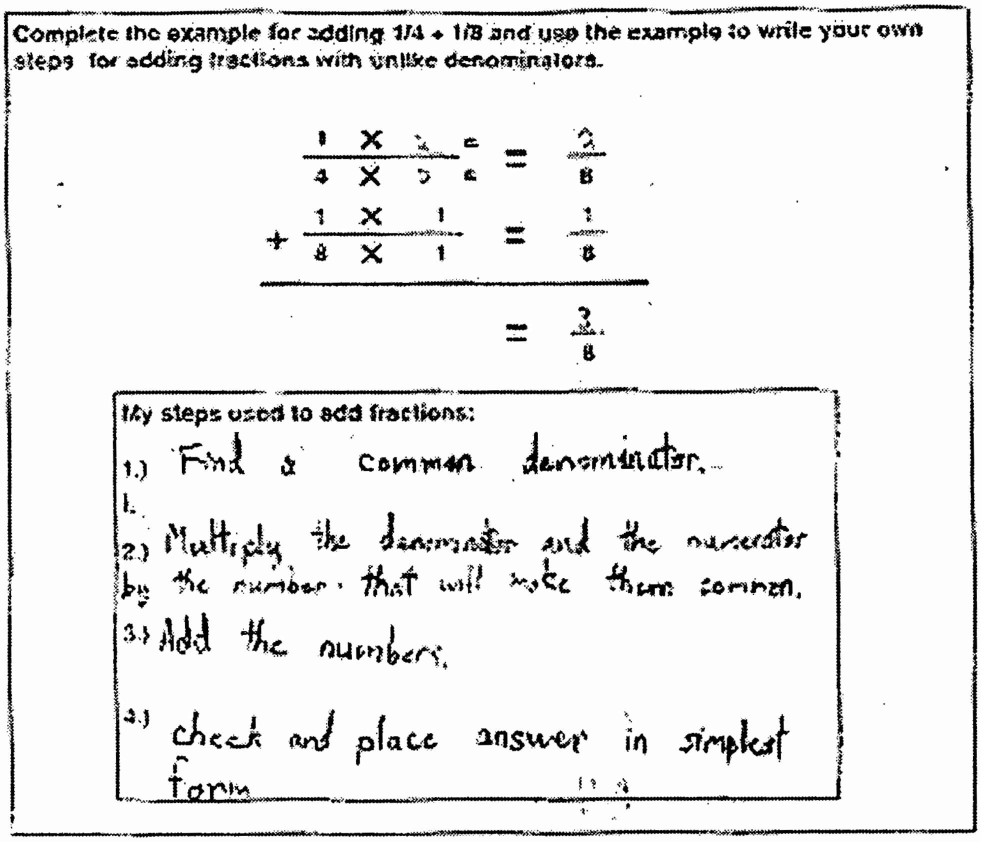


Figure 8. A student's procedure for addition of unlike fractions.

procedure seems to be working they write it down. Figure 8 shows one student's written procedure.

Note the progression of activities in this the application of the Action-Generalization principle. First students are "doing with instances" by using measurement to physically find fraction sums. Next, students are "doing with a generalization" by attempting to test an idea for a fraction addition procedure (which is not yet articulated) with new instances. Finally students "explain with a generalization" by writing a procedure for fraction addition in their own words. This approach is applied for other concepts in the curriculum.

An Initial Full Course Evaluation of Cognitive Tutor Math 6

We began research and development of Cognitive Tutor Math 6 in 1999 and immediately began testing in classrooms. In the 2000-2001 school year, a complete alpha version of Cognitive Tutor Math 6 was used at two Pittsburgh-area schools. To evaluate the effect of the course on raising student achievement, we contrasted end-of-course performance of students in Cognitive Tutor classes with students in schools with comparable demographics. Two types of assessments were used, a "Standardized Skills" test made up of selected test items from State and National standardized tests and a "Problem Solving and Concepts" test that we developed to assess students' problem solving abilities and conceptual understanding. The test items were selected or created to be challenging and with the goal that average performance would not be far from 50%.

Table 5 summarizes students' average percent correct on the two assessments in Cognitive Tutor and comparison classes in two school districts. All four differences between Cognitive Tutor and comparison student performance are statistically reliable.

It was not feasible to randomly assign students or teachers to tutor versus comparison classes given the logistics of agreements with the districts. Thus, there is some chance that the differences are due to other factors. However, it is encouraging that students in Cognitive Tutor classes consistently outperformed students in comparison classes on both assessments and at both schools.

Conclusions and Future Work

Cognitive Tutor courses have achieved considerable success both in terms of raising student achievement and in reaching large numbers of students. So have other research-based mathematics curricula. Nevertheless, it is critical we remain humble about our theories and beliefs about appropriate and effective mathematics instruction. There is still plenty of room for novel ideas in mathematics instruction. Moreover, there is even more room for firm evidence that existing ideas are effective in increasing student learning and do so in ways that can scale up to widespread cost-effective use. A particular gap exists between theories of general cognitive function and theories of mathematics content. We need theories and principles in this intermediate terrain that provide pedagogical leverage. Ma's (1999) work and the more general effort to characterize pedagogical content knowledge (e.g., Shulman, 1986) are extremely valuable toward filling the gap, but better still if such work makes contact with theories of general cognitive function. From the psychological side, efforts to characterize general learning and instructional principles (Bransford, et al., 1999) are also useful. However, better still if we had more specific principles and associated methods that make contact with specific pedagogical content issues.

Table 5. Results of an Initial Full Course Evaluation of Cognitive Tutor Math 6

School District	Test	Cognitive Tutor	Comparison Classes	T-test p-value
District A	Standardized	63% (N=47)	54% (N=50)	< 0.001
	Solving & Concepts	50% (N=48)	41% (N=48)	0.007
District B	Standardized	71% (N=37)	64% (N=132)	0.046
	Solving & Concepts	69% (N=38)	62% (N=116)	0.045

I have made some, I hope humble, attempts to articulate some principles within this intermediate terrain: Situation-Abstraction, Action-Generalization, Visual-Verbal, and Conceptual-Procedural. They bear considerable similarity to a number of other approaches that I am aware of and probably many more that I am not. My point is not to argue that these principles are particularly new. Nor is it to argue that I have provided firm proof for any them. Indeed, I think attempts to "prove" such principles are premature and perhaps even wrong-headed. I do suggest that we need to accumulate evidence that indicates guidelines for when and how to apply principles like these. I also suggest that such principles cannot be applied purely analytically – they require associated empirical facts, for instance, about whether particular situations or visualizations ease understanding.

This paper presented an overview of our new Cognitive Tutor Math 6 course, some samples of text and tutor activities within it, and some early evaluation data that seems to indicate that students may be learning more from this course than alternatives. I actually have not said much about the unique features of Cognitive Tutors, but have written about them elsewhere. Perhaps the most important feature is the capability to approximate the kind of moment-by-moment individualized assistance that a good human tutor can provide. The classroom teacher is a critical component in Cognitive Tutor classes, but when teachers are in the computer lab they have the support of what is, in effect, an automated teacher's aid for every student. This extra support allows them to give more individualized attention to the students who most need it (Schofield, Evans-Rhodes, & Huber, 1990; Wertheimer, 1990). Such automated support is not possible without theory-based analyses of learners and learning.

Looking forward, I would like to see mathematics educators and cognitive scientists explore the feasibility and advisability of a large program of research, on the order of the human genome project, to identify and catalog the knowledge structures or cognitive objects that underlie mathematical thinking and learning. This knowledge base would help drive new instructional innovations and provide for a rich cause theory of how, when, & why they may work. Like the human genome project, such an ambitious program would yield major applied and basic research outcomes.

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LINKING RESEARCH AND PRACTICE: KNOWLEDGE TRANSFER OR KNOWLEDGE CREATION?

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In this paper, it is argued that the failure of educational research to impact on practice stems from a failure to understand the nature of expertise in teaching, and that traditional models of knowledge *transfer* can only be effective for those at a relatively limited level of competence. Instead, it is suggested that teachers need to be involved, collaboratively, with researchers in a joint process of knowledge *creation*. In the King's-Medway-Oxfordshire Formative Assessment Project (KMOFAP) a group of 24 secondary school teachers (grades 6 to 12) of mathematics and science were supported in developing action plans of how they wanted to develop their classroom assessment practice with a single class, through a series of four day-long workshops, and by observations of their teaching. Comparison with local controls (established on a case-by-case basis for each teacher) on curriculum-based tests, showed an average effect size of +0.32.

Introduction

Although the amount of money spent on educational research in most countries over the last hundred or so years has only been a tiny fraction of the expenditure on education (ranging from one-third of a percent to one percent in most developed countries in recent years) a large amount of money has certainly been spent on educational research, and yet its impact is very hard to discern.

The failure of educational research to have any real impact on educational practice in general, and on mathematics education in particular, has been lamented for many years. Today, there are, it seems, two broad strands of criticism of research in education. The first is that educational research is unnecessary. This manifests itself either in a belief that expert practitioners already know 'what works' in mathematics classrooms and so novice mathematics teachers can learn all they need to know by watching experienced teachers, or that pedagogical practice will always be weak, and that the solution lies in prescribing curricula and teaching methods in 'teacher-proof' schemes.

The second strand regards educational research as necessary, but of poor quality. Too often, it is said, educational research produces results that are ambiguous or contradictory, perhaps best summed up by Robert Kennedy's furious reaction to the ambiguous evaluation of the impact of additional money provided for the education of socioeconomically disadvantaged students: "Do you mean that you spent a billion dollars and you don't know whether they can read or not?" (Lagemann, 2000, p. 202). On those few occasions when research does produce unambiguous results, these are generally felt to tell us what we already knew. The lament continues: If educational researchers could only agree how to go about research properly, then educational research could join the elite club of 'hard' sciences producing reliable knowledge (these people have in the past rather unkindly been described as suffering from 'physics envy').

In this paper, I want to suggest is that by its very nature, by the nature of the things it studies, educational research cannot produce 'reliable knowledge' in the sense that Physics—the paradigm case—has done. Modern Physics may be complex, but its successes have been secured because the things it studies, complex as they are, are actually rather simple compared to educational processes. In education, the pursuit of Grand Unified Theories that provide reliable descriptions of what to do in every situation is doomed to fail. Instead, I want to argue that educational research should be about the pursuit of 'practical wisdom' about how educational processes can be improved, and a necessary corollary of this will be that educational research cannot be done 'on' teachers, but only with them, and that this should be not a process of getting teachers to do what we want them to do (co-operation) but of creating knowledge, with teachers (collaboration).

In doing so, I hope to show that such a shift does not entail a down-grading of educational research to a pseudo science, but that, as was first argued by Aristotle the pursuit of practical wisdom is actually a higher-order goal than the pursuit of pure knowledge. The remainder of the paper then goes on to describe how these ideas about collaborative research were put into practice with a group of teachers in the King's-Medway-Oxfordshire Formative Assessment Project (KMOFAP).

The (Troubling) History of Educational Research

The history of educational research can be viewed as a search for disciplinary foundations. At the beginning of the last century, educational research, to the extent that it existed at all, was either historical or an aspect of philosophy. One of the earliest attempts to use empirical methods in educational research was the 'School Survey' movement in the United States. Beginning around 1910 this movement sought to gather 'objective evidence' about factors influencing the educational progress of school students. However, due to the sheer diversity of the United States education system, with over 100,000 school districts each free to determine its own education policy, there was little agreement about the purpose or scope of education, and meaningful comparisons of educational outputs was almost impossible.

In *An Elusive Science* (whose subtitle is the title of this section) Ellen Condliffe Lagemann (2000) shows that the search for ways of producing high-quality research in education has been, in effect, a search for secure disciplinary foundations for the production of reliable knowledge. At first, philosophy and history provided those foundations but, around the turn of the century, these were supplanted by psychology, which dominates to the present day, although since the 1970s sociology and social anthropology have also been influential.

Lagemann argues that the failure of educational research to deliver what has been wanted has arisen from three main causes—the isolation and low status of educational research in the academy; its tendency to focus too narrowly on particular aspects of education rather than looking at education systems; and the weak governance and regulation of educational research. These three causes are of course intimately entwined.

In the United States, teaching had been regarded as 'women's work' since early in the nineteenth century, so that educational research was accorded low status by association. Lagemann also points out that being an applied subject served to marginalia education within the academic mainstream. No doubt partly in an attempt to raise its status, educational research attempted to emulate the hard sciences through the quantification of educational processes, which of course entailed focusing on those aspects of education that could be easily quantified. And while most teachers were female, most school supervisors and district administrators were male, so that the emerging field of educational research emphasized educational administration almost from the outset.

This lack of agreement about not just how to undertake educational research, but also what should be researched continued to plague attempts to establish 'what works' in education over the next half-century, but Lagemann's history closes with an ironic twist. In the final quarter of the last century, educational research finally began to get on the right track with two key realizations. Firstly, the complexity of educational settings requires that insights from all of the 'foundation disciplines' (and not just one) are required to make progress in educational improvement. Secondly, it slowly became clear that centre-periphery models of dissemination are simply ineffective in education. The result was a blossoming of multi-disciplinary research, involving teachers in real innovation and improvement. However, at the same time, it seems that the politicians gave up on educational research, and by 1991, federal funding for educational research in the USA stood at approximately one-third the level provided in 1971.

While Lagemann's analysis is persuasive, I want to suggest here that the goal of educational research as a science is not just elusive, but impossible. This is in part a philosophical claim, but it is also in part an empirical claim. The phenomena that are studied in educational research are, in the first instance, far more complex than those that are studied by the 'hard' sciences—just imagine trying to set boundary conditions for the initial state of a typical mathematics classroom. However, in addition, it is important to realize the autonomy that individual students bring to lessons is not a problem with which physical sciences have to grapple. Bars of iron do not behave differently because someone has been testing them. Or more precisely, while bars of iron may behave differently depending on how they have been treated in the past (ie whether they have been annealed or subject to repeated stress and strain), we know what kinds of treatments matter, and we know how to find out in advance how the bar will perform under tests. Even those who believe that there is no such thing as free will, and that all human behavior at time T1 is actually determined by the state of the system at T0 have to concede that it is too difficult to specify the starting conditions precisely enough to determine the outcome. Chaos theory, and, at a smaller scale, Heisenberg's uncertainty principle, renders Laplace's dream of being able to predict all behavior from initial conditions a non-starter.

Expertise

There are also reasons to suspect that the nature of expertise in teaching is not reducible to the kinds of recipes used in the physical sciences. Flyvbjerg (2001) reports an experiment that was conducted on a group of paramedics (Klein & Klein, 1981). Six short video extracts of a person administering cardiopulmonary resuscitation (CPR) were shown to experienced paramedics, students being trained as paramedics, and people who taught life-saving techniques. They were then asked which of the six they would choose to resuscitate them if they needed CPR. Five of the six video extracts were of inexperienced trainees just learning CPR while the sixth was a highly experienced paramedic. Of the experienced paramedics, 90% chose the experienced paramedic, while only 50% of the students did so. However, only 30% of the instructors chose the experienced paramedic.

Flyvbjerg argues that we can understand this apparently paradoxical result by considering the five levels of expertise in learning proposed by Dreyfus and Dreyfus (1986). At the *novice* level, the individual is guided by rules that are applied irrespective of context. The novice teacher tends to try to apply the same sets of rules to all the classes they teach. The *advanced beginner* begins to take situationally-specific factors into account, and personal experience is often relied on more than context-independent rules. However, as experience accumulates, the number of recognizable elements or 'chunks' increases, and threatens to overwhelm the individual. For example, the need for the school teacher to attend to the learning needs of her or his students, controlling the behavior of some, while also trying to make sure that they interact as much with female and male students, can lead to a feeling of 'plate spinning'—dashing from one imperative to the next to try to attend to all. The *competent performer* is characterized by performance in which conflicting priorities are resolved through the use of strategies, usually derived from conscious problem-solving behavior. In contrast, the *proficient performer* acts quickly and intuitively often doing the 'right thing' without conscious awareness. In this context, it is important to realize that 'intuition' is used here not as some irrational prejudice, but rather as the result of the sedimentation and synthesis of vast amounts of experience. Finally, in the *expert*, the ability to act quickly and intuitively, in a range of contexts and settings, is unified into a 'feeling' of the right thing to do. The use of an emotive term—'feeling'—here is not coincidental. Experts 'feel' the best course of action, not just with their mind, but in their whole body. Expertise is therefore not the culmination of rationality, but rather transcends it.

Expertise involves going beyond what can be done through rationality, not irrational, but meta-rational (i.e. beyond rationality).

Therefore, Flyvbjerg argues, it appears that the paramedic trainers identified the trained paramedics less successfully because they looked for paramedics who followed the rules that they themselves taught. In other words, they were looking for those at the level of competent performers, rather than proficient performers or experts. If we accept that the classification proposed by Dreyfus and Dreyfus also applies to teaching, then it seems likely that the failure of educational research to impact on educational practice stems from a similar limitation.

The kinds of prescriptions given by educational research to practice have been in the form of generalized principles, that may often, even usually, be right, but are sometimes just plain wrong. The expert can see that a particular recipe is inappropriate in some circumstances, although because their response is intuitive, may not be able to discern the reason why. What the practitioner learns is that the findings of educational research are not a valid guide to action.

But research findings also run foul of the opposite problem—that of insufficient specificity. Many teachers complain that the findings from research produce only bland platitudes that are insufficiently contextualized to be used in guiding action in practice. Put simply, research findings underdetermine action.

Knowledge Transfer and Knowledge Creation

If we accept that the prime (although not the only) purpose of educational research is the improvement of educational processes, then research findings must be taken up by teachers and incorporated into their practice. There are other ways that educational research might influence practice—through the improvement of textbooks for example—but without some change in those who use them, innovations are unlikely to have much effect. In the past, this process has been called dissemination, and is now more often called knowledge transfer—both interesting metaphors, suggesting that all that needs to be done is to inform practitioners about the latest findings and they will be used. If expertise transcends rationality, as I have argued above, however, then the process of knowledge transfer cannot be one of providing instructions to novices, advanced beginners, or competent performers in the hope that they will get better. Rather what is needed is an acknowledgement that what teachers do in 'taking on' research is not a more or less passive adoption of some good ideas from someone else, but an active process of knowledge creation:

Teachers will not take up attractive sounding ideas, albeit based on extensive research, if these are presented as general principles, which leave entirely to them the task of translating them into everyday practice—their classroom lives are too busy and too fragile for this to be possible for all but an outstanding few. What they need is a variety of living examples of implementation, by teachers with whom they can identify and from whom they can both derive conviction and confidence that they can do better, and see concrete examples of what doing better means in practice. (Black & Wiliam, 1998b, pp. 15-16)

The different ways in which knowledge is transferred and created within organizations has been studied by Nonaka and Takeuchi (1995) who have proposed a simple framework for knowledge creation in organizations. In their model, there are four modes of knowledge conversion, depending on whether knowledge is converted to or from implicit or explicit knowledge (see figure 1).

The traditional kind of knowledge conversion practiced by educational researchers is what Nonaka and Takeuchi call *combination*. Knowledge in an explicit form is converted to more knowledge in explicit form. At the other extreme, *socialization* is their name for the process by which new practitioners become enculturated into new practices which are not known explicitly to those who are learning, nor to those from whom they are learning. Tacit knowledge becomes explicit knowledge through a process of *externalization*, and explicit knowledge becomes implicit by *internalization*. A learning cycle can then be set up in which knowledge is created, transformed, and circulated around an organization. Through *learning by doing* systemic knowledge becomes operationalized, which can then be *shared* with other practitioners. In dialogue with others conceptual knowledge is built up, which is then combined with that of others through networking. It was this knowledge cycle that we attempted to implement in the King's-Medway-Oxfordshire Formative Assessment Project (KMOFAP) funded initially by the Nuffield Foundation (as the *Developing Classroom Practice in Formative Assessment* project) and subsequently by the United States National Science Foundation through their support of our partnership with the Stanford CAPITAL project (NSF Grant REC-9909370)

Collaborating with Teachers: The KMOFAP Project

Reviews of research by Natriello (1987) and Crooks (1988) and more recently by Black and Wiliam (1998a) had demonstrated that substantial learning gains are possible when teachers integrate assessment with classroom instruction. However, it is also clear from these reviews, and from other studies (see Black and Atkin 1996) that achieving this is by no means straightforward. As Black and Wiliam (1998b) point out:

Thus the improvement of formative assessment cannot be a simple matter. There is no 'quick fix' that can be added to existing practice with promise of rapid reward. On the contrary, if the substantial rewards of which the evidence holds out promise are to be secured, this will only come about if each teacher finds his or her own ways of incorporating the lessons and ideas that are set out above into her or his own patterns of classroom work. This can only happen relatively slowly, and through sustained programmes of professional development and support. This does not weaken the message here—indeed, it should be a sign of its authenticity, for lasting and fundamental

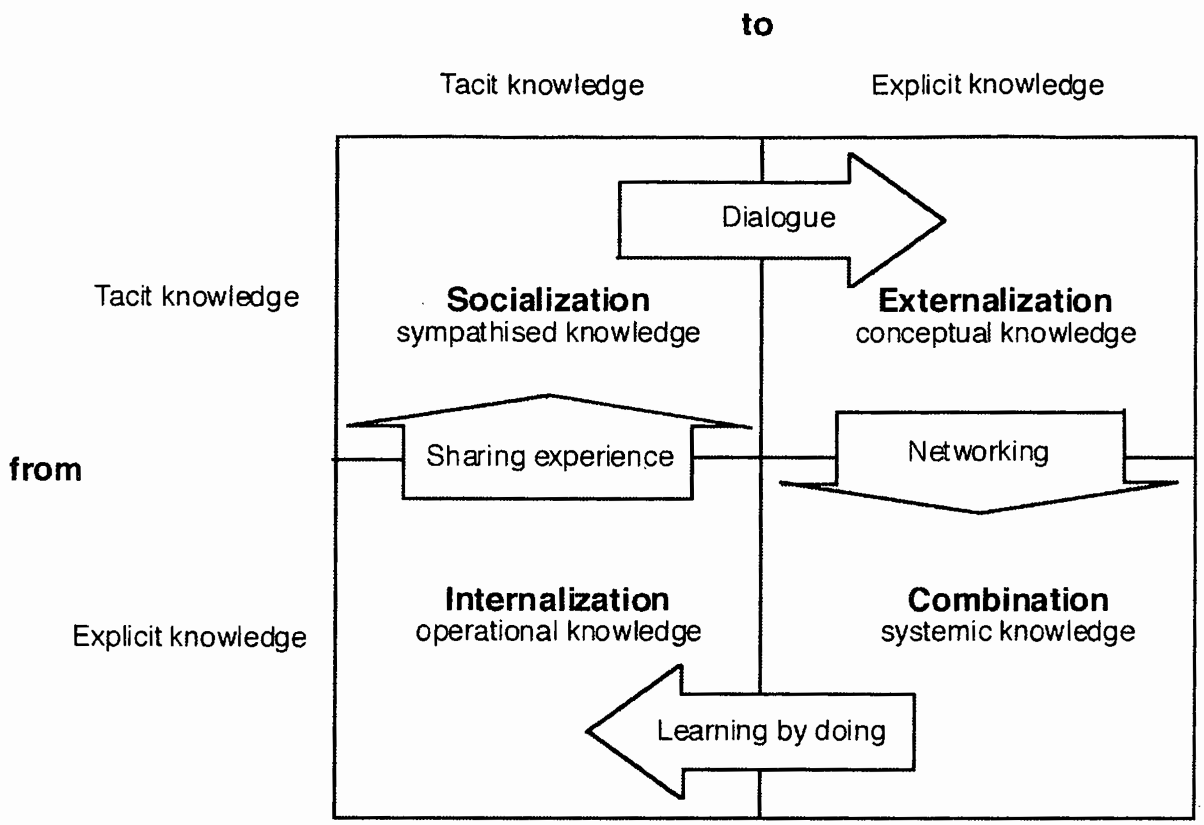

Figure 1. Four modes of knowledge conversion (after Nonaka & Tageuchi, 1995).

improvements in teaching and learning can only happen in this way. (p. 15)

The challenge for us, then, was how could teachers be supported in incorporating formative assessment (or assessment for learning as it is sometimes called) into their classroom practice, not as a 'bolt on' series of tactics, but integrated into planning and teaching?

The Research Strategy

The central tenet of the research project was that if the promise of formative assessment was to be realized, traditional research designs—in which teachers are 'told' what to do by researchers, for all the reasons discussed above—would not be appropriate. We therefore decided that we had to work in a genuinely collaborative way with a small group of teachers, beginning in the bottom left-hand corner of Nonaka and Takeuchi's model, by sharing with them our understanding of the research literature. We then invited the teachers to explore some of these ideas for themselves, by trying them out in their own classrooms (internalization). At first, they were hesitant. Although we told them that we did not have a clear plan for what they should do, the teachers did not believe this. They seemed to believe that we were operating with a perverted model of discovery learning in which we knew full well what we wanted the teachers to do, but wouldn't tell them, because we wanted the teachers 'to discover it for themselves'. However, after a while, it became clear that there was no prescribed model of effective classroom action, and each teacher would need to find their own way of implementing the general principles of high-quality classroom assessment in their own classrooms. We then planned that they would share their experiences with other teachers in the group, and develop a common way of thinking about classroom assessment (socialization). Through extended dialogue, we hoped that they could then develop a common language of description (externalization) thus yielding findings that could be made explicit, so beginning another cycle (combination).

The Sample

We began by selecting two school districts where we knew there was support from the authority for attempting to develop formative assessment, and, just as importantly, where there was an individual officer who could act as a link between the research team and the schools, thus providing a local contact for ad hoc support for the teachers. In this regard, we are very grateful to Sue Swaffield from Medway and Dorothy Kavanagh from Oxfordshire who, on behalf of their authorities, helped to create and nurture our links with the schools. Their involvement in both planning and delivering the formal inservice sessions, and their support 'on the ground' were invaluable, and it is certain that the project would not have been as successful without their contributions.

Having identified the two districts, we asked each district to select three schools that they felt would be suitable participants in the project. We were very clear that we were not looking for 'representative' or typical schools. From our experiences in curriculum development—for example in graded assessment (Brown, 1988)—we were aware that development is very different from implementation. What we needed were schools that had already begun to think about developing 'assessment for learning', so that with these teachers we could begin to produce the 'living examples' alluded to earlier to use in further dissemination.

Each district identified three schools that were interested in exploring further the possibility of their involvement, and the project directors visited each school with the officer from the school district to discuss the project with the principal and other members of the senior management team. All six schools identified agreed to be involved. Brief details of the six schools are shown in table 1.

In our original proposal to the Nuffield Foundation, we had proposed to work only with mathematics and science teachers, partly because of our greater expertise in these subjects, but also because we believed that the implications for assessment for learning were clearer in these areas. In order to avoid the possible dangers of isolation, our design called for two mathematics and two science teachers at each school to be involved.

Table 1. The Six Schools Involved in the Project

School	Abbreviation	Description
Brownfields	BF	Boys
Century Island	CI	Mixed
Cornbury Estate	CE	Mixed
Riverside	RS	Mixed
Two Bishops	TB	Mixed
Waterford	WF	Girls

The choice of teachers was left to the school, and a variety of methods was used. In some schools, the principals nominated a faculty chair together with a relatively inexperienced teacher. In other schools, teachers appeared to be selected because, in the words of one head, "they could do with a bit of inset [professional development]". In the event while our schools were not designed to be representative, there was a considerable range of expertise and experience amongst the 24 teachers selected.

The Intervention

The intervention had two main components:

a) a series of half-day and one-day professional development days, during which teachers would be introduced to our view of the principles underlying formative assessment, and have a chance to develop their own plans;

b) visits to the schools, during which the teachers would be observed teaching by project staff, and have an opportunity to discuss their ideas, and how they could be put into practice more effectively.

The pattern of professional development sessions is shown in table 2 (subsequent events were held as part of the NSF-funded work on the CAPITAL project, but the data reported here relate to the original project, from January 1999 to August 2000.

The key feature of the sessions was the development of action plans. Since we were aware from other studies that effective implementation of formative assessment requires teachers to re-negotiate the 'didactic contract' (Brousseau, 1984) that they had evolved with their students, we decided that implementing formative assessment would best be done at the beginning of a new school year. For the first six months of the project, therefore, we encouraged the teachers to experiment with some of the strategies and techniques suggested by the research, such as rich questioning, comment-only marking, sharing criteria with learners, and student peer- and self-assessment. Each teacher was then asked to draw up, and later to refine, an action plan specifying

Table 2. Pattern of Professional Development Events

Inset	Held		Format	Focus
A	February	1999	whole-day, London	introduction
B	May	1999	whole-day, London	developing action plans
C	June	1999	whole-day, London	reviewing and revising action plans
	September	1999	half-day, district-based	reviewing and revising action plans
D	November	1999	whole-day, London	sharing experiences, refining action plans, planning dissemination
E	January	2000	whole-day, London	research methods, dissemination, optional sessions including theories of learning
F	April	2000	whole-day, London	integrating learning goals with target setting and planning, writing personal diaries
G	June	2000	whole-day, London	action plans and school dissemination plans, data analysis 'while you wait'

which aspects of formative assessment they wished to develop in their practice and to identify a focal class with whom these strategies would be introduced in September 1999. Although there was no inherent structure in these plans, the teachers being free to explore whatever they wished, we did find that they could be organized under the broad headings shown in table 3. In all the 24 teachers included a total of 102 activities in their action plans—an average of just over four each—and while there were a small number of cases of teachers of the same subject at the same school adopting common plans, there was no other clustering of teachers discernible. Inevitably the clear phases suggested by Nonaka and Takeuchi's model became increasingly blurred over the course of the project, with discussion frequently involving all four modes. While it has not been useful for analysis of the data arising from the project, nevertheless, we believe that the model provided a useful framework for shaping our initial interventions.

Most of the teachers' plans contained reference to two or three important areas in their teaching where they were seeking to increase their use of classroom assessment generally followed by details of strategies that would be used to make this happen. In almost all cases the plan was given in some detail, although many teachers used

Table 3. Frequencies of Activities in the Action Plans of 24 Teachers

Category	Activity	Code	Frequency
Questioning	Teacher questioning	TQ	11
	Pupils writing questions	PWQ	8
	Existing assessment: pre-tests	EAPT	4
	Pupils asking questions	PAQ	4
Feedback	Comment-only marking	COM	6
	Existing assessment: re-timing	EART	4
	Group work: test review	GWTR	4
Sharing criteria with learners	Course work: marking criteria	CWMC	5
	Course work: examples	CWEG	4
	Start of lesson: making aim clear	SoLMAC	4
	Start of lesson: setting targets	SoLST	1
	End of lesson: teacher's review	EoLTR	1
	End of lesson: pupils' review	EoLPR	4
	Group work: explanation	GWExp	2
	Involving classroom assessment	ICA	2
Self-assessment	Self-assessment: traffic lights	SATL	11
	Self-assessment: targets	SAT	5
	Group work: test review	GWTS	6
	Self-assessment: other	SAO	7
	Pupil peer-assessment	PPA	5
	Group work: revision	GWRev	1
General	Including parents	IncP	1
	Posters	Post	1
	Presentations	Pres	1
Total			102

phrases whose meanings differed from teacher to teacher (even within the same school).

Practically every plan contained a reference to focusing on or improving the teacher's own questioning techniques although not all of these gave details of the particular way in which they were going to do this (for example using more open questions, allowing students more time to think of answers or starting the lesson with a focal question). Others were less precise (for example using more sustained questioning of individuals, or improving questioning techniques in general). Some teachers mentioned planning and recording their questions. Many teachers also mentioned involving students more in setting questions (for homework, or for each other in class). Some teachers also saw existing national tests as a source of good questions.

Using comment-only grading was specifically mentioned by nearly half the teachers, although many foresaw problems with this, given school policies on grading. Four teachers planned to bring forward end-of-topic tests thus providing time for remediation.

Sharing the objectives of lessons or topics was mentioned by most of the teachers, through a variety of techniques (using a question that the students should be able to answer at the end of the lesson, stating the objectives clearly at the start of the lesson, getting the students to round up the lesson with what they had learned). About half the plans included references to helping the students understand the grading criteria used for investigative or exploratory work, generally using exemplars from students from previous years. Exemplar material was mentioned in other contexts such as having work on display and asking students to correct work using a set of criteria provided by the teacher.

Almost all the teachers mentioned some form of self-assessment in their plans, ranging from using red, amber or green 'traffic lights' [stop lights] to indicate the student's perception of the extent to which a topic or lesson had been understood, to strategies that encouraged self-assessment via targets which placed responsibility on students (eg one of these twenty answers is wrong: find it and fix it!). Traffic lights (or some equivalent) were seen in about half of the plans and in practically all cases their use was combined with strategies to follow up the cases where the students signaled incomplete understanding.

Several teachers mentioned their conviction that group work provided important reinforcement for students, as well as providing the teacher with insights they into their students' understanding of the work.

The choices of activities by the different teachers also showed no particular pattern, as the multidimensional scaling (Schiffman, Reynolds, & Young, 1981) of these data in figure 2 shows.

The other component of the intervention, the visits to the schools, provided an opportunity for project staff to discuss with the teachers what they were doing, and how this related to their efforts to put their action plans into practice. The interactions were not directive, but more like a holding up of a mirror to the teachers. Since project staff were frequently seen as 'experts' in either mathematics or science education, there was a tendency sometimes for teachers to invest questions from a member of the project team with a particular significance, and for this reason, these discussions were often more effective when science teachers were observed by mathematics specialists, and vice-versa.

We aimed for each teacher to be observed six times over the school year from September 1999 to July 2000, although releasing teachers to discuss their lessons either before or afterwards was occasionally a problem (and schools that had guaranteed teacher release for this purpose at the beginning of the project were sometimes unable to provide for this).

Research Design

Given the nature of the intervention, which was designed to build on the professionalism of teachers (rather than imposing a model of 'good formative assessment'

```
  -+--------------+--------------+--------------+-------------
 2 +
   |                                GWREV
   |                                          SOLST
   |
 1 +
   |                           PPA   COM
   |         POST    CWEG     TQ             CWMC
   |                EART            PWQ GWTS GWTR
 0 + EOLTR                  SAO  SATL            PAQ
   |          GWEXP  SOLMAC SAT
   |                          EAPT          INCP
   |                           EOLPR
-1 +                                               PRES
   |                                     ICA
   |
   |
-2 +
  -+--------------+--------------+--------------+-------------
  -2             -1              0              1
```

Figure 2. Multidimensional scaling of teacher action plan data.

on them), we felt that to utilize a traditional research design on the teachers would have been inconsistent. Furthermore, it would have been impractical. Since each teacher was free to choose which class would be the focus for this work, there was no possibility of standardizing either the 'input' or 'output' variables. For this reason, the collection of empirical quantitative data on the size of effects was based on an approach, which we have termed 'local design'. Drawing more on interpretivist than positivist paradigms, we sought to make use of whatever assessment instruments would have been administered by the school in the normal course of events. In many cases, these were state-mandated assessments such as the national tests for 14-year-olds or grades achieved in the national school leaving examinations (the General Certificate of Secondary Education or GCSE). In other cases we made use of scores from school assessments (particularly in science, where 'modular' approaches meant that scores on end-of-module tests were available). For each teacher we therefore had a focal variable (i.e. dependent variable or 'output') and in all but a few cases, we also had reference variables (i.e. independent variables or 'inputs). In order to be able to interpret the outcomes we discussed the local circumstances in their school with each teacher and set up the best possible control group consistent with not disrupting the work of the school. In some cases this was a parallel class taught by the same teacher in previous years (and in one case in the same year). In other cases, we used a parallel class taught by a different teacher and, failing that, a non-parallel class taught by the same or different teacher. We also made use of national norms where these were available. In almost all cases, we were able to condition the focal variable on one or more reference variables, although in some cases the reference variables were measures of aptitude (e.g., NFER's Cognitive Abilities Test) while in others they were measures of achievement (e.g., end-of-year 8 tests).

In order to be able to compare the results, raw differences between experimental and control groups were standardized by dividing by the pooled standard deviation of the experimental and control scores.

Results

Table 4 provides the results achieved by the 19 teachers for whom controls were available and the standardized effect sizes are summarized in stem-and-leaf form in figure 3. As can be seen, the majority of effect sizes are around 0.2 to 0.3, with a median value of 0.27 and a mean of 0.34. Since the effect sizes were not normally distributed, the jack-knife procedure recommended by Mosteller and Tukey (1977) was used which provides an estimate of the true mean as 0.32 and a 95% confidence interval of the true effect size as (0.16, 0.48).

In order to examine the relationship between a teacher's practice and the effect sizes, we classified teachers into one of four groups, according to their use of formative assessment strategies in their classrooms:

Experts	Formative assessment strategies embedded in and integrated with practice.
Competent performers	Teachers who were successful with one or two key strategies, but having routinized these, were looking for other ways to augment their practice.
Advanced beginners	Teachers who were successful with one or two key strategies, and had restricted themselves to these.
Novices	Teachers who had attempted strategies, but had not embedded any strategies into their practice.

These characterizations had emerged from our observations of each teacher's practice, and were based on their use of key strategies during the main period of the project. Independent classification of the 24 teachers by the two main researchers produced identical classification for all but two teachers, and these were resolved after discussion. The effect sizes by teacher type are shown in table 5. Although there is no obvious trend in terms of average effect size, as one moves from less to more expert teachers, the interquartile range of effect sizes reduces, indicating further support for the attribution of the effects to the quality of classroom assessment.

A comparison of the effects by different forms of control in the form of side-by-side stem-and-leaf diagrams (figure 4) shows that no significant difference in effect sizes for the different form of controls is apparent.

Table 4. Experimental Results for the 24 Teachers

School	Subj	Teacher	Yr	Set	n	Focal variable	Reference variables	Control group	n	SD	Raw effect	*d*	p
BF	M	Iwan	7	1	25	SE7	C7	D	95	17.54	+6.63	**+0.38**	0.0299
BF	M	Iwan	9	1	27	KS3	C7, S8	D	94	33.84	12.25	**+0.36**	0.0081
BF	M	Lily	7	3	25	SE7	C7	D	95	14.96	-5.22	-0.35	0.1434
BF	S	Rose	7	5	8	SE7	SB7	D	25	24.80	38.44	**+1.55**	0.0001
BF	S	Peter											
CE	M	Belinda	8	1	21	SE8	KS2	P	26	10.61	2.76	**+0.26**	0.3604
CE	M	Angela	9	3	23	KS3	KS2	D	21	15.93	19.12	**+1.20**	0.0001
CE	S	Sian	8	-	26	SE8	SE7	P	169	0.889	0.342	**+0.38**	0.0113
CE	S	Carl	8	-	27	SE8	SE7	P	169	0.911	0.417	**+0.46**	0.0018
CI	S	Derek	9	2	27	KS3	C7	D	56	0.666	0.183	+0.27	0.1984
CI	S	Philip	9	1		KS3	C7	P	56	0.695	0.169	+0.24	0.2305
CI	M	Greg	9	4	24	KS3	SE7	P	20	0.0379	-0.025	-0.07	0.8045
CI	M	Eva	9	1	29	KS3	SE7	P	28	0.4916	-0.127	-0.26	0.3997
RS	M	Nancy	8	1	32	KS3	C7	P*	34	38.7	-12	**-0.31**	0.0019
RS	M	Nancy	8	1	32	KS3	C7	S	30	27.8	+32	**+1.15**	0.0001
RS	M	Nancy	9	1	34	KS3	KS2	N	-	0.50	0.13	+0.26	0.0669
RS	M	Patrick	9	1	30	KS3	KS2	N	-	0.58	0.38	**+0.66**	0.0001
RS	M	Lisa											
RS	S	Jerry	8	2									
RS	S	Tom	8	2	32	SE8	-	P	34	43.38	+10.02	+0.23	0.3852
TB	S	James	11	1	32	GCSE	-	S	32	0.879	0.255	+0.29	0.2628
TB	S	James	11	1	32	GCSE	-	P	32	1.013	0.375	+0.38	0.1038
TB	S	Robert	9	-	30	KS3	SE8	I	56	15.33	2.95	+0.19	0.1438
TB	M	Steve	11	2	32	GCSE	KS3	P	31	0.941	0.380	+0.40	0.1093
TB	M	Steve	11	4	24	GCSE	KS3	D	87	1.48	0.222	+0.15	0.2849
TB	M	Kerry	11	4	23	GCSE	KS3	D	87	1.54	0.309	+0.20	0.1348
TB	M	Kerry	11	1	32	GCSE	KS3	D	82	1.95	0.4786	**+0.25**	0.0276
WF	M	Gwen	9	2	23	KS3	-	L	24	0.462	0.158	**+0.34**	0.2469
WF	M	Alice											
WF	S	Susan											
WF	S	Kieron											

Key

Focal variables

KS3 Key stage 3 tests
SBn School-produced test at beginning of year n
SEn School-produced test at end of year n

Reference variables

Cn CAT scores in year n
SEn School produced tests at end of year n

Controls

I Parallel track taught by same teacher in same year
S Similar track taught by same teacher in previous year
P Parallel track taught by different teacher in same year
L Similar track taught by different teacher in previous year
D Non-parallel track taught by different teacher in same year
N National norms
* Non-representative control

1.5	5
1.4	
1.3	
1.2	0
1.1	5
1.0	
0.9	
0.8	
0.7	
0.6	6
0.5	
0.4	0 6
0.3	4 6 8 8 8
0.2	0 3 4 5 6 6 7 9
0.1	5 9
0.0	
-0.1	3
-0.2	
-0.3	4
-0.4	5 9

Figure 3. Overall standardised effect sizes.

There was no difference in the mean effect size for groups of different ages, although it is worth pointing out that the year 11 (grade 10) focal groups all had positive effect sizes. There was no systematic variation in effect size by track. Analysis by subject shows that all the negative effect sizes were found for the mathematics groups, although the median effect sizes for the mathematics and science groups were almost identical.

Discussion

By its very nature, the quantitative evidence provided here is difficult to interpret. The controls are not equally robust. In some cases, we have comparisons with the same teacher teaching a parallel class in previous years, which, in terms of the main question (ie has the intervention had an effect?) is probably the best form of control. In other cases, we have comparisons only with a different teacher teaching a parallel class, so it could be that in some cases a positive effect indicates only that the teacher participating in the project is a better teacher than the control. In other cases, the control is another class (and sometimes a parallel class) taught by the same teacher, and while there are examples of positive effect sizes here (in the case of Robert, for example) it is also reasonable to assume that the observed size of such effects will be attenuated by what we have termed ‘uncontrolled dissemination’. In some cases, the only controls available were the classes of different teachers teaching non-parallel classes, and given the prevalence of ability-grouping in mathematics and science, and its effect on achieve-

Table 5. Effect sizes Classified by Teachers’ Use of Formative Assessment Strategies

Group	Count	Median	Interquartile Range
Experts	7	0.25	0.07
Moving pioneers	10	0.31	0.25
Static pioneers	2	1.38	0.35
Trialers	6	0.15	0.64

	N I S		L P		D
1.5		1.5		1.5	5
1.4		1.4		1.4	
1.3		1.3		1.3	
1.2		1.2		1.2	0
1.1	5	1.1		1.1	
1.0		1.0		1.0	
0.9		0.9		0.9	
0.8		0.8		0.8	
0.7		0.7		0.7	
0.6	6	0.6		0.6	
0.5		0.5		0.5	
0.4		0.4	0 6	0.4	
0.3		0.3	4 8 8	0.3	6 8
0.2	6 9	0.2	3 4 6	0.2	0 7 5
0.1	9	0.1		0.1	5
0.0		0.0		0.0	
-0.1		-0.1	3	-0.1	
-0.2		-0.2		-0.2	
-0.3		-0.3	4	-0.3	
-0.4		-0.4	9	-0.4	5

Figure 4. Standardised effect sizes by control type.

ment (see Wiliam & Bartholomew, 2001) disentangling the effect of our interventions from contextual factors is quite impossible. However, given the fact that the outcome variables were either national tests and examinations, or assessments put in place by the school, rather than devised by the teacher, we have some confidence that these measures have some validity in terms of what the teachers were trying to achieve, and, more importantly, that teachers do not have to choose between teaching well and getting good results.

However, the improvements in the achievements of students are not the only (nor perhaps the most important) outcome of this project. While a small number of our teachers did view involvement in the project as a short-term commitment, after which they would return to teaching 'normally', for the vast majority of our teachers, involvement in the project has not just spread to all their classes, but has fundamentally altered their views of themselves as professionals (Black, Harrison, Lee, Marshall and Wiliam, 2002). They not only enjoy their teaching more, but also have become ambassadors spreading the message to other teachers.

Conclusion

In this paper, I have argued that by trying to emulate the 'hard sciences' educational research has taken a wrong turn. Expertise in teaching does not consist of more and more complex explicit schemes for determining action (as is the case for example, in quantum physics), but is, rather, beyond rationality. Expertise is the ability to 'feel' what is the right thing to do, not after long deliberation, but immediately, and intuitively and this intuition is not instinctive, but is the result of the sedimentation of vast numbers of examples of experience. The role of the researcher in supporting the development of such expertise is not to attempt to distil expertise down to its essence, but to encourage its development in others. We cannot 'bottle' this for widespread distribution, but we can support communities of teachers by highlighting profitable directions in which they might develop their practice. At the end of the project, we are left with a final irony. In allowing the teachers to choose what they developed in their practice (so that each teacher was, in effect, engaged in a unique experiment) we have given up the ability to say what worked. We know that the process in which the teachers were engaged was productive, but we cannot say which elements worked, and which did not. In allowing the teachers to create their knowledge, we have given up the ability, as researchers, to make our own particular knowledge claims. So be it.

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Working Groups

WORKING GROUP ON GENDER AND MATHEMATICS: GATHERING REFLECTIVE VOICES

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This paper reviews the history of the work of the PME-NA Gender and Mathematics Working Group, and the progress made toward the intended end product of the Working Group. The first section, Introduction and Review, outlines the history and work to date of the group. The second section, PME-NA XXIV Session Goals, delineates the plans of the group for the PME-NA IV sessions. The third section, Review of Contributed Voices, includes abstracts of the13 papers submitted for the monograph project and being reviewed and discussed in this year's conference sessions. These papers are organized into four categories: *Reflecting on Voices in the Literature* includes three literature reviews; *Voices of Inquiry and Adolescent Girl* includes three papers reporting on adolescent girls' experiences with mathematics; *Voices from Post Secondary Classrooms* includes three papers reporting on work in post secondary mathematics and mathematics education classrooms; and *Voices (Re)Questioning* where four scholars raise questions about the work we do and future directions for inquiry around gender and mathematics.

Introduction and Review

In all prior PME-NA Gender and Mathematics Working Group sessions from 1998 through 2001, we took the role of the "working group" to heart. We planned for the inclusion of *all* of participants in terms of being informed of the work to date and in participating in the work at hand. We plan and conduct the 2002 sessions similarly.

In 1998, at PME-NA XX in Raleigh, North Carolina, sessions centered around "Gender and Mathematics: Integrating Research Strands." We explored why we do the work we do and what we know from this work; what the compelling topics for future study are; and how we might further this work. Participants presented short papers and discussed connected issues. Organizer analyses of discussions revealed two main strands in the discussions: the "sex-gender system" and the "doing of mathematics." Participants created a web (see Erchick, Condron, & Appelbaum, 2000 for a visual representation of the web) to represent on-going work in gender and mathematics that illuminated for us the multidimensionalities of our scholarship. We concluded that the web represented research strands in gender and mathematics as well as new ways in which the group was thinking about the research strands.

In Cuernavaca, Morelos, Mexico in1999, our PME-NA XXI Gender and Mathematics Working Group sessions were devoted to discussion, with the organizers responsible for synthesizing and analyzing each day's work. The first session was framed by the summary of work at and since PME-NA XX and by work contributed by Dawn Leigh Anderson, Peter Appelbaum, Susanne K. Damarin, and Diana B. Erchick. In the second session, goals were set by the group: keep our work visible in the mathematics education community and work toward integrating our research findings into mathematics education and mathematics teacher education. Toward that end, we generated suggestions for how we might accomplish our goals: developing a monograph of our work; conducting research exploring practice; and, as suggested by Fennema and Hart (1994),pursuing research that is feminist and qualitative in nature. Participants developed an action plan and an organization for a call for papers for our monograph. Categories for the monograph included, but were not limited to the development of epistemological voice; the integration of gender research into the mathematics classroom, K-16; the integration of gender research into the mathematics education classroom; the role of the content in addressing gender issues in mathematics education research; mathematical success in fast-track and other programs for girls in mathematics; and mathematical success for women in mathematics and math-using fields.

At the 2000 PME-NA XXII Gender and Mathematics Working Group in Tucson, Arizona, we discussed in depth the ongoing work of all of the participants present at the sessions. This allowed us to understand some of the research that was already in progress by these working group members and to place that work within the context of the emerging themes of our intended monograph. We also developed a working thematic structure for the monograph where the concept of "Reflective Voices" guided the structure. We organized work around multiple perspectives that include researcher, historical, teacher, student and critical perspectives and these perspectives include feminist, methodological, self-reflective and empirical standpoints. We1) generated ideas for the distribution of a call for papers; 2) defined group member participation in the creation of the monograph; and 3) planned continued connection in professional meetings over the upcoming year.

At the 2001 PME-NA XXIII Gender and Mathematics Working Group sessions at Snowbird, Utah, we shared draft papers intended for the monograph and welcomed a number of new participants into our group. We discussed the emerging work in the sessions and many of us had the opportunity to continue those discussions beyond the confines of our sessions. We planned guidelines and timelines for progress toward the monograph and have been able to meet the goals set in that planning since we left the conference in Snowbird. We now have 13 scholars with papers in preparation or completed and submitted to our monograph editors.

PME-NA XXIV Session Goals

For this, the 2002 PME-NA XXIV Gender and Mathematics Working Group in Athens, Georgia, we are organized into two sessions. In these sessions, we anticipate that, as has been true in the years of the Gender and Mathematics Working Group at

PME-NA, new members will join our ranks; some with the intention of submitting work to the monograph; others seeking the opportunity for scholarly discussion We find these new members, their input, critique, questioning and insights, vital to the growth and development of our efforts in the group.

Session One

During the first session a review of the progress of the PME-NA Gender and Mathematics Working Groups is in order. The focus of that review is the working outline of our developing monograph, *Research, Reflections, and Revelations on Gender and Mathematics: Multiple Perspectives & Standpoints* (Erchick & Condron, 2001). We expect to critique the outline, review our submitted work for the project and determine how it reflects the essence of the monograph. We also plan to begin to review the submitted manuscripts. Working subgroups developed in this session will read, discuss and begin to provide feedback on and delineate needs of submitted papers.

Session Two

Session two participants will attend with a familiarity and working knowledge of the monograph outline and contributed work. Subgroups will devote the second session to peer review and feedback on the structure, analysis, conclusions, and implications of the submitted papers. It is the goal of the group that all submitted papers have this internal review, feedback and support before final revisions, editorial reviews, and full manuscript submission.

Review of Contributed Voices

When Suzanne Damarin and Diana Erchick started this project in 1998, an early result of the working group sessions was a graphic that revealed two conceptions determined by the scholars working within the group. One determination of the group was that the structure of our examination of the scholarly work of gender and mathematics was nonlinear and very complex. The other determination of the group was that there were absences in the field of study, and it would be part of our mission as members of the working group to pursue scholarly inquiry in directions that would begin to contribute to the field in the areas of those absences.

As we collect, review and organize the work our members have been conducting over the past few years, we do so with an "attunement " to the voices of the teachers, students and scholars participating in the work. As Erchick & Kos (in press) explain, "attunement implies a deliberate focus on particular concepts and the voiced representations of them. It is a special way of listening, a special way of hearing. It requires knowledge and understanding of the concepts heard as well as an effort to hear." Our attunement reveals both our committed effort to focus on participant voices in our work and the scholarly understanding we bring to that hearing. Following are examples of the work we now bring to this project. This project is not intended to be a comprehensive satisfaction of the need to address the absences we have identified in the field of gender and mathematics; it is, however, intended to be a beginning.

Reflecting on Voices in the Literature

In her contribution, Research and Reflections on Women Leaving Post-Secondary Mathematics, Abbe H. Herzig writes from the Graduate School of Education at Rutgers University. In her work she writes how women and members of some racial and ethnic groups are underrepresented at higher levels of education in science, mathematics, and engineering (SME). That is, at each educational stage, disproportionately fewer members of these groups persist in the study of SME disciplines. This is a concern both because of our need to ensure equitable and fair opportunities for all individuals, and because of the loss of the contributions that these students might otherwise make. Herzig explains how research effort has been invested in examining the loss of these students—particularly women—from particular stages of the SME educational "pipeline," but few studies have focused particularly on mathematics. In her review, she draws out the parts of that research that are relevant to mathematics in particular, and reviews the research about females leaving mathematics. This review synthesizes research from various educational stages to present a longitudinal picture of women leaving mathematics, beginning in high school and throughout college and graduate school. Herzig includes her own reflections as a woman who left mathematics, and kept coming back, several times.

Linda Condron, from The Ohio State University, contributes a review of literature "stories" about women and the technical realm. There are stories throughout history that demonstrate women have always been active with/in science, mathematics, and technology. For example, Hypatia of Alexandria is famous for her work in algebra; Ada Lovelace wrote the first computer programs; and Grace Hopper was instrumental in the creation of modern computers and programming languages (Alic, 1986; Rossiter, 1982; Stanley, 1983). There are also stories of the consistent absence of women from the technical realm. Less than a hundred fifty years ago, many people believed that women's reproductive organs would be damaged if they studied mathematics and science (Solomon, 1985). Women were excluded from such study by official policies barring their access to formal education in many high schools, colleges, and engineering schools (Hacker, 1983). A story encountered by women in technical professions in our own times is that of marginalization. In the 1970s, women entered technical professions in record numbers. However, statistics indicate these well-educated women, working in the technical realm of today experience lower status and pay than their male counterparts (NSF, 1996). A compelling story can be gleaned from the writings of women like Evelyn Fox Keller (1985) who showed how science and the scientific method was conceptualized in 17th century Europe. Examining the metaphors used for science and technology, Keller reveals a story of science defined as male, a story within which women have no place from which to engage with/in the technical realm without calling into question their very identities as women (Harding, 1986; Keller, 1985).

Another story about the relationship of women to mathematics and technology is one of how mathematics came to be a school subject, to be important in engineering education, and to be used as a "critical filter" that excludes women and certain other groups from technical educational opportunities (Cohen, 1982; Hacker, 1983; Sells, 1992). Condron interprets this vast and complex literature as numerous competing stories, sometimes overlapping, sometimes contradicting one another (Condron, 1997; Haraway, 1991; Lather, 1991).

Dawn Leigh Anderson, from California State University at Fullerton, contributes a theoretical framework that draws upon the literature on feminist standpoint theory. In A Theoretical Framework for Inquiry: Feminist Standpoint Theory and Its Application to Mathematics Education, Anderson discusses a feminist epistemological framework by beginning with the point that there is no one feminist epistemology. She admits to the many and diverse feminist epistemological frameworks and that nowhere will we find one true feminism or feminist epistemology. The feminist theoretical framework that Anderson relies upon in her work is feminist standpoint theory, one that most aptly describes the lens through which she approaches her work. She first provides an overview of feminist standpoint theory in order to familiarize readers with its main tenets; then she discusses the application of feminist standpoint theory to mathematics education.

Anderson cites Nancy Hartsock's (1983) introduction of a "feminist standpoint" and traces the origins of standpoint theory in feminism back to Marxist thought, where the idea that the proletariat maintains a standpoint that is unique to the working class. Her points are made with a rich literature of feminist scholarship as she explains the tenets of feminist standpoint theory through discussion of women as constructors of knowledge; privileged epistemic viewpoint; agency; objectivity; and multiplicity. As she moves her discussion into mathematics education, Anderson includes topics like females as the focus of inquiry, mathematics as a gendered process, diversity, learning as a reciprocal activity, voice, agency, and authorship. She includes a discussion of epistemological perspective of mathematics within feminist standpoint theory and addresses the issue of "the risk of essentialism." She draws on the work of scholars such as Burton (1995; 1999), Confrey (1994; 1995a; 1995b; 1999), Damarin (1990; 1995), Fuss (1989), and Rogers and Kaiser (1995). Anderson is committed to the idea that there is no single feminist perspective of doing, knowing, and learning mathematics. She strives to understand the diverse and multiple experiences of girls in feminist mathematics classrooms. She explores patterns and examines how those patterns affirm girls' experiences in a feminist mathematics class.

Voices of Inquiry and Adolescent Girls

Jae Hoon Lim, from the University of Georgia, contributes her paper, Sociocultural Contexts of Learning School Mathematics: Impact of Social/Cultural Capital on Girls' Motivation and Identity to this project. The theoretical framework of this study is derived from the recent accomplishments of social constructivism and critical ethnography in education. Lim identifies critical perspectives that argue the case that education helps to maintain the status quo. Practices and structures marginalize or deny groups of people from society's influential positions. These groups of people include women, minorities, and members of low socioeconomic classes. She cites scholars such as Apple (1988), Calhoun (1993), Harter, Waters, and Whitesell (1997), Oakes (1990; 1992) in her arguments.

The work she reports upon here is a cross case study. Lim explores two young adolescent girls' experiences with school mathematics. The impact of the sociocultural context upon their motivation and mathematical identity is a focus. Lim conducted repeated in-depth interviews and ethnographic observations of the girls' mathematics classroom, and portrays two contrasting pictures of young adolescent girls who come from different ethnic, economic, and cultural backgrounds. The interview data revealed three themes. One is that the girls' experienced an anxiety that was grounded in the school mathematics culture. A second theme is that social and cultural capital contributed to the girls' motivation and identity construction. The third emergent theme for the girls is the "problematic dislocation between their social world and their experience with school mathematics."

Lim reports that "[t]he overall data analysis reveals the profound impact of social/cultural capital upon the girls' experiences with school mathematics well as their construction of identities in the discipline" and " illuminates the ways in which various sociocultural factors and forces dynamically impact aspects of their identities and motivation for learning school mathematics."

Ann C. Howe and Sarah B. Berenson report from the Center for Research in Mathematics and Science Education at North Carolina State University. In their paper, Talented Girls Talking About Their Attitudes, Experiences and Expectations in Mathematics, they describe work that focuses on twelve middle class, mathematically talented girls' as they talk about their attitudes, experiences and expectations related to mathematics. They also ask whether teachers see the girls as the girls see themselves. The paper is based on interviews of girls who participated in Girls on Track, a summer and follow-up program for ethnically diverse middle school girls who are enrolled in upper level math courses, and the results of a survey of teachers and counselors in the project. Howe and Berenson interviewed the girls and coded their data for interest, motivation, confidence, parental (and others') support, readiness, general self esteem, usefulness of mathematics and career plans. They also asked the girls to respond to a written questionnaire; teachers and counselors responded to two similar forms of the instrument, one to assess attitudes toward girls and mathematics, the other to assess attitudes towards boys and mathematics.

The results of the interviews confirm what is expected for girls who are on the upper math track. In their own words they tell us that they like math, their parents expect them to do well and they are willing to put forth the necessary effort to maintain good grades. Most of them are self confident, optimistic, have many interests, are involved in a variety of activities, have thought about their futures and expect to have professional careers. When Howe and Berenson analyzed the teachers' and counselors' responses to the survey instrument they found no significant difference in attitudes toward boys and girls in math. However, they found that the girls scored themselves somewhat higher in every category than teachers and counselors scored girls. That is, the girls are more interested, confident, motivated, and aware of priorities, stereotypes and gender than their teachers believe girls to be. The paper explores and reflects on these findings.

Diana B. Erchick contributes a report from an evaluation of a summer mathematics camp project implemented through The Ohio State University. Her paper, *Matherscize Camp for Middle Grades Girls: Reflections on Content and Process*, describes the perceptions of girls and teachers who participate in the camp, and the parents of the girls who participate. The camp is a weeklong summer project where lessons integrate mathematics with science, art and literacy. The Matherscize program intends to support the mathematical development of its participants through implementation of research-based curriculum and instruction and informed pedagogy.

Erchick explains how, to support understanding and connection in mathematics, the Matherscize camp focuses on providing meaningful mathematics experiences based on three criteria: 1) a curriculum and pedagogy grounded in recommendations of the learned societies and their research-based recommendations both for pedagogy and equity (NCTM 2000; Erchick, 2002a); 2) support for a process-based epistemology (Erchick, 2002b) in understanding the content of mathematics; and 3) development of community by girls during their school experiences with mathematics. Erchick conducted the evaluation from an interpretivist perspective (Denzin, 1989; Schwandt, 1994), a methodological focus that was particularly relevant in terms of generating findings grounded in students' interpretation of the experience of mathematics education – what role the content plays in their development; how meaningful the content is in their lives; how it does or does not contribute to the students' quality of life; how selected pedagogies and supports serve their needs; and how the girls interpret efforts to support their continued networking around mathematics.

All data collected for evaluation purposes was a part of the usual implementation and evaluation of the camp and consisted of application essays, daily writing samples, work products from the camp, feedback questionnaires from parents, and students and teachers participating in the camp. Erchick's analysis revealed the presence of a process-based epistemology and the ways in which the camp's pedagogical focus supported that epistemology. Analysis also revealed how the girls attending the camp perceived the social connections made in the camp to be important and meaningful. It is not inconsequential that the social connections centered on mathematics experiences.

Voices from Post Secondary Classrooms

Dorothy Buerk, from Ithaca College, contributes work entitled "Listening to Women in College Mathematics Classes." In quotations from remarkably articulate women, she hears a metaphor of math as a stainless steel wall, offering no handhold, on which are written innumerable God-given rules of mathematics. She also hears "'math is not a place for ideas in process'" (Buerk & Szablewski, 1993, p.152); and "'there seemed no room for interactions with the content, no possibility of connection with the ideas'" (Buerk & Szablewski, 1993, p.151).

Many students believe that mathematics is made up only of rules, formulas, and proofs to be memorized; skills to be practiced; and methods to be followed precisely. They believe that mathematics is a discipline where certainty is secure; where all questions have answers, which are known to authority (mathematician, professor, TA, textbook); where memorization, hard work, and some mystical quality called the mathematical mind are required. Buerk hears mathematicians report that the way mathematics is taught in the traditional classroom, in textbooks, and in their professional writing is the public image of mathematics. However, the way mathematicians do mathematics–the private world of mathematics–is intuitive, contextual, and narrative, involving experiencing the problem, relating it to their personal lives, and examining and resolving ambiguities (Buerk, 1985). Buerk relates this gap between the conception of mathematics of many students and the conception of mathematics often held by mathematics educators to the theories of Perry (1970, 1981) and Belenky, Clinchy, Goldberger & Tarule (1986). In her research, she presented women a series of mathematical experiences to encourage growth in conceptions of mathematical knowledge through successive positions in Perry's scheme, and observed parallel progression toward personal responsibility for their own learning, with mathematics becoming more approachable for them (Buerk, 1981, 1982). In the absence of such classroom experience of mathematics, Buerk is compelled to return to the provocative question posed by Elizabeth Fennema, "Is it possible that females have recognized that mathematics, as currently taught and learned, restricts their lives rather than enriches them?" (Fennema, 1994).

Kathleen L. Bonn, from Michigan Technological University has contributed a paper entitled What Factors Affect Women's Decisions to Pursue Graduate Degrees in Mathematics? In this work Bonn discusses nine college seniors, all women majoring in mathematics at a mid-size Midwestern research university. She points our that currently 50% of the bachelor's degrees in mathematics are now going to women; but given that so many women now major in mathematics, it is unusual that not one of these nine women had plans to pursue a graduate degree in mathematics. They had come to dislike mathematics, yet planned to get a job teaching high school mathematics.

Bonn's objective in this study was to understand how senior female mathematics majors made decisions about continuing on in mathematics. She conducted in-depth individual interviews and focus group interviews. Through the interviews Bonn asked participants questions regarding their educational histories in high school and college mathematics. She asked also questions about role models and questions about future plans. In additional interviews, seven particpants with high grade point averages, and thus the background to enroll in graduate-level mathematics programs, were asked to focus directly on the questions, "What are the factors that directly or indirectly influenced *your* decision on whether or not to attend graduate school in mathematics?" and "What are the factors that could directly or indirectly influence *other female math majors*' decisions on whether or not to attend graduate school in mathematics?" Bonn found three major factors which affected these women's decisions: (lack of) confidence in one's ability to do graduate-level mathematics, (lack of) perceived usefulness of a graduate degree in mathematics, and (lack of) enjoyment in mathematics. Bonn makes two recommendations: encourage more women to attend graduate school in mathematics and support mathematics education majors so that they will reenter high school classrooms enthusiastic about mathematics and confident in their abilities to foster this love of mathematics in their own students.

Hea-Jin Lee, from The Ohio State University at Lima, contributed The Evolution of Prospective Teachers' Perceptions of Teaching. She reports on work with preservice elementary and middle school teachers. Lee analyzed interviews, journal entries and concept maps, to investigate changes in the total number of items on the maps; the number of item streams (superordinate concepts close to the central concept); hierarchical organization; increased similarity to one another; use of technical vocabulary and frequently used terms introduced in the program.

Lee's findings point to the importance of understanding preservice teachers' prior beliefs to inform supervision and university course design; the need to routinize classroom management knowledge before attending to subject-specific pedagogy; and the importance of the academic task as part of the teaching knowledge base. Many students listed terms such as lesson preparation, attention, enthusiastic teaching, and teaching aids. Other frequent item streams included humor, reinforcement, and classroom management. There was a general lack of technical vocabulary evidenced on the maps, as well as a lack of detail and hierarchical organization.

Lee cites Pajares (1993) with suggestions for several approaches to challenging beliefs; and the work of Feiman-Nemser and Buchmann (1989) to discuss the tension between challenge and support, assimilation and accommodation, tensions that must be tolerated and cultivated. Lee discusses preservice students' evolution; connections between teacher beliefs and how they choose to teach (Anning, 1988); and research on changing beliefs and teaching patterns to make them more constructivist and student oriented. This work becomes particularly significant when considering the demographics of preservice programs. With females dominating the profession, the issue could very well become the ways in which women in the profession perpetuate beliefs about mathematics and how they use those beliefs to shape instruction for student learning. Indeed, how might these teachers teach the young in their care to learn to see mathematics? If the teachers approach mathematics as they do in Lee's study, women in elementary and middle grades education may model and teach their own beliefs and understanding, found in Lee's work to lack technicality, detail and hierarchical organization.

Voices (Re)Questioning

In her work at the University of Wisconsin-Madison Ólöf Björg Steinþórsdóttir studies students' strategies in solving proportion problems and the influence of problem semantic type and number structure on the use of strategies. Steinþórsdóttir also examines gender differences in strategy use. For the work she discusses for this project, she interviewed twenty-seven females and twenty-six males - all eighth grade students in one school in Reykjavik, Iceland. The problems used in her study represented four semantic structures, four problems of each structure. Each problem represented a distinct number structure. Steinþórsdóttir finds that number structure influenced strategy use and success to a greater extent than semantic type.

In her paper, *Less sophisticated girls? Or less sophisticated analysis,* Steinþórsdóttir's analysis does not end with strategies and successes. With a cognitive lens in analysis, Steinþórsdóttir focuses on "sophistication" of strategy and strategy use. She finds no gender differences identified in the overall success rate at solving these problems. Girls were more successful than boys in associated sets and symbolic problems; and boys more successful than girls in part-part-whole problems. Girls and boys used different types of strategies for all semantic types except the symbolic. Data suggest the semantic type influences females' choice of strategy more than that of males.

Steinþórsdóttir problematizes her analysis and looks at how this traditional cognitive analysis portrays a deficit model of girls. It therefore sustains the common belief that girls don't do as well as boys in math. After analysis she critiques her own analysis, presentation of results, and the discourse used in her reporting. Her intention is to reconstruct the terms less and more sophisticated and less and more mature strategies, toward more gender-fair implications and a new discourse for discussing girls' and boys' achievement in mathematics.

Suzanne K. Damarin, from The Ohio State University, in her paper, As the World Turns: Salient Issues on the Study of Gender and Mathematics, discuss both a linear progressions model and a "world turning" metaphor to discuss models of research around gender and mathematics. The linear progressions model begins with a set of findings and uses that one set of findings to build upon another. It includes the idea of progress, of one day coming to a final solution. The world turning metaphor is a cyclic one that Damarin finds more suitable for the study of gender and mathematics. In this model, each new day/season/year researchers in the field of gender and mathematics find the same issues and experiences as they have always found. Instead of this indicating an endless circle, the model indicates constant (re)encounters with concepts that are changing in the always new situations, always evolving contexts for returning issues of equity, power, agency, and voice.

Damarin starts her discussion with multiple critiques: of sex versus gender, essentialism in multiple venues, the denial of agency, and mathematics as a male domain. She then moves to a discussion of postmodernism as "both a condition and a philosophy, a mode of thought, a way of knowing." From this perspective, she returns to the critiques in the earlier part of her paper. Damarin deconstructively reconsiders those critiques in one reprise after the other entitled Both Sex and Gender, Anti-Anti Essentialism, Subjected Positions, and Maleness as a Mathematical Domain. She closes with thoughts on the promise for the future with regard to Technology, Cyber Feminism, The Information Age, and Cultural Change.

Bob Klein, from The Ohio State University, in his paper, *Computer Calculus: Integrating Technology (with respect to sex),* interrogates issues that lie at the intersection of technology, gender issues, and pedagogy. He asks the questions: How are pedagogies identified as being "female-friendly" or "feminist" enacted within a calculus reform classroom; How are feminist transformative desires approached by the enactment of these pedagogies? How are pedagogies contextualized? What implications arise from looking at ways in which pedagogies reflect (or not) the inclusive aims of feminist pedagogies and how accurate is the repeated use of "for all" as a label of inclusivity within educational discourse? Klein asks his questions of the site in terms of how the pedagogy was mediated by the use of technology. Klein collected data from a variety of sources surrounding a computer calculus course at a large Midwestern university in fall 2000. He identifies "Female-friendly" or "feminist" pedagogies are educational practices as having great potential to transform existing social relations thereby making our classrooms environments that embrace human diversity. Yet, taken uncritically or applied superficially, these techniques, when blindingly applied "for all" can serve ends that are counterproductive to feminist transformative desires.

Peter Appelbaum, from Arcadia University, contributes "What Do We Learn From Critical Theory and Media Studies?" and asks us to construct questions about power, knowledge, and social change. He discusses the increasing sophistication of our comprehension of gender issues in mathematics and how that gets reduced to popular, common sense pronouncements on the need of female students for special help; and the need for funding for innovative programs that "solve the problem." Appelbaum explains that the" increasing sophistication" comes from a consistent program in feminist epistemology and critique of mathematics as a disciplinary project and curricular encounter (e.g., Barton, 1998; Damarin, 1995). Strides made in reframing questions get translated in the mass culture as a threat to the commonsense notion that a neutrality and skill-based body of facts and procedures define mathematics. Any attempt to make mathematics meaningful, relevant or intellectually engaging for all becomes "a discursive interjection in the math wars debate between two constructed poles: traditional and "fuzzy" math." Special classes for girls sustain beliefs that the end result must be the "real," or traditional curriculum and girls should be compared as a category to norms outside the category. "Even if the objectives of a special program for girls adhere to standards of 'workplace readiness' and basic skill levels, however, the program itself can never be a model for the standard curriculum, but is doomed to special 'remedial' or 'epidemiologically preventive' status." Efforts to promote "girl power" and celebrate the accomplishments of girls and women in mathematics, serve to recreate glass ceilings for participation and performance while introducing new infrastructures and challenging the goals of traditional school programs.

Appelbaum points to an unbalanced view of the scenario on cultural change and social action, a view once posited by the anthropologist Marshall Sahlins (1985). People act to maintain social structure, but must change things to maintain that structure. Appelbaum makes the point that "what critical theory can help us with is the importance of power in the acting out of such action and social change. It is essential that we ask how knowledge is constructed as canonical or central to school practice, and why other knowledge is not."

Closing

In pursuing inquiry around the absences in the research on Gender and Mathematics, the PME-NA Gender and Mathematics Working Group participants have committed themselves to an interpretation of the field of gender and mathematics as complex and nonlinear. We have also chosen to investigate the absences we encounter with a respect for the reflective voices of the researchers, teachers, students, women and girls who contribute to the work. In the papers and processes of this project, we work consistently to respect the structure and voices that emerge.

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MODELS AND MODELING

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In this workshop, we will continue to reflect on *a models and modeling perspective* to understand how students and teachers learn and reason about real life situations encountered in a mathematics and science classroom. We will discuss the idea of a model as a conceptual system that is expressed by using external representational media, and that is used to construct, describe, or explain the behaviors of other systems. We will consider the types of models that students and teachers develop (explicitly) to construct, describe, or explain mathematically significant systems that they encounter in their everyday experiences, as these models are elicited through the use of model-eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000). During the workshop we will continue to explore these aspects of learning, teaching, and research by continuing our work in smaller groups focusing in: Student Development, Teacher Development, Assessment, Curriculum Development, Problem Solving, and an emphasis on Research Design.

Highlights of a *Models and Modeling Perspective*

According to this perspective, models are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other systems. A mathematical model focuses on structural characteristics of the relevant systems (Lesh & Doerr, in press). These models reside inside the minds of the learners *and* are embodied in the equations, diagrams, computer programs, or other representational media. Because models are conceptual systems, they are partly internal and are similar to the conceptual systems that cognitive scientists refer to as cognitive structures. Nevertheless, the conceptual systems that are most powerful and useful seldom function in sophisticated ways unless they are expressed using spoken language, written symbols, concrete materials, diagrams or pictures, computer programs, experience-based metaphors, or other representational media.

Students have preconceived notions about many important mathematical constructs. However, it is impossible for anyone to know exactly what's inside a students' mind. But, when students are asked to develop a model (which is expressed in some kind of representational media), many inferences can be made about the nature of their mathematical knowledge and its development. These representational systems encourage students to externally present their ideas on paper or other media. Therefore, the external representations that these students create are a means for researchers and teachers to view how students are thinking. In an analogous situation, when teachers are asked to develop a model about their students' ways of thinking, this model becomes a powerful window to view how teachers are thinking about this situation.

In order for students and teachers to externalize their understanding of complex situations, we have created different tools that elicit this type of thinking. Model-eliciting activities require students or teachers: (1) to develop a model that describes a real-life situation, (2) that encourages the solver to describe, revise, and refine their ideas; and (3) that encourages the use of representational media to explain (and document) their conceptual systems. These activities are similar to many real life situations in which mathematics is useful. Model-eliciting activities can be designed to lead to significant forms of learning because they involve mathematizing –by quantifying, dimensioning, coordinating, categorizing, algebraizing, and systematizing relevant objects, relationships, actions, patterns, and regularities.

For example, model-eliciting activities designed for students are intended for the solvers to reveal the way they are thinking about a given real-life situation that can be modeled through mathematics. These activities present real-life problem, to be solved by small groups of 3-5 students. The solution calls for a mathematical model to be used by an identified client, or a given person who needs to solve the problem. In order for the client to implement the model adequately, the students must clearly describe their thinking processes and justify their solution. Thus, they need to describe, explain, manipulate, or predict the behavior of the real world system to support their solution as the best option for the client. Like in real life, there is not a single solution, but there are optimal ways to solve the problem.

Model-eliciting activities are designed to engage students in developing math concepts. They set the students into a familiar context in which they are able to understand a need to develop powerful math ideas in order to solve a problem that is meaningful to them. That is, they are given a purpose to develop a mathematical model that best explains, predicts, or manipulates the type of real-life situation that is presented to them. Thus, students are forced into a cognitive situation where they can refine their mathematical ideas iteratively until they develop a construct that is useful and meaningful for them and for their client. Students' descriptions, explanations and constructions given to the client reveal how they are interpreting the mathematical situations they encounter by disclosing how these situations are quantified, organized, coordinated, and interpreted. In this way, model-eliciting activities allow students to document their own thinking and learning development.

A more complete description of *a models and modeling perspective*, written by many of the leading participants in this workshop, can be found in the book *Beyond Constructivist: A Models & Modeling Perspective on Mathematics Teaching, Learning, and Problems Solving* (Doerr & Lesh, in press). This book including chapters that focus on student development (Aliprantis & Carmona, in press; Dark, in press; Johnson & Lesh, in press; Kardos, in press; Lehrer & Schauble, in press; Lesh & Carmona, in press; Lesh, Zawojewski, & Carmona, in press; Oakes & Rud, in press;), teacher development (Doerr & Lesh, in press; Koellner Clark & Lesh, in press; McClain, in press; Schorr & Lesh, in press; van Reeuwijk & Wijers, in press), curriculum development (Carlson, Larsen, & Lesh, in press; Cramer, in press; Lamon, in press; Lesh, Cramer, Doerr, Post, & Zawojewski, in press; Shternberg & Yerushalmy, in press), and problem solving (English & Lesh, in press; Harel & Lesh, in press; Kehle & Lester, in press; Lesh, Lester, & Hjalmarson, in press; Lester & Kehle, in press; Middleton, Lesh, & Heger, in press; Zawojewski & Lesh, in press; Zawojewski, Lesh, & English, in press).

Research Design from *a Models and Modeling Perspective*

A models and modeling perspective has proved to be a rich context for research and development. One of the main points of convergence from the conclusions achieved by each of the groups in the past workshops resided on the need for innovative designs for research and assessment that can help answer questions that involve the understanding of complex situations that are dynamic and iterative. Through the work that main participants of this workshop have been doing in each of the areas: Student Development, Teacher Development, Assessment, Curriculum Development, and Problem Solving, collaborative work has been done in innovative research design.

The *Handbook of Research Design in Mathematics and Science Education* (Kelly & Lesh, 2000) describes a variety of innovative research designs that have been developed by mathematics and science educators to investigate interactions among the developing knowledge and abilities of students, teachers, and others who influence activities in mathematics and science classrooms. This handbook has helped in setting the foundations to identify several characteristics that distinguish the type of research design needed to answer they types of questions we are most interested in, which at the same time, lead to the need of new research designs in mathematics and science education:

First, it is important to radically increase the relevance of research to practice –often by involving many levels and types of practitioners in the identification and formulation of problems to be addressed- or in the interpretation of results, or in other key roles in the research process. So, instead of having only one-way transmission of research into practice, research methodologies that are proving to be most useful in mathematics and science education often involve bi-directional interactions and iteratively evolving feedback loops among many levels and types of participants (students, teachers, researchers, curriculum designers, policy makers).

Second, in mathematics and science education, most of the things that need to be understood and explained are complex systems –not necessarily in the strict mathematical sense, but at least in the general sense that they are dynamic, interacting, self-regulating, and continually adapting. That is, they do not simply lie dormant until they are stimulated; rather, they initiate action, and, when they are acted on, they act back. In particular when they are observed and when information is generated about them, changes often are induced that make researchers (and assessments) integral parts of the systems being investigated. Furthermore, among the most important systems that mathematics and science educators need to investigate: (a) many do not occur naturally (as givens in nature) but instead are products of human construction, (b) many cannot be isolated because their entire nature tends to change if they are separated from complex holistic systems in which they are embedded, (c) many are not observable directly but are knowable only by their effects on other agents or events, and (d) most include a variety of interacting communities of agents whose interactions lead to:

- Feedback loops that produce second-order which may outweigh or change the influence of first-order effects,
- Emergent characteristics of the system-as-a-whole that cannot be derived from characteristics of the agents within these systems.
- Behaviors that are often inherently unpredictable.

Third, the mathematical models that are needed to describe and explain the preceding systems are not restricted to linear equations or other kinds of simple input-output rules that presuppose the existence of independent variables that can be isolated, factored out, or controlled (Lesh & Lamon, 1992). For example, because of recent advances in fields such as those focusing on *geographic information systems*, there has been an explosion of new software and technologies that are capable of using graphic, dynamic, and interactive multimedia displays to generate simple (but not simple minded) descriptions of complex systems ranging from weather systems, to traffic patterns, to biological systems, to dynamic and rapidly evolving economic systems. Consequently, it is no longer necessary for educational decision-makers to rely on reports that involve nothing more than simple-minded unidimensional reductions of the complex systems that characterize the thinking of students or teachers –or relevant communities.

Fourth, research is about knowledge development; and, not all knowledge is reducible to a list of tested hypotheses and answered questions. In particular, in mathematics and science education, the products that emerging new research designs are intended to emphasize often focus on the development of models (or other types of conceptual tools) for construction, description, or explanation of complex systems. When producing these latter types of products, distinctions are being made between: (a) model development studies and model testing studies, (b) hypothesis generating studies and hypothesis testing studies, and (c) studies aimed at identifying productive questions versus those aimed at answering questions that practitioners already consider to be priorities.

Based on the term and the characteristics described by Brown (1992) and Collins (1992), we will call such research design a "design experiment". This type of research design can be characterized through four general principles that apply to Design Experiments focusing on the development of constructs and conceptual systems used by students, teachers, or researchers.

We will use the term "participant", to refer, generically, to students, teachers, curriculum developers, program developers, software developers, and other types of researchers, developers, or practitioners. This terminology is appropriate because, to investigate the nature of the developing constructs and conceptual systems used by any of these participants, the following principles should be expected to apply.

1. **The Externalization Principle**. Situations should be identified in which the relevant ways of thinking that are desired to investigate (and/or develop) are expressed in forms that are visible to both researchers and to relevant participants. Design activities naturally tend to lead to *thought-revealing artifacts*, like the model-eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000) described in the previous section. That is, the underlying design often is apparent in things that are designed; the underlying constructs often are apparent in complex artifacts that are constructed; and, the underlying models often are apparent in conceptual tools that embody them. In other words, in the process of designing complex artifacts and conceptual tools, participants often externalize their current ways of thinking in forms that reveal the constructs and conceptual systems that are employed. In this sense, then, the products may be referred to as *embodiments* of the relevant conceptual systems. Therefore, as the tools or artifacts are tested, revised, or refined, the underlying ways of thinking are also tested, revised, and refined. This tends to be true especially if the products are *conceptual technologies* in the sense that they include not only procedures for *doing* something, but also conceptual systems for *describing* and *explaining* the situations in which the artifacts or conceptual tools are intended to be useful. That is, the reason for developing these tools has to do with interpretation, description, explanation, or sense-making –as much as transformation, construction, or computation.
2. **The Self-Assessment Principle.** Design "specs" should be specified as criteria that can be used to test and revise trial artifacts and conceptual tools (as well as underlying ways of thinking) –while discerning products that are unacceptable, or that are less acceptable than others. The design "specs" should function as Dewey-style "ends-in-view". That is, they should provide criteria so that formative feedback and consensus building can be used to refine thinking in ways that are progressively "better" based on judgments that can be made by participants themselves. In particular, ends-in-view should enable participants to make their own judgments about: (a) the need to go beyond their first primitive ways of thinking, and (b) the relative strengths and weaknesses of alternative ways of thinking that emerge during the design process. Productive ends-in-view also should require participants to develop constructs and conceptual systems

that are: (a) powerful (to meet the needs of the client in the specific situation at hand), (b) shareable (with other people), (c) re-usable (for other purposes), and (d) transportable (to other situations). In other words, both the tools and the underlying ways of thinking should be shareable and generalizable.

3. **The Multiple Design Cycle Principle (or the Knowledge Accumulation Principle).** Design processes should be used in such a way that participants clearly understand that a series of iterative design cycles are likely to be needed in order to produce results that are sufficiently powerful and useful. If design processes involve a series of iterative development>testing>revision cycles, and if intermediate results are expressed in forms that can be examined by outside observers as well as by the participants themselves, then auditable trails of documentation are generated automatically; and, this documentation should reveal important characteristics of developments that occur. In other words, the design processes should contribute to learning as well as to the documentation and assessment of learning.
4. **The Diversity and Triangulation Principle.** Design processes should promote interactions among participants who have diverse perspectives; and, they also should involve iterative consensus building –to ensure that the knowledge, tools, and artifacts will be shareable and reusable- and so that knowledge accumulates in ways that build iteratively on what was learned during past experiences and previous design cycles. In general, to develop complex artifacts and tools, it is productive for participants to work in small groups consisting of 3-5 individuals who have diverse understandings, abilities, experiences, and agendas. By working in such groups, communities of relevant constructs tend to emerge in which participants need to communicate their current ways of thinking in forms that are accessible to others. Once diverse ways of thinking emerge, selection processes should include not only feedback based on how the tools and artifacts work according to the ends-in-view that were specified –but also according to feedback based on peer review. In this way, consensus-building processes involve triangulation that is based on multiple perspectives and interpretations. So, the collective constructs that develop are designed to be shareable among members of the group; and, they are designed in ways so that knowledge accumulates.

These characteristics and principles are not at all unfamiliar to scientists who investigate complex systems in fields such as astronomy, biology, chemistry, or physics –or in design sciences such as architecture or artificial intelligence. Thus, a new project has been supported by the National Science Foundation, in which input from leading scientist from outside the fields of mathematics and science education who are experienced in the use of Design Experiment methodologies in their fields will be considered for appropriateness to answer some of the questions math and science educators are demanding from research. This new project will produce a sequel to the earlier

Handbook of Research Design in Mathematics and Science Education (Kelly & Lesh, 2000). Not only will the new book focus on design research methodologies, but it will also describe on new types of dynamic and iterative assessments that are especially useful in design research –where rapid multi-dimensional feedback is needed about the behaviors of complex, dynamic, interacting, and continually adapting systems.

The Working Group at PME-NA XXIV

It is clear that mathematics and science educators have become increasingly aware of the fact that few of the most important problems they need to address are going to be resolved by only a single isolated study. Instead, understanding the types of complex systems we are interested in research nearly always require communities of researchers and practitioners, representing a variety of theoretical and practical perspectives, working together collaboratively over extended periods of time and across a variety of sites. The discussion during this workshop of the use of Design Experiments in Mathematics Education in each of the groups: Student Development, Teacher Development, Assessment, Curriculum Development, and Problem Solving, will provide an opportunity for all of the participants to begin or continue the development of these greatly needed communities of researchers and practitioners, to expand our focus of research and answer the types of questions we are being challenged by our field for this new century.

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CURRICULUM MATERIAL AND ITS USES IN TEACHING AND LEARNING MATHEMATICS

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As an outline of school educational activities, curriculum has been the focus of educational reforms in the past several decades (e.g., National Council of Teachers of Mathematics, 1980, 2000). In contrast, there are very limited research efforts given to examine the nature of mathematics curriculum and its roles in teaching and learning mathematics. Efforts to change curricula in the past have focused on revising and developing curriculum materials that were used in classrooms. Because curriculum materials have been a mainstay in mathematics classrooms in many education systems (McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers et al., 1987; Schmidt, McKnight, & Raizen, 1997), it is often assumed that the quality of curriculum materials matters. However, previous reform efforts in changing curriculum materials have not been as successful as educators might expect (e.g., Kline, 1973; McKnight et al., 1987; Project 2061, 1999). Previous efforts and results in reforming curriculum, in fact, suggest the importance of developing research on mathematics curriculum. As a step toward a better understanding of the nature and effects of curriculum materials in teaching and learning mathematics, a discussion group was organized at PME-NA XXIII (Snowbird, UT; October 18-21, 2001). General issues on the quality and role of curriculum materials in mathematics education were discussed. This working group is proposed as a means to further discussions by drawing on continued investigation and collaboration on this topic. In particular, this working group will focus on the following two issues:

(1) How can we examine the quality of curriculum material in terms of its content selection, presentation, and organization?

(2) How can we examine the use of curriculum material in teaching and learning mathematics in classrooms?

Research Background

Existing studies on the quality of curriculum materials have derived from two concerns: (1) possible contributions of curriculum materials to students' mathematics achievement, and (2) instructional features and functions embedded in curriculum materials. Examining possible contributions of curriculum materials to students' achievement was often taken in a direct and quantitative way in the 1950s and 1960s

(Walker & Schaffarzick, 1974) and was developed as measuring students' Opportunity-To-Learn (OTL) in more recent international studies (e.g., Robitaille & Garden, 1989; Westbury, 1992). Findings from previous studies suggest that curriculum materials are one of the key contributing factors to students' achievement. However, students' achievement cannot be solely explained by the differences in curriculum materials, and there are also much more in curriculum materials that need to be examined than simply measuring students' OTL. In particular, the instructional functions of curriculum materials are one aspect that has received research attention recently. Relevant studies have shown the feasibility and importance of examining instructional features embedded in curriculum materials in the United States (e.g., Project 2061, 1999) and in materials from different national education systems (Mayer, Sims, & Tajika, 1995; Schmidt et al., 1997). However, further efforts are needed to review, explore and discuss the types of features that are essential for comparing curriculum materials and determining their functions in facilitating the teaching and learning of mathematics.

Although curricular materials are a mainstay in mathematics classrooms in many education systems (Eisner, 1987; McKnight et al., 1987), the curriculum implemented in classrooms often combines with teachers' own thinking and planning (Doyle, 1993; Remillard, 1999). Examining the interaction between curriculum materials and teacher is a relatively new endeavor in curriculum studies. A few existing studies have examined teachers' use of textbooks in classrooms (e.g., Freeman & Porter, 1989; Stodolsky, 1989). For example, Stodolsky (1989) focused on comparing teachers' selections of content topics and instructional suggestions. She found that teachers consistently adhered to the content topics given in their textbooks, but departed from many accompanying instructional suggestions. Similarly, Freeman & Porter (1989) reported that teachers did not follow exactly what curriculum materials suggest they teach. Although few would disagree that there is a discrepancy between curriculum materials and what is implemented in classrooms, remarkable differences can be found from existing studies with regard to the extent curriculum materials affect teaching and learning mathematics in classrooms. Further research efforts are needed for a systematic and in-depth examination of the use of curriculum materials in teaching and learning mathematics in classrooms.

Plan for Involvement of Participants

The working group will be organized as a two-part activity. During the first part, participants will introduce each other and then the two organizers will present brief (about 15 minutes) overviews and/or examples of relevant research. The purpose of this short presentation is to outline the historical development of curriculum studies and bring participants up to date about relevant studies. Samples of curriculum materials from different education systems will be brought to the working group and shared with all participants. After the presentation, the participants will be organized to join the small-group discussions that will constitute the second part activity. The discussion in small groups will center on the two focal issues and will be facilitated with the examination and comparison of sample curriculum materials. After small-group discussions, all participants will come together to generate a collective summary and synthesis of the small-group discussions.

Anticipated Follow-Up Activities

Based on activities to take place at Athens, a list of potential research questions will be selected and interested participants will be organized to develop further research activities on this topic after the meeting. The issues of curriculum materials in mathematics education can and should be examined with a variety of points of view. Collaborative work, based on participants' research interests, can be developed either cross-nationally or within the United States. Participants will be strongly encouraged to share their research and come together again in future PME-NA meetings.

Connections to the Goals of PME-NA

This Working Group has emphasized research into the quality and uses of curriculum materials in mathematics education. The topic is developed from a long and broad range of studies on mathematics curriculum, instruction, and students' achievement, both within and across educational systems. Research on this topic draws on a rich background of psychological, pedagogical, and mathematical ideas, and it can open great opportunities for further study and collaboration. Thus, this Working Group connects to all three goals of PME-NA.

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THE COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

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Over the last several years, we have given serious attention to how students learn to reason probabilistically; that is, how learners construct mathematical models, and how these models interrelate with each other and with data. A special focus has been on how students build and work with information. Some of research along these lines has been reported and discussed at Singapore (ICOTS-5, June 21-26, 1998), at PME-NA 20 (North Carolina State University, Raleigh, North Carolina, October 31–November 3, 1998), at the third Robert B. Davis (RBD) Working Conference (Snowbird, Utah, May 22-26, 1999) at PME-NA 21 (Cuernavaca, October 23-26, 1999), PME-NA 22 (Tucson, October 7-10, 2000) and PME-NA 23 (Snowbird, October 18-21, 2001). Continuing discussion, investigation and collaboration draw on work at sites around the world.

Issues

At PME–NA 20 (Raleigh, 1998), the Working Group began to formulate a joint agenda for research, discussion and investigation. At Cuernavaca, the Working Group at PME-NA 21 developed this informal research direction further, together with presentations of data and research results. Discussions at PME-NA 22 (Tucson) and at PME-NA 23 (Snowbird) continued and extended the past discussion, with particular emphasis on identifying important mathematical and psychological issues that our work suggests should be addressed by learners, researchers and teachers. Central issues which the group discussed include:

- Gaining more detailed understanding of how learners work with data, through analyses of learners' images, representations, models, arguments and generalizations. Attention to learners' successes as well as learners' difficulties, in the unifying context of research on the development of understanding.
- Attending to how models, reasoning and thinking function in communities of learners, teachers and researchers. Examination of the roles of given tasks, of classroom environments, of student-teacher interactions, and of how learners, in a range of settings, share ideas, reasoning, and information.
- Placing more fundamental emphasis upon the development of mathematical ideas through time, with learners of different cultures, ages, and social backgrounds;

and with different prior mathematical and scientific experience. Recognition of the importance of detailed analysis of learners' and researchers' changing views of underlying psychological, mathematical and scientific issues.

To help focus and develop this agenda, the Cuernavaca discussion took as starting points the interplay of combinatorial and probabilistic reasoning for constructing images and models in the course of task investigations. At Tucson, and at later meetings, the complex role of such key ideas as sample spaces and data distributions came into the foreground, as well as very stimulating questions and discussion centered on the role of simulations and experiments in the building, by learners, of important models and ideas.

Theoretical Framework

Recent research emphasizes the complexity and subtlety of probabilistic reasoning, even in very basic situations. Models and representational strategies can easily extend distortions, even while they help support the growth of understanding. Indeed, the variety of representations which learners find useful, and the complex relationships among the models learners build and the data which they seek to explicate, provide rich opportunities for research investigations focused how learners, in social settings, construct, present, revisit and reconsider key ideas and ways of working. Indeed, given its complexity, the development of probabilistic thinking seems to *demand* reflective building over time. The tools available, the ways the tools are used, the ways in which ideas and information move among the learners, the teacher's questions, ideas and interventions, all contribute (or perhaps fail to contribute) in important ways. In our group's emerging view, both research and teaching need to take the need for long-term building, as well as the complexity even of very basic tasks, into account.

Background

Related cross-cultural research on particular dice games, by researchers in several countries, using different methods of analysis across a range of settings and learner populations, was reported in joint sessions at the International Conference on the Teaching of Statistics (ICOTS-5, Singapore, June 21-26, 1998). The Singapore reports (Amit, 1998; Fainguelernt & Frant, 1998; Maher, 1998; Speiser & Walter, 1998; Vidakovic, Berenson & Brandsma, 1998) helped motivate the work at Raleigh. Further discussions at the third RBD Working Conference (Snowbird, Utah, June 1999) addressed important aspects of the Working Group's agenda in the context of the growth of understanding.

The present Working Group, first at Raleigh, then at Cuernavaca, and again at Tucson, built upon this shared research, enlisted new collaborators, and helped continue an evolving and quite lively conversation. An incomplete but perhaps representative list of active members of the Working Group would include Sylvia Alatorre, Fernando Hitt and Araceli Limon Segovia from Mexico; and Alice Alston, Sally

Berenson, George Bright, Hollylynne Stohl, Susan Friel, Regina Kiczek, Clifford Konold, Carolyn A. Maher, Bob Speiser, Pat Thompson, Draga Vidakovic, and Chuck Walter from the United States, and Claude Gaulin from Canada. Further colleagues, in several countries, are known to be engaged in work related to the Group's agenda and concerns.

Plan for Involvement of Participants

At Raleigh, the Working Group considered data drawn from sixth-graders' work on two dice games (Maher, Speiser, Friel & Konold, 1998) which led to an extremely rich discussion. Based on this experience, a list evolved which at Tucson had come to include four tasks, which we invited participants at different sites to explore with diverse learner populations. Here are current versions of these tasks.

A Game For Two Players

Roll one die. If the die lands on 1, 2, 3 or 4, Player A gets one point (and Player B gets 0). If the die lands on 5 or 6, Player B gets one point (and Player A gets 0). Continue rolling the die. The first player to get 10 points is the winner. Is this game fair? Why or why not?

Another Game For Two Players.

Roll two dice. If the sum of the two is 2, 3, 4, 11 or 12, Player A gets one point (and Player B gets 0). If the sum is 5, 6, 7, 8 or 9, Player B gets one point (and Player A gets 0). Continue rolling the dice. The first player to get 10 points is the winner. Is this game fair? Why or why not?

The World Series Problem

In a "world series" two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the "world series." Assuming that both teams are equally matched, what is the probability that a "world series" will be won: (a) in four games? (b) in five games? (c) in six games? (d) in seven games?

The Problem of Points

Pascal and Fermat, in correspondence, discuss a simple game. They toss a coin. If the coin comes up heads, Fermat receives a point. If tails, Pascal receives a point. The first player to receive four points wins the game. Each player stakes fifty francs, so that the winner stands to gain one hundred francs, and then they play. Suppose, however, that the players need to terminate the game before a winner is determined. Further, suppose this happens at a moment when Fermat is ahead, two points to one. In correspondence, Pascal and Fermat discuss the question: How should the 100 francs be divided?

The first two tasks, and extensions, for example with tetrahedral dice, were developed for sixth-graders in the Rutgers-Kenilworth longitudinal study by Carolyn A.

Maher and her collaborators. Related work includes (Kiczek & Maher, 1998; Maher, Davis, & Alston, 1991; Maher & Martino, 1996; Maher & Martino, 1997; Maher & Speiser, 1997; Martino, 1992; Martino & Maher, 1999; Muter, 1999; Muter & Maher, 1998). The last two tasks were developed initially for eleventh-graders in the Rutgers-Kenilworth study.

Research on several of these tasks has already taken place at several sites around the world. Work in Brazil (Fainguelernt & Frant, 1998), in Israel (Amit, 1998), and in at least four places in the United States (Berenson, 1999), (Kiczek & Maher, 1998), (Maher, 1998), (Speiser & Walter, 1998), (Vidakovic, Berenson & Brandsma, 1998) has already been reported. Closely related findings, including (Alatorre, 1999) and (Berenson, 1999), were discussed in detail by the Working Group at Cuernavaca.

At the Tucson session of the Working Group, still more recent research was presented. In particular, the group discussed an extended videotaped student discussion of the World Series Problem, from the Kenilworth long-term study, directed by Carolyn A. Maher. At the Snowbird meeting, the group's continuing discussions were further broadened and extended. Points of special emphasis included (1) the roles played by experiments and simulations in the building of ideas by learners, (2) the multiple ways in which learners present and reason about data, and (3) the complex use, in actual practice, of software and related tools. Much discussion concentrated on the following new task, introduced by Maher:

The Boy-Girl Problem

There are two children in the house. You knock on the door, and a girl answers. What is the probability that the other child is a girl?

An extended written analysis of this problem by one student in the Kenilworth study was presented at Snowbird, after a very lively interchange among the members of the working group. At the Athens meeting, we look forward to further conversations based on task investigations, including recent work by Amit (in Israel) on children's thinking about the dice game tasks described above.

Anticipated Follow-Up Activities

Collaborative work, based on case studies drawing on a focused set of tasks, and upon related research from a variety of points of view, at different sites in several countries, has already helped to focus and extend discussion and collaboration. Based on our continuing experience, further work with learners, in a range of settings, as well as further sharing and collaboration, are expected. In this spirit, we cordially invite new participants to join a growing and productive enterprise.

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GEOMETRY AND TECHNOLOGY WORKING GROUP

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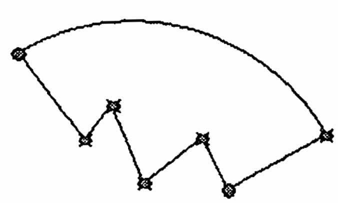
The working group on Geometry and Technology has met since the PME-NA XX conference at Raleigh, North Carolina in 1998. The objectives were to explore: the technology environment, student perspectives, and teacher perspectives. Since this initial year, we have discussed

- The student in an environment created and caused by micro worlds (PME-NA XXI)
- Preservice teachers' understanding about the role of proof in mathematics and the impact of dynamic geometry software on their understanding of proof and proof-writing skills (PME-NA XXII),
- A focus on proof with the investigations of the role of software in moving students from conjectures about drawings to the theoretical work with sketches (PME-NA XXIII)

Correspondence with the network established in the working group indicated that participants in 2001 wanted to continue sharing instruments and methods from ongoing projects. Instead of organizing the group sessions as a set of presentations, the group leaders used the network's suggestion to develop a format that would require participants to actively work on instruments and discuss potential student responses that would be meaningful to research. They presented two tasks from the Balanced Assessment Project and used *Rethinking Proof* (de Villiers, 1999) as a model for developing new performance tasks in research.

The group leaders divided participants into small groups and asked each group to discuss the mathematics in each Balanced Assessment task and to examine sample student responses. The essential parts of a high school level task involving properties of circles is shown below:



The circular glass top of your neighbor's coffee table breaks. Your neighbor is very upset and would like to replace the glass top but does not know exactly how big it was. He brings you a piece of broken glass that contains part of the boundary of the original top. Describe exactly what you would do in order to figure out the exact size of the original glass tabletop.

Figure 1. Task 1: Glass Top.

Students who attempt this problem know that a center and a circumference point determine a circle, and in this case the center is missing. A typical student response is to extend the outer straight edges of the glass piece and to mark the intersection as the center of the circle. Another student response shows that the student estimated that the arc of the glass piece was ¼ of the circumference. If the length of the arc is measured, the circumference and subsequently the diameter measure can be determined so that the full glass circle can be ordered. Some students remember that chords are somehow used to find the center of the circle but do not recall using the perpendicular bisectors of the chords to find the center. The first two students used an eyeball estimate to propose a solution. They used a method to produce a circle that looked good enough but did not check this solution for accuracy. The third student vaguely remembered some properties of a circle, but her memory of the logical relationships among the circle, chords, and the center was incomplete.

These responses demonstrate the struggle students have because they separate activities such as application and drawing from proof and reasoning. Schoenfeld (1986) describes students as "naïve empiricists whose approach to straightedge-and-compass constructions is an empirical guess-and test-loop" (p. 242). In proof, the objects are hypothetical and theoretical; the standard of correctness for the student is logic. On the other hand, objects in constructions and diagrams have spatial-graphical properties and are real; the standard of correctness is accuracy of the drawing.

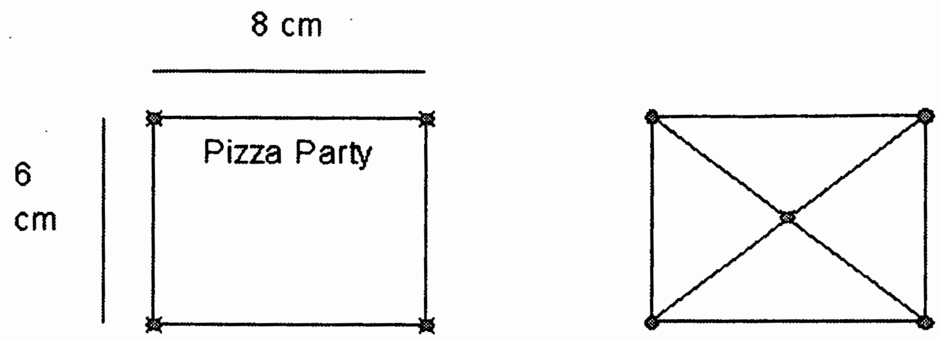
Theoretically, geometry software should address the gap between empirical work and deductive thought because the dynamic features of the software make it a philosophical tool that allows spatial-graphical diagrams to behave in a theoretical way. The group leaders posed these questions:

1. How would students perform differently if this task were in a computer environment?
2. Do students demonstrate the use of deductive thought with the software as a philosophical tool?

An obvious response was that students could more easily check their solutions to see if the proposed circle fit the curve of the glass. However, checking is still empirical work. It was noted that students could cheat by selecting three points on the given arc in a certain order to draw the major arc and complete the circle. While this response demonstrates knowledge of the software, the student may not know the theoretical relationships to find the center. It was noted that very different and interesting solutions could be found that student would probably not attempt with pencil and paper.

A second task involved the properties of quadrilaterals and measurement calculations (See Figure 2). The working group liked this task because it encompassed a wide variety of geometric topics. It requires students:

- To use spatial reasoning with properties of the rectangle and rhombus to explain why most envelopes make a rhombus when unfolded.



You work for an envelope manufacturer. You have been given a job by your boss: investigate different envelopes that will hold this invitation. There will be three parts to this investigation:

1. Explain why most envelopes make a rhombus when they are unfolded.
2. Design a "rhombus" envelope to fit the card giving all measurements of sides and angles.
3. Design an envelope to fit the card that makes a rectangle when it is unfolded and give the measurements of the sides and marked angles.

Note: Diagrams on two more pages of the task suggested the geometry involved for students in this task.

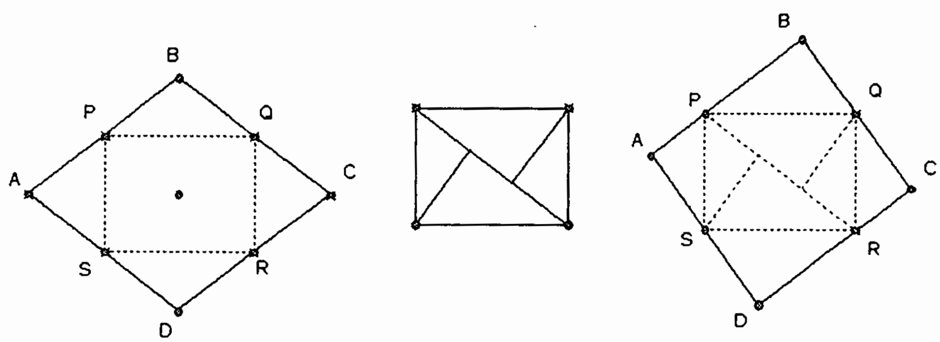


Figure 3. Task 2: Design an Envelope.

- To use the Pythagorean theorem and trigonometric ratios to design an envelop and provide measurements.
- To use similarity and/or trigonometry to design an envelope that makes a rectangle when it is unfolded.

In a computer environment students could find points A, B, C, and D by using reflections over the edges of the invitation. Of course students could cheat with the measurement tool, but they could be required to justify the measurements. The following samples show student work:

Student A: "ABCD is a rhombus because it is a sloping square."

Student B: He/she used the Pythagorean theorem appropriately to calculate the side measurements of the rhombus but believed that the diagonals of the rectangular invitation bisected the right angles. He/she also knew that the triangle formed with the rectangular envelope were similar but could not proceed to make calculations with ratios.

Student C: "The diagonals of a rectangle bisect each other, and the endpoints of the diagonals form the midpoints of the sides of the outershape. Since all of the segments formed by the midpoints are congruent, the outer figure is a rhombus." This student also used the Pythagorean theorem appropriately and recognized the similar triangles in the rectangular envelop without calculating the measurements.

Student D: This student did not have as nice an explanation as student C for the rhombus figure, but he/she did systematically and accurately perform all measurement calculations.

The computer software would have allowed students to test statements and measurements. For example, Student B could have tested the conjecture about angle measure. A more interesting and appropriate task for the computer would be to require students to construct the envelopes without showing them the unfolded diagrams.

After the discussion of these Balanced Assessment tasks, the group leaders introduced the article *Rethinking Proof* (de Villiers, 1999). De Villiers relates students' separation of application and drawing from proof and reasoning to van Hiele theory. He emphasizes *constructive defining*, a process in which concepts and theory are constructed by "changing a given definition by excluding, generalizing, specializing, replacing, or adding properties to the definition." (p. 15) This process progresses through the van Hiele levels. At level 1 students use visual definitions and recognize whole figures; at level 2 they use uneconomical definitions, recognizing all properties for a given figure; and at level 3 they use correct, economical definitions and recognize logical relationships between figures; level 4 students use longer sequences of logical statements and recognize a system of definitions, axioms, and theorems. Levels 1 and 2 students also have partitional definitions as opposed to the level 3 hierarchical definitions that allow inclusion of subsets of figures in more general sets of figures (squares are subsets of rectangles). De Villiers states that teachers should allow students to construct and refine definitions at all levels. He demonstrates how the dynamic nature of software can aid in the development of hierarchical definitions and advocates that teachers first provide ready-made constructions of figures so that properties can be thoroughly explored in the causal relationship between a premise condition and a conclusion. For example, "perpendicular bisecting diagonals" is a consequence of constructing "all four sides equal" in a quadrilateral. "A tangent line at a point on a circle" is a consequence of constructing "the perpendicular to the radius determined by that point." Eventually students learn to make their own constructions. Citing an older but nonetheless relevant article, Smith (1940), de Villiers describes the importance of using "if-then" thinking in learning to do constructions:

> Pupils saw that when they did certain things in making a figure, certain other things resulted. They learned to feel the difference in category between the relationships they *put* into a figure—the things over which they had control—and the relationships which *resulted* without any action on their part. Finally the difference in these two categories was associated with the difference between the *given* conditions and *conclusions*, between the if-part and the then-part of a sentence.

The computer environment allows students to explore which properties or conditions are necessary for a construction and to reduce the properties until they have a sufficient set for construction. When a sketch is created with logical properties that students *put* into a figure, other relationships *result and remain* invariant when the students drag the defining objects of the sketch. It would seem that the students who responded to the Glass Top task missed the prerequisite experiences of exploring the logical relationships among radii, chords, tangents, and perpendicular lines. Without a known center, only general chords could be constructed in the glass piece. Perpendicular bisectors of chords necessarily contain the circle center, and two intersecting perpendicular bisectors are sufficient to determine the center point. The task itself could be done with or without the computer, but perhaps the conceptual substructure needs to be developed with the geometry software.

Because the logical "if-then" syntax is integral to geometry software and should provide a connection between drawing and proof, the group leaders extended the work with Balanced Assessment tasks to de Villiers' activities categorized by functions of proof. In *Rethinking Proof*, de Villiers (1999) has listed these functions of proof:

- Verification (concerned with the truth of a statement)
- Explanation (providing insight into why it is true)
- Systematization (the organization of various results into a deductive system of axioms, major concepts and theorems)
- Discovery (the discovery or invention of new results)
- Communication (the transmission of mathematical knowledge). (p. 4)
- Intellectual challenge (the self-realization/fulfillment derived from constructing a proof)

De Villiers' research (1991) indicates that explanation, discovery, and verification can be meaningful to level 2 students while challenge and systematization are appropriate for level 3 and 4 students. If students have experiences with the first three functions, teachers help them make a transition from level 2 to levels 3 and 4. This result opens investigations to study samples from the middle grades (5-8) to college level.

The working group leaders assigned each of 6 small groups one of these functions of proof so that fertile ideas could be developed for any convenient sample of study and new performance tasks for research could be modeled. Each group described deVilliers' interpretation of this function, added a group opinion, and reported an example of a task that would demonstrate this role. The following summaries review the group reports:

Group 1-Verification: A convincing argument bolsters the conviction that a conjecture is true. Testing whether or not a conjecture makes sense and fits with other logical relationships and properties is an important aspect of mathematical activity. In one example, "Area," (de Villiers, 1999, p. 73) students are given a pre-made sketch of the following quadrilateral. They are asked to make a conjecture about the relationships between the area of IJKL and ABCD. Of course, students will most likely use measurement tools to investigate, but the figure on the right provides a way for students to be convinced that the dissection actually divides the original figure into four equal areas.

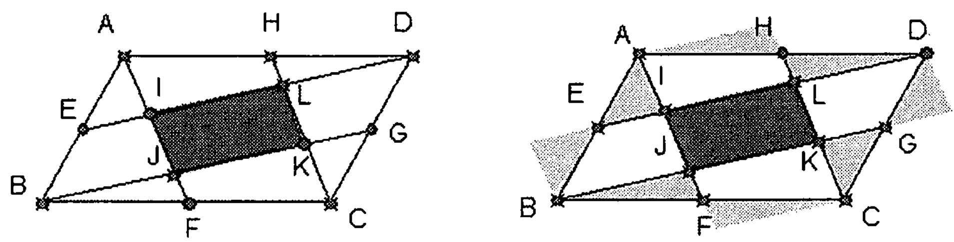


Figure 3. Area comparison of quadrilaterals.

Making the figure on the right requires transformations and is more rigorous than testing with measurements.

Group 2-Explanation: Accurate constructions and measurements confirm a conjecture but do not explain why it is true; they are convincing but do not provide insight into how the conjecture fits with other familiar results. For example, a visual proof that the triangle sum of angles is 180° occurs when the angles are cut out and placed on a straight line. However, the proof provides insight into the necessity of the parallel postulate to establish the relationship. This group also examined the concurrency of the medians at the centroid of a triangle. Dynamic software provides a convincing demonstration of the validity of the conjecture. The group showed how de Villiers' activity, "The Center of Gravity of a Triangle," (1999, p. 51) guides students to form a proof that explains the result with the use of auxiliary constructions, the midpoint connector theorem, and the properties of parallelograms. Students also gain insight into the technique of using the intersection of two segments to show that a third segment determined by this intersection and a triangle vertex also has the properties of the first two segments, namely that it is a median also.

Group 3- Systematization: Tasks that investigate systematization use properties that students have previously studied. They are not designed to determine whether these properties are true or not, but to investigate their underlying logical relationships. Students develop an understanding of the role of definitions in an axiomatic system and investigate alternative but equivalent definitions. Student definitions become both economical and hierarchical. An example studied by this group was "Systematizing Rhombus Properties." (de Villiers, 1999, p. 133)

Group 4-Discovery: Empirical work can lead to discovery of logical relationships when students participate in the explore/discover/conjecture/prove model of teaching geometry. However, some results can be discovered in a deductive manner, i.e. proof may lead to proof. When writing a proof, one may realize that the argument can be modified somehow to lead to other relationships and conclusions, and hence, other proofs. This group shared the example of how students can discover the law of cosines in the process of exploring chord relationships in a circle using geometry software. A participant showed how the theorem, "If two chords intersect in the interior of a circle, then the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other chord." leads to the law of cosines. The group also discussed the value of having students investigate false conjectures in order to underscore the idea that a valuable part of exploration is the search for counterexamples. Consider: Given a rectangle, the area increases as the perimeter increases. Finding and characterizing counterexamples of this statement show why a proof of the conjecture is impossible.

Group5-Communication: Both the "internal" communication required when one is thinking though a problem and the "external" communication required to share work with others encompasses all the roles of proof. The group agreed with de Villiers (1999, p. 7) that proof is a form of social interaction and involves negotiating both the meaning of mathematical concepts and the criteria for acceptable argument. However, they questioned isolating this role of proof in tasks because any task that could isolate use of any of the other roles would necessarily demonstrate communication skills. In fact, de Villiers does not develop separate activities for this role.

Group 6-Intellectual Challenge: Sometimes we seek to prove a conjecture simply because it is there to be proved. The Pythagorean theorem is perfect example for thinking about proof in this way. Although one proof was certainly sufficient, there exist hundreds of proofs of this famous relationship. The challenge might be to find a proof that is more elegant more intuitive, or more direct. Proof as intellectual challenge also includes the mathematical activity of extension—changing initial conditions, removing or adding constraints, asking "what-if" questions and thinking outside of the box. This group shared an example of student work from a summer workshop in which inservice teachers explored proofs of the Pythagorean theorem and then extrapolated variations involving equilateral triangles and semi-circles on the sides of the right triangle. They also showed how teachers extended their investigations by connecting the vertices of the squares on the sides of the right triangle to form three new triangles that each have the same area as the original right triangle. In this manner, experimenting with a well-known proof provided opportunity to explore then make and prove other conjectures. The proof itself served to stimulate further mathematical activity and proof.

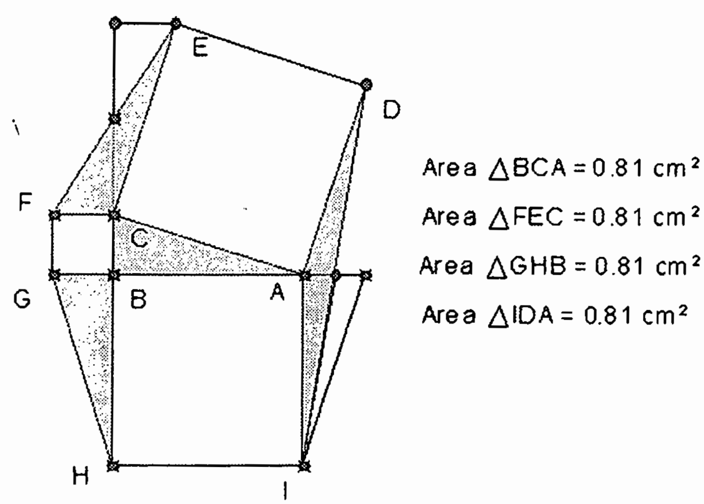


Figure 4. Inservice teacher extension of the Pythagorean theorem.

The culminating questions that occurred to several participants were, "How will students benefit from computer tasks that focus on the functions of proof?" and "Will these tasks transfer to design tasks so that students apply more theory to constructions?" The group leaders' charge to the working group participants was to investigate these questions by using an existing or developed assessment with a targeted group and report in 2002.

In addition to participant reports, the working group for PME-NA XXV will focus on an underlying theme in de Villiers' work, the idea of making sense of geometry through verification, explanation, discovery, and systematization. A key reading for this working group session will be "Sense Making: Changing the Game Played in the Typical Classroom." (Flewelling, 2002). On the first day, the group will briefly summarize past work and call for participant reports. We will review the new article and relate it to past work and tasks. Student learning in elementary levels through college geometry will be the focus of the second day. Participants will create and share sense-making tasks for various levels of students and discuss the implications such tasks/activities have for research. On the third day, we will discuss how the tasks/activities might differ in design for preservice and inservice teachers when there are both mathematical and pedagogical learning objectives. Working group participants will be encouraged to find ways to use the tasks/activities in their research and teaching over the coming year.

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INVESTIGATING AND ENHANCING THE DEVELOPMENT OF ALGEBRAIC REASONING IN THE EARLY GRADES (K-8): THE EARLY ALGEBRA WORKING GROUP

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The Early Algebra Working Group is interested in investigating and enhancing the development of algebraic reasoning in young children, primarily in grades K-8. The group is NOT interested in investigating or promoting the *early* introduction of the standard high-school algebra course in the elementary or middle grades. Our focus is on investigating and describing what we construe as the possible geneses of algebraic reasoning in young children, and in developing and investigating ways to enhance that reasoning through innovative instruction, applications of appropriate technology and professional development for teachers.

Brief History of the Group

The Early Algebra Group (EARG) has been meeting since 1998 and, more recently, in response to a call from the International Commission on Mathematical Instruction (ICMI), constituted the Early Algebra Working Group (EAWG), one of the eight groups in the 12th ICMI Study Conference *The Future of the Teaching and Learning of Algebra* (Chick, Stacey, Vincent & Vincent, 2001) held in Melbourne, Australia, in December of 2001. The EAWG held a preparatory meeting before the Melbourne Conference and conducted an email discussion of points to be raised at the conference. During the conference, the EAWG met for a total of 14 hours over 5 days. A report was given to the whole conference that summarized the results of our discussions. This report outlined the following points:

1. What Early Algebra is NOT (it is not the early introduction of standard HS algebra).
2. Historical overview (the different origins of arithmetic and algebra).
3. Overview of research in Early Algebra (from 1970's through present day with some concrete illustrations using video clips from various research projects).
4. Implications of this research for algebra in grades 6-14.
5. Work to be done: Research and Policy Issues needing attention.

Plans for Sessions at PME-NA

During the PME-NA meeting we plan to hold at least three sessions (one on each of Sunday, Monday and Tuesday). The opening session will begin with a report from the ICMI EAWG discussions that will provide an overview and vision for the Working Group and share the conclusions of the ICMI EAWG with the wider PME-NA community. Jim Kaput will lead this overview. We shall invite discussion of these conclusions and outline the goals for the rest of the sessions. Les Steffe and Andrew Izsak will make brief presentations on their proposed research on students' quantitative reasoning and how this relates to students' algebraic development. At the Monday session, Analúcia Schliemann will present results from the TERC/Tufts longitudinal study on early algebra in 2nd through 4th-grade classrooms, Teresa Rojano will present her research on an early approach to functions using computing technologies and Joanne Lobato will present a report on her research concerned with transfer and generalization in the context of quantitative reasoning. The final session on Tuesday will begin with a report from Megan Franke on the work of the *National Center for Improving Student Learning and Achievement in Mathematics and Science* on issues related to the professional development for elementary teachers with respect to algebraic thinking. We shall end the session with a discussion of the issues raised during the meetings of the WG at the conference and plans for future activities, including discussion group meetings at PME 27 in Hawaii.

Brief papers describing each of the scheduled presentations follow this introduction.

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RESEARCH ON THE DEVELOPMENT OF ALGEBRAIC REASONING IN THE CONTEXT OF ELEMENTARY MATHEMATICS: A BRIEF HISTORICAL OVERVIEW

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The Algebra Problem

The US has an increasingly difficult "Algebra Problem" to solve. Standard indicators of student achievement – international comparisons, NAEP and other national measures, as well as state test results—all point to decreasing achievement as grade levels increase. Well-known TIMSS curricular analyses point towards repetitious computational skill building and superficiality of topic coverage as important contributors to the problem. The repetition is followed by a delayed, abrupt and isolated first year algebra course, with continued tell-and-drill pedagogy that puts students in an overly passive position as learners. High failure rates in first year algebra courses, especially in urban districts, contribute to student alienation, high drop-out rates, and low teacher morale. Increasingly political responses to the problem involve a wide variety, but largely ineffective tactics in the form of middle and early high school pre-algebra courses of various kinds, screening and diagnostic tests, district and state algebra mandates, lengthened courses, diluting the definition of "algebra," creating multiple versions of "algebra," end-of-course & other high-stakes exams, among others. Each adjustment produces small local improvements, although, since the tactics tend to be ad hoc and implemented independently, improvements do not accumulate or multiply. NCTM recommendations for integrating the development of algebraic reasoning across K-12 are implemented only in a few of the standards-based curricula, and these mainly at the middle school level. And most importantly, algebra's breadth & depth are not systematically reflected in school mathematics experience, so that School Algebra does not serve School Mathematics in the positive ways that Algebra has historically served Mathematics.

Research Toward Solutions

Several research teams studying the development of algebraic reasoning in the early (K-5) grades formed an informal research collaborative in the late 90's which renews a prior collaboration funded by the US DoE OERI in the early/mid 1990's through the National Center for Research in Mathematical Sciences Education that in turn grew out of an earlier "Algebra Working Group" in the latter 1980's that examined the nature and purposes of algebra in school mathematics. Today's line of work shares broad hypotheses regarding the large positive potential of building algebraic reasoning in the context of elementary mathematics, the need to exploit untapped student learning capacities among younger students, the powerful role of generalization and formalization in deepening students' experience and understanding of elementary mathematics – especially arithmetic—and the potential for using changes in the curricular relationships between algebra and elementary mathematics as a catalyst and vehicle for faculty development of elementary teachers. These researchers engage in several complementary lines of work focusing on different aspects of the problem—student learning, classroom practice, curriculum design, professional development, capacity building, and so on.

Reflections on Why the Algebra Problem Is Difficult

Several aspects of the problem contribute to its depth and complexity beyond uncertainty regarding its place in the curriculum or its relevance and importance due to continuing changes in technology and applications. We see 2 categories of reasons: (a) involving the inherent complexity of algebra content, and (b) institutional/historical reasons.

Content Reasons:

The pervasiveness of the roots of and uses of algebraic reasoning mean that it is not just a topic or topic strand. It has multiple interacting aspects, two of which underlie the others. Root Aspect #1: Algebra as generalizing and systematically expressing patterns & constraints, especially, but not exclusively, algebra as generalized arithmetic & quantitative reasoning. Embodying generality and the systematic expression of generality, it is an intrinsic way of being mathematical. Root Aspect #2: Algebra as reasoning from the *forms* of syntactically-defined statements and the syntactically guided manipulation of those formalisms to build insight. Other aspects of algebra, include algebra as the study of patterns, functions & relations, as the study of the structure of operations & systems (abstract algebra), and algebra as a web of representation systems.

Institutional/Historical Reasons:

School algebra is hard to change because: (1) Algebra is a Socio-Political-Economic Institution as well as a web of skill & knowledge that complexly interact with other knowledge & skill. (2) Algebra the Institution co-evolved with our systems of education, so is deeply intertwined with them and cannot change independently of them. (3) Algebra & arithmetic had different historical origins, served different social purposes, served different populations and were historically instantiated as separate curricula for demographically separate populations.

THE ONTOGENESIS OF ALGEBRAIC KNOWLEDGE

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Mathematics education as a profession is in a period of reform to improve the teaching and learning of school algebra. Furthermore, "algebra for all" is a popular mandate in schools and mathematics education policy documents that signals desires for and movement toward more equitable school practices. However, without rethinking what algebraic reasoning is and how students learn to reason algebraically, these mandates effectively legislate that everyone must come to know a univocal algebra that the researchers feel does not necessarily respect individuals' ways of knowing and does not necessarily delve deeply into conceptual ways and means of operating algebraically. Interpretation of "algebra for all" mandates within the reform movement is crucial because our prior research (Olive, 1999; Steffe, 2002; Steffe & Olive, 1998) indicates that students come to middle school with distinctly different ways of operating quantitatively and numerically, ways that we believe will significantly impact the students' development of algebraic knowing.

Thus, to interpret such mandates, we plan to conduct an ontogenetic analysis of algebraic knowing by examining how 6th and 7th grade students operate quantitatively and by investigating how to bring forth their algebraic reasoning as a reconstitution of their quantitative operating. Based on prior research on children's number sequences (Steffe & Cobb, 1988; Steffe, 1992) and children's fractional knowledge (Steffe & Olive, 1998) we have identified three categories of students that are fundamental to this work: *pre-fractional* students, *additive fractional* students, and *reciprocal reasoners*. Students in the different categories reason in significantly different ways: pre-fractional and additive fractional students are yet to construct multiplicative fractional concepts. Since quantitative and numerical operating forms a basis for algebraic reasoning, the nature of the quantitative operating of these students presents a serious and as yet unsolved problem in mathematics education. While reciprocal reasoners operate multiplicatively, many have not yet constructed fractional operations that are fundamental to reasoning algebraically. The goal of this research is to construct an algebraic learning trajectory for each of these categories of students based on their initial concepts and operations and how these concepts and operations are used in their further constructive activity. An ontogenetic analysis is made as we construct actual learning trajectories of students using hypothetical learning trajectories as a guide in the process of teaching students. Thus, an essential orientation in this research is to bring forth students' algebraic reasoning out of their quantitative operating in the context of teaching-learning interactions. We hypothesize that the pre-fractional and additive fractional students will make progress toward constructing an additive algebra, including unknowns and equations, as well as multiplicative operations and ratios. Once the reciprocal reasoners have constructed fractional operations, they will move toward the construction of rates, variables, covariation, and linear functions.

We plan to conduct a three-year constructivist teaching experiment with four students in each of the three categories starting in the spring semester of the students' 6th Grade and working with them twice a week over the course of two years. As we use teaching as a source of scientific investigation, our intention is to teach the students with the goal of promoting the greatest possible progress we can in all of the participating students. To this end and to serve our central goal, we formulate and test local hypotheses within teaching episodes. The work of the teaching experiment will include: ongoing conceptual analysis of quantity, quantitative operations, and algebraic knowing; a mapping of actual learning trajectories based on hypothetical learning trajectories; and extensive use of representations and technology tools.

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COORDINATING STUDENTS' AND TEACHERS' ALGEBRAIC REASONING (COSTAR)

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Student learning in algebra is a perennial concern in U.S. mathematics education. In recent international studies, U.S. middle-school students' achievement in algebra lagged behind that of their peers in high-achieving countries (National Center for Education Statistics, 2001b; Peak, 1996). Despite increases in average scores on the National Assessment of Educational Progress (NAEP) exam in 2000, 34% of the nation's eighth graders scored below a "basic" level of performance in mathematics (National Center for Education Statistics, 2001a). Moreover, improving the teaching and learning of school algebra in the United States has become central to achieving equity in mathematics education because difficulties with algebra can constrain students' career choices and perspectives on the surrounding world. In response to claims that algebra predicts success in college, there have been recent calls for "algebra for all."

Alternative responses to the algebra problem take into account expanded notions of what school algebra can be, similar to those articulated by Kaput (1995), National Council of Teachers of Mathematics's (NCTM) *Principles and Standards for School Mathematics* (2000), and the National Research Council (NRC) (Kilpatrick, Swafford, & Findell, 2001). For example, the NRC committee characterized algebra as consisting of: *Representational* activities, *transformational* activities (i.e., operating on and transforming algebraic symbols), and *generalizing and justifying* activities. Some versions of an expanded algebra are more explicit about algebraic structure and relations among operations. All versions stress that algebra includes not only operating on and transforming algebraic symbols but also ways of representing and reasoning about problem situations that contain features such as *unknowns, patterns,* and *covariation.*

Expanded notions of algebra have set the stage not only for changing high-school algebra courses but also for instructional approaches intended to make important aspects of algebra accessible to younger students. The NSF has supported the development of middle-school materials and high-school materials that are aligned with the NCTM *Standards*, but the NRC (Kilpatrick et al., 2001, p. 279) has pointed out that more research is needed on the role that expanded activities can play in the development of "symbol sense." The proposed work is an instance of such research in the context of an expanded algebra. In particular, the mathematical content at the center of our proposed research lies at the intersection of algebra, problem solving, and representations.

Algebra. We start with a general characterization of algebra that is consistent with that articulated by the NRC. That is, we take algebra to be a subject concerned not only with operating on and transforming algebraic symbols but also with representing and solving problems, generalizing and justifying arguments, and reasoning about relations among operations. We focus on multiplicative comparisons—including ratio, rate, and slope—in problem situations that contain unknown quantities, patterns, or covariation.

Problem solving. We ground our perspective on problem solving in those perspectives previously developed. Schoenfeld (1992) summarized five aspects of mathematical thinking and problem solving around which there is considerable agreement: (1) The knowledge base, (2) Problem-solving strategies, (3) Monitoring and control, (4) Beliefs and affects, and (5) Practices. A mathematical task is a problem only when it is not routine for the person trying to accomplish that task; that is, when the person cannot quickly generate a path by which to complete the task. In particular, we take algebra problem solving to be neither the routine manipulation of symbols—although such activities can play an important role in algebra problem solving—nor the solution of routine word problems.

Representations. Considerable theoretical and empirical work has been done in mathematics and science education on both internal and external representations. We use the word "representation" to refer to external ones. Empirical work has made clear that students often struggle to understand and use standard mathematical representations found in algebra courses to solve problems (e.g., Kieran, 1992). Some recent research has focused on how students can construct their own representations to successfully solve problems about physical situations with algebra (e.g., Izsák, 2000, in press).

The proposed research will focus on teachers' and students' understandings of shared classroom interactions and on ways that teachers and students work together to shape the teaching and learning of middle-school algebra through representing and solving problems. To gain access to and analyze teachers' and students' understandings of shared classroom interactions, the project will coordinate analyses of taken-as-shared classroom problem-solving practices with individual teachers' and students' understandings of those practices. Thus, the project will examine the sense that students make of their opportunities to learn and teachers' sensitivity to the core learning issues for their students.

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FROM UNKNOWN AMOUNTS TO REPRESENTING VARIABLES

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Our work in Early Algebra has been built on the ideas that (a) generalizing lies at the heart of algebraic reasoning; (b) arithmetical operations can be viewed as functions; and (c) students' spontaneous representations of mathematical relations can be nurtured to evolve into meaningful notation based on mathematical conventions. We have also assumed that mathematical learning is an inherently social activity, closely tied to the discussions and contexts where it takes place. We will illustrate how such cognitive and social ideas underpin the design and implementation of third grade algebraic-arithmetic tasks related to the operations of addition and subtraction.

In planning the lesson we will describe, we had several specific considerations in mind, among which were the following:

1. Additive comparisons tended to be somewhat challenging to third graders;
2. Although the amount in each box was unspecified, the difference was fixed; from earlier work, we knew that a difference quantity is an important component of additive structures;
3. We wondered whether and how children would express the idea that there could be many possible values for two compared amounts;
4. The problem would possibly provide an opportunity to introduce letters to stand for unknown amounts and for variables; and
5. The activity would best be promoted through intellectually playful dialogue in a setting where students were encouraged to express their ideas and show their insightfulness into the problem.

Our example is part of our second to fourth grade longitudinal investigation with the students of four classrooms in a public elementary school from a multi-cultural working-class community in Greater Boston. Descriptions of our class materials are available at www.earlyalgebra.terc.edu. The lesson we focus upon in this paper was the first one we taught to third grade students we had been working with since second grade.

The lesson is structured around a situation in which one child, Mary, is said to have three more candies than another child, John. We actually show two boxes, one Mary's, the other John's, and assert that there are equal amounts of candies in each. We then take an additional three candies and place them on top of Mary's box and state that her total amount includes both the candies inside and on top of the box. We begin by asking the students to express what they know about John and Mary's amounts.

The problem can be viewed both as a particular empirical state of affairs and as a set of logical possibilities. The former viewpoint gains prominence when one wonders how many candies are actually in the box. The latter emerges as one focuses on the verbal description of the problem and attempts to find multiple "solutions". Each viewpoint offers its version of truth or correctness. There can only be one empirical truth regarding the number of candies John and Mary have. By this standard, only students who guess the numbers of candies in the boxes can be right. However, the logical standard accepts all answers consistent with the information given, regardless of whether they correspond to the reality present in the classroom. As one student expressed it, "Everybody [in her class] had the right answer... Because everybody... has three more. Always." This second viewpoint supports the idea of a variable: there are multiple possibilities that would make the basic premise of the problem true.

By asking the students to make predictions about numbers of candies, we may have encouraged some of them to construe their task as having to guess accurately. However, this served as an opportunity to discuss "impossible answers", such as the suggestion that one child had 8 candies and the other 10 candies. These answers are of course "impossible" only by virtue of the verbal information provided. Furthermore, once the prediction table was set up, students could try to describe what features were invariant among the (valid) answers. In a sense, the data table allowed students to make a generalization.

The mere fact that they entertained an expression such as "N+3" as a reasonable way to express Mary's candies as a function of John's was encouraging. We believe students accepted this because the notation was a natural extension to the students' own experience. One student had introduced question marks to represent John and Mary's amounts. This led the teacher-researcher to point out the potential confusion due to using the same symbol, "?", to represent different amounts. This puzzle was advanced when students suggested that Mary's amount should be "any number plus three", which the teacher embodied first as "? +3" and later as "N+3".

Another teacher-researcher outright proposed that students adopt the letter N to specify an unknown amount. Once the students accepted this convention they came to the conclusion, through suggestions by two of the students, that Mary's amount should be called "N plus three." This is admittedly a small step in the direction of using algebraic notation; but we consider it an important one.

The second day of the candies activity took the discussion further into the territory of logical possibilities and constraints. In this class we asked the students to consider the totals of John's and Mary's candies, given the "fact" that Mary had three more candies than John. Could they have 15 candies, together? What about 12 candies? Or 2 candies? Why or why not? We gave them a number of problems to figure out based on varying assumptions about the totals. After working through several of these examples (with odd numbered totals), they began to see that there was a pattern: they could determine the number of candies in each box systematically. However, the fact that students systematically solved the problem (subtracting 3 and dividing the result by 2) did not guarantee that they could state their procedures clearly: at least two pairs of students in one class described their approach as dividing by 2 *and then* subtracting 3. Eventually, we would like to have seen them produce answers such as (N-3)/2, but they would need considerably more experience with using letters to stand for unknown amounts before such an expression would make sense to them.

AN EARLY APPROACH TO FUNCTIONS: THE MEDIATING ROLE OF COMPUTER-BASED ALGEBRA-LIKE SIGN SYSTEM

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Conceiving that the simultaneous variation of two variables in a numeric function table is governed by a general rule is an idea that many secondary pupils find very difficult. Even more difficult they find the idea that such variation can be analyzed and "manipulated" through expressing the corresponding rule in a symbolic (algebraic) way. The latter presupposes an appropriate understanding of numeric substitution in a formula as well as a basic notion of numeric variation domains. This means that from the perspective of the symbolic algebra, the mathematical sign system of algebra plays a crucial role as a mediator of subject's action, to build up the concept of functional variation. In some previous studies, this role of the notation system has been seen more as a constraint for pupils' thinking, rather than as a thought mediator.

However, in other studies, there has been gathered evidence with regards how algebra-like sign systems like spreadsheets can help 10-15 year olds to express and manipulate a functional relationship (Rojano & Sutherland, 1994). Results of this sort suggest that when working with a spreadsheet, children can resort to their numeric knowledge at the same time that they have to deal with a language that captures the generality of variation and functional dependence. This can be interpreted in the sense that the spreadsheet language takes on a mediating role in the pupils' development of an algebraic notion of function.

Cases from the spreadsheets study mentioned above corresponding to the youngest pupils (10 year olds) can be used as examples of how an early introduction to algebraic thinking can take place through a numeric grounded approach to functions. Whereas, issues from the same study with "algebra resistant pupils" (of 15 years of age) suggest that a connection between spreadsheets language and the algebraic code is feasible. In this way, a possible rout to algebra can be conceived, in which algebraic symbolism can be recovered as the language of generality and variation in school mathematics. Issues arising from a couple of cases will be used in this presentation to illustrate how this can be possible. A reference to Balacheff's paper on "symbolic arithmetic" (Balacheff, 2001) and to Filloy's work on transitional processes from arithmetic to algebra (Filloy, Rojano, & Rubio, 2001) will be made, in order to provide a framework for discussion about the nature of the mathematical thinking that is produced by experiencing variation in a spreadsheet environment. In a recent publication, Yerushalmy and Chazan bring in an analysis of the use of multiple representations of functions in technologically supported approaches to introductory algebra (Yerushalmy & Chazan, 2002). These authors emphasize the influence that such a multi-representational treatment has had in the curricular role of functions in school algebra. This perspective could enrich our discussion, in terms of implications for the algebra curriculum.

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RETHINKING TRANSFER AND GENERALIZATION IN THE CONTEXT OF QUANTITATIVE REASONING

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Two issues will be explored in this presentation—one involving generalization activities in algebra and the second involving a reconceived view of transfer that emerged from studies of algebraic thinking. Quantitative reasoning plays a critical role in each of these explorations. According to Thompson (1994), quantitative reasoning involves the construction of new quantities and relationships among quantities in a situation, where a quantity refers to one's conception of measurable attributes of objects, such as height or distance. Quantitative is not a synonym for numeric, as the term is often used in statistics; quantitative reasoning can be nonnumeric in nature. The two themes in this presentation are also informed by empirical studies conducted as part of a continuing 5-year research project, the Generalization of Learning Mathematics in Multimedia Environments. One of the major goals of the project is to examine how instruction based on quantitative reasoning about ratios, rates of change, and linear functions affects the nature of individual students' generalizations when reasoning about novel "real world" situations (such as wheelchair ramps, speed, and heart rates). Although the studies have been conducted with middle and high school students, many of the findings are applicable to discussions of early algebra.

Many researchers have argued that the development of generalization and the systematic expression of generality are key elements in distinguishing algebraic from arithmetic activity (e.g., Bednarz, Kieran, & Lee, 1996). However, the generalization activities in reform mathematics materials typically involve the generalization of number patterns. Lobato and her colleagues demonstrate the limitations of generalizing based on numeric patterns alone and the importance of generalizing based upon quantitative reasoning (Lobato & Ellis, in press; Lobato & Thanheiser, 2002). Specifically, they investigated students' reasoning in a modified *SimCalc Mathworlds* computer environment (1996), in which students entered distance and time values for one animated character so that the character would walk the same speed as a given character. Students generalized numeric patterns, for example noting that multiplying both distance and time values by the same number will produce "same speed" values. However, closer investigation revealed that students could not explain why the patterns worked. Furthermore, they produced visual representations that failed to capture the proportional nature of the situation. In contrast, the classroom discussion encouraged students to develop a justification for why doubling both time and distance did not change the speed, which was based upon reasoning with the quantities in the situation—distance, time, and speed—rather than based solely upon numeric reasoning. As a result, students demonstrated evidence of constructing ratios as a measure of speed, which appeared to lead to more powerful generalizations than did the numeric reasoning. This finding suggests a number of questions to be explored in the session: (a) How might justifications that rely upon quantitative reasoning differ given a variety of phenomenologies, such as a heart rate situation rather than speed situation?; (b) Is there a common characteristic among justifications and generalizations that have a quantitative as opposed to a numeric basis?; and (c) How can teachers and technological environments support a focus on quantitative reasoning?

A second issue related to quantitative reasoning in algebra involves rethinking key aspects of the transfer construct, in particular the surface/structure distinction of traditional models of transfer. In transfer experiments researchers typically provide subjects with paired tasks that share "structural" features, usually the same solution method, but differ in terms of "surface" features, usually the situational aspects of the word problem. For example, the two tasks of calculating the slope of a line and finding the slope of a wheelchair ramp are typically conceived as sharing the same structural or mathematical relationship between slope, "rise," and "run," but differing in terms of a graphical versus a "real world" context. At the heart of the structure/surface distinction is an assumption that the paired tasks share a similar level of complexity, but for whom—the researcher or the subjects? Researchers operating within a traditional model of transfer adopt an observer's perspective since they predetermine what counts as transfer and what constitutes structural similarity between tasks using normative models of performance. In contrast, a reconceived model of transfer, called *actor-oriented transfer* (Lobato, 2002) will be presented. Actor-oriented transfer rests on a definition of transfer as the personal creation of relations of similarity, or how "actors" see situations as similar. One consequence of adapting an actor-oriented view of transfer is demonstrated by Lobato and Siebert's (in press) work, which suggests that what researchers typically consider a surface feature can present conceptual complexities for students that are more structural in nature than previously understood. Specifically, Lobato and Siebert document how an eighth-grader was unable to "transfer" his understanding of the slope formula to a typical transfer task involving a wheelchair ramp, due to the quantitative complexity of the transfer situation. At the end of a teaching experiment, the researchers presented the student with the same wheelchair ramp situation again. Once the student reconstructed his understanding of the relationships among the quantities in the situation, he was able to provide evidence of actor-oriented transfer using previous proportional-reasoning experiences from the teaching experiment. This study provides some understanding of the types of quantitative reasoning that students need to develop in order to succeed in complex transfer situations. The work also raises important questions regarding how to provide instructional experiences that enable students to make sense of quantitatively complex situations without directly providing instruction related to each target situation.

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RETHINKING PROFESSIONAL DEVELOPMENT FOR ELEMENTARY SCHOOL TEACHERS AROUND ALGEBRAIC THINKING

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A growing consensus suggests that it is necessary to reconceptualize the nature of algebra and algebraic reasoning and to provide students opportunity to engage in algebraic reasoning earlier in their education (Kaput, 1998; National Council of Teachers of Mathematics, 1997, 1998). The artificial separation of arithmetic and algebra traditional in school mathematics curricula deprives students of powerful schemes for thinking about mathematics in the early grades and makes it more difficult for them to learn algebra in the later grades (Kieran, 1992; Matz, 1982). For the last five years we have been working with teachers and students to understand how to create classroom environments that provide opportunities to engage in algebraic thinking. We have developed ways of thinking about both the content of algebraic thinking for the elementary school grades and how to provide support for teachers as they consider engaging their students in algebraic thinking.

Drawing on Robert Davis's (1964) seminal work on the Madison Project, we have used the solution and discussion of true, false, and open number as a primary context for helping teachers understand and focus on student thinking. These number sentences provide a context to initiate conversations that can lead to generalization and to introduce discussion of notation that might be used to express those generalizations. This context has proved particularly fruitful in engaging students in algebraic thinking and making that thinking visible to teachers. Specific ideas that we have addressed include: (a) equality as a relation, (b) generalization about number and properties of operations, (c) representation of generalizations, and (d) the progression of forms of argument that students use to justify generalizations. Previous research suggests that students in traditional classrooms are not aware of the underlying structure and properties of arithmetic operations (Chailkin & Lesgold, 1984; Collis, 1975; Kieran, 1989). In our research we have found that young children are capable of making such generalizations, constructing ways of representing them, and justifying them if they are provided appropriate opportunity (Carpenter & Levi, 1999; Carpenter et al., 1999;). These findings are consistent with other recent studies (Bastable & Schifter, 1998; Davis, 1964; Kaput, 1999; Schifter, 1999; Tierney & Monk, 1998).

We have taken what we ourselves and others have learned about students' capabilities in algebraic thinking and developed a set of ideas about engaging teachers. Our work with teachers builds on the conception of teacher learning as a generative process interactively constituted in complex learning environments. We see that we cannot separate learning from the context in which it occurs and that we need to capture the evolutionary character of teacher learning. Our goal then is to provide teachers' opportunities to engage in inquiry about their students' mathematical thinking. What we have learned as we have used these ideas to work with teachers is that content plays a central role in what and how the teachers engage. The mathematical content plays into teachers' notions about their students and what they can accomplish as well as into their notions of mathematics.

We have found that it is not just the structure of the professional work that makes a difference but the interactions around those structures. (We have described the structures and processes elsewhere.) So while we can enumerate some key principals for supporting professional development (a) creating communities where the teachers are engaged in inquiry (b) student thinking provides a tool that engages teachers, makes explicit their and their students participation with the mathematics and provides a trace of the communities ideas, (c) understand the histories of cultures of the communities that we enter. What we did not realize was how these ideas interacted with the mathematical content on which the professional development is based. We have found that we may need to think differently about professional development for elementary school teachers around algebraic thinking.

We piloted our algebraic thinking work this past year in an urban elementary school. We wanted to use the work around algebraic thinking as a lever for changing teachers' mathematics practice and the ways they engaged with each other as a community. What we found was that the teachers at Lincoln could easily co-opt the algebraic thinking work to fit their notions of the teaching and learning of mathematics and the algebraic thinking work on the surface did not explicitly keep at the forefront a different way of thinking about the teaching and learning of mathematics.

We went into Lincoln Elementary anticipating the teachers would find the algebraic thinking "fit" with what they needed to focus on for their students. On the surface, the algebra looks much like what the teachers already do; the algebra work focuses on solving number sentences and understanding the relationships within and across mathematical expressions. It looks symbolic in nature. It involves problems that teachers see on the standardized test. It does not look like "fluffy" problem solving". We hypothesized that it would provide a reasonable entry point. What we did not anticipate was how the content of algebraic thinking we identified and the ways in which that mathematics was made explicit fed into teachers existing notions in ways that did not help them think about how they might engage students in talking about their thinking or representing their ideas within the group. In our previous work the word problems did serve this purpose. We have found a number of ways in which the content plays out in the professional development differently than we would have anticipated. We need to think carefully about the mathematical content and its relation to teacher learning and professional development.

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Discussion Groups

THE MESSY WORK OF STUDYING PROFESSIONAL DEVELOPMENT: THE CONVERSATION CONTINUES

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At the 2001 National Council of Teachers of Mathematics (NCTM) Research Presession, the facilitators of this discussion group led a work session titled "Studying Professional Development is Messy Work: What are the Research Issues?" This PME-NA discussion group is designed to continue the conversation begun in 2001 and will focus on generating group goals for continued work. A second product of this discussion session will be group-generated research questions that will help guide future investigations of professional development for mathematics teachers. Participation in the 2001 Research Presession work session IS NOT a prerequisite of attendance at this discussion group.

As interest in study groups, lesson study, and other alternatives to traditional forms of professional development increases, the mathematics education community needs to have better descriptions of what these alternative forms of professional development entail and also what the effects on teachers are. But investigating these forms of professional development is often more complex than studying traditional forms such as workshops and courses (Zeichner, 1999). Researchers are frequently directly engaged as participants in the professional development activities; researcher-teacher relationships are sometimes intense and long-term; and some of the professional development activities themselves are hard to distinguish from the everyday work of teachers. All of this suggests a need to develop new ways to think about the characteristics of "good research" on professional development. Ultimately, designing "good research" on professional development will help us understand on a deeper level how teachers learn.

A Brief History of Prior Work

At the 2001 NCTM Research Presession, the facilitators of this discussion group led a work session titled "Studying Professional Development is Messy Work. What are the Research Issues?" The description of this session read: "This work session will address research issues related to studying forms of professional development that involve intense, collaborative work by groups of teachers, teacher educators, and others." Approximately 50 people attended the session. After a short presentation of research issues that have arisen in the facilitators' research, participants were actively engaged in small work groups. They were given the following charge:

- Further refine the issues raised
- Raise additional issues
- Discuss ways to address the issues
- Record the 1 or 2 most interesting/pressing issues and proposed ways to address them
- Select a reporter to share these with other participants

After the small groups reported back to the whole group, we asked the question "Where do we go from here?" The following list contains the suggestions made by the group:

- Prepare a monograph in which research issues surrounding professional development are discussed.
- Sponsor a small conference that focuses on research on professional development.
- Begin a working group to set important variables and measurements for studying professional development, including measurements of "where teachers start."
- Take up the issue of how to assess the impact of professional development on student learning.
- Develop ways talk about and link the research literature on teacher education, teaching, and student learning.
- Develop a framework, or frameworks, of the "bigger picture" of professional development, given that teachers "sit" within a culture.
- Design longitudinal framework(s) that allow for publishing/reporting out along the way.
- Think about new data collection sources.
- Think more deeply about the "phase out period," that period of time when the grant money is ending and the researcher/professional developer begins to "phase out" involvement with the teachers.

The participants of the 2001 Research Presession Working Group brainstormed many important issues related to studying professional development, as indicated by the list above. It is now time to begin the work suggested by that group.

Focus Issues for 2002 PME-NA Discussion Group

Participants in this discussion group will focus on two issues that will support continued work of the group: 1) generating goals for the group; and 2) generating research questions regarding studying forms of professional development that involve intense, collaborative work by groups of teachers, teacher educators, and others. Each focus is expanded below.

Group Goals and Future Direction

Given the list of "where do we go from here?" responses generated at the 2001 Research Presession, it is likely that the goals of this discussion group might include some items from that list (see previous section). As organizers of this session, we hope that this discussion group can become a PME-NA working group that collaborates to generate and share research on professional development. It is important that we use the time in this discussion group session to organize and prioritize future work for the group, work that begins by setting goals.

Generating Research Questions

At an AERA session in April 2002, Deborah Ball challenged us to think about questions that should guide our work when studying professional development involving groups of teachers who work collaboratively to improve their practice. As a part of her challenge, she provided a set of questions that she feels are important for the community to consider. Those questions are reproduced here with her permission and include:

- How do teachers learn to collaborate?
- How do teachers learn to study their own and others' teaching?
- What do we do after making teaching public to make it actually VISIBLE and LEARNABLE?
- How do we help teachers ask good questions about their practice?
- What does it mean to have rigor in research on practice?
- What does it mean to scrutinize or analyze practice?
- What is crucial about collaboration?
- What is the key element of collaboration?

Ball's questions provide a "jumping off point" for this discussion group to generate a more comprehensive list regarding studying professional development involving groups of teachers collaborating to improve their practice.

Where Do We Go From Here?

As we did at the end of the 2001 Research Presession Working Session, we will ask the question "Where do we go from here?" Based on the group goals generated earlier in the session, Arbaugh, Brown, and McGraw will take initial responsibility, under advisement of the group, for organizing participants for future work. 2002 PME-NA discussion group session notes and participant list will be compiled and distributed through email. If interest is strong, the facilitators of this discussion group are committed to organizing the group again for the 2003 PME-NA meeting.

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MATHEMATICS TEACHING ASSISTANT PREPARATION AND DEVELOPMENT

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Teaching assistants (TAs) play a vital role in the mathematics education of undergraduates and may go on to become professors of mathematics. From the K-12 literature, we know that patterns of teaching practice, as well as beliefs about teaching and learning, form early in a teacher's careers. Yet, while there is a large body of research about K-12 teachers, researchers are only beginning to consider the development of TAs. To date, the small body of work that exists addresses TAs in two distinct ways: professional development, usually taking the form of week- or semester-long introductory programs to help TAs during their first teaching assignments; and research into the graduate student experience and the challenges TAs face, both in and out of the classroom. This discussion group will bring research- and practice-focused people together and provide a forum to begin identifying researchable questions and establishing agendas for future work. In addition, it will foster collaboration on these issues between K-12 and undergraduate mathematics educators.

Introduction

At present, we know very little about the characteristics (including teaching practices, beliefs, challenges, needs, and understandings of mathematics and teaching) of graduate students who are mathematics teaching assistants (TAs) or about the factors that shape those characteristics. Over the past five years, mathematics educators have begun to address these issues in two ways: (1) by designing preparation and development programs that target mathematics graduate students who are or will be teaching undergraduates and (2) by beginning research programs focusing on the TA experience. As the community becomes more aware of the importance of TA development and of understanding the TA experience, there is a growing need to organize what has been done, to facilitate cooperative efforts at understanding these issues, and to discuss and determine needs for future work. In particular, in order that research and practice not be disjoint, it is critical that participants from multiple perspectives have opportunities to share their ideas, articulate challenges, report on progress, and establish further agendas for moving the field forward. To that end, this paper is meant to fuel a discussion by briefly describing each of the following related areas: the recent interest in undergraduate mathematics education that provides a context for research about TAs; the fundamental need and exciting opportunities for research about TAs; some of the research already in progress; and connections to TA preparation and development programs as well as to the K-12 literature. In the last section of this paper, we present our expectations for the discussion group session at PME-NA XXIV in Athens, GA, and suggest potential research directions that the discussion group might pursue.

Undergraduate Mathematics Education

Fueled by research and national reports about the state of the teaching and learning of college mathematics (National Science Foundation, 1986; Seymour & Hewitt, 1997), over the past two decades the amount of attention paid to undergraduate mathematics education has increased substantially. Efforts have been made to examine and to improve teaching and learning. These efforts have taken the form of instructional design projects (e.g., new textbooks, computer-based curricula), research on student learning, and (to a lesser extent) research on teaching at the undergraduate level. Although the focus of this paper is on research, the relative novelty of innovation in mathematics instruction at the college level is an important part of the context in which the research is conducted. Research in the teaching and learning of mathematics at pre-college levels (as well as some research on college teaching in general) has been ongoing since early in the 20th century, but research specifically in undergraduate mathematics education has only been active since the mid-1980's. These factors combine to make undergraduate mathematics education an arena flush with rich contexts for research.

Undergraduate mathematics education research is intimately related to research into mathematics teaching and learning at the pre-college level, but there are two important distinctions: the focus on questions surrounding undergraduate mathematics content and the individuals involved in conducting research in this area (Schoenfeld, 1994; Selden & Selden, 1993). First, undergraduate mathematics education research assumes students are enrolled in college courses, thus there tends to be a focus on student learning of mathematical topics typically presented in such courses, i.e., calculus, differential equations, linear algebra, and abstract algebra.

The second distinction is the different demographics of the researcher populations in each area. Pre-college mathematics education research is typically carried out by Ph.D. mathematics educators, and often involves some level of collaboration with teachers in schools. In some cases, researchers from the K-12 community have expanded their agendas to include examination of issues at the undergraduate level. Undergraduate mathematics education has also attracted the interest of many Ph.D. mathematicians, who have come to questions about student learning during their own teaching careers. Unlike their colleagues in mathematics education who are professionally interested in pre-college education, these mathematicians are most directly concerned with the success of undergraduate students in mathematics courses. Another way in which K-12 education researchers and mathematicians with an interest in education differ is that the latter group often begins with little formal exposure to the existing body of mathematics education research and learning theory. They succeed through a kind of on-the-job training and, thus, their experiences, competencies, backgrounds, and perspectives are different from those of mathematics educators and are an important factor to consider. It is often these mathematicians who take responsibility for the design and implementation of TA preparation programs.

Need for Research about TAs

At many universities, graduate student teaching assistants play a critical role in the education of undergraduates. TAs are often given significant responsibility for teaching lower-division courses, including courses for mathematics majors, service courses (e.g., engineering calculus), and content courses for prospective K-12 teachers. This responsibility comes in many forms: some TAs have sole responsibility for teaching a course, some teach recitation or discussion sections that accompany large lectures given by faculty, some work as homework graders, and some provide tutoring services to students. Whatever the form, contact with TAs can constitute a significant portion of undergraduates' instructional time in lower division courses. Thus, the potential influence that TAs have on undergraduate students' experiences with mathematics is tremendous.

In addition to their importance as TAs, the current pool of graduate students is also the source of mathematics faculty of the future. Several authors (e.g., Brown, 1985; Eisenhart, 1995; Lacey, 1977; Lortie, 1975; Zeichner & Tabachnick, 1985) have discussed the significance of early experiences in solidifying beliefs, developing practices, and setting patterns of social learning for new teachers. Thus, the time spent as a TA is the time during which young mathematicians will discover teaching in a way they will likely carry with them into their careers as faculty. A TA's first teaching experience provides rich opportunities to support and shape emerging instructional practices. Yet, traditionally, support structures and guided enculturation experiences have not been available, let alone an expected part of a graduate student's professional development. Although over the past two decades more attention has been directed at undergraduate education than was in the past, the focus of this attention has mostly been on areas other than instructional practices (e.g., curriculum development, uses of technology, assessment). These features of the context provide great challenges as well as interesting opportunities for research that has the potential to improve both current and future undergraduate mathematics education.

Connections to TA Preparation and Development Programs

TAs often have a great deal of responsibility for instruction but little opportunity to learn how to teach in ways other than those they have experienced themselves or those modeled by their faculty mentors. Even for college and university faculty, early and ongoing instructional development opportunities tend to be somewhat limited. TAs (future faculty) will need a rich and reflective understanding of their teaching practices if they are to make informed decisions and respond professionally to changes in mathematics education. Yet, the preparation and support TAs typically receive is minimal. Most arrive in graduate school with little or no prior teaching experience. Before they start to teach, they may attend a department- or campus-wide orientation session, lasting a few hours or a few days. The range of topics that are addressed in these sessions is broad. In a typical program, new TAs learn about campus, department and course policies and procedures. They may receive information about the specific course they are teaching and a list of tasks they are expected to perform (such as grading homework, administering quizzes, holding exam review sessions, etc.). They might also receive information about teaching, learning, and ways of interacting with students. In some cases, new TAs have the opportunity to practice teaching (often very briefly) and receive feedback (often very superficial) from their peers and/or people running the orientation sessions.

In efforts to provide ongoing development opportunities, some institutions run semester- or year-long seminars to introduce graduate students to their roles as teachers as they are beginning to function as TAs. In 1999 and 2000 the American Mathematical Society (AMS) - Mathematical Association of America (MAA) Joint Committee on Teaching Assistants and Part-Time Instructors sponsored Joint Meetings sessions about the preparation and development of mathematics graduate teaching assistants. At both of these sessions (designed for an audience of practitioners, not researchers), presenters described programs ranging from quick orientations to semester-long courses on teaching. Activities included reading about teaching and learning, microteaching, keeping journals, observing others teach, and participating in discussions (Murphy, et. al., 2000). Only a few presenters at these sessions referred to the literature on teacher preparation or professional development.

Additional ideas for curricula and reports of efforts were gathered in Case (1994). Rishel (2000) proposes mathematics-specific, basic strategies for new (and experienced) TAs to be successful. To fuel conversations about deeper issues related to undergraduate teaching, Friedberg, et al. (2001) offers crafted vignettes of common situations that challenge instructors' perspectives and behaviors. With support from the Fund for the Improvement of Postsecondary Education, these case studies were pilot-tested with groups of TAs over a period of years at a variety of institutions. Furthermore, recognizing that the mathematicians typically in charge of TA development programs lack experience with case study discussion as an instructional strategy, Friedberg, et al. held training workshops in the Boston area in 2000 and 2001 (see http://www.bc.edu/casestudies for more details). Despite the wealth of valuable ideas collected here, none of these resources was specifically based on research about teaching assistants or related issues.

More recently, Meel (2000) spoke at PME-NA XXII about using journaling and the reading of journals to help new TAs become reflective about their classroom experiences, as the beginnings of a project relating research and practice. The TAs who participated in Speer (2001) were teaching a particular kind of calculus discussion section for the first time and were simultaneously participating in a professional development program for which Speer was partially responsible. These efforts to connect research and practice are impressive because, for the most part, researchers and TA educators have worked separately.

TAs themselves contribute significantly to college instruction and they may become faculty members with lifelong contributions to education, yet the historic lack of communication among players has led to a paucity of program models based on systematic investigations of TAs' contributions and challenges. TA educators have developed courses for beginning TAs to help address their first semester needs; they have published course materials; many have participated in orientation sessions at the start of the semester for incoming TAs. These projects are most frequently limited in scope and in publication to the department where the TA educator works. Rarely do TA development course projects lead to collaboration with members of the larger mathematics and mathematics education communities (Friedberg's (2001) contribution is a notable exception). Similarly, mathematics educators studying the experiences, beliefs, and contributions of TAs are distinctive in the field because of the relative isolation in which they have worked. While many are tangentially aware of the work of others, collaboration in the field has been rare. Communication has so far been limited, and there is a clear need for TA educators and for researchers to communicate, discuss, plan, and organize ideas, both within the groups and (perhaps more importantly) between the groups. The current climate is in part due to the emerging nature of the interest in the field, but its affect has been, thus far, to limit understanding and progress. The PME-NA discussion group was motivated primarily by a need for greater communication and opportunities for collaboration.

Connections to the K-12 Literature

Literature addressing teaching and learning at the K-12 level includes a rich base of information about how preservice teachers think about mathematics and about the teaching and learning of mathematics. This base is extended to document how established teachers practice in and reflect upon their own classrooms. In particular, there is an emerging base of information about how teachers' beliefs and practices shape the implementation of reform. There is no comparable base of information about TAs, neither about their beliefs about mathematics, nor about the teaching or learning of mathematics, nor about how they understand and think about their own classroom practices. Indeed, although the research on general undergraduate mathematics teaching and learning is increasing, the body of literature that exists is still quite small relative to that at K-12 levels. Attention has yet to be focused on TAs specifically. One important gap is the lack of scientific documentation of what TA teaching typically looks like. Informed by efforts at the K-12 level, this basic research is needed to guide efforts to help TAs develop their teaching practices in productive ways.

K-12 teachers and TAs are different from and similar to one another in some interesting ways. Unlike most elementary and secondary schoolteachers, college instructors do not typically participate in compulsory, extensive teacher preparation programs. Like elementary and secondary schoolteachers, TAs (and other college instructors) can become isolated from colleagues, with little or no opportunity to grow from interactions with other instructors (Lortie, 1975; Murphy, in press). For many, their first jobs as TAs will be the only time in their careers when they (may) participate in (essentially minimal) professional development about teaching. K-12 teachers, on the other hand, may encounter a variety of opportunities for professional development throughout their careers. Since faculty are unlikely to receive any further guidance regarding their teaching, the practices they develop as TAs may well shape their teaching for the rest of their careers. Support and professional development associated with their initial teaching may be the only formal opportunity to help college instructors develop their teaching practices in effective ways.

The similarities may point to ways in which the existing research base on K-12 teacher development can be applied to TAs. Differences may help identify areas where additional research is especially needed. In both situations, making use of and building on what is known from research in K-12 teacher development could be an important component of the design and implementation of professional development for TAs.

Research in Progress

Thus far, most of the attention to TAs has been on professional development activities and programs. Now, however, the community is poised to engage in research in substantial ways and will be able to build on the momentum gained by research in undergraduate mathematics education and by the increasing investment in TAs as current and future key players in undergraduate education. Ideally, research directions will be based on the needs of TAs and TA educators and findings will inform the design of increasingly effective preparation and development programs.

Some groundwork research is already in progress. For example, DeFranco and McGivney-Burelle (2001) reported on TAs' beliefs about the nature of teaching and learning mathematics. Speer (2001) also conducted a study about TAs' perspectives, focusing on the instructional practices of TAs teaching calculus and the relationship between collections of beliefs and moment-to-moment teaching decisions in class. A broader approach is to consider the entire graduate experience. For example, Herzig (2001) investigated reasons for attrition among mathematics graduate students, finding a mismatch between the preferred work styles of students who left (e.g., collaborative) and the perceived culture of the discipline. Related work includes research that addresses the cultural climate in which mathematicians work. Examples of this approach are Gutmann's (2000) ethnographic case study of a small, teaching-focused mathematics department; Enzensberger's (1999) essay, which addressed the attitude toward mathematics common in the general population; and Damarin's (2000) sociological observation that mathematicians represent a marked category and, like other minority groups, operate in a world partially defined by their otherness.

The issue of enculturation is especially important. Graduate students somehow learn to function in their departments. To succeed, they need, or at least believe they need (Lacey, 1977), to adopt the habits and attitudes of their faculty mentors. Pressures to become part of the existing culture are strong. Even TAs who arrive in graduate school with substantial concern for undergraduate education and strong motivation to teach may find that holding on to those ideals is incompatible with success as defined by their department, their faculty mentors, and the discipline as a whole. With proper support and enculturation, it may be possible to build on and nurture good intentions and practices.

The Discussion Group at PME-NA XXIV and Potential Research Questions

Inaugurated at PME-NA XXIV in Athens, Georgia, this discussion group is intended to generate ongoing conversation about potential research paths. The discussion group will facilitate dialogue among and between researchers and practitioners, at all levels K-16. Participants will be asked to speak for a few minutes to the group about their own interests and to summarize their connections to work with or about TAs. These introductions should assist group members in meeting potential colleagues and in beginning collaborative projects of mutual interest. Further time will be spent generating researchable questions and considering how best to organize the diverse contributions in the field for future productivity. As part of this process, we offered above an overview of some research-in-progress. We offer also the following sample of potential research questions that the discussion group might consider pursuing:

(1) In what ways and to what extent does research about pre- and in-service teachers apply to TAs? In particular, what are the similarities and differences in knowledge and beliefs about teaching and about mathematics?

(2) What are TAs' expectations about teaching? What are their self images in terms of thinking of themselves as teachers? How do these expectations/images evolve as TAs gain experience?

(3) What are TAs' conceptions of mathematics? conceptions of teaching? conceptions of how students learn mathematics?

(4) In terms of professional preparation and development, what can be transferred from the K-12 research? What adaptations need to be made for K-12 models to work in the TA setting?

(5) What effects do students' perceptions of TAs have on them? What effects do faculty members' perceptions have on them?

(6) What challenges/opportunities does cultural context present? How do TAs learn how to value teaching?

(7) How are TAs similar to/different than K-12 pre-service/in-service teachers? As a result, how should the preparation and development of TAs be similar to/different than K-12 programs?

The community needs large-scale investigations to identify critical, widespread issues. Small case studies will provide in-depth understanding of TAs' perspectives and of the challenges they face. Research with longitudinal designs will inform the design of exemplary programs that have lasting impact on instructional practices. Given that TAs are critical agents in college instruction, such research is especially vital at this time of increased attention to the quality of education at the undergraduate level.

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ACHIEVING EQUITY AND IMPROVING TEACHING IN MATHEMATICS EDUCATION THROUGH TEACHER EDUCATION AND PROFESSIONAL DEVELOPMENT

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This discussion group explores equity in mathematics education with an initial focus on integrating equity in mathematics teacher education and professional development. The group will analyze existing research regarding equity in mathematics education with the goal of identifying research findings important for P-12 teacher education and professional development. In addition, the group will make plans for further discussions at future PME-NA meetings. Four papers with a variety of perspectives on equity will serve as the starting point for the group's discussions. Laurie Hart and Martha Allexsaht-Snider will set the stage with a summary of recent research. Brian Lawler and Amy Hackenberg will explore the nature of equity and what it means to work for equity. Jae Hoon Lim will discuss implications for teacher education from a recent study of sixth grade girls' motivation for learning school mathematics. Vilma Mesa and Shari Saunders will describe a new secondary teacher preparation program with a social justice perspective for teachers in urban settings.

This discussion group explored equity in mathematics education with an initial focus on integrating equity in mathematics teacher education and professional development. This was the first meeting of the group. The discussion group analyzed existing research regarding equity in mathematics education with the goal of identifying research findings important for P-12 teacher education and professional development.

In the discussion group, four papers were presented with each discussing findings related to equity in mathematics education that are important for teacher education and professional development. Following these papers, the participants considered the following concerns: (a) how the research findings may be used in teacher education and professional development and (b) additional research findings about equity that extend the findings presented. Finally, participants generated ideas for future research that could inform teacher education and professional development. Organizers recorded the ideas generated in the discussion group and produced a summary to be distributed to the participants. Participants were invited to make plans for a follow-up discussion group at PME-NA 2003.

Overview

Disparities in mathematics achievement and enrollment in higher level mathematics between White middle class males and other groups of students such as females, African Americans, Latinos, Native Americans, and poor students have concerned educators at all levels for many years. Policy makers and researchers have acknowledged these differences in their calls for improving mathematics learning for all students. What has not been clear is how teachers, administrators, and counselors in individual schools and districts throughout the country might accomplish equity in mathematics with their students. Equally unclear has been how teacher educators and professional developers might support teachers, administrators, and counselors in understanding the barriers and supports to accomplishing high levels of mathematics learning for all students.

Our review of research conducted during the past decade suggests that important findings and perspectives related to achieving equity in mathematics education are available in the literature and need to be more accessible to teacher educators and professional developers (Allexsaht-Snider & Hart, 2001). Knowledgeable and committed teachers who have developed their own understanding of the ways in which racism, classism, and sexism affect mathematics teaching and learning are critical to equitable outcomes for students in mathematics (Allexsaht-Snider & Hart, 2001; Apple, 1992; Campbell & Silver, 1999; Love, 2001; Weissglass, 2000). Both beginning and experienced teachers need to develop capacity in mathematics education reform and skills for providing equitable instruction in order to support diverse students and their families in mathematics learning. Teachers require preservice preparation and professional development support in the areas of:

- mathematics education reform
- mathematics content for equity
- mathematics pedagogy
- identifying inequities and strategies to achieve equity in mathematics education
- building partnerships with families and communities.

Research discussed by Love (2001), Oakes and Franke (1999), and Weissglass (2000) has shown that beliefs that are pervasive in society can interfere with the attainment of equity in mathematics education. All those involved with and interested in the education of children may hold and convey many of these beliefs that can inhibit students' access to and attainment of high levels of learning in mathematics. In addition to beliefs, classroom processes and teaching practices are essential for understanding how schools and teachers contribute to differences in mathematics achievement by race, gender, and social class. Hart and Allexsaht-Snider (1996) have suggested that the concepts of resistance, engagement, and a sense of belongingness for underrepresented students in the mathematics classroom are important new ideas for mathematics educators to consider. Culturally relevant pedagogical strategies described by Gutstein, Lipman, Hernandez, and de los Reyes (1997), Ladson-Billings (1997), and Tate (1995) offer promise as a means for promoting engagement and a sense of belongingness for poor students, students of color, and female students who are too often disengaged from mathematics.

The above provides a brief summary of some of the research that Hart and Allexsaht-Snider have identified as important for teacher educators and professional developers in mathematics education. The following sections provide summaries of three papers by mathematics education researchers from the United States, Asia, and Latin America.

EXPLORING THE ROOTS OF EQUITY IN TEACHER EDUCATION

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Characterizing equity as equal access to learning opportunities and resources (NCTM, 2000) is necessary but not sufficient when working towards an equitable mathematics education. We believe this characterization is insufficient because it may not adequately problematize how grouping happens. That is, it may take for granted common ways people identify other people (or themselves) and generalize them into groups, namely by race, class, and gender. We do not dispute that the phenomenon of grouping is necessary in order to operate in the world, or that race, class, and gender are powerful and pervasive ways that people group themselves or are grouped. What concerns us is that the phenomenon of grouping itself may often be ignored and that how we group requires deep examination (Secada, 1992) when considering the nature of equity and what it means to work for equity.

In order to reconstitute a definition of equity that takes into account rethinking how grouping happens, we believe we must begin with each person's responsibility to herself and others, what we will call "roots" of equity. While we cannot say categorically what is and isn't equitable in ways that transcend humans in particular contexts, we believe that the potential for realizing equity is heightened when the focus begins with these two roots. First, confidence and competence in one's own ideas and thinking are essential, coupled with an awareness of the fallibility and the perpetual incompleteness of these ways of knowing. This aspect of equity is related to Allexsaht-Snider and Hart's (2000) emphasis on belonging and engagement. Second, valuing others' confidence and competence in their knowing and regarding their knowing as not identical to one's own is necessary. We refer to this aspect as conferring an independent existence on others (Hackenberg & Lawler, 2002; Kamii & Housman, 2000). These roots of equity allow people to problematize how grouping happens because they open the way to forming judgments and generalizations about people founded on qualities other than how one looks or where one lives (White, 2002). Furthermore, they acknowledge the tentativeness of all judgments and generalizations, thus contributing to efforts to rethink how we group.

This vision of equity for mathematics classrooms informs teacher education. Teachers need experiences that prepare them to examine and act upon these roots of equity. To accomplish this goal, the emphasis of teacher education should be to develop the ability to know learners. First, the teacher should develop her ability to listen hermeneutically (Davis, 1997) and construct models of students' ways and means of operating (Steffe & Tzur, 1994; Steffe & Wiegel, 1992). Eleanor Duckworth (1996) develops listening by instructing teachers to learn the thinking and reasoning of small groups of children. However, learning to listen is insufficient without developing an awareness of the nature of the constraints of one's socio-historical experiential reality. Thus, second, we believe mathematics teachers need to have experiences that allow them to know themselves, which involves having and investigating ideas about their environment, mathematics, and others. Julian Weissglass (1996) suggests techniques to know oneself that require intense, trusting relationships with colleagues. Learning to listen and to know oneself can then be practiced by the teacher and moreover, brought forth in learners.

We believe that when the teacher begins with learning the thinking of the student and growing aware of her own constraints, her classroom actions can become more equitable. By listening to students and constructing models of their thinking, the teacher develops cognitively informed and rich categories of students from which to operate. She values student's thinking and also enhances her competence and confidence in her own knowledge of students. This kind of teacher learning opens the way for making judgments and generalizations about students in ways that do not involve superficial or stereotypical aspects. Working toward awareness of the constraints of her own experiential reality opens the way for a teacher to continually rethink her judgments and generalizations. She develops her knowledge about the fallibility and incompleteness of all knowing, and at the same time can value students because her knowing about them is always subject to rethinking. We believe development of the roots of equity fosters a critical perspective at a personal level that has significant implications for a critical perspective on society and individuals' actions in society.

IMPACT OF ETHNICITY AND CLASS ON YOUNG ADOLESCENT GIRLS' MOTIVATION FOR LEARNING SCHOOL MATHEMATICS: IMPLICATIONS FOR MATHEMATICS TEACHER EDUCATION

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The relatively low achievement of female students in school mathematics has been of concern to many educators for the last few decades. Compared to male students, female students report lower self-confidence in their abilities and performance in mathematics and, as a result, are more likely to avoid taking advanced math courses in high school (Eccles et al., 1983; Fennema & Sherman, 1978; Leder, 1992; Meece, Wigfield, & Eccles, 1990). Researchers suggest that various social and cultural factors may influence female students' experiences in learning mathematics and affect their motivation to acquire advanced mathematical knowledge (Reyes & Stanic, 1988).

However, a very few educational researchers have investigated the impact of various socio-cultural factors, particularly the combined impact of ethnicity and class, upon girls' mathematical learning experiences (Campbell, 1989). Few studies have explored how ethnicity and class influence girls' experiences with school mathematics and how girls understand and respond to the direct or indirect socio-cultural influences developing their motivation and academic identities in the mathematics domain.

This project was an ethnographic case study seeking to illuminate adolescent girls' experiences in learning school mathematics and to define the socio-cultural context of their motivation. In particular, this study examined the mixed impact of ethnicity and class upon their learning experiences and motivation. Case studies of five sixth-grade girls were conducted in a rural middle school in the Southeastern United States. Data analysis revealed a striking contrast between the perceptions of high-achieving girls and those of less well achieving girls. For the high achieving girls, the instrumental value of mathematics within the current educational system appeared to be the most profound reason for learning, as well as enjoying, school mathematics. These high-achieving girls successfully internalized school ideology while developing a positive academic identity, in contrast to others who were less well achieving.

In contrast, the majority of participating girls who were not fully persuaded by school ideology, including the instrumental value of mathematics learning, struggled more with various practices in their mathematics classrooms even though they continuously searched for the meaning of their learning and the feeling of "belongingness" in their mathematics classroom (Ames, 1992). In particular, minority students and those from lower socioeconomic backgrounds strove for meaningful mathematics learning in their classroom. They often posed more profound and critical questions about the meanings and values of school mathematics, expressing their desire for a deeper understanding of mathematical knowledge in the world. Yet, their questions were rarely answered or addressed in their mathematics classes, resulting in decreased motivation to learn school mathematics.

Various socio-cultural factors were found to influence each individual's experience with school mathematics. Among these factors are the mathematics streaming practices in their school and their socio-economic backgrounds, including the amount of parental support they receive for mathematics learning. Each participant also showed a variation in her ways of understanding and reacting to the inhospitable culture of mathematics classrooms, especially to female students.

Even though the above findings are based on a limited number of young adolescent girls and their teacher within a particular school setting, they provide teacher educators with insights into the complexity of issues of ethnicity and class, which have been minimally addressed in mathematics teacher education programs. First, the findings of this study pose a need for a critical discussion of various types of school-related ideologies, such as the instrumental value of learning school mathematics. While white middle-class society tends to accept these ideologies as given, ethnic minorities and working-class people are not fully persuaded by those concepts. The educational system in the United States has functioned against the hopes and interests of minority and working- class people, and this constitutes their experiences with and knowledge about schools (Spring, 1997). At the same time, it is important to understand that inculcating those school-related ideologies, such as the instrumental value of learning mathematics, in the hearts of minority girls and girls from working-class backgrounds is not the answer. Rather, educators have to accept their critical point of view as a stepping stone to developing a more meaningful way of learning school mathematics, one which is connected to students' everyday lives and ultimately leads them to become lifelong learners with the "continuing impulse to learn" (Oldfather & Dahl, 1994).

Second, it is important to note that some minority students, particularly high-achievers, experience a high level of anxiety and social pressure because they feel that their desires for school success and peer acceptance are contradictory to one another. This study illuminates the absurdity or ineffectiveness of fostering only "a few good students" out of many underachieving students with whom the few good students share their ethnic and cultural identities. What is most needed is not an individual teacher's isolated effort to "save" the few good students, but rather a collective, consistent, and self-reflective effort of the community of educators directed toward the entire group of minority and working-class students. Without such realization and commitment, it would be hard to make a significant difference, a positive long-term lasting effect, in the students' motivation and academic achievement in mathematics.

The third implication of this study is the need to help teachers recognize and counteract the classism historically plaguing our society and our schools. Because of the current economic disparity between whites and other ethic minorities, classism is intertwined with racism itself. Some school practices and decision-making processes, such as mathematics streaming, often exacerbate the inequality in education by tending to prevent minority and working-class children from achieving a higher level of mathematical knowledge (Oakes, Gamoran, & Page, 1991). We need educators who eagerly and consciously monitor their school practices and raise their voices in resistance when their schools make a decision that may disadvantage girls, minority students, or students from working-class families. Ideally, a critical task of those who educate our educators would be to help pre-service and in-service teachers see themselves as advocates for girls, minority students, and students from working-class backgrounds, and practice their every day "teaching as social activism" (Gutstein, Lipman, Hernandez, & de los Reyes, 1997, p. 732).

CRAFTING EXPERIENCES WITHIN A SOCIAL JUSTICE PERSPECTIVE FOR TEACHERS IN URBAN SETTINGS

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Urban public schools in the United States today have special characteristics that make them unique. Diversity is one of the dimensions in which schools in urban areas differ from rural or sub-urban schools and, in consequence, the dilemmas that "urban" teachers face are quite particular and distinct from the dilemmas that other teachers face. For people involved in teacher preparation and development the most pressing question is what should be the nature of the experience that teachers interested in teaching in urban settings need to have in order to be able to (1) deal with diversity—especially given that the overwhelming majority of people entering the teaching force are still white, female, and not likely to have grown up very poor—and (2) persuade them that they can (i.e., give them legitimate, earned, grounded reasons to believe that they can) teach successfully in "urban" settings?

The Master of Arts with Secondary Certification program (MAC) at the University of Michigan is a very intensive field-based program for students seeking certification in secondary school teaching. The program seeks to integrate theory and practice and gives the students the opportunity to work with professionals on extensive field-based projects; the students are placed in urban public schools for the public school academic year. Shari Saunders has just begun coordinating one of the cohorts, whose students are placed in Detroit and surrounding areas. She is in charge of developing an urban education with a social justice orientation strand in the program. Together, we are developing a database of resources and activities that have been used in different subject matters (e.g., science, reading and language arts, mathematics) and that have been documented in the literature (Ayers, Hunt, & Quinn, 1998; Ayers, Klonsky, & Lyon, 2000; Calabrese-Barton, 1998; Calabrese-Barton & Osborne, 2001; Frankenstein, 1983, 1990; Gutstein, 2002; Howes, 1998) with the purpose of selecting those that would fit better in the program. Whereas we are interested in activities that deal with social justice issues, we are also interested in highlighting the connection with the content areas and in making sure that the activities help teachers realize that those activities can have an impact on students' lives—that students can "write the world" (Freire & Macedo, 1987). Emerging from this work is a framework for selecting activities we think might be helpful in assisting other teacher educators in the process of designing an experience that will deal with other dimensions of diversity in urban school settings.

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FOSTERING THE MATHEMATICAL THINKING OF YOUNG CHILDREN, PRE-K-2

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At root, mathematics is the science of relationships. The young child's world, like the teacher's/caregiver's, is a world of relationships. So mathematics is pregnant in every situation. It could be argued that teachers/caregivers are "teaching" mathematics, either consciously or unconsciously, in any social interaction with a child, whether they are aware of it or not. Experiences and vocabulary laid down in the early years will consolidate a huge fund of experience already begun before birth, and paves the way for fluent and accurate ways to communicate about relationships, as language arts, mathematics, science, and so on, are taught in K-12 programs.

There has been a renewal of interest in the mathematical capacities of young children following recent advances in cognitive science, convincing evidence that young children are more capable learners than current practices reflect, and evidence that good educational experiences in the preschool years can have a positive impact on school learning.

Evidence is also mounting that prior to formal schooling, children's potential for engaging in powerful and diverse mathematical behavior is not being recognized or realized by their teachers and caregivers (Ginsburg, Inoue, & Seo, 1999; Greenes, 1999). Balfanz (1999) noted that linkage of early childhood education—championed in the 19th century by Froebel and Montessori—with the public schools had the effect of suppressing child-centered curriculum. Since a kindergarten experience was not universal for all children, textbook authors assumed that young children started school with no prior mathematical knowledge or experience. Also public education shifted from a primarily rural system with flexible age-groupings to become dominated by the age-graded classroom of urban settings. Serious and sustained work in mathematics was delayed until the second or third grade. Thus an examination of representative mathematics material for preschoolers provides evidence for a minimalist curriculum—activities for teaching numbers to 10 and some basic geometric shapes.

It is important to differentiate the learning of connections between representations of mathematical symbols and representations of perceptual material such as patterns of items, sequences, and spatial phenomena, on one hand, and development of mental schema for dealing with mathematical relationships and problem solving, on the other. Both are important, so to emphasize one and neglect the other is wrong. One reason for the imbalance is that one is better understood than the other. Another reason is a commonly held view that basic skills should be taught before higher order thinking.

An example of the first might be learning to associate the symbol 5 with a collection of five items, or labeling with a symbol 5 the fifth item in a sequence. An example of the second might be mental coordinations allowing an understanding that a set of 5 items can be partitioned in various ways—one and four, two and three, three and two, four and one, two and two and one, and so on. It is this kind of knowledge which is the fuel for powerful problem solving. Piaget (1977) called it logico-mathematical knowledge; Skemp (1971) called it schematic learning.

Participants in this discussion group will be invited to contribute informal reports of recently completed research, research in progress, and/or assist in identifying problems and questions worthy of future investigation. Current and future collaborations between participants interested in common problems will be encouraged. Some areas that seem to offer fruitful avenues for investigation, together with some possible questions, include:

I. Investigations of the nature of young children's mathematical thought and capabilities.

   a. What emotional and affective factors influence young children's engagements in mathematical problem solving?

   b. How powerful do affective factors appear to be in the early years?

   c. How do verbal interactions between child and teacher/caregiver facilitate mathematics learning?

   d. What strategies do young children use to solve particular mathematical problems?

   e. What are the foundations of young children's reasoning in core areas such as probability, combinatorics, multiplicative thinking, part-whole relations, spatial relations, and so on

   f. What is the relation between logico-mathematical development and "school" mathematics? How can one serve the other?

   g. What are the mathematical features of children's play? Are some play activities more likely to provoke mathematical reasoning than others?

II. Investigations of the role of teachers/caregivers in fostering mathematical thought.

   a. What do teachers/caregivers of young children know about mathematics? What is their background, particularly with respect to providing mathematical experiences for young children?

   b. What do teachers/caregivers of young children believe about mathematics, particularly the mathematics that is appropriate for young children?

   c. How do teachers/caregivers interact with young children during a mathematical learning activity? What interactions are conducive to learning?

d. What kinds of support materials and other assistance do teachers/caregivers need to bring forth mathematical growth in young children?

e. What sorts of interventions (e.g., workshops, courses) are needed to assist teachers/caregivers provide meaningful mathematical experiences for young children?

III. Investigations of what mathematics young children can learn using computer-accessible materials.

a. Can young children learn mathematics using computer-accessible materials? What is the role of the adult in computer-situated mathematical problem situations?

b. What role can computer technology play in the mathematical development of young children?

c. Can young children's ability to learn mathematics using computer-accessible materials be enhanced by exposure to conventional materials first?

d. Is there evidence of transfer from knowledge gained from computer-accessible materials to problems set in non-computer-based environments?

e. Are some kinds of games more efficacious for enabling transfer to other problem contexts? If so, what are their features?

IV. Investigations into the nature and role of mathematics curriculum and professional development.

a. What early childhood mathematics programs work, and what are their characteristics?

b. What insights, theories, and practices from primary mathematics education are transferable to the preschool situation?

c. How can mathematical thinking be fostered through music, story books, outside activities, movement, etc?

d. Are there cultural or social class differences in the ways young children engage in mathematical experiences? If so, how can early childhood teachers and authors of mathematics curricula take account of such differences?

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AN INTEGRATED APPROACH TO THE PROCEDURAL/ CONCEPTUAL DEBATE

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This discussion group will be pursuing the clarification of the relationship between the conceptual and procedural aspects of mathematical knowledge. The organizers propose an important shift of focus in the debate, from the question of developmental primacy of one aspect over the other to the question of an integrated development of both.

The organizers of the discussion group represent two distinct approaches to mathematics education, that of "pure" research and that of teaching-research. Dr. Mindy Kalchman and Dr. Bethany Rittle-Johnson are new young researchers in Mathematics Education and Educational Developmental Cognitive Psychology. Kalchman's interest in procedural/conceptual knowledge developed out of work on micro-genetically analyzing sixth grade students' progression from a procedural to an object (or reified) understanding of basic functions. The procedure and concept involved in this process seemed inextricably linked in ways worth investigating. Rittle-Johnson's interests in procedural/conceptual knowledge developed as a result of theoretical efforts to understand mechanisms of cognitive development..

On the other hand, Dr. Baker and Dr. Czarnocha are teacher-researchers from a community college in New York City, whose interest in the subject is motivated by their students' problematic performance in Remedial and Developmental Mathematics courses in the bilingual, Spanish/English context. Equally important in this motivation is the public evidence of serious classroom misunderstandings concerning the instruction along the procedural/conceptual divide in public schools. [See appendix.]

The majority of past research and theory on these relations has focused on whether conceptual or procedural knowledge emerges first (e.g., Rittle-Johnson & Siegler, 1998; Hapaasalo & Kadijevich 2000). The developmental precedence of one type of knowledge over another has been hotly debated and fueled by contradictory results (e.g., Gelman & Williams, 1998; Siegler, 1991; Siegler & Crowley, 1994; Sophian, 1997). This debate over which type of knowledge develops first tends to obscure the gradual development of each type of knowledge and the interactions between them during development. Rather, we propose an alternative view in which we posit that throughout development, conceptual and procedural knowledge influence one another in mutually supportive and integrated ways. This view allows for the simultaneous and intertwined instruction of both conceptual understanding and computational fluency called for recently by (Kilpatric, Swafford & Findell, 2001).

The purpose of this discussion group is to bring together researchers, teacher-researchers and mathematicians to move beyond the debates on "which come first" and "which is more important" to focus on the bi-directional relations and inner structure within the integrated conceptual/procedural mode of knowledge. We see the discussion as the beginning of a longer interdisciplinary process involving cognitive and developmental psychologists, education professionals, mathematicians and teacher-researchers to formulate a comprehensive research/instruction program to deal effectively with the procedural/conceptual divide across the full spectrum of K-16 curriculum. In terms of a longer terms strategy, we are aiming for the publication of a volume articles which will expand and develop issues signaled in the important Hiebert book (Hiebert, 1986) while focusing on the integrated approach to p/c issues. The development of themes and research questions to be discussed in the volume will be the primary way in which the attendees of the discussion will be pulled into active participation. The creation of the collaborative research teams addressing the research issues formulated during the discussion, is one of the ways through which the discussion group intends to transform itself into a Working Group or Research Forum at the next joint PME/PME-NA meeting in Hawaii. If this dream actualizes, we envision the collaborative teams working together throughout the intervening year, possibly aided by a small grant.

The Program of Work at the Meeting

The sessions will have four parts. First, we will provide a sketch of the historical development of the procedural/conceptual issue drawn from the integrated point of view. Second, we will discuss the language of elementary algebra where procedural and conceptual aspects are correlated yet independent variables in promoting academic growth. Here, "independent" signifies that neither can be derived from the other one; they are irreducible (Baker & Czarnocha, 2002). Then, we will present two models of procedural/conceptual integration. In the iterative model, Rittle-Johnson proposes that conceptual and procedural knowledge develop iteratively, with increases in one type of knowledge leading to increases in the other type of knowledge, which trigger new increases in the first (Rittle-Johnson, Siegler & Alibali, 2001). In the coordinated model, Kalchman suggests that conceptual and procedural knowledge are co-active elements in the development of students' more general numeric and spatial understandings of any domain in mathematics.

Summaries of Presentations

Degrees of Independent Activation of Procedural and Conceptual Modes of Acquiring Knowledge- The Case of Algebra

This study of the relationship between written mathematics and procedural problem solving knowledge grew out of the need to address pedagogical needs of bilingual, Spanish/English students of an urban community college. The focus on the mutual positive influence of the conceptual and procedural modes of thinking, translated in this context into two research questions: (1) to what degree one can positively influence learning of language through the learning of mathematics (Czarnocha & Prabhu, 2000), and connected one, (2) to what degree one can positively influence the mastery of mathematical procedures with the proper use of written mathematical language. The answer to these two questions depends, naturally, on the degree to which these procedural and conceptual modes of mathematical thinking can be independently activated during an instruction cycle. This is the main focus of this study. Of separate interest is to observe that the answer to this basic question is also a definite statement in the context of Piaget/Vygotsky debate (Baker & Czarnocha, 2002).

In order to employ writing as a tool to both measure and promote conceptual development we coordinate the three-step model due to Sfard (1991, 1992, & 1994), which is based upon the work of Piaget with the theoretical framework of (Shepard, 1993) who matches levels of conceptual development with the appropriate writing categories due to Britton (Britton et al., 1975). In the model of Sfard, concepts are assimilated into the schema in the last stage of a three-step abstraction process.

"A constant three step pattern can be identified in the successive transitions from operational to structural conceptions: first there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired. These three steps will be called interiorization, condensation and reification." (Sfard, 1992, pp. 64-65)

Interiorization. According to Sfard, a procedure is interiorized when it, "can be carried out through mental representations, and in order to be considered, analyzed and compared it needs no longer to be actually performed." We match the interiorization step, with the late initial learning phase and generalized narrative writing category in Shepard's work. In this phase Shepard recommends writings that produce, "personal examples of concepts" or that explain, "definitions of procedures in one's own words." In this phase we continually asked our students to translate algebraic expressions and expressions back and forth between language and symbolic language.

Condensation. According to Sfard, "at this stage a person becomes more and more capable of thinking about a given process as a whole without feeling an urge to go into details." In describing condensation, Sfard makes an analogy to computer algorithms when she writes that condensation allows the individual to look at a procedure as autonomous, "from now on the learner would refer to the process in terms of input-output relationships rather than by indicating any operations." On the effect that condensation has on an individual's ability for abstraction she writes, "Thanks to condensation, combining the process with other processes, making comparisons, and generalizations become much easier."

We match the condensation step of Sfard with the intermediate learning phase and the low-level-analogic and analogic writing categories in Shepard. In this phase Shepard suggests, "explaining how to solve a problem" and further, "explaining how concepts are related," or explaining why, "concepts and procedures either do or do not apply." Thus, at this phase students' were required to turn their meta-cognitive reflection away from the definitions or rules of the procedures and towards the conditions that govern their use, as well as the difference and similarities between procedures or conceptual objects.

Reification. In the words of Sfard (Sfard, 1992), "the condensation phase lasts as long as a new entity is tightly connected to a certain process." Of reification she notes, "the new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category." In Sfard's model of conceptual development, like all models based upon Piaget's work conceptual development takes place in the framework of a cognitive schema. Thus, the last step of reification is identified with structuring and organization of one's cognitive schema, a step necessary for conceptual development. As explained by Sfard, for an individual who has not organized their schema, "information can only be stored in an unstructured sequential cognitive schemata." In contrast for an individual with a structural understanding, their cognitive schemata has a "compact whole" thus through a process of ordering or restructuring, it becomes a "hierarchical schema." Furthermore without such an ordering, "there is hardly the place for what is usually called meaningful" (Sfard, 1992).

We match the reification step of Sfard with the early terminal phase and the analogic-tautologic writing category used in Shepard, who recommends writing categories that involve, "speculating about several different ways to solve a novel problem." More specifically in our work we required students to focus their meta-cognitive reflection not on the procedures and not on the rules that govern their use but instead on the strategies involved when applying procedural knowledge in problem solving. Our objective was to encourage students' organization and structuring of their cognitive schema.

Results

For slightly over 180 students (n = 183) the correlation between course average and GPA was $R = 0.398$, which was significant at the 0.01 level (high degree of significance). The corresponding R-square value was 0.158 and thus approximately 15.8% of the GPA was determined by course average. The correlation between writing scores and GPA was 0.402, which was also significant at the 0.01 level. When we used both course average and writing scores as independent variables together to explain GPA, the R-value was 0.455 and the corresponding R-square value was 0.207. Thus, approximately 20.7% of the GPA was explained using course average and writing scores. This represents an increase of 37% over the 15.8% explained using only course average. It is not to be expected that course average and writing scores in one class would explain most of a student's GPA throughout their college career. However, the 37% increase explained GPA when writing scores were added to course average is an indication of the important role written mathematical thought has in learning and cognitive development. The F-value of this multivariate model was 23.952, which had a 0.000 significance rating, thus the use of writing scores and course average resulted in a very significant model in which neither course average nor writing scores dominated the other. In particular, writing scores were not dependent upon the ability to apply one's procedural knowledge.

Analysis

We have argued that written mathematical thought by its reflective nature is predominately composed of conceptual thought and meta-cognitive reflection upon procedural knowledge, both of which characterize the more advanced stages of development in Sfard's model. In contrast, we have argued that computational proficiency is predominately composed of the ability to apply procedural knowledge, which epitomizes the initial stage of development. Our result that, written meta-cognitive reflection and conceptual thought are independent of an individual's ability to apply his or her procedural knowledge provides evidence against Piaget's position interpreted as the "genetic view" (i.e., the more advanced stages of cognitive development are dependent upon completion of the first "interiorization" stage).

This result indicates that reflection upon procedural knowledge is not always a by-product of the repeated actions that characterize the "interiorization" stage. Instead, *meta-cognitive reflection can proceed during the act of writing about mathematics as well as through the process of repeated actions*. Moreover, this result provides evidence in support of Vygotsky's position that development can proceed through reflection, while writing, upon existing conceptual knowledge independently of the "interiorization" process (i.e., reflection due to repeated actions).

Iterative Model

Rittle-Johnson proposes that conceptual and procedural knowledge develop iteratively, with increases in one type of knowledge leading to increases in the other type of knowledge, which trigger new increases in the first (Rittle-Johnson, Siegler, & Alibali, 2001). For example, limited understanding of some domain concepts could guide students' attention to important problem features and facilitates generation and use of a correct procedure. In turn, developing this procedural knowledge could lead to improvements in conceptual understanding, perhaps by freeing cognitive resources for noticing patterns and relations, highlighting the importance of certain problem features, or revealing certain misconceptions. Improvement in problem representation is one particularly promising mechanism underlying these iterative relations.

Evidence from 4 studies on children's learning about the equal sign and about decimal fractions support this iterative model for the development of conceptual and procedural knowledge. These studies raise important new questions for curriculum design about the timing and sequencing of instruction on concepts and procedures.

Study 1

The goal of Study 1 was to provide causal evidence for the bi-directional relations between conceptual and procedural knowledge. Students received instruction on a concept or a procedure, and we assessed the impact of instruction on knowledge of the other type (Rittle-Johnson & Alibali, 1999). The domain was mathematical equivalence, or the idea that quantities on both sides of an equation are equal. The target problems were equivalence problems such as $3 + 4 + 5 = 3 + \_\_\_$.

Participants were 59 fourth and fifth graders who solved equivalence problems incorrectly. Students completed assessments of conceptual knowledge (e.g., What does the equal sign mean?) and procedural knowledge (e.g., $7 + 4 + 8 = 7 + \_\_\_$ and $3 + 6 + 2 = \_\_ + 2$) before and after a brief intervention. For the intervention, students were randomly assigned to receive conceptual instruction, procedural instruction or no instruction on equivalence problems in an individual session.

Students in both the conceptual and procedural instruction groups showed greater gains in conceptual understanding and use of a correct procedure than the control group. The conceptual instruction group was better able to adapt a correct procedure to solve transfer problems. Overall, these findings indicate that improvements in conceptual knowledge can lead to improvements in procedural knowledge and that improvements in procedural knowledge can lead to improvements in conceptual knowledge.

Study 2

The goal of Study 2 was to use individual differences in prior knowledge and learning to assess the iterative relations between conceptual and procedural knowledge and whether problem representation is a link from conceptual knowledge to improved procedural knowledge (Rittle-Johnson, Siegler & Alibali, 2001). The domain was decimals, with a focus on decimal magnitudes. The target problems were placing decimals on number lines.

Participants were 74 fifth graders. The intervention was a computer game the author designed called "Catch the Monster." To catch the monster, students needed to choose the correct location for a given decimal on the number line or to choose the decimal that corresponded with a given location on the number line. Before and after participating in the individual intervention session, students completed assessments of conceptual knowledge (e.g., What numbers are worth the same amount as 0.6?) and procedural knowledge (e.g., Place 0.54 on the number line).

Students began the study with some prior conceptual knowledge of decimals. Students' prior conceptual knowledge predicted improvements in their procedural knowledge. In turn, these improvements in procedural knowledge predicted improvements in students' conceptual knowledge. To understand the link from conceptual knowledge to improved procedural knowledge, we coded students' explanations of the answers to the intervention problems for whether students represented each problem correctly (e.g., noted the value of the digit in the tenths). As expected, prior conceptual knowledge predicted frequency of correct problem representation during the intervention, and correct problem representation predicted improvements in procedural knowledge. Overall, these results support the iterative model and suggest that improved problem representation may be one mechanism in the model.

Study 3

The goal of Study 3 was to manipulate problem representation and examine the causal influence on improved procedural knowledge (Rittle-Johnson, Siegler, & Alibali, 2001). The intervention, assessments and design were the same as in Study 2 with minor modifications. Two manipulations of problem representation were used during the intervention: 1) Prompts to notice the first digit after the decimal point, along with highlighting this digit in red, and 2) marking the tenths on the number line. Participants were 117 fifth and sixth graders. Students were randomly assigned to one of four conditions based on the 2 by 2 crossing of the manipulations.

Both manipulations helped students to represent more problems correctly during the intervention. Both manipulations also led students to make greater gains in procedural knowledge, although it was only the combination of the two manipulations that led to sustained gains in procedural knowledge. These results indicate that improved problem representation leads to improved procedural knowledge and suggests that problem representation is an important mechanism within the iterative model.

Study 4

The goal of Study 4 was to evaluate the instruction implications of the iterative model. We compared learning from iterative instruction, where conceptual and procedural lessons were interleaved, to concepts-first instruction, where all the conceptual lessons were presented before the procedural lessons (Rittle-Johnson & Koedinger, this volume). The domain was decimal fractions, focusing on concepts of place value and re-grouping and procedures for adding and subtracting decimals.

The intervention was part of a sixth-grade curriculum we are designing that incorporates a computer-based intelligent tutoring system. This study focuses on a manipulation within the tutoring system where we varied the order of 3 conceptual lessons and 3 procedural lessons. In the conceptual lessons, students were asked to enter a decimal number in a place value chart and then to show the value of the number in novel ways using regrouping. In the procedural lesson, students were given word problems that required adding or subtracting two decimal numbers. The order of these lessons varied by condition. In the *concepts-first* condition, all three conceptual lessons were presented, followed by the three procedural lessons. In the *iterative* condition, the first conceptual lesson was presented, followed by the first procedural lesson, followed by the second conceptual lesson, and so forth. Participants were 72 sixth graders.

Students in the iterative condition made greater gains in procedural knowledge, and comparable gains in conceptual knowledge, compared to the concepts-first condition. In particular, students in the iterative condition were less likely to make digit alignment errors when adding and subtracting decimals (e.g., 8.41 + 25 = 8.67). Introducing the procedural task early, and interleaving it with conceptual instruction, seemed to help link and strengthen knowledge shared by the conceptual task and the procedural task.

Open questions

This research supporting an iterative model for the development of conceptual and procedural knowledge raises many new questions about the timing and sequencing of instruction on concepts and procedures: What is the optimal time-scale for interleaving instruction on concepts and procedures – minutes, hours, days, weeks, months? What is the smallest change in one that produces noticeable change in the other? Why does improvement in one type of knowledge sometimes not lead to improvements in the other and how can instruction help to avoid this? When can conceptual and procedural knowledge not be separated? Audience discussion of these issues will be invited.

The Coordinated Model

In this presentation, Kalchman will focus on the design and implementation of mathematics curriculum that attends to the simultaneous development of procedural and conceptual knowledge in a domain of learning. Her presentation will have four parts. First, she will present a grounding theoretical framework that guides the development of curriculum for her research. Second, she will present a curriculum for the teaching and learning and mathematical functions that facilitates the development of procedural and conceptual knowledge for students from middle school through to secondary school. Third, she will present empirical evidence of students using both conceptual and procedural knowledge as a result of engaging in the experimental curriculum. These studies involved students from Grades 6 through 11. And fourth, she will open up discussion about designing curriculum, which simultaneously develops conceptual and procedural knowledge in a domain of mathematical learning.

Theoretical framework

At the 2001 PME-NA meeting in Snowbird, Kalchman and Fuson presented a paper in which they attempted to apply issues of conceptual and procedural development to Case's theory of central conceptual structures as it relates to children's learning of mathematical functions (Kalchman & Fuson, 2001). Very generally, in Case's theory (Case, 1992, 1996), children develop understanding in any domain of learning by progressing through a four-stage sequence of elaborations and integrations of two primary mental schemas. One of these schemas is primarily numeric for mathematics and the other is primarily spatial. When these schemas are sufficiently elaborated and then integrated, a new psychological unit, which he called a central conceptual structure (CCS) is formed.

In Kalchman and Fuson (2001), they suggested that within each of the primary numeric and spatial schemas, conceptual and procedural knowledge were being co-developed at each stage of elaboration and integration. That is, there is initial and distinct conceptual and procedural numeric understandings as well as initial and distinct conceptual and procedural spatial understandings. When the initially separate numeric and spatial schemas integrate, so does the conceptual and procedural knowledge previously connected only to one particular schema. Thus, the resulting CCS is an intricate weaving of conceptual and procedural knowledge within a larger framework of a child's newly integrated understanding of, in this case, functions.

Although the relationship of a CCS to the ongoing and simultaneous development of conceptual and procedural knowledge in a domain of mathematics was begun for the 2001 meeting, there are still many issues that need to be further explored. For example, was is the difference between the "small-c" conceptual understandings that are considered within each of the primary schemas and the "big-c" Conceptual understanding of a domain that is achieved upon the formation of a central conceptual structure?

The curriculum

The functions curriculum referred to earlier was designed in the traditions of Case's theory of cognitive development. Essentially, that means that curriculum was designed for the express purpose of first elaborating students primary numeric and spatial understandings of functions and then integrating them. Empirical studies of students' learning with this curriculum have consistently and repeatedly shown through task analyses that such elaborations and integrations occur following instruction. The question, which was asked of me, and which I continue to pursue, is: Which parts of the curriculum develop and elaborate the primary numeric versus the spatial schemas in general? And furthermore, which parts of the curriculum develop conceptual numeric versus procedural numeric knowledge and which parts develop conceptual spatial versus procedural spatial knowledge?

Empirical results

Empirical results come from pre and posttest responses to a functions test developed by Kalchman and Case (see Kalchman, 2001). Classes of students in Grades 6 through 12 participated in the treatment-comparison studies. The purpose in sharing these results is not so much to "present" results as to work with the participants of the discussion groups to understand the proportion of conceptual and procedural indicators in students' responses to individual items. Responses in general will be examined as well as the differences between responses from students who learned functions with the experimental curriculum and those who learned with a modern textbook approach.

Discussion

Questions driving this open discussion include: ways of improving current curriculum (including the functions curriculum) to improve students procedural fluency and conceptual organization in a range of domains; participants ideas and opinions on this model of ongoing co-development of procedural and conceptual knowledge both within each of the primary numeric and spatial schemas associated with Case's theory of intellectual development and within the integrated central conceptual structure; differences between "small-c" and "big-c" conceptual understanding in a mathematical domain.

Initial Observations

The inter-domain comparative analysis implicit in the organization of this discussion group suggest some interesting observations. First, the results of Rittle-Johnson and Kalchman seem to represent two conceptually independent solutions to the same problem of finding the dynamics for the integrated procedural/conceptual development, each coming from a different research paradigm: the strategy - choice approach and the schema development (Baroody, 2001; Geary, 1994). The presence of theses two solutions, while providing the answers to the original question: What is the integrative relationship between the two modes of mathematical thinking, creates, of course, a new question: What is the relationship between these answers, between the strategy - choice approach and schema development approach and how does it manifest itself in instruction and learning?

Second, there is an interesting division in the meaning of the "conceptual" along the K-12 curriculum. While it seems that arithmetic as well as pre-calculus and calculus research utilize numerical/visual relationship to express the procedural/conceptual character of the domain, algebra procedural/conceptual research has utilized numerical/verbal relationship (algebra has been seen here as generalization of arithmetic). Does this difference reflect the inherent difference between mathematical domains or the difference in the developmental progression of students, or both? Would the composition of numerical, visual and verbal modes of thinking be useful from the instructional point of view? How would it affect the procedural/conceptual integration?

The expression of the procedural/conceptual duality through procedural/language interaction brings one closer to Vygotsky's duality between thought and language (Vygotsky, 1986), where the fundamental unit of analysis, which motivated the thrust of his thinking, was the "verbal thought". What could be the corresponding unit of analysis for the numerical/spatial manifestation of the procedural/conceptual divide and its integration?

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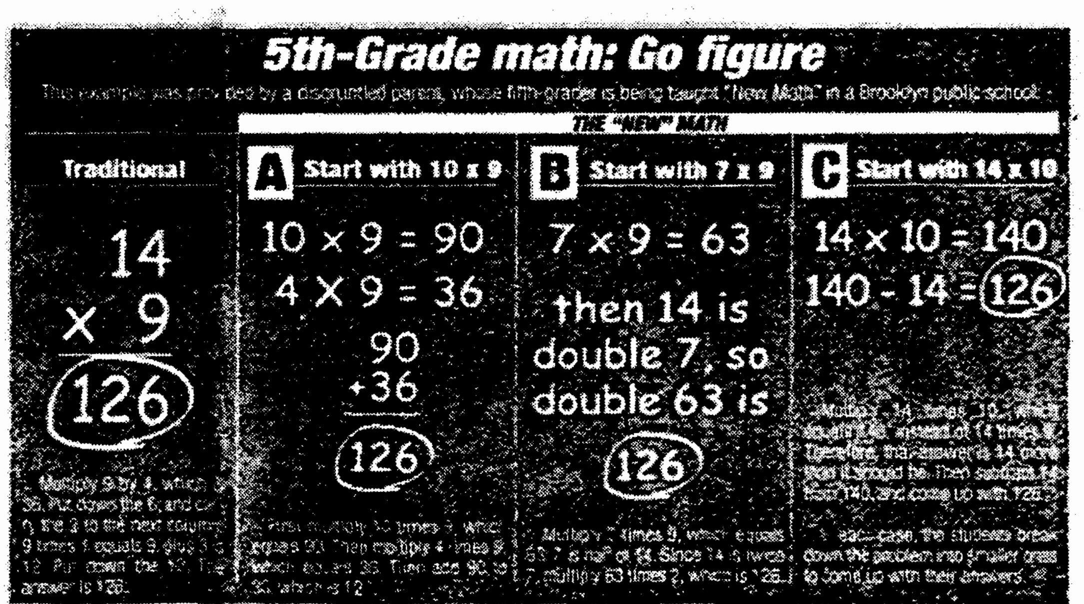
Appendix

New York Post, Tuesday, April 17, 2001

This inset appeared in the New York Post article as an example of the reform motivated instruction and was provided by a disgruntled parent from a public school in Brooklyn.



From the cognitive point of view (as distinct from the political one), it takes some time to realize what could possibly be the reason for that feeling on the part of a parent, especially since computationally everything seems OK. A closer look, however, and taken especially from the point of view of the procedural-conceptual dichotomy, gives a lot of food for thought. It becomes clear why all three "New Math" columns A,B and C might be quite confusing in this presentation as opposed to the traditional vertical multiplication based on the positional notation. For example, in the column A, the presentation starts from $\begin{matrix} 10\times9=90 \\ 4\times9=36 \end{matrix}$. To an inexperienced mind, an immediate question comes to mind, as to where 10 and 4 came from. Before, there were two digits 1 and 4 in one number 14 being multiplied by 9 while now there are two numbers 10 and 4. In other words, there is an important step missing, the step which explicitly shows the interaction between the positional notation and the use of the distributive law $14\times9=(10+4)\times9=10\times9+4\times9$. Without this step the whole procedure outlined in this column is intuitively rather weak. The standard manner of multiplying is able to bypass that point by virtue of the role of positional notation and visual/spatial organization partaking in the vertical method of multiplying numbers. A similar problem appears in column C (distributive law) and in column B (associative law). Looking together at columns A, B and C, the pedagogical problem in the design of the technique becomes clear. In these columns only the procedural aspect of the techniques was utilized, while, in fact, both conceptual and procedural aspects are needed. Moreover, since the whole instruction here seemed to be geared towards showing the conceptual richness of multiplication process, it seriously missed it by focusing purely on its procedural aspect. Clearly, the understanding and the coordination of the relationship between the two aspects of knowledge in this particular mathematical situation was missing in the presented instruction and, quite possibly, contributed to the confusion. This little analysis underlines the urgency with which the research community centered around the reform in mathematics education needs to attend to the integration of the p/c divide and, through the coordination of research with teaching, introduce it into instruction.

INVESTIGATING PRESERVICE ELEMENTARY TEACHERS' ATTITUDES AND BELIEFS TOWARD MATHEMATICS

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This paper explores research completed on preservice teachers' attitudes toward mathematics. Our findings suggest that after implementing curricular changes in the areas of cooperative learning, writing, technology, and increased usage of hands-on teaching methods, we were able to positively impact preservice teachers' attitudes. In addition, this paper is an investigation /discussion into best practices in conducting research on preservice teacher attitude.

As mathematics educators preparing students to teach mathematics, we have often thought that the beliefs, attitudes and experiences our students have had learning mathematics may have an impact on their learning and performance in mathematics, which ultimately may affect how they teach mathematics. Beliefs that students hold about mathematics and their abilities to perform mathematically are critical in the development of their responses in mathematical situations. This discussion group provides a forum to share best practices in conducting research on attitudes and beliefs.

Affective variables related to the learning of mathematics play an important role in the development of preservice teachers. Throughout the process of learning mathematics, preservice teachers collect a wide range of experiences. Both positive and negative, these experiences have led to the development of their beliefs and attitudes about mathematics. Recent research has found that teachers' beliefs about mathematics and the teaching of mathematics are significantly influenced by their mathematical experiences as a student (Brown & Borko, 1992; Brown, Cooney & Jones, 1990; Raymond, 1997). Furthermore, research has found that success in solving mathematics problems is not based solely on one's knowledge of mathematics. It is also based on metacognitive processes related to mathematics strategy usage, the emotions an individual feels when doing a problem and the personal beliefs of one's mathematical abilities (Garafalo & Lester, 1985; Schoenfeld, 1985; Mcleod, 1988).

For the purposes of discussion, beliefs will be defined as the personal assumptions from which individuals make decisions about the actions they will undertake. This notion is consistent with research that indicates that actions are motivated by what individuals perceive are the outcomes of their actions (Kloosterman, 1996). Students, in the process of learning mathematics, experience both positive and negative emotions, which influence the development of their attitude towards mathematics as a whole. These beliefs about mathematics, about what they need it for, and how strong they are as mathematics students are related to learning and can significantly affect what students do in a mathematics classroom (Kloosterman, 1996).

Teacher educators are charged with the daunting task of shaping and reshaping the attitudes, beliefs and content knowledge of preservice teachers. Teacher education programs must sometimes help participants to deconstruct, and reconstruct their views on teaching and learning (Brown & Borko, 1992; Wilson & Ball, 1996). It is important for mathematics to be learned in a supportive community of learners (Brown & Campione, 1994). This environment can also provide an arena for discussion of these important affective issues.

The authors' primary purpose of this research was to investigate the beliefs, attitudes and experiences that preservice teachers have had while learning mathematics. This paper will describe 1) students' perceived attitudes, beliefs and experiences related to teaching and learning mathematics 2) methodology we used to impact attitude of preservice teachers, and 3) future investigations for this research.

Methodology

The first component of this research plan was completed by collecting data from preservice teachers, who planned to teach at either the elementary, middle, or secondary level, and were enrolled in a mathematics or mathematics education course required for students seeking teaching certification in their respective level. Data were gathered in the form of written responses called "Mathematics Autobiographies" during the first week of the semester. We requested that students respond in writing about any experience in mathematics that they felt contributed to their mathematical development. A total of seventy-two mathematics autobiographies were gathered from three different levels of college mathematics courses and two secondary mathematics education courses. In the second session of each of these courses students were asked to share with the class any experiences they were comfortable sharing, and then as a group we discussed the implications for mathematics instruction.

After analyzing the previous data, we collected baseline quantitative and qualitative data (both pre and post) from forty- one preservice elementary teachers regarding their attitudes, beliefs, and experiences. This data was collected at the beginning of the first course (Number Systems) and at the end of the second course (Concepts in Mathematics, Probability, and Statistics). In addition, we gathered qualitative data about how these prospective teachers view themselves as mathematics students; and, their thoughts on their future usage of technology, writing, and cooperative learning in the teaching mathematics.

Curricular modifications were made and implemented in the courses following baseline data collection. These changes include an increased emphasis on: 1) multiple concrete representations and appropriate conceptual models of mathematics, 2) cooperative learning and mathematical discourse, and 3) technology.

Concrete Representations and Appropriate Models

Learners of mathematics who do not have exposure to concrete representations of mathematics may tend to view mathematics as a collection of abstract unconnected topics. Through the use of manipulatives, preservice teachers are presented with a different view of mathematics. Our sequence provides preservice teachers with the opportunity to investigate mathematics through the use of concrete, pictorial and symbolic representations.

Cooperative Learning and Mathematical Discourse

Mathematical discourse is one of the most important facets in preparing preservice teachers to teach mathematics. Effective teachers not only need to understand the concepts thoroughly, they also need to be able to discuss the mathematics in a way that is understandable to elementary-age children. Preservice teachers in our courses investigate mathematics daily, in a cooperative learning atmosphere that allows them to share their ideas with others as well as listen to different approaches used by their peers. Journal assignments are integrated so that preservice teachers are able to work on expressing their thoughts in writing and emphasize the importance of communication in mathematics.

Technology

When including technology in a mathematics course, it is essential that one look at *why* and *how* technology is used. One of the goals in our three-course sequence is to use technology in meaningful ways, as a means to develop a deep understanding of a topic, not just as a tool to avoid computation. Technology is used as a tool to investigate concepts and real-world situations.

Data Analysis

Analysis was completed on all data collected. Quantitative data were analyzed by computing statistical means and qualitative data were analyzed by conducting a comparison across all participants and coded according to emerging themes (Miles & Huberman, 1994).

Results

Mathematics Autobiography

Mathematics autobiographies were written by K-12 preservice teachers at the onset of this research study. Presented below is a summary of key findings.

The following were found to be important in students' experiences learning mathematics: the role of the teacher; support and influence of family; challenge; issues of fear, failure and avoidance; learning strategies; and content issues. All of these issues mentioned are in some fashion tied to one variable, the teacher. Since the teacher generally guides classroom instruction, many of these issues focus on a teacher's approach to classroom instruction.

Typically preservice elementary teachers had more negative feelings, attitudes and less confidence toward learning mathematics. They felt that mathematics "was just not their strong suit", "was a horrible subject" and that they were "just plain stupid at math" and thus "hated math." These students also expressed their fear of their current mathematics class. In addition, they experienced increased failure in their entire mathematical experience from kindergarten through college.

Middle school preservice teachers felt some of the same anxiety as their elementary teacher counterparts. However, this anxiety was mixed with enough successful experiences in mathematics to encourage a greater commitment to the teaching and learning of mathematics. For the most part, their experiences in mathematics class were positive, although the love/hate relationship with mathematics was evident in their self reports: "As for my relationship with math today, I hesitate to call it dysfunctional, but it is definitely wrought with anxiety and at times exhilaration."

High school preservice teachers' memories of their mathematics classes were the least tainted by frustration and failure as depicted by this student, "I like to think I've always been good at math and never had to struggle." They all spoke of influential teachers who helped them develop a love of math, teachers who also often singled them out and instilled a belief that they were talented in doing mathematics. They had a desire to emulate the characteristics of this beloved teacher. Many spoke of great admiration for their parents' understanding of mathematics and the help they received growing up which helped them learn mathematics.

The major difference we found in each of these groups was in confidence level. It seems obvious that students who experience less failure would have more confidence in their abilities. The preservice elementary students generally reported more anxiety over learning mathematics. They also spoke, as a group, of more failure and self-doubt toward their abilities to learn and do mathematics well than secondary preservice teachers.

Attitude Data

As part of our ongoing research project, we chose to focus on preservice elementary teachers enrolled in the mathematics for elementary school teachers course. Quantitative data were collected from students at the beginning of the first course and at the end of the second course. A total of forty-one elementary preservice teachers who took both courses consecutively completed both pre-course and post-course instrumentation. See Appendix A for the Instrumentation.

Future Teaching Methods

At the beginning of the semester, students were asked to respond to the following: "Please describe how you would teach mathematics in an elementary classroom?" We found that responses to this question fell on a continuum, with one end of the continuum representing traditional teaching methods, the other end representing hands-on teaching methods, and the middle representing a group of preservice teachers that we consider to be emerging. A total of thirty-nine participants responded to this question (two left it blank); nine percent fell in the traditional category, thirty-seven percent were in the emerging category, and fifty four percent were in the hands-on category. After two semesters of Mathematics for elementary school teachers' courses, Sixty-four percent of the teachers fell into the hands-on category, thirty-five in the emerging category and one percent in the traditional category. We describe the characteristics of the three categories below.

The "traditional" teacher is one who stated that their primary mode of teaching would be lecture. An emphasis was also placed on using a step-by-step, repetitive approach to teaching as opposed to a conceptually- based focus. One preservice teacher described their future classroom teaching in the following manner: "If I were to teach to an elementary classroom I would first lecture to the class, explaining the concepts, definitions, as well as going over previous problems on assignments, home-work or quizzes. Afterwards, I would have every student try out a few practice questions that they will be working on that day."

The "emerging teacher" had many characteristics of a traditional teacher, yet also demonstrated a progression toward teaching in a conceptually-based manner. These individuals were more concerned with engaging their students in learning but they restricted the engagement to cooperative learning. They also placed some importance on representing material in a visual manner. One participant stated: "I would lecture in such a way that the students would be involved. After lectures, I would have games with the new material so that the student get practice with it (math concepts) in a fun way."

The "hands-on" teacher seems to be aware that students learn in a variety of ways, and that concrete representation is very important when first introducing a concept to students. These individuals focused on engaging their individual students in learning mathematics by using visuals, concrete manipulatives, and cooperative learning. Note the sharp contrast to the traditional teacher who focused on one presentation to the group of students but did not emphasize student participation and engagement. One preservice teacher in the "hands on" category described her vision of teaching mathematics in the following manner: "Use simple concepts to make it fun; collaborative and hands-on activities to teach math. In groups, pairs, and as a whole, whatever it takes."

Cooperative Learning

Preservice teachers were asked about their experiences learning mathematics cooperatively and if they would use group learning in their future teaching. Ninety-five percent of the preservice teachers responded that they enjoyed learning mathematics cooperatively while five percent reported that they did not. An overwhelming ninety-five percent would teach their students using cooperative learning. Sixty-two percent said they had previously been involved in learning mathematics cooperatively, however, thirty-eight percent had not.

Among the ninety-five who enjoyed working cooperatively, the motivation differed. Some believe that cooperative learning sessions afforded them the opportunity to strengthen their understanding of mathematics by seeing other perspectives. Others used cooperative learning simply as an opportunity to get help with their mathematics homework.

Following completion of the two courses using cooperative learning techniques, all participants had experience with this method of teaching. Preservice teachers then responded that ninety-nine percent of them would teach using cooperative learning, and one percent would not.

Writing in the Mathematics Classroom

Preservice teachers were asked about the role writing played in their learning of mathematics; if they had to explain their answers in writing in previous mathematics classes; and if they thought explaining their answers in writing would help them learn mathematics.

Prior to taking both mathematics courses, eighty-three percent of the participants stated that writing had been a part of their experience of learning mathematics, seventeen percent did not. Eighty-three percent thought explaining their answers in writing would help them learn mathematics, six percent did not and eleven percent were not sure.

As with cooperative learning, the courses these participants engaged in for two semesters were taught with an emphasis on writing. All participants had experience writing in the mathematics curriculum and ninety-four percent reported that they feel writing is important method of helping them learn mathematics.

Perceived Mathematical Ability

We asked students to self report their mathematical abilities and whether they believed that a weak student could become a strong student. The participants described themselves on a continuum from a weak to a strong mathematics student. Seventy-four percent described themselves as weak students, twenty percent said they were average, and six percent said they viewed themselves as strong mathematics students.

Overwhelmingly, ninety-nine percent of preservice teachers stated a weak student could become a strong student, one percent said no. Several factors mentioned that participants believed would contribute to the strengthening of a student include practice, studying hard, determination and encouragement from a good teacher or tutor.

As in the pretest participants on the posttest described themselves on a continuum from a weak to a strong mathematics student. Twenty-four percent described themselves as weak students, sixty-three percent said they were average, and thirteen percent said they viewed themselves as strong mathematics students. They all stated that a weak student could become a strong student.

The Role of Technology

Information was also gathered about preservice teachers' experience using technology in learning mathematics. And their responses to the following question, what role do you believe technology should play in the teaching and learning of mathematics?

Eighty-eight percent of our participants had previously used technology in learning mathematics, and twelve percent had not. The types of technology that were utilized were computers, graphing calculators, and non-graphing calculators. Preservice teachers were very specific about how technology should be used and the role it should play in teaching and learning mathematics. Most participants held a strong belief that calculators should not be used for basic math, that students should have a deep understanding of a concept before using the calculator. Overwhelmingly they were concerned with using technology without understanding; they did not want technology to be used as a crutch but as a learning tool.

All participants were exposed and engaged in learning mathematics using technology in both of the courses. Ninety-one percent of the participants viewed technology as an important component to teaching mathematics and eight percent believed that it was not a necessary component and should only be used minimally, especially in lower level mathematics such as elementary mathematics.

Quantitative Questionnaire on Attitudes

A Likert-scale questionnaire on attitudes and beliefs was administered at the beginning of the first course and end of the second course. The same forty-one participants from the qualitative instrument completed the quantitative instrument. See Appendix B for the instrument. Although the instrument contains both positively-and-negatively- worded questions, responses were ordered so that a higher mean represents a more positive attitude. Responses ranged from 1 to 5.

Data collected at the beginning of the first course suggest that the preservice teachers held a fairly neutral view on whether they liked mathematics and looked forward to taking more mathematics classes as well as whether they looked forward to teaching mathematics courses in the future. Slightly more positive responses were reported on the use of technology, the use of manipulatives, and the belief that mathematical topics are connected. Preservice teachers generally agreed that 1) understanding mathematics does not require special abilities, 2) technology is not only for slow learners, 3) investigating a problem is more important that getting the correct answer, 4) time spent thinking about mathematics is important, and 5) college mathematics courses are helpful for their future teaching of mathematics. For each respondent, a composite mean was computed from all responses, representing an individual's overall negative or positive attitude. They also believed that students must solve many problems following examples provided. The overall mean score for attitude was 3.57, representing a slight positive attitude toward mathematics.

Of the fourteen items, three resulted in statistically significant differences in mean values upon comparison of the questionnaires administered at the beginning and end of two mathematics courses. Students had a more positive attitude towards taking more mathematics courses, liked mathematics more, and believed more strongly that mathematics topics were connected to each other. In addition, preservice teachers' overall mean score on attitude increased significantly from 3.57 to 3.75.

Discussion

Experts in attitudinal measurement warn of the difficulties of obtaining statistically significant differences in attitude over a relatively short time period, such as a ten-month period spanning two semesters. Yet many of us would like to believe that our courses have some long-term effects in preservice teachers' attitudes towards mathematics. Our study has provided us with some hopeful results. Recall that the quantitative instrument resulted in statistically significant differences in mean values for the following questions: I like mathematics, and I look forward to taking more mathematics classes. There was also a statistically significant difference on the item referring to the connectedness of mathematical topics. This significant difference indicates that students believe more strongly that mathematical topics are connected. The overall mean score from the questionnaire increased significantly as well. All of these significant differences suggest that students are having positive experiences in their mathematics for elementary teachers course sequence. The qualitative results corroborate this statement.

According to our qualitative data on attitudes, we have discovered that preservice teachers, at the end of two mathematics courses, have a much greater tendency towards hands-on learning in their teaching of mathematics. That can be interpreted as a positive reaction to the many experiences with hands-on learning provided in the preservice mathematics courses. Furthermore, preservice teachers responded that they would use a greater variety of teaching methods in their mathematics classes that they will teach including cooperative learning, writing, and technology. In addition, preservice teachers' confidence in their mathematical abilities increased substantially.

Issues for Discussion

Overall, the data point to an improvement in attitude. Yet, we realize there are limitations to this study and that further research is needed. For discussion, we would like to focus on several issues, related to the following questions. First, how do we as mathematics educators positively impact preservice teachers' attitudes towards mathematics? Secondly, how do attitudes affect both the teaching and learning of mathematics? And lastly, what are best practices and research methods that can be used to effect greater visible change in attitudes and beliefs?

How Do We Positively Impact Preservice Teachers' Attitudes Towards Mathematics?

Using *Principles and Standards for Teaching Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000) and current research on best practices we designed our courses with a goal of developing a mathematical community of learners. The following sections describe techniques we have used to positively impact preservice teachers' attitudes.

Development of a Mathematical Community

Mathematics autobiographies. Mathematics autobiographies can be used to facilitate a positive classroom environment, one in which all students feel that their experiences and background are valued. Sharing autobiographies in a classroom discussion can help students identify with others' similar experiences as well as enlighten students on the diversity of student backgrounds. For example, as our students shared their experiences, a common discussion that developed focused on difficulties learning mathematics. The discussion progressed into exploring a variety of different learning strategies used to learn mathematics. We have found that an important component of using mathematics autobiographies is having our students verbally share and discuss their experiences. Shared experiences begin to develop commonalties such that they begin to connect to one another and often realize they are not alone with their feelings, beliefs, experiences, and approaches toward learning mathematics. In addition, such discussions facilitate the development of a supportive mathematical community, one in which students can construct their own ideas, find their own representations, and connect mathematical ideas in their own ways while doing mathematics together. Students can share their work in a comfortable atmosphere where discourse and collaboration are valued (Brown & Campione, 1994; Bruner, 1996).

Instructional strategies. One premise is our research is the belief that engaging preservice teachers in mathematical investigations can bring about positive change in attitude. This belief brings into question the whole issue of what it means to "do" mathematics. A well-designed activity contains one or more of the following elements: 1) investigation of a mathematical concept; 2) cooperative learning; 3) connections to other mathematical concepts, other disciplines or real-world problems; and 4)-student discourse. In each of our courses concrete representation of each concept being taught is utilized before moving to the pictorial and symbolic representation. In addition, these concepts are investigated with technology when appropriate. Furthermore, preservice teachers investigate these topics through different lenses; first through the lens of a student an then through the lens of a elementary teacher. With such an activity, the focus is on active learning rather than passive reception. Furthermore, the emphasis is also removed from the rote repetition of procedures and is shifted towards grappling with major concepts of elementary mathematics. It is the authors' belief that these activities serve a great purpose in redefining what mathematics is for these preservice teachers and the different ways it can be learned.

The strategies we implemented were based on National Standards (NCTM, 2000) and recent research. We are interested in making further curricular changes and pursing other means to impact attitude.

How Do Attitudes Affect Both the Teaching and Learning Of Mathematics?

Since prior experiences appear to have shaped preservice teachers' attitudes and self-efficacy about mathematics, it is important to consider how their experiences may affect their future teaching of mathematics. Although our preservice teachers did not explicitly mention the impact their learning experiences might have on their future teaching, we believe there is an impact. There seems to be a range in possibilities of how this might impact teaching. There is the possibility that these students may teach just as they were taught. Conversely, there is the possibility that preservice teachers, when taught using appropriate methodologies, may embrace these methods and utilize them in their own teaching. The impact that a positive or negative experience learning mathematics has on future teaching is an area that needs to be explored further. Without the opportunity to experiment with new methods of teaching and learning, to examine their own beliefs and knowledge (Kloosterman, Raymond, & Emenaker, 1996), teacher educators are likely to fall into past patterns of teaching and learning (Wilson & Ball, 1996).

What are Best Practices and Research Methods that Can be Used to Effect Greater Visible Change in Attitudes and Beliefs?

In order to fully investigate teacher attitude, longitudinal studies need to be developed. We are interested in the attitudes of those teachers currently teaching mathematics; specifically, their attitudes, beliefs, and experiences with regard to teaching and learning mathematics. We would like to investigate preservice teachers' attitudes prior to their coursework and follow them through their first few years of teaching, measuring their attitude throughout this process. This information could be helpful when developing courses for preservice teachers and inservice professional development throughout a teacher's career.

Our data suggests preservice teachers' views of mathematics as a connected body of knowledge increased. As such we are interested in the possible connection between attitude and mathematical understanding. Specifically, we are interested in assessing the knowledge base of both preservice and inservice mathematics teachers and how they view mathematics and the connections between certain topics.

The onset of this study we began collecting data from 105 preservice teachers. At the end of the second semester, only forty-one students completed the second (required) course in a three-course sequence to prepare preservice teachers. The initial data we collected was different for 105 participants, versus data from the thirty-nine who completed the second course in the following semester. Of significance to the researchers are several questions: 1) do the preservice teachers with more positive attitudes take the courses sequentially? 2) Would the results be significantly different if taken after the third course in the sequence 3) what is the impact on attitude if all three courses are taken at once or if they are taken over a longer period of time? 4) What are the best practices for assessing change in preservice teachers' attitudes'?

Summary

As mathematics educators we believe it is essential for preservice teachers to have a positive attitude toward mathematics. Research supports our belief that future teachers need to have a positive view of themselves as mathematicians, and as competent future mathematics educators. First we need to identify their attitudes and then work to provide positive experiences that will effectively impact their future teaching of mathematics. We also need to insure that throughout their teaching careers they maintain a positive view of themselves as mathematicians and as effective teachers of mathematics. Further research in this area needs to be completed to successfully achieve this goal and support our future mathematics educators.

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Appendix A: Qualitative Instrumentation

1. Please describe how you might teach mathematics in an elementary classroom (e.g., how would you instruct students, lecture, use cooperative learning, etc.).
2. Do you enjoy learning mathematics while working in groups? Would you teach your classes using group work? Have you previously been taught mathematics while learning in groups?
3. What role, if any, has writing played in your past learning of mathematics? Have you had to explain your answers in writing in previous mathematics classes? Do you think explaining your answers in writing will help you learn mathematics?
4. How would you describe your mathematical abilities? Explain. Do you believe a weak student can become a strong student? Explain your answer.
5. How much have you previously used technology in learning mathematics? What role do you believe technology should play in the teaching and learning of mathematics?

Appendix B: Quantitative Instrumentation

Below, there is a series of sentences. Indicate in your bubble sheet the degree to which you agree or disagree with each sentence. There are no right or wrong answers. The correct responses are those that reflect your attitudes and beliefs .

A Strongly Agree B Agree C Not sure D Disagree E Strongly Disagree

1. I am looking forward to taking more mathematics courses.

2. I enjoy learning how to use technologies (e.g., calculators, computers, etc.) in mathematics classes.
3. I like mathematics.
4. In grades K-9, truly understanding mathematics in schools requires special abilities that only some people possess.
5. The use of technologies s (e.g., calculators, computers, etc.) in mathematics is an aid primarily for slow learners.
6. Mathematics consists of unrelated topics (e.g., algebra, arithmetic, calculus, and geometry).
7. To understand mathematics, students must solve many problems following examples provided.
8. Students should have experiences manipulating materials in the mathematics classroom before teachers introduce mathematics vocabulary.
9. Getting the correct answer to a problem in the mathematics classroom is more important than investigating the problem in a mathematical manner.
10. Students should be given regular opportunities to think about what they have learned in the mathematics classroom.
11. Using technologies s (e.g., calculators, computers, etc.) in mathematics lessons will improve students' understandings of mathematics.
12. I expect that the college mathematics courses I take will be helpful to me in teaching mathematics in elementary or middle school.
13. I want to learn to use technologies s (e.g., calculators, computers, etc.) to teach mathematics.
14. The idea of teaching mathematics scares me.

THE NATURE AND FUTURE OF CLASSROOM CONNECTIVITY: THE DIALECTICS OF MATHEMATICS IN THE SOCIAL SPACE

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New theoretical, methodological, and design frameworks for engaging classroom learning are provoked and supported by the highly interactive and group-centered capabilities of a new generation of classroom–based networks. This discussion group situates networked learning relative to a dialectic of (1) seeing mathematical and scientific structures as fully situated in socio-cultural contexts and (2) seeing mathematics as a way of structuring of our understanding of and design for group-situated teaching and learning. Acknowledging (1), significant classroom examples are then used to illustrate the reciprocal process (2) of mathematics structuring the social sphere (MS3). The mathematically informed ideas of *space-creating play* and *dynamic structure* are then used to update our ideas of generative teaching and learning and to situate these classroom examples. Then, returning again to the dialectic, this current work is critiqued from a socio-cultural perspective (1). *Participation* and *agency* are highlighted in this critique. The session closes with a discussion of future possibilities for classroom connectivity.

The highly interactive and group-centered capabilities of a new generation of classroom-based networks are helping both to support and to provoke the development of new theoretical, methodological and design frameworks for engaging classroom learning. This discussion group situates networked teaching and learning relative to a dialectic of (1) seeing mathematical and scientific structures as fully situated in socio-cultural contexts and (2) seeing mathematics as a way of structuring our understanding of and design for group-situated teaching and learning. The idea is that if mathematical and scientific structures are seen to fully participate in the social plane, then not only are they structured by the social plane (i.e., (1)) but they also structure social activity (i.e., (2)), including learning and teaching. Due to the group-focused interactivity and data collection capabilities of next-generation networking, we now have a new tool to explore the dynamics of—and design for—classroom learning. A number of projects have responded to the challenge of learning in a network space by using mathematical/scientific ideas to organize and analyze classroom activity. Some of these recent projects focus on student learning and one is focused on teacher understanding. All of these projects have begun to use mathematics itself to organize their classroom-based work. This use of domain-related "big ideas" to organize and analyze group learning is what is meant by mathematics structuring the social sphere (MS3). The mathematically informed ideas of *space-creating play* and *dynamic structure* are then used to update our ideas of generative teaching and learning and to further situate the previous classroom examples. Our argument is that to take full advantage of these notions, and of the new classroom tools, researchers and educators must acknowledge explicitly the dialectic that exists between the domains of mathematics or science as structuring agents and the structuring functions of the social, cultural, historical milieu in which classroom learning and teaching in the domains exist. Returning again to the dialectic, this current work is then critiqued from a socio-cultural perspective focusing on ideas of participation and agency. Consistent with this dialectic framework, an overall notion of what is called generative teaching and learning is clarified in a way that both draws on previous work and uses the mutually constitutive relations captured by the dialectic for extending the prior analyses.

What's New about Next Generation Classroom Networks

This paper is not about technology *per se*, but rather about a specific instance of the interaction and co-evolution of design, technological affordance, and cognitive theory (Dewey, 1938; Hickman, 1990). Networks and ideas of generative teaching and learning can be and have been discussed without specific reference to technology (Wittrock, 1978, 1991; Learning Technology Center [LTC], 1992). The idea is that next generation networking can better support or "resonate" with generative practices and also, in iterating back to theory, allow us to further develop our ideas of what generativity is. To start this conversation, we may need to understand first how these new network designs are functionally evolved, and thus distinct, from the kinds of networking that has been with us for a long time. In this section the question becomes: What might be said to be "new" about next-generation network functionality in classrooms?

With a few important exceptions (among them, *CSILE*, associated with Scardemalia (1993); Scardamalia and Bereiter (1991); Scardamalia, Bereiter and Lamon (1994); and Scardamalia, Bereiter, McLean, Swallow and Woodruff (1989); collaboratory notebook associated with Edelson, Pea and Gomez (1996); *ClassTalk* as discussed by Abrahamson (1998); Mestre (1996), or Hake (1998), or some of the projects discussed herein), networking in the classroom has served primarily in two ways: (*i*) As a portal to sources of information or interactivity having their "centers of balance" outside the classroom (e.g., visiting the *CNN* web site or filling out a web-based questionnaire), or *(ii)* To implement Computer Assisted Instruction (CAI) or tutoring environments. Two top-level features characterize these uses of networking in classrooms. First, the experience is fundamentally individual – most of the activity could be carried out at home or in a local library as little or no use is made of the social space of the classroom. Second, the knowledge and/or the trajectory of learning is owned by someone who is not a member of the classroom (e.g., a distant "expert" for web content or the computer programmers for CAI/tutoring environments). Especially for low socio-economic status (SES) students who have had their experience of school-based technology skewed heavily toward the use of CAI environments, a fundamental message of mathematics and science reform—that the classroom should be a community of inquiry characterized by joint ownership and construction of understanding—is undermined. Traditional uses of classroom networks place the locus of knowledge making outside the classroom (even as the individual experience is often described as highly "customizable").

The next-generation network functionality, provoking a new round of theorization and design, does mark a significant break with this tradition in terms of pedagogical vision and technical capability. To begin with, these systems are typically designed "from the ground up" with the classroom in mind. Rather than constrain the learning experience to be narrowly individualistic, this technology supports socially situated interaction and investigation. Moreover, the learning trajectories and the processes of knowledge construction are owned by the group itself. Software and hardware work together in supporting this "group-oriented" design. A significant number of these networks are about to become widely available and are poised to become a major presence in classroom learning. The implementations may vary but at the top level the design features of these systems are remarkably similar and typically include: individual devices or "nodes" that are relatively unobtrusive; the network supports a range of topologies for real-time or near-real time interaction (peer to peer, peer to group including whole-class, or group to group); the network is "wireless" and hence very flexible and portable; there is a core set of meaningful functionality in each device (e.g., at least that of a graphing calculator); there is a mixture of public and private display spaces (e.g., the public space can be a computer projection system as with participatory simulations (Wilensky & Stroup, 1999) or a calculator Viewscreen™ with the SimCalc materials (Kaput, Roschelle, Tatar, & Hegedus, 2002) and the private space can be the students' own individual displays on a calculator, Palm Pilot™ or laptop computer); the network experience is "author-able" in that it allows teachers or others to create new activities; the network can readily support the exchange of a range of kinds of both group and individual artifacts/data-types including text, strings, numeric values, ordered pairs, lists, matrices, individual and whole class graphs, images and, in some cases, sounds or video; and the network enables new (as well as old) forms of gesture (e.g., turning to the left with an arrow key on a calculator).

An interesting facet of working with these kinds of networks is how traditional research methodologies and theories, based on some notion of simply "scaling up" to the group methods and models based on individual cognition, are overwhelmed. This kind of challenge has always been present in analyzing classroom learning, especially as we move toward a reform-based ideal of classrooms being communities of inquiry (Brown & Campione, 1996). The theoretical and methodological disconnect seems even more pronounced for researchers attempting to make sense of what happens in these highly interactive networked spaces. All the projects discussed in this paper are struggling to situate and organize what they are doing in terms of activity design, learning theory, and research methodology. The good news is that new patterns of analysis and design are beginning to emerge. One of these new frameworks is the use of mathematical ideas not just as "content" to be learned but also as an interpretive framework for learning analysis and activity design. This is what will be discussed as mathematics structuring the social sphere (MS3). And then, in turn, the mathematically informed ideas of space-creating play and dynamic structure are used to clarify and update our understandings of generative teaching and learning. Here we can begin see how technological affordance can loop back to change the shape of cognitive theory and activity design. Returning to the dialectic, we then use insights from a socio-cultural perspective to critique and extend the ideas of participation and agency related to generative teaching and learning.

Mathematics Structuring the Social Sphere: Examples From Networked Classrooms

During a recent session at the Annual Meeting of the American Educational Research Association, a group of investigators presented results from working in classrooms with this kind of next generation classroom network (Kaput, Roschelle, Tatar & Hegedus, 2002; Mack, 2002; Stroup, 2002a, 2002b; Wilensky). A striking features in common to these presentations was the way in which ideas from mathematics proper were either starting to become, or already were, central to the organization of their work. This engagement with mathematics went well beyond thinking of it as content to be learned by individual students. These presentations are discussed briefly in this paper as part of helping us understand what it might mean to think of mathematics as a socially structuring tool in learning research and design. These examples illustrate how thinking of mathematics as an active participatory agent in social activity, might be considered a reasonable "next step" in our ever-evolving understanding of the nature of mathematical and scientific activity itself. That is, once we relinquish the assumption of the truly solitary mathematical or scientific researcher – as either a practical possibility or theoretical ideal – we are then free to fully engage how mathematics is a socially structured form of activity (part (1) of the dialectic). With this freedom also comes the possibility of investigating how mathematics is, or can be, socially structuring (part (2) of the dialectic). Each of the research efforts discussed in this section is responding to the challenges of creating learning activities in next generation networks. Each illustrates the directions being taken in trying to say something meaningful about the learning and teaching that takes place in relation to these activities.

Parametric Space

Moving their work from developing software for learning calculus ideas with individual computing devices to now situating the use of this kind of functionality in a network space, Jim Kaput, Jeremy Roschelle, Deborah Tatar and Stephen Hegedus from the SimCalc Project have been pushed to think anew about the design of the activities themselves (2002). Prior to the availability of classroom connectivity, they had presented students or small groups of students with individual learning tasks related, for example, to giving in advance a *velocity* graph that controls the motion of one animated character and then asking them to author a *position* graph for a second character such that its motion would now match that of the given character (thereby using the simulation and graphs to obtain the integral of the original velocity function). Given the network, they have begun to use the idea of a parametric space to organize the network-supported activity.

For example, in the curricular context of linear functions in y = mx + b (slope-intercept) form, to help understand the roles of the "m" and "b," each group of students is assigned its own value of "b" (the y-intercept), which controls the starting point of the group's "mascot" in a "race." In this race they are asked to finish in a tie with all the other groups' mascots at a given time (six seconds) and position (twelve meters). They then must determine the velocity (and hence slope) that accomplishes this task for their given initial conditions. In submitting their solutions to a public display space the resulting aggregation of graphs has properties that none of the individual graphs have: A "star" of lines all meeting at the same place is created. When simulated characters associated with each of these lines are animated, the characters all come together at (6, 12) and, if continued for an additional two seconds, they then spread apart again in a kind of mathematical dance. In this way, the SimCalc Project has begun to use the mathematical idea of parametric space as both the "content" and the cognitive organizer for the network-supported learning activity to raise the level of organization of mathematical objects as well as the focus of mathematical attention in the classroom. For further specific examples, see Kaput & Hegedus (these proceedings).

Proof

In a related way, Andre Mack (2002) has begun significant network-based research related to how teachers' notions of mathematical proof structure and organize their real-time decision making about where to "go" with the classroom learning, especially as related to moments of mathematical uncertainty (e.g., when students ask questions that the teachers (and/or the students) are uncertain about). Just as moments of uncertainty in mathematics proper provoke a need to construct proofs, some idea of how valid reasoning "works" in mathematics animates teachers' organization of learning in their own classrooms. Teachers' notions of aesthetics for mathematical proof serve as an instructional resource for making sense of and validating (or disproving) student-generated mathematics claims. A wide range of understandings of mathematical proof is represented in Mack's investigations. What is important here is the way in which the mathematical idea of proof is used to organize his analysis of classroom learning. Much of this work moves well beyond traditional understandings of pedagogical content knowledge (Shulman, 1987) centered on the interaction of pedagogy (typically understood as *how* to teach) and content (understood as *what* gets taught), the content—proof in this case—*becomes* or is *enacted* as the pedagogy in moving to the group as the unit of learning analysis.

The network-supported activities bringing about significant instances of uncertainty in the classroom include participatory simulations (discussed next) where, for example, each student uses the arrow keys on a networked calculator to move an individual point around on his/her screen. At the same time, this point and all the others from the class move on a computer screen projected at the front of the class. The teacher asks the students to move according to a rule like "move until your y-value is two times your x value." When a line forms, the points are then aggregated and sent back to the students with the challenge using the calculator to find equivalent functions that go through these points. The student functions can then be collected and displayed (Wilensky & Stroup, 1999). Mack has found (2002) that often, in the course of this activity, students will submit functions where the teacher is unsure of their equivalence. The teacher then has to make decisions about how to proceed based on his/her notions of how proof works in mathematics. The trajectory of the group learning experience is structured by the teacher's ideas of how mathematics reasoning is carried out.

Complexity

In a similar way, content becomes pedagogy for the Participatory Simulations Project (Wilensky & Stroup, 1999). Participatory simulations are activities where learners act out the roles of individual system elements and then observe how the behavior of the system as a whole can emerge from their individual behaviors. These emergent results then become the focus of in-class discussion and analysis. Using network technology with a public display space, students can, for example, become agents in a population where a disease is introduced and be part of the system when the disease spreads. Or they can control a stoplight in a simulated city's traffic grid and work toward improving the traffic flow. Not only is dynamic systems modeling the content being introduced into the curriculum but also the learning itself is organized in terms of the classroom *becoming* the dynamic system. By assuming roles in a system, mathematical ideas like emergence, feedback, and complexity are literally embodied by the network-supported learning activity. The traffic control or "gridlock" activity helps to illustrate both how students already have ideas about how complex systems may work and how – in using this embodied learning activity – students develop significant insights and conjectures related to emergent phenomena like traffic.

When introduced to the gridlock activity, students are presented with the following scenario: The mayor of the City of Gridlock is unhappy with the traffic congestion in town and she has commissioned the class to improve the situation (Wilensky & Stroup, in review). The goal of the activity is for the students to find ways of optimizing traffic flow for the simulated city. The Gridlock activity has been run in a variety of settings, from middle school science classrooms, to secondary social science classrooms, undergraduate and graduate education classes, and at research conferences. The responses reported below are from one of these instances of running the activity in a class made up of twenty-four seventh graders taking science at a relatively low income, ethnically/racially diverse school located near Austin, Texas.

After the goal of the activity was introduced students are asked what they know about traffic flow. In response to the teacher asking, "What are some of the things that you guys listed that would be indications that traffic would be good or bad to you?," the students presented ideas that covered more than two chalk boards. Ideas ranged from "slow drivers" and "barrier walls too close for cars" to "too many cars" and "special events" like "crashes." The sense is that learners can articulate a wide range of factors that can impact complex phenomena like traffic. Some of these responses are behaviors of individual drivers and others are related to the structure of the roadways or the context for the behavior (e.g., lights too long). Taken collectively, these lists suggest that learners do have an initial appreciation of how agent behavior in an environment can have consequences for the emergent features of a complex system. For traffic this insight comes from their first-person experience. Part of the purpose of participatory simulations is to leverage the learners' first-person perspective in a way that can lead to more robust, incisive and powerful understandings of complex phenomena.

Next the students engage in the participatory simulation using the network. After a number of rounds of "play" starting with getting familiar with controlling the lights and then moving to improving the traffic flow, the teacher stopped the class and asked:

Teacher: Has anybody started to think of some ideas like what are you doing at your stop light that you think is working really well? Who has some ideas of why they think maybe their stoplight is working better than somebody else's stoplight. Anybody have any ideas? Anything that you're trying to do yet? Yeah (pointing to student).

Student 1: Letting one go and then the other go.

Teacher: Okay, so you're letting one go ... are you letting it go for a certain amount of time?

Student 1: Nah, just go this and that way.

Teacher: So you're just doing one then enter, two then enter....

Student 2: Pick a space between the line of cars to turn the light red, so that way, so then all the cars will be stuck together.

Teacher: So get the cars staggered and spaced out and pick a spot between ... yes, sir (pointing to a third student).

Student 3: Have it like every place where I saw the cars go like this (gesturing downward) and once they're done have them all go like that (gesturing to the side).

Teacher: So have the down ones green (gesturing downward with fingers extended) and then all the side-to-side ones green (similar gesturing to the side).

These are just some of the strategies students generate. The first strategy is simply alternating the lights at a regular interval. The second strategy looks for open spaces to shoot for in turning the light red in that direction. Other sections of classes at this school articulated what we call a "traffic cop" strategy where you simply look locally to see in which direction there are the most cars at your intersection, and then let that direction go (i.e., like what a traffic cop might do at a busy intersection). Some students discussed the possibility of developing "smart cars." With smart cars there might not be a need for lights, *if* a way could be found to "coordinate" with cars coming from the "other" direction. As is true for all the groups we have worked with, the Austin area students came up with a phase-related strategy for "synch-ing" the lights. The actual classroom exchange related to this strategy highlights some of the pedagogical challenges related to engaging and extending the developing strategies of students. These issues help point to possible future directions for developing the HubNet functionality and are under development.

Self-Organization

In a closely related line of research based on using participatory simulations (Wilensky & Stroup, 1999), Andy Hurford is looking at the learning in a network through a "self-organizing, critical systems" (SOCS) lens (1998). SOCS characterizes both the behavior of the simulated system and the learning that occurs in relation to *living in* that system. Bak and Chen (1991) provide an accessible real-life example of a SOCS and identify some general attributes of these systems. SOCS are: 1) sensitive to initial conditions; 2) scale invariant; 3) self-organizing; and 4) evolve in a way that may be characterized as a one-over-f distribution (Bak, & Chen, 1991, p. 48). SOCS are said to be "critical" because they alternate between relatively stable and relatively chaotic phases (Peterson, 2000). An application of SOCS theory to a learning system where learning was defined in terms of reorganization of individuals' conceptual structures (Hurford, 1998) argued that the first SOCS attribute, sensitivity to initial conditions, might be compared with the importance of prior knowledge in learning. That is, what one learns depends upon what the student knows– learning is sensitive to initial conditions. Conceptual reorganizations may occur in many sizes from small "ahas" to grand gestalt moments (Strike & Posner, 1992; Demastes, Good, & Peebles, 1996; Kuhn, 1996) and large reorganizations "look" just like small reorganizations except for their scale. Self-organization, "a process in which pattern at the global level of a system emerges solely from numerous interactions among the lower-level components of the system... executed using only local information, without reference to the global pattern" (Camazine, Denoubourg, Franks, Sneyd, Theraulaz, & Bonabeau, 2001, p. 8), is what happens as the learner interacts with the subject matter. In this analysis it is the learner's conceptual structures, evolving according to the learner's own sets of rules and in the individual's unique context that are self-organizing. The final attribute of a SOCS as discussed here is the notion of a one-over-f distribution, and here we can only assert that conceptual change may actually occur as a "continuous spectrum of learning events" (Hurford, 1998, p. 21) that may be characterized by the distribution function.

Although the above example is based on an investigation of individual learning, SOCS are scale-invariant, and as such, the model contends that learning should be self-similar across several orders of magnitude. Hence it seems reasonable to assume that learning in an individual and learning in small groups, classrooms, or school systems should share significant similarities. Beyond this attribute of self-similarity the fundamental assumption of this research is that learning can be profitably viewed as a SOCS, and all self-organizing critical systems share attributes as described above. The challenge now is to "watch" for these SOCS attributes in classrooms and to use that information to promote student learning. At least initially the justification for attempting to look at learning through a SOCS lens is based on a plausibility argument: In contrast to viewing a classroom as the simple linear summation of the individual learning units, it seems at least plausible that 1) the classroom is better understood as a complex dynamic system and 2) that a SOCS perspective might stand to "fit" as an account of this complexity. This "plausibility" argument is similar to that advanced at an early stage of theory development in other areas of cognitive science (Riesbeck & Schank, 1989; Kolodner, 1993). In now moving beyond the individual, SOCS rubrics and real-time computational tools are being developed for use with network-supported participatory simulations. The network functionality can carry out simultaneous analyses of student activity, such that it should be possible to look for SOCS signatures at various learning scales *while students are engaged* in network-mediated participatory simulations. Next generation connectivity can support new, real-time, analyses of student learning.

MS3 Overview

What all these projects have in common is a sense in which mathematics is more than content. In the social space of network-supported interactivity, mathematics is seen as structuring the classroom activity and the related learning. A way of situating this development is to say a "next step" may follow from accepting that mathematical and scientific learning is fully socially situated. The insight that learning is socially situated has – certainly in the last twenty years – brought to the table of learning analyses perspectives and approaches from disciplines as diverse as anthropology, sociology, ethnic studies, and critical theory. Methodologically, case studies and ethnography have become commonplace both in analyzing scientific activity itself, and in making sense of interactions in math and science classrooms. The inclusion of social analyses has heralded a new era of not just greater subtlety in the ways we look at learning, but also in what it means to learn. Learning has come to be understood as a form of participation in activities and processes much larger than individual comprehension. In a related way, the design of classroom activity is being increasingly situated relative to socially-derived frameworks and analyses. Many learning researchers now embrace the idea that cognitive structures or big ideas exist in, and as a result of, social activity.

What may follow as a next step in this analysis, then, is this: If mathematical ideas participate fully in the social space, they are not just *organized by* the social space they are also *organizing of* this social space. Just as the move to situating mathematics and science in the "larger" social plane brought new theoretical, methodological, and design perspectives to the table of understanding learning, so too the move to seeing mathematics as structuring of social space may invite to the table perspectives from, among others, formal mathematics, complexity theory, mathematical biology and computer science. Extending this MS3 idea to previously existing ideas of generative teaching and learning allows us to add new facets to activity design and learning theory, especially as they come to be supported by next generation networks.

Updating Generative Teaching and Learning from an MS3 Perspective

We begin to move from that part of the dialectic focused on using mathematics to structure the social space (2) to use mathematically informed ideas of space-creating play and dynamic structure as a way to move back to situating mathematics in the social sphere (1). In a sense, this analysis of generativity begins to situate these various projects relative to the dialectic of mathematics in a social space. To be sure, models of generative teaching and learning have been around for a while. Some aspects of space-creating activity and dynamic structure are nascent in previous analyses of generativity, but this paper is pushing to make these ideas more visible and more explicitly about designing for highly interactive group spaces.

Generative teaching as discussed by Wittrock is "a model of the teaching of comprehension and the learning of the types of relations that learners must construct between stored knowledge, memories of experience, and new information for comprehension to occur" (1991, p. 170). What Whittrock means by the learners' active construction of new "relations" is close to what we might call constructivist teaching pedagogy. Consequently, generative learning in his framework involves students' ability to create artifacts that embody their constructed understandings. In a closely related way researchers from the Learning Technology Center at Vanderbilt emphasize aspects of creating "shared environments that permit sustained exploration by students and teachers" in a manner that mirrors the kinds of problems, opportunities, and tools engaged by experts (1992, p. 78). Teaching involves "anchoring or situating instruction in meaningful, problem-solving contexts that allow one to simulate in the classroom some of the advantages of apprenticeship learning" (1992, p. 78). Extending these practices to the new functional capabilities associated with next generation networking may push us to update these existing ideas of generative teaching and learning. Although each of these previous theories is generative at the level of the individual learner (or even at the level of a small group) there is not enough of a picture of how to structure the cross individual or cross small-group learning. Using mathematically structured ideas to organize classroom learning helps us to augment these understandings of generative teaching and learning in ways that can be well-supported by next generation network capabilities. Moreover this update is intended to include a wider range of activities and allow for greater precision in saying what it means for learning and teaching to be considered "generative." Using an MS3 stance the ideas of *space-creating play* and *dynamic structure* are highlighted.

Space-Creating Play

In a technical sense mathematical "space" refers to, "An arbitrary collection of homogeneous objects (events, states, functions, figures, values of variables, etc.) between which there are relationships similar to the usual spatial relations..." (Union of International Associations, 2002). If students are asked to create and display functions that are the "same as 4x," they are generating a *space* of homogenous objects. In a network, this space is the result of students' play-full explorations of possibilities, using their individual computing devices, and then using the network to share one's own examples and learn from the examples of others. Worth emphasizing is the fact that play is not an "anything goes" state of affairs. Indeed, when considered carefully, "anything goes" is not usually a reasonable account of how play actually works for children:

The premise that Durkheim, Vygotsky, and Piaget share ...is that thinking and cognitive development involve participating in forms of social activity constituted by systems of shared rules that have to be grasped and voluntarily accepted. ... The system of rules serves, in fact, to constitute the play situation itself. In turn, these rules derive their force from the child's enjoyment of, and commitment to, the shared activity of the play-world (Nicolopoulou, 1993, p. 14).

Thinking and cognitive development related to play involve participation: the activity is structured by a "system of shared rules" that need to be "grasped" and "voluntarily accepted." The power of this form of activity comes precisely from the children's "enjoyment" and "commitment" related to being part of the "play-world." Generative teaching and learning as discussed earlier has had this space-creating play aspect, but it has been implicit in a way that has not utilized the "space" of potential mathematical behaviors and artifacts. The student-centered projects discussed earlier (Section 3.0) advance a sense in which learners "play" in an activity "constituted by systems of shared rules." In turn, this participation creates a space of objects or emergent behavior that embodies students' understandings of the mathematical or scientific content.

Dynamic Structure

When students in a networked space create and then display functions that are "the same as 4x," the structure of the activity is itself *lived* or brought into being by what the learners do. Unfortunately in mathematics research in particular, the idea of structure has come to be understood in a relatively static way. "Process-object" analyses of mathematics learning have tended to view "structure" as a relatively static kind of object. Social constructivists have tended to respond critically to this idea of structure.

When Soviet psychologists speak of the 'structure of an activity,' they have in mind something very different from what has come to be known as 'structuralism' in Western psychology (and mathematics education). The units are defined in terms of the function they fulfill rather than of any intrinsic properties they possess (Wertch, 1979, p. 19).

Dynamic structure is intended to point to a functional or operational sense of structure, not fixed or intrinsic attributes. A space of functions is created by the "4x" activity, and in a significant sense this space is animated by the learners' understanding of the mathematical ideas of equivalence. Their mathematical ideas are what *dynamically structure* their learning activity. This socially situated understanding of structure fits well with the ideas Soviet psychologists – and especially Vygotsky (1934/1962) – bring to learning. What may come as a surprise to some, however, is the sense in which this "lived" and larger-than-the individual meaning of dynamics structure may be exactly what Piaget (1970) was pointing to in suggesting constructs like the algebraic group could serve as the "prototype" of what he meant by cognitive structure. Structure, understood in this dynamic way, is a *patterning* or *coordination* in the kinds of operations on elements of the system. It is this larger dynamic sense of structure that allows Piaget to talk about group learning as "co-operation" (Montagnero & Maurice-Naville, 1997, p. 140) and may be what qualifies Piaget as the first major learning researcher to think of mathematics as structuring the social experience of coming to know. Viewed this way, there are striking parallels and forms of complementarity between Vygotsky's analyses of language and Piaget's analyses of operational thought that can be brought to the task of thinking about, and designing for, activity in classrooms supported by next generation networked functionality. Generative learning and teaching come to be understood as organized by space-creating play and dynamic structure.

Generative Teaching and Learning from a Socio-Cultural Perspective

This paper proposes that exploring the dialectic that exists between mathematics as a socially constructed domain and the domain-structured, concrete activity of teachers and students in math and science classrooms (MS3) provides researchers and educators an important framework for understanding the implications for design, pedagogy, notions of content, and research methods in classroom-based networks in particular, and classroom teaching and learning more generally. The projects reported on earlier represent important examples of the unique affordances of such networks and designs in that they highlight one aspect of the dialectic, namely the structuring of the classroom social space by mathematics or science proper. However, concomitant examination of these projects in terms of how mathematics is socially constructed by students and teachers engaged in sociocultural activity is necessary to extend the prior analyses of generativity within this framework.

Acceptance of the social construction of mathematics that results from the interactions of teachers and students with content, and with each other, affords a particular, powerful understanding of the nature of what is learned and taught in classrooms. It is also necessary to situate those interactions and the domain more explicitly and in more sophisticated ways within sociocultural activity to truly capture these complex phenomena. Many of the projects using networked classroom environments are focused on traditionally underserved students (e.g., students of color and/or living in poverty), but such things as historical notions of teacher, student, and content (Moll, 1990; Wells, 1999); cultural ways of knowing, interacting, and communicating (Gay, 2000; C. Lee & Smagorinsky, 2000); and the mediating function of language (Vygotsky, 1987; Wertsch, 1991) are under-specified. Thus, agency and participation; what kind of mathematics or science gets constructed, how, and by whom; and the mutually constitutive relation between structure and language need to be more fully elaborated. With this criticism we look to highlight opportunities to build on the important benefits inherent in a focus on the dialectic between mathematics or science as a domain that structures activity and the sociocultural activity through which mathematics or science is constructed. The work described herein does represent significant advancement relative to levels of agency and participation but in each we need to push again relative to situating mathematics in a richer sense of socio-cultural analysis.

Levels of Agency

As detailed earlier, students' opportunities to assume agency in these new classroom social spaces are markedly different from traditional classrooms. Certainly, to be a visible and necessary participant in public construction of knowledge entails a significant form of agency that has been largely missing in mathematics/science classrooms.

Recent work by Sarah Davis looks to make visible some important aspects of student and teacher agency. In the networked classroom, students can submit responses to be considered by the class without their identity being associated with that information. Both a sense of "that's me" and a "right level" of anonymity combine to give students new forms of agency and situated identity. Anonymity facilitates the ability to explore mathematical behaviors in a non-threatening way. Freed from *who* sent in a response, students are able to explore *what* the mathematical activity represents, whether it is sending in functions that are the same as 4x or controlling a traffic light in a simulation. Davis has found this anonymity helps advance both the students' and the teachers' sense of agency.

From the teacher's perspective: "It just promotes a lot of discussion and everybody's free to discuss it because kids can be criticizing an equation that they themselves wrote and nobody would know" (Davis, 2002). Students identify with their responses, icons, and data that show up in the group display. As can be seen in the next quote, in time the representation of self in relation to the group space can give the students a sense of how they are doing relative to the class as a whole.

Interviewer: Does that, in the other classes where you don't know how other people are doing (Student 12: Right), you don't know if you're the only one (Student 12: Right), does that raise your anxiety level any...?

Student 12: Yeah it's scary, because I think I'm the only one...I'm looking at my test, I think I'm the only one who got a 60 or whatever. And the couple of kids around me I'll know what they got but then I have no idea how anyone else is doing, because it's all privately done. Not that I need to know their test grades, but I'd like to know, how I stand. Am I the only one who needs help? And then you feel embarrassed to be the one raising your hand all the time, be the one staying after class because you think you're the only one. So, here, it's a lot more comfortable. You're not embarrassed in front of the other kids. (Davis, 2002)

The ability to gather responses on all questions from all students gives important knowledge to the teacher. That knowledge then gives the teacher options for how to proceed in class. And again from the teacher's perspective:

It's great to know, where the kids are, actually it's not always great because sometimes it's pretty depressing to see where the kids are. There was something I did this year in one of my classes and I asked if there were any—I thought I had done a fine job—I asked if there were any questions, nobody had any questions and I just had an inkling, And I said okay well log on and lets check. And I believe two kids got it right so obviously they didn't have a clue what they were doing and I went back and re-taught. (Davis, 2002)

Agency here includes opportunities for both the teacher and students to anonymously reflect on and situate their respective states of understanding. Understanding centers both on the construction of content as such and, just as importantly, on "where everybody is" individually and collectively.

As significant an improvement as all these network-supported forms of interaction are—where historical notions of teachers' and students' relation to content are challenged—historical notions of content itself are still being maintained, tacitly, in the work to date. One might observe that an under-examined, embedded assumption is that math is homogeneous, monolithic; there is no ambiguity in what math is. The very notion that mathematics is socially constructed, especially given attention to the concrete, localized construction processes in classrooms, problematizes that homogeneity or lack of ambiguity. Viewed in this way, even an updated notion of generativity can become an aspect of classroom activity that serves only to lead to a predetermined outcome, rather than something that produces potential insights and growth possibilities for both actors *and* content. Generativity needs to be pushed to have us become more open to the possibility that what it is that ends up mattering to a group, and how this comes to be understood, might not have a greater point of comparison other than that of the group's co-constructed agency.

Participation

Careful consideration of what kind of participation in the construction of what kind of content is important so that classroom learning is generative in the sense that students not only learn in ways that foster powerful, dynamic understandings of mathematics, but that they develop knowledge and skills that foster their successful participation in mathematics in the larger world. If one of the aims of these projects is to open mathematics and mathematical reasoning to students, particularly students who have traditionally been under-served, in more powerful ways, then questions about the nature of the sociocultural activity from which such knowledge emerges are critical. Whose community, culture, history is the focus of the design activities (e.g., PartSim's elevators, traffic gridlock)? What connections to students' lives—lives that are situated in social, cultural, historical, political arenas—do design activities include? If the goal is design of generative activities, how do they facilitate students' development of meaningful mathematics that can help them take action on their world? Linguistic and cultural diversity in classrooms push notions of participation, as well as historical notions of student and content. (O. Lee's (1999, 2000) work in inquiry-based teaching is germane in science education.) What kinds of considerations need to be made in inviting the contributions of, for example, English as a second language learners as legitimate participants whose language and culturally grounded knowledge are viable, important resources for assessing proof (for example), and for teaching and learning?

Further, attention to the fundamental influences of culture and language on learning (similar to Kaput's work (Kaput & Shaffer, in press) on externalized manipulation of formal systems changing the very nature of cognitive activity) may make visible heretofore unexamined notions of appropriate interaction and participation in the new social spaces created within networked classroom environments.

And finally, relative to developing more culturally situated notions of participation, a somewhat uncritical focus seems to be on mathematics as a universal language. Treating language as a mediating tool through which structures emerge poses interesting challenges to universality. To explicate what is truly meant by claims that mathematics is constructed through sociocultural activity, the mutually constitutive relation between the language of mathematics and the structure of classroom activity, as well as between the language of the classroom and the structuring of mathematics, must be explicated more fully. In addition, in networked classrooms, the expansion of language through the production of new artifacts (e.g., real-time public displays of jointly constructed knowledge) and the nature of the conversations that have public construction and display as their focus offer unique glimpses into the development of mathematical knowledge and reasoning. They also have potential to invite diverse modes of communication and interaction into the process. Careful examination of the new avenues for socially and culturally specific communication and interaction can optimize the generative potential of the new social spaces created in networked classroom environments.

To be fair, while a critique of mathematics education as acultural, ahistorical, or apolitical is possible, critiques of sociocultural theoretical frameworks for ignoring content are also important in extending prior analyses, as we begin to do in this paper. Socioculturalists have helped researchers and educators focus on the reality of the social construction of mathematics, however, they have done less well in attending carefully to the dialectic proposed here because of a disconnection from content. There has been (with some exceptions, e.g., Gauvain's (1998) work on cognition being shaped over time by various number systems) one-way attention to the fundamental connections between activity and learning, looking more at activity structuring learning rather than also looking at domains structuring learning through activity. Thus, the dialectic framework proposed here has potential to enrich both sociocultural research and theory and mathematics education by bringing them into conversation with each other.

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THE ROLE AND NATURE OF SYMBOLIC REASONING IN SECONDARY SCHOOL AND EARLY COLLEGE MATHEMATICS

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Activity for this Discussion Group will center on framing ways to think about the role and nature of symbolic reasoning in secondary school and early college mathematics. Perspectives on this issue will be through presentations during the first session and through papers shared prior to the meeting. One possible goal for the group will be the production of a collection of more refined papers on issues such as the nature of and relationships among symbol sense, symbolic reasoning, symbolic manipulation, symbolic insight, and symbolic disposition.

The advent of CAS calculators in a growing number of schools, the international attention (Australia, Austria, France, England, etc.) being focused on CAS in schools, and the recent attention to CAS issues by the National Council of Teachers of Mathematics are factors that contribute to a growing need to develop a better understanding of the nature and role of by-hand and technology-assisted symbolic work. Past attention to symbolic work had focused on development of symbolizing in younger children (Cobb, Yackel, & McClain, 2000) and on beginning to define what might be meant by symbol sense (Arcavi, 1994; Keller, 1993/1994; Pimm, 1995). This symbolic reasoning discussion group is a follow-up on a very small invited conference on symbolic reasoning held several years ago in Bethany Beach, Delaware. The Bethany Conference focused on symbolic work at the high school and early college levels.

The Bethany Conference was assembled to consider the role and nature of symbol use in school and early college mathematics. Participants included mathematics and mathematics education faculty and doctoral students from Penn State, University of Iowa, University of Delaware, and Iowa State University. That conference generated questions and sought clarification on the issue of the nature of and relationships among the following: symbol sense, symbolic reasoning, symbolic manipulation, symbolic insight, and symbolic disposition. Questions that were raised include:

- What is the role of symbolic manipulation programs in the development of an ability to work with algebraic symbols? What new understandings of symbols will be required in a CAS-intensive mathematics classroom?
- What are the relationships between conceptual understanding and symbol sense, symbolic reasoning, symbolic manipulation, symbolic insight, and symbolic disposition?

- What is the impact of work with symbols on the disposition to verify mathematical claims symbolically?
- What is the impact of particular symbol systems on the development of symbol sense?
- To what extent can the development of symbol sense be informed by the metaphor of enculturation into a language?

Some Thoughts on Key Questions

Understanding and use of algebraic symbols is a topic of prime importance in mathematics education, and one for which essential questions remain unanswered. In this section, we make some observations from extant research literature, point to issues raised in this literature, and raise some questions that may be important in furthering our understanding of work with symbols in mathematics. Our observations are organized around the aforementioned list of questions.

What is the role of symbolic manipulation programs in the development of an ability to work with algebraic symbols? What new understandings of symbols will be required in a CAS-intensive mathematics classroom?

Constructs pertinent to the role of symbolic manipulation programs include the white box/black box principle and how it may be used to think about the relationship of students' use of CAS to the didactic approach taken in the classroom (cf. Heugl, 1997). This immediately raises the question of how and to what extent by-hand symbolic manipulation contributes to students' understanding of symbols. One perspective on how this might be addressed is the issue of technique, as discussed by Lagrange (1999) and others.

What are the relationships between conceptual understanding and symbol sense, symbolic reasoning, symbolic manipulation, symbolic insight, and symbolic disposition?

Herscovics and Linchevski (1994) observed that students without formal instruction in algebra (as in symbolic manipulation) used strategies based on inverse operations to solve equations in one unknown. The students, however, abandoned manipulations of the equations and resorted to guess-and-test strategies when the equations involved two unknowns. Herscovics and Linchevski connected this work with prior work on the meaning of the equal sign. It seems to us however that this work also suggests that students' sense of a symbolic item is related to their use of symbolic manipulations.

Mason (1980) described the conjecturing process in mathematics in terms of three components. It begins with doing particular examples that one "can manipulate easily while part of [one's] attention remains focused on [one's] primary goal" (p. 10). One then possibly calls on diagrams or metaphors as one tries to get a sense of the underlying pattern or relationship. Lastly, one continually articulates and refines the pattern. Mason's notion of starting with manageable examples seems to underlie the task structures suggested by Drijvers (in press). The structure of Drijvers' CAS classroom examples with systems of equations involves approximately two questions similar in structure but with different numbers. The last part of the task then moves beyond specific numbers and into the use of variables. This structure seems to parallel Mason's notion of starting with the more manageable and then changing focus to the underlying pattern. Classroom tasks such as these may capture the essence of Mason's point that students need supports that give meaning to symbolic forms. We also raise the question of to what extent the specific, manageable instances should be examples done by hand or by head, in the absence of CAS or other supporting technology.

An understanding of work with symbols requires understanding what it means to be fluent in the use of symbols, and such fluency would involve understanding the relationships among conceptual understanding, symbol sense, symbolic reasoning, symbolic manipulation, symbolic insight, and symbolic disposition. One way researchers have observed students' fluency with symbols is to study how students use symbols in the solution of non-routine problems. Two decades ago, Rosnick (1982) observed that participants in his study had an inclination to use algebraic symbols as labels for "broad, undifferentiated concepts." He remarked on their abilities to shift among different interpretations of a symbol within their work on a single task.

In her study of college calculus students' nonroutine problem solving, Kinzel (2000) added to the conversation about fluency with algebraic symbols when she characterized students' work with symbols in her study of the interpretation and use of algebraic notation by college calculus students. She observed several categories of factors that typified students' use and interpretation of symbols. First, she observed that a key factor in students' work with algebraic notation was the extent to which students attended to the definitions of the symbols with which they worked. For example, one of the participants in Kinzel's study shifted among interpretations of symbolic expressions; her loose definition of *d* allowed her to use *d* to represent "number of days" and simultaneously to use $\frac{20}{d}$ as an abbreviation for $\frac{\text{20 miles}}{\text{1 day}}$. Kinzel pointed out that "Such conflict obscures the mathematical relationships between the quantities and thus compromises her [the student's] representation of those relationships." (p. 333). However, such loose attention to definition of symbols does not always compromise the mathematical meaning of the symbols, however, as one participant in Kinzel's study seemed to operate from a loose definition of symbols while maintaining an awareness of the verbal interpretation of the formula. Another of Kinzel's categories of factors centered on the influence of students' attention to symbolic manipulation. Although students may have paid careful attention to defining particular notation when they first introduced it, they lost sight of that meaning once they became involved in symbolic manipulation and changed the meaning of the symbols to suit the manipulation, At times, students' intention to use particular symbolic manipulations influenced the form of their representations. To Kinzel, key to fluency with algebraic symbols was attending to and integrating the selection, manipulation, and interpretation of algebraic notation. We believe that the understanding of the nature of this balance is crucial to understanding the nature of symbolic work in school and college mathematics.

What is the impact of work with symbols on the disposition to verify mathematical claims symbolically?

Heid, Blume, Iseri, Flanagan, Kerr, and Marshall (1997) observed juniors and seniors in college as they solved non-routine problems. The researchers observed a tendency among three of the four target students to reason toward symbolic representations but not to reason from them. Seeming to view the production of a symbolic representation as a terminal goal, students resorted to the numerical or graphical forms as representations from which to reason.

In summarizing the role of visualization in doing mathematics, Noss, Healy and Hoyles (1997) drew on Hadamard (1945) for evidence of the absence of typical symbols in mathematicians' initial work with problems. Healy's group noted the importance of the mathematicians' skill in moving between the formal symbolic work and more informal modes of thinking. Healy and colleagues paired observations of mathematicians with the findings of Hillel, Kieran, and Gurtner (1989). The resulting stance contended that students prefer not to draw on mathematical knowledge when they can reason by perceptions when diagrams are present. Healy and colleagues raised the question of what happens when microworlds are used. We would tend to ask the related question, how might computer algebra systems and other computer environments be used to challenge students' non-symbolic tendencies while still respecting the potential for non-symbolic work in the initial stages of problem solving and of learning mathematics.

To what extent can the development of symbol sense be informed by the metaphor of enculturation into a language?

If mathematics is a language, the symbols used may be the written word. We not only raise the question of how individuals construct and interpret mathematical symbols but we also ask to what extent this question may be informed by drawing on research and theory in reading and foreign language education. Constructs in these fields such as encoding and decoding capture the change of thoughts into written forms and conversely from written word into thought. There seem to be parallels to these in work with mathematical symbols. For example, Kirshner (1989) concluded students depended on cues in the surface features (e.g., spacing between individual symbols) of ordinary notation in order to evaluate algebraic expressions. Ericksen (1991) noted that students saw more surface-level features but struggled to recognize deep patterns in algebraic expressions, sentences and sequences. She raised questions about how what is obvious to the mathematically experienced is not obvious to the learner. Ericksen also questioned the role of conceptual understanding in seeing structures. English and Sharry (1996) also questioned how students become able to identify relationships and to unpack symbolic expressions. In making the connection with the language metaphor, we ask questions about the extent to which research on individuals' comprehensive of Chinese symbols for native and second-language speakers may inform how learners and more mathematically experienced individuals perceive and give meaning to strands of mathematical symbols.

The Work of the Discussion Group

This PME-NA Discussion Group seeks to expand the conversation about symbolic work to a larger pool of researchers and to develop collaborative papers addressing the concepts of symbol sense, symbolic reasoning, symbolic manipulation, symbolic insight, and symbolic disposition. Our goal would be the collaborative production of a series of papers for consideration as a special issue of the *International Journal on Computer Algebra in Mathematics Education* (IJCAME).

Invited short talks will be used as a vehicle to begin the conversation, addressing issues related to the following: symbol sense, symbolic reasoning, symbolic manipulation, symbolic insight, and symbolic disposition. In addition to the invited talks, Discussion-Group participants may submit additional papers addressing issues related to the work of this discussion group to post on a web site. To obtain the URL for the web site, participants should contact M. Kathleen Heid at mkh2@psu.edu. Papers should be sent to rmz101@psu.edu. An initial goal of the discussion group is to define topics and provide possible frameworks for potential papers and to form groups of researchers who could meet and work in the intervening year. Subgroups may communicate electronically and meet at the Annual NCTM Meeting, at the AMS-MAA Joint Meetings, at the annual AERA meeting, and/or at the MAA MathFest, so that drafts of papers on the topic could be ready for critique by the group at the 2003 PME-NA meeting. As drafts are posted, a list-serve will be created for this group and discussion of the papers would be ongoing. Those drafts would be revised during the 2003-4 academic year and would be prepared for submission to IJCAME soon thereafter.

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Advanced Mathematical Thinking

SOME CHARACTERISTICS OF MENTAL REPRESENTATIONS OF EXPONENTIAL FUNCTIONS – A CASE STUDY WITH PROSPECTIVE TEACHERS

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Starting point is the hypothesis that spontaneous retrieval of knowledge graphs (= concept maps) entails relevant information on the organization, abstractness and the accessibility of mental representations of mathematical objects and allows conclusions on the structure of human long-term memory. In the following case study knowledge graphs on exponential functions from prospective teachers in their third academic year were spontaneously created and recorded, then concluded with interviews in which predominantly the steps of creating the graphic representations were retrieved. Comparison of the concept maps with the interview data revealed structures of the generation process of the knowledge maps, pointing to the meaning creation process of the original mathematical terms.

Objectives and Theoretical Framework

The fact that for the past 10 years – declared by the Congress of the United States as the 'Decade of the Brain' – cognitive science has been 'beginning to understand how the brain works and how it gives rise to the mind' (Kosslyn &, Koenig, 1995) is not least due to the contributions that the rapidly developing discipline of cognitive neuroscience has been making to this field. As the central concern of didactics is describing, understanding and influencing cognitive processes, we believe that theoretical cognitive research (Sigel, 1999), also in the framework of mathematics didactics, cannot ignore the 'neuronal dimension'. As this 'hardware level' is not accessible for classroom research, we believe that at least the 'software level' should be given more attention. This central research dimension entails, amongst other aspects, the question of the *mental representation* of mathematical contents (Davis, 1987).

Our starting point, which brings mental representations immediately into play, is the basic hypothesis that the data collected in our observations entail more or less clear traces of active independent mental constructs from the long-time memory of our research subjects (prospective teachers). If learning is linked to mental rearrangement processes, then we can assume that a 'new order' is based on subjective consistent logic at least partially shaped or even controlled by these learning processes. Of course, autonomous processes may also have influenced rearrangement results. Our research interest focuses on these autonomous processes. Which retrieval processes are initiated under spontaneous access appears to us to be an open research question.

The number of contributions in mathematics education on mental representations has reached a two-figured number, but a uniform definition for mental representations is not in view. After all, context-dependent (see Ball & Bass, 2000) as well as subjective experiences within and outside of mathematics (science, technology, everyday life etc.), which constitute the basis for mental representations, increase the difficulties in striving for a uniform approach. In the last instance individual research questions determine the favored model. Whereas Confrey (1991) views representations more as microstructures, we are more interested in the macrostructure aspect (cf. Pines, 1985). In agreement to this last approach we attempt to understand mental representations of mathematical knowledge as 'network structures' (directed or non-directed graphs) of knowledge that individually guide human information processing (Hasemann &, Mansfield, 1995; Williams, 1998).

Thus, our data gathering process focuses on what may be described in terms of graph nodes together with vertices producing relations between these nodes. We thereby adopt the tool of concept maps developed by Novak & Gowin (1984) (also known in the literature as conceptual graphs), whereby we linguistically prefer the term 'knowledge graphs' (Zwanefeld, 2000). Novak & Gowin wanted to have an external representation of the way people store information in their minds. In the present study, a knowledge graph is viewed as a structured representation of acquired and (by the investigation) retrieved mathematical knowledge and skills. In the German academic world similar graphic representations (however with more limiting directives) are described as 'mind maps'.

It is clear to us that the structures described here as mental representations are scientific constructs, i.e. models with certain inherent basic assumptions. A description can furthermore only be undertaken indirectly, as we conclude the original representation from the information reproduction. It would be of great interest here to gain data from nuclear magnetic resonance tomography. The quality criteria for our approach are *adequacy* and *viability* of the chosen model, which subsequently justifies the approach or rejects it as unsuitable.

We will, however, extend this static 'two-dimensional' graph structure by a dynamic 'third dimension', namely time. Just as informative as the structure of the graph (the mind map) appears to us the process of its (re-)production in the interview context (when creating the mind map). Indeed our observations point to the assumption that the elements of an individual graph (a person's mind map) belong to different (background-) levels, and that these come into the foreground step by step, i.e. enter the consciousness level of the individual (generic aspect). It is hereby interesting to note whether, and if so, which patterns recognizably appear in these processes.

The chosen theme of *exponential functions* appears to us to be particularly suitable, and even to be of a high degree of relational content for this purpose, and to be an 'ideal critical research site' (Törner, 2001). If we understand the term *exponential function* as the 'conceptual root' of the mind map – as an 'evocational starting point' initiating an individual (re-)production process of a complex knowledge-network –, then this reproduction stemming from the various nodes and vertices – so our hypothesis – follows certain 'priority rules'. Such rules have already been postulated by Green (1971) in relation to beliefs, where he discusses the functional-generative categories *'primary versus derivative' beliefs* as well as the ego involvement categories *'central versus peripheral' beliefs*.

Our research objectives within the context of mathematical knowledge can thus be summarized into the following research questions: *What are observable representations? What are patterns of representation* on the background of the reproduction (retrieval) processes? How is then human long-term memory retrieved? It hereby remains an open question insofar this retrieval process is controlled by an object-independent retrieval program, or already contains context-dependent information packages which are unpacked when spontaneously activated (comparable to the memory of a zip file). Through our observation window the arguments seem to point more to the first alternative. The contributions from Sigel (1999) are partially devoted to related questions; only one contribution, however, is specific to mathematics (Lesh, 1999).

Methods of Inquiry – Data Sources

Our data sampling integrates two corresponding data sources from voluntary test persons: *knowledge graphs* and *open interviews* (45–60 minutes, video-taped) on the theme of exponential functions. The survey was undertaken in 2001. Both authors jointly – in the form of a mutual 'peer debriefing' – did the work of data collection and evaluation. The interview partners were prospective teachers (upper secondary grades) in their third university year, so that one can principally assume they possess sufficient familiarity with exponential functions. We would like to point out that that in Germany it is obligatory for a prospective teacher to study two school subjects at the same level and scope. We will call the 6 students (2. academic discipline in brackets) Andrew (social sciences), Berta (chemistry), Chris (chemistry), David (physics), Ed (physics) and Fred (history).

An open interview style was chosen and carefully designed to avoid the impression of an examination situation. The interview participants were familiar with the concept-mapping method. We presented them for a short time a graphic representation of the Pythagoras theorem. We deliberately did not inform them of the theme of this survey beforehand. After revealing this theme to the students, they then had 10 minutes time to produce (unobserved) a knowledge graph on our theme in question (cf. appendix). As we were concerned with the macro-structuring of this subject matter, we deliberately did not give the candidates ample time to produce their knowledge graphs. We wanted to ensure that each candidate would be able to reconstruct the steps of the knowledge graph generation in the subsequent interview. Moreover, the produced graphic representations would have become too complex and thus their spontaneous quality would have been suppressed.

Following this, in the videotaped interviews the students were questioned in detail on the creation of their knowledge graphs (cf. the numbering of the nodes in the concept maps). We specifically questioned the students about the aspects: definition and application of exponential functions, their relations to logarithm functions, and their graphic representations – had they not commented on these aspects themselves.

Results

The results presented here focus on the discussion of the graphic representations as a data source; we only marginally refer to the detailed transcriptions of the interviews. It would go beyond the scope of this paper to discuss the complementary, however not contrary, interview results here. We agreed in future to relate the name of the student to the name of the graphic representation produced by that person, as long as this does not lead to any misunderstandings. The mind maps of the six interviewees possess quite a broad spectrum of content richness and structural complexity standing in a certain correlation to the other subjects being studied by the individual students for their teacher degrees. The spectrum reaches from elaborated mind maps (chemistry [Berta], physics [Chris], [Davis]) across reasonably elaborated ones (physics [Ed], social science [Andrew]) down to less elaborated ones (history [Fred]).

As we are interested in spontaneous retrieval processes, we assumed that the global 'locating' of the initially unknown theme is of central relevance.

The Affective Dimension

Immediately after presenting the theme to the students we asked them how they would affectively categorize the theme: Andrew = rather rejective, Berta = positively loaded, Chris = neutral, Davis = neutral, Ed = "feel insecure" und Fred = neutral, "not very secure". It is surprising how reserved the many different facets this central theme in the academic curriculum are expressed. In particular the students Andrew and Fred, who both cannot relate this theme to their second academic discipline, reacted more rejectively than not.

The 'home-localization' of the Theme

The spontaneous subject-specific localization of the theme plays a key role, as the following knowledge maps demonstrate. We have to assume that relating a theme to mental information fields activates different retrieval processes. Illuminating was the short question asked in response to our question in the interview of Berta: "May I also use physics examples?" Sources of subjective localization are rather implicit information from the maps as well as verbal statements (e.g., quotations) from the interview. The following table lists these data.

home...	Andrew	Berta	Chris	David	Ed	Fred
map	mathe-matics, calculus	mathe-matics, functional equations	processes of growth in physics	physics	mathe-matics, functions	Mathe-matics, Eulerian number
inter-view	mathe-matics	coaching maths stu-dents	everywhere in physics	physics	mathe-matics	mathe-matics

When we speak of home localization we implicitly define in a dual mode a so-called outer location which is not necessarily explicitly mentioned but is nonetheless exactly for this reason present; Fred for example does not like to address aspects from physics: "I am absolutely not very good at physics!". The inner-outer view (Green employs the terms central and peripheral) underlines the relevance of the home localization, which could be confirmed by the interview statements.

Around the Mathematical World

It has been noted by a number of authors that beyond the dynamics of the conceptual network of mathematics there is a world of stabilized expectations and beliefs which deeply influence the reception as well the use of mathematical and scientific knowledge (see e.g., Fishbein, 1987). Thus it is fundamental to identify these intuitive forces and to take them into account in analyzing the pattern of the retrieval process. The mathematical worldview of a physics-oriented student will perceive mathematics in a trivial fashion differently than a "pure" mathematician. In particular, mathematical schemata and systematics such as definition, theorem and lemma appear to fade away.

The Pattern of Retrieval

The present six knowledge graphs point to a few retrieval heuristics, which are nonetheless explainable. Basically, when drawing a graph one is generating another node, thus one is probing in a suitable direction towards a new aspect. One of the fundamental principles is the 'variation of a constant' - to employ a mathematical principle as a metaphor. A number of approaches appear to be obvious here: First of all there are the completely mathematically neutral classical variation principles: from the *specific to the general* (David) as well as from the *general to the specific.* (Ed). Here switching backwards and forwards between the exponential functions (logarithm function) with a random basis a and the classical exponential functions (natural logarithm function) occurs. Alongside these heuristics one can also list variations on the linguistic level when searching for suitable associations (cf. Fred): Here the theme exponential function is also varied over the letter e, synonymous for the Eulerian number. One has the impression of a rather helpless non-directed search. Fred does not leave his mathematics home location; a fruitful link to his parallel academic discipline history cannot occur for content reasons.

More successful, as it is more mathematics-specific, is the *variation on the linguistic level.* Here term representations undergo variations, and formula representations are manipulated. The *variation on the conceptual level* can be viewed as a further development of the aforementioned principle; here we are dealing with associations that can also be understood as dualisms: term representation, graph representation, function, inverse function. Our interviewees represented their knowledge contents – at least in the context of the exponential function – often very clearly in the form of dualisms (graph - term, function - inverse function, application - theory, figurative - formal, approximately - precise, reality - school, learn - forget, know it - look it up). Notably, even dualisms that are in fact symmetric (and when asked, the interviewees are aware of this symmetry) are nonetheless represented as a rule in a hierarchical form. Thus logarithm functions are conceptualized as inverse functions of exponential functions, but not vice-versa. The opposite direction is felt to be 'unnatural": logarithm functions, perceived as 'complicated', are 'reconstructed" from the simpler concept of exponential function by inversion. Other dualisms also appear to be conceptualized as dichotomies with inherent assessment in the forms: 'simple-difficult', 'important–unimportant', 'good–bad' or similar. We tend to interpret these observations in such a manner that representation (even) of mathematical knowledge is 'emotionally charged' (cf. central versus peripheral mathematical beliefs).

Dominance of context-relevant aspects

(Aspect: anti-didactic, objective-logical nature of mathematics.)

Characteristic for the (mathematical!) definition of the exponential function is that it fulfills the differential equation f' = f. The identity of f to its differentiation is insofar constitutive and unique for this function. It is surprising that this central differentiation aspect in the mind maps is allocated only secondary priority - a tendency more or less strongly supported in the interviews. The same can be said about the interviews, in which conscious mentioning of the f'=f-property (if at all) occurs at a very late stage. We explain this fact by noting that the systematics of a function discussion mentions the f'=f-property later, and this consequently influences the retrieval process. The logical nature of mathematics has insofar an anti-didactic effect here.

Door-opener resp. dead end

We would just like to shortly mention this aspect. Individual nodes of the graphs point to a high networking index, which strikes one's attention immediately. They are equally starting points for (interdisciplinary) associations in various directions, for example the aspect of exponential growth links to physics (see Andrew, Chris, David, Ed), to economics (see Andrew, Berta), to chemistry (see Berta), to geography (see

Chris) and to biology (see Chris). On the other hand there appear to be nodes that can be called 'dead ends', when for example Fred reflects the theme domain under the didactic aspect of how to teach it in schools.

Aspect: Piggyback

Sigel (1999) throws up the question: 'Does emotion involve a separate representational system or does piggyback affect onto other systems?' We believe in the productive nature of this piggyback aspect: Our interviews reveal numerous instances of colorful emotionally affected details.

Conclusions

It is apparent to our authors that the observations made here cannot be uncritically generalized without limitations. On the other hand they agree to preceding implicit observations and feature patterns of generalized structures. We consider home localization, which occurs quite spontaneously, to be of decisive importance. The home domain functions as a reference level to which the interviewee always finds his way back, and which also controls retrieval processes to a great extent. In the case of the interviewees, those students akin only to mathematics presented only limited knowledge maps, whereas those students who were science oriented presented more multifaceted aspects. The networking of science contents with mathematical contents and viewpoints, however, had its deficiencies. We have to assume that interdisciplinary linking of content is not automatically executed by the learner but has to be explicitly taught. Moreover, home localization determines the sense-making of mathematical objects und terms (differentiation, growth etc.). It seems as if the university course for mathematics does not take this aspect sufficiently into consideration. Spontaneous reporting of memorized information from the long-term memory does not seem to be determined by the primarily requested content, as if one were to expand a stuffit file. On the contrary: 'home`-specific routines seem to considerably influence retrieval. Home localization of a theme is insofar a key factor; decisive for home localization is in particular the domain that guarantees sense-making of the mathematical object.

Our observations reveal that the classical way of imparting mathematical knowledge (lectures) does not guarantee the emergence (or construction) of a relevant and sense-making 'mental network of mathematical knowledge' in our students – at least in the context of exponential functions. It seems that natural science subjects (chemistry, physics) are allocated specific functionalities here that (regretfully) cannot be mediated solely within mathematics studies. The effects of other subjects upon mathematical knowledge that we observed (a neglected aspect in research) appear to be a rather ambivalent but nonetheless relevant phenomenon for the development of mental representations of mathematical knowledge, which we would like to investigate in further going studies.

From a methodological point of view our survey has demonstrated that research into mind maps only allows correct analyses if in the research context also the production process of each map is illuminated, for instance by a subsequent interview. The mere graph structure of a mind map often leads to improper results, as 'nearness on the paper' may conceal the actual (chronological) distance of mind map elements. While the static graph structure essentially depicts the 'stored representation' of knowledge, the interview offers insight into the dynamics of the 'working memory'.

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THE OPERATION TABLE AS METAPHOR IN LEARNING ABSTRACT ALGEBRA

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This paper presents an analysis of the ways that one particular student used operation tables to support her reasoning about groups, subgroups, binary operations, and their properties. Based on interviews in the context of an undergraduate course in abstract algebra, the student's reasoning processes were external and based in the operation table, so that in large measure the operation table *was* the group for this student and, to a lesser extent, for other students as well. This result provides insight into the role of multiple representations in supporting the emergence of abstract mathematical objects in students' thinking.

Introduction

Among the growing research literature in undergraduate mathematics education, there is comparatively little attention to the learning of abstract algebra. This paper presents selected results from a dissertation (Findell, 2001) on learning and understanding in abstract algebra. To complement to the extant literature describing action, process, and object conceptions of topics in elementary group theory (e.g., Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997; Brown, DeVries, Dubinsky, & Thomas, 1997; Dubinsky, Dautermann, Leron, & Zazkis, 1994; and related literature describing difficulties with particular concepts in group theory (Hannah, 2000; Hazzan & Leron, 1996; Leron, Hazzan, & Zazkis, 1995; Zazkis, Dubinsky, & Dautermann, 1996; Zazkis & Dubinsky, 1996), the study sought to identify prominent features and components of students' concept images (Tall & Vinner, 1981) in the context of an undergraduate course in abstract algebra.

Analysis of interviews and written work from five students provided insight into their concept images of topics in elementary group theory, revealing ways they understood the concepts. In attempting to describe and explain the data, well-known theoretical constructs seemed insufficient. Thus, analysis became a process of theory generation, and much of the conceptual and analytic framework emerged during the analysis through use of the constant comparative method (Cobb & Bauersfeld, 1995; Glaser, 1992; Glaser & Strauss, 1967). Because many salient episodes in the data involved seemingly idiosyncratic uses of language and notation, the analysis was essentially semiotic, using the students' linguistic, notational, and representational distinctions to infer the students' conceptual understandings and the distinctions they were and were not making among concepts. The perspective on semiotics was based in the work of Peirce (1955) as well as recent contributions by Sfard (2000) and others (e.g., Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Pimm, 1995). The framework also took advantage of the distinction between process and object conceptions (e.g., Dubinsky, 1991; Sfard & Linchevski, 1994) as well as various perspectives on abstraction and generalization (Dienes, 1961; Frorer, Hazzan, & Manes, 1997; Hazzan, 1999; Wilensky, 1991) and the metaphorical nature of mathematical thinking (Lakoff & Núñez, 1997, 2000). As the conceptual and analytical framework emerged, the data were synthesized under two themes: making distinctions and managing abstraction. This paper discusses the latter theme, focusing in particular on the ways that students used operation tables to manage abstraction.

Analysis

This research report concentrates on one student, Wendy, whose reasoning was particularly tied to operation tables. All interview excerpts are taken from the first of four interviews with Wendy. The guiding question for this interview was, "Is Z_3 a subgroup of Z_6?" (see Hazzan & Leron, 1996). Before presenting data and analysis, I discuss Wendy's use of language, which bears on the interpretation of the data presented.

Wendy often misused words, saying one word while meaning another, such as multiplication for addition, identity for inverse, or associativity for commutativity. Sometimes she corrected herself immediately, other times she corrected herself when I asked for clarification, and occasionally her linguistic missteps suggested deeper conceptual difficulties. In any case, such slips-of-the-tongue were shown to indicate strong connections among the concepts that the words represented (see Findell, 2001). Because slips-of-the-tongue are not a focus of this paper, I use brackets in the transcripts to indicate the word that she likely meant.

What is the operation in Z_6?

As Wendy began the interview, she was uncertain of the meaning of Z_6 but surmised that it was "integers mod 6" (line 11). She wanted to create "a total table" (line 13) and then realized that she would need to determine the operation. She discussed both addition and multiplication as possibilities and began constructing a table for multiplication in Z_6 (see Figure 1).

Wendy: Okay. Well, Z_6 is not going to be, when I start with my chart, and I do the first row, 0 times any element is going to equal 0, so if you look at that.... Actually, okay. Let me just.... It's not going to have.... You have to.... I'll just finish it [the row]. Okay, now it has to hold four properties to be a group. Let's write these down. It has to have an identity, an inverse, it has to be closed, and it has to be associative, which we're going to leave for last. [Laughs] (line 20)

Wendy: If you look at this row [the "1 row"], you multiply.... If I.... I am calling 1 the identity. If you multiply 1 by every element, you get the element back, get the original element back. So, like 1 multiplied by this row gives you the same row back. (line 26)

Wendy's inclination to finish rows and to write down names of the group axioms suggests that she preferred to have visual supports for her reasoning. She reasoned about both the action of the 0 and 1 on individual elements and the action of the identity on an entire row of the table. As she constructed the next row in the table, she considered the inverse property.

Wendy: So if you look at the second row [the "2 row"], there is no number when you multiply.... If you take *m* equaling 2, if you take a number equaling 2, when you multiply, there is nothing to multiply by 2 to get—in mod 6, cause it has to be an element, to be closed, you can only work with the elements within mod 6. And I have tried every element, 0, 1 ... 0 through 5, multiplied by 2 to see if I can get the identity, 1, and I can't get it. So therefore, Z_6 is not a group under multiplication. (line 34)

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3				
4	0	4				
5	0	5				

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Figure 1. Wendy's tables for addition and multiplication in Z_6.

Thus, Wendy had developed a procedure that could be used in verifying the inverse property. To find the inverse of any element in a set, she multiplied that element by every element of the set to see whether the result was ever the identity. Although the operation table may be considered as merely providing an organizational scheme for the necessary calculations, Wendy's procedure was based "in the table" in the sense that she preferred the whole row to be present. In fact, she used this procedure to reconsider the "0 row":

Wendy: Actually, up here, in multiplication, I didn't even have to look at the second row [the "2" row] because if you look at 0 there is nothing you can multiply by 0 to get the identity element back, 1, because 0 times every element is going to equal 0. (line 38)

Having abandoned multiplication modulo 6 as the operation in Z_6, Wendy began constructing a table for addition in Z_6 (see Figure 1) and quickly stated, "I can see by filling out the first table [row] that the identity" is zero (line 47), suggesting that her recognition of the identity was tied more strongly to visual aspects of the table than to mental reasoning about the operation. She went on to consider the other group axioms.

Wendy: So next I am going to check the inverse property. And 0 has an inverse so 0 + 1, or.... Excuse me. Since 0 is the identity we have to check that when you add 0 to 0 you get the identity 0. So 0 is the inverse element for itself. And then 1. When you multiply, when you add 1 and 5 it equals 6, but that equals 0 (mod 6) cause 6 is divisible by 6. That's pretty obvious, but.... So 1 has a inverse. 2 has an inverse because $2 + 4 = 6$, which equals 0. 3 + 3 has an in-.... equals 0 (mod 6). $4 + 2 = 0$ (mod 6). And $5 + 1 = 0$ (mod 6). So each element has an inverse. So you know that Z_6 is a group under addition.

Wendy: And then it's closed. You can see that there are no elements other than 0 through 5, looking at the chart, because we have all possible combinations on elements in Z_6. So it is closed also.

Wendy: And associative. You can see, because the chart has symmetry, that the group will be, is associative. This is how I look at it, anyway, because if you look at 2×5 you are going to get 1 and if you look at $2 + 5$ you get 1. (lines 50-52)

Wendy correctly verified the inverse property by using the table to find the inverse of each element. She supported this process by making a check mark alongside each row of the table as she identified the corresponding inverse. In verifying the closure and associative properties, Wendy explicitly referred to the table to support her reasoning, but she made several errors in her attempt to verify associativity. First she stated that she was comparing $2 + 5$ and 2×5 when, based on her statement about the symmetry in the table, she probably was comparing $2 + 5$ and $5 + 2$. A more significant error was that she was describing commutativity but calling it associativity. Finally, the penultimate sentence implies that she thought that Z_6 is a subgroup of Z. During the interview, I pursued and she quickly corrected the first two errors.

Observations

Although the students had been working with the groups Z_n, all of the students who were interviewed expressed some uncertainty about the operation in Z_6, and most of these students used operation tables to support their reasoning. To verify the group axioms, Wendy had created procedures based in the operation table. Her reasoning was largely external, however, in that it seemed to require the presence of the table. Other students developed similar procedures, but none were as tied to the table as Wendy.

The above excerpts provide some explanation for the fact that Wendy sometimes confused the words and syntax while discussing the group axioms and related properties. The identity and inverse properties were closely related in Wendy's thinking partially because finding an inverse for an element involves looking for the identity in a particular row or column of the table. Similarly, when using an operation table, commutativity is more salient than associativity, leading to misstatements by Wendy as well as other students. Noting the symmetry in an operation table, they claimed that the operation was associative when the symmetry indicated instead that the operation was commutative. Thus, although an operation table supports students' reasoning, it may also contribute to conceptual and linguistic difficulties.

Is Z_3 a subgroup of Z_6?

After determining the operation in Z_6, Wendy was able to take on the question that had been posed at the beginning of the interview.

> Wendy: Now is Z_3 a subgroup of Z_6? Now, we have to check that Z_3 is going to be a group because it has to have all of the elements [axioms] of a group, which means it has to have identity and inverse; it has to be closed. So I am going to start checking Z_3. Z_3 would consist of 0, 1, and 2 under addition. But Z_3, the table is going to be different. See I am going to have to explore right now whether or not.... When you say Z_3 is a subgroup of Z_6, whether it means you are taking Z_3 out of Z_6, or if you are just looking at Z_6 [Z_3] and seeing whether it's a group. See when you say something is a subgroup of something else [pause] I am not quite sure what way to look at it. Like how it exactly, like how Z_3 ties into Z_6, like to be a subgroup of Z_6. What, that.... Like I know how to check whether or not Z_3 itself is a group and whether Z_6 is a group, but to check whether Z_6, Z_3 is a subgroup of Z_6, I don't know exactly what to look at. (line 76)

Wendy had a sense that the operation table for Z_3 would be different, depending upon whether it was constructed on its own or taken out of the operation table she had just constructed for Z_6. Consistent with the hypothesis that Wendy's reasoning was highly dependent on *looking* at an operation table, her statement "I don't know exactly what to look at" suggests she didn't know what *table* to look at.

The literature on the learning of group theory indicates that although students think of a group as a set, they are not always sufficiently aware of the operation (Dubinsky et al., 1994). For Wendy, however, the operation remained prominent because of her reliance on the operation table, which suggests a much stronger observation than has been made thus far. Rather than saying the operation in Z_3 is different, Wendy said, "But Z_3, the table is going to be different," implying that the table was not merely supporting her reasoning but rather was substituting for the group in her thinking. The phrase "taking Z_3 out of Z_6" implies again that, for Wendy, Z_6 was not merely a list of elements that appeared on the edges of the table but was in fact the table. This observation is further supported by the following excerpt in which Wendy referred not to the group Z_6 but again to the table:

Wendy: Because if you use the elements of Z_3, which is 0, 1, and 2—are the elements of Z_3. But if you look at them in terms of Z_6, like if you just look at this section of the table Z_6 [dotted square in Figure 2], this isn't going to be a group.... Because it is not closed.... Because 4 isn't an element of Z_3. (lines 78-82)

Thus, through her reliance on the table, Wendy had correctly identified the central issue behind the interview question: whether the addition was to take place based on the operation in Z_3 or in Z_6. Nonetheless, she was not ready to come to a conclusion.

Wendy: See, it doesn't make sense. Like, I started over here to do, to look at whether or not Z_3 was a group itself, but that didn't make sense to me ... [t]o look independently to see whether Z_3 was a group under addition. (lines 86-88)

Wendy was bothered by a contradiction that was implicit in the interview question. Z_3 and Z_6 are each groups if they are considered separately, but the concept of subgroup prohibits such separate consideration. Her thinking coalesced later in the interview:

Wendy: I think we have to look at it Z_3 as like part of the set of Z_6, which, like subgroup, like as a group in Z_6. So if you look at ... which is why I kind of choose the elements Z_3 out of the Z_6 table. (line 111)

Wendy was clearly thinking of Z_6 as more than a set and of Z_3 as more than a subset. She was choosing entries out of the Z_6 table that corresponded to the restriction of the binary operation to the subset Z_3. On the conviction that this was the appropriate method for thinking about a subgroup, Wendy decided that Z_3 is not a subgroup of Z_6 because the subset was not closed under the operation.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Figure 2. Wendy's table for Z_6, annotated version. [Circles added to clarify transcript]

Are there subgroups of Z_6?

The fact that Wendy had answered the interview question was apparently of little concern, for she immediately began looking for subsets of Z_6 that could be subgroups. In particular, she considered the set {3, 4} and saw from the table that 4 wouldn't have an inverse. She generalized the question "Is Z_3 a subgroup of Z_6?" to consider whether Z_n might be a subgroup of Z_6 for other n, but saw that the closure property wouldn't be satisfied.

I asked explicitly whether she could find a subset that was a subgroup.

Wendy: See Z_6, it's hard to take a subset because you have to make sure you include the identity element in the set that you pick. So let's, just for instance, I'm going to take this. Because if I am looking in the fact that you have to have an identity element. Here, if you look at 1, 2, and 3 they each have and 3, 4.... You can't do that. 'Cause now it's not closed, really. You can't take 3, 4, 5 and 1, 2, 3. It wouldn't work. (line 144)

Wendy saw that she needed to include the identity, but she was simultaneously considering "blocks" in the operation table, and she saw that this would not work. After I suggested that she consider nonadjacent elements, such as 3 and 5, I noticed that she was covering up the 4 in the table. She explained:

Wendy: It distracts me. Technically you are only looking at the 0, 2 ... 2, 4 [circled in Figure 2]. Right? Because, in other words, you can make that table.... You can't look at the other elements. You can't look at the whole row, 3 and 5. You know what I mean? Because you can only look at the addition of those two. (line 150)

Wendy indicated how she was restricting her view of the table, listing precisely those entries inside the table (0, 2; 2, 4) that are relevant to whether {3, 5} is a subgroup. Furthermore, she justified this view by noting that "you can make that a table." From this view, she noticed that 5 does not have an inverse in {3, 5} because "when you add 3 or 5 to 5, you can't get 0" (line 152).

Wendy next decided to begin with the set {1, 5} to see whether it is a subgroup of Z_6 because "if you took those two separately, it upholds the inverse property" (line 162). Realizing that she also needed an identity element, she then considered the three-element subset {0, 1, 5}. She chose to "move this over" (line 166) to create a new uncluttered table (see Figure 3).

After completing the table for {0, 1, 5} and verifying the identity and inverse properties, Wendy noticed, with some frustration, that "It's not closed. [Laughs.] Oh, no. It's got 2 and 4 in it. This is just getting really difficult" (line 173).

With the failure of the closure property, Wendy focused on closure and realized that any subgroup that contains 1 must also contain 2, then 3, 4, and 5. I asked her what would happen if she started with something else.

Wendy: If you start with 2 you are going to need 0. You are always going to need 0, 'cause, like you said. Okay. So, things are getting kind of messy. I need a new piece of paper. If you start with 2, you're going to need 4.

Wendy: And when you're doing 4, you need 0. Well.... Ooh.

+	0	1	5
0	0	1	5
1	1	2	0
5	5	0	4

+	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

Figure 3. Wendy's tables for {0, 1, 5} and {0, 2, 4}.

Wendy: You need 0, anyway. You need 4 though. [inaudible] So, 2 ... 'cause 2 and 4 is going to equal 0. Uh oh.

Wendy: It works! You don't.... It's closed. It's got an id-, everything has an identity element ... 0 is the identity element for all, each element. Well, they have to have the same identity element, but.... And it's got an inverse. (lines 196-202)

Wendy had found a subgroup of Z_6, but she was able to conclude that it was a subgroup only after she had completed both the operation table and her table-based verifications of the group axioms. For the first time in the interview, she considered several axioms in quick succession, although her explanations were somewhat muddled. Her language suggests that she was surprised—an impression that she confirmed when I asked whether there are any other subgroups of Z_6: "I don't know; I'd have to play and try. I just found one; I didn't think that we could find one, but I just found one" (lines 212).

Observations

Wendy's reasoning about groups and subgroups was largely external, often requiring that relevant portions of the table be present before her eyes without extraneous information interfering with her perception. When considering whether $\{3, 5\}$ was a subgroup, she covered up the 4, and when building a subgroup with $\{1, 5\}$, she created a new table separate from the Z_6 table. In large measure, the operation table *was* the group for Wendy, although later in the interview she began to separate the group from the table, as evidenced by a suggestion that the table she called $4Z_3$ was a rearrangement of the table for $\{0, 2, 4\}$, which she called $2Z_3$. The operation table both supported and limited Wendy's ability to reason about groups and subgroups. On the one hand, the table helped her see quickly the central issue about considering Z_3 to be a subgroup of Z_6. On the other hand, her reliance on the table made it difficult for her to find subgroups.

A symptom of the external, table-based nature of Wendy's reasoning was that she often considered only one group axiom at a time when reasoning about groups and subgroups. Toward the end of the interview, however, she had developed more fluency and was able to move more quickly among the axioms. Considering the axioms multiply and flexibly might be described as a matter of proficiency and fluency with the group axioms and with the particular examples, which may be a result of gradual internalization of some of the external processes that were based in the table.

In a later interview, Wendy continued to struggle with the distinction between a group and its representation in a particular operation table. She recognized isomorphisms among various groups of order 4 and had a strong sense that there are only two such groups (up to isomorphism), but her methods remained largely external, relying upon the appearance of the operation tables and upon processes of renaming and reordering elements in those tables.

Results

Operation tables served to mediate abstraction for the students in this study in that they worked with a concrete representation to gain access to abstract objects and their properties. A group's operation table makes the group more concrete by making aspects of its form directly visible. Furthermore, by squinting one's eyes or coloring the operation tables by elements or by cosets, the abstract group—of which the particular table is an instantiation—can almost become visible. Abstracting the essence of a group from an instantiation seems a quintessential example of an activity that requires reflective abstraction—abstraction based on action (i.e., operations) alone. With the help of the operation table, however, perhaps only empirical abstraction is required. Thus, the table becomes both a tool for reasoning and an object of reflection. Under Wilensky's (1991) view of abstractness as a measure of one's familiarity with a situation, the table serves to increase one's familiarity, thereby making the abstract more concrete.

Operation tables served a metaphorical role for many students in that the tables supported their reasoning and helped them think of groups as objects. For some of the students in the study, the table *was* the group rather than a representation—a metonymic substitution of the concrete for the abstract. Their reasoning seemed to be largely external, in the sense that it was based *in* the table and in procedures that required that the operation table be present rather than in reflection on the binary operation. Most of the group axioms, for example, were verified through such table-based procedures. The cancellation laws (e.g., $ab = ac$ implies $b = c$) became embodied in the requirement that each element appears exactly once in any row or column.

The table served also to heighten the students' sense of anticipation about the way the calculations should turn out, similar to Boero's (1993) observation about the role of anticipation in algebraic manipulation. One student, for example, expected the coset $\{5, 7\}$ to be its own inverse. Many students came to expect certain patterns in their operation tables and likened those patterns to cycles, which caused some potentially problematic connections with the cycle representations of permutations.

The prominence of the operation table helped explain the fact that students demonstrated strong connections between distinct concepts. For example, students sometimes confused the identity and inverse properties in part because finding the inverse of an element requires looking for the identity element in a particular row or column. Students confused commutativity and associativity in part because when an operation table is present, commutativity is more salient than associativity. Commutativity is quite visible through symmetry in the operation table, whereas associativity is more difficult to see and much more difficult to verify when the group is given by an operation table.

The operation table as metaphor has other limitations as well. First, it becomes cumbersome for large groups, and extending the metaphor to infinite groups requires some sophisticated patterning abilities since it is not possible to write out the whole group table. Second, the students expected subgroups to occupy a corner of the table, probably because of reliance on Groups-Are-Sets and Set-Are-Containers metaphors. Third, writing down a group table requires one to choose an ordering of the elements, which sometimes made it difficult for the students to recognize isomorphisms and to think of the order as nonessential. Nonetheless, through experiences in renaming and reordering operation tables, the students began to separate the table from the group—the signifier from the signified—and thus began to develop concepts of abstract groups.

Discussion

The results of this study suggest that the operation table can play a useful metaphorical role in students' thinking about group theory because of the conceptual support that the metaphor can provide. This paper discusses three implications. First, the results of this study provide insight into and support for Peirce's (1955) semiotics, in which a sign becomes a sign when it is interpreted by someone as a representation of something else. In the case of Wendy, at first each operation table *was* a group. She began to develop understanding of groups as abstract objects only through considerable conceptual struggle, supporting Sfard's (2000) contention that "The transition from signifier-as-object-in-itself to signifier-as-a-representation-of-another-object is a quantum leap in a subject's consciousness" (p. 79).

At the same time, the results of this study call into question the developmental relationship between process and object conceptions. Through use of the operation table, some of the students appeared to have strong object conceptions of groups with relatively weak process conceptions. One possible explanation is that the tables were not objects but pseudo-objects for these students (Sfard & Linchevski, 1994; Zandieh, 2000). The processes were largely external in the sense that the students used procedures that depended upon the presence of the operation table. Another possible explanation is that through the use of the operation table, the developmental trajectory was reversed, with process conceptions emerging slowly from object conceptions.

Finally, the results of this study provide additional support for the use of multiple representations as a way of building understanding of abstract mathematical concepts and objects. Thompson (1994) observes:

> Our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one representational activity to another, keeping the current situation somehow intact. Put another way, the core concept of *function* is not "represented" by any of what are commonly called the multiple representations of function, but instead by our making connections among representational activities. (p. 39)

In this study, the students similarly developed abstract conceptions of objects and concepts in group theory by translating among what are taken to be equivalent representations of the same object.

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CHARACTERIZING SECONDARY STUDENTS' UNDERSTANDING OF MEASURES OF CENTRAL TENDENCY AND VARIATION

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The purpose of the study was to begin the construction of a framework to describe secondary students' statistical thinking. Semi-structured clinical interviews were used to assess four secondary students' thinking about measures of central tendency and variation. It was found that the research-based Middle School Students' Statistical Thinking (M3ST) framework adequately characterized much of the thinking exhibited by students during the interview sessions. However, as a result of the analysis of the clinical interviews, three descriptors related to students' understanding of the relationship between sample means and population means were proposed for addition to the M3ST framework for use with secondary students. Also, a fifth level of thinking was proposed for addition to the M3ST framework for use with secondary students, since some of the interview subjects demonstrated a higher level of thinking than the highest level described in the existing framework.

It is becoming more widely recognized among the mathematics education community that in today's information-driven society, no student should leave high school without engaging in the study of statistics. It is no longer adequate for high school students to take a sequence of courses designed only to prepare them for the study of calculus. Statistical reasoning is essential for all students, regardless of what occupation they may choose to pursue (Gal & Garfield, 1997). Statistics play a key role in shaping policy in a democratic society, so statistical literacy is essential for all citizens in order to keep a democratic government strong (Wallman, 1993). Recognizing the key role statistics play in modern society, in its *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM, 2000) recommends that "by the end of high school students have a sound knowledge of elementary statistics" (p. 48).

The study of statistics in high school has gradually become more common over the course of the past decade. On the student questionnaire given with the 1990 National Assessment of Educational Progress (NAEP), 87.6 percent of all twelfth graders indicated that they had not taken probability or statistics, but by the 1996 NAEP that number had dropped to 79.2 percent (Shaugnessy & Zawojewski, 1999). In the late 1990's, reform curricula funded by the National Science Foundation (NSF) emerged at the secondary level integrating the teaching of statistics with the teaching of traditional topics from algebra and geometry (Alper, Fraser, Fendel, & Resek, 1998; Hirsch, Coxford, Fey, & Schoen, 1998). Even traditional algebra textbooks have begun to include statistical topics alongside algebraic topics (e.g., Collins, 2000). The subject of elementary statistics appears to be gradually taking root at the secondary level.

As the study of statistics becomes more common at the high school level, it is imperative that the research base for advising curriculum design, instruction, and further research be expanded. Extensive research has been done of students' understandings of topics traditionally included in secondary school curricula, such as algebra and functions (Chazan, 2000; Sfard & Linchevski, 1994; Tall, 1992; Vinner, 1985), geometry (Burger & Shaugnessy, 1986; Fuys, Geddes, & Tischler, 1988; van Hiele, 1986; Usiskin, 1982), and calculus (Ferrini-Mundy & Guadard, 1992; Heid, 1988; Orton, 1983; Thompson, 1994). Since statistics is a relatively new topic in secondary school curricula, research about how high school students think about the subject is not as well developed. In fact, it is only within the past decade that research investigating how students handle data has accumulated in any substantial amount (Shaugnessy, Garfield, & Greer, 1996).

One way to help the various emerging efforts to expand the teaching of statistics at the secondary school level to be successful in enhancing students' understanding is to conduct research directed toward constructing a framework that describes the statistical thinking of high school students. Such research-based frameworks are vital for advising curriculum and instruction (Cobb et al., 1991; Resnick, 1983). The purpose of this study was to begin the construction of a framework to describe secondary students' statistical thinking by characterizing secondary students' understanding of measures of central tendency and variation. Research-based frameworks have already been constructed for describing the statistical thinking of elementary and middle school students (Jones et al, 2001; Mooney, 2002). This study marks the beginning of the process to extend that line of research to the secondary level.

Theoretical Perspective

This study seeks to begin the process of extending the Middle School Students Statistical Thinking (M3ST) framework (Mooney, 2002) to the secondary level. The M3ST framework describes levels of statistical thinking across four processes: describing data, organizing and reducing data, representing data, and analyzing and interpreting data. The organizing and reducing data section of the M3ST framework (non-italic print in fig. 2) describes how middle school students think about measures of central tendency and variation. The focus of the current study was to investigate the adequacy of the organizing and reducing data section for describing secondary students' thinking about measures of central tendency and variation. The two major research questions in this study are: 1) To what extent do the descriptors in the M3ST framework characterize secondary statistics students' understanding of measures of central tendency and variation? and 2) What changes in the descriptors, if any, are necessary to characterize the understanding of secondary statistics students?

The M3ST framework is based upon a Neo-Piagetian model of development described by Biggs and Collis (1982). According to the Biggs and Collis model, human development follows the stages, or developmental "modes," of: sensorimotor, ikonic, concrete-symbolic, formal, and post-formal (Biggs & Collis, 1991). As humans attain each mode, their thought processes are qualitatively different from those of previous modes. Within each mode, three cognitive levels (unistructural, multistructural, and relational) cycle and represent shifts in the complexity of students' reasoning. There are also two cognitive levels serving as "bookends" for each mode: the prestructural, which is related to the previous mode, and the extended abstract, which is related to the next mode. The M3ST framework contains four levels of thinking which are isomorphic to cognitive levels described in the Biggs and Collis (1982) Structure of the Observed Learning Outcome (SOLO) Taxonomy. More precisely, level 1 in the M3ST framework, described as "idiosyncratic" thinking, corresponds to the prestructural level in the SOLO Taxonomy; Level 2 in the M3ST framework, called "transitional" thinking, corresponds to the unistructural level (within the concrete-symbolic mode); Level 3, "quantitative," corresponds to the multistructural level; and Level 4, "analytical," corresponds to the relational level. Mooney (2002) hypothesized the existence of a fifth level of thinking, which would correspond to the extended abstract level in the SOLO taxonomy, but observed no such level of thinking among the middle school students he interviewed. This fifth level of thinking, in terms of the SOLO taxonomy, would represent a shift from concrete-symbolic to formal reasoning, and most students are not likely to function at this level of thinking until their secondary school years.

Method

Instrument

Interview questions designed to elicit students' thinking about measures of center and variation were constructed by consulting the existing body of literature concerning students' thinking about measures of center (Mokros & Russell, 1995; Watson & Moritz, 2000a) and variation (Shaugnessy, Watson, Moritz, & Reading, 1999; Tversky & Kahneman, 1982). A total of four interview tasks (fig. 1) were written. The first task was designed to assess students' understanding of the relationship between population means and sample means. The second task required students to compare three data sets, each of which had the same mean and median. The third task consisted of three problems which could possibly be solved using various different measures of center and spread. The final task asked students to approximate the mean, median, mode, and standard deviation for each of four different distributions.

Participants

Semi-structured clinical interviews using the written tasks were used to assess the statistical thinking of two male and two female students from an AP Statistics class. At the time of the interviews, the students had just completed the study of the first two

1) The average ACT score at Clintonville University is 23. The university only admits students with a score of at least 18. The university gives scholarships to those students who score 36, the highest possible score on the ACT. A random sample of 10 students' scores is taken. Which data sets are possible scores of the 10 students?

   A) 19, 21, 27, 20, 18, 31, 34, 21, 20, 19
   B) 18, 24, 19, 36, 27, 23, 20, 18, 24, 18
   C) 32, 36, 28, 31, 29, 28, 30, 32, 29, 29
   D) 18, 23, 25, 16, 27, 24, 19, 32, 28, 18
   E) 19, 18, 20, 19, 18, 21, 18, 18, 18, 19

2) A consumer protection group determined the number of hours three different brands of batteries lasted in five different cassette recorders. Below are listed the number of hours each brand of battery lasted in the five different recorders. Judging from the data given below, which two brands do you think are most similar in performance? How would you convince someone else of your choice?

	Recorder 1	Recorder 2	Recorder 3	Recorder 4	Recorder 5
Brand A:	10	15.5	20	24.5	30
Brand B:	14.0	19	20	21	26
Brand C:	11	16	20	24	29

3) How would you determine:

   a) The typical height of people in a third grade class?
   b) The typical height of people in an elementary school?
   c) Which two classes had the most similar performance on a test given to 10 different classes in a school building?

4) Give an approximation for the mean, median, mode, and standard deviation for each of the following distributions.

   a)

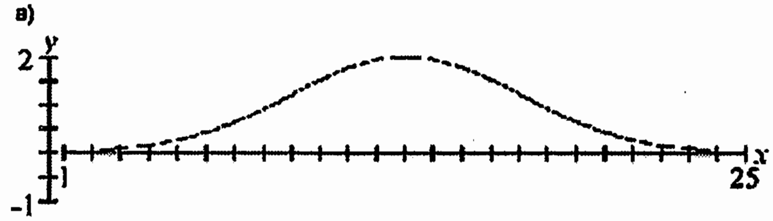

Figure 1. Interview tasks (Part 1).

chapters of a popular AP Statistics textbook (Yates, Moore, & McCabe, 1999). The first two chapters of the text introduce various formal measures of center and spread. The students were recommended for interviews by their teachers, who felt that they represented a fairly wide range of statistical thinking ability. AP Statistics students were chosen for the current study in the hope that their studies would allow some of them to display higher levels of thinking than the middle school students studied by Mooney (2002), and therefore serve to extend the M3ST framework to characterize higher levels of thinking.

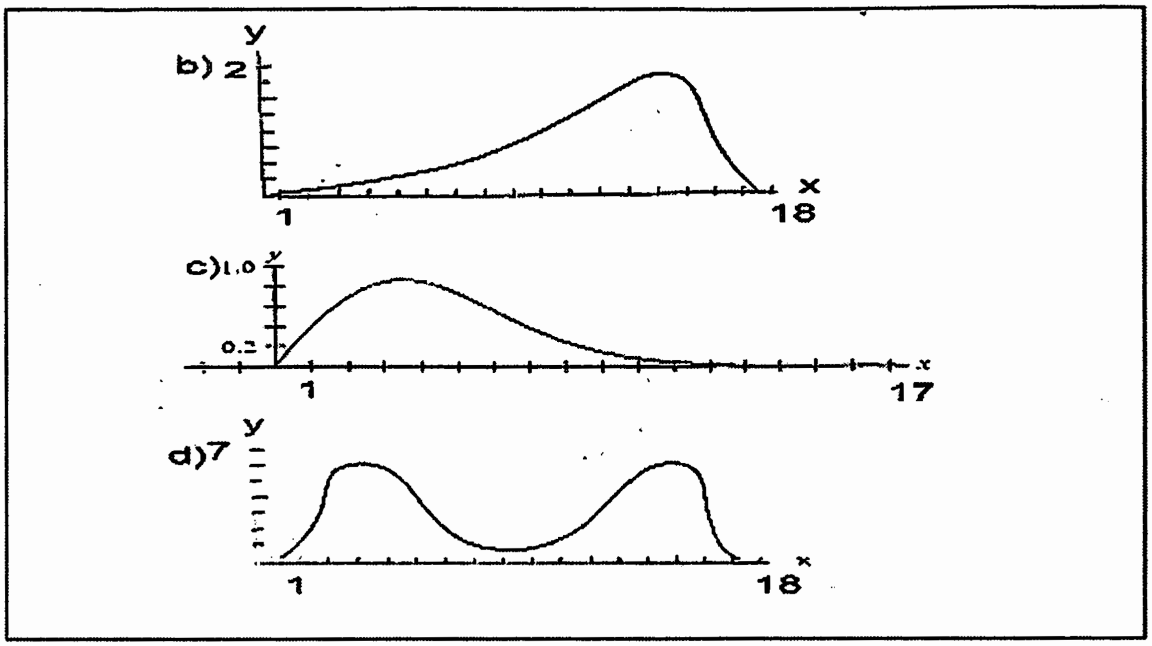

Figure 1. Interview tasks (Part 2).

Procedure

Each interview session lasted approximately 45 minutes. Each interview item was presented on a separate piece of paper to each student one at a time. I read each question aloud in an attempt to ensure no tasks were misread or misinterpreted. Students were told they could write as much or as little as they pleased on each task sheet they were presented. Students were given a TI-83 calculator with all lists cleared and told they could use the calculator as much or as little as they pleased in answering the interview questions. I took field notes as students gave responses, audiotaped each interview, collected students' written work at the end of each interview, and then transcribed each interview.

Data Sources and Analysis

The primary data for the study came from the clinical interview sessions. Analysis of data from the interviews followed methods described by Miles and Huberman (1994). To begin the analysis, I wrote reflective notes on interview transcripts shortly after each session. The interview transcripts, reflective notes, field notes, and the students' written work were used to write vignettes of each interview session. Next, a 4 x 4 matrix was constructed with a different student's name heading each column, and a different level of thinking from the M3ST framework heading each row. Individual responses to interview questions were cut from the interview transcripts and placed within the matrix. In this manner, each response was assigned a level of thinking within the M3ST framework. I refined the assignment of responses to levels by consulting the written vignettes and the descriptors in the existing M3ST framework. Responses which did not fit within the scheme of the M3ST framework were not placed within the matrix, but were set aside and categorized to aid the formulation of essential components not currently included in the M3ST framework.

Results

The M3ST framework adequately characterized much of the thinking exhibited by students during the interview sessions. The existing descriptors in the organizing and reducing data section of the M3ST framework proved to be accurate descriptors for many of the interview responses. Therefore, data gathered from the current study did not result in the revision of any existing descriptors.

However, three descriptors were proposed for addition to the M3ST framework for use with secondary students. During the interviews, three distinct levels of thinking about sampling situations emerged in response to interview question 1 which were not described by the existing framework. At the lowest level of thinking observed in response to the first interview task, two students felt that the mean of a small sample must exactly match the mean of the population from which it is drawn. This was considered a "multistructural" response within the SOLO Taxonomy, since "in multistructural responses, closure is determined when (two or) more aspects are perceived, but since those aspects are not interrelated, inconsistency may result" (Biggs & Collis, 1982, p. 28). In this case, students perceived something of the nature of random sampling and measures of center, but came to an incorrect conclusion from not being able to relate the two concepts. At the next level, two students acknowledged that the mean of a small sample may vary slightly from the mean of the population from which it is drawn. This was considered a relational response, since a student responding relationally "will come up with a definite answer (closure), possibly an excellent answer for that context, but it will not do for other contexts (i.e. an over-generalization may be made)" (p. 28). In essence, answering that the mean of a sample will vary only slightly from the mean of the population from which it is drawn is an excellent answer for most situations. However, it is an over-generalization to think that the two means will vary only slightly in any given case. None of the students interviewed exhibited a higher level of thinking, at which one would acknowledge that the mean of a small sample may vary greatly from the mean of the population from which it is drawn. Such a response would be typical of an extended abstract response, which "sets out principles and heavily qualifies their application to a given situation" (Biggs & Collis, 1982, p. 28). Descriptors for each of these three levels of thinking were formulated and proposed for addition to the framework. These new descriptors and how they may fit into the current M3ST framework are shown in italics in figure 2.

A fifth level of thinking was also proposed for addition to the M3ST framework for use with secondary students. In the existing framework, at the fourth level of thinking in the process of organizing and reducing data, students are able to "describe data using a valid and correct measure of center." One student interviewed demonstrated a higher level of thinking by going beyond using just one valid and correct measure of center to describe data. He consistently gave correct discussions of the relative merits of using different measures of center to describe data sets. None of the students interviewed were able to consistently give correct discussions of the relative merits of using different measures of spread to describe data sets, but the thinking of students who are able to do so would not be adequately described by the fourth level of thinking in the M3ST framework. In essence, in order for the M3ST framework to be useful in characterizing secondary students' thinking about measures of central tendency and variation, a fifth level of thinking must be added. This proposed fifth level of thinking,

Organizing and Reducing Data **Organizing and reducing data involves arranging, categorizing, or consolidating data into a summary form. In general, students performing or explaining questions, tasks, or activities which involve organizing and reducing data will...**	
Level 1: **Idiosyncratic**	-Not attempt to group data -Not be able to describe data in terms of representativeness or typicalness. -Not be able to describe data in terms representative of the spread.
Level 2: **Transitional**	-Group data but not in summative form. -Describe the typicalness of data using invented measures that are partially valid. -Describe the spread of data using a measure from a flawed procedure or a valid and correct invented measure.
Level 3: **Quantitative**	-Group data in summative form or group data by creating new categories or clusters. -Describe the typicalness of data using a measure of center from a flawed procedure or a valid or correct invented measure. -Describe the spread of data using a measure from a flawed procedure or a valid and correct invented measure. *-Expect that the center of a small sample will always exactly match the center of the population from which it was drawn.*
Level 4: **Analytical**	-Group data in summative form by creating new categories or clusters. -Describe the typicalness of data using a valid and correct measure of center. -Describe the spread of data using a valid and correct measure. *-Acknowledge the possibility that the center of a small sample may vary from the center of the population from which it was drawn.*
Level 5: ***Extended*** ***Analytical***	*-Have a number of valid and correct measures of center at disposal for describing typicalness of data, and recognize the relative power of each for describing contextualized and decontextualized data sets.* *-Have a number of valid and correct measures of spread at disposal, and recognize the relative power of each for describing contextualized and decontextualized data sets.* *-Acknowledge the possibility that the center of a small sample may vary greatly from the center of the population from which it was drawn.*

Figure 2: Possible revisions (in italics) to the organizing and reducing data section of the M3ST framework (Mooney, Langrall, Hoffbauer, & Johnson, 2001) for use with secondary students.

with its associated descriptors, is shown in italics in figure 2. It is isomorphic to the extended abstract level of thinking in the SOLO taxonomy, since, according to Biggs and Collis (1982), "The student giving an extended abstract response can entertain alternative outcomes: He is not forced, as are the others, to come to a definite closure or conclusion" (p. 28). In this case, students at the fifth level of thinking show the ability to consider various measures of spread and central tendency as appropriate, and not necessarily settle on only one. The name "extended analytical" to describe this level of thinking has been borrowed from Mooney (2002).

Discussion

The current study raises a number of issues for further investigation. First of all, further research must be done to investigate secondary students' notions about sampling from a population. It is quite startling that two students involved in the study felt that the mean of a random sample must exactly match the mean of the population from which it is drawn. It is impossible for students to learn about inferential statistics in a meaningful manner if they cling to such a misconception. In order to comprehend inferential statistics, students must understand that the means of all possible samples drawn from a population are not all the same, but rather form a sampling distribution. Current curricular guidelines for secondary statistics (NCTM, 2000; College Board, 2001) call for secondary students to understand and work with sampling distributions. Watson and Moritz (2000b) suggest that students be exposed to sampling concepts across their years of schooling in order to develop understanding. Future studies need to examine the effectiveness of activities designed to dispel students' misconceptions about sampling situations.

It is also striking that none of the students interviewed exhibited higher than level 4 thinking about measures of spread. Some students even viewed standard deviation simply as a formula, and did not think of it as a measure of spread. This situation is analogous to Mokros and Russell's (1995) findings about children's conceptions of the arithmetic mean. Some of the children studied by Mokros and Russell did not think of the mean as being an indicator of the center of the data set. Instead, they viewed the mean simply as an algorithm. This view prevented them from attaining a level of thinking at which they would understand various measures of center to describe a data set, and then choose which one was most appropriate to use for any given situation. Some students in this study were in a similar situation in their thinking about standard deviation, since they did not conceive of it as a measure of spread, and hence could not possibly evaluate how the measure could be used in order to gauge spread in any given situation. Teachers and researchers need to design and implement appropriate instruction about measures of spread in order to prevent students from developing impoverished procedural understandings of concepts such as standard deviation.

Finally, studies involving tasks designed to investigate the adequacy of the other areas of the M3ST framework for describing secondary students' thinking must be conducted. The current study focused mainly upon investigating aspects of the Organizing and Reducing Data section of the framework. We now need to gather information about the adequacy of the other sections, which describe how students analyze, represent, and describe data. In investigating the other sections, it may be found that some of the descriptors characterizing levels of thinking formulated in this study could be refined, or perhaps even moved to other sections of the framework. A more complete framework for describing the statistical thinking of secondary students needs to be developed in future studies.

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EXPLORATION OF PATTERNS AND RECURSIVE FUNCTIONS

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It is important that, during the study of algebra at school, students have opportunities to deal with situations in which symbols, operations and rules make sense. The formulation of functions, as the result of the exploration and recognition of patterns, is a process that is intrinsically linked to modeling and in which students can learn new forms of expressing their mathematical thought and develop a better understanding of the ideas involved. This paper makes some reflections about the process of recognizing patterns and reports the work done by students who were given tasks related to the construction and analysis of recursively defined functions and explicitly defined functions. The results of the analysis show that students have some difficulties in the process of symbolizing successions and patterns, especially when these emerge in the process of solving a task that involves a contextual situation.

Introduction

One aspect on which mathematics education has centered its attention is in the design and implementation of tasks that permit the student to become actively involved in a real mathematical experience. Tasks that motivate reflection and communication of important mathematical ideas and in which students' resources and previous experiences are the basis for extending and strengthening their knowledge and understanding of mathematics.

The type of tasks that are proposed to students in the classroom contributes significantly to form a vision that students have of the discipline (Romberg & Kaput, 1999). The tasks that allow students to reflect, explore and communicate, besides strengthening the understanding of ideas, can promote a new vision of mathematics as an interesting, powerful discipline, in which knowing makes sense.

In *Principles and Standards for School Mathematics* of NCTM, the standards for high school state that students should begin to experiment and understand aspects related to the structure and form of mathematics, including the analysis and use of a wide range of functions to model real-life situations (NCTM, 2000, pp. 287-288). The formulation of recursively defined functions and explicitly defined functions, as a means to express the properties and characteristics of a succession, permits students to extend and strengthen their understanding of a variety of mathematical concepts. These functions are also essential processes for the study of calculus, discrete mathematics, and other sciences.

The recognition of patterns is not an unfamiliar process for high school students. It is taught from elementary levels, beginning with the ordering of natural numbers and counting by *k*s integers (by 1s, by 2s, etc.) to promote processes of reasoning that will permit later development of the skills of generalization and abstraction. These previous experiences can constitute important resources for exploring and constructing models based on recursively defined functions (recursive models) as well as a powerful way of expressing relationships and analyzing properties of mathematical processes, operations and notions involved in the recognition of patterns.

The objectives of this paper are to reflect on the characteristics of pattern recognition itself and to report the work that students do when they face contextual situations in which the generation of numerical successions is a resource for the solution, and students are invited to construct recursive or explicit models for those situations.

Theoretical Framework

Many studies (Carpenter & Lehrer, 1999) support the assumption that when students learn with understanding, they can use their knowledge flexibly in new situations. Heibert et al. (1997) define mathematical understanding in terms of the mental activity that unfolds when an individual makes connections between mathematical ideas. Carpenter and Lehrer (1999) propose five forms of mental activity from which understanding emerges: i) building relationships, ii) extending and applying mathematical knowledge, iii) reflecting on experience, iv) articulating what is already known, and v) appropriating mathematical knowledge.

In the design and implementation of task, opportunities must be taken so that students become involved in these forms of mental activity, in which knowledge and skills develop together as necessary resources in the process of solving a problem.

One of the expectations posed in the NCTM *Principles and Standards for School Mathematics* (2000) for high school algebra students is that they must have opportunities to "generate patterns using both recursively and explicitly defined functions" (NCTM, 2000, p. 395). One pattern is a rule with which elements of a succession of mathematical objects (for example, numbers or figures) are constructed, and, therefore, recognizing a pattern means discovering the rule.

In the process of recognizing a pattern, the student reflects and, in the first try, attempts to identify common characteristics and qualitative differences among elements that form the succession. She (or he) then tries to systemize these characteristics and *generalizes her ideas* to propose a pattern and test it to see if it is valid for the elements of a succession. Later, she tries to express the results in natural language and/or utter the rule mathematically. In this process, the generalization of ideas is a very important step, since it is at this moment that the student can be considered to have attained the systemization of her observations to describe a rule.

Of course, a numerical succession, expressed by a only a few terms, as is commonly done, can have different rules of construction that can lead to mathematical expressions that are not necessarily equivalent. For example, the expressions $F_n=3*2^n$ (n=1, 2, 3, 4, ...) and $G_n=n^3-3n^2+8n$ (n=1, 2, 3, 4, ...) are valid models for generating the terms shown explicitly in the succession: 6, 12, 24, 48, . . .; however, the following term in the succession differs from one case to another ($F_5=96$ y $G_5=90$). In fact, Krutetskii (1976), in studies on the development of mathematical skills in the early 60s, states that if a student is capable of recognizing different forms of generating all of the given elements of a sequence and indicate some possible variations for the following elements, "this is an indication of a very high level of mathematical thinking" (Krutetskii, 1976, p. 151). We must acknowledge that in the process of pattern recognition there can emerge different rules of construction that, if they are not discussed and analyzed carefully, they can cause confusion for both students and teachers.

Pattern exploration can serve as a context for the introduction of different forms of expressing relationships and show that such relationships can be described correctly using different methods. That is, the regularities discovered in a succession can be systemized and uttered by means of rules or formulas that are algebraically different, some of which may be equivalent. For example, in the succession of the numbers 2, 4, 6, 8, 10, ..., a rule for the construction of the following term is "add two to the previous result"; another might be "multiply the number corresponding to the position by two." Both of these are surely identified easily by high school students. Symbolically, they can be expressed as a recursively defined function, $T_n=T_{n-1}+2$, with $T_1=2$, and n=1, 2, 3, 4, . . ., and as an explicitly defined function, $T_n=2n$, with n=1, 2, 3, . . .

Although students can discover a pattern, they seem to have difficulty when they try to express it symbolically (Franzblau & Warner, 2001). Even though the algebraic notions involved in the mathematical formulation of patterns of arithmetic and geometric successions are part of the basic algebra courses, it seems that the difficulty in expressing them lies not in the systematic generation of the objects of the succession, nor in not having algebraic notions, but in the *use* of symbols appropriate for the expression of the relationships and properties identified. Particularly, the recursive models are generally more natural forms of expressing the characteristics of a pattern (NCTM, 2000) and, therefore, can give meaning to the relationships between processes and functions. However, the symbols used to express these models mathematically (use of subscripts, or subscript notation) is not frequently incorporated and used in high school algebra courses, although it seems to be a very important resource for the development of a better understanding of the notions of both dependency and independency of variables and the concept of function (Franzblau & Warner, 2001).

Subjects, Methods and Procedures

In this study we attempt to explore the ability of high school students to recognize and symbolize patterns. A qualitative research method was used, consisting of an analysis of the work done by students to solve each of four tasks proposed and notes of the observations made during the sessions of discussion.

These four tasks were given to seven students with average achievement who volunteered to participate in the exploration at the beginning of December 2001. These students studied the topic of functions as part of their algebra course at the high school Rector Hidalgo in Morelia, Michoacán. Since our purpose was to promote the mathematical formulation of numerical patterns of arithmetic and geometric successions that emerge in specific situations, the participating students were encouraged to discover relationships and their mathematical formulation by asking them specific questions during the tasks.

The students first worked individually, reporting their work in writing and, later, they discussed and analyzed their solutions with the group and their instructor. Two sources of data were used to gather the information: (a) the individual work reported in writing by the students and (b) the notes of observations made on the participation of the students during the discussion after solving the tasks. In both cases, special attention was paid to the following aspects:

i)The interpretations that the students give to the situations posed in the tasks.

ii) The systemization and organization of the data in tables or graphs.

iii) The identification of operational characteristics and properties of the process.

iv) The possible proposals of solution and the symbols used.

v) The recognition of equivalences and differences among the functional relationships proposed.

The application of the tasks was done in four sessions of approximately one hour for a period of two weeks, combined with sessions of discussion after each of the sessions of application.

Results

Some of the characteristics we tried to give to the tasks were the following:

(a) that no sophisticated mathematical resources be required for the solution;

(b) that the students be given the opportunity to construct and reflect on their own methods of solution;

(c) that the use of different strategies and procedures be permitted;

(d) that calculators be permitted as a tool for exploration and testing the results; and

(e) that the tasks provided be sequenced in the construction and analysis of recursive and explicit models, progressing gradually as the students advance in the solutions.

In general terms, although the students use recursive type procedures and construct lists of data, they tended to focus on the calculated results instead of the process.

Only after instructing them on the use of notation with subscripts and encouraging them to look for common operations and procedures, did they try to give a pattern, exhibiting some difficulty in expressing it symbolically. Also, although at the end of the exploration some students could give recursive and explicit patterns, they justified the relationship between these saying that "in both cases we get the same results," without giving a reasonable explanation of why these results are equal. The use of subscripts and symbolic formulation of patterns seems to be strengthened as students progress in the solution of the tasks, although the relationship between recursive and explicit expressions could not be justified mathematically. In the following section we present the results of the analysis of each task.

Task 1. Population growth.

This first task served as context to explore the ways in which the students generate and name the elements of a succession; that is, an attempt was made to inquire into the resources students use when they work on tasks that require generating a numerical succession. All of the students used a recursive process to arrive at the solution, constructing lists of numbers that, in some cases, they organized in a table.

Although for this problem students were not asked explicitly to obtain a general solution, three of them showed a tendency to try to express a symbolic relationship based on previous numerical calculations.

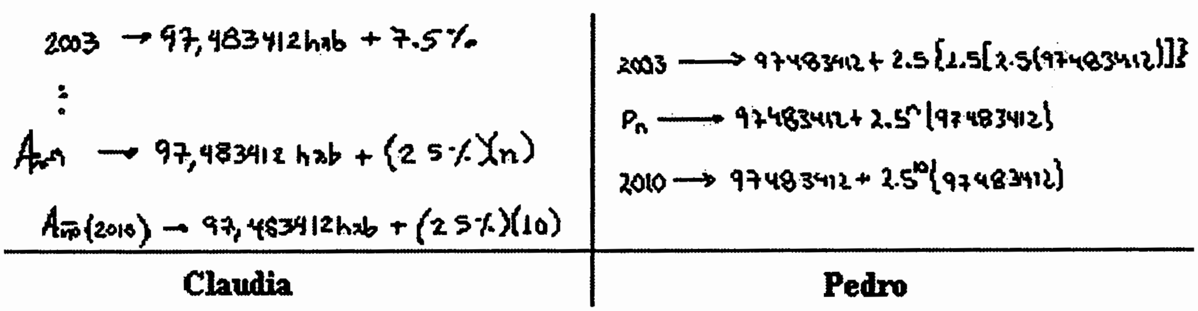

Figure 1. Solutions (partial) by Claudia and Pedro for task 1

One of the students who organized the information in a table made a graph of his results and during the discussion session commented that the graph looked like a curve that "is becoming horizontal," but without recognizing a specific analytical relationship to represent it.

Claudia and Pedro, for example (See Figure 1), constructed explicit relationships in which, in the first case, Claudia uses a notation that tends to be more functional, [Año(2010)   97,483412 hab + (2.5%)(10), with which she means to express: Population in the Year(2010) = 97,483412 + (0.025)(10)*(97,483412)]. In contrast, Pedro uses subscripts to denominate the general term. Both solve the problem with their models, although they exhibit some difficulty in interpretation and symbolization.

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An important part of the activity was discussion and analysis of the solutions they obtained. This gave the instructor opportunity to use and show possible ways to denote the terms that were generated by the students during the process of solution and to compare several solutions while encouraging students to reflect on how reasonable their solutions were. In these discussion sessions, other arithmetic and geometric successions, proposed by both students and instructor, were analyzed.

Task 2. Putting nails.

The application of the second task (See appendix) was an attempt to explore the ability of the students to obtain the result of addition of a succession resulting from the situation presented in the task. Although the students, in the discussion sessions previous to the application of this task seemed to be aware of the usefulness of the symbolic denomination of the terms to generate a pattern, several of them (four) did not use symbolic notation during the process of solution. As in the previous task, they generated an ordered list of calculations and/or numerical results recursively, adding one by one to obtain the solution.

In contrast, two students (Pedro and Roberto in Figure 2) use subscript notation to symbolize the terms of the succession. Pedro, using only a few terms, could discover and symbolize an explicitly defined pattern [C_n=10+15(n-1)], which he used to calculate the terms and, later, their sum (as can be observed in Figure 2, Pedro uses the technique of reversibility of the addends to calculate the sum). However, he treats the two parts of the task separately and, in the second part, in which a pattern is explicitly asked for, he proposes a different relationship and expresses the sum with each of the terms symbolically. It seems that the contextual situation, in itself, led him to construct a correct model, while, when a relationship is explicitly asked for, Pedro has difficulty with symbolization.

Roberto, on the other hand, first generates the terms of the succession in numerical form and responds to the first part of the task, adding one by one. In the second

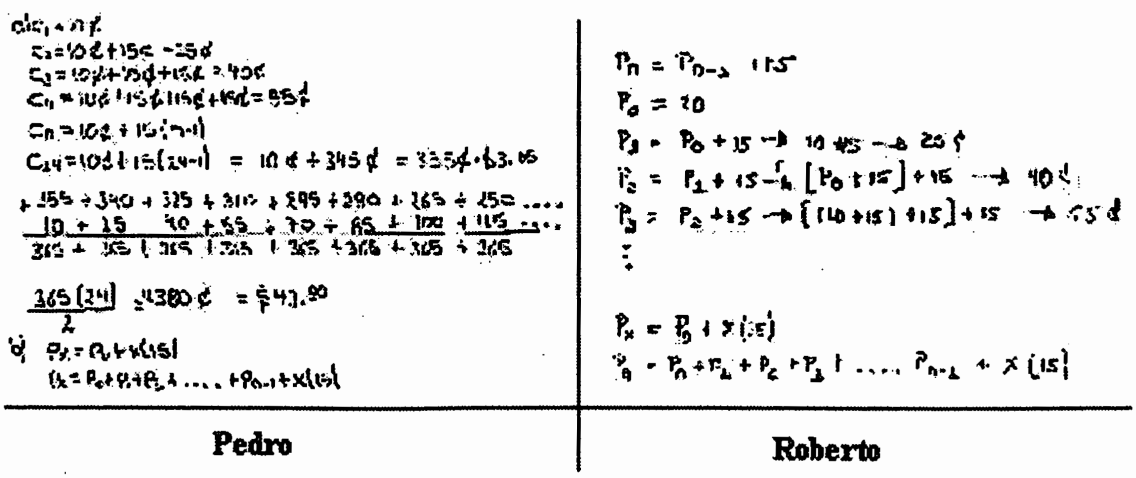

Figure 2. Pedro and Roberto's solutions for task 2.

part (shown in Figure 2), he proposes $P_n = P_{n-1} + 15$ as the pattern and test his proposal by applying it to a few cases. This proof seems to lead him to an explicit relationship, but he has difficulties in symbolizing. Like Pedro, he expresses the sum with each of the terms symbolically. The use of "x" to express the pattern, in both cases, is due to the fact that in each task the situation asked for is given in terms of this letter (See appendix).

In this task, only one student attempts to symbolize the recursive process to establish a pattern (See Figure 3). However, like Pedro, she needs to calculate each of the terms to arrive at establishing the sum.

The processes carried out by the students afforded the possibility that the discussion following the application center mainly on the analysis of the procedures and models proposed, since these involved both recursive and explicit rules for the succession (not so for the sum of the terms of the succession). Special attention was made to the variety of models that can be generated for one situation as well as to the advantages that, in some cases, have the explicit over the recursive models. Also the processes that are followed to obtain the terms of the succession were explored, as much in a recursive model like in an explicit one. One of the students pointed out: "[in the recursive model,] *the operations are easier, but it is necessary to work more* [than in the explicit model]", concluded that, for the examined cases, the explicit models were "faster".

$$T_1 = 10¢$$
$$T_2 = T_0 + 15¢ = 10 + 15 = 25$$
$$T_3 = T_1 + 15¢ = 25 + 15 = 40$$
$$T_4 = T_2 + 15¢ = 40 + 15 = 55$$
$$T_5 = T_3 + 15¢ = 55 + 15 = 70$$
$$T_6 = T_4 + 15¢ = 70 + 15 = 85$$
$$T_7 = T_5 + 15¢ = 85 + 15 = 100$$
$$\vdots$$
$$T_n = T_{n-1} (.0.15 + 0.10)$$
$$T_{24} = T_{24-1} (0.15 + 0.10) = 3.55 \$$$

$$\begin{array}{l} 3.55 + 3.40 + 3.25 + 3.10 + 2.95 + 2.80 \ldots 0.10 \\ \underline{0.10 + 0.25 + 0.40 + 0.55 + 0.70 + 0.85 \ldots 3.55} \\ 3.65 + 3.65 + 3.65 + 3.65 + 3.65 + 3.65 \ldots 3.65 \end{array}$$

$$\frac{3.65\ (24)}{2} = \$\ 43.80$$

Figure 3. Adriana's solution for task 2.

Task 3. Printing costs.

This task involves a situation that, at first, looks similar to the previous one (See appendix). While the process of counting the terms in the case of task 2 was one by one (1 nail, 2 nails, etc.), in this task counting 500 by 500 (1000 flyers, 1500 flyers, etc.) imposes certain difficulties for the students during the solution since they need to generate not only a succession for the cost of production but also for the number of flyers involved. Figure 4 shows Adriana's process of solution, a case that is typical of the work done by five of the students.

$$P_0 = 100 = 1000$$
$$P_1 = 100 + 20 = 1000 + 500$$
$$P_2 = 100 + 20 + 20 = 1000 + 500 + 500$$
$$P_n = 100 + n(20) = 1000 + n(500)$$
$$P_0 = 1000 \qquad P_1 = P_0 + 500 \qquad P_2 = P_1 + 500$$
$$P_n = P_{n-1} + 500$$

Nota: Para determinar el numero de series de volantes extra se divide el numero de volantes extra entre 500. ó $n = \dfrac{P - 1000}{500}$

Figure 4. Adriana's solution for task 3.

In this solution, Adriana makes an attempt to match the number of flyers to their cost of printing, trying to equate the numbers (incorrectly). She generates two explicit models: one to quantify the number of flyers [P_n=1000+n(500)] and the other for their cost of production [P_n=100+n(20)].

In this task, two students again carry out a numerical process to calculate the cost and produce a pattern (explicit) as a solution.

Although all of the students use a variable (n) that increases 500 by 500, they propose a solution that can be equivalent to P_n=100+20(n) without mentioning additional conditions for n. Only two students (for example, see Figure 4) try to express symbolically the fact that n (number of flyers) is a quantity that increases 500 by 500, although they have problems expressing it.

During the discussions following the application (in three sessions of approximately one hour), the models proposed in the third and fourth tasks were analyzed. The characteristics of each situation were analyzed to highlight the differences and the usefulness of the models. Processes of transformation from a recursive form to an explicit form were explored, in function of the observed operational characteristics as well as in terms of the notion of linearity in the recursive models in the case of some arithmetic and geometric successions.

Task 4. Recursive and explicit functions.

In this task the word *progressions* (See appendix) was used to refer to numerical successions since some of the students remembered that they had studied them with that name at some moment in their junior high algebra course. In general, most of the students (six) begin with the denomination of the terms as P_0, P_1, etc. (or some equivalent form), to later propose a pattern and test its validity with a few terms.

It seems to be that because of the fact that the terms of the succession are given explicitly in the task, the denomination of the terms is an important step in the solution (not so in the other tasks). The students, with no difficulty, propose a recursive model for the successions and, although for the second succession, the explicit model can be identified easily, the fact that a recursive formula was asked for first seems to have affected the process, since most of the students do not manifest, in their written work, having recognized the model. The students attempt to arrive at a recursive formula and, based on this, construct an explicit formula. Only one student (Pedro) shows an explicit pattern as a construction rule (See Figure 5), although he does not recognize it as such and tries to arrive at a formula by applying the algebraic algorithm (taught by the instructor in previous sessions) to transform a recursive relationship into an explicit one.

During the discussion following the application, it was necessary for the instructor to give an example of the solution to this task, emphasizing the process itself since all of the students attempted to use the algebraic algorithm (illustrated by Pedro's work in Figure 5) without reaching a conclusion.

$P_1=1 \quad P_2=2 \quad P_3=3$

$P_0=0 \quad P_1=0+1=1$

$P_2=1+1=2 \quad P_3=2+1=3$

$P_0=0 \quad P_1=P_0+1$

$P_2=P_1+1 \quad P_3=P_2+1$

$\underline{P_n=P_{n-1}+1}$ /

¿Podrío ser $P_n=n$?

función explícita

$P_n=P_{n-1}+1 \quad P_n-P_{n-1}+1=0$

$P_n-P_{n-1}+1=0$

$\underline{P_{n-1}-P_{n-2}+1=0}$

$P_n-2P_{n-1}+P_{n-2}+0=0$

Si $P_n=x^n$, $x^n-2x^{n-1}+x^{n-2}=0$

$x^{n-2}(x^2+2x-1)=0 \quad x^2+2x-1=0$

$x_1=1 \quad x_2=1$

Figure 5. Pedro's solution for task 4.

Conclusions

The participating students had no difficulty in recognizing that the generation of numerical successions is a feasible procedure for arriving at a solution for the first three tasks. Although in this exploration, the students use recursive procedures when they attempt expressing a general result, they tend to do so in terms of an explicitly defined function. It seems that this tendency is due to the fact that the students had not had sufficient opportunities to use recursive models as mathematical resources to express their mathematical ideas and thought.

The students, although they pay attention during the process of solution, say that it "is boring because many operations are repeated," focus mainly on the exactitude of their calculations and not on the procedure itself. Only after they reflect on this aspect can they perceive that this is an important aspect of the tasks.

The organization of the data generated in lists and tables was a common characteristic in the work of all of the students during the process of solution and, even though at that time they were learning to graph linear and quadratic functions using tables in their algebra course, only one student attempted representing the data with a graph.

One important aspect that encouraged reflection and communication of ideas was the variety of models generated by the students for the same task. The discussion of analogies and differences and the viability of the models generated gave the students the opportunity to appreciate the power of mathematics as a tool that allows a diversity of forms to express the same situation.

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Appendix

Task 1

The population in Mexico in the year 2000 was 97,483,412. If the population is growing at a rate of 2.5% per year, what will be the expected population of Mexico in the year 2010?

Task 2

A carpenter charged 10 cents for the first nail hammered in the construction of a fence. For each additional nail he charged 20 cents more than for the previous one. (Adaptation of the problem in Algebra, by Ress and Sparks, 1968)

(1) How much money did the carpenter receive for hammering 24 nails?

(2) If the carpenter wants to know how much to charge for an x number of nails, how could he figure it out?

Task 3

A print shop charges $100.00 to print 1000 flyers and, for every 500 additional flyers, the charge is $20.00 more than for the previous amount. Write a formula that would allow the printer to give quick estimates for the cost of printing *n* flyers. (The instructor indicated that a quick estimate consists of determining the cost directly from a given number of flyers, as was mentioned in the above discussion).

Task 4

Arrive at a recursive function and an equivalent explicit function for the following progressions:

(1) 1, 5, 9, 13, . . .

(2) 1, 2, 3, 4, . . .

THE NATURE OF REFLECTIVE THINKING IN MULTIVARIABLE CALCULUS

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This paper reports on the vital role that reflective thinking plays in solving problems involving the mathematics of change and variation particularly multivariable Calculus. We report on how reflective thinking is one kind of self-regulatory (in the sense of metacognitive) thought mechanism. We present an outline of the results of a larger study, which investigated the metacognitive behavior of undergraduates solving problems in the Calculus.

Theoretial Perspective

> Metacognition refers to one's knowledge concerning one's own cognitive processes or anything related to them, e.g., the learning-relevant properties of information or data ... Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of those processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete [problem-solving] goal or objective. (Flavell, Friedrichs, & Hoyt, 1970, p. 323)

This introductory quote by Flavell et al. set the scene for the next 30 years of inquiry into a complex cognitive mechanism which has received much attention as well as skepticism from various research traditions – from Neuroscience to Mathematics Education.

In Mathematics Education, Schoenfeld (1985, 1987) introduces metacognition through the Artificial Intelligence notion of *Control* as a cognitive strategy for resource allocation. Similarly, Garofalo & Lester (1985) introduce Control as related to metacognition:

> Metacognition has two separate but related aspects:
>
> (a) knowledge and beliefs about cognitive phenomena, and
>
> (b) the regulation and control of cognitive actions. (p. 163)

Later Schoenfeld (1985) added a social dimension to the inquiry by observing a Vygotskian perspective on regulation-of cognitive strategies as a process by which we observe and internalize other people's similar thought mechanisms. Through operating in the *Zone of Proximal Development*, learners observe and replicate overt examples of how other people regulate their own problem-solving behavior. Through this practice, we proximally appropriate the behavior as part of own cognition in solving mathematical problems and specifically as part of our own revised regulatory behavior. Brown (1987) also observes this kind of "other-regulation" as being one of four historical roots for metacognition. So, in attending to studying the metacognitive behavior of students learning mathematics it soon becomes apparent that the relevant literature in Mathematics Education and Cognitive Science (and the overlap) create a multifarious if not multi-headed beast that can lead to confusing lines of inquiry. While it is evident that metacognition is still studied in the fields of Neuroscience (Shimamura, 1994) the word is more so used in mathematics education rather than studied. It is evident in the National Council of Teachers of Mathematics (NCTM) *Standards* (2000) that instructional programs from pre-kindergarten through grade 12 should enable all students to *monitor* and *reflect* on the process of mathematical problem-solving:

> Good problem solvers become aware of what they are doing and frequently monitor, or self-assess, their progress or adjust their strategies as they encounter and solve problems. Such reflective skills (called metacognition) are much more likely to develop in a classroom environment that supports them. (NCTM, p. 54)

It is not evident firstly, what this process of mathematical problem-solving is, hence enabling teachers to observe this behavior in their own students; nor secondly, what kind of classroom environments do actually support these skills as the Standards report. In our work, we have regarded the term metacognition as an umbrella term to embrace multiple perspectives such as Control (from AI), socially-constructed regulation (Vygotskian perspective) and self-regulation (Piagetian perspective).

Regulation-of thinking has played a significant role in Piaget's work (cf. 1978) and that of his contemporaries (Inhelder, Sinclair, & Bover, 1974; Karmiloff-Smith, 1979) where they refer to it as self-regulation and conceptual reorganization. Karmiloff-Smith & Inhelder's ideas explain how these mechanisms are characterized by the internal pressures to systematize, consolidate and generalize knowledge. Following our initial pilot work and study, our analysis highlighted the necessity to concentrate on this one strand of metacognition which offered us a suitable theoretical perspective given the nature of the learners we were studying and our research design.

Garofalo & Lester (1985) cite some evidence of *knowledge-of* thinking. They claim that older children realize that memory ability differs between pupils and that some realize that they can recall better than other individuals. Our pilot work showed similar evidence of this type of thinking and that self-regulatory cognitive strategies were far more apparent.

The main piece of empirical work (Hegedus 1998) further enhanced the meaning of self-regulation in advanced mathematical thinking by observing and verifying four particular strands of this type of metacognition: *Reflection, Organization, Monitoring* and *Extraction* of mathematical resources. In Hegedus (2001), we reported on the role of organizational thinking in the integral Calculus. This refers to the planning behaviors that a problem-solver engages in, in both the exploratory phases and the execution phases of a problem. We now outline the method of data collection and analysis and then report on another strand of self-regulatory thinking outlined above; the nature of *reflective thinking* particularly when dealing with problems in the Calculus of many variables. As stressed in our previous report, our main theoretical perspective here is grounded in the nature of the mathematics that our students are learning. In detailing the nature of the students' problem-solving behavior in terms of their self-regulation we are examining how particular aspects of the mathematics of multivariable Calculus calls for particular mathematical cognition. Hence our study interprets psychological behavior in terms of mathematical procedure, conceptualization and execution of method.

Methods of Data Collection

Building on the work of Schoenfeld (1985), a pilot study was completed using a think-aloud protocol analysis to observe the role of *knowledge-of* and *self-regulatory* thought processes by undergraduate mathematicians. Our initial research aims focused on highlighting what kind of metacognitive activities the undergraduate students exhibited and how effective these were in their work. The Schoenfeld methodology proved most effective in attending to our primary research aims. We used a method of protocol analysis to focus on the decision-making processes at the executive level or control level. Schoenfeld highlights, in chapter 9 of his book, how this method provides ways of 'identifying three classes of potentially important decision making points in a solution':

1. Decisions at the control level are those that affect the allocation or utilization of a substantial amount of problem-solving resources (including time). It is thus appropriate to look for executive decision-making at points in problem solutions where there are major shifts in resource allocation.
2. The second type takes place when either new information or the possibility of taking a different approach comes to the attention of the problem solver(s).
3. The third category of decision-making points is far subtler. These are times in a solution where nothing has gone catastrophically wrong, but when a string of minor difficulties indicates that it is probably time to consider something else.

We adhere to Schoenfeld's method of data collection and analysis by obtaining verbal data of think-aloud accounts by our respondents, which we then record and transcribe. Each item of dialogue is then numbered. We then parse the protocol into six episodes and using the numbering system make the data more referable. An illustration of the parsing of the solution process (see http://merg.umassd.edu/heg/thesis) was derived by a consensus opinion of a team of three undergraduates working with Schoenfeld. The team had been trained in parsing the data in this way and the results were found to be very reliable. These are then plotted on a time-line diagram to offer a representational image of how the solver changed problem-solving states with respect to time (see figure 1) completed in Excel. The six states are *Reading, Analyzing, Exploring, Planning, Implementing, and Verifying*. Triangles are used on top of the bars to show times of overt management activity – we did not attend to this final detail, as the respective activities were too numerous to illustrate.

Schoenfeld (1985, p. 297-301) outlines a description of the six episodes and associated questions, which are relevant to our analysis. These episodes refer to particular behavioral states, under which Schoenfeld believed that most problem-solving activity fell.

In his analysis Schoenfeld includes local assessments and introduction of new information with other overt managerial behaviors. He describes managerial behaviors to include:

> [S]electing perspectives and frameworks for working a problem; deciding at branch points which direction a solution should take; deciding in the light of new information whether a path already embarked upon should be abandoned; deciding what (if anything) should be salvaged from attempts that are abandoned, or adopted from approaches that were considered but not taken; monitoring and assessing implementation "on-line" and looking for signs that executive intervention might be appropriate; and much, much more. (Schoenfeld, 1985, p. 152)

We use the time-line diagrams to give an overview for the solution attempts of our students across a variety of problems as well as comparison with expert's solutions we similarly categorize and analyze. We then utilize the questions set out by Schoenfeld for analyzing the times of transition between episodes and the times where new information and local assessments occur. These include:

New-Information and Local Assessments

N1: Does the problem-solver assess the current state of his knowledge? (Is it appropriate to do so?)

N2: Does the problem-solver assess the relevancy or utility of the new information? (Is it appropriate?)

N3: What are the consequences for the solution of these assessments or the absence of them?

Transition Points

T1: Is there an assessment of the current solution state? Since a solution path is being abandoned, is there an attempt to salvage or store things that might have been valuable in it?

T2: What are the local and global effects on the solution of the presence or absence of assessment as previous work is abandoned? Is the action (or lack of action) taken by the problem-solver appropriate or necessary?

T3: Is there an assessment of the short- and/or long-term effects on the solution of taking the new direction, or does the subject simply jump into the new approach?

T4: What are the local and global effects on the solution of the presence or absence of assessment as a new path is embarked upon? Is the action there appropriate or necessary?

This method of analysis highlighted further categorization of what self-regulation means in the problem-solving behavior of Calculus students and its effect on mathematical efficacy.

Data Sample

Primarily, two classes of Calculus and Advanced Calculus were observed for two semesters at a leading British University. Working closely with the students in problem-solving classes certain students were chosen for their ability to think-aloud well and their problem-solving behavior. This said, the students chosen were still of varying ability and gender. We then completed a pilot study with three groups of 2 to 3 students per group to explore the existence of various metacognitive behaviors. Over the course of the following year, these three groups met periodically through two stages of the main data collection. Both sessions involved solving problems from the integral calculus. They had covered the relevant material that same semester yet they had not seen the problems previously. Examples of the problems include:

Problem 1

$$\text{Evaluate} \iint_R x^2 \sin(x^4 + 2x^2y^2 + y^4)\,d(x,y)$$

$$\text{where } R \text{ is the region satisfying } x^2 + y^2 \le 1 \text{ and } y \ge 0.$$

Problem 2

$$\int_0^1 dx \int_x^{x^{1/3}} \sqrt{1 - y^4}\,dy$$

These questions call for a degree of spatial awareness and flexibility, manipulation of co-ordinate systems and the geometric consequences, and retrieval of algorithmic and algebraic techniques from other areas of the Calculus.

Stage 1 of the empirical work comprised of two exploratory studies. These studies illuminated certain characteristics of metacognitive thinking which affected their problem-solving skills and led to the development and refinement of the four main items of self-regulatory thinking - one of which we report on now. Stage 2 aimed at verifying and further developing the four main areas of self-regulatory thinking evident in stage 1. The observer intervened in stage 1 to assist in thinking-aloud which was in direct opposition to Schoenfeld's methodology. In stage II, a strict interventionist agenda was set up to direct the line of inquiry. In both stages an interventionist analysis was conducted to observe any bias in contriving the reported metacognitive behavior. Limitations of this report prevent us from detailing this method but it is incorporated in our analysis.

Analysis of Reflective Thinking

Throughout the study it was evident that the students' metacognitive behavior in the form of self-regulation was intimately bound up with the mathematics.

Three types of reflection describe this behavior:

1. Forward-reflection,
2. Backward-reflection,
3. A-temporal reflection

The first refers to looking forward into the solution and how it might develop; predicting outcomes or expecting changes based upon the solution so far or through experience. The second refers to checking and verifying the algebraic and geometric structure in the solution so far; why these were chosen and how they have an affect on how the solution might develop. The third is line-by line and on-the-fly checking often performed without close regard but nevertheless contributing to the coherence or development of the solution. This latter one is harder to distinguish yet more understood as part of problem-solving. The former two are what we particularly concentrate on in our analysis.

It is evident that each type of reflection is largely guided by the form and shape of the various mathematical signs and symbols in the students' problem-solving domain.

In general, the students involved in the study often only examine the variables of integration when they are substituted for another rectangular variable or converted into polars. Reflection on such a change of variable is often in the form of backward reflection. The students reflect upon the suitability of the change of variable with respect to simplification of the algebra. The change is not a function of the theoretical suitability or effectiveness of the new variable, which is a form of effective forward reflection. This has repercussions on the limits of integration that were formulated through the region of integration.

It is evident in the students' behavior that changing the order of integration could minimize the levels of difficulty in the algebra (i.e. forward reflection) but the theoretical implications of such actions again are often not acknowledged.

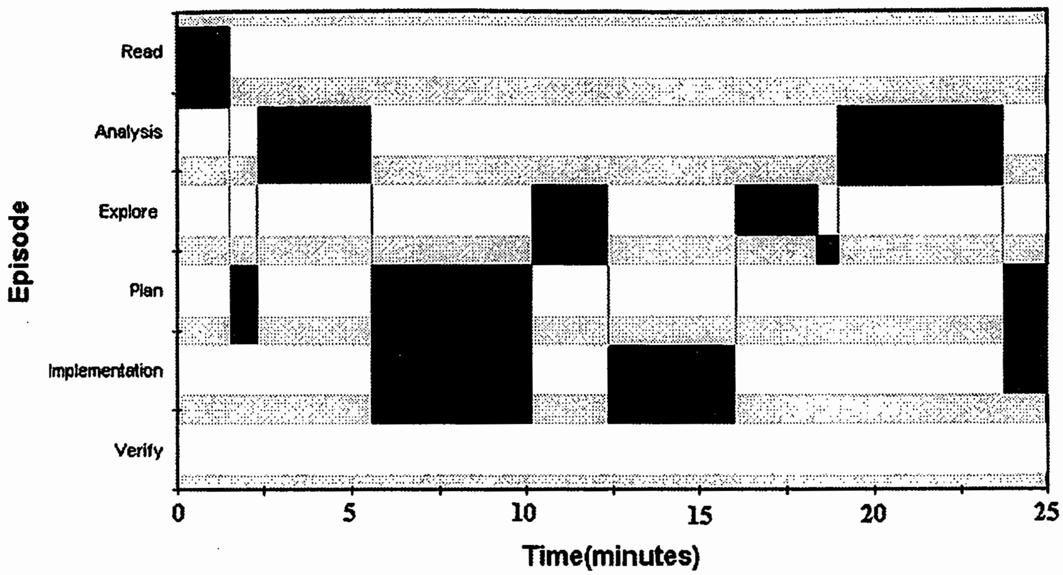


Figure 1

In general, it is reflection on the shape of a region and the form of the algebra that leads to a change in the variables used. We offer extracts of various pieces of work where two average students are working on the first and second problem above.

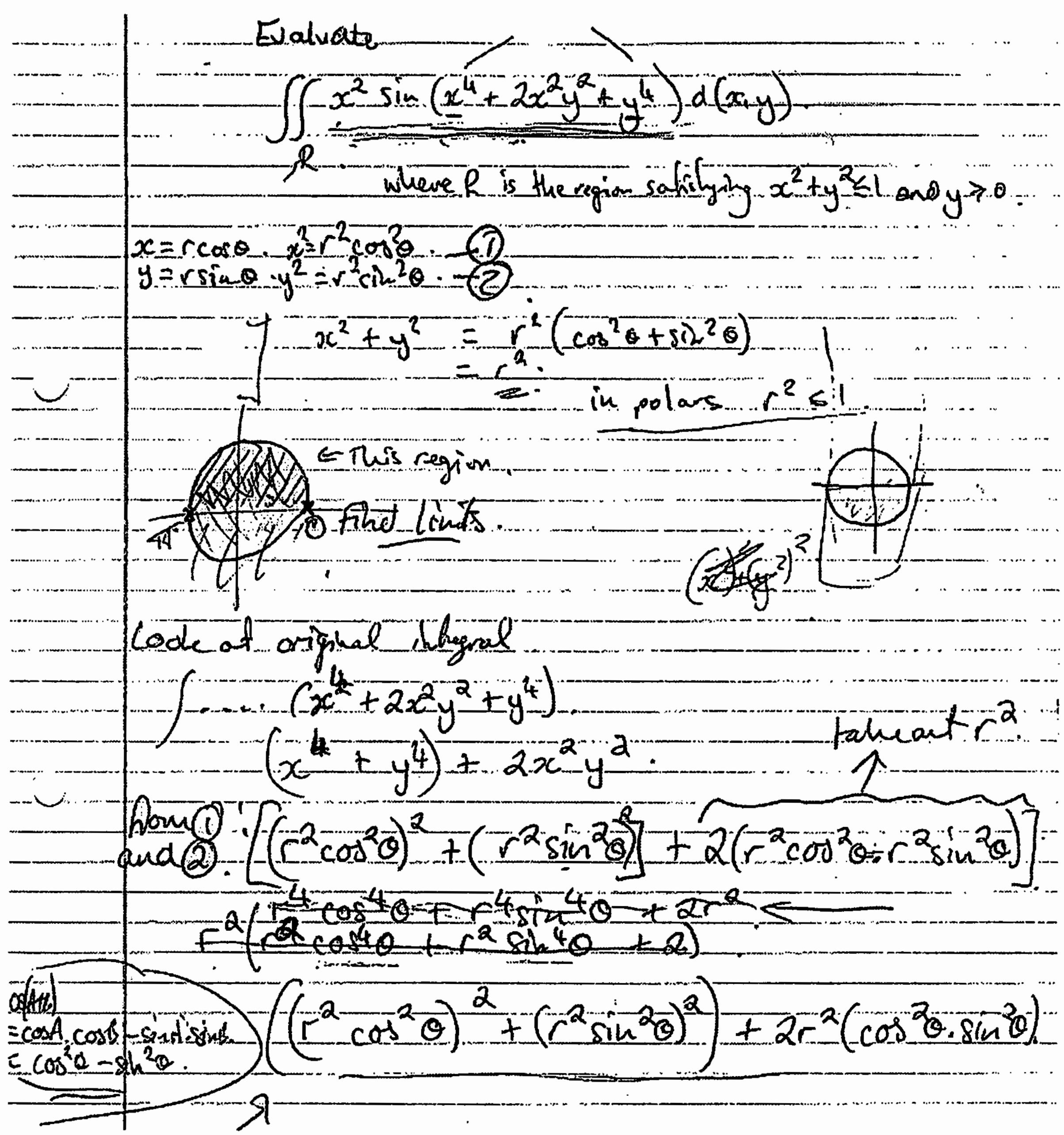
Following a Schoenfeld protocol analysis we constructed time-line diagrams for each of our sessions. One of the groups completing problem 1 is given in figure 1. It is evident that the students' work followed many periods of transition (grey block areas) and circulated through many plans. We performed the same analysis for expert problem-solvers (Mathematics Professors at the same university), and what is comparative with their solution and the student's work at the meta-level is the extended use of analysis time.

Our protocol analysis has highlighted particular aspects of their work (albeit ineffective) which relate to their reflective thinking as a self-regulatory thought mechanism through analysis of the verbalizations of the student with their written work. This was intimately bound up with the mathematics they were engaged with and their comprehension of the various concepts, algebraic and geometric techniques that they struggled with.

We offer one example of the verbal data which highlights the students' reflective behavior in attempting example 1. Backward reflection on the circularity of the region R: $x^2 + y^2 < 1$, already drawn in their work (see figure 2), leads one of the students to change the rectangular variables to polars (i.e. cylindrical):

G: Erm I think we might have to change the er ... the variables to polars because er ...

S: Why do you say that?



Evaluate

$$\iint_R x^2 \sin(x^4+2x^2y^2+y^4)\, d(x,y)$$

where R is the region satisfying $x^2+y^2\le 1$ and $y \geq 0$.

$x = r\cos\theta$. $x^2 = r^2\cos^2\theta$. 1

$y = r\sin\theta$. $y^2 = r^2\sin^2\theta$. 2

$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta)$
$= r^2$.

in polars $r^2 \le 1$

← This region.

find limits.

$(x^2+y^2)^2$

Look at original integral

$\int \ldots (x^4+2x^2y^2+y^4)$

$(x^4+y^4)+2x^2y^2$.

take out r^2.

from 1 and 2: $\left[(r^2\cos^2\theta)^2+(r^2\sin^2\theta)^2\right]+2(r^2\cos^2\theta\cdot r^2\sin^2\theta)$

$\cancel{r^4\cos^4\theta + r^4\sin^4\theta + 2r^2}$

$\cancel{r^2(r^2\cos^4\theta+r^2\sin^4\theta+2)}$

$\cos(A+B) = \cos A.\cos B - \sin A.\sin B = \cos^2\theta - \sin^2\theta$.

$$\left((r^2\cos^2\theta)^2+(r^2\sin^2\theta)^2\right)+2r^2(\cos^2\theta\cdot\sin^2\theta)$$

Figure 2.

G: Well because we're doing a double integral we going to end up with a volume over that the base .. which is $x^2 + y^2$.. and because it's less that or equal to 1 it's the disc not a circle so .. we've got a circle and everything inside of it so you've got like this base .. horizontal and you're going to be evaluating up in the air and you need to find what the limits are .. and polars are a bit easier to work with.

S: Why would they be easier? What would they convey? What would that expression be in polars? The $x^2 + y^2$...?

G: Erm .. I'll write it down. Because it's $x =$ ... r cos$q$, I think .... y's r sin. [Pause]

G: Right so that's r^2 ... should have known that ... but it's in polars so you've got r^2 less that equal to 1.

S: So that region has led you to say polars?

G: Yeah. [G - Student S - Researcher]

Having completed this stage, the students then thought about the implications of their substitution on the solution.

The region of integration plays an important role in determining the limits of integration and suitable methods of integration. It is evident in many of the students' work that the provision of a diagram for the region of integration is useful to these effects, but it is only effective if a theoretical understanding of the region is evident.

Reflection on the region often concerns the method of 'slicing', either by using washers or discs, in single integration. In double integration, it affects the construction of the limits of integration. It is not evident in their reflective behavior, though, by what means the region could effectively establish a suitable procedure of integration that would decrease difficulty in algebraic manipulation. Whilst reflection on the integrand in geometrical terms is not evident (i.e. as a surface $w = f(x, y)$, say, in double integration, or a hypersurface $w = f(x, y, z)$ in 4 space for triple integration) this behavior does not have any direct effects on their problem-solving processes. Reflection on the form and shape of the integrand and its algebraic construction is evident and this affects the choice of method, and the algebraic manipulation of it. In the example below, G & D (students) are attempting to solve the integral in problem 2:

S: Why can't you integrate it?

D: Well ... its in terms of y's?

S: Yeh.

D: Because .. I can't remember why but the function is er ... erm ...

[Pause]

G: Is it just because it's improper because it's y to the 4. I know you can do it if it's like something squared .. minus something squared square rooted.

S: Yes you saw that last term .. last semester, didn't you?

D: Yeh.

G: You've got to make a substitution with the y to the 4 .. you've got to substitute like er $z = y^2$. So like you then you got $1 - z^2$.

D: Which is $1 - z^2$ which equals er .. a standard integral, which is the back of the book. which I can't remember what it is?

The students then debate whether to change the order of integration and thus integrate the integrand with respect to x first rather than y. This, of course, would be the most suitable method, as long as suitable changes in limits are made.

In other examples the acknowledgement of oddness/evenness or symmetry in the integrand lead students to effectively reduce the levels of algebra in the integration procedures.

Reflection on the limits of integration seems only evident in the substitution of them. Even then, reflection is minimal. Furthermore, line-by-line reflection is not evident which results in algebraic difficulties and mistakes.

The shape and form of the algebra is often the driving force behind the organization and retrieval of procedural tools, but the functionality and efficiency of these tools is poorly reflected upon. These tools are both algebraic and algorithmic, and include for example, methods of anti-differentiation, trigonometric substitution, and the substitution of trigonometric identities.

Conclusion: Links to Practice

In general, reflection on the form and shape of the algebra lead to a variety of conceptual and procedural thinking. In conclusion, backward reflection caused the manipulation of algebra and the interpretation and reconstruction of geometric objects, which were part of the Calculus problem. Such behavior was conducted to varying degrees of efficiency. Progress occurred when reflection was also in a forward sense that looked critically at the choices made and understanding the theoretical implications (in terms of the Calculus) of their implementation. In addition, reflection is supported by the implementation and integration of the three other main self-regulatory behaviors (organization, monitoring and extraction) highlighted earlier.

This work has shaped a theoretical perspective to observe how self-regulatory thinking is a function of the mathematics being investigated, particularly the Calculus, and our present line of inquiry is exploring how these insights help inform pedagogy and how they might be implemented.

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UNDERSTANDING OF FUNCTION AS SEEN IN UNDERSTANDING OF MULTIVARIATE FUNCTION: THE CASE OF PROSPECTIVE SECONDARY MATHEMATICS TEACHERS

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Key to the current discussion about the mathematics education of today's youth are concerns about the mathematical knowledge and background of secondary school mathematics teachers (CBMS, 2001). Current research suggests that teachers with robust understandings of the mathematics they teach are capable of teaching for understanding. For example, Lloyd and Wilson's (1998) case study of one teacher suggested that "teachers' comprehensive and well-organized conceptions contribute to instruction characterized by emphases on conceptual connections, powerful representations, and meaningful discussions" (p. 270). Current research (Manouchehri & Goodman, 2000) also suggests that teachers without such robust and connected understandings may be unable to capitalize on opportunities to enrich the discussion of mathematics in their classrooms.

Although deep understanding of mathematics is not all that is needed for teachers to teach for conceptual understanding and mathematical meaning (Ebby, 2000; Wilson, 1994;), such understanding does seem to be central to teacher's effectiveness, and so it is important to understand the nature of prospective secondary teachers' understandings.

Central to the organization of mathematics is the concept of function (Bidwell & Clason, 1970; MacLane, 1986), a notoriously difficult concept for students (Dreyfus, 1990; Markovits, Eylon, & Bruckheimer, 1988; Yerushalmy & Schwartz, 1993). The word "function" often evokes the notion of $\Re$ to $\Re$ functions, and research on understanding of function focuses on $\Re$ to $\Re$ functions. With the renewed emphasis on mathematical modeling in new school curricula and access to interactive three-dimensional environments, however, multivariate functions command increased attention. One can examine the depth of student understanding of the function concept by examining the transferability of this idea from the familiar, real-valued functions of one variable, to the unfamiliar, real-valued or vector-valued functions of several variables. As Yerushalmy (1997) suggested, as students deal with multivariate functions and related new notation systems, new notions of dependency (coordination of multiple dependencies), and the notion of families of curves as a single object, their concept of function is stretched. Our study characterized prospective secondary mathematics teachers' understandings of function as seen through their work in multivariate settings.

The Study

This study examined the mathematical understandings of prospective secondary mathematics teachers of the concept of function as shown through their work in the context of their participation in a course in which one focus was multivariate functions. The context was an elective mathematics/mathematics-education course at a large university. For two consecutive semesters, two mathematics education researchers, one research mathematician, and several graduate students designed, implemented and studied student understanding in the context of students' work within the courses. Although the two courses consisted of different material, both courses adopted as a central focus functions of several variables. This report will focus on the data gathered during the second of these two courses.

The Course

The course consisted of 15 three-hour class sessions each semester. One of the major foci of the course was to develop students' understandings of function. In particular, students studied a range of manifestations of function, including multivariate functions, transformations, and contextually embedded functions. In this paper, we will focus on their work with multivariate functions. Students studied multivariate functions through examination of graphical, analytical and contextual/data and model-driven representations of functions of several variables. During the course, we engaged students in investigating functions of several variables using electronic technologies (Geometer's Sketchpad, Mathematica , DPGraph, TI-92+, etc.), manipulatives (Zometool and DoodleDome), and office-supply materials. Students studied the decomposition of graphs of functions of two variables into curves obtained by slicing the graphs with 'standard' planes (those parallel to the coordinate planes). Students also studied the construction of the entire graphs using a few such slice curves. Students were called upon to make connections between graphical and analytic representations of such functions.

A feature of our approach to teaching about multivariate functions was to help students reduce the study of multivariate functions to the study of a collection of functions of one variable obtained by fixing a number of the variables at constant values. Students who learn to view functions of two variables this way would presumably be able then to use their understanding of graphs of real-valued functions of one real variable to identify the resulting curve. Generally, the graph of a real-valued function of two variables is a surface. Geometrically, fixing one variable at a constant value corresponds to slicing the surface with a plane. The intersection of the plane and the surface is then a curve in that plane. By fixing the variable at different constant values, one gets different planes and hence a family of slice curves. One could also vary which variable to hold constant, producing a different family of slices the collection of which also describes the surface. If the students could understand the family of curves so obtained, it was our assumption that they would then be able to work on the problem of trying to understand how those curves could be assembled into the original surface. They also could work on understanding how attributes of the slice curves determine attributes of the surface, and of the original function in question. Students in these courses worked on the graphical to analytical (given the graphs, work on developing analytical formulas) and the analytical to graphical (given the analytical formulas, work on developing graphs) implementations of these strategies.

Data Sources

The study in which this investigation lies relies on four data sources: videotapes of whole class sessions, videotapes and annotated transcripts of small group work in class (during the Fall class only), videotapes and annotated transcripts of individual interviews, and copies of out-of-class assignments (used primarily for planning of subsequent class sessions). The Fall 2001 data included beginning-of-semester, mid-semester, and end-of-semester interviews with eight students. Each interview lasted 60 to 90 minutes. The whole class sessions, small group work and individual interviews provide different insights into students' understanding. Our primary analysis for this paper entailed examination of the interview data, corroborated by the data from the other sources.

Whole group discussions were based on inquiry and exploration, and shifted the authority to the student. This enabled us to examine evidence of understanding as seen in students' reactions to other groups' conjectures or conclusions, and in the ways in which students provided what they saw to be convincing arguments in a group setting.

Small group discussions were a regular part of the class sessions. Each group generated results that each member was responsible for being able to explain/support. Although the instructors sometimes posed clarifying and extending questions to individual groups as they watched the group's work, for the most part, the groups worked without interference. This enabled us to examine evidence of understanding as seen in the ways in which the groups approached and solved problems and in reactions of group members to other students' ideas and reactions to other groups' conclusions/conjectures as well as in interaction with other students and other groups.

Individual interviews provided us with evidence of the arguments that were convincing to the student him/herself. They provided us with the opportunity to probe and examine students' understandings more deeply than what is possible in small group and whole group settings. We could observe what students did as they generated ideas on their own and reacted to mathematical ideas and situations. Throughout the course, students encountered a range of examples of functions. It was through these experiences that we expected students to develop a broad and connected concept of function that would generalize beyond an ability to work with a prototypical $\Re$ to $\Re$ function. From our perspective, a function is a mapping from one set (the domain) to another set (sometimes called the codomain) so that each element of the domain is assigned to one well-defined element of the codomain. A relation is a mapping from one set (the domain) to another set (sometimes called the codomain) without the restriction that each element of the domain is assigned to exactly one element of the codomain The domain elements are called inputs and an element of the codomain corresponding to such an input is called an output. The set of inputs and the set of outputs need not be confined to numerical values. The set of output "values" associated with the domain is called the range of the function, and may be a proper subset of the codomain. The vertical line test is a strategy for ruling out particular relations as functions. Families of functions are functions with a common characteristic, frequently with a function rule that has a common form with parameters whose values vary among different members of the family. The meanings of fixed points, maxima and minima, domain, and range all draw on the general meaning of function as a mapping.

Through the interviews, we sought to understand students' understandings of the function concept. We asked questions about function and related ideas in a range of contexts with which students were expected to have varying familiarity. In this paper, we will focus on student responses to questions asked during the third interview.

The third interview was conducted at the end of the semester. During the third interview, students were asked about their concept of function with questions like:

- What is a function?
- Can you give an example of a function?
- Here is a common definition of function: A function is a set of ordered pairs no two of which have the same first element. How can you tell whether something is a function?
- Can you use the vertical line test to tell whether the graph I am showing you [graphs of $x = 9y^2 + 36$ and $y = 9x^2 + 36$ are shown] is that of a function?
- Some functions have what we call fixed points. Can you tell me what a fixed point is?
- Here is a common definition of a fixed point: If x is an input value for which $f(x) = x$, then x is called a fixed point of f. Can you tell me whether the function $f(x) = x^2 - 3x - 12$ has any fixed points?

Portions of students' understandings of the function concept were disclosed in their work with functions of two variables. During the interview, three particular settings were provided as occasions for the students to talk about functions of two variables. The first setting was an article appearing in a nationally distributed newspaper reporting a new formula for the wind chill factor. The article had several typographical errors. The first typographical error was the omission in the function rule of one of the variables; the new wind chill factor was reported as a function rule with only the air temperature (and not the wind speed) as input. The rule appearing in the article was "$WC(°F) = 35.74 + 0.6215T - 35.75(^{0.16}) + 0.4275T(^{0.16})$", instead of the correct rule, "$WC(°F) = 35.74 + 0.6215T - 35.75(V^{0.16}) + 0.4275T(V^{0.16})$". The second typographical error was the omission of information from a graph related to the function rule. The graph was a two-dimensional slice of the three dimensional graph, but no information was given identifying which particular slice was the one that was shown. Students were asked to make sense of the article and its representations of the function rule. The second setting was an $\Re^2$ to $\Re^2$ function, the transformation of points in the plane to other points in the plane by a mapping 45 degrees counterclockwise about a given point C. Students were asked whether the rotation was a function, and why. They were asked to argue from a given definition of fixed point whether the rotation had any fixed points. Third, the students were given a dynamic graph (one that rotated) of the function, $f(x,y) = 5xy + 4x + 4y$, with the function rule appearing on the graph. One frame of the rotating graph animation is shown in Figure 1. Students were asked questions like the following about this function of two variables: Is that relationship a function? Why? Can you use the definition of function to show that the relation is or is not a function. Does this relation (or function, if they decide it is a function) have any fixed points? Can you use the definition of fixed point to convince me that (insert their conclusion)?

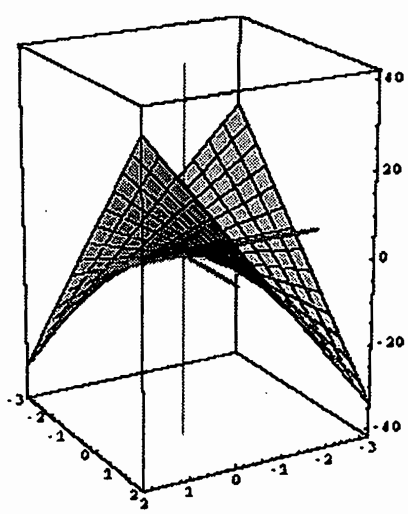

Figure 1. One frame of a dynamic graph (one that rotated) of the function, $f(x,y) = 5xy + 4x + 4y$

Data Collection Methods/ Analysis Methods

In all three settings we video-taped and audio-taped to capture students' verbal statements and written work, and we directly captured the results of their work when using computer algebra systems (CAS) and dynamic geometry software using screen-capture technology (scan converters, TI Presenters).

Our ongoing analysis began with weekly reviews of the class sessions. We identified evidence related to understanding of key mathematical ideas, and reviewed them in planning for subsequent lessons and data collection. From annotated transcripts of interviews and small group work, we identified excerpts that involve functions of several variables. We are in the process of building models of students' understandings of functions, particularly as seen through functions of several variables. In building these models we are describing the nature of understanding of functions of several variables as shown in different contexts (e.g., dynamic three-dimensional graphs, symbolic representations of real world quantitative relationships).

Results

Based on the data we had gathered throughout the course, we made the following general observations:

- The prospective teachers with whom we have worked at first found the concept of multivariate functions difficult in spite of having successfully completed courses in multivariate calculus. They especially found it difficult to visualize graphs of functions of two variables. By the end of the course, many of them grew more comfortable with graphical representations and contexts representable by functions of two variables.
- There was some indication in our data that slicing planes enabled students to modestly improve their understanding of analytic/algebraic formulas for functions of several variables.
- At the end of the semester they had tools and strategies they could use to examine and study functions of several variables in an applied setting described in a current newspaper article.
- Students' sensitivity grew to the ways in which functions of several variables can describe real-world situations.

Several of the interview questions were particularly fruitful in revealing student understandings of function in that they suggested the nature and limits of students' understandings of the function concept. In the remainder of this paper we will present observations about the mathematical understandings of three of the students in our study (Bob, Ned, and Violet). We have selected these students in order to present a range of understandings of function among students in the same class.

Students' Approaches to New Concepts as Seen in Their Work with Functions, ParticularlyMultivariate Functions

Students' approaches to understanding new concepts are revealed through their work with new ideas about functions, including functions of two variables. For example, Violet frequently confined her reasoning to the information that was currently in front of her. When presented with the newspaper article about the wind chill situation, she frequently used the empirical data presented in the article as her primary authority rather than reasoning from the function rule that was given. It was the fact that the article refers to the wind speed and air temperature being measured and used in the new wind chill formula that alerted Violet to the need for two variables in the formula. While she confined her reasoning to the given, she used the given information carefully and accurately. When given the graph appearing in the article, she immediately remarked that the equations in the graph differed from those in the article. She quickly determined that the graph depicted the wind chill when the temperature is held constant, and she recognized that the air temperature was 5 degrees from reading the website graph. She was careful not to reason beyond the information she was given, and this tendency was sometimes coupled with a tendency not to reason across representations. In the dynamic graph situation, she hesitated to make conclusions about values that exceeded the limits of the window provided, in spite of the fact that she was also given the function rule for the graph.

Violet's reasoning about function seemed to be in a transition between rote understanding and sense-making. She displayed almost a stimulus-response reaction when asked to describe slope, by immediately responding "rise over run" without specifying what the "rise" and "run" represented. A similar phenomenon occurred when she was faced with a quadratic function rule; her immediate reaction was, "... my first thought was to factor it." Violet was able to reason using critically conceptual prototypes to clarify her own concepts. She recalled somewhat vaguely that one could determine that something was not a function by using either a vertical or a horizontal line test, and reasoned from prototypes to decide which rule made sense. She used rich prototypes to clarify her understanding of this and other concepts (domain and range; whether a function needed to be one-to-one). Whereas Violet was able to clarify her understandings by consulting prototypes, her reliance on algebra conventions sometimes muddied her understanding. For example, with her reliance on the convention that x is an input variable and its associated "vertical line test," she was not able to view the graph of $x = 9y^2 + 36$ as a function (with x as a function of y). Violet did not reason from definitions even when they were supplied, but rather tried to fit these definitions into her own concepts. For example, when she was given a definition for function, "A function is a set of ordered pairs no two of which have the same first element," she debated the definition by thinking about the vertical line test and some critically conceptual examples of functions on which she had learned to rely.

Students' Understandings of Function as Seen in Their Work with Multivariate Functions

Students' understanding of function is revealed through evidence of their understanding of functions of two variables. Ned seems to understand function as a mapping from a given set to another set, and displays that generalized understanding in a range of encounters with $\Re$ to $\Re$ functions and even with geometric transformations (types of $\Re^2$ to $\Re^2$ functions). He had difficulty, however, in applying the definition of function to a graph of f with rule $f(x,y) = 5xy + 4x + 4y$ or $f((x,y)) = 5xy + 4x + 4y$. He seemed to need to think of two numerical inputs for the function instead of thinking of the input as an ordered pair of values, (x,y). Ned may have been uncomfortable with treating entities other than numerical values as input for functions.

It's just that it seems, it just sort of seems to follow that, you know, I have this idea of what a function is in two-dimensional space. So then I go to three-dimensional space, I have an extra value in my ordered set. Now I have an order of triples instead of ordered pairs. I'm just sort of following along with that function and just making a logical step in saying that. And they can't have the...the first two elements can't be the same in order for a three-dimensional figure to be a function.

This example of thinking of input as numerical was unusual for Ned. Bob, on the other hand, seemed almost always to think about input for functions as having to be real numbers, thus inhibiting his transition to the idea of fixed points for a function with a three-dimensional graph. After seeing the definition of fixed point, Bob changed his way of approaching the idea of fixed points and came to an understanding that a fixed point is an input with an equal corresponding output. However, his only way of deciding on whether an input value was a fixed point or not was to calculate and compare numerical inputs and outputs, and he was unable to determine fixed points symbolically. Although Bob's numerical bent served him well in his work with $\Re$ to $\Re$ functions, it served as a cognitive obstacle when Bob needed to generalize to other types of functions.

Unlike Bob's and Violet's understandings, Ned's understanding of function was flexible enough that he did not need to rely on a single strategy for determining whether a relation was a function. Bob and Violet relied on the vertical line test and were able to apply it to $\Re$ to $\Re$ functions. When they were asked how it might apply to a function of two variables, however, the fragility of their understanding was uncovered. Bob struggled to construct a way to apply the vertical line test in three dimensions.

Bob: I'm trying to think of ...how you're not going to get two outputs for each input. But you do have two outputs here for the same...I don't know, whatever value that is. ...You're getting two outputs. Maybe you're not gonna get three outputs when it changes to that.

Interviewer: Why three?

Bob	Because now it's a function of two variables. Because these two points [pointing to two points on the graph of the surface in Figure 1 with his small and index fingers] are definitely the same z. And so are these two and so are these two and yada yada. Umm... Or wait, no, that's... They wouldn't go this way. I want to look at it this way. I want to look at it this way. Yeah, well, I guess you could do the same test [moving his small finger from up to down].
Interviewer:	When you say the same test...
Bob	Maybe vertical line test? I don't...I've never done that three-dimensional. I haven't seen much three- stuff. ... And you'd only get one value. And I'm trying to think of a graph, if I've ever seen it where you get more than one, but I don't think so. Hm-mm. I mean at zero you'd only get one value for z...for each x y, your values for x and y.
Interviewer:	Okay. So if that test held true, then you're saying this would...that would be an argument that this represents a function.
Bob:	I would say. Okay.

Violet, on the other hand was somewhat intrepid in venturing into new territory with her concept of the vertical line test. In making the transition between working with functions of one variable and functions of two variables, she noticed the grid lines on the graph of $f(x,y) = 5xy + 4x + 4y$ and decided that this was a function because each of these lines passed the vertical line test. Her conclusion seemed more based in modeling the procedure she had learned than in applying the concept.

Students' Understandings of Multivariate Functions

Each of the students found a way to talk about multivariate functions. It turned out that a hands-on activity in which we had groups produce a model of a graph of a multivariate function in class was a powerful cognitive device. Each student drew on his experience in this activity in a different way. Bob thought about multivariate functions by evaluating slices of the surface numerically. Violet called upon the model which was created in class and talked about "frames" (her name for slices). Ned seemed to have the most robust understanding, tailoring the model produced in class:

> If I were to plot it all on an X Y plane, I would just get a whole bunch of graphs like, you know, ... And they're just gonna get this whole range of lines, just you know an infinite really amount of lines. Well, you could do it, you know, like we did it in class where we used the three dimensional, the "Office Max" model or whatever, ...And you could say that we're gonna call this axis. We're gonna have this be the changing T along this axis [draws hash marks along axes]. And you could graph, you know, and you would have one single graph that looks like that [picks up paper with graph of a slice

> and holds over each hash mark one at a time] and you could plot it, you know, and you could do the overhead thing and you would get some sort of...[takes hand and moves from "ball" of graph curving down following axis with hash marks] you know, as T increased and decreased you would get some sort of curve and, you know... If this is this curve by my hand, you know, you would get something like this as T increases and decreases [uses hand to trace surface in air] I believe as some sort of wave looking guy or something. I don't know. But, yeah, to really get a good look at that, you should have some sort of three-dimensional thing.

Conclusions

In spite of their having completed a multivariate calculus course, students had entered our class with little ownership of their understanding of multivariate functions. As we investigated students' understandings of multivariate functions, we realized that understanding their understanding was a two-way street. As they encountered functions of two variables in our class, we witnessed their approaches to generalizing their mathematical understandings. Moreover, their work with function concepts in the context of multivariate functions revealed much about their understanding of function. Our analysis has just begun. It will continue with the examination of other data about these students and with data about other students in the class. We expect that our work will draw us closer to building viable models not only of how prospective secondary mathematics teachers come to understand new mathematics but of how they understand the concept of multivariate function.

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STUDENT STRATEGIES FOR REINVENTING LINES OF EIGENVECTORS[1]

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The purpose of this research report is to delineate a student invented method for locating lines of eigenvectors associated with systems of two first order linear ordinary differential equations with constant coefficients. Our research has both pragmatic and theoretical significance. Pragmatically, we are using the results of our findings to revise a sequence of instructional tasks and to inform our instructional decision making. Theoretically, we are using these results to inform the creation of a conceptual framework for how students' develop important mathematical ideas in differential equations. The work reported here lays a foundation for both these endeavors.

Introduction

Advances in technology and mathematicians' evolving interests in dynamical systems are currently prompting changes to the introductory course in differential equations. Traditional approaches in differential equations emphasize analytic techniques for finding closed form expressions for solutions whereas current reform efforts are emphasizing graphical and numerical approaches for analyzing and understanding the behavior of solutions. In the treatment of systems of differential equations, the use of phase plane analyses for describing and predicting solutions is receiving increased attention (Kallaher, 1999). The little existing research related to students learning of systems of differential equations (Artigue, 1992; Rasmussen, 2001; Trigueros, 2000) indicates a variety of difficulties with basic ideas and methods of analysis. The research reported here complements this small but growing body of research in student learning by highlighting students' intellectual achievements rather than their shortcomings.

The purpose of this research is twofold. First, we wish to document a student-invented approach for locating solutions that, when viewed in the phase plane, fall along straight lines. From a mathematical point of view, straight-line solutions are determined by the eigenvalues and eigenvectors for the 2×2 matrix corresponding to the system of differential equations. Typical approaches for finding the closed-form solution for these straight ine solutions employ techniques from linear algebra where students first find eigenvalues, then find the corresponding eigenvectors, and then form the analytic solution. Although students can often find eigenvalues and eigenvectors, the meanings of these mathematical ideas tend not to be well understood. Second, we view the research reported here as foundational because our analysis focuses on student invented methods for first finding eigenvectors and then eigenvalues (a reversal of the typical order) through their interest in locating straight-line solutions. The instructional design of the materials is based on students' mathematical goals rather than a top-down approach. As such, we document students' conceptual development of straight-line solutions as they reinvent the ideas of eigenvectors and eigenvalues from their own sense-making.

In our work with students, we treat graphs of solutions in the phase plane as inscriptions that potentially emerge out of students' reasoning. This perspective builds on recent theoretical advances in symbolizing (Cobb, Yackel, & McClain, 2000) and differs from typical reform-oriented approaches in differential equations. For example, Smith (1998) posits that after drawing a slope field for a single differential equation, "the meaning of a solution is then clear: We seek a function whose graph 'fits' the slope field" (p. 12). We start from the premise that meaning is not inherent in an inscription like a graph or vector field, but rather grows out of students' interactions and experiences with such inscriptions (Blumer, 1969).

Methodology

Our methodological approach follows the classroom teaching experiment as described by Cobb (2000). In brief, this approach involves the development of a tentative and revisable set of instructional activities, classroom enactment of these activities with ongoing revision of the activities based on day to day analysis of classroom events, and retrospective analysis of all data sources once the experiment is completed. Data sources include videorecordings of every class session, videorecorded interviews with students, copies of all written work, instructor's journal, researcher fieldnotes, and audiorecordings of project meetings. Our instructional planning about students' conceptual development was informed by research on students' learning about rate of change in general and about differential equations in particular (Thompson, 1994; Rasmussen, 1999, 2001) and by our understandings of the Realistic Mathematics Education instructional design theory (Gravemeijer, 1999). The method of data analysis follows the constant comparison method outlined by Glaser and Strauss (1967) and elaborated by Cobb and Whitenack (1996) for classroom teaching experiments. We carried out our research in two different classrooms. In one classroom, we conducted a 15-week modified classroom teaching experiment where we collected copies of student work, held weekly project meetings, and keep field notes of classroom events. In another classroom, we conducted a 15-week classroom teaching experiment where we also videotaped each class session and conducted interviews with students.

As RME instructional design is founded on student motivations and understandings, listening to student questions and ideas becomes a critical tool to the development of curricular materials. The motivation for choosing this particular instructional sequence was based on questions students had from previous semesters. A particular student comment during a class when small groups were investigating how vector fields change as a parameter changed in the differential equations using technology. A student had been looking at the vector field associated with a linear system of differential equations which had straight-line solutions. The vector field for a system with a straight-line solution, or eigensolution, would appear to have a collection of vectors lying on a line through the origin, where each vector is parallel to the line (see figure 2 below for an example). Although the student hadn't had any experience with analytic forms of the straight-line solutions, he had been perplexed as to why the vector field seemed to have a "line" running through the middle of it. Traditionally, these lines are determined analytically by finding the eigenvectors associated with the 2×2 matrix associated with the system. This student's question was used as a motivation for a class investigation into whether there was a straight line was in the vector field and if so, which line was it. To the students' surprise, by developing analytic techniques, they found a second straight-line solution that wasn't as visibly clear as the previous one. This created questions for students to study its implication for solutions to the system, and its use for modeling graphical and analytic solutions to the system. In addition, the ideas and concepts that were used answer the students question for this specific system and could be used to analyze other systems. As a result, the instructional sequence was modified to reflect student questions and address eigenvectors first and then develop eigenvalues.

Results

Students in these differential equation classes were primarily engineering students, although there were a few mathematics and science majors, mostly in either their sophomore or junior year. During the videotaped class in this study, students were analyzing a system of differential equations modeling the motion of a mass on the end of a spring with friction, as seen in figure 1.

The classroom topics covered up to this point in the course included detailed investigations of first-order differential equations as well as some qualitative experiences with non-linear systems of differential equations. This included looking at solutions as three dimensional curves and as pairs of functions of time, discussions of the

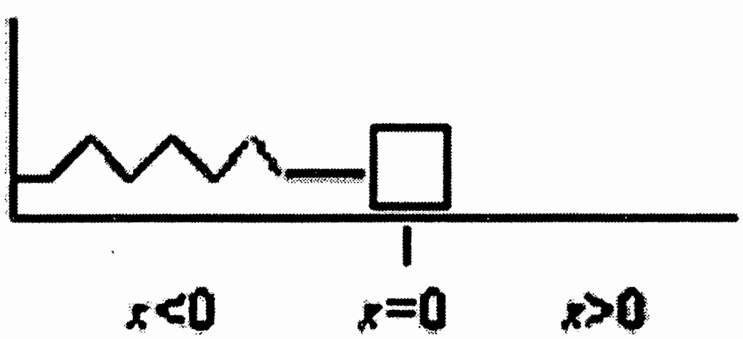

Figure 1. Diagram of the spring-mass system.

phase plane and how it related to the functions, as well as qualitative and quantitative aspects of equilibrium solutions.

To investigate a specific spring-mass problem with variable kinetic friction, we had chosen the mass to be 1 and the spring constant to be 2. As a result, these system of differential equations were

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -2x - by$$

where x represents the position of the spring mass relative to the equilibrium point, y represents the velocity of the mass, and b is the coefficient of friction. Before doing any analytic or graphical investigations of these equations, students had conjectured that the possible motions of the mass were either oscillations with a fixed amplitude if $b = 0$, or oscillations with a decreasing amplitude over time if $b > 0$. Students drew diagrams of what curves in the phase plane would be for each of their physical predictions. Rather than introducing students to analytic methods for obtaining solutions to differential equations, we had students use technology to graph the different possible vector fields in the phase plane as the friction coefficient varied. More importantly, the technology we used only plotted the vector field, it did not actually plot out graphs of solutions in the phase plane. It did, however, allow students to interactively move a point with its associated vector about in the plane and allowed students to dynamically change the vector field for different amounts of friction. They confirmed that both conjectured situations actually did arise for particular b values. Using the program NuCalc ®, students plotted the following vector field when $b = 3$ as seen in figure 2.

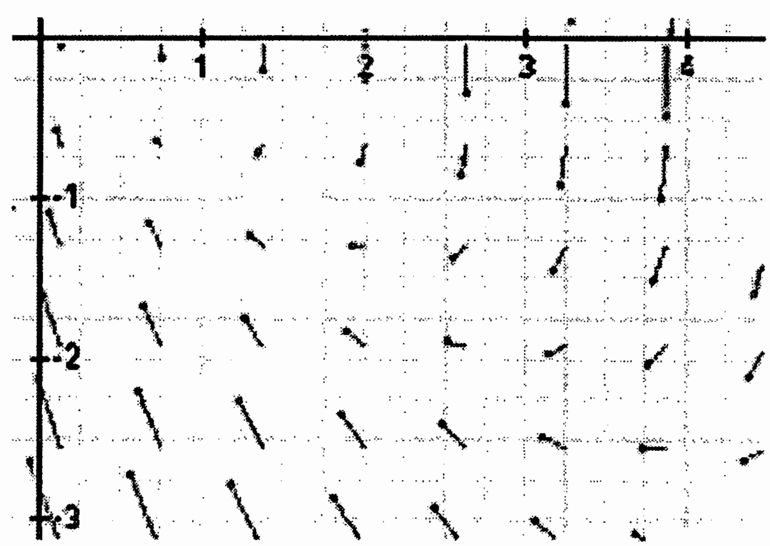

Figure 2. Vector field associated with the spring-mass system.

This is an over-damped oscillator, meaning that not only does the system have real-valued eigenvalues and eigenvectors, but that the mass will not cross the point $x = 0$ after having initially been displaced from it.

Students were asked to describe the motion of the mass at $b = 3$ to see if it matched what they had predicted. Students weren't sure as to whether the mass was oscillating, as they had previous predicted, or whether the behavior was different. To help facilitate classroom conversation, the image of the vector field on the computer screen was projected up at the front of the room. Student A made the following argument that the motion was different.

Student A: I mean, if it was just one circle, like when [b] was 0, then these would all link up [pointing to the vectors in the third quadrant], you know what I mean. But you can see these [pointing to the second quadrant] are way different, uh, uh, slopes than over here [pointing to the third quadrant], so it's not going to work that way. It almost seems like it's going to come up almost like that [draws the path on image of the phase plane in the fourth quadrant with his finger seen in figure 3] and it will do the same over here. [draws the reflected path in the second quadrant seen in figure 3]

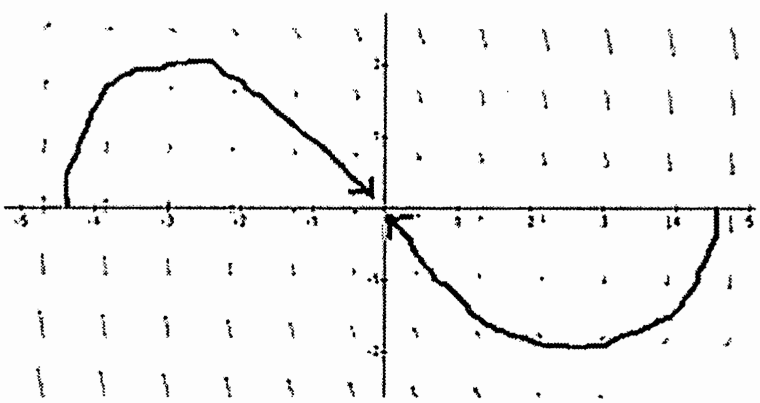

Figure 3. Student suggestions of possible paths in the vector field.

Student D: Like an eight?

Student A: It's not a total... Two arcs almost.

Student A used the inscription of the phase plane as a reasoning tool. Because the phase plane has particular characteristics, Student A noted that the vectors drawn in the plane do not match the vectors that would be necessary for simple harmonic motion as discussed before. There was no discussion as of yet as to how his conclusions related to the physical movement of the mass.

The class then discussed what the physical implications of such solution curves in the phase planes were. After establishing that for this friction coefficient, the mass would not oscillate, further discussion ensued in the classroom as to whether larger initial starting positions of they would actually oscillate rather than return to the origin. Student S argues that the straight line solution extends to infinity.

Student S: In this case, if you start here [points to approximately 1 on the x-axis], it should, you're not going to have a lot of velocity towards the equilibrium, so it'll go like this [makes a similar path to what student A did in figure 3, starting at x = 1]. But here, if you start here, [points to about 5 on the x-axis] you're moving faster then you slow down...You can make a line out of [the arrows which directly to the origin], so how can it ever overshoot this line? So even if you started at fifteen, it'll all go like this [draws a similar path to what was shown before].

It is likely given the context of the conversation that Student S is coordinating the physical movement of the mass ("you're not doing to have a lot of velocity towards the equilibrium," "you're moving faster then you slow down") to inscriptions within the vector field and associated solutions (starting at places on the vector field and drawing out particular paths). For Student S, the vector field is a tool for reasoning, as he uses the vector field to describe physical phenomenon and makes a claim that oscillations can't occur because of aspects of this particular inscription.

As most of these students had some background in physics or engineering, the situation of a spring on a mass is one which is experientially real. We do not use this example to motivate the study of simple harmonic motion, but rather to capitalize on the students' familiarity with this phenomenon as a starting point for mathematical progression. Students were able to use their physical intuition to anticipate which motions for the mass should be possible and to translate the physical motions into inscriptions in the phase plane. When students are faced with solutions which are not one of the ones they had predicted, the investigation of the straight-line solutions becomes a question for them rather than one that is imposed on them.

The significance of the straight-line solution became clear as they were viewed as the limiting behavior of all solutions to the system of differential equations. We will refer to Student S's observation of a line that existed in the phase plane as the straight-line projection, as it is the projection of all straight-line solutions into the phase plane. Graphical interpretations then led to analytic investigations. What was the equation of the straight-line projection in the phase plane? Was this the only straight-line projection for this system of differential equations? How could you get the equation for the components of the straight-line solution? As students use graphical and physical interpretations as sense-making tools before using algebra, they are less distracted from the symbol manipulation that can distance them from the original meanings. Furthermore, the algebra expressions emerge as a way of expressing what has already been found and a means to adding insight to the analysis (Arcavi & Hadas, 2000).

Before continuing to address these questions, the class discussed what the graphical representation of the three dimensional curve of the triple (t, x, y) would look like. Although the fact that solutions to a system two differential equations could be represented by curves in three-space had been discussed in previous classes, a student offered that the graph of the solution should be a plane. His argument developed as follows:

Student C: I'm going to say that, because if you look at this plane, this is t = 0 and it's a straight line. You can take any t, all the lines should be the same. So if we stack all the t's on top of each other, you'll have a straight-line solution. So I'm going to say it's going to be a plane.

The plane student C referred to was perpendicular to the phase plane and intersected the phase plane exactly through the straight line projection, as seen in figure 4.

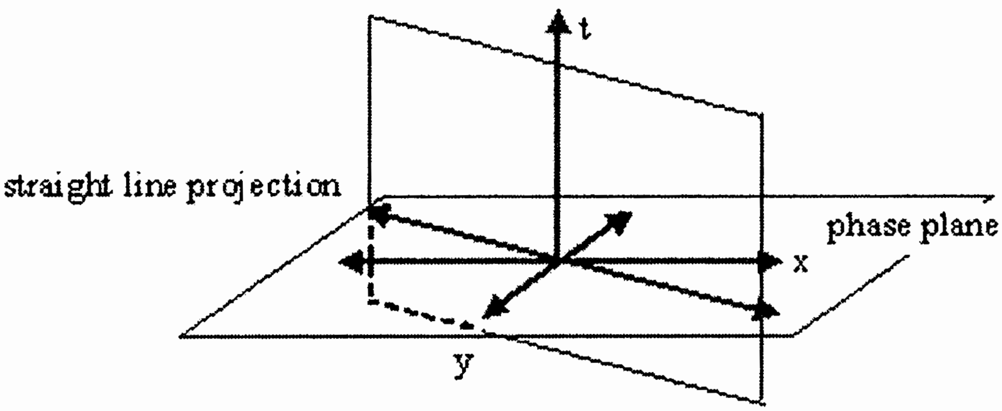

Figure 4. Student C's image of a straight-line solution.

Upon further discussion, student C told the class that he was using the fact that the system was autonomous, namely that the rate of change of both the position and the velocity didn't depend upon time. As a result, he believed that the straight line seen in the phase plane was just one of many straight lines, and all would be successively stacked on top of each other creating a plane. The class had previously studied autonomy of a single differential equation and student C was using analogies from this context to make sense of systems of differential equations. However, in class discussion, students brought up many points to respond Student C. First, they noted that while the rates of change can be determined solely by position and velocity, these quantities themselves vary with time. Furthermore, the fact that for each time there's one position and velocity implied that the solution would be a curve, rather than a plane. Students debated whether the curve would be straight or curved in three-space. They came to the conclusion that the graph would be curved since the vectors had decreasing magnitude they approached the origin, and that this lead to the mass slowing down as it came closer to the equilibrium point.

This curriculum places emphasis on graphing of solutions of systems of two differential equations not only in the two-dimensional phase plane, but also in the full three dimensional system including time. The phase plane emerges as a result of students needs to express three-dimensional graphs in an efficiently but still maintain the essence of original graph. The connections between the phase plane and the three-dimensional graphs are traditionally difficult for students. The instructional activities allow students to use experientially real contexts to develop meaningful inscriptions and then use these inscriptions to analyze the contexts from which they emanated.

In the next class period, students developed a method for finding the slopes of the straight line projections in the phase plane. Many students had speculated that the line was $y = -x$ and, using NuCalc, had checked particular components of vectors on the line to make sure that it agreed with their guess. The instructor asked if there was an analytic means that would work so that we didn't have to guess each time. A student offered the following method to developing an analytic means of finding equation of the straight-line projection: Since the line had to go through the origin as (0,0) is an equilibrium solution, we knew the straight-line projection would be of the form $y=mx$. Student A suggested using the fact that the slopes of the vectors and the slope of line they resided on were the same. Since the slope of the vectors could be determined by the ratio of the two rate of change equations, namely

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2x-3y}{y} = m$$

,

by replacing y with mx, one could get and equation for the slope of the straight line by substitution, as seen below:

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2x-3mx}{mx} = \frac{-2-3m}{m} = m$$

$$m^2 + 3m + 2 = 0$$

This yielded *two* solutions for the slope of the straight-line, namely –1 and –2. While most had conjectured that the straight line projection with slope – 1, the second one wasn't apparent and was a surprise to the students. The equations of lines students found are equivalent to the eigenvectors associated with the linear system, as any particular eigenvector is a representative from an equivalence class of a line.

Students began by analyzing the situation with regard to the physical characteristics of the spring mass system. For example, students looked at the effect of the straight-line solution on the motion of the mass. Gravemeijer (1999) refers to this as referential activity, as it is based on interpretations based on the particular context. As students continued to develop their understandings of the straight-line solutions in that context, they were able to make statements and use formal mathematical reasoning independent of the context of the solution. Gravemeijer (1999) refers to this as general activity, where students begin to reflect collectively on the referential activity. For example, eventually the students explored relationships between x and y that were algebraic in nature rather than as the position and velocity of the mass on a spring. These analyses led students to develop not only techniques but understandings of straight-line solutions which transcended the particular situation first posed.

The development of the analytic form of straight-line solutions as parametrized curves ($x(t)$, $y(t)$) followed from graphical investigations of these curves using the program PPLANE authored by (Polking). This program allows one to enter in a system of differential equations, choose an initial condition, and examine the solution curve from many different angles and projections. By observing the two-dimensional graphs of $x(t)$ versus t and $y(t)$ versus t, as well as the three dimensional plots of both $x(t)$ and $y(t)$ versus t, students were able to conjecture that both $x(t)$ and $y(t)$ were of exponential type, namely that $x(t) = C_1 e^{\lambda_1 t}$ and $y(t) = C_2 e^{\lambda_2 t}$. With this assumption in mind, students were asked what they could determine about the constants C_1 and C_2 as well as the exponential constants λ_1 and λ_2. We assumed an initial condition of $x(0) = 2$ and $y(0) = -2$.

To answer both of these questions, most students first began by looking at the graphs of $x(t)$ and $y(t)$ simultaneously using PPLANE, (as seen in figure 5). Students

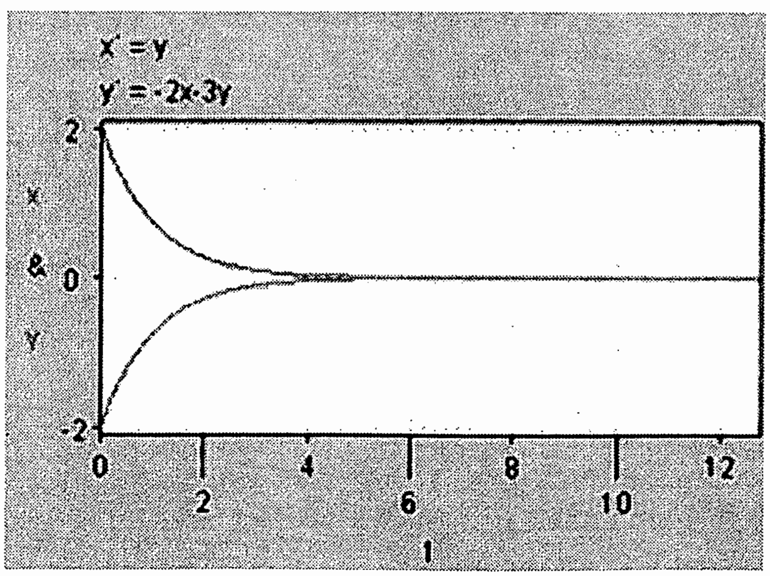

Figure 5. Graphs from PPLANE program.

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developed a number of ways to look at the relationship between the constants. One group plugged in $t = 0$ and found that C_1 and C_2 were the initial conditions 2 and –2. Another group conjectured that the graphs were reflections across the t-axis of each other, so that this would be true for all t. When asked to explain why this was the case, a student offered that the relationship $y=-x$ implied that $y(t) = -x(t)$ for the straight-line solution. This meant that $C_1 = -C_2$ for all initial conditions we chose for our straight line solution. Given that $x(t)$ and $y(t)$ were proportional, students also developed arguments that $\lambda_1 = \lambda_2$.

The last question that remained was how to find the $\lambda = \lambda_1 = \lambda_2$. In previous semesters, students had suggested plugging in $y(t)$ or $x(t) = Ce^{\lambda t}$ into the equation for the rate of change of x and solving for λ. This semester, students had recommended using substituting $y = -x$ into the differential equations and getting closed form solutions for $x(t)$ and $y(t)$ as seen here:

$$\frac{dy}{dt} = -2x - 3y = -2(-y) - 3y = -y$$

$$y(t) = Ce^{-t}, \lambda = -1$$

A critical distinction of this curriculum from that which is traditionally taught is that eigenvectors develop before eigenvalues. Traditionally, one rewrites the linear system of differential equations using a matrix, finds the characteristic equation of the matrix and solves for the eigenvalues. The eigenvalues will give important information about whether solutions converge to or diverge from the equilibrium at the origin and whether there is any rotational motion in the phase plane as time evolves. However, eigenvalues are transparent in a particular context without fairly sophisticated mathematical understandings and techniques based on hundreds of years of mathematical development. Our goal was to provide students an opportunity to engage in guided reinvention of eigensolutions. The instructional design needed, therefore, to answer students questions about solutions rather than ones posed to them. Students investigated straight-line solutions to explain discrepancies from their predicted behavior of the motion of a dampened spring-mass system. This lead to questions that students could continue to address: What is the motion of the mass? How can we find what the limiting behavior is? Are there analytic solutions to these behaviors? This process mirrors student activity in the discovery of existence of straight-line projections. Straight-line solutions emerge first in this sequence and have meaning for the students related to the situation. Students originally discussed these straight-line solutions in reference to the motion of the mass. As the students developed understandings of these solutions, they were lead to using less context-specific ideas from which concepts and computation of eigenvalues emerged.

Another distinction is the opportunity for students to conjecture and test their conjectures formally. This appears many times in the instructional sequence. Students conjectured what types of motion were possible for the spring-mass system. After their conjectures proved to be missing some cases, students then conjectured about the nature of these unpredicted limiting cases, such as whether they were lines and which lines they were. After making conjectures based on emerging intuition, students developed techniques to analytically prove their conjectures about finding the eigenvectors. Similarly, students made conjecture followed by more formal analysis for eigenvalues as well. This sequence gave students an opportunity to engage in the mathematical activity of reinventing the concepts and calculations behind eigenvalues and eigenvectors. Instead of supplying students with an algorithm and conducting analysis from there, students were able to come up with their own personal knowledge surrounding this topic and develop answers to questions posed by the instructor as well as their classmates.

Students practiced this technique with other linear systems of differential equations, and in the teaching experiment without videotaping, they were introduced to the technique of finding eigenvalues first and then getting eigenvectors. By the end of the semester, a majority of the students were finding the eigenvalues first as a means of finding analytic forms of solutions to linear systems. In both classrooms, most students demonstrated computational proficiency finding the eigensolutions for a generic linear system of differential equations in homework and on the final examination.

Conclusion

RME has been used to develop a number of curricula at the elementary school level. It is only recently that these ideas have been used to develop undergraduate mathematics courses aimed at science, mathematics, and engineering majors. The principle of guided reinvention could be viewed with skepticism if applied to these courses, as the concepts and computations are more complicated than at the elementary level. Finding eigensolutions to linear systems of differential equations is considered one of the most difficult tasks for students in an introductory course to differential equations, both conceptually and computationally. Our research provides some evidence that students can develop both conceptual understanding and computational proficiency from this instructional sequence.

Note

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PROGRESSIVE MATHEMATIZATION IN ELEMENTARY GROUP THEORY: STUDENTS DEVELOP FORMAL NOTIONS OF GROUP ISOMORPHISM

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The purpose of this paper is to report on a college-level teaching experiment in elementary group theory. The experimental "classroom" consisted of two students and the teacher/researcher. The teaching experiment was guided by the instructional design theory of Realistic Mathematics Education. Activities were designed to promote the guided reinvention of the concept of group isomorphism. The goal was to have the formal definition emerge from the mathematical activity of the students. The students were able to invent and apply procedures for determining whether two groups were isomorphic using group tables. They also demonstrated some progress in constructing a formal definition of isomorphism.

Introduction and Theoretical Perspective

Leron, Hazzan and Zazkis (1995) described isomorphism as a "complex and compound concept, composed of and connected to many other concepts" (p. 153). Leron et al. advocated introducing isomorphism in the form of a naïve conception in which two groups are seen as isomorphic if "they are the same except for notation" (p. 154).

DeVilliers (1998) suggested that students should be actively engaged in the defining of mathematical concepts. My goal was to engage students in the defining of isomorphism. Consistent with the philosophy of RME, the aim was to have the formal definition emerge from the mathematical activity of the students. The instructional plan consisted of three phases. Note that although I wanted the students to *understand* the concept of isomorphism, the thrust of the project was to have the student reinvent the concept. (The theory is that students are more likely to understand and successfully use concepts that they have (re)invented themselves.)

Step 1: Encourage the students to develop a naïve conception of isomorphism (Leron et al., 1995).

Step 2: Encourage the students to invent procedures for determining whether two groups were isomorphic based on this naïve view.

Step 3: Encourage the students to generalize, abstract, and formalize these procedures in order to formulate a definition of isomorphism.

The instructional design and subsequent data analysis was guided by the instructional design theory of Realistic Mathematics Education (RME). RME is based on the notion that "mathematics can and should be learned on one's own authority and through one's own mental activities" (Gravemeijer, 1999). This idea is embodied in the first instructional design heuristic of RME, the *reinvention principle.* This principle does not suggest that students should reinvent everything for themselves. The more accurate term "guided reinvention" is often used to express the desired learning process. The goal is for students to feel ownership of, and responsibility for, the mathematics. Mathematizing (organizing from a mathematical perspective) is seen as the way to accomplish this reinvention (Gravemeijer, 1999). This includes horizontal mathematization in which one mathematizes a context-specific situation and vertical mathematization in which one's own mathematical activity is mathematized (Gravemeijer & Doorman, 1999).

One of the instructional design heuristics of RME is the notion of an *emergent model.* These models are called "emergent" because they emerge from the students' mathematical activity and because they support the emergence of more formal mathematical knowledge. Initially an emergent model surfaces as a 'model of' the students mathematical activity. Subsequently an emergent model is used as a 'model for' supporting more formal reasoning. In the teaching experiments described here, the group table is seen to have initially emerged as a model of students' activity with geometric figures and then evolved into a model for supporting more formal reasoning including the construction of the concept of isomorphism.

Methodology

The research design consisted of a sequence of three teaching experiments (Cobb, 2000; Steffe, 1991), each involving the researcher/teacher and two students. The retrospective analyses of each teaching experiment informed the instructional design of subsequent teaching experiments. Each of the participants had completed their first proof-writing course during the previous semester. Table 1 briefly describes the participants, including their academic major and the grade they received in the proof-writing course.

The teaching experiments consisted of seven or eight sessions each lasting 90 to 120 minutes. The sessions were videotaped and the students' written work was collected. The focus of this paper is on the later sessions of the teaching experiments in

Table 1

	Participants	Major	Grade
Teaching Experiment #1	Sandra	None	C
	Jessica	Math & Math Ed.	A
Teaching Experiment #2	Melissa	Computational Math	A
	Leah	Math	A
Teaching Experiment #3	Debbie	Math Education	C
	Nicole	Math Education	C

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which the students constructed notions of the concept of isomorphism. To set the stage for this discussion, I briefly describe the first phase of the teaching experiments during which the group concept was developed. It was in these sessions that the group table emerged as a model of the students' activity with geometric figures and other physical objects.

The first few sessions of each teaching experiment were dedicated to the development of the group concept. The first activity required the students to identify the symmetries of an equilateral triangle and determine the result of performing any two of these transformations consecutively. The students physically manipulated the triangle and recorded their results in the form of a list. Eventually the students reorganized this information into a table. Additionally the students developed rules that could be (and were) used to complete the chart by performing calculations rather than manipulating the triangle. These rules included the group axioms and the dihedral group relations. Note that closure under the operation was not one of the rules the students developed at this point.

This process was repeated for the set of symmetries of a square in the first teaching experiment. For the second and third teaching experiments, the students also worked with the set of symmetries of a rectangle and the rotations of a square. Following these geometric activities, the students worked on two activities based on the game "It's a SNAP" described by Huetinck (1996). These activities involved a game board consisting of a rectangular array of pegs and some rubber bands. These activities provided physical examples of the symmetric group, S3, and the Klein 4-group.

After completing the SNAP activities, the students identified the group axioms as the features that were common to all of the task settings. I introduced the term 'group' as a name for systems that satisfy the axioms. A discussion followed in which we developed a definition together. The most difficult aspect of this process was defining binary operation.

The students were then asked to verify that each element of a group must appear exactly once in each row and column of a group table. (This was an observation that all of the groups made early in their work with the geometric figures.) This led to the conjecture and proof of the cancellation law for groups.

Results From Teaching Experiment #1

Step 1: Developing A Naïve Conception Of Isomorphism

My initial plan was to motivate the isomorphism concept by asking Sandra and Jessica to determine how many different groups there are with four elements. This question immediately sparked a debate.

Sandra: Why is it not infinite? Cause you have four elements, you know one of them has to be *I* because it has to have an identity. You have three other elements. Why can't it be any number in the real number system? I mean why can't it be infinite. Because these could be twelve, two, and six or something or a and b and z. How many groups have four elements? I mean isn't it infinite number of groups?

Jessica: Well you can look at something with four elements like you said you take one is I. So you know this is going to be your chart. So basically how many ways can you fill this in so it would work and it would satisfy all the requirements of a group? How many different ways can you fill in these nine boxes so that it would have all the different ways to be a group?

Sandra: But that would just be with a, b, and c as the elements.

Jessica: There's four different elements, those are just random elements.

Sandra: ...It's a different question. Do you see? This is how many groups have four elements. Well there's an infinite number of groups that can have an infinite number of four elements. Because you can take any element from anywhere. But this question is how many different configurations of a table can you get with four specific elements.

Jessica: I'm saying you have group with four, these four elements. Right? This is one group with these four elements. Now how many other groups can I make with these specific elements?

Sandra: Yeah. See it's a different question. I just don't like the way the question is phrased.

Eventually Sandra agreed that if the names of the elements of a group were changed, "a mathematician wouldn't think that it was any different."

Step 2: Inventing A Procedure For Determining Whether Two Groups Were Isomorphic

Sandra and Jessica then constructed four different tables that satisfied the identity and inverse group axioms. To save time and tedious calculations, I assured them that the associative property also held for these tables. Then the question became whether some or all of the tables were really the same. The students identified the inverses for all of the elements in each group. It was agreed that the group in which each element was its own inverse was most likely different from the other three groups. At one point Jessica felt confident that two of the tables represented the same group.

Jessica: They are the same.

Teacher: Why? In what sense are they the same?

Jessica: Because if you just put $c\ b\ a$ in here, $c\ b\ a$.

Teacher: So if I put $c\ b\ a$ in here would you think it would be the same as...

Jessica: But you have to change these (motions to the interior of the table) accordingly as well.

It appeared that Jessica had a workable notion of what it means to change the names of the elements of a group. She appeared to be aware that one must change the interior of the table as well as the elements themselves. However, the students had difficulty developing a procedure for verifying that two of the remaining three groups were actually the same. In fact as they began to develop their procedure, their intuitive conception of isomorphism became unstable.

Jessica: I have a question. If we're changing things, we don't change the inverses. Right? Do we change the inverses? Or do we just have to...?

Teacher: Why do you ask?

Jessica: Does that mean we are just going to change these, or do you want us to change the inverses? Are there any restrictions, basically? Like you can't change inverses?

A moment later Sandra also indicated that she was unsure what it really meant to "change the names".

Sandra: I am still confused when you say change the names? Do you mean the set your starting out with to make the table or do you mean what's actually inside the table? And does it matter?

Later Sandra suggested that one should not change the names of the elements at all, but merely change the interior of one of the tables to match the interior of the other. She claimed that "there's nothing in the rules of the question that says you can't do that." I attempted to redirect Sandra back to the notion of naïve isomorphism by drawing two two-element group tables on the board, one with the set $\{I, a\}$ and the other with the set $\{I, b\}$. She agreed that these two groups were the same and that this could be demonstrated if you "change the b's to a's". However, she protested that this was not the same situation as in the original problem because two different sets were being used while in the problem setting the same set was being used for each group.

After a lengthy discussion, the students' naïve conception of isomorphism appeared to stabalize. The students agreed that if the names of the elements were changed, the interior of the table must be changed as well. Eventually, the students were able to verify that the two groups were indeed isomorphic and wrote a description of their procedure (See Figure 1).

Sandra and Jessica were also able to use this procedure to determine that the two groups they had constructed with six elements (the symmetries of a triangle and one of the SNAP groups) were also isomorphic.

Step 3: Formulating a Definition of Isomorphism

Sandra and Jessica made limited progress in formulating a definition of isomorphism. After their first renaming of the symmetries of a triangle did not result in the

1. Rename

   a: = b.
   b: = c.
   c: = a

2. Fill out chart
   switching a, b, c
   with new values.

3. organize chart
   by switching columns and rows.

Figure 1.

table they had generated for the six element SNAP group, I challenged them to figure out why this renaming did not work. (The tables that the students generated for these groups are presented in Figure 2.)

Sandra: Shouldn't flip be able to be matched up with this one and this one and 6 because it's kind of the same relationship?

Teacher: Yes, that's what Jessica said. But it might matter which one you choose.

Jessica: It has to do with its combination with other things too.

The students were unable to be more specific, so I asked a more specific question.

Teacher: So you gave f a name and you gave r a name. You gave f 2 and r 4 right. Isn't f times r already determined in this table.

Sandra: That's the problem, the combinations. So you gave 2 a name of *flip* and you gave r a name of 4 but then fr is a name of 3 which is a problem.

Teacher: What do you think it should be?

Triangle Group

•	I	R	R^{-1}	F	FR	FR^{-1}
I	I	R	R^{-1}	F	FR	FR^{-1}
R	R	R^{-1}	I	FR^{-1}	F	FR
R^{-1}	R^{-1}	I	R	FR	FR^{-1}	F
F	F	FR	FR^{-1}	I	R	R^{-1}
FR	FR	FR^{-1}	F	R^{-1}	I	R
FR^{-1}	FR^{-1}	F	FR	R	R^{-1}	I

Snap Group

S	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	5	6	3	4
3	3	4	1	2	6	5
4	4	3	6	5	1	2
5	5	6	2	1	4	3
6	6	5	4	3	2	1

Figure 2.

Sandra: It should be…

Jessica: 6

Sandra: 6. Yeah.

In the hopes of encouraging the students to explicitly use a function to rename the elements of a group, I asked them whether the additive group of integers was isomorphic to the additive group of integer multiples of five. Jessica was able to make some progress in writing a definition. In the excerpt below, Jessica is explaining what she wrote to Sandra.

Jessica: You could multiply by five. Cause if you take everybody in here and you want to get something in 5*Z* you just have to multiply by 5.

Sandra: That's right.

Jessica: And I was thinking it has to work the other way. So I said that they are isomorphic if there is a function from *A* to *B*…it has to be bijective.

At this point I reminded the students that their first (unsuccessful) attempt to rename the elements of the triangle group was also a bijective mapping. Jessica responded by saying that it had to be a bijective function that "fits". The students then realized that the definition needed "something else". This missing property proved to be elusive; they had no trouble verifying in practice, however they were unable to articulate *what* they were verifying.

The discussion in which the definition was finally formulated was almost entirely teacher driven with the students' contributions limited to murmurs of assent. While the students were able to accept that a bijective function could be used to rename the elements of a group, they appeared to lack facility with function notation. This appeared to hamper their ability to participate in the final formulation of the definition.

Teacher: Jessica's already got us halfway done. She thinks that it would be a good idea if instead of using the word rename we have a bijective function. Does that make sense that a bijective function is the same as renaming?

Sandra: [Murmurs assent]

Teacher: OK, so we have a function. We want to say what it means. What did you call yours, *A* and *B*?

Jessica: [Murmurs assent]

Teacher: Group *A* is isomorphic to *B* if and only if. You said there exists an *f* from *A* to *B* such that *f* is bijective (one to one and onto right) and we figured out that bijective isn't good enough.

[Additional discussion]

Teacher: Let's suppose I had *a* and *b* in the set *A*. What would I check to see if the renaming worked? First of all what are they renamed as? What is the new name for *a*? How does the function rename it? Our function takes something in *A* to an element in *B*. Which element is that?

Jessica: This that transitive thing?

[Additional discussion in which *a* and *b* are changed to a_1 and a_2 to avoid confusion with the set *B*.]

Teacher: I take an element of *A*, it's going to be set to an element of *B*. Don't you know exactly what element that is or at least how to write it?

Jessica: [Quietly] *f* [pause, laugh] of a_1?

Teacher: Yes!

Discussion

The group table appeared to be somewhat useful as a model for supporting Jessica and Sandra's construction of the isomorphism concept. They were able to use group tables to articulate a naïve conception of isomorphism and to invent a procedure for determining whether two groups (given by tables) were isomorphic. However, the group table seemed to have limited value when it came to formulating a definition of isomorphism. In short, the students were more successful in using the group table for horizontal mathematizing than they were for vertical mathematizing.

Recall that I attempted to motivate the concept of isomorphism by asking the students how many different groups there were with four elements. This approach seemed to have introduced additional difficulties for the students. First, although the students were able to produce four group tables with four elements, these tables did not seem as meaningful to the students as the group tables they constructed from their own activity with physical objects. This may be part of the reason the students' naïve notion of isomorphism was unstable in this context. Second, the tables produced in this activity all used the same set of symbols for the elements. This seemed to be quite confusing for the students. As a result, this activity was delayed until near the end of the second and third teaching experiments. Instead I motivated the concept of group isomorphism by presenting them with a new group table and asked them whether it could actually be one of the groups they had constructed. This appears to have been a more effective approach. Preliminary analyses of those teaching experiments suggest that the students' naïve conception was much more stable and that their invented procedures were more meaningful to them from the beginning. Interestingly, the problem of the sameness of symbols did reappear (in milder forms) in second and third teaching experiments when the students finally were asked to determine the number of groups of order four.

The three groups of students produced more than three different procedures for determining whether two groups are isomorphic from group tables. This variety that emerged suggests a strategy for facilitating the formulation of a definition of isomorphism. If different groups of students presented different procedures during class discussion, the subsequent comparison of procedures might serve to bring the operation preserving property (that is implicit in each of them) to the surface. The three groups of students also used different sets of symbols for the elements of their groups and their binary operations. In a whole class setting, these different representations of the same group could provide an early entrance into a discussion of isomorphism.

Summary

The concept of group isomorphism is one of the fundamental ideas of abstract algebra. Based on the theory of Realistic Mathematics Education, I am working to develop an instructional approach that can make this concept personally meaningful to students. This paper reports on part of the initial stage of this project. A number of interesting observations emerged from the analysis of these first three teaching experiments. These findings indicate that such an approach can be successful while highlighting the complex nature of both the concept of isomorphism itself and the process of defining it through a process of progressive mathematization.

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INQUIRY INTO THE INTERLOCUTION OF STUDENTS ENGAGED WITH MATHEMATICS:APPRECIATING LINKS BETWEEN RESEARCH AND PRACTICE[1]

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For either to be useful, links between research and practice are critical. Just as important are connections between the practice of students engaged in mathematical activity and research that seeks to understand that practice. This research report explores lessons that researchers and practitioners can learn from an inquiry into the interlocution of students working collaboratively in small groups when engaged in talking and listening to each other. We use the term interlocution to denote discursive practices of learners in conversational exchanges. Questions that motivate this research included the following: What discursive practices do interlocutors employ as they work collaboratively to understand and resolve mathematical tasks? How do these practices influence the growth of their mathematical ideas? In what ways do their discursive practices help them move from a contextualized, situated task to generalize the task or their solution? Do students' discursive practices assist them to connect and generalize ideas from a new problem to others on which they have worked?

Background and Setting

This report focuses on the development of mathematical ideas by individual students as they work collaboratively in small groups. More particularly, we analyze how students through their discursive practice structure their own investigations. The data come from an ongoing, fourteen-year longitudinal research project of the Robert B. Davis Institute for Learning, directed by Maher (Maher, in press, and for references to relevant reports), that has been conducted in public elementary and secondary schools in a suburban, working-class, and immigrant town of New Jersey. Overall, our longitudinal study aims to contribute basic scientific understanding of cognitive behaviors as well as pedagogical conditions for which mathematics learning occurs as a process of sense making. The participants in the present study are four students—Brian, Jeff, Michael, and Romina—in their senior year of high school who, from their entry into first grade have participated in mathematical activities of our longitudinal study. Over the years, these students have engaged tasks from several strands of mathematics, including algebra, combinatorics, and probability, both in the context of classroom investigations as well as in after school settings. For the present research report, these four students participated in an after-school session of mathematical problem solving. Data came from a culminating task, "The Taxicab Problem," in the research strand on combinatorics. The task has an underlying mathematical structure and contains concepts that resonate with those of other problems on which the students have worked during their involvement with the longitudinal study. The participants potentially revisit and deepen mathematical ideas they have already built as well build new ideas. After the students worked on the problem for about an hour and a half, we interviewed them concerning the nature of their work so that we might follow movements in their discourse toward further justification of their solution.

Theoretical Framework

Our research questions emerge from our ongoing, longitudinal investigation into the development of mathematical ideas by individual students as they work collaboratively in small groups (Davis & Maher, 1990, 1997; Maher & Martino, 1996, 2000; Maher & Speiser, 1997). The theoretical underpinnings of this investigation come from principally three sources: research on the development of representations (Davis, 1984; Davis & Maher, 1990, 1997; Speiser & Walter, 2000), models of the growth of understanding (Pirie, 1988; Pirie & Kieren, 1989, 1994) and theories about the generation of meaning (Dörfler, 2000). This report builds on these theoretical notions and explores the applicability of an additional conceptual framework concerning a particular discursive practice with the aim of understanding how students structure their own investigations through such a practice.

Grounded in philosophical ideas of Levine (1989) concerning the social evolution of human's capacity for listening and a movement toward listening that is an antidote for "alienated meaning" (p. 8), Davis (1996) develops a theory of pedagogical listening, according to three different modes of listening, not necessarily mutually exclusive: evaluative, interpretive, and hermeneutic. Evaluative listening occurs when a teacher maintains a "detached, evaluative stance" (p. 52). In contrast, a teacher listening interpretively endeavors "to get at what learners are thinking...to open up spaces for re-presentation and revision of ideas—to *access* subjective sense rather than to merely *assess* what has been learned" (pp. 52-53, original emphasis). Finally, Davis posits hermeneutic listening as "more negotiatory, engaging, and messy, involving the hearer and the heard in a shared project...an imaginative participation in the formation and the transformation of experience through an ongoing interrogation of the taken-for-granted...the unfolding of possibilities *through collective action*" (p. 53, original emphasis).

Recently, drawing on the Pirie-Kieren theory for the growth of mathematical understanding (Pirie & Kieren, 1994) and Davis's (1996) interpretive and hermeneutic listening, Martin (2001) observes teacher interactions in an elementary classroom to analyze how mathematical understanding grows at the individual and whole-class levels. He elaborates and provides evidence for how the listening patterns of a teacher can occasion opportunities for students "to construct and modify their own images in response to her interventions" (p. 251).

Both Martin (2001) and Davis (1996) inquire into teacher listening and its consequent impact on the growth of student understanding. Our study broadens this scope of inquiry as well as applies and extends Davis's category to analyze not just listening but rather discursive practices of learners in conversational exchanges. This we call interlocution. Our category has four properties and guides the inquiry into how learners' discursive exchanges structure their investigation as well as contribute to the mathematical ideas they build and their growth in mathematical understanding. Emerging from new data within our longitudinal study, we analyze them in light of four properties of interlocution, three of which are adapted from the three listening modes that Davis (1996) proposes, which we define as follows:

- *Evaluative*: an interlocutor maintains a non-participatory and evaluative stance, judging statements of his or her conversational partner as either right or wrong, good or bad, useful or not.
- *Informative*: an interlocutor requests factual data to satisfy a doubt, a question, or a curiosity without evidence of judgment.
- *Interpretive*: an interlocutor endeavors to tease out what his or her conversational partner is thinking, wanting to say, expressing, and meaning; "to open up spaces for re-presentation and revision of ideas—to *access* subjective sense rather than to merely *assess*" it (Davis, 1996, pp. 52-53, original emphasis).
- *Hermeneutic*: an interlocutor engages and negotiates with his or her conversational partner; the interlocutors are involved in a shared project; each interlocutor participates in the formation and the transformation of experience through an ongoing interrogation of the taken-for-granted and the prejudices that frame perception and actions; the interlocutors "engage in an unfolding of possibilities *through collective action*" (Davis, 1996, p. 53, original emphasis).

Interlocution as a conceptual category in our research enables us to track the participation, growth in understanding, and autonomy of learners in their construction of mathematical ideas. Over the course of our longitudinal study, we have observed how individual learning occurs as learners participate in meaningful activities with other learners (Maher & Martino, 2000). Characteristics of these activities include not just the conceptually rich and open-ended nature of the mathematical tasks but also the research setting as well as how and when researcher-teachers intervene or not while learners engage in doing mathematics (Maher, 1998a, 1998b; Maher, Davis, & Alston, 1992). Over the course of our work, the role of researcher-teachers has evolved, increasingly becoming less interventionist and, after presenting a problem task and answering clarifying questions, being physically absent from where participants work.

Method

Our data sources consist of video images collected from the different perspective of three cameras, each of the research session video's is about two hours in length; transcript of the videotapes combined to produced a fuller, more accurate verbatim record of the research sessions; student inscriptions; and researcher field notes. Our analytic method employs a sequence of phases, informed by grounded theory (Charmaz & Mitchell, 2001; Creswell, 1998; Pirie, 1998), ethnography and microanalysis (Erickson, 1992; Goldman-Segall, 1998), and approaches for analyzing video data (Cobb & Whitenack, 1996; Pirie, 1996, 2001; Powell, Francisco, & Maher, 2001). Specifically, our method of data analysis involves the following nine non-linear, interacting phases: (1) attentively viewing the videotapes several times without intentionally imposing a specific analytic lens; (2) describing consecutive time intervals; (3) identifying critical events; (4) transcribing the video record; (5) inductive and deductive synchronous coding of transcript, videotape, and inscriptions; (6) writing analytical memoranda; (7) categorizing codes, identifying properties, and dimensionalizing properties within categories; (8) constructing a storyline; and (9) composing a narrative. (For an elaboration and examples of these phases, see Powell et al., 2001).

Results

The following is the combinatorial investigation on which participants were invited to work collaboratively, "The Taxicab Problem":

> A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.
>
> What is the shortest route from a taxi stand to each of three different destination points? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

With this problem statement, we gave the participants a map, actually, a 6 x 6 rectangular grid on which the left, uppermost intersection point represents the taxi stand. The three passengers are identified at three different intersections as blue, red, and green dots, respectively, while their respective distances from the taxi stand are four units south and one unit east, three units south and four units east, and four units south and five units east. (To more fully appreciate the episodes that we present below, you may wish first to work on the problem.)

Clockwise from the left, seated on three sides of a trapezoidal-shaped table are the four participants, Michael, Romina, Jeff, and Brian. At the start of the session, Researcher 1 pulls up a chair, sits on the right side of the table between Jeff and Brian, thanks the four students for coming, distributes the Taxicab Problem, and asks them to read and see whether they understand the task. Afterward, the researcher stands up and, backing away from the table, removes her chair. With his head bent downward, facing the problem statement, Jeff asks aloud whether one has to stay on the grid lines and whether they represent streets. The researcher responds, "Exactly." Romina, Brian, and Jeff discuss that 5 is the number of blocks it takes to reach the blue destination point and that different routes to blue are the same length as long as one does not go beyond it. Brian says that they should prove it.

Researcher 1 walks back over to the table and asks the students for their understanding of the problem. Jeff says that the task is to find the shortest route while "staying on the streets." The researcher adds that it is about finding whether there is more than one shortest route. Both Brian and Romina voice agreement. The researcher goes on to say that if there is more than one shortest route, they have to determine how many there are. Jeff inquires whether the researcher is asking how many different shortest routes? Researcher 1 says that not only do they have to find the number of shortest routes but also that they will "have to convince us" that they have found all of them. The researcher then walks away from the table.

Jeff asks for colored markers. Jeff, Romina, and Brian choose to each work on different destination points. Romina says that it is five blocks to the blue point. Brian suggests counting them and being sure. Jeff asks why the length of each route to blue is the same. Michael explains that to get the blue point one has to go "four down and right one" since one cannot move backward or diagonally. Romina asks how to devise an area for that. Jeff and Michael tell her that it's not area, it's perimeter with each segment of the grid considered as one unit.

The above descriptive account is of the first four minutes and forty-four seconds of the research session. It is worth noting that the researcher spends little time at the table with the students and responds only to student questions in a tailored yet sparse manner. Also, the students rather quickly organize themselves by requesting colored markers and assigning subtasks to each other. When Jeff inquires about why the length or routes to the blue destination point are the same, Michael explains. When Romina requests help in devising an area, Jeff and Michael respond and inform her that the applicable notion is perimeter not that of area. In general, they also carefully and respectfully listen and respond to each other's questions, statements, and ideas.

After almost fourteen minutes into the research session, there is an interesting and pivotal interchange among Romina, Brian, and Jeff:

Episode 1: Break Apart and Do Easier Ones Time interval: 0:13:42 to 0:13:54

Romina: I think we're going to have to break it apart and draw as many as possible.

Brian: Yeah, //that's what I'm going to do.

Jeff: //And then have that lead us to something? What if we do- why don't we do easier ones? You know what I'm saying? What if the- the thing- Do you have another one of these papers?

In Episode 1, An agenda for action emerges from this interlocution among the three students. Brian and Jeff accept the task implied in Romina's statement and act on her heuristic. Moreover, Jeff refines her suggestion in his interrogative: "why don't we do easier ones?" Romina's statement and Jeff's interrogative establish a new agenda for the group's actions. Importantly, this action agenda represents a milestone in their mathematical investigation. From this point onward, the students no longer work on the combinatorial problem as given and, instead, pose and work on simpler situations to glean relevant information and extract insights from those situations so as to inform their understanding and resolution of the given problem.

Following the heuristic approach that the participants articulated in the pivotal episode presented above, the participants continue their investigation with a renewed sense of purpose. They chose to work on a more general problem than the one the researcher posed but felt that it was simpler and sensed that it would, in the words of Jeff, "lead us to something." This new approach developed out of a brief but focused negotiation between Romina and Jeff. Romina presents an idea and Jeff listens, and responding to her idea, participates in an unfolding of possibilities. What emerged from their interlocution was a plan for collective action and illustrates one of four interlocutory mode or property observed within the data. Their discursive exchange is hermeneutic in the sense that both Romina and Jeff participate in a negotiation, an unfolding of possibilities, each listening to the other and co-constructing an idea.

The data suggest that particular interlocutory modes influence differently the progress of the group. We will first describe aspects of the work that the participants had accomplished by the time they were midway through the session and then quote from a long episode to illustrate further the different interlocutory modes.

In Episode 1 above, Romina, Brian, and Jeff decided to count the number of shortest routes by starting with simpler cases for intersection points nearby the taxi stand. Afterward, Romina and Jeff negotiate a method for counting. Starting with an intersection point that is 2 blocks east and 2 blocks south of the taxi stand (forming a 2 by 2 sub-grid), they count the number of shortest routes of several nearby points and record their results in the taxicab grid. Then, working with their 2 by 2 sub-grid for which they found 6 shortest routes, they work on a 3 by 3, finding 15 shortest routes. They also work on 2 by 4, 2 by 3, and 4 by 3 sub-grids. In this way, they control for variables, a heuristic that they had developed and employed in several earlier tasks in the longitudinal study.

At the start of the Episode 2 quoted below, Michael is double-checking that the number of shortest routes for the 3 by 3 sub-grid is 20. There is a brief, evaluative exchange of information between Michael and Brian, where the latter wants to know whether the former has included in his count a route whose trace has the shape of a staircase.

Episode 2: The Staircase One Time interval: 0:55:31 to 0:56:42

Brian: Did you figure out the five by five?

Michael: Five by five? I'm doing three by three right now.

Brian: Let's just agree. If we already know what it is then we have to figure out-

Michael: I just want to make sure that's twenty. So- [Michael counts routes, moving his pen on his grid.]

Michael: I'm missing two. That's probably right though.

Brian: Did you get the, uh, staircase one?

Michael: Which one? For the three by three?

Brian: Yeah. [Inaudible]. [Romina returns.]

In the above, Brian, after Michael says that he is determining the number of shortest routes for a 3 by 3 sub-grid, inquires whether Michael found the "staircase one." The exchange involves passing on information and not evaluating responses. Brian lets Michael know the route whose shape is a staircase should not be overlooked. In this episode, the interlocutory property that characterizes the exchange between Brian and Michael is informative and leads to no externally expressed awareness or construction of mathematical ideas.

The next episode illustrates another mode of interlocution and points to its contribution the development of mathematical ideas. Romina had gone to the bathroom and rejoins the group.

Episode 3: Now It's Working Time interval: 0:56:42 to 0:57:04

Brian: Did Jeff tell you?

Romina: What?

Brian: That this one-

Romina: For which one?

Michael: //For-

Brian: //Four by two.

Romina: So you did get fifteen? So now it's working? [Meaning that the pattern of shortest routes corresponds to Pascal's triangle.] And then the two by four has to be fifteen too. Now if we do three by three and that's twenty, then we're done.

In Episode 3, Brian announces to Romina that Jeff and he have verified that 15, not 12, is the number of shortest routes in a 4 by 2 sub-grid. Romina notes that 15 must also be the number of shortest routes for a 2 by 4 sub-grid. In so doing, she voices her implicit awareness of a symmetrical property for the numerical pattern of shortest routes that Jeff and she have developed. Moreover, she observes that the pattern resembles that of Pascal's triangle. Romina listens, evaluates the information ("So now it's working?"), and interprets its meaning ("And then the two by four has to be fifteen too."). This episode establishes that a discursive exchange may involve more than one interlocutory property. In the present case, the interlocutor's words reveal that she engages both evaluative and interpretive interlocution.

At the end Episode 3, Romina suggests that if they can show that the number of shortest routes in a 3 by 3 sub-grid is 20, they can feel assured that their numerical pattern is Pascal's triangle. Michael and Jeff will assert that they will have to explain why numerical pattern is Pascal's triangle, and Romina will suggest relating their result to another problem they have worked on. However, at this point in their investigation, she, Michael, and Brian work individually to produce inscriptions of the different shortest routes in a 3 by 3 sub-grid. After about three minutes, she announces to Jeff that she is stuck.

Episode 4: How Do You Know It's Nothing Else?
Time interval: 0:59:49 to 1:00:34

Romina: I'm already stuck. [Brian draws a 3 by 3 rectangle on his paper. Romina draws in shortest routes for the "imaginary" 3 by 3 on her grid. Romina's pen stops when drawing a route.]

Jeff: You shouldn't be. Where you going?

Romina: Three by three. [She shows the paper to Jeff.]

[...]

Michael: Yeah I got twenty for that one.

Jeff: For a three by three?

Michael: Yeah.

Jeff: Alright well then- I mean can't we explain why we think- well- alright. [Jeff waves his hand.]

Michael: //They're going to ask us-

Jeff: //Alright then the next question is why- //why-

Romina: //Now-

Michael: //How do you know-

Romina: //Just relate this back to the //blocks. [Jeff points to the grid on the transparency with his marker.]

Jeff: //But wait- Why is this- why does the Pascal's Triangle work for this is the question.

Romina: //Exactly. Relate it to the blocks. [The word "blocks" here refers to the Towers Problem.[2]]

Michael: //If it is how do you know it's twenty? How do you know it's nothing else?

[...]

Jeff: If we can explain why- why this is the Pascal's Triangle up to here [He points to the transparency grid.], we don't need to explain how we're positive this is twenty. //You know what I'm saying? [Jeff waves his hand again.]

A few minutes before the start of Episode 4, Romina voices concern about the group's datum for the 3 by 3 sub-grid. Brian offers to recount the routes for that one, and too creates inscriptions of the routes in a 3 by 3 sub-grid. Meanwhile, Michael has been counting the shortest routes for the 3 by 3 sub-grid and, in Episode 4, announces that he found 20. Suspecting that this confirms that their numerical pattern follows Pascal's triangle, Jeff states, "Why does Pascal's triangle work for this is the question." Michael echoes this question as well as states that the researchers are "going to ask us" this. Romina suggests relating their result to "the blocks," which is actually refers to "The Towers Problem," a class of related problems some of which are indeed isomorphic to the "Taxicab Problem."[1] Similar to Episode 3, in this one, the participants' conversational exchange has both an interpretive and hermeneutic interlocutory structure.

Discussion

The four episodes presented above point to key mathematical ideas that the four students generate as well as heuristic and content connections that they make to other problems on which they have worked. Episode 4 illustrates that the students seek to understand and explain reasons why Pascal's triangle underlies the mathematical structure of the Taxicab Problem. Moreover, in the episode, Romina suggests that they relate (find an isomorphism between) this problem to the Towers Problem, a problem they have already met and resolved. As well, at the end of Episode 4, Jeff points to the numerical pattern on his and Romina's grid for the shortest routes for sub-grids smaller than a 3 by 3. Implicitly reasoning by induction, he then asserts that if they "can explain...why Pascal's Triangle works up to here [a diagonal of numbers above the entry 20 that represents a 3 by 3 sub-grid], we don't need to explain how we're positive this is twenty."

This and the other three episodes contribute and further our understanding of how learners develop mathematical ideas through their thoughtful engagement with task situations that they mathematize and of problem solving. Under the pedagogical conditions of our research, the data suggest that learners collaborate to create agendas for action, that is, structure their method of investigation; co-construct an understanding of the task as its underlining structure; and build mathematical ideas synchronously as individuals and as collective members of a community of practice. Their co-constructions of heuristics and mathematical ideas emerge from informative, interpretive, and hermeneutic interlocution. A further outcome of the dominance of these interlocutory modes or properties is that over time learners build an *esprit de corps*.

Importantly, through specifically interpretive and hermeneutic interlocution learners do structure, monitor, and conduct their problem-solving investigation. In these interlocutory modes, the four participants of our study suggested subtasks for each other and ways of working, identified their need to verify expected outcomes and actual results, as well as prompted their need to explain and justify the underlining mathematical structures that emerged. As seen in Episodes 1 and 4, hermeneutic interlocution is a discursive practice in which learners structure their investigation of a mathematical task. As seen in Episodes 1, 3, and 4, learners engaged in interpretive and hermeneutic interlocution construct heuristics as well as mathematical ideas. Typically, evaluative and informative interlocution lead to no externally expressed awareness or construction of mathematical ideas. Tracking the properties of learners' interlocution yields insights into their mathematical ideas, growth in understanding, and autonomy as learners.

Though we have presented an analysis of episodes of a single problem-solving session, our observations of participants' mathematical engagement are informed by previous analyses in our longitudinal study (Kiczek, 2000; Kiczek, Maher, & Speiser, 2001; Maher & Martino, 2000; Martino, 1992; Muter, 1999). For instance, the evolved cultural norms of our research setting, most particularly the questioning patterns and expectations of researchers for explanations and justifications have been assimilated by research participants and infused into their ways of collaborating with each other. This is manifest in Episode 4 when Michael comments "they're going to ask us," and Jeff says "why does Pascal's Triangle work for this is the question."

Conversational interactions among learners can advance their subsequent individual and collective actions. Our data suggests that, among our four interlocutory properties, (1) interpretive and hermeneutic interlocution have the potential for advancing the mathematical understanding of individual learners working in a small group, (2) the personal or individual understanding of a learner is intermeshed with the understanding of his or her interlocutors, and (3) the mathematical ideas and understanding of an individual and his or her group emerge in a parallel fashion.

Conclusion

The results of this study show that learners can autonomously structure, monitor, and conduct their investigation of a mathematical task. They can sustain their engagement with a task for long periods of time and do so essentially without researcher-teacher intervention. We examined closely student interlocution and traced the development of their heuristic and mathematical ideas and growth of mathematical understanding. We have demonstrated how the discursive practices of learners influence how they structure their investigations, the mathematical ideas they develop, and how their understanding grows.

Notes

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[2] *n*-Tall Towers Problem: Your group has two colors of Unifix Cubes to work with. Work together and make as many different towers *n* cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers *n* cubes tall.

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DUELING DEFINITIONS: THE ROLE OF DEFINITION IN ANDREW'S UNDERSTANDING OF CONGRUENCE OF INTEGERS

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Developing an understanding of how to use definitions in mathematics has been regarded as a major hurdle for students at many levels. While studying modular arithmetic in a number theory course, undergraduates were observed using an incorrect pseudo-definition for the concept of congruence of integers, in addition to a standard definition given in class. In particular, Andrew seemed to understand the importance of definitions in mathematics, and recognized the inconsistency between the two primary definitions that he used for congruence. However, rather than reject one of the definitions, he chose to use both, inventing rules for when each was valid. Despite his success in the course, it appeared that he did not recognize that mathematical definitions should be consistent; that is, he did not appear to believe that all definitions for a concept should be logically equivalent.

Understanding the role of definition in mathematics has been characterized as one of the primary sources of difficulty for undergraduate mathematics students (Tall, 1992; Vinner, 1991; Dreyfus, 1991). Definitions are crucial in mathematics because of their ability to unify the related ideas of a concept across many fields of mathematics. Gray, Pinto, Pitta, and Tall (1999) argue that the transition from constructing a definition inductively from examples of a concept or mathematical object to constructing a mental object from a verbal definition alone is an extremely difficult one for students to make.

In an introductory number theory course at a large state university, students were introduced to the concept of congruence of integers as a tool for solving some classical problems in number theory. Congruence of two integers a and b modulo an integer n was defined in class in two ways:

1. $a \equiv b \pmod{n}$ if and only if $n \mid a - b$ [Standard definition]
2. $a \equiv b \pmod{n}$ if and only if a and b have the *same* remainder upon division by n [Division-remainder definition]

The equivalence of these definitions was proved in class and the instructor made use of both definitions fairly equally. In the larger study from which the results presented in this paper are drawn, it was found that the students consistently interpreted statements of congruence incorrectly, using a pseudo-definition that they appeared to have collectively invented (see Smith, 2002). The pseudo-definition was generally stated as follows: "$a \equiv b \pmod{n}$ if and only if b is the remainder of a upon division by n." Note that if one replaces the phrase "if and only if" with "if", this is indeed a true statement. It represents a sufficient condition for two integers to be congruent modulo n, though not a necessary one.

The fact that this pseudo-definition is not equivalent to any other given definition for congruence did not appear to cause any conflict for most of the participants in the study. The case of Andrew, however, was different in that he acknowledged that the pseudo-definition was not equivalent to either of the definitions given in class. Despite the fact that he seemed to appreciate the significance of the role of definition in mathematics, he ignored the inconsistency between the two definitions.

Theoretical Perspective

Vinner (1991) described the *concept image* as "something non-verbal associated in our mind with the concept name" (p. 68). In other words, forming a concept image for a particular concept is an inherent part of coming to understand that concept. Tall (1992) describes the notion of concept image more succinctly as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 496).

Definitions in mathematics, according to Vinner (1991), form the crux of the "conflict between the structure of mathematics as conceived by professional mathematicians, and the cognitive process of concept acquisition" (p. 65). Vinner draws a distinction between "everyday life contexts" and "technical contexts," noting that in most everyday experiences, one does not need to consult definitions in order to understand a sentence occurring in conversation. Indeed, some terms commonly used in everyday life, such as "red," have no real definition. Instead, we each have a concept image for "red," some aspect of which is evoked when we encounter this word. In general people have no need to revisit the definition of an understood concept; in fact, definitions in everyday life are quite dispensable. Vinner offers a "scaffolding" metaphor for this role of definitions; the definition may serve as a useful framework in learning a concept, but once the "building" is finished, the "scaffolding" is removed. In technical contexts, however, definitions play a much larger role.

According to Tall and Vinner (1981, as cited in Tall, 1992), the *concept definition* is "a form of words used to specify [a] concept" (p. 496). However, there does not appear to be a consensus in the literature as to precisely what this means. Vinner (1991) separates the concept definition from the concept image and allows for the possibility that an individual's concept definition may conflict with the external formal definition used by the mathematics community. That is, the concept definition is a specific wording used by an individual to define a concept and thus is unique to the individual. This definition may or may not agree with the formal mathematical definition agreed upon by the mathematical community or taught to the student, but it is the definition that corresponds with the individual's concept image. Tall (1991), on the other hand, uses the terms "concept definition" and "formal definition" interchangeably. For the purposes of this paper, the term "definition" will refer to a formal definition that is accepted within the mathematical community, while the term "pseudo-definition" will refer to an incomplete verbal description that is regarded by students as a definition.

The definition for a concept clearly plays an important role in the learning of mathematics, especially in the development of proof. Tall (1999) writes that at the undergraduate level, "the introduction of axioms and proofs leads to a new kind of cognitive concept – one which is defined by a concept definition and its properties deduced from the definition" (p. 117). Students must begin to integrate their informal and formal concept images with these definitions in order to make the shift to the deductive nature of undergraduate level mathematics.

Method

In the larger study, an exploratory case study design was used to investigate the conceptions of the concept of congruence of six undergraduates enrolled in an introductory number theory course. The participants, primarily pre-service secondary mathematics teachers, were chosen based on high scores on a first exam, in conjunction with Dr. Thomas, the instructor of the course. The researcher's role was that of a participant-observer and unofficial teaching assistant in the course.

The six students participated in three semi-structured phenomenological interviews over the course of the semester. The first interview took place one week prior to the exam covering congruence, a second interview took place approximately two weeks later, and a third approximately three weeks after the second. In addition, class sessions during the unit covering congruence and applications of modular arithmetic were videotaped and transcribed. The data from which the results described in this paper are drawn come from the first two interviews with Andrew.

The first part of the first interview focused on the students' experiences with and perceptions of mathematics as a discipline and of themselves as mathematics students. In the second half of the interview, the students' responses to mathematical tasks on a previously completed worksheet were discussed. The second interview focused primarily on the students' solutions to problems on the exam covering congruence.

Results

Andrew was a quiet, introspective 20-year-old who planned to teach high school mathematics after graduation. His perspective on mathematics was unique from that of his peers, but like many of them he did not consider himself to be particularly good at mathematics. However, Andrew was extremely successful in the course, consistently earning the highest scores on assignments and exams. He intensely studied the text and other number theory books in order to gain further insight into the material. He was generally regarded by Dr. Thomas and the other participants as the best student in the class.

When asked what he liked most about mathematics, Andrew spoke first of his appreciation for the power of mathematics to unify concepts and ideas in many unrelated fields. He was aware that a shift in emphasis had occurred in his undergraduate mathematics courses in which the use of sophisticated theory and precise definitions played a larger role than it had in earlier courses. Andrew stated that understanding a mathematical concept entailed "being able to apply definitions very flexibly [...] comprehending an idea in its most general and abstract form and being able to apply the idea in a variety of situations." Andrew's use of language in describing his solutions to mathematical problems and tasks also revealed this emphasis on the importance of definitions in mathematics.

When asked in the first interview to explain the meaning of congruence of two integers a and b, modulo n, Andrew offered the following three definitions:

1) n divides $a - b$
2) a divided by n leaves a remainder of b
3) $a = b + nk$ for some integer k

When presented with a specific statement of congruence he tended to rely on the first [standard] definition, his language frequently revealing the significance he placed on definitions: "I'm sort of just re-churning again what the definition is saying, finding a number that meets these conditions." Note that the second item Andrew offered is the pseudo-definition, and the third is algebraically equivalent to the standard definition. Like most of the participants, Andrew never used the definition of congruence that Dr. Thomas used in class – that a and b have the *same* remainder upon division by n.

One of the items on the worksheet used in the first interview consisted of four congruences, each with a variable in either the a or b position. The students had been asked to determine the possible values for the variable. Andrew's written responses to these items are shown below.

Item:	Andrew's written answer:
(a) $x \equiv 3 \pmod 7$	$x = 3 + 7k$
(b) $17 \equiv x \pmod 4$	$x = 1, 5, 9, \ldots, 4j - 3$
(c) $x \equiv 23 \pmod{11}$	$x = 23 + 11m$
(d) $2 \equiv x \pmod 9$	$x = -7, -16, -25, \ldots, 2 - 9m$

He had appeared to solve items (a) and (c) differently from items (b) and (d), and when asked if there was a difference between these congruences, he indicated that the position of the variable in the congruence was a factor, saying, "you're talking about remainders here [when the x is on the right side] and you're talking about finding numbers here [when the x is on the left side]." He explained that if one uses the second definition he gave [the pseudo-definition], there are multiple answers when the x is on the left, since there are infinitely many numbers that have a given remainder when divided by n. When the x is on the right, however, he stated that there is a *best* answer, whose form was given by the division theorem. This reasoning was quite common among the participants in the study. In fact, several of the participants had insisted that there was only one value for x in items (b) and (d), while there were infinitely many values for the x in items (a) and (c) (see Smith, 2002).

When asked why he had in fact given multiple answers for items (b) and (d) in light of this explanation, Andrew explained that if one applied the first [standard] definition, then many numbers could be substituted for the x in these congruences. The interviewer then asked Andrew to consider what would happen if the positions of the numbers in item (b) were switched, so that the congruence read $x \equiv 17 \pmod 4$. Would this change the congruence? Would there still be a *best* value to choose for x? Andrew seemed to believe that it depended on the choice of definition.

"I mean, clearly when you take a number and divide it by four, especially this number [17], you will get a remainder of one, and not five or nine. But the congruence still works because of this [standard] definition. [Having one best value] would make more sense in terms of this [pseudo-] definition because by the division theorem, you couldn't have a remainder that's bigger than your divisor."

He seemed to believe that both the standard definition and the pseudo-definition could be applied in the situation, even though these were clearly not equivalent, and produced different solutions. Andrew appeared to believe that there were two different definitions for the concept of congruence, and the fact that they were not equivalent was not problematic.

A few weeks later during the second interview, Andrew remained firm in his belief that there were two different definitions for congruence. One of the exam problems asked the students to "fill in the blank" in two congruences: (1) $33^{25} \equiv$ ____ (mod 8), and (2) $19^{19} \equiv$ ____ (mod 7). Dr. Thomas' intention was that the students would use a "cycling" strategy[1] on the first item and Fermat's "little" theorem[2] on the second, though most of the participants described their initial attempts to solve these items in terms of using their calculators to find the decimal representation of the large integer, dividing this by the modulus, and finding the remainder (see Smith, 2002).

For the second item, Andrew had used Fermat's "little" theorem to find an answer of 19. However, he lost 2 points on his exam for not reducing this result to 5. He claimed not to have been bothered by losing points for this, but he was clearly convinced that his answer of 19 was correct.

A: I mean, it was like, once I saw what I'd written down, I instantly recognized what the mistake was. I don't think it was necessarily a mistake. I mean, the congruence is true. I don't know.

J: Do you think that's... that's a really good question now. I mean, what you've written here is true, right? But you didn't write 5.

A: Right, it doesn't make sense though ... in the world of mod 7, you're only talking about numbers zero up through six. So [19] doesn't really make much sense. But... you could really argue the point though that this is a true congruence. So, I mean... but no, I understand.

J: So, it's sort of analogous to leaving a fraction as 2/4? I mean, is that sort of the level of wrongness?

A: Yeah, I mean, I guess... I don't want to say that it's wrong. It's just... I don't know. I guess in some sense it's sort of a matter of convention. It's a very minor thing and I'm not too bogged down by it. I mean, as far as I can think, I mean, it's still right, you know. The congruence is still true, just like 2/4 is still ½.

At this point, Andrew clearly was struggling with the inconsistency between the two definitions for congruence[3].

Another item on the exam asked the students to solve for x in the congruence $5x \equiv 1 \pmod{11}$. Whereas all of the students had given a single number as an answer for items (1) and (2) described above, a *set* of answers was given by all students for this item. Andrew's answer was $x = -2 + 11k$. When asked if $1 + 8k$ would have been a better answer for item (1), he responded that, though it would not have been wrong, it was not really an appropriate answer.

A: Well it wouldn't make sense in mod 8... If you're talking about the world of mod 8, it wouldn't be true that you'd get a remainder greater than 8. But the congruence would still be true.

J: When you say the congruence is still true, what does that mean to you exactly?

A: It means that if you took the difference between [the two numbers], 8 would still divide it.

Again, Andrew's use of two definitions for congruence was causing conflict. According to the pseudo-definition, the integer on the right side of the congruence was *the* remainder, and by definition, the remainder had to be smaller in absolute value than the modulus. On the other hand, the standard definition only required that the difference between the two integers be divisible by the modulus. By the end of the second interview, Andrew seemed to have become more entrenched in his belief that one could use two different definitions for congruence, even though these were at times contradictory.

Discussion

The students' use of the pseudo-definition for congruence was an interesting phenomenon. Careful study of the transcripts of class sessions revealed that the instructor never used such language when referring to congruences in class, and though he generally showed a preference for reducing the integers on the right side of a congruence, he also frequently did the opposite—substituting larger integers for smaller ones when this action facilitated the solution of a problem. All of the participants struggled with the role of the position of the integers in their understanding of congruence, though only Andrew recognized the importance of definitions in mathematics. He was the only student of the six who continually referred to definitions when trying to prove or verify statements involving congruence. Whereas the others tended to rely on their concept images when working with congruences, Andrew's first response was typically to refer to the concept definition – or definitions, in this case

In Andrew's experience, circumstances frequently arose in which the pseudo-definition was not satisfied, though the standard definition was. He did not seem particularly disturbed by this fact, merely confused and mildly frustrated. Rather than reject the pseudo-definition, he tried to understand the complex set of rules under which each definition could be applied. The fact that there were two conflicting definitions for a single concept was dealt with not by comparing the definitions, recognizing that they ought to be equivalent, and then rejecting the inferior one. Instead, Andrew accepted both and believed that use of one definition or the other was dictated not by logic, but by convention. It is interesting to note that he carefully applied each definition in the context that he considered appropriate, making certain that the conditions of the definition were satisfied.

Andrew was by all measures a successful mathematics student, and was one of the few in his peer group who had learned how to use definitions and mathematical formalism. However, he did not recognize that there should be a *single* definition for a concept, and that any alternative definitions should be equivalent. How had Andrew missed this important fact, which clearly contradicts his belief that the power of mathematics lies in its ability to unify concepts across many fields?

Definitions of concepts in "real life" can be contradictory; language can be twisted and words defined in such ways that their meaning in or out of a particular context is completely changed. However, this is not the case in mathematics. Mathematical definitions *are* consistent. The distinction made by Vinner (1991) between the use of definitions in real life and technical contexts applies to the perceived role of definitions in these contexts as well. Perhaps this subtle issue causes difficulty for students when they are learning to prove in mathematics. This research demonstrates that, at least in Andrew's case, not understanding the role of definition in mathematics caused a good deal of conflict and confusion. In his struggle to make sense of the concept of congruence, Andrew seemed to have come to the conclusion that he had to determine the nature of a set of arbitrary rules that clouded the concept of congruence in order to understand it.

Andrew was the only participant of the six in the study who seemed to appreciate the role of definition in mathematics, though all of the participants displayed the tendency to use contradictory definitions. Further research could be conducted with advanced undergraduates in order to determine the extent to which mathematics students struggle with the role of definition, and to understand how a student's conception of definition affects him or her when learning to prove.

Notes

[1] Using a "cycling" strategy would entail raising the integer 33 to a sequence of powers and reducing the result modulo 8 in order to find an exponent k such that $33^k \equiv 1 \pmod 8$. In this instance, $k = 5$ works, and so one can substitute 1 for 33^5 to get $33^{25} = (33^5)^5 \equiv 1^5 \equiv 1 \pmod 8$.

[2] Fermat's "little" theorem states that if a is an integer and p is a prime that does not divide a, then $a^{p-1} \equiv 1 \pmod p$. In item (2), since 7 is prime, note that $19^6 \equiv 1 \pmod 7$, so $19^{18} \equiv 1 \pmod 7$, and then $19^{19} \equiv (1)19 \equiv 5 \pmod 7$.

[3] It should be noted here that Dr. Thomas' grading scheme may have reinforced this difficulty. Though he did not specifically instruct the students to reduce integers on the right side of a congruence, most of the students did indeed do so. When asked about this incident, Dr. Thomas noted that he had been grading the exams quickly and had not immediately recognized that 19 was a correct answer. He noted that if Andrew had asked him about it directly, he would have re-awarded the points immediately. Andrew, however, decided not to pursue the issue. Another participant also gave 19 as answer, and like Andrew, decided that she must have made a mistake and did not ask Dr. Thomas why 19 was "incorrect." For more discussion of the social construction of the students' notions about the position of integers in a congruence, see Smith, 2002.

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UNDERSTANDING COMPLEXITY: LINKING PHENOMENA TO REPRESENTATIONS

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This paper reports on results of an exploratory study on undergraduate students' understanding of complex systems' ideas across varied representation systems. The study described two groups of students' explorations of a physical device which exhibits complex dynamic behavior. Students could manipulate the device and observe it both directly and through graphs. Through their interaction with the device, their use of familiar types of representations, as well as analogies to familiar events and devices, students gained entrance to the complex phenomena and the ideas that underlie complexity. As their familiarity with the device increased, students used it to understand new representations and to reconstitute elementary mathematical ideas. While in the first case students used a previously mastered representation system as a means to understand complexity, in the second case they developed a new representation system by experiencing it in relation to previously understood phenomena.

The ideas of complexity are increasingly becoming an integral part in the mathematical, physical, biological, and even social sciences. The conceptual basis of complex systems ideas reflects a change in perspective about our world that is important for students to develop, as it corresponds to the scientific environment that will exist when they graduate (Chen & Stroup, 1993; Kaput, Bar-Yam, Jacobson, Jakobsson, Lemke, & Wilensky, 2001). This perspective emphasizes both the limits of predictability as well as the possibility of understanding indirect consequences of actions taken through modeling the interdependence of our world (Hall, 1992; Prigogine, 1999; Tufillaro, Abbott & Reilly. 1992).

Despite the wide recognition of the broad applicability of complex systems' models in our world, these ideas have not yet found their position in our curricula. Except for the education of a few highly trained specialists, little of the conceptual power embodied in the rapidly developing tools and perspectives of complex, dynamical systems has reached the educational experience of our citizenry at any level. Recent research in student learning has shown that many of the core ideas of complex systems and nonlinear dynamics are challenging for most students (Casti, 1996; Feltovich, Spiro & Coulson, 1989; Jacobson, et al., 1998; Kaput et al. 2001; Resnick, 1996; Roberts, 1978; Wilensky, & Resnick, 1999). For example, many students, even at the college level, believe that chemical reactions stop at equilibrium (Kozma, Russel, Johnston, & Dershiimer, 1990). Further, many concepts related to complex systems may be counter-intuitive or conflict with commonly held beliefs (Casti, 1996). Students often expect a linear relationship between the size of an action and its corresponding effect: a small action is expected to have a small effect, while a large action a large effect correspondingly. However, it is now well-understood that in complex and dynamical systems a small action may have interactions in the system that contribute to a significant and large-scale influence – the "butterfly effect". Other researchers have proposed that people tend to favor reductive explanations that assume central control and deterministic single causality and that there is a deep-rooted resistance towards ideas describing various phenomena in terms of self-organization, and decentralized processes (Feltovih, et al., 1989; Wilensky & Resnick, 1999).

Further study is warranted to determine how students develop an understanding of these ideas, and what are appropriate learning experiences and instruction activities to support students' learning. This study, whose purpose includes an investigation of the ways that students approach and come to understand complexity, aims to provide insight into this issue. Specifically, we examine

(i) How can interaction with complex physical systems lead into the core ideas, representations and phenomena of system dynamics.

(ii) How a physical complex system can act as a medium for reconstituting certain key traditional ideas, some of which are at the heart of classical calculus and basic kinematics.

Method of Inquiry

We designed a study to begin to explore the above issues based on prior work and apparatus developed by Ricardo Nemirovsky. Our prototypical tool is the Bouncing Car device (Figure 1), a small car sliding on an inclined plane and colliding with a moving piston on the lower end of the plane. The piston is driven by a motor and it moves periodically. As the piston moves, it may interact with the "car" that can bounce against the piston and move up and down the inclined rack. Each time the car is bounced up the rack, its velocity after impact determines the distance (and time) the car will travel up the rack and return to the piston again, which determines how far the car will travel (bounce) the next time. This is the origin of the non-linearity of this model.

This device provides a manipulable and visually explicit version of a nonlinear dynamical physical system studied by Tufillaro et al. (1992), which involved a ball bouncing on a vibrating table. The amplitude and the frequency of the oscillating piston can be modified as it moves. Based on a motion sensor, a computer generates real-time graphs of various kinds for both the car and the piston. These graphs can display various data (velocity or position vs. time, velocity vs. position, etc.) which can be displayed separately or simultaneously on the same screen if desired.

The Bouncing Car is one of a family of physical devices developed such that they can exhibit a wide range of dynamic behaviors, their inner workings are visible and

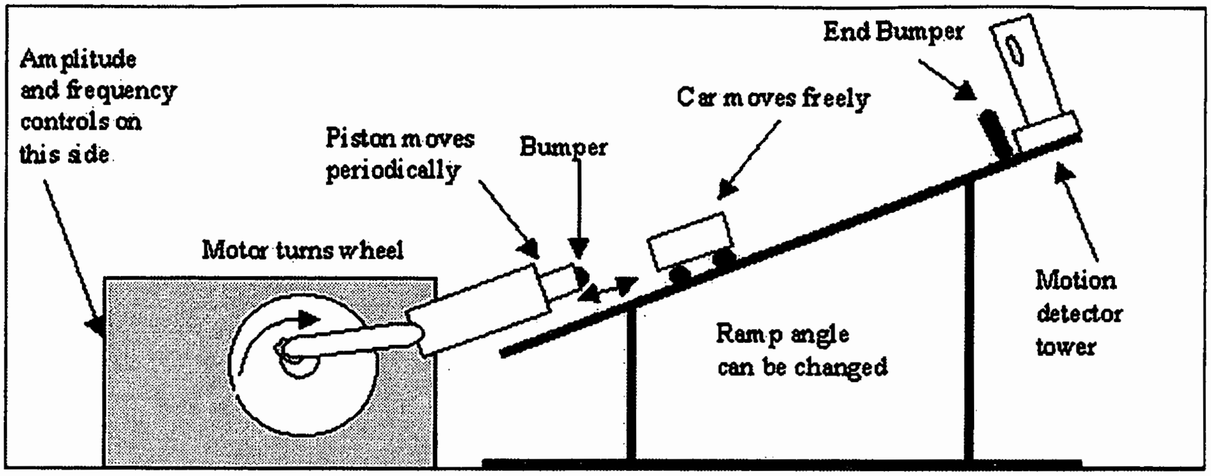

Figure 1. The Bouncing Car.

open to examination and control, and their motion is symbolized with real-time graphs generated on a computer screen. These three features, coupled with the fact that these systems are physical, tangible, are critical to students' engagement with the science and mathematics of system dynamics (Corriea, 1997; Noble, Nemirovsky, Wagoner, Solomon, & Cook, 1996).

We used the device to interview two groups of undergraduate students. The first group consisted of 5 students and the second consisted of 3 students. All students had a strong background in mathematics (each participant had completed on average five college-level mathematics courses). Each interview lasted approximately 3 hours. Students were asked to explore the device in order to understand its function. The interviewer kept a non-interfering stance and allowed students to pursue the questions they found interesting. Both sessions were videotaped and transcribed and were analyzed using a qualitative methodology. The method of analysis involved inductively deriving the descriptions and explanations of how the students interacted with the device and approached selected dynamical systems ideas with a focus on periodic vs. chaotic behavior, and sensitivity to system parameters, which in this case were the period and amplitude of the piston's motion and the slope of the inclined plane. Using related behaviors and comments within each topic, we wrote descriptions of the students' operations and actions. These descriptions formed the findings of the study.

Results

Complexity can be found in almost any physical system – both in a technical and informal sense. In the technical sense, complex systems are sets of interconnected elements whose collective behavior arises in a non-obvious way (and often counter-intuitive and surprising way) from the properties of the individual elements and their interconnections. In the everyday, informal sense, complexity suggests that many phenomena are happening simultaneously in a way that is difficult to comprehend. Clearly, students experience complexity in both senses in their everyday lives. However, students seldom, if ever, are exposed to a situation where they have to understand and represent complex phenomena in the technical sense. Throughout their schooling students are only asked to study instances within phenomena that are carefully chosen to *avoid* complex behavior. In some ways, this follows the epistemological pattern of Western Science of the past several centuries – avoid and idealize away phenomena that are not subject to simple, universal and often closed-form mathematically expressible laws (Casti, 1989). In our interviews students were faced with complexity in both senses. Thus, it was not surprising that both groups initially expressed a need to "regulate" the motion of the car. Students attempted to adjust the device and the car to achieve what they tried to describe as a "consistent", "periodic", "equilibrium", "regular" and, "harmonic" motion. It was these attempts to regulate the motion of the car that formed the context of their explorations and their discussions of their explorations of complexity. Here, we describe the ways in which students first came to develop insights into certain core ideas of system dynamics (which, of course intends simplicity of description and modeling as well, but in a way that includes many of the phenomena that were, in principle, ignored by traditional science (Hall, 1992). Secondly, we describe how students' understanding of other mathematical concepts interacted with the development of their understanding of these ideas.

Understanding the Core Ideas, Representations and Phenomena of System Dynamics

Throughout the interviews students used a variety of ways to organize the complex behavior they witnessed. Over time students used different viewpoints in their attempt to decipher the complexity they faced: experimenting with the device itself, studying the graphical representations of the motion, and forming analogies to other experiences with physical devices and functions.

Studying graphical representations

During interviews students often used a visual approach in which they qualitatively analyzed the graphs that represented the motion of the bouncing car. Students questioned the local and global meanings of graphs, and attempted to form conjectures on the nature of the motion and possible functions that model the motion. When the students in the first group first saw the distance vs. time graph of the motion of the piston and the bouncing car (Figure 2), they attempted to qualitatively analyze the graph and gain insights into the motion of the bouncing car. As experienced users of distance vs. time graphs, they could see the graph as a description of an event. Don had predicted that the piston's movement would be described by a sine function and was surprised to see the graph that the motion detector gave him:

Paul: "The blue graph [lower graph in Figure 2] is constant. It's the piston going back and forth"

Don: "It doesn't go below zero! It's like a vertical shift...

Paul: "No I don't think. It's different, it stops [...] Where is zero?

Don: "It's here, at the end. It comes out and then back again. And there's these stops..."

The two students attempted to use the graph to understand the motion of the piston, which, initially, appeared to be periodic. They trusted their reading of the

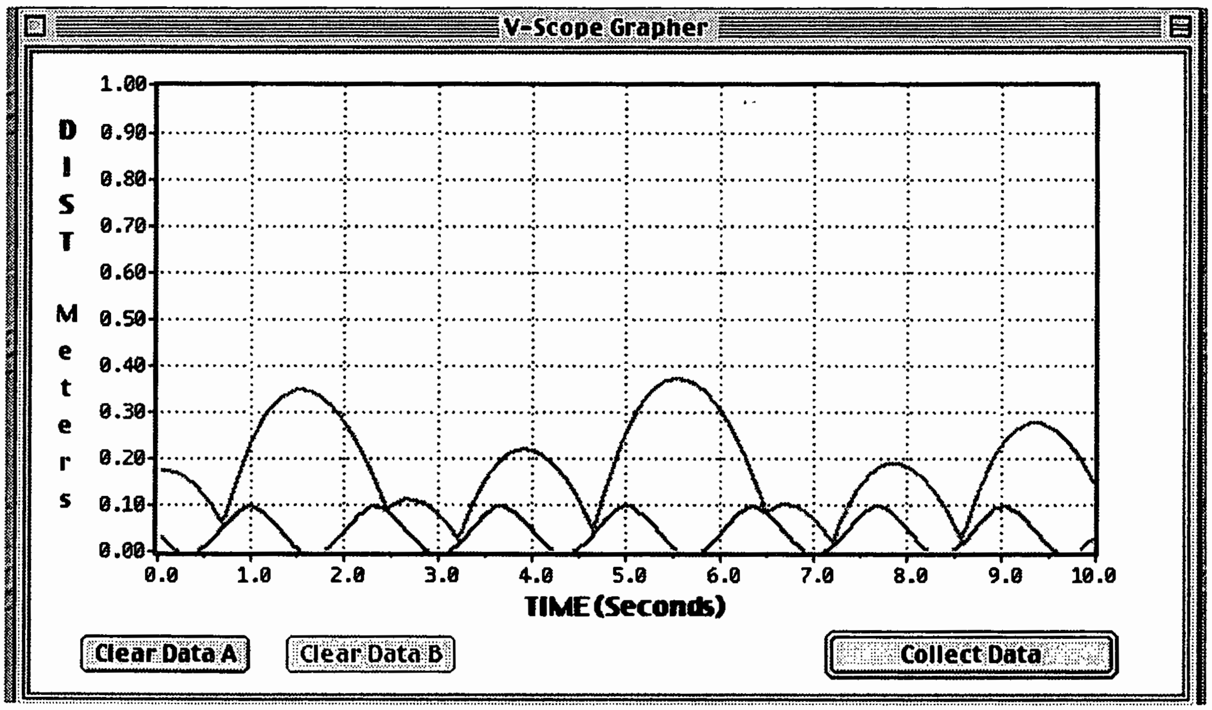

Figure 2. The distance vs. time graph of the piston (lower graph) and the car (upper graph).

graph to interpret the motion of the piston. Similarly, they attempted to understand the motion of the car using the same graph:

Paul: "Those casps... where it's changing direction. It's when the car hits the piston. It goes "boom" and it bounces back."

Joyce: "It's changing sharply. That's obviously when it's hitting the bumper"

Joyce and Paul once again interpreted the motion of the car using the well-understood graphical representation. The second group followed a similar path and moved further to attempt to predict the distance the car travels based on its interaction with the piston – but as mediated by the graphs rather than the phenomena directly:

Andrew: "See, how far it [the car] goes depends on whether the piston is going inward or backward"

Mike: "Yes, see when it hits it when it's coming out, it goes farther"

Andrew: "This is the way I see it [moves toward the white board and starts drawing a graph]: When it [the car] hits it [the piston] when it's [the piston] decreasing it doesn't go very far.

Ilda: "Yeah, sometimes it just bounces back..."

Mike: "I want to change the speed and the amplitude of the piston. I want the car to hit always at the same time. We should time it. Then it [the car] will always go up the same distance"

The second group put aside the device and worked directly with the visual representation which appeared to be a powerful tool in their hands. While they responded to the graph as an object, they saw through the graph to what it represented and felt confident in forming predictions about the behavior of the device based on the relationships they could extract from the graph. Furthermore, contrary to common expectations, students in both groups did not reduce the complexity of the situation into isolated pieces with which they felt comfortable. Rather, students often *chose* to use the visual complexity of the graphs to aid them gain insights into the situation they were exploring.

Forming analogies between the device and previous experiences

In several occasions students drew analogies between the bouncing car device and other physical situations where collisions occur. The following excerpt from the first group's interactions illustrates the use of analogies as a tool to manage complexity:

Paul: "See, when it [the car] hits it [the piston] and it follows through, it goes farther away. Like a baseball bat [imitates the swing of a bat]. If you hit it at the end of the swing it doesn't go far, but when you hit it during the motion you follow through. The motion goes all the way through."

Don: "Yeah, like in golf..."

The game of baseball or golf is another familiar situation and students used it to understand the factors that determine the motion of the car relative to the piston. In other cases, students attempted to find similarities and differences between this device and objects that exhibit periodic motion. These findings resonate with earlier work of Clement and Brown (1988).

Experimenting with the device itself and the interactions with the device

During the second part of the two group interviews, students were presented with a second representation of the bouncing car motion, the velocity vs. distance graph (Figure 3). It has been observed that students have little familiarity with the velocity vs. distance representation of motion (Leinhardt, Zaslavsky & Stein, 1990). Students' discomfort with this representation became immediately obvious in their behavior towards the graphical representation and the Bouncing Car device. In this case, students not only did not utilize the graph to gain information about the device, but they turned their attention to the device as a means to gain i insight into the nature of the graph – a complete reversal. As one student in the first group commented

Joyce: "I'm not familiar with this graph; it'll be harder to understand the car's motion by looking at the graph... I'm trying to think of what is happening with the car to understand the graph".

Similarly, Andrew in the second group asked his teammates to turn on the bouncing car device for him to observe the formation of the graph while the car was moving:

Andrew: "I want to see how those circles are formed. Turn the thing on again and slow it down. I want to see it drawing the graph"

The direct and observable connection between the graph of the motion of the car and the piston on the computer screen and the actual motion of the car and the piston on the Bouncing Car device allowed students to interpret the graph directly, in terms of their own actions and the behavior of the device. Andrew and his group attempted to interpret each element of the velocity vs. distance graph by watching it while it was

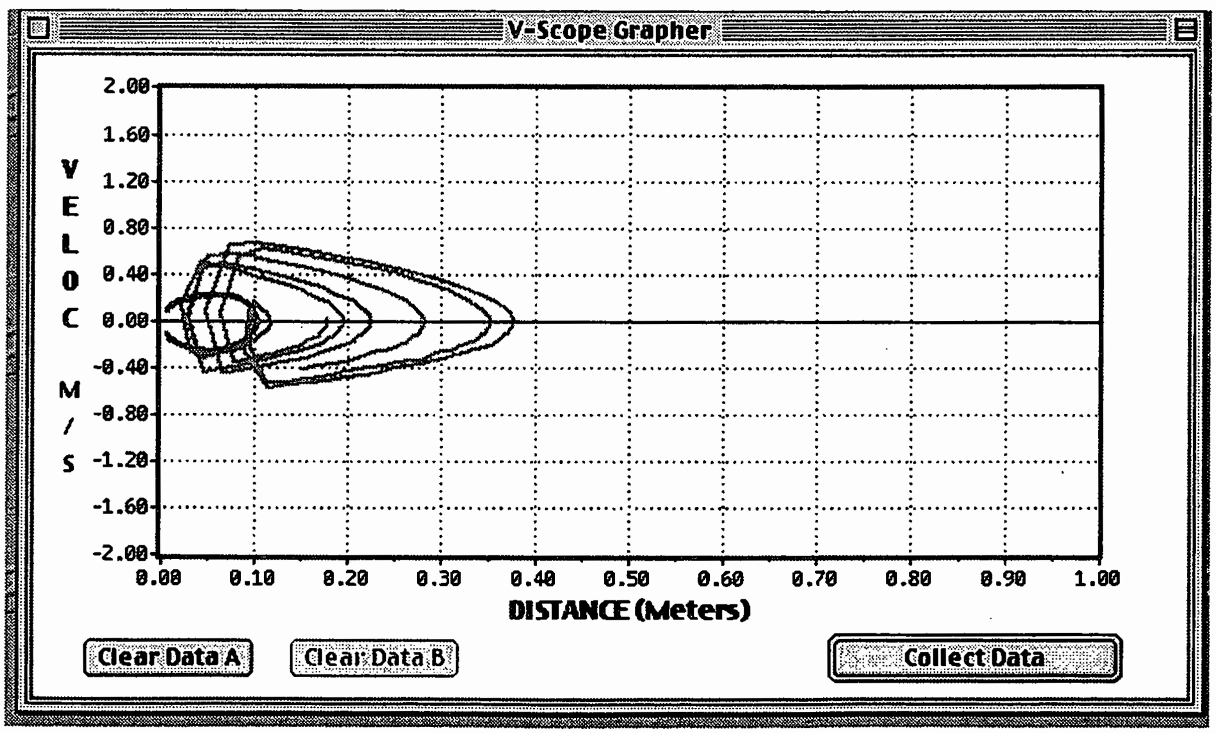

Figure 3. The velocity vs. distance graph of the piston (small ellipse on the left) and the car.

being formed. He watched the device carefully to see what the intersections of the two graphs meant and how the changes in the car's direction, velocity and distance were represented on the graph.

The ways with which students operated on the device differed qualitatively across students, which indicates differences in the underlying conceptualizations of the functions they are working with (a finding that resonates with earlier work by Monk, 1994). For example, a number of students worked in a playful, spontaneous mode with the hands-on device and watched the graph being formed, while, others carefully manipulated the various features to achieve certain outcomes and to test explicit conjectures – a qualitative but systematic exploration of the situation.

Interaction Among Ideas Underlying Complexity and Elementary Mathematical Ideas

Using elementary mathematical ideas to understand complexity

As discussed above, students often used elementary mathematics to gain insights into the complex behaviors. In our case, it was the contrast of the graph of the periodic motion of the piston to the graph of the non-periodic motion of the car that helped students start understanding the variables that were possibly affecting the motion of the car. In addition, with experience of various kinds of quasi-periodic motion, students gradually came to develop insights into the concept of various types of periodic orbits in phase space representations (plots of velocity against position in this case).

Using new representations of the complex system to reconstitute elementary mathematical ideas

The study of the phase-space graph of the bouncing car motion involved the use and, subsequently the re-examination of elementary mathematical concepts such as the concept of a function. Students questioned the idea of a periodic function and the meaning of the intersection of graphs (ideas which become problematic in the context of velocity/position graphs) and attempted to interpret these concepts and representations in the context of these new representations of the bouncing car device. Their experience with the physical device exhibiting a complex behavior in a relatively novel notation system revealed in some cases weaknesses in their understanding of key mathematical concepts, or provided a new perspective and context to their previous knowledge.

Conclusions

As with Noble et al. (1996), we feel that complexity, in the informal sense, is often what makes a topic interesting. Further, complexity allows multiple ways to enter an investigation, and, hence, multiple ways to connect with the topic, especially when the topic is complex systems in the technical sense, bringing in and connecting previous experiences in the new context. Students in this study explored the device, and subsequently concepts of complex systems using multiple perspectives, and, even though their understanding of the specific ideas was rudimentary, it not only became richer as their explorations progressed, it provided a basis for revisiting core ideas from their traditional mathematical background. As students connected the motion of the device to previously learned mathematical concepts and to other physical phenomena, their previously established knowledge was destabilized and began to be reconstructed in new and richer ways as was argued by Noble, Nemirovsky, DiMattia, and Wright. (2002).

Our findings suggest that one of the critical sense-making elements was the link between the graphical representations and the actual device or phenomena that took place. Students resorted to their strong understanding of certain types of representations to interpret the new phenomena. Graphical attributes were interpreted as actions of the bouncing car. However, when the representation system was less familiar, students used their newly acquired knowledge of the device to give meaning to the representation.

Overall our results suggest that students, instead of avoiding complexity, embrace it and use it to build new understandings. Indeed, we see an enormous opportunity to approach an important topic, currently very underrepresented in students' educational experience, while simultaneously reconstituting traditional concepts. A major initiative is needed to determine how students can learn these ideas, as well as what are appropriate learning experiences and instruction activities that support students' learning. Of particular interest is determining how to integrate traditionally important topics as well as new topics in ways that are broadly implementable. We should use complex systems ideas and methods to unify and render more coherent students' educational experiences in the same ways that these ideas and methods are helping unify and organize knowledge across traditional boundaries.

Note

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COLLEGE STUDENTS' CONCEPTUAL LINKS BETWEEN THE CONTINUITY AND THE DIFFERENTIABILITY OF A FUNCTION

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In this paper, we report results from a study that researched the way in which college students relate two important properties of a function: its continuity and its differentiability. In particular, we investigated students' interpretations of part of the Fundamental Theorem of Calculus (*"If a function f is differentiable in a closed interval, then f must be continuous in such interval")*, and the influence and use of different representational registers. Fifty-five Mexican college students answered 3 questionnaires designed to get insights into the use of the algebraic and graphic representational registers when determining whether a function is continuous and differentiable. Eleven of these students were also asked to answer similar questions to those on the questionnaires, but using the software DERIVE, in an attempt to compare their approaches with and without the use of technology. In both cases there was a predominant use of the algebraic representational register, with students tending to consider that continuity implies differentiability.

Introduction and Theoretical Framework

The focus of our research was to investigate the way in which college students relate two important properties of a function: its continuity and its differentiability. When researching the literature, we were surprised by the lack of research dealing with the ways in which these two mathematical properties are conceptually linked by students. There is a wealth of research on students' conceptions of continuity and of the concept of derivative, and some evidence that many students are not clear about the relationship between those two properties of a function; however, there is scarce research on the ways they connect these two interlinked concepts. Thus, the purpose of our study was to get insights into the ways in which students use the continuity or non-continuity of a function in order to determine the differentiability of that function: specifically, on how they use or misuse the theorem *"If a function f is differentiable in a closed interval, then f must be continuous in such interval"*, that is part of the first form of the Fundamental Theorem of Calculus. We were particularly interested in analyzing the influence and use of the different representational registers during the determination process, since both the continuity and the differentiability of a function can be studied through algebraic or graphical approaches.

For example, the concept image of a continuous function, often involves the thought of a function whose graph can be sketched without lifting the pencil, a graph without "holes" or "jumps;" and that of a differentiable function could evoke the image of a function with a "smooth" graph "without spikes." These are the visual characteristics that can be considered in relation to the continuity or differentiability of a function. We wanted to see how much students use or depend on these visual characteristics, and how much they connect them with the results of algebraic analysis. In his work on representational registers, Duval (1993) points out that the power to move between different registers for a same object, seems to be a necessary condition for the mathematical objects not to be confused with their representations, and so that they can also be recognized in each of them. He points out that for the proper understanding of a concept it is necessary to coordinate the representations in (at least two) different registers of that concept. The absence of coordination between different registers does not hinder all understanding, but limits the understanding to the use of a single representational register, which in turn is not optimal for learning.

A Historical Analysis of the Notions of Continuity and Differentiability

As a background for our research, we carried out a historical analysis of the notions of continuity and differentiability. We consider this historical aspect relevant since it has been observed (see Piaget & Garcia, 1989) that often students give similar arguments and exhibit similar cognitive processes to those that have appeared in the historical development of science.

To discuss in detail the historical analysis is beyond the scope of the paper, but we would like to point out some interesting events.

In relation to the concept of continuity, it is interesting to note some of the evolution of that concept. In the 18th Century, Euler and Lagrange considered that a function was continuous if it was defined by a single algebraic expression. In 1767, Euler defined discontinuous functions in the following way: *all the curves not determined by a defined equation from the usual class, that are sketched by a free-hand movement* (Youschkevitch, 1976). This indicates he considered as discontinuous those curves that are produced by what are today called functions defined by parts; also, those curves were not considered genuine functions (Monna, 1972). But what is also interesting in this, is the relationship between the graphical representation and the algebraic one: it was not enough for a curve to look continuous; it had to be defined by a single equation. It wasn't until the work of Bolzano, Cauchy and later Weierstrass that the notion of continuity evolved to its present meaning.

In terms of the relationship between continuity and differentiability, it is interesting to note that up to the 19th century it was considered that the continuity of a function was sufficient for a function to be differentiable almost everywhere— Ampère, among others, believed in 1806 that any continuous curve had to have tangents everywhere except maybe in some isolated points (Ribnikov, 1987). Even Cauchy, who later, along with Bolzano, contributed to construct the formal definitions of the concepts of continuity and of derivative also originally believed, like most mathematicians of that time, that a continuous function had to be differentiable (Kline, 1972). But in 1817, Bolzano ended with this conception, when he presented a continuous function that was nowhere differentiable. After Bolzano, many mathematicians gave examples of non-differentiable continuous functions: e.g., Riemann, Cellérier and Weierstrass (whose example, the Weierstrass' function, is the most well known, since Bolzano's example remained generally unknown for a long time). The historical significance of such discovery was the realization that continuity does not imply differentiability.

Literature Review and Background for our Research

In the mathematics education literature, we have found that many works that have focused on the notions of continuity and/or of the derivative of a function show as common result that many students are not clear about the relationship between the continuity and the derivative of a function (e.g., Artigue, 1991, Lara, 1998): for instance, some students consider that the visual characteristics of the graph of the derivative of a function are determinant factors for the continuity of the function (Tall & Vinner, 1981). These were the kind of results that motivated our interest in studying the ways in which students link the two notions of continuity and differentiability.

Other results also point to a lack of integration of the different representational registers and a prevalence of the algebraic register. It has been found that many students are fixed in the use of the algebraic derivative algorithms (Orton, 1983); and that they avoid working with graphs, always looking for the algebraic expression (Artigue, 1991). In general, students are unable to sketch the graph of the derivative a function from the graph of a function, or vice versa, and show confusions between the meaning of continuity and of differentiability (Artigue, 1991).

Research related to the notion of continuity has found that, in general, students don't take into account the domain of definition; they tend to consider that a function is continuous if its graph is connected. Thus, they believe that functions with the following characteristics (regardless of the domain of definition) are not continuous: those whose graphs are not in a single piece, those not defined in the origin, those that diverge to infinity at the origin (Tall & Vinner, 1981). Sometimes students consider that a function is continuous if it is defined for all the real numbers. On the other hand, often students think that a function that is defined by parts cannot be continuous (Tall & Vinner, 1981), which is one of the conceptions also found in history (as explained above, both Euler and Lagrange considered that a continuous function had to be defined by a single algebraic expression).

Design and Methodology of Our Study

As stated above, the purpose of our study was to get insights into the ways in which students use the continuity or non-continuity of a function in order to determine the differentiability of that function; we were also interested in analyzing the influence and use of the different representational registers during the determination process. There were two phases to the research: the main one using paper and pencil questionnaires, and a small limited second phase implementing computer-based activities.

The subjects were fifty-five Mexican college students studying towards a degree in computer engineering. These students had all passed a first Calculus course where they had studied all the notions that we deal with in our research. That course was taught in a traditional way without the use of technology such as computers or graphing calculators, except in rare occasions for producing sketches of graphs. The research was carried out at the beginning of their second semester of studies: these students were enrolled in two groups, randomly selected, of a second Calculus course.

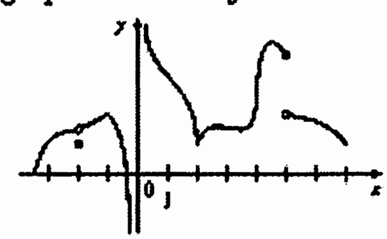
The Questionnaires

In the first part of the research, we used three written questionnaires: the first two were used to give us background information that could be correlated for analysis purposes with the answers of the third questionnaire, which was our main interest and which is the one we describe below. These first two questionnaires were designed to get insights into students' conceptions on, respectively, the continuity and the differentiability of functions; the third questionnaire focused on how students associated the two notions. All three questionnaires presented functions in different ways, in search of insights into students' conceptions with reference to the two different representational registers (the algebraic and graphical) and the ways they coordinated the information in the two registers.

The third questionnaire had 7 items, which included: Four questions where students were asked to determine the continuity and differentiability of several functions given in both algebraic and graphical ways: $f(x) = -x^2 + 1$; and $f(x) = |x|$, both defined in the real numbers; a function defined in parts, with points of discontinuity in its domain; and another function also defined in parts but continuous in its domain of definition. The fifth question (see Fig. 1) only showed the graph of a function and students were asked to determine the points of discontinuity and the points without derivative.

In the last two questions the functions were given only through their algebraic representation: one (in question 6) was a continuous function but defined in parts,

Question 5. This is the graph of function f.



(a) Is function f continuous in every point? If not, where? And why?
(b) Is function f differentiable in every point? If not, where? And why?

Figure 1

which had some non-differentiable points $f(x) = \begin{cases} 0 & if\ x \leq -2 \\ x+2 & if\ -2 < x < -1 \\ 1 & if\ -1 \leq x \leq 1 \\ -x+2 & if\ 1 < x < 2 \\ 0 & if\ x \geq 2 \end{cases}$; and the other (in question 7) was $f(x) = \frac{1}{x+1}$ defined in R - {-1}, whose graph (not given to students) is visually disconnected.

Many of the questions used, were taken or inspired from previous research (e.g. Tall & Vinner, 1981; Orton, 1983; Artigue, 1991; Eisenberg & Dreyfus, 1991; Dolores, 1998), which allowed us to compare, as far as possible, our results with theirs.

As an analysis methodology, we classified the answers to the three questionnaires in several categories, some of them referring to the use that students made of the different representational registers. For example, we classified answers where students referred to formal definitions (either correctly or incorrectly) in the category named "formal idea"; if students gave visual arguments, we classified their answers in the "visual idea" category. We also had a mixed category called "formal-visual" for answers that used both types of arguments.

The Computer-based Activities

In the second phase, we attempted to observe if there were differences in the use of the representational registers when determining the continuity and differentiability of a function, using only paper and pencil or using a computational tool: the software DERIVE. The use of technology and of multi-representational systems has sometimes shown (e.g., Jones, 1993) to produce a better learning by helping to integrate the different representational registers (Duval, 1993).

Eleven students, randomly selected from the 55 who answered the questionnaires, participated in this phase. They were asked, in particular, to determine whether several functions were continuous and differentiable, using DERIVE. The computer-based questions were similar to those given in the first questionnaire phase, so that the approaches with and without the technological tool could be compared, and the answers were classified using the same analysis categories from the first phase.

Prior to the activities, the students participating in this second phase, were given a two-session, 3 hour-workshop on the use of DERIVE, where they learned to use the main commands (with emphasis on commands such as *differentiate, sum;* and plotting commands including *Trace*, *Range y Zoom).* Students were also shown how to define functions by parts. They then practiced with activities related to the topic under study.

In a third session, students were asked to solve, using DERIVE, a series of questions. Students were asked to write down the DERIVE commands they used and what they had done to arrive at their conclusions, as well as a justification for their answers. Like in the questionnaires, the computer-based questions referred to (i) the continuity of functions (the first question asked students to determine the continuity of two functions: one disconnected but continuous in its domain, and the second defined in parts); (ii) the differentiability of functions (three questions tried to determine students' use of the derivative-slope relationship, both algebraically and geometrically, and whether they realized that the derivative is not just an "algorithmic computation"); and (iii) the relationship between the derivative and the continuity of a function.

In particular, students were asked to determine the continuity and differentiability of each of the following functions: the first was the same function defined in parts that we presented in question 6 of the third questionnaire (see above). Most students, in the first phase, had considered that this function was differentiable, when in fact it is not. We expected students in this second phase to make use of the graph produced by the software, and notice the spikes it has; we were surprised when 7 of the 11 students disregarded the visual characteristics and claimed the function was differentiable everywhere. This indicates a lack of knowledge and understanding of the visual characteristics of a differentiable function.

The other functions were the following:

$$f(x) = \begin{cases} x^3 & \text{if } x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

$$f(x) = \sum_{n=1}^{50} \cos(nx)$$

$$f(x) = \sum_{n=0}^{5} \left(\frac{2}{3}\right)^n \cos(9^n \pi x)$$

The last two functions are functions whose graphs (which students can only produce using a technological tool) are very irregular and have many spikes; we wanted to see if students were able to analyze the geometric characteristics and determine the points of continuity and differentiability. These two functions have very similar graphs, but we noticed that students approached the two functions differently, with most of them claiming the first one as continuous and the second one as not: it seems that students were influenced by the complex algebraic representation of the latter one, and by its irregularly shaped graph.

Discussion and Summary of Some Results

Findings from the Questionnaire Answers

- In relation to the continuity of functions defined in parts, it is interesting to note that many students tend to regard these functions as discontinuous (even when they aren't) because of the way they are algebraically defined (in parts). This shows a lack of analysis and understanding of the definition of continuity, but also a non-reliance of the visual representation. It is also consistent, however, with the view that was held in the 18^{th} Century by Euler and Lagrange, as explained at the beginning of the paper.

- In answering the second questionnaire, most students (80%) relied on the algebraic rules and algorithm for finding the derivative (even when the graph of the function was given); a result consistent with the works reviewed (e.g., Artigue, 1991, Orton, 1983). For many students, the reasoning appeared to be: *if the derivative can be computed, then the function is differentiable; if not, then it isn't.*
- But in answering the third questionnaire, most students (70%) used the continuity or non-continuity of a function for determining the differentiability. For instance, when considering the function $f(x) = |x|$, 23 of 41 students who justified their answers, used arguments based on its continuity (it is interesting that 8 of them answered that the function was discontinuous at the origin – see further below); and 18 of those students stated that the function was differentiable because it was continuous (an incorrect interpretation of the Fundamental Theorem – see below).
- 70% of students' answers refer to the Fundamental Theorem of Calculus. Some answers were: "*every differentiable function is continuous*", or "*the first condition for a function to be differentiable, is that it be continuous*". However, although students are familiar with this theorem, it is generally misunderstood, misused, or contradicted: in many cases students asserted that discontinuous functions were differentiable; and half of the students, see below, erroneously considered as valid the reciprocal theorem to the fundamental theorem:
- 49% of the students consider that if a function is continuous, then it is differentiable. Among the answers, were: "the function is differentiable because it is continuous", "the function is differentiable because every continuous function is differentiable"; some even wrote "a function is differentiable, if and only if, it is continuous". In other cases, the relationship is not explicitly expressed, but they refer to characteristics of continuity to justify the differentiability; for example, one student wrote: "the function is differentiable because the function exists for every x".

In analyzing the influence of the different representational registers, we found that:

- In 95% of the cases, students give either "formal" arguments (making reference, either correctly or incorrectly, to theorems or definitions) or use the algebraic representation. There was thus a prevalence of the use of the algebraic register, as has been noted by Eisenberg & Dreyfus (1991).

- In determining the continuity, there was more use of visual arguments than for the differentiability (some arguments were that the graph had a hole, a jump, or else was uninterrupted). Whereas for determining if a function is differentiable, students tended to ignore the visual aspects. In fact, students constructed the graphs of the functions for less than a third of the cases – a result that contrasts greatly with the findings of Artigue (1991). Only in cases where the graph of the function was given, a few students (10%) used some visual aspects: they determined the non-differentiability by identifying spikes or saying that the curve was not smooth.
- However, even when the visual aspect was used, there was confusion between the notions of differentiability and continuity: some students claimed that a function was discontinuous if its graph had a spike: a characteristic of non-differentiability not related to continuity. This is consistent with the findings of Lara (1998) and Artigue (1991) who have found confusion in students between the characteristics of continuity and those of differentiability.

Findings from the Computer-based Activities

It is important to begin by pointing out that the second phase did not turn out as we had hoped, since we had expected that the use of the software tool, with its visual and computing capabilities, would lead students to approach the problems in a different way. But the findings showed very little difference in the way the problems were approached and conceived with and without the use of technology. Nevertheless, we are cautious in emphasizing that this does not necessarily imply that the use of technological tools, such as DERIVE, cannot bring conceptual changes and a better integration of the different representational registers. We believe our results were mainly due to the way that the computer-based activities were presented (doing the same type of activity with paper and pencil, as was done with the computer, without incorporating a different pedagogical approach), the lack of experience in the use of the tool, and the lack of experience in using technology as an exploratory tool. These findings, however, do stress the importance that should be placed on the ways that new technologies should be implemented.

Despite all of this, we would like to summarize some of the findings from this phase. From their written work, we observed the following:

- for all the questions, the students gave the graphs of the given functions
- the continuity of those functions was almost always determined using the graph (some students even said: "*the function is continuous because it can be seen on the graph*") with the main justifications given by students were that the graph did not have "jumps", but also that the function was "defined for every x".

- in 63% of the answers, the differentiability of the functions was justified using the continuity, i.e. using, either correctly or incorrectly, the theorem "*every differentiable function is continuous*".

The software tool was used to visualize the function and to determine the continuity, particularly for the last two functions (see above). On the other hand, when determining the differentiability, there was a lack of visual analysis and the same arguments given on the paper and pencil questionnaire were used (e.g. students used only the continuity of the function or its algebraic expression, to determine its differentiability): students did not take advantage of many of the useful tools provided by the software, such as the *zoom* command, and did not explore the graphs in any way. Thus, the students limited themselves to using the software only to construct the graph, ignoring the possibilities for exploring visual characteristics and carrying out a possible visual analysis.

The only observed advantage was in the fact that they could produce an accurate graph for each function, and this led to fewer mistakes when determining the continuity of the functions, than had been observed in the answers to the questionnaires.

But as in the first phase, it was observed that students lack a visual understanding of the concept of derivative, with a predominant use of the algebraic representation. In general, students tend to rely, whether correctly or incorrectly, on the continuity or non-continuity of the function for justifying the differentiability.

Conclusions

The findings, from both phases, confirm one of our hypotheses: although students are familiar with the theorem: "*every differentiable function is continuous*", they misuse it to determine the differentiability of a function, not recalling it properly or applying it in an incorrect way (they often use the invalid inverse theorem).

In terms of the use and influence of the different representational registers, we observed the following:

When determining the continuity of a function, students used both the visual and the algebraic registers, but there was a predominance of algebraic one, and their answers focused on the analysis of features from only one of the registers.

When determining the differentiability of a function, it was found that students do not use the graphical register, even when the function is given through its graph. A majority of students tend to focus only on the algebraic register (in the second questionnaire, only 2% gave answers based on visual characteristics, and only 0.5% used both registers).

However, when students were asked to determine both the continuity and the differentiability, there was a greater use of the graphic characteristics with some students referring to the smoothness of the graph or presence of "spikes", even though the use of the algebraic register still dominated.

Thus, we observed a predominant use of the algebraic register, as has been found by Eisenberg & Dreyfus (1991), with a greater use of visual arguments when determining the continuity than when determining the differentiability.

In relation to the second phase, as is explained above, the use of the software tool did not produce the results we expected, since hardly any differences were detected between the approaches used with or without technology. We blame this on the way the computer-based activities were presented, but also on the school culture from which these students come, where software is only used as a tool for sketching a graph or making complex computations, and no emphasis is placed on the use of technology as a mathematical exploration tool.

As in the first phase, in the second phase we observed: a lack of understanding of the visual characteristics that help determine the differentiability of a function, and an over reliance on the continuity or non-continuity of the function for determining the differentiability. The only difference between the two phases was in the possibility to accurately sketch the graphs of the given functions, which did lead to fewer mistakes, particularly when determining the continuity. But even with this advantage, students did not exploit the possibilities offered by the software and for exploring the visual representations.

To summarize, we found that in general (i) there are confusions regarding the relationship and characteristics of the continuity and differentiability of a function (in accord with the findings of other researchers —e.g., Artigue, 1991; Dolores, 1998; Orton, 1983; Tall & Vinner, 1981); (ii) although students are familiar with the first form of the Fundamental Theorem of Calculus, it is generally misunderstood: it is used for determining the differentiability of a function, but it is not properly recalled and is used incorrectly; and (iii) there is a preference for the algebraic register of representation.

More importantly, there seems to be a disassociation in the use of the representational registers: they are seldom related or combined. This points to a lack of integration of, and appropriation of the concepts which, following Duval's theories, is not optimal for learning. There was some hope that the use of technology, which facilitates the use and exploration of the visual representational register, would help in this respect. But with only a limited use of the computer software, the computer-based activities did not change the results, since students approached the activities with the technology, in the same way as they did the paper and pencil ones. Therefore, clearly more research is needed in this last direction; in particular, in the way that technology is implemented.

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EXPLORING THE ROLE OF METONYMY IN MATHEMATICAL UNDERSTANDING AND REASONING: THE CONCEPT OF DERIVATIVE AS AN EXAMPLE

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This paper discusses the role of metonymy in a structured derivative framework (DF). The DF consists of three layers of process-object pairs, one each for ratio, limit and function. Each of the layers can then be illustrated in any appropriate context, for example graphically as slope, verbally as rate of change or kinesthetically as velocity. Metonymy functions within this structure in terms of 1) the use of the contexts of the concept of derivative as paradigmatic examples, 2) the use of the layers of the concept of derivative to represent the whole concept (a type of individual metonymy) and 3) misstatements using part-whole or part-part references that could be described as metonymic but are mathematically inaccurate.

Purpose and Objectives

The purpose of this research report is to explore how metonymy functions in both powerful and problematic ways in student reasoning about and understanding of the concept of derivative. The study of natural language structures such as metaphor and metonymy in the learning of mathematics is a growing research area (cf. English, 1997; Lakoff & Nunez, 2000). Since there has been much less work done on metonymy than on metaphor, this paper will add to that body of research. The concept of derivative is a particularly illustrative example because of the richness of its conceptual structure, and because the types of reasoning that occur in this example will have parallels in other contexts.

Theoretical Perspectives

Metonymy

Lakoff and Johnson (1980) spend one chapter of their book, *Metaphors We Live By,* focusing on metonymy. They describe metonymy as occurring when we use "one entity to refer to another that is related to it" (Lakoff & Johnson, 1980, p. 35). This may mean that we use a part to stand in for the whole as in their example, "I've got a new *set of wheels*" (Lakoff & Johnson, 1980, p. 36). Here the phrase "set of wheels" stands for the whole car. In other cases we use a part of a conceptual structure to refer to another part of the same conceptual structure. For example, "We need a better *glove* at third base" (Lakoff & Johnson, 1980, p. 38). Here the glove stands for the user of the glove.

Lakoff and Johnson (1980) point out that although metonymy is primarily a referential device, it also influences understanding. There are many parts that might be chosen to represent the whole or to represent another part of a conceptual structure. These choices are not arbitrary. Using one part (set of wheels) to stand for the whole (car) draws attention towards one part of the conceptual structure and away from other parts. This usage may be a choice of the speaker, in which case it shows her/his focus. On the other hand it may be that the usage is conventional and that the speaker could have chosen no other part and been understood. For example, "I've got a new set of windows," would not convey the appropriate meaning. However, the choice of letting the wheels stand for the car is not neutral; it demonstrates a lens through which our culture understands the relationship between these entities.

The preceding examples of metonymy from Lakoff and Johnson (1980) are of the type that Presmeg (1997) calls "metonymy proper." She states, "Metonymy proper [occurs when] a salient or related entity is taken to stand for another entity" (Presmeg, 1997, p. 270). She then describes a second type of metonymy where "one element for a class may be taken to stand for the whole class" (Presmeg, 1997, p. 271).

These two types are both emphasized in Lakoff's (1987) book, *Women, Fire and Dangerous Things.* Lakoff states that a metonymic model consists of two entities A and B that are in the same conceptual structure. B is either part of A or closely related to it. "Compared to A, B is either easier to understand, easier to remember, easier to recognize, or more immediately useful for the given purpose in the given context" (Lakoff, 1987, p. 84). The metonymic model describes how B is used to represent A in the conceptual structure. Metonymic models may model a single entity or a category.

Metonymic models of categories are equivalent to the second type of metonymy referred to by Presmeg (1997). They take an example or submodel and use it to stand for the entire category. These are what Fischbein (1987) calls paradigmatic models. Paradigmatic models describe our tendency to see a whole class of objects or an entire concept through the knowledge of particular examples or a submodel that exemplifies the concept or class. Not all examples are paradigmatic models, only those that provide enough variety of features to be representative of the entire group, yet are simple enough to be easy to use in reasoning. These representative examples are sometimes called exemplars or prototypes. For the concept of derivative, velocity is a particularly important example, a paradigmatic model of the concept of derivative in a physical context. Velocity is an exemplar because it is an extremely familiar phenomenon for which we have additional natural language structure. For example, increasing velocity is called acceleration and decreasing velocity is called deceleration.

If the model, B, is an example of the original category or concept, A, then we may call B a paradigmatic model. If the model, B, is a part of A that is not an example, I will call this *individual metonymy* to emphasize that the metonymy refers to an individual conceptual structure rather than a collection of examples of a category or concept.

One further example of individual metonymy provides insight into the derivative concept. Lakoff (1987) uses the example of going to a party. The trip consists of a precondition that you have a way to get to the party, embarkation, the travel itself, arrival and an end point. If someone asks you how you got to the party, you would not recount the entire scenario. You might say, "I drove", letting the center stand for the whole. Alternatively you might say, "I have a car", letting the precondition stand for the whole.

Functions (including the derivative function) have a structure similar to a trip. There is a domain of starting values, a rule or correspondence, the calculation of which is analogous to traveling, and an end point or value of the function for each starting value. We sometimes name a function by referring only to its rule or correspondence without reference to its domain or range. This short hand is a type of individual metonymy, letting the part stand for the whole. The short hand also provides an emphasis on one aspect of the whole over other aspects.

It must be remembered that the existence of a metonymic (part-whole or part-part) relationship between two entities is not enough to guarantee that these entities may be properly referred to using the same word or that the part may properly be used to stand in for the whole. The use of the same word for both the part and the whole is motivated by the part-whole relationship, but is not implied by it. If a student or novice uses a part to stand for a whole (or another part) in a way that would not be considered conventional or proper usage in her/his culture, then this may be called a misstatement. For example, a student might shorten the phrase "the derivative is the slope of the tangent line at a point" to "the derivative is the slope," a phrase which is considered appropriate usage by the mathematical community. On the other hand, the student shorten the phrase to"the derivative is the tangent line," a phrase which would be considered mathematically incorrect. Examples of such misstatements for the concept of derivative will be discussed near the end of this paper.

Lakoff (1987) argues that we can determine the structure of a concept by examining the language we use to talk about the concept. His case study is the concept of anger. He and his colleague Zoltan Kovecses began with a systematic study of the expressions we use in English to talk about anger based on methods developed in Lakoff and Johnson (1980). Lakoff (1987) argues that "there is a coherent conceptual organization underlying all these expressions and that much of it is metaphorical and metonymical in nature" (p. 381). This notion was the impetus for my developing the structure of the concept of derivative that is in the next section (and previously described in more detail in Zandieh, 1997, and Zandieh, 2000). Like Lakoff (1987), I began the process by analyzing all the ways that I heard students and faculty talk about the concept of derivative at the freshman calculus level. Even though I did not follow Lakoff's method in detail and used many influences from the mathematics education literature that would not have applied to his case study, the notion of metaphoric and metonymic models was implicit in its development and in my use of it for analysis. In this paper I will make more explicit some of the metonymic connections.

However, first I will give an overview of the derivative framework without regard to metonymy.

Derivative Framework

The derivative framework, DF, has two main components: multiple representations or contexts and layers of process-object pairs. The concept of derivative can be represented (a) graphically as the slope of the tangent line to a curve at a point or as the slope of the line a curve seems to approach under magnification, (b) verbally as the instantaneous rate of change, (c) physically as speed or velocity, and (d) symbolically as the limit of the difference quotient. Many other physical examples are possible, and there are variations possible in the graphical, verbal, and symbolic descriptions.

Consider the formal symbolic definition of the derivative:

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative of f, f', is a *function* whose value at any point is defined as the *limit* of a *ratio*. I will call these underlined aspects of the concept of derivative (ratio, limit, function) the three "layers" of the framework. At this point we may combine the layers and the contexts or representations to form a matrix outlining the structure of the concept of derivative (Figure 1). Although only four columns are named specifically, there can be many other columns, one for each situation in which there is a functional relationship for which one may discuss the concept of derivative.

Each empty box in the matrix represents an aspect of the concept of derivative. For example, the box in the ratio row and the graphical column represents the slope of a secant line on a graph of the function whose derivative we are concerned about. More details about the layers in terms of slope, velocity and the difference quotient may be found in Zandieh (2000).

Mathematically it is clear that the concept of derivative involves a ratio, a limit and a function. However, thinking of these as "layers" comes out of my thinking about Sfard's (1992) framework of processes acting on previously established objects. Her notions of operational and structural conceptions add a third dimension to my framework. Each empty box in the matrix in Figure 1 may denote either a structural or operational understanding which I denote as a process or object understanding for each square in the matrix[1].

Methods

The data reported here are from five semi-structured interviews conducted with each of nine high school students during their senior year Advanced Placement

	Contexts				
	Graphical	Verbal	Paradigmatic Physical	Symbolic	Other
Process-object layer	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

Figure 1. Outline of the framework for the concept of derivative.

(AP) calculus course. This data is part of a larger study reported in detail in Zandieh (1997).

For the data analysis, transcripts of student interview responses for each of the five interviews were color-coded for each of the contexts and layers. In addition, connections a student made between contexts and student misstatements were noted on the transcript. Most of the coded transcripts were then transformed into a tabular format similar to Figure 1 (Zandieh, 1997; Zandieh, 2000). These tables documented and were used for further analysis of students' use or lack of use of specific contexts, layers and process-object structures. In addition, a place on the table was designed to note misstatements.

Result and Discussion

There are a number of ways in which metonymy occurs in student reasoning with the context of the derivative. This section will discuss four issues: 1) general metonymic structures that occur within this framework and that would occur in many other mathematical settings, 2) metonymic use of the contexts of the concept of derivative as paradigmatic examples, 3) metonymic use of the layers of the concept of derivative to represent the whole concept (a type of individual metonymy) and 4) misstatements using part-whole or part-part references that could be described as metonymic but are mathematically inaccurate.

Metonymy Embedded in the Derivative Structure

As Presmeg (1998) notes, whenever we use a letter to represent a set of numbers, then we are using metonymy. Lakoff and Nunez (2000) refer to this as the Fundamental Metonymy of Algebra. Another fundamental metonymic process is the chaining of signifiers (Presmeg, 1997, 1998) which is closely related to the reification which occurs within each process-object layer of the DF and allows for the chaining of the layers. In addition, metonymy occurs when students do not reify an operational "process" conception into a structural "object" conception but use a pseudostructural conception (Sfard, 1992) or a pseudo-objects (Zandieh, 2000) to stand for or stand in the place of the incompletely understood conception. This allows for some chaining of the layers even when each layer is not well understood.

Consider a student who says that the derivative is "the instantaneous velocity." This phrase may represent for the student a complex set of processes including a ratio of distance over time and a limiting process to convert average velocities over specific times to an instantaneous velocity which occurs at a moment, i.e. over 0 time. Thus the student uses the brief phrase to metonymically stand for the chaining of the first and second process-object pair layers of the derivative concept. On the other hand the student may only mean that the derivative is the speed that the car is going at a moment in time, without any knowledge of the underlying mathematical structure of these concepts. The notion of speed at a moment of time can be understood experientially even by very young children. In either case, the student's knowledge of instantaneous velocity can be used to help the student understand the velocity function as a speedometer that provides a speed value for any moment in time. This chaining of the second layer to the third layer takes only the metonymic representation which stands for the first two layers, whether or not the first two layers are fully or only partially being represented.

A Context Stands for the Derivative: Paradigmatic Examples

One way in which we see a context as standing for the whole derivative concept is when we ask students to display a personal concept definition (Vinner, 1991) by asking, "What is a derivative?" Another way of noting metonymic use of particular contexts is in noting what context a student reaches for when asked to reason about the concept of derivative in a variety of settings.

The nine students in this study had a wide variety of preferred contexts at the time of the first interview. Some students mentioned slope and rate of change most often while other students mentioned velocity or the process of taking the derivative more often. By the final interview all but one of the students mentioned slope and rate most often. These results may be influenced in this study by the questions that the students were asked and by the instruction the students were given. However, given that the questions were broad and wide ranging and that the instruction was fairly standard, I believe that there is also a more fundamental reason for this result.

The best paradigmatic models are familiar, well understood and easily applied to understanding the derivative as a whole. In the first interview, students do not have a very complete understanding of derivative so they mention models that are most familiar. As the students broaden and deepen their knowledge of derivative, the students become more familiar with other possible models and applicability becomes more of an issue. Students see that rate of change is a general phrase that may be used in many different situations without as much explanation or interpretation as a specific context such as velocity. Slope continues to be a model that is particularly effective for interpreting other contexts because of the special characteristics that graphs provide. In particular, graphs are useful as synoptic, global representations.

However, not all students who mention slope often can always use it appropriately. Ingrid, does not see that her answer to "what is a derivative?" need play a strong role in her reasoning about derivative in other contexts. The most striking example of this occurs in her second interview. In both of the first two interviews Ingrid repeatedly states that the derivative is "the slope of the tangent line." In the second interview she is asked to give a 'real world example.' Her example is a slightly faulty example regarding change in volume over time, which she represents as dV/dt. When asked how the derivative plays into her example she indicates the dV/dt. Then the interview asks whether slope is related since she has said that the derivative is slope of the tangent line. Her answer is that it is not, and then, "Are all derivatives that? Slope?" It hasn't occurred to her that each context in which we use derivative should have an isomorphic structure with each other context. In natural language it is common for a word to refer to many things which are not isomorphic in structure, nor even have any one thing in common to all examples, but rather that the examples have something in common with each other. This is the notion of family resemblances (Wittgenstein, 1958; Rosch & Mervis, 1975).

Mathematical structures, unlike natural language structures, are organized more rigidly. We expect to have a definition for a mathematical term like derivative and to be able to determine immediately whether something fits by whether it has the same structure as the definition. My data indicates that although many students do see the parallel structure across contexts, they are least likely to use the formal symbolic context as the primary context that they reason from. The best students can reconstruct the formal definitions from the structure of their informal understanding and the worst students ignore (or memorize and then ignore) the formal definition completely.

In summary, each of the contexts of the concept of derivative has the potential to be a metonymic model or paradigmatic example for the concept as a whole, but some contexts are much more useful than others for this purpose. As we saw with Ingrid, the use of a paradigmatic example such as slope does not mean that a student understands that there is an isomorphism between that example and the other contexts for derivative. An exploration of how a student views and uses the potential isomorphisms between pairs of contexts would be a study of metaphoric rather than metonymic models.

A Layer Stands for the Derivative

The first layer of the derivative concept is the average rate of change, which cannot by mathematical convention be called "the derivative." The second and third layer can each be called "the derivative." We can see Ingrid's confusion with this issue when she is asked a subtle question, "Is the derivative a function?" Her response: "No ... It's just a slope. It's not like y equals something. ... cause you can't graph a slope. ... Or you can't graph a limit. But then ... Like on a test, 'This is the graph of the derivative.' I guess it has to be. It could be a function."

From Ingrid's response we can see that she is struggling to clarify the conflict between calling something a derivative that is a slope or limit value (second layer) with something that is a graph or equation of a function (third layer). Here we see metonymy play out in two different ways. First, the phrase "the derivative" is being used to stand for a longer phrase "the derivative value" or "the derivative function." If a longer phrase had been used as opposed to a metonymic short cut (part of the phrase to stand for the whole phrase), then there would have been no confusion. A second metonymic relationship is that between the two meanings that can be associated with "the derivative." The derivative function is a complex process like the trip described in the Theoretical Perspectives section. The derivative value is just one part of the trip, the destination for one starting value. So the second layer derivative is a part of the third layer derivative whole. Lakoff (1987) explains that metonymic relationships are common in these cases. "Related meanings of words form categories and the meanings bear family resemblances to one another" (Lakoff, 1987, p. 12)

Metonymic Misstatements

The two most common misstatements in my data are "the derivative is the tangent line" and "the derivative is the change." One way of viewing these metonymic misstatements is purely syntactic. An example with slope was provided in the Theoretical Perspectives section. A second example is the phrase "instantaneous rate of change". If a student shortens this phrase to say that "derivative is a rate," then the statement is acceptable. However, if the student says, "derivative is a change", then the statement is not mathematically accurate. So some cases of letting part of the phrase stand for the whole phrase are considered acceptable by the mathematical community whereas other metonymic short-cuts are not.

I will provide two examples for rate of change and then two for slope of the tangent line. In each case the first of the two examples is primarily syntactic whereas the second example has a more conceptual focus.

In the first interview, Carl makes several syntactic misstatements involving rate of change. Although he does state that the derivative is the instantaneous rate of change in the following sequence, his explanations focus more on the derivative as change instead of rate of change. Carl says, "If you were to take the derivative, you'd end up with dx/dt or Δx. The change in x over time. So that's basically like Δx." When asked for clarification, Carl explains further, "When you take the derivative of something you find the change in that. ... And it changes the equation from how much water is in it to how much it's changing at that instant, how much is leaving or going in at that instant. It's the instantaneous rate of change."

In the fifth interview, two students make a conceptual metonymic error. Each student is told that f is a function that gives the outside temperature at a given time. The student is then asked to interpret $f'(3)=4$ and later $f'(3)=4$ for 0 x 3. Both Carl and Derick initially interpret $f'(3)=4$ to mean that the temperature changes instantaneously by 4 degrees at the 3 hour mark, instead of the correct interpretation that the rate of change of temperature is 4 degrees per hour. Each of these two students fail to see their error until they make an analogy with distance and velocity. Derick makes the connection on his own; Carl responds to the interviewer's suggestion.

Carl and Derick's initial error is that they let the change (per one unit) stand for the rate of change. Instead of taking a part of the phrase for the whole phrase, they take a part of the concept, the change in output values, for the whole concept, the change in output values divided by the change in input values. Often we calculate average rate of change or approximate instantaneous rate of change using one-unit intervals, which obscure the role of the denominator. Even in stating that the derivative is 4 the nature of the derivative as a ratio is implicit.

Another type of conceptual individual metonymy occurs when discussing slope of the tangent line. Consider the complex process by which we describe the derivative at a point graphically. Whether we describe a sequence of secant lines approaching the tangent line at a point or we describe zooming in on a point until the curve appears to be a line, in each case the derivative is the slope of that line. However, a student who has focused on this image may say that the derivative is this whole process or that the derivative is the most obvious image or endpoint of this graphical process, the tangent line. The slope is implicit in both graphical images, but the tangent line is explicit, visible and thus more easily remembered. Even without the idea of a limiting process a student may remember a single image, a curve with a line tangent to it, when asked what a derivative is. Again one might pick up on the tangent line as the explicit, visual representation of the derivative instead of remembering that the derivative is the slope of that line.

Carl provides syntactic metonymic examples for slope of the tangent in the following. When asked if derivative is related to slope, Carl says, "Derivative is the tangent line to the function," before correcting this a few lines later to "The derivative is the slope of the tangent line to the graph." When asked if derivative is related to line or linear, Carl again confuses the two, "It's the line, the tangent line. It's the slope of the tangent line is the derivative, so the tangent line to the graph is the derivative as well."

Grace's misstatements are subtle and in the end a bit more conceptual. She says that the derivative is "tangent to the slope at a given point", "tangent of your slope" and "tangent of slope". At first these statements sound like a simple transposition of words. That these statements refer to the tangent line to the curve (slope) is made more clear in Grace's second interview. In the second interview she says that the derivative is "tangent of the slope" and when questioned clarifies by saying "the line that's tangent of the slope". She also says that the derivative is a line when asked to relate derivative to line or linear. "I mean your derivative is like a line. I guess that's kind of the context, isn't it, of derivative. When you have a point and you're finding tangent lines getting closer to that point, approaching that point."

In each of these examples, inappropriate metonymic connections seem to come naturally to these students. The students at times also seem to believe both versions of the statements, e.g. that the derivative is the slope of the tangent line and that it is actually the tangent line itself. The two contradictory ideas may be compartmentalized by a student so that the student does not notice the contradiction. However, even without compartmentalization, a student may think that it is legitimate to use derivative to mean both slope of the tangent line and the tangent line itself. If "the derivative" can refer to both a value and a function, perhaps derivative could refer to both slope value and tangent line. Slope and tangent line have a metonymic part-whole relationship that is not too unlike the derivative value to derivative function relationship.

Conclusion

As we have seen, metonymic models may function both in powerful and problematic ways for students as they come to understand and work with a complicated concept such as the derivative. It is the goal of this paper to illuminate and clarify these relationships in the hopes of providing insights to teachers and researchers.

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PSEUDO-EMPIRICAL ABSTRACTION IN ADVANCED MATHEMATICS

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This presentation is based on one student's attempts to argue that all convergent sequences of real numbers are bounded. It is suggested that his approach may be indicative of learning via pseudo-empirical abstraction. This leads to questions about the relationship between the objects constructed through this type of learning and their formal mathematical counterparts.

This presentation is based on interviews with students during the seventh week of a first course in Real Analysis. During these interviews, students were required to justify the statement that if a sequence is convergent, it must be bounded.

Some students answered this by sketching a convergent sequence and generalizing. This may be seen as a concept-image based argument (Tall & Vinner, 1981). For such students, the image is representative of the set of convergent sequences; in their work, such objects exist on a perceptual basis (Tall, Thomas, Davis, Gray, & Simpson, 2000).

Other students reason by introducing the definition of convergence and making deductions from this. In their work, the set of convergent sequences exists on a formal basis (Tall et al., 2000.), and their arguments apply to all the objects in the defined set.

Here, we consider the work of a student for whom the set of convergent sequences does not appear in the argument at all.

Xavier begins by introducing elements from the definition.

X: *Isn't it something to do with that...a_n minus l, don't you remember?*

Although this sounds rather instrumental, he gives a reasonable approximation to the entire definition, albeit not a logically well connected one.

X: *Er, the term in the sequence, minus the limit...the modulus of that is less than epsilon where epsilon...is...any real number...and n is beyond any big N in the sequence...*

Although Xavier is not fluent in speaking about this definition (in the next excerpt he says "e" when he means "epsilon" and confuses his "*l*" for the digit "1"), he appears to be on his way to making a sensible argument.

X: ... And then if you get rid of the modulus sign, you can write that as minus epsilon is less than a to the n minus one, is less than epsilon, and that can be rearranged so that a to the n is less than, e plus one, but it's also greater than one minus epsilon...

However, from this he concludes:

X: So therefore a to the n is bounded above and below, and therefore this is the definition for the convergent sequence. So it converges.

It appears that once he begins working with the definition, Xavier becomes side-tracked into a routine of showing that a given sequence is convergent. This persists when the interviewer asks him what happens for the terms before "big *N*".

X: Erm, well they'll be...erm, the modulus will be bigger than epsilon. So you've got to go far enough along, so that you're less than epsilon.

This last statement also assumes that for a given epsilon, all the terms before the associated "big *N*" must be further from the limit than this epsilon. Hence, while his argument initially looks more "mathematical" than those using concept images, he does not have a solid an understanding of what objects belong to the set.

It would be easy to dismiss this as rote learning, but we believe that this would do Xavier an injustice. Students who did attempt rote learning could not recall even parts of the definition, much less manipulate an almost complete version (cf. Moore, 1994). In Xavier's case he began appropriately, but his work does not seem to have any link with the set of convergent sequences, either as conceived by Xavier as an individual, or as a formal set delimited by the definition.

Instead the definition was associated with a process. This, for us, brought to mind the idea of pseudo-empirical abstraction, that is, the construction of new mental objects not from perception, but from repeated experience with actions upon others (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996). Perhaps this case could be characterized as pseudo-empirical abstraction in progress, the action being that of demonstrating that a sequence is convergent by showing that it satisfies the definition.

If this characterization is reasonable, then this raises some interesting questions regarding how Xavier's learning will proceed. If he encapsulates this process to form an object, will this object be the definition? If so, will this then be available for proof problem like this one? Will it ever be associated with a category containing the individual convergent sequences? Does this matter? Could Xavier learn to take advantage of image-based thinking, in which the range of these objects can be more apparent? Should he be encouraged to?

It is hoped that this presentation will generate discussion of these issues.

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STUDENTS USE DIFFERENT REPRESENTATONS FOR THE CALCULUS DERIVATIVE FUNCTION: TWO CONTRASTING CASES

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Traditional instruction in calculus concentrated primarily on manipulation of symbolic or analytic representations while reform efforts over the past decade have promoted an understanding of both analytic and graphic representations of functions and derivatives. For example, graphic representations convey mathematical information visually, whereas expressions represented symbolically can be easier to manipulate, analyze, or transform. Instructional representations have been treated as a desired *end* in mathematics curricula; students' progress has been measured by how closely they are able to express their internal representations as accurate manifestations of instructional representations. Our work differs in that external representations as an instructional activity constitute a *starting* point for students' mathematical constructions. In this paper, we report on the development of understanding of two students for the calculus derivative function and the implications for learning and teaching calculus.

Generally, the ability to visualize in mathematics is thought to be advantageous, but Aspinwall, Shaw, and Presmeg (1997) reported on the case of a calculus student whose vivid mental images hampered his ability to think about the graph of an algebraic function with infinitely increasing slopes. Moreover, studies (e.g., Vinner, 1989; Tall, 1991) have consistently shown that students' understanding is typically analytic and not visual. Two possible reasons for this are when the analytic mode instead of the graphic mode is pervasively used in instruction, or when students or teachers hold the belief that doing calculus is skillfully manipulating symbols and numbers. It is clear from the literature (e.g., Lesh, Post, & Behr, 1987; Janvier, 1987; NCTM, 2000) that having multiple ways – for example, graphic and analytic – to represent mathematical concepts is beneficial.

In this paper we report on the development of understanding of two students for the calculus derivative function and the implications for learning and teaching calculus. Our work is framed by Krutetskii (1976) who distinguished among main types of mathematical processing by individuals: analytic, geometric, and harmonic. A student who has a predominance toward the *analytic* relies strongly on verbal-logical processing and relies little on visual-pictorial processing. Conversely, a student who has a predominance toward the *geometric* relies strongly on visual-pictorial processing and has above average verbal-logical processing. A student who has a predominance toward the *harmonic* relies equally on verbal-logical and visual-pictorial processes.

Our work also is framed by the view that posing and analyzing rich tasks for students provides windows into their thinking with ramifications for curriculum and instruction. As a result of observations of what students say and write, and how they represent mathematical situations, researchers make decisions about appropriate ongoing investigations. Generally, subsequent tasks are designed to clarify or validate early assertions. Here, we illustrate the contrasting thinking processes of Tim, whose representational scheme is predominately geometric, and Lil, whose representational scheme is predominately analytic.

The qualitative methods for the study consisted of analyses of participants' responses to tasks designed to probe their different understandings of the calculus derivative function. The framework for the analyses is a theoretical perspective we assembled that is inspired by research in both mathematics education and cognitive psychology. The theory generated, interpreting these differences, is explicated by references from the data to these perspectives in the literature. We explored the students' thinking on twenty tasks, one of which is depicted in Figure 1. For this task, students were directed to "Draw the graph of the derivative of the function whose graph is shown." The students were asked to explain their thinking. Additional tasks will be shown during the presentation.

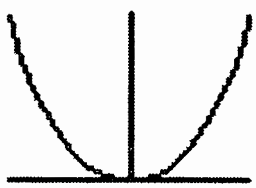

Figure 1.

This research is a component of a larger line of research aimed at a better understanding of how students learn elementary calculus viewed through their personal and idiosyncratic representations of the derivative function. The data sets consist of a large collection of videotapes of students working on tasks, verified transcripts of those videotapes, researchers' field notes, students' work, and researchers' journals. Analyses of videotaped sessions included coding of events, triangulation of qualitative data, and identification of distinct strands. Task design and implementation were critical: a limited number of tasks were created before data collection began; the vast majority of tasks were the result of prior tasks designed to clarify or validate early assertions. Frequently, tasks were revisited to determine if students thinking had changed as a result of subsequent tasks. The data were recorded over a period of six months as the students met weekly with the researchers.

Results

Tim's and Lil's instructional experiences in Calculus I and II were similar, but their unique, contrasting representational schemes for the derivative function were quite different. Tim processed information by virtue of analysis of his graphic representations of the derivative function. Tim did display an ability to combine his graphic schemes with an increasingly robust analytic representation that he used to validate his conclusions; however, overall, he showed a predominance toward a geometric type of processing.

Lil, who tends to process information analytically, made her sketches of the derivatives for these tasks by first translating to analytic representations. Lil appeared not to be comfortable with constructing a graphic connection between the slopes of tangent lines to the graph of a function and the behavior of its derivative. Thus she rarely used such constructions. She produced sketches and written and verbal responses to questions that revealed her facility with analytic representations.

Tim's and Lil's work, and their descriptions of their work, for the task in Figure 1 are typical examples of the differences in their thinking. Lil's preference for analytic representations in the form of equations soon became evident. She began by inferring that the graph in Figure 1 was $y = x^2$. She then computed $y' = 2x$ as its derivative. Using analytic representations in the form of equations, she calculated ordered pairs [(1,2) and (2,4)] for the function $y' = 2x$ and produced the drawing shown in Figure 2. Lil's procedures suggest that her cognitive structures for the derivative function are predominately analytic.

Analysis of Tim's thinking suggests he tends toward Krutetskii's geometric type of processing. For the task in Figure 1, he worked quickly and explained how he produced the drawing in Figure 3 based on his estimates for slopes: "It's [the slope is] a greater negative here [pointing to the negative part of the x-axis of Figure 1], and then it [the slope] slowly becomes more positive [meaning less negative] near 0. As we go to quadrant I over here, from quadrant II, it [the slope] goes from 0 to more and more positive values. So, I knew the line [graph of the Derivative] had to go from quadrant III to quadrant I and on up [as seen in Figure 3]."

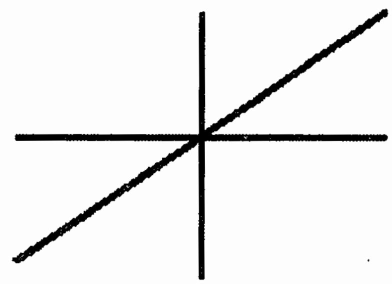

Figure 2.

Tim's drawing of the derivative as a curved line from quadrant III to quadrant I illustrates his preference for using a graphic representational scheme, typical of individuals with an ability and a preference for geometric processing. In his work he made no attempt to describe his work analytically, for example, by inferring the function $y = x^2$, computing $y' = 2x$, and determining ordered pairs to complete the graph of the derivative.

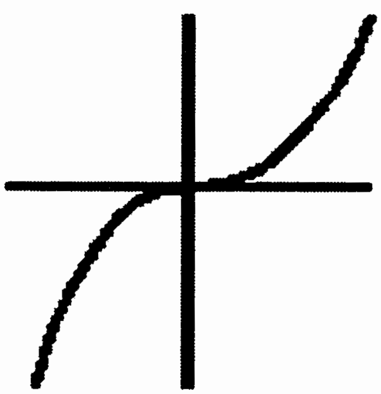

Figure 3

Conclusion

Although Tim and Lil utilized two different types of representations, we believe students develop mathematical power by learning to recognize an idea embedded in a variety of different representational systems and to translate the idea from one mode of representation to another. A positive result of multiple instructional representations of concepts is that students learning to present their representational schemes will reveal aspects of their understanding

that might not otherwise emerge. Otherwise, students are merely confronted with and must construe a teacher's representational preference, and the task for students becomes one of memorizing the presentation rather than learning to select or create representational schemes for problems they are trying to solve.

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TO STUDY MATHEMATICS IN AN ENGINEERING PROGRAM

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A significant number of students who study mathematics after graduating from the gymnasium in Sweden are students in engineering programs. Typical of the engineering programs at the Chalmers University of Technology is that most first-year students take courses in unfamiliar areas of mathematics. For many students, meeting with abstract or linear algebra is like a culture shock. An interuniversity project, CDIO,[1] contains several assessment efforts, including the study reported here, which should be seen as an attempt to describe how engineering students in their first linear algebra course apprehend the organization of the course, the teaching, the course goals, the assessment procedures, and the course context.

Introduction

Several researchers have attempted to categorize advanced mathematical thinking. Robert and Schwarzenberger (1991) describe in the following statement how learning in advanced mathematics is different from learning in elementary mathematics:

There is a quantitative change: more concepts, less time, the need for greater power of reflection, greater abstraction, fewer meaningful problems, more emphasis on proof, greater need for versatile learning, greater need for personal control over learning. The confusion caused by new definitions coincides with the need for more abstract deductive thought. Taken together these quantitative changes engender a qualitative change, which characterizes the transition to advanced mathematical thinking. (p. 133)

It is unclear from this statement how much of the difference is due to the mathematical content itself and how much to the way the courses are taught. To date, there seem to be no agreement in the community of mathematics and mathematics education on what advanced or elementary mathematical thinking really is. In *The Nature of Mathematical Thinking* (Sternberg & Ben-Zeev, 1996), Sternberg wrote in the culminating chapter, "In reading through the chapters of this volume, it becomes clear that there is no consensus on what mathematical thinking is, nor even on the abilities or predispositions that underlie it" (p. 303).

The Algebra Course

The algebra course in the mechanical engineering program is built around a common core of linear algebra, including such concepts as vectors, matrixes, determinants, complex numbers, polynomials and algebraic equations. The course is a 7-week course, organized into what are called "theme weeks," with each week organized as follows, beginning with Day 1 on a Thursday:

- Day 1: The instructor gives an introductory 2-hour lecture, with an introduction to the theme—the study domain for the rest of the week—the objectives and goals, examples, and important theorems and relations.
- Days 2 and 3: The students work in groups of 4 within a section of about 30 students and 1 teaching assistant. The teaching assistant serves as a coach but can also demonstrate further examples in the content domain to the whole section. So-called huge questions are left for the post-lecture.
- Day 4: The students are examined on the week's work.
- Day 5: The instructor gives a concluding post-lecture for all students.

An important feature of the organization is that the students work in small groups, thereby allowing discussions and, according to the instructor, prepare for "learning by explaining to someone else" but also for stimulating group discussions. Another important idea is the two assignments each week; these are larger problems on which the students are examined on Day 4, both orally and in writing. The students are also encouraged to keep a journal during each theme week in order to reflect on their learning. The journal is counted as part of the assessment. The 14 homework problems (2 every week) and the 6 journal entries (journal entries every week but one) yield a maximum of 20 points. To pass the course, the student needs to have at least 12 points from this part of the assessment. In addition, the student needs to get at least 12 points on the 30-point final exam.

The Students

In the fall of 2001, 189 students were admitted to the mechanical engineering program at Chalmers. They ranged in age from 19 to 33 years, with a median age of 22 and a mode of 20. The median value of their gymnasium grade was VG (well passed; i.e., 4 on a 5-point scale). Thirty-two were women, and 157 were men. They began their studies in mathematics with the algebra course, which started with an introduction on September 4 and ended with the final examination on October 25. Thereafter the students continued the program with courses in analysis and classical mechanics.

Method

The study was conducted with two surveys, interviews, and lecture hall and classroom observations. The survey instrument was based on an indicator instrument from Australia called Course Experience Questionnaire (Johnson, Ainley, & Long, 1996) and given during and after the algebra course. The interviews were conducted with volunteers who responded to a general request for students who would like to participate in an interview regarding their studies in mathematics. Fifteen students contacted me and came to the first round of interviews, one at a time. Ten students came to the second round of interviews, seven came to the third interview, and five were interviewed a fourth time. The first interview began with me providing information about the CDIO project; then I asked each student about his or her rationale for studying at Chalmers, the time spent on studies, the way the student experienced studying mathematics at this level in general, and so forth. The interviews also included conceptual and technical questions about linear algebra. Finally, we talked about the algebra course in a more detailed way. The students were also asked to draw conceptual maps to describe how they saw linear algebra concepts linked together. All this happened during the first interview, the following interviews concerned the analysis course, and the course in classical mechanics although we always discussed the algebra course as well. I also sat in on lectures in algebra (as one "student" among all the other students) and visited classroom activities.

Results

A final report containing the full set of surveys, responses to each item, interviews, and student results for linear algebra, analysis, and mechanics should be available by the end of 2002. The results of the study need to be seen in relation to one another, but as a single and isolated result from the surveys, there was a trend for students' beliefs in their generic skills to decrease during the course. The fact that as many as 40 % (72 out of 180) of the students failed to pass the first exam was probably responsible for much of the deterioration in confidence.

Another interesting result concerns the students' opinion of the reflective writing assignment. During the course there was a polarization of students' opinions about the usefulness of reflective writing. After the course, more students expressed the opinion that they saw no benefit whatsoever in reflective writing when learning mathematics.

The affective part of the interviews revealed many surprising findings, together with some more expected ones. As with the surveys, there is no room here for a detailed report.

The interviews also revealed rather surprising gaps in the students' knowledge of linear algebra. Only a third of the students could, for instance, describe the concept of *rank,* and no one was prepared to give an answer to the somewhat challenging question, "What is linear algebra?" After some time I usually added the questions: "Is it a formal game?" "A set of abstract structures?" "A language?" "A tool with which to investigate natural phenomena?" The great majority of the students selected the last alternative. I find it intriguing to see the difficulties these students experienced when trying to see an overall picture and how they apparently have chosen to study only to learn facts and procedures.

Conclusion

Given the efforts by mathematicians and mathematics educators in the CDIO project to give this algebra course a greater focus on understanding, the results of this study were disappointing. Linear algebra is hard to learn and has for years had the reputation of being one of the principle obstacles in introductory university mathematics.

During the sixties, at a conference in Zürich, I made the acquaintance of a charming old man who was none other than Plancherel – of Plancherel's theorem – and who, during a very interesting conversation, insisted on the fact that of all the teaching he had done that of linear algebra seemed to be by far the most difficult for students to understand. Thirty years later the situation does not seem to have changed very much and we can assure Plancherel that he is in good company. (Revuz, 2000, p. xv)

Nevertheless, this situation is not satisfactory, and educators and researchers must continuously work on improving the teaching, learning, and assessment of linear algebra to increase its accessibility to more students.

Endnotes

[1]CDIO is a project in cooperation between Chalmers University of Technology (Gothenburg), The Royal Institute of Technology (Stockholm), Linköping Institute of Technology and Massachusetts Institute of Technology (Boston). This cooperation is directly enabled by the support of the Wallenberg Foundation. The CDIO project aims at improved engineering education by making the Conception-Design- Implementation-Operation—CDIO—of systems and products the main context of engineering education.

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STUDENT'S DIFFICULTIES WITH Z_{12}, THE CYCLIC GROUP OF ORDER 12

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In many abstract algebra classes, the cyclic group Z_n is introduced as a straightforward example of a factor group. It is generally assumed that this example is quite accessible to students. Our own experience teaching abstract algebra led us to question this assumption. In this paper we discuss students' misconceptions about Z_{12} and how these misconceptions point to the important concepts underlying an understanding of factor groups.

The data we discuss is from a one-semester study of an undergraduate abstract algebra class. Data include students' written work, field notes, videotapes of class instruction, and six semi-structured interviews with each of six target students. Beginning with the interview data, we looked first for examples of tasks that were difficult for students. Comparing successful and unsuccessful approaches, we looked for differences in their underlying conceptions of factor groups. This helped us identify important underlying concepts that led to successful responses.

When trying to make sense of Z_{12} as a factor group, our students had to engage in three different constructions: the factor group as sets, the factor group as cosets, and the factor group as representative elements. When approached from a formal mathematical perspective, all three of these constructions were equivalent. However, from a psychological perspective, each of these constructions requires fundamentally different understandings. Furthermore, our claim is that for students to truly understand Z_{12} as a factor group, they need to build and coordinate all three constructions.

STUDENTS' UNDERSTANDING OF DERIVATIVE AND FUNCTION IN DIFFERENTIAL EQUATIONS

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This research informs us about university students' understanding of the concepts of function and derivative. The analysis considers both theoretical and practical issues. Theoretically, we add to an understanding of how students progress in understanding rate of change (Rasmussen, in press). Pragmatically, baseline information on students' understanding of function and derivative inform instructional decisions and curriculum design for a classroom teaching experiment in differential equations.

Methodology

Individual interviews were conducted with eight students as part of a classroom teaching experiment in differential equations (Cobb, 2000). These eight students were chosen from a class of 30 on the basis of their willingness and the diversity of their answers in an intake survey that was administered the first day of class. There were two parts to the semistructured interview. First, we sought clarification and elaboration of the responses on the intake survey. Second, students worked through two tasks meant to reveal information about their conceptual understanding of rate of change and rate of change equations. These interviews were videotaped and transcribed; thus the data consists of the videotapes, transcriptions of the discussions, and copies of written work done by the students during the interview. To analyze our data, we reviewed it several times together, looking for patterns and trends in their verbal responses and written work.

Discussion

Our analysis revealed two principal findings with respect to the students understanding of function. First, the students' concept image of function (Tall & Vinner, 1981) revolves around knowing function in a static sense. By this we suggest that students conceptualized function as a completed unchanging graph with one output for each input value. For example, when asked to define function, one student states "its a continuous line, increasing or decreasing Â...each x-value can only have one y-value." Second, university students often maintain a more instrumental understanding of function as opposed to a relational understanding (Skemp, 1978). When asked why a graph is a function, the majority of students asked mentioned the vertical line test, a rule taught in algebra to determine functionality. When asked to elaborate, these same students were often unable to explain its basis, or mention more than a memorized answer.

Regarding students' understanding of derivative, the interviewees showed that they had a non-dynamic perception of rate of change. They have internalized the concept of derivative as the slope of a tangent line, but do not see it as a changing quantity. For example, when given the differential equation dT/dt=0.3(T-70), students replaced T with the initial value and drew an entire line with that same slope. This seems to confirm our hypothesis of how students' understanding of rate of change develops and suggests that in differential equations, students may learn to conceptualize derivative in more dynamic ways.

These results have ramifications for our work in the development of a differential equation course where concepts are primary and conceptual understandings imperative. These findings develop at both an individual and classroom level and what we learn will impact both arenas.

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Algebraic Thinking

CONSIDERING THE ROLE OF COMPOSITE UNITS IN REIFYING EQUIVALENT EXPRESSIONS

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When solving equations of the form $ax + b = cx + d$, students struggle with the ontological essence of the expressions which form the equation. They may or may not have made the conceptual leap from an operational to a structural mode of thought. The authors site research and argue that part of the process of understanding students dilemmas involves observing students difficulties with composite units that are made up of both unknown and known quantities. These observations lay the groundwork for the further study of students' reification of algebraic expressions.

Purpose

Several studies have noted the ease with which students are able to solve linear equations of the form $ax + b = c$. (see English & Sharry, 1996, Kieran, 1992, Fillory & Rojano, 1989, Sfard & Linchevski, 1994). In equations of this form, there is a clear distinction between the calculational processes, $ax + b$, and the result of the process, c. To solve tasks of this type procedurally, it is not necessary for the learner to have reified the expression $ax + b$ as a quantity.

These same studies have also illustrated the difficulty students have in solving equations of the form $ax + b = cx + d$. To provide a theoretical grounding for understanding the reasons that underlie this difficulty, Sfard and Linchevski cast this inconsistency as the difference between students having an *operational* rather than a *structural* conception of an equation. In this paper, we use the term structural thinking to indicate that the learner has moved from viewing $ax + b$ as only a calculational process to include viewing the expression as a mathematical object. When confronted with a situation such as $3x + 24 = 2x + 70$, the structural thinker will move beyond treating the right side of the equation as the "answer" to the left side of the equation. The learner who has made the conceptual leap to structural thinking must be adept at treating both sides of the equation operationally and structurally, as both computations and as mathematical objects. In addition, the equality sign must signify equivalence for the learner and a perfect balance of the quantities.

The purpose of our research is to assert that even when students can successfully solve such tasks when presented symbolically, they may have, at best, what Sfard called a pseudo-structural conception of algebraic expressions. In other words, if a student is able to correctly solve the task $3x + 24 = 2x + 70$, they may or may not be thinking about the equation structurally. Some students successfully act upon the expressions in appropriate ways because they are closely following a predetermined set of rules. We endeavored to construct a scenario in which students could demonstrate the duality of their thinking. We argue that students' difficulties in doing so are closely related to their inability to perceive expressions of the form $ax + b$ as *composite units* consisting of both a known and an unknown quantity.

We use the term composite unit in the sense developed by Steffe (1992). According to Steffe, when a child has developed a notion of composite unit, she can coordinate units of different rank. For example, she can treat a number, such as 23, as a single unit comprised of 23 ones, or as a group of 23 individual units and can move back and forth between these conceptions and coordinate them in flexible ways. She can think of a group of 23 combined with a group of 25 more as two groups of 20 and 8 more, or possibly think of them as 2 fewer than two groups of 25. This is the type of flexibility that is needed to have an understanding of place value numeration. In an analogous way, conceptualizing an unknown as a composite makes it possible to think of X+X as 2X where 2X is the result of counting X units, followed by counting X units again. That is, 1, 2, 3, ..., X, X+1, X+2, ..., X+X, where X+X is now seen as 2X. In the same way, an algebraic expression such as 3X-2 can, depending on one's current needs, be viewed as a single composite unit, as 3X-2 individual units, as 2 fewer than 3 units of (size) X, or as 2 units of (size) X and 2 fewer than another unit of (size) X. In this way, this conceptualization of algebraic expressions is analogous to conceptions of place value numeration.

Research Questions

The following research questions guide our investigation:

- How do students approach problem situations of the form $ax + b = cx + d$ when these problems are not represented using conventional symbolism?
- Are students acting upon algebraic equations operationally or structurally?
- Have students reified both the unknown x as a quantity and expressions such as $ax + b$ as a quantity?
- Do students comprehend that the equivalence of two expressions is maintained when equal amounts are removed from each expression?
- When unknown quantities are composed of both known and unknown amounts, can students break apart that composite unit into convenient pieces?

Data Sources and Methods of Inquiry

The students who participated in this study were enrolled in a developmental-level beginning algebra course for university students. All of the students enrolled had completed high school algebra at some point in their schooling. An inquiry approach to mathematics instruction was employed, and instructional materials were developed as part of a classroom teaching experiment. We conducted pre- and post-semester cognitive interviews with 11 students. The students' responses on tasks related to our research questions were analyzed using qualitative research methods. The interviews of two students (one who completed the course at the top of the class and one who struggled unsuccessfully) are highlighted in this study because they best represent the range of difficulties that students face.

Because the students had previously completed high school algebra we assumed that many would be familiar with algebraic equations of the form $ax + b = cx + d$ and might be able to solve tasks using methods that they had learned in high school. To ascertain if these students would perform in a less ritualistic manner when the problem was presented within a context, we posed tasks such as the two described below.

> Question #1. Cindy and Bobby have envelopes containing the *same amount* of money in all of the envelopes. Cindy has three envelopes and 24¢. (Actual envelopes containing money were placed in front of the students, as well as the "extra" change.) Bobby has two envelopes and 70¢. If Cindy opens all of her envelopes and counts her money, and Bobby opens all of his envelopes and count his money, they both have the same total amount of money. Can you use this information to help you figure out how much money is in each of the envelopes?

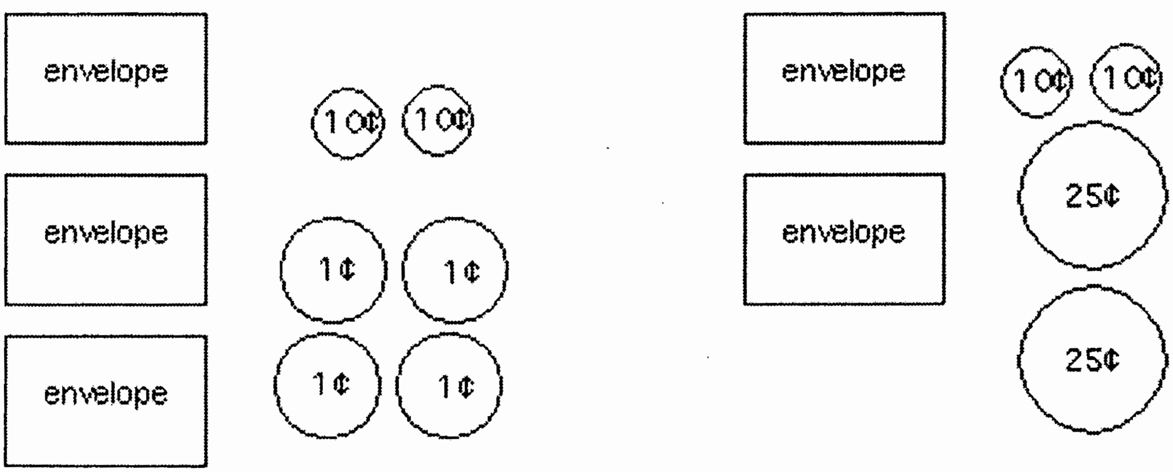

Figure 1.

> Question #2. Cindy and Bobby have envelopes containing the same amount of money in all of the envelopes. Cindy has three envelopes and 80¢. Bobby has four envelopes and 24¢. If Cindy opens all of her envelopes and counts her money, and Bobby opens all of his envelopes and count his money, they both have the same total amount of money. Can you use this information to help you figure out how much money is in each of the envelopes?

Results and Discussion

Holly's Dilemma

Holly finished the semester as one of the top students in the class. She had done very well in high school, graduating in the top 4% of her class. When asked question #1, Holly quickly began writing $3x + 24 = 2x + 70$. She solved this equation procedurally by subtracting $2x$ from each side and then subtracting 24 from each side and finished with $x = 46$. When asked if there was a way to confirm her answer, Holly wrote $3(46) + 24 = 2(46) + 70$, and said that "[We] just substitute it in and it should *equal*." While it was clear that Holly was able to represent the task symbolically and solve it procedurally, we wanted evidence that she had truly reified x, that is, the unknown amount in the envelope as a static quantity and understood that the $x = 46$ meant that there was 46¢ in each of the envelopes. We also looked for evidence that she had developed structural knowledge of the unknown quantities $3x + 46$ and $2x + 70$ in addition to her procedural knowledge of those quantities.
(The interviewer took away one envelope from each 'child')

Int: *Who* has more now?

Holly: Cindy has more. I think Cindy does. I have to figure it out. (Holly proceeded to write two new expressions, $2x + 24$ and $x + 70$, to represent the new situation. Next, she substituted 46 for the unknown x into each expression and proceeded to evaluate each.) Oh, they have the same amount. Oh, look at that. I wouldn't have guessed that!

The interviewer then removed another envelope from each collection, leaving Cindy with one envelope and 24¢ and Bobby with 70¢. Once again she asked Holly, "Who has more money?" Holly substituted 46 for the unknown x in $x + 24$ and declared that each child had the same amount. When the interviewer asked Holly to explain why this occurred, Holly appeared to contemplate this question for several seconds before exclaiming, "Well Duh!, Because there's 46¢ in each of those and if you take away that *equal* amount there will obviously be an equal amount left over."

Those who have already reified algebraic expressions and equations may find it difficult to understand the crux of Holly's dilemma. If she knew that there was 46¢ in each envelope, and that Cindy and Bobby had the same amount of money to start, how could she not understand that the equality of Cindy and Bobby's amounts was maintained when one envelope (i.e., 46¢) was taken from each one? To understand the complexity involved in shifting one's thinking from an operational approach to a structural approach, one must consider all of the following:

1. The unknown static quantity x.
2. The unknown total quantity of money that each child had was composed of known quantities (the change) and unknown quantities (the envelopes)

3. The unknown quantities of money ax+b and cx+d are each static quantities themselves in this situation and not just a series of operations on an unknown quantity.
4. The same quantity can be represented in different ways even when they do not appear on the surface to represent the same thing.
5. The issue of conservation of equality. That is, if we remove the same amount from both expressions, each quantity has changed, but the equality between the two new quantities is preserved.

Holly appeared to have reified x as a static quantity in this situation. However, one must consider that there are *two additional* unknown quantities *and* an equality to be considered in tasks such as these. The quantity in the envelope was unknown to the students, and the quantity of money that Cindy and Bobby had in all was also an unknown, but equal to each other, quantity. The envelope quantity could be represented by x. The unknown quantity that Cindy and Bobby had was composed of an unknown quantity, x, and a known quantity, the change, and is represented in two different ways, $3x + 46$ and $2x + 70$. In these expressions, we have the same composite unit, the total unknown quantity, broken into pieces that do not immediately look like they represent the same amount. So when Holly had difficulty answering our questions, we needed to search for evidence of a reified x, a reified $3x + 24$ and $2x + 70$ as quantities, and a reified '=' sign.

Pat's Dilemma

Pat was a middle-aged returning student. She struggled to make sense of most of the topics that she encountered over the semester. Pat was given question #1 and asked to figure out how much money was in the envelopes. Pat began by notating the task as shown below:

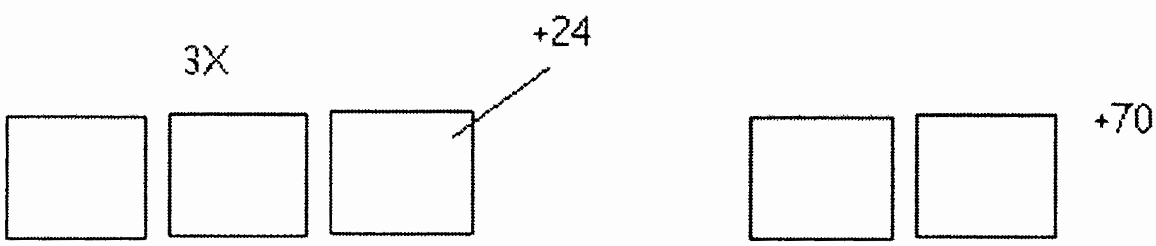

Figure 2.

Pat indicated that she understood that both Cindy and Bobby have the same amount of money, and yet insisted that "We have to somehow make them *even.*" She then stated that she was unable to complete the task because she had forgotten the procedure for solving problems of this type.

The interviewer continued by taking one envelope away from both Cindy and Bobby and asked Pat who had more money now. Pat replied that, "We don't really know who has more money *now.*" She was convinced that the interviewer had taken more money away from Bobby because "there was more money in each of his two envelopes than there was in each of Cindy's three envelopes." She indicated that this information would not help her in deciding who had more money. Pat was asked question #2, and she solved it in the same way. She was convinced again that by removing one envelope from each child, more money was being taken away from Cindy because Cindy had fewer envelopes.

The interviewer began the first task by reiterating that there was the same amount of money in each of the envelopes, and Pat shook her head in agreement and said, "okay." But Pat was unable to comprehend this idea. The money in the envelopes was not a static amount upon which she could act. She had not reified the unknown amount, x, as a static quantity.

Pat ignored the extra coins on the table in both tasks. She seemed unable to coordinate a quantity composed of a known and an unknown amount. Ignoring the 24¢ and the 70¢ allowed her to see the total amount of money that Cindy and Bobby had as a composite unit made up of only the envelopes. She then deduced that if the total amount in the envelopes, three for Cindy and two for Bobby, have the same value, then Cindy's envelopes must have less money in them than Bobby's envelopes. When the interviewer took one envelope away from each person, Pat said that more money was taken away from Bobby because the total amount divided by two meant more money in each envelope than the total amount divided by three for Cindy. Pat has not yet reified x as a quantity, nor $2x + 24$ as a quantity, but she has reified the *unit.* For Pat, the unit was the total amount of money in the envelopes. She was able to decompose that unit into Cindy's 1/2 unit + 1/2 unit and Bobby's 1/3 unit+ 1/3 unit +1/3 unit, and she knows that the envelopes with 1/2 unit must have more money in them than the envelopes with 1/3 unit in them.

Future Research

Holly was able to take the task and symbolically represent it using literal symbols. She displayed evidence of the reification of x because it was an object on which she operated. She knew that $x = 46$ meant that there was 46¢ in each of the envelopes. It wasn't until the interviewer broke apart the $3x + 24$ into $2x + x + 24$ and the $2x + 70$ into $x + x + 70$ that Heidi lost her grasp of the situation.

At that point, were those x's no longer the same quantity, the $3x + 24$ no longer the same quantity as $2x + x + 24$, or was $3x + 24$ no longer equal to $2x + 70$? We can't make a definitive judgment about Holly's thinking. If we had taken only one side of the equation, say $3x + 24$, and asked Holly how that quantity would change if we removed one envelope, would she have been able to say that it would decrease by 46¢? If we had done that, it would have given us a clue as to the point at which her understanding fell apart. Asking that question would have provided us with information about Holly's reification of $3x + 24$. I believe that if we had established that Holly had not yet reified 3x + 24, we could conclude that she would be incapable of the reification of the equality 3x + 24 = 2x + 70. The reification of the individual parts is necessary before a reification of the equality is possible.

It is interesting to note what Holly said after she procedurally figured out that the total amounts remained equal after one envelope was removed from each. She said, "I wouldn't have guessed *that!*" We wish we had asked her *why* she wouldn't have guessed that.

In our discussion, we will further elaborate on the complexity involved in developing a structural conception of algebraic equations and expressions. In addition, we will explain how the construct of composite units might be used as a potential pathway for designing instruction that would facilitate students' movement toward viewing algebraic expressions as both a process and an object.

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ON STUDENTS' FORMULATION OF SIMPLE ALGEBRAIC WORD PROBLEMS: SYNTACTIC TRANSLATION AND REVERSAL ERRORS AMONG TURKISH STUDENTS

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The purpose of the study was to investigate how a group of Turkish high school students formulate algebraic equations from word problems. The study was designed as a replication of MacGregor and Stacey's study (1993), which provided evidence that students do not necessarily use syntactic translation, which is commonly cited as the main source of students' errors and difficulties in translating from natural language to algebraic syntax. We aimed to investigate students' errors and observe whether or not syntactic translation is dominantly used in the context of Turkish schools. We found that students generally use syntactic translation while forming equations from verbal statements, and that reversal errors are the most persistent errors in forming equations.

Many students have difficulty formulating linear algebraic equations from information presented in words (Clement, 1982; Clement, Lochhead & Monk, 1981; MacGregor & Stacey, 1993, 1997a, 1997b; Rosnick, 1981). *Syntactic translation*, in which translation of a statement in English into an equation occurs by replacing key words by mathematical symbols sequentially from left to right, is accepted as a procedure frequently used by students for formulating equations from natural language expressions, and is thought to be an important cause of errors, particularly the *reversal error* (Clement, Lochhead, & Monk, 1981). Although syntactic translation was the first source blamed for reversed equations, Clement et al. (1981) observed that another approach to writing equations was used frequently. In this approach, which they called *static comparison* and Herscovics (1989) called it *semantic translation*, the equation is used to represent an association of related groups, rather than equal numbers. On the other hand, MacGregor and Stacey (1993) reported that the majority of secondary school students did not use a syntactic translation procedure for writing simple linear algebraic equations; instead they tried to express the meaning of the statement and wrote incorrect equations. MacGregor and Stacey suggested that the majority of the incorrect equations, particularly the reversal equations, are consequences of cognitive models attempting to represent compared unequal quantities. In a different cultural context, Mestre (1989) reported that many Hispanic students have misconceptions in solving the Student-Professor problem (Clement, 1982) such as "6S = 6P and 6S + P = T," which are originated in language differences. He concluded that differences in language cause Hispanics to commit the same types of errors as Anglos, but with a higher frequency. Therefore, we can hypothesize that types of errors in word problem translation are similar in different contexts and linguistic differences do not seem to qualify as a major source of errors. As a part of a larger research concerning teaching and learning of elementary algebra in Turkish schools (see Erbas, 1999; Ersoy & Erbas, 2000), we investigated how a group of Turkish high school students formulate algebraic equations from word problems. In particular, we were interested in whether Turkish students used syntactic translation and performed reversal errors as frequently as reported in the research literature.

Method

We selected 217 ninth-grade students from 4 high schools, two public academic (PuAc), one private (PrSc), and one vocational-technical high school (VoTe) in a socio-economically middle-class district of Ankara, Turkey using "simple random sampling method" (Fraenkel & Wallen, 1996). Two classes from each school (eight in all) were chosen based on availability and permission of participant schools. Students' frequencies and percentages according to their gender and schools can be seen in Table 1.

Table 1. Distribution of Students According to Their Schools and Gender

Schools	Female		Male		Total
	n	%	*n*	%	*n (%)*
PrSc: Private School	15	*43*	20	*57*	**35 *(16)***
VoTe: Vocational-technical High School	6	*12*	44	*88*	**50 *(23)***
PuAc-1: Public Academic-1	36	*49*	37	*51*	**73 *(34)***
PuAc-2: Public Academic-2	23	*39*	36	*61*	**59 *(27)***
Total	**80**	***37***	**137**	***63***	**217 *(100)***

We administered a simple paper-and-pencil test consisting of five items, four of which were open-ended (classical) and one was multiple-choice. All of the five items used can be seen in the Appendix. Four open-ended items (A, B, C and E) required students to formulate a simple linear equation from information given in words. The multiple-choice item (D) required students to identify the correct equation from a set of alternatives. Four items were translated replicas that MacGregor and Stacey (1993) used. Instead of the fifth item used by MacGregor and Stacey, we decided to use "The Students and Professors Problem" (Clement, 1982). In rewriting the items in Turkish format, we tried to preserve the essential characteristics of the items including the order of algebraic symbols/letters/variables and the numerals, sentence structure, and the meaning. However, the difference between the grammatical structure of a sentence in English and Turkish made preserving the location of keywords (i.e., verbs such as equal, be; comparatives such as more than, less than, etc.) difficult if not impossible. For example, in item A, although the order and structure of key words is *z, "is equal to", "sum of", 3, y* in English (MacGregor and Stacey, 1993), it is *z, 3, y, "sum", "is equal to"* in the Turkish translation. Readers may compare the keyword orders of the items from the tables we provided below. In the analysis of the answers, we took the differences between two languages into consideration and made the comparisons accordingly.

After we prepared the mathematics teachers, they administered the test to their own students in the fall semester of the 1998-99 academic year. The students had no time restrictions. In the data analysis and presentation, we followed the same pattern described in MacGregor and Stacey (1993). The alpha reliability of the test (Cronbach, 1951) was calculated as 0.73.

Results

The percentages of students correctly answering each item can be seen in Table 2 and Table 3. Except for the Item A, all items were more difficult than had been expected, with success rates ranging from 40% on item D to 83% for Item A. Because of high omit rates, the percentages of success rates for the whole sample was even smaller.

Table 2. Percentages of Correct and Incorrect Responses for the Students Answering the Items

Item*	n	Correct (%)	Incorrect (%)
A	121	83	17
B	122	60	40
C	109	49	51
D	150	40	60
E	90	62	38

n shows the number of students answering the items
* See Appendix for Items

To investigate the use of syntactic translation, like MacGregor and Stacey (1993) we followed the strategy of classifying students' responses as "possibly syntactic" and "definitely not syntactic" (p. 224). In that process, the key idea was that if the students mainly use syntactic translation then they would use the symbols, numerals, and variables in the same order that they appeared in the wording of the question. For example, in item A, if students used a syntactic translation then they would construct the equation as "$z = 3 + y$" perceiving "is equal to" as an indicator of equality sign and keeping the order of variables. Like MacGregor and Stacey, we allowed some "cosmetic changes" (p. 224) in the position of numerals when deciding if a response indicated word-order matching. As MacGregor and Stacey pointed out

Table 3. Percentages of Correct, Incorrect, and Omitted Responses for All of the Students

Item*	N	Correct (%)	Incorrect (%)	Omit (%)
A	217	46	10	44
B	217	34	23	44
C	217	24	26	50
D	217	28	42	31
E	217	26	16	59

N shows the whole number of students taking the test
* See Appendix for Items

...many students are thought that 3 + y is normally written as y + 3 and y × 8 is normally written as 8 × y or 8y. Therefore, it was considered that the responses z = y + 3 for Item A and s = t + 8 for Item C could have been produced by the syntactic method. For Item B, a student might use syntactic translation to get the sequence y.8 = z and then move the "8" to give 8y = z. We have allowed this and similar expressions as an evidence of possibly syntactic translation. Furthermore, allowance was made for incorrect translation of the operation. For example, y + z8, y + 8 = z were judged as indicating syntactic translation of Item B. By allowing for cosmetic changes and mistranslation of the operation, we hoped to identify every response that could have resulted from an initial syntactic translation. (p. 224)

The numbers of responses classified as "possibly syntactic," "definitely not syntactic," and "unclassified" are shown in Table 4a and Table 4b. The corresponding percentages of attempted responses for each item are shown in Table 5. As MacGregor and Stacey did, we classified responses as definitely not syntactic if the first variable in the item was used as the second variable or an inverse operation was used in the response. However, there were several responses that we could not classified either as "possibly syntactic" or "definitely not syntactic." For example, summing every numeral and letter used in the item (i.e., s + 8 + t in Item C). We classified those as "unclassified."

Contrary to MacGregor and Stacey, we observed, overall, that classifiable responses were likely to match word order. Only the data for Item E in Table 5 and Table 6 provide no support for the proposal that syntactic translation was a procedure commonly followed by the students in the sample. However, in total, the data in Table 5 and Table 6 provide support for the proposal that syntactic translation was a procedure commonly followed by the students in the sample. As shown in Table 5, more than half of the correct responses were syntactic. This does not look surprising because syntactic translation seemed so easy for the items. However, we cannot

Table 4a. Students' Responses to Items A and B

Possibly Syntactic		Definitely Not Syntactic		Unclassified	
Format	Frequency	Format	Frequency	Format	Frequency
Item A (z, "is equal to", "sum of", 3, y) - Key word order in English (z, 3, y, "sum", "is equal to") - Key word order in Turkish translation					
z=3+y	73	3+y=z	18	z + y = 3	4
z=y+3	9	y+3=z	1		
z=3y	1	yz=3	2		
3z=y	3	y-z=3	1		
z+3=y	4	Other	5		
Item B (y, 8, "times", z) - Key word order in English (y, z, 8, "times") - Key word order in Turkish translation					
y=8z	54	8z=y	5	8/y=z	1
y=z8	15	z=y8	4	y = z	1
y+z8	2	z=8y	6	y + z/8x	1
y+8=z	2	8y = z	17	8y + z	1
$y=z^8$	2	Other	7		
y.8=z	5				
yz = 8	2				

Table 4b. Students' Responses to Items C and E

Format	Frequency	Format	Frequency	Format	Frequency
Item C (s, 8 "more", t) - Key word order in English (s, t, 8, "more") - Key word order in Turkish translation					
s = t + 8	40	t + 8 = s	11	s = t - 8	2
s + 8 = t	20	t = s - 8	1	8s = t	2
s.8 = t	1	t = s + 8	10	s = t8	2
8 + s = t	3	t = 8 + s	2		
8s + t	2	Other	6		
s + 8t	5				
s = 8t	1				
s + t = 8	1				
Item E (6, "times as many", "Students", "Professors") - Key word order in English ("Professors", 6, "times as many", "Students") - Key word order in Turkish translation					
P = 6S	6	S = 6P	29	S.P	2
6P = S	14	6S = P	9		
P6 = S	5	6(S+P)	2		
P6S	1	S6 = P	2		
		S/6 = P	1		
		S = P6	9		
		S = P	2		
		SP = 6	1		

say that syntactic translation produces a greater error rate than those of non-syntactic translation. Contrary to MacGregor and Stacey, Table 5 and 6 do not provide strong evidence to reject the hypothesis that the words "*is equal to*" (p. 226) prompt syntactic translation. As may be seen in Table 5, many more students gave possibly syntactic responses for Item A than for the others and the number of incorrect responses is less than that of the non-syntactic.

Table 5. Frequencies of *Possibly Syntactic*, *Definitely Non-syntactic*, and *Unclassified* Responses to Items A, B, C, and E

		Possibly Syntactic		Not Syntactic		Unclassified		No Attempt
Item	n	Correct	Incorrect	Correct	Incorrect	Correct	Incorrect	
A	121	82	8	19	8	0	4	97
B	122	69	13	5	33	0	5	93
C	109	40	33	12	18	0	6	110
E	90	19	7	38	24	0	2	129
Total	442	210	61	74	83	0	17	429

n: The number of students answering the items

Table 6. *Possibly Syntactic* and *Definitely Non-syntactic* Responses as % of Total Responses to Items A, B, C, and E

Item	N	Possibly Syntactic	Not Syntactic	Unclassified
A	121	90 (74%)	27 (22%)	4 (3%)
B	122	82 (67%)	38 (31%)	5 (4%)
C	109	73 (67%)	30 (28%)	6 (6%)
E	90	26 (29%)	62 (69%)	2 (2%)
Total	442	271 (61%)	157(36%)	17 (4%)

n: The number of students answering the items

We observed that, for all items, most of the errors were reversal. Table 7 shows the number of reversed responses for items A, B, C, and E and the percentages within the total and incorrect responses.

In total, 68% of the incorrect results are reversal errors. This seems surprising since the test items (except the Item E: The S-P problem) were designed (by MacGregor and Stacey) to reduce or overcome those previously published causes of reversal. The percentages of reversal in total responses were lower than those in MacGregor and Stacey. Furthermore, the percentage of reversals in Item E was greater than for the other items. This might be attributed to the fact that it was not constructed to eliminate all previously published causes of reversal. This observation might also be thought of as the structure of the Items A, B, C being effective in reducing reversal errors.

Table 7. Reversal Errors in Responses to Items A, B, C, and E

Item	Total Responses	Reversed Responses		
		Frequency	% of total responses	% of incorrect responses
A	121	9	7	45
B	122	34	28	67
C	109	40	37	68
E	90	26	29	79
Total	442	109	25	68

In most curricula and classroom practices, students are often experienced in solving equations, but not necessarily experienced in constructing or writing equations. This point lead us to consider that in a multiple choice item/test environment students might do better than open response format. Item D, constructed as a multiple-choice item, was included for this reason, as did MacGregor and Stacey. As seen in Table 8, contrary to expectations, only one-third of the students in total were successful in identifying the correct equation from among the choices. A comparison of schools shows a success rate ranging from 16% to 57%. Considering that the fourth and fifth alternatives (6x = y and 6 + x = y), definitely, included reversal errors, about one-third of all the students chose reversed equations. This situation is parallel to the other items.

Table 8. Students' Correct and Incorrect Choices for Item D

Schools	N	Correct $x=6+y$		Incorrect $x=6y$		$x=6-y$		$6x=y$		$6+x=y$		Omit	
		f	%	*f*	%	*f*	%	*f*	%	*f*	%	*f*	%
PrSc	35	20	57	0	0	1	3	2	6	7	20	*5*	*14*
VoTe	50	18	36	1	2	2	4	2	4	13	26	*14*	*28*
PuAc-1	73	12	16	0	0	1	1	6	8	18	25	*35*	*48*
PuAc-2	59	12	20	3	5	4	7	10	17	18	31	*13*	*22*
Total	**217**	**62**	**29**	**4**	**2**	**8**	**4**	**20**	**9**	**56**	**26**	***67***	***31***

Discussion and Conclusions

In contrast to MacGregor and Stacey (1993), our results do not cast serious doubt on syntactic translation as a major cause of errors in formulating equations. However, the reversal error has been shown to apply to a wider class of equations than was previously considered as pointed out by MacGregor and Stacey. Like MacGregor and Stacey, the present study has shown that the reversal error occurs not only in multiplicative items but also in additive items that relate to very familiar situations.

Furthermore, as MacGregor and Stacey (1993) have pointed out, the high percentages of reversal errors in Items B, C, D cannot be attributed to previously identified causes such as *static comparison of associated sets of objects, the use of algebraic letters as abbreviated names, the interpretation of numerals as adjectives, frame retrieval errors, and complex or misleading English syntax* (here Turkish). Although all these factors could have had an effect on the items previously reported in the literature, they were eliminated as much as possible from the items used in the present study. Moreover, we found that a high percentage of students committed reversal errors in Item E for which no such precautions were made in its adaptation into Turkish. The percentage of reversal errors in this item was higher than that in other items. Clearly, the precautions taken against previously identified causes were significantly effective to avoid the reversals. Since we cannot reject that students used syntactic translation, we cannot ignore the effect of it on reversal errors here. As we hypothesized at the beginning, types of errors in word problem translation are very much on a same line in different contexts and linguistic differences do not seem to qualify as a major source of errors. However, this claim needs to be verified with more research focused on language differences and their affect on algebraic problem statement, formulation and solution.

We conclude that without the change in cultural context, syntactic translation appears to be widely used among algebra students. We hypothesize that this is mainly the result of instructional practices and classic textbook emphasis that focuses on the procedures rather than meanings. Students often are taught to solve word problems by seeking the key words and word order matching techniques, which are key in syntactic translation.

As Herscovics (1989) proposed, students are taught the syntax of a language without the semantics. Namely, they know the rules in the grammar but they do not understand the words. As a result, students often develop what Skemp (1976) calls an "instrumental understanding" of algebra: "It is what I have in the past described as 'rules without reasons,' without realizing that for many pupils *and their teachers* the possession of such a rule, and the ability to use it, was what they meant by 'understanding' "(p. 153). However, it is well recognized that to educate a generation for tomorrows' world where demands will be placed on problem solving and communication skills, we need to shift instruction away from routine symbolic manipulation and procedures toward an in depth, "relational understanding" (Skemp, 1976) of algebra (Booth, 1989). Thus, students should be provided with meaningful tasks that develop critical thinking skills and enhance appreciation for the usefulness of algebra. As Davis (1989) cautioned, these mathematical tasks should be of a genuine nature, not the artificially contrived word problems found in many texts.

Note

* The items A, B, C, and D were taken from MacGregor and Stacey (1993) and the item E was taken from Clement, Lochhead & Monk (1981).

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Appendix: Test items used in the study*

A) "z is equal to the sum of 3 and y." Write this information in mathematical symbols.

B) "The number y is eight times the number z." Write this information in mathematical symbols.

C) s and t are numbers. s is eight more than t. Write an equation showing the relation between s and t.

D) I have \$$x$ and you have \$$y$. I have \$6 more than you. Which one of the following equations must be true?

a) $x = 6y$ b) $6x = y$ c) $x = 6 + y$ d) $6 + x = y$ e) $x = 6 - y$

E) Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university. Use S for the number of students and P for the number of professors.

WARRICK'S SECRETS: TEACHING MATHEMATICS THROUGH A 3D MASSIVELY MULTI-PLAYER ROLE PLAYING GAME

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The project leverages students' fascination with 3D digital entertainment and medieval fantasy stories to teach mathematics, including beginning algebra, to middle school students by situating the mathematics in an appealing computer environment based on massively multi-player role-playing games (MMPRPG). The development team for *Warrick's Secrets* consists of individuals from the information technology (IT) industry and researchers from three universities. *Warrick's Secrets* is the result of a Phase I Small Business Innovation Research (SBIR) grant completed for the US Department of Education in April 2001 and Physitron, Inc. internal research and development funding. Second stage research and development continues following the September 2001 awarding of a Phase II SBIR by the U.S. Department of Education. Phase I and II versions of *Warrick's Secrets* are constructed as single-player role-playing games (RPG). The RPG is used to deliver National Council of Teachers of Mathematics (NCTM) *Standards*-based educational content. The system includes advanced 3D computer rendering capabilities and high quality 3D artwork since both have proven crucial to maintaining players' attention in the gaming community. The results of the Phase I prototype study are reported along with the progress of the Phase II program.

Background and Theoretical Framework

What are Massively Multi-Player Role Playing Games?

Warrick's Secrets is modeled on currently popular MMPRPG's such as Ultima Online (UO), Everquest, and Asheron's Call. Currently, more than 150,000 customers play Electronic Arts Ultima Online-- including players from 114 countries (Walton, 2000). Sony's EverQuest and Microsoft's Asheron's Call each have a substantial customer base as well. A MMPRPG is a game with a thousand or more players gaming concurrently in real time in the same virtual world. Role-playing indicates that a human player controls an in-game character's speech and actions. The virtual world in the most successful games contains aspects of the historical middle ages, including some degree of fantastic creatures (e.g., dragons) and magic. Due to limited resources and time, Phase I and II prototypes of *Warrick's Secrets* only support single-player activity. With future funding, we will add multi-player capabilities. At first glance, some people mistakenly think the educational tool described and the total system planned differs little from existing products which attempt to make learning into a game. Our approach is to tightly integrate the educational content with a MMPRPG.

This seemingly subtle difference (integrating educational content with a particular game genre versus attempting to make learning into a game) has an enormous impact on the way students react to the tool. There are several differences in our approach that increase the student acceptance rate, potential learning success rate, and age range for which the teaching tool can successfully be used. Some of these advantages are listed below.

- *Warrick's Secrets* is a mathematics educational tool that is based on currently popular commercial MMPPRG concepts.
- *Warrick's Secrets* is based on NCTM *Standards* for middle school mathematics and provides teachers with a mapping of each mathematical challenge to the corresponding topic/skill within the standard.
- Educational content imbedded into realistic situational context.
- Inherent promotion of cooperative learning.
- Ability for students to interact with remote students (In Phase III).
- Vehicle for covert and overt learning.
- Advanced motivation and reward system.

Related Research

Mathematics education research areas germane to this project include situated perspectives on learning and constructivist views of representation. Mathematics viewed as socially constructed knowledge has strongly influenced other learning theories. Many researchers (Kieren, 2000; Steffe & Thompson, 2000; Sfard, 1998; and Cobb & Yackel, 1996) describe the theoretical underpinnings of their work in terms of "social constructivism." The learning theory that grounds many of these researchers, including those in the present project, may be traced to von Glasersfeld's (1991) radical constructivism, and Piaget's (1970) genetic epistemology. Lave and Wenger (1991) describe situated learning as developing as a function of the activity, context, and culture in which it occurs. One of the primary benefits of integrating educational content with a virtual environment and storyline is that educational content can be imbedded into a realistic situational context. These ideas are important because we consider how such theories apply to the development of educational interactive multimedia. An example of such multimedia research is the work of Herrington and Oliver (1997).

Mathematical representation is considered from a constructivist perspective. Representation describes the form in which a problem is encountered—such as graphical, numerical, verbal, or symbolic representations. It also describes the learner's mental construct of the problem. One of the major advantages of the virtual environment for problem solving is that students have many opportunities to attempt solutions of mathematical tasks. The virtual environment affords a natural mechanism for providing hints and alternative teaching representations based on a student's previous success with a concept or skill. This may increase the number and quality of the mental representations that students apply to the problems. The project relied on the notion of reflective abstraction to explain how, as students work through multiple attempts to solve a problem, they may develop progressively more abstract mathematical conceptions. It has previously been observed (Cifarelli, 1988; Goodson-Espy, 1998) that if students cope with recurrent mathematical themes, they are more likely to develop higher levels of reflective abstraction, and thus attain more powerful mathematical concepts.

Research Questions

The project examines several research questions, including the following:

- Does playing this type of game raise the performance level of students in a regular mathematics classroom?
- Can the notion of reflective abstraction be used to describe mathematical concept building in the MMPRPG setting?
- Will our collection of situation-based and "scaffolded" mathematics pathways encourage students to apply problem-solving skills that encourage the development of higher order thinking?
- If allowed to create their own character, what kind of character do they choose in terms of gender, race, ethnicity, facial appearance, and clothing?
- Will this educational resource be popular with females?
- Do students display an increased motivation to try to solve problems? Do students with math anxiety behave differently when they assume a character within a virtual world to solve problems?
- Can standards-based mathematical content be integrated into an MMPRPG in a manner that allows students to learn without forcing the student to step out of the role of their online character?

Warrick's Secrets Development Process

Game Play Design and Storyline

The game world of *Warrick's Secrets* is set in what is often called the "fantasy" genre. The fantasy genre has historically proven considerably more successful than other alternatives (e.g., science fiction, western) across the total spectrum of games—table-top, single player computer, and multiplayer computer. The fantasy genre is akin to earth's historical "middle age" with the addition of magic and monsters. Given that it is a fictitious world, creating an engaging "back-story" to draw the player into the game world is a primary challenge. Our story traces the footsteps of a boy named

Palus as he travels from his home in the "Old Country" to attend a magical academy in the Northlands. Being sensitive to gender-issues in math education, Phase II and subsequent editions vary the name, sex, and ethnicity of the back-story protagonist. Although the central purpose of the back-story is to immerse the player into the game world, a variety of subtle features exist. First, the character is of an age to which players can relate. Second, and perhaps analogous to the experience of many underprivileged children in America, the character is an immigrant. Third, the character is portrayed as self-reliant and resourceful. Palus is both willing and able to undertake this journey on his own initiative. Fourth, while cautious, Palus is nevertheless friendly and hospitable to strangers in need. Fifth, the story includes several different races of peoples. From the back-story, players are introduced to the game world, a world where monsters and magic exist, in a crisp and effective manner. Another important consideration for an edutainment system is the integration of gameplay with content. A base requirement for any game system is that it pass the "fun test." That is, is it enjoyable to play? As such, for this short prototype we were challenged to showcase not just a potential for math education delivery, but also a fun game within a concise timespan. In order to give students at least a taste of how fun the gameworld and game engine could be, we employed a wide range of user activities. For example, although the demo features a very small portion of the Warrick Academy, students playing as Palus do see multiple levels, diverse art and architecture, use stairs, and meaningfully interact with a variety of objects (e.g., potion flasks). The "first-person" style engine allows the players a tremendous sense of opportunity and control as they move through their environment. Although the students' actions are being directed, good game design creates the persistent illusion that the character could "go anywhere" or "try anything." The back-story is literally just the introduction to the larger issues of game content and gameplay.

Software Development

The Phase I and II prototypes of *Warrick's Secrets* were developed as "mods", (modification) to the computer game, *Deus Ex,* that was developed using the popular *Unreal Tournament* game engine. The *Deus Ex* Editor and *Deus Ex* Conversation Editor were used extensively to create the prototype. The only avatar (player maneuverable character) for the Phase I prototype was Palus, a male student character. Additional avatars, to include females and characters of different ethnicity are being developed for Phase II. To achieve the same capabilities as commercial MMPRPG's we plan in Phase III, to incorporate the ability for students to choose the following character attributes: gender, race, facial appearance, and clothing.

Problem Pathways and Hints

A quest in *Warrick's Secrets*, embedded within the storyline and directly involving the action of the characters, illustrates a mathematical concept or skill through a collection of problems, called a problem pathway. The more advanced structure for our Phase II quests is illustrated in Figure 1. The path through a quest is structured based on the student's responses and includes a collection of scaffolded hints that are provided to the student, when necessary. The structure of our Phase II scaffolded hints is illustrated in Figure 2. In order to complete a quest, the student must navigate through the virtual world, interact with the conversation tools, mentor characters, and the objects involved in the quest. Because this is a fantasy environment, interacting with the objects involves conversations and active responses with the quest objects.

Example Quest and Problem from the Phase I Prototype

The Phase I prototype included a basic quest. The collection of problems in the quest was directed at helping students develop or refine the skills necessary to solve problems involving mixtures. The student character, Palus, meets a magical mentor, Adel. The first challenges required him/her to solve a series of problems involving representations of percent. For example, in the introductory meeting between Palus and his mentor, Adel, Palus sees a projection of Adel's image in an entry hallway of the castle. Adel gives Palus instructions on how to navigate in the castle in order to find the room where Adel is waiting for him.

> **Adel Projection:** It is one thing to talk about how understanding the nature of reality affords those with the 'gift' the ability to harness magical energies, and quite another to experience it. Let us consider a sample challenge, and you can demonstrate your knowledge.
>
> **Adel Projection:** Imagine that I give you a collection box like the one projected onto the window of enlightenment behind me. An alchemist uses such a box to help gather and organize various alchemical components. I might, for example, ask you to put a fire moth wing into every bin. After an hour of capturing fire moths in the forest, perhaps your box would look like the projection behind me. What is the percentage of filled cells in the collection box?

If the student was not successful, he or she was provided with a series of hints and provided with opportunities to try again. The next series of challenges required Palus to interact with a series of talking flasks in the laboratory. The flasks helped Palus learn to read the measurements on a flask and, in a progressive manner, helped him/her gradually learn how to solve a mixture problem and learn how to represent it algebraically.

> **Adel:** Welcome to the laboratory. Now that I see you in person, I have something for you. Take this wand lying before me. (After I've told you about it, right-click on it.) The wand's crystal can be treated with a potion to make it shine like a lit torch. The first formula for such a potion was developed long, long ago by a Darslang War-Wizard named Kazuhide. He used it with great

success to search inside deep caverns where gas and winds would extinguish an ordinary torch. It was for generations a closely guarded secret, handed down only to his descendants. But, these days it's a very common potion, and our students call the mixture "liquid stars".

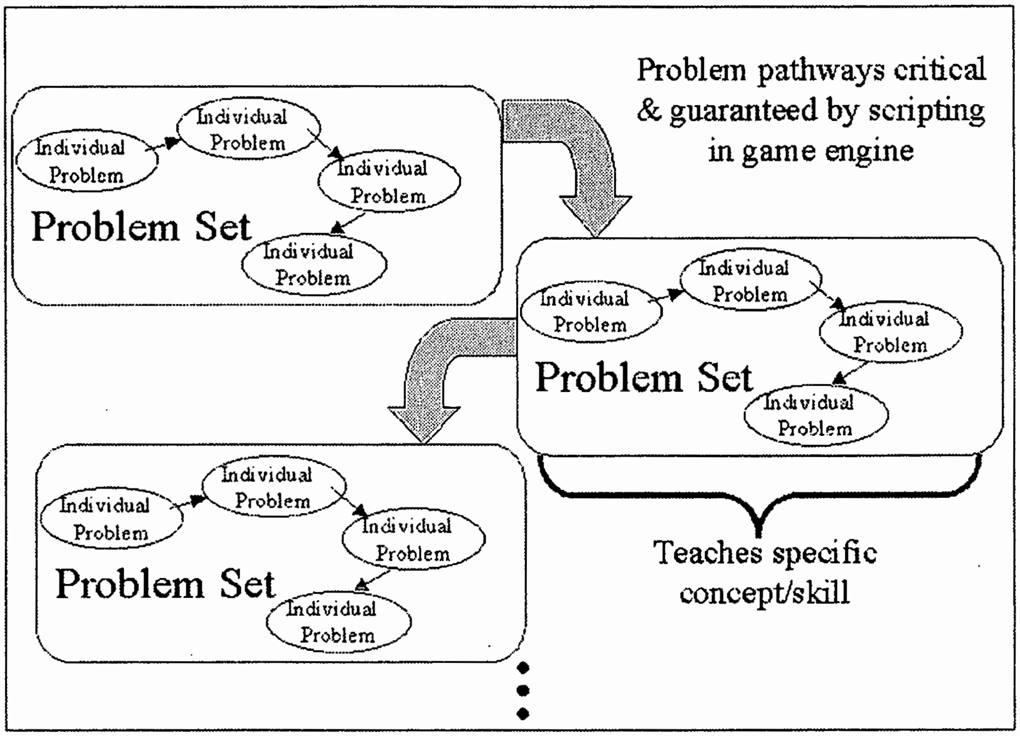

Figure 1.

Adel: To make the liquid stars potion, we start with a fire moth solution—what we alchemists call 'fire-water'. Earlier this morning I ground up a batch of fire moth wings to make some fire-water just for you to work with, Palus. I want you to learn how to make the 'liquid stars' potion, so you can use the wand to produce light when needed. Let us begin with the first flask on the bench. Oh, and these flasks are rather special. They will help you with your potion creation and review of mixtures. Each flask will assist you with your potion creation skills if you get near it, Palus. Now, approach the flask closest to this end of the bench, please. After you have picked up the wand, you should approach each flask in turn.

Action Triggers Flask1: Each fire-moth wing produces 1 mL of liquid fire-water. When I'm full I hold 100 mL. How much fire-water do I currently contain? (Visual representation of flask appears and a collection of multiple choice answers is offered.)

Action: *If Palus gets an incorrect answer, a hint is provided and he is allowed to try again.*

Flask1: Great. You are one fourth of the way done with the liquid stars potion. Let's consider the next step in making your potion. Fire-water can be used in many different potions, but often has to be diluted; so let's move on to the second flask.

Action Triggers Flask 2: I can hold 100 mL and contain fire-water diluted with flying fish oil. If the fire-water solution has a 50% concentration of flying fish oil, then tell me, Palus, how many mL of flying fish oil do I have? (Visual representation of flask appears and a collection of multiple choice answers is offered.)

Action: *If Palus gets an incorrect answer, hints are provided and he is allowed to try again.*

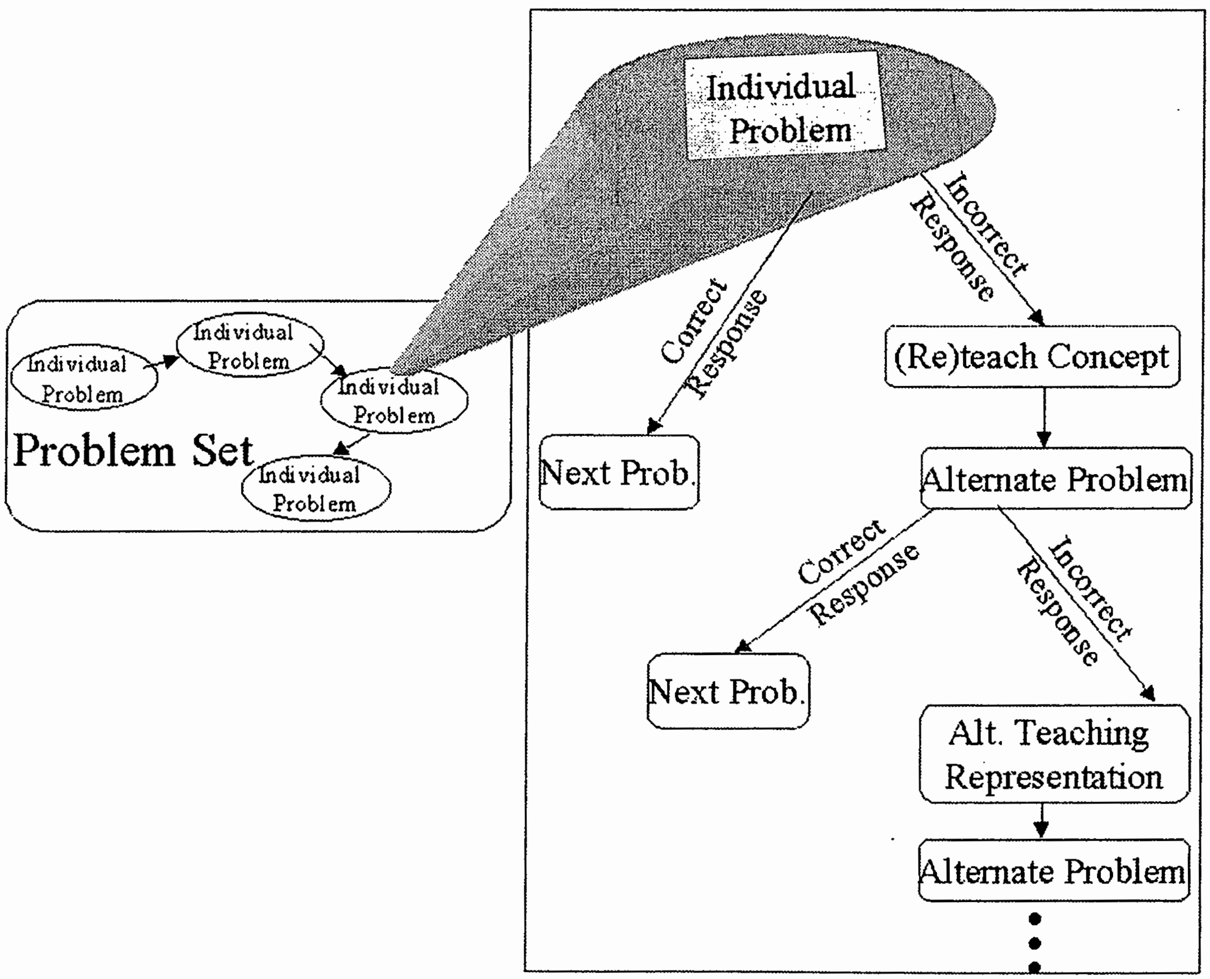

Figure 2.

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Flask2: Excellent work, Palus. We diluted that 25 mL batch of fire-water with an amount of flying fish oil. Let's do the same thing, but in a slightly larger quantity with the third flask.

Action Triggers Flask3: Once again we have a fire-water solution that has been diluted to 50% concentration with flying fish oil. Notice how many total milliliters I contain, and tell me how much flying fish oil I contain. (Visual representation of flask appears and a collection of multiple choice answers is offered.)

Action: *If Palus gets an incorrect answer, hints are provided and he is allowed to try again.*

Flask3: That is wonderful, Palus. You are indeed a very promising pupil. And, we have now done three-fourths of the steps needed for your liquid stars potion. The mixture of fire-water and flying fish oil in even parts is sometimes called "hotfish soup" by the students in the lab. "Hotfish soup" is used in many different potions that produce heat or light, or even fire. Actually, the hotfish soup is still a bit too volatile for just making light, and needs further dilution. So, you should move to the last flask to complete the potion.

Action Triggers Flask4: I have 75 milliliters of liquid, which is hotfish soup diluted with the spittle of glowworms. If there is a 30% concentration of glowworm spittle, then how many milliliters of glowworm spittle do we have?"

Through the use of the backstory, students are provided with a rationale and motivation for completing the quest. As is customary in MMPRPGs, the successful completion of a task results in the player being rewarded with a desirable object or new character skill. The successful completion of Quest 1 results in Palus receiving a lighted wand that is necessary to navigate the underwater corridors required to complete the next quest. The prototype, including this series of challenges, was pilot tested with middle school students, as described below.

Methods

Subjects

The subjects included seven students from a middle school in the southern US. These included two sixth graders, two seventh graders, two eighth graders, and one ninth grader. The subjects included one African-American, six Caucasians, four males, and three females.

Data Sources and Analysis Method

The game prototype testing took place at school on April 6, 2001. The individual sessions each lasted between 45-60 minutes. Students attempted all the problem scenarios developed in the game, completed a questionnaire and participated in a concluding interview. Students were given a pen, paper, and a calculator. The problem sets included two paths called "challenges." The first "challenge" consists of a set of problems involving percents. The second "challenge" consists of a series of progressive problems involving concepts involved in mixture problems. The data sources for the analysis of the pilot study include: (1) videotapes of the prototype testing sessions, (2) student-written artifacts, (3) completed questionnaires, (4) observational field notes, and (5) researcher notes from the post-game interviews.

Results

Summary of Results

- *MMPRPG Environment.* All subjects expressed increased interest in working on mathematics problems that were situated in a computer gaming environment. Students explained that the game made learning math easier for them because it placed the mathematics into a meaningful context for them. Students indicated that they were motivated to solve problems because they wanted to find out what would happen next in the game or because they wanted to gain access to the next magical tool that was described in the story.
- *Tool Use and Concept Development.* The structure of the game and the quests actively encourage tool use in representing problems. Students indicated that they were motivated to persist in problem-solving due to the constant availability of tools, such as the in-game help system. These may be viewed as self-generated representation tools that may have a bridging effect, providing opportunities for students to build on their current conceptions by suggesting seeds of ideas that the students may not have thought of, but may prove useful for exploration in order to solve the problem.
- *Expressions of Student Confidence.* In surveys, problem-solving sessions, and post-session interviews, subjects reported confidence in solving mathematics problems in the game setting but expressed less self-confidence in terms of solving word problems in a classic classroom or textbook setting. All of the students attempted all of the problem scenarios. Two of the subjects were able to solve all of the problems on the first try. Three of the subjects were able to solve all but one of the problems on the first attempt. Because of the visual representations of the problems and the scaffolded hints that could be accessed, students were willing to attempt what they considered to be difficult problems.
- *Role of Graphics.* High quality graphics, in conjunction with an interesting backstory for the game, kept students interested in the game and in working on the challenges. The students indicated that they found the collection bin representations for the initial percent problems to be very helpful in solving these problems.

The students were emphatic that the visual representations were extremely useful to them. The students' interactions with the flasks in the laboratory appear to have influenced their problem-solving success with the mixture-related problems. Students indicated that it was easier to solve the problems because the in-game flasks were displayed with measurement calibrations visible. In the videotapes and post-session interviews, the students indicated that the visual nature of the representations were also useful in reducing their anxieties. The students positively evaluated the hints that were provided for each task.

- *Character Creation.* Students indicated that they enjoyed being an active character in the game and that this improved their ability to interact with the mathematics. Students wanted to control the character's gender, skin color, clothing, ethnicity, character name, and skills. This is a standard option in most MMPRPGs. Students made numerous positive comments about details such as being able to see the characters breathe. They were greatly attracted by maneuvering their character arbitrarily in the virtual world and by character head motion, lateral motion, climbing stairs, walking underwater, etc. Interactivity with objects in the game environment is a crucial factor in gaining and maintaining student interest.
- *Concept Development.* Students demonstrated solution activity that indicated that they had developed their conceptions in important ways. Three of the students were able to complete the second problem pathway by building on their prior solution strategies for the previous problems. The students' problem solving activities and successes suggest that carefully crafted problem pathways can be a successful tool for encouraging students to build on their previous notions.

Conclusions

Development and testing of the Phase II prototype is ongoing. The Phase II prototype includes more complex quest and help system structures, expanded mathematical quests, and additional avatars. A complete MMPRPG will be developed in Phase III and only then can the full benefits of the game genre hope to be realized.

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EXPLORING THE PHENOMENON OF CLASSROOM CONNECTIVITY[1]

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We describe highly generative and affectively powerful classroom activity structures that are made possible by applying new levels of connectivity across diverse hardware platforms. Based on teaching experiments involving core topics in basic algebra (slope-as-rate, linear functions, simultaneous conditions), we examine 3 kinds of activity structures exploiting a common display of student-produced mathematical objects: (1) Construction and sharing of personally meaningful and executable mathematical objects in "mathematical performances," (2) Aggregation and display of student constructions that are systematically varied based on classroom social and/ or physical structure, and (3) The *Where Am I?* aggregation activity structure, where students' constructions are aggregated in ways that require students to perform careful analyses to "find themselves" in the aggregation. Strong learning achievement pre/post-test results suggest considerable promise in such activities, especially with low-performing students.

Context and Aims of the Sstudy

We are currently investigating the impacts and potentials of recent advances in connectivity technology in grades 7-9 mathematics classroom, particularly linking diverse hardware platforms such as the TI-83+ graphing calculators and larger computers.

The work builds on earlier SimCalc research (for summary see Roshelle et al., 2000) which aimed to democratize access to the Mathematics of Change and Variation underlying the Calculus (Kaput, 1994) using a variety of new representations, links to simulations and new curriculum materials. The software enables students to interact with animated objects whose motion is controlled by visually editable piece-wise or algebraically defined position and velocity functions. One form of the software has been developed for the TI-83+ (Calculator MathWorlds) and the other is a cross-platform Java application (Java MathWorlds) which exploits higher screen resolution, with the ability to pass MathWorlds documents between the two platforms—see www.simcalc.umassd.edu for further details. The new ingredient is classroom connectivity that enables students to share mathematical functions across diverse hardware platforms and teachers to collect, aggregate on a common classroom display, and otherwise work with students' constructions on the teacher's workstation, as well as to distribute functions to the students. Hence we combine the two root affordances of the computational medium, representation and communication.

In this paper, we describe the phenomenological space of a SimCalc connected mathematics classroom that arises from preliminary studies in the Spring and Fall Semesters of 2001, and more tightly controlled empirical work currently underway.

Theoretical Framework

Classroom connectivity (CC) opens a large and richly endowed opportunity space for teaching, learning, assessment and curriculum activity design, a space jointly structured by the structures of mathematics, and the social and physical structures of the classroom in a dialectical relationship (Stroup, et al., these proceedings). The social structure plays a direct role in the structuring of mathematical activities, and vice-versa in a dialectical fashion. In some cases, the interplay of social and mathematical structures lead to an elevation of organization of mathematical structure, as when students organized into groups build functions that vary parametrically across the groups, yielding structured families of functions reflecting the structure of the classroom, an elevation of the organizational structure of the mathematical objects, from functions to families of functions. At the same time, the focus of student attention is likewise elevated to the level of what the group is doing rather than what the individual is doing. In addition, for certain of the activities that we explore, students' personal identities are intimately involved in their building and sharing of mathematical objects in the public space of the classroom. While in this brief paper we focus on teaching and new activity structures, we note that an enormous range of foundational educational issues are raised by CC because the radically increased bandwidth of CC, supporting the direct sharing of mathematical objects such as functions, directly affects the heart of what happens in classrooms, among students and between students and teacher. See our Discussion Group paper with Stroup, et al. (these Proceedings) for further discussion of our theoretical framework.

Technological Connectivity

The prototype Navigator network from TI allows us to connect TI-83+ graphing calculators together. As many as four calculators, including the teacher's calculator, can be physically connected to a hub which wirelessly communicates to an Internet gateway in the classroom that in turn communicates with a remote server. This server acts as an active storage buffer between students and teacher. The teacher can also send and retrieve information and display received information on a standard TI-Viewscreen. This information includes Calculator MathWorlds documents. For example, a student can construct or edit a piece-wise defined position function and send it to the teacher who can then display the function, run its animation, and base a classroom discussion on it using the TI-83+ with ViewScreen.

Since Java MathWorlds can pass, collect and collate students' documents, the teacher can aggregate student constructions on the teacher's computer as well, including constructions produced on a computer rather than a graphing calculator and communicated via a standard computer network using standard intranet protocols. Finally, using the ability to pass MathWorlds documents between the two types of platforms, virtually any mix of the platforms can be used in the classroom (or computer laboratory). Our empirical work increasingly exploits these connectivity options.

Data Sources

The ideas and data below reflect work in three venues, where we collected video data, field notes, and student work, as well as pre-post test data in 2 & 3. The mathematical topics across each venue included slope-as-rate, linear functions, simultaneous conditions, and modeling of linearly changing phenomena in both motion and non-motion contexts. In addition, in 2, we covered the full SimCalc curriculum, which involved rate-accumulation connections across a wide variety of rates of change and modeling contexts using graphical, algebraic and numerical descriptions. A fourth intensive teaching experiment was carried out with high-performing middle school students and which focused on peer-peer applications of connectivity, but will not be reviewed here.

1. We piloted the Navigator system in two local high school Algebra I classes in Spring 01 taught by their regular teacher (a SimCalc-experienced teacher). Topic:
2. We used a more stable form of the system in combination with a teacher workstation in a required year-long course taught by the PI at UMass-Dartmouth for 12 academically weak entering College Freshmen (who mathematically and demographically are comparable to typical urban high school algebra students).
3. We used a mix of calculators and computers in a 5 week Spring 02 after-school course for 35 grade 7-9 students in a second local high school taught by a SimCalc novice teacher from that school and assisted by two other SimCalc novice teachers. This course took place in a computer laboratory.

The first two venues helped generate and prototype activity structures that were then used more intensely in the 3rd. And since the 3rd also included pre-post test data where the majority of items were chosen from a required 10th grade state assessment, we will focus on task-types used in this intervention and follow with a brief summary of the results. Space limitations prevent transcript segments from sessions where these activities were utilized, but annotated video will be offered as part of the PME-NA presentation.

Three Basic Activity Structures

We will illustrate three kinds of activity structures, each of which uses the social space of the classroom, and engages students' identities, in a different way. We will, however, limit the examples to topics associated with linear functions and slope-as-rate.

(1) Creating and Sharing a Personally Meaningful Mathematical Object—Mathematical Performances

This is a relatively simple type of activity, but one that we feel has enormous pedagogical potential because of the ways it taps into adolescent students' personal experience, their personal identity, their need for recognition, and their creativity in expressing their unique personal experience. It also serves to focus class attention, which leads to opportunity for intense follow-up engagement by the teacher to exploit issues raised, for pedagogical and curricular purposes. We provide an example that was used across all venues using the TI-83+ version of MathWorlds and, with links to instructional material as well as graphics and student scripts illustrating the activity.

Create an Exciting Sack Race That Ends in a Tie (Slope-As-Rate-of-Change)

We provide students the graph of a constant velocity position vs. time function which controls the (horizontal) screen motion of one object (A, the "car-like box" Above), which has the given constant velocity position vs. time graph in Figure 1 - A travels for 10 seconds at 2 m/sec. With the short stub of a starter-graph for B, we ask the student (1) to write a race-script for an "exciting race" with A; (2) to create a position vs. time graph for B that enacts the race; and (3) Send the race-document to the teacher who replays the race in front of the class on a large-screen display while the student author of the race "calls the race" by reading their narrative script as it runs.

We have seen both a large variety of uniquely personal student creations in response to this task and clear indicators of the "mathematical performance" aspects of the task—for example, in most cases, the classroom audience breaks into spontaneous applause when the race and story are complete. We offer a teacher's model race for simplicity's sake, and, in order to give a sense of how this activity is introduced to students, we embed the race-story in the form of directions to a teacher who is introducing the activity to a class. The teacher raises issues of steepness of line segments, zero slope, negative slope as well as intersection of function graphs and their interpretation as simultaneous position.

Imagine the teacher adding and adjusting one segment at a time (by stretching left or right and "tilting" it to adjust its slope) to produce the composite race shown in Figure 2 while asking a series of questions as follows. (i) Assume that B gets off to the

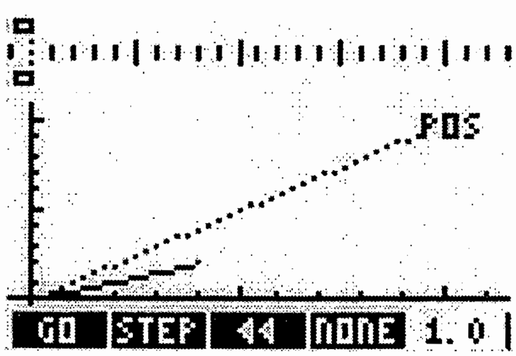

Figure 1.

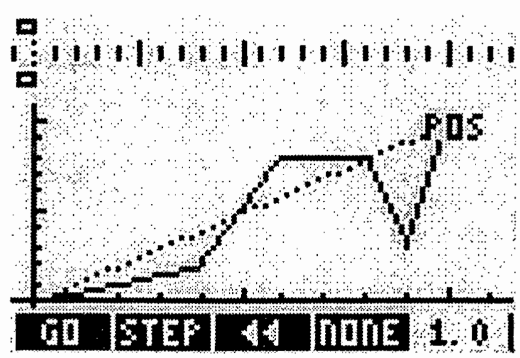

Figure 2.

slow start as indicated by the first segment. Where is A relative to B at the end of that first segment? (ii) We want B to move faster than A and to pass A in a burst of speed. What slope do we need for our new (2^{nd}) segment? (iii) Now, B went so fast he falls down! What kind of slope do we now need so that B does not move for 2 seconds? (iv) A has now caught up and is passing A, and B gets up confused and runs backward! What kind of a segment do I need now? (v) Finally, B gets an amazing burst of energy and finishes the race in a tie! How should I make my last segment? (This raises the issue of what a "tie" means graphically, etc.) Finally, the teacher runs the race and narrates it at the same time, to model what the students will be doing.

(2) Aggregation & Display of Student Constructions, Systematically Varied-Based on Classroom Structure

Here the broad goal of this very general application of classroom connectivity is to generate and examine important mathematical structures and relationships, and to elevate the abstraction-level of mathematical attention from individual constructions to publicly displayed aggregates of these. The underlying idea is to engage students, or groups of students, in building mathematical objects that systematically vary in ways that depend on their place in the social (and perhaps physical) space of the classroom, and then to upload and aggregate these in a common classroom display. One obvious example is the elevation from functions to parametrically varying families of functions. While this vertical flexibility is a powerful pedagogical resource not only for supporting abstraction to parametrized families of objects but for many more general purposes, we will offer only a basic pair of examples.

A Flexible and Generative Group Structure for the Class

Typically, the class is subdivided into groups, where the size of the group is determined by the teacher or activity designer to fit both the given size of the class and the mathematical activity (so the group might simply be the whole class, or each group might have only two members, meaning students are organized in pairs). Then the students count-off inside the group. In this way, each student has a two-number identity that then serves as the value of a "personal parameter" that thus systematically varies across students. The students then create mathematical objects that depend in some critical way on their respective parameter values and then upload these to the teacher where they are aggregated and displayed to the class. Sometimes important variation occurs within a group and sometimes across groups, depending on the activity designer's learning objective and how (s)he chooses to tap into students' identities (e.g., as colleagues, classmates, friends, fellow-sufferers, etc.) Moreover, if members of a group are physically adjacent, then varying the count-off number allows students to see the variation in their group's productions. On the other hand, if we vary group number and not the count-off number, then group members are creating the same object and can help each other, be part of a team, etc. Again, choice of which to vary depends on the goals of the activity.

We will assume for our examples that students are formed into groups with 3-5 members, so each student has a count-off number ranging from 1 to 5, and the number of groups will depend on the size of the class, say 24 in this case.

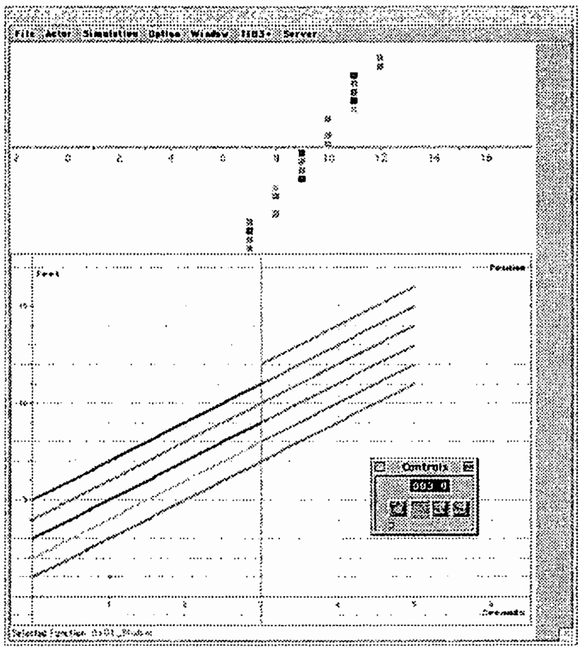

Figure 3.

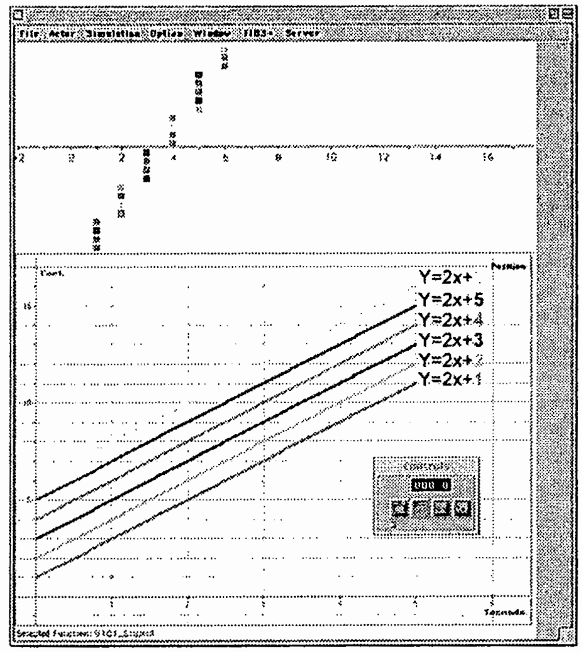

Figure 4.

Linear Functions—The "Staggered-Start, Staggered Finish Race" (varying "b" in y=mX+b)

In the simplest cases, students make a linear position vs. time Y=mX+b function where either m or b is their count-off number. In the latter, they make a 2 ft/sec motion defined by the position vs. time function Y=2X+b where "b" is their count-off number. We give Y=2X as a reference point. (Nobody has count-off number zero, although we can make activities where students subtract, say 3, from their count-off number, so someone gets to have parameter value equal to zero.) The resulting set of parallel lines and staggered starting points help reveal the invariance of slope (2 in this case), and how the systematically varying y-intercept relates to initial position. A companion activity involves using their group number as a starting point, so everyone in a group travels side-by-side, as shown in Figure 3, where we see the screen after 3 seconds of the 5 second race, and all persons in a group travel together. Furthermore, the position vs. time graphs of a given group are coincident, while the respective graphs of the 6 groups are all parallel. Lastly, in Figure 4 we can see the equation of each function and hence the parametric variation reflected in the seven values of "b" in y = 2X+b

Linear Functions—The "Staggered-Start, *Simultaneous* Finish Race"

In this activity, one dot (A) starts at 0 m and travels at 2 m/sec for 6 seconds. Each student starts at 3 times their group number and is to finish in a tie with A. Here each student in a group is solving the same problem, but may do so in many different ways. Furthermore, since the group numbers vary from 1 to 6, the starting points vary from 3 to 18, which means that the slopes of the graphs (see Figure 5) vary from positive, through 0, to negative, with all members of a group traveling together. The coefficients of X vary, along with b, descending by 1.5 as Group number increases from 1 to 6. Group 4, interestingly, starts at the "finish line" (3*4 = 12), has zero velocity, has X coefficient of 0, and has formula Y= 0X+12. This strongly contextualizes Y=12 in a family of functions in 3 ways—algebraically, graphically and in terms of motion (where slope as rate of change is likewise in a central role). Groups 5 & 6 move backwards!

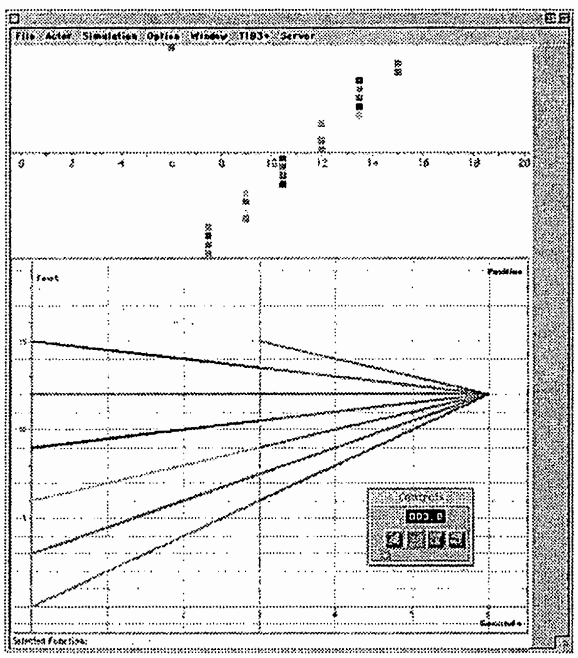

Figure 5.

(3) The *Where Am I?* Aggregation Activity Structure

In this genre of activities, both group and count-off numbers typically are allowed to vary, so each student in the class produces and sends up a unique object. However, the display of the aggregate is deliberately ambiguated to put the student in the position of needing to focus and reason in generally predictable ways to "find themselves" in the common display. We see two sources of pedagogical power in this type of activity: (1) The control of mathematical focus and reasoning based on the specific design of the activity (usually through the variation of representational elements), and (2) The engagement of the student's personal identity at the mathematical heart of the activity via the student's personal projection of their identity into the publicly visible display – students and their peers quickly come to refer to the objects as directly indexing the members of the class, referring to a dot via a person's name, rather than indirectly (e.g, using phrases such as "John is ahead of Mary," or "Is that you?" rather than indirect references such as "John's dot is ahead of Mary's dot," or "Is that your dot?"

Our repeated experience with this activity structure convinces us that it has enormous power to energize a class, to focus students' attention on specific and important mathematical relationships and infuse it with affect based on the fact that students' personal identity is projected into the shared public space . We offer a simple example with linear functions.

Linear Functions—Varying Starting Position (Group Number) and Velocity (Count-off Number)A

Start at your group number & go for 5 seconds at a velocity (whose numeric value is) equal to your count-off number.

(a) Which graph is yours? Explain your reasoning. (See Figure 6.)

(b) Based on *your motion only*, Where Are You? Explain your reasoning. (See Figure7.)

(c) Which formula is yours? Explain your reasoning. (A list of all possible formulas is shown.)

In versions (a) and (c), respectively, students must relate the given initial position and velocity information to vertical intercept and slope of the graphs, or the constants in the formulas. In (b) they must relate the given initial position and velocity information to the motion, with the graphs hidden. Note that the teacher

has control of what information that is made visible to the students, hence can hide the graphs. In figure 6, we have displayed all the functions and representational elements simultaneously. However, we could display the motion with "Marks" dropped on a per-second basis, as shown Figure 7.

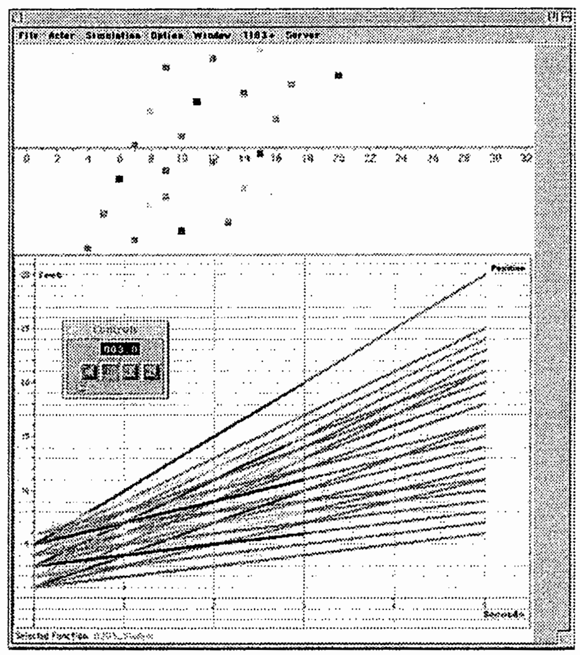

Figure 6.

In Figure 6, we included an "outlier." The potential role of errors is enhanced, as is the potential for student embarrassment—hence the teacher has the option of hiding any functions she chooses. We have seen great excitement and excellent logical reasoning occur as students attempt to track down the author of an erroneously produced object.

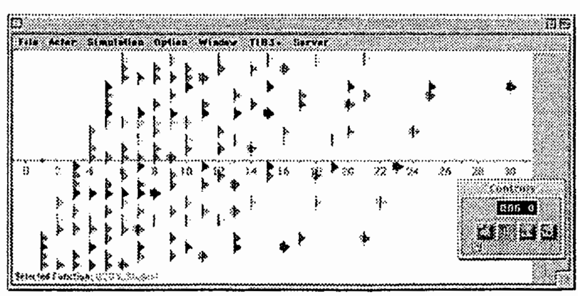

Figure 7.

Results and Conclusion: What Are We Learning?

A pre/post-test comparison for a 15 hour intervention with grade 7-9 students that applied the above activity structures as well as others involving simultaneous equations showed strongly significant gains on a battery of items drawn from a rigorous state 10^{th} grade examination. Two thirds of the students were 9^{th} graders who had previously failed or nearly failed the 8^{th} grade version of the test a year earlier and the remaining students were 7^{th} and 8^{th} grade volunteers. All students gained on almost all items, and statistically strong gains summed across items for each of the groups. Of special note were strong gains on open-ended modeling items which most students find especially difficult.

We are in the very earliest stages of applying classroom connectivity and the illustrations offered above barely scratch the surface of what we foresee. As noted in Stroup, et al. (These Proceedings), the relationship between mathematical and classroom social structure has been radically strengthened, as has the potential for engaging aspects of students' identity and personality. Not only are our traditional expectations regarding classroom technology use being challenged, but our theories and accounts of teaching and learning are being challenged as well.

Note

[1]This work was funded by National Science Foundation Grant # REC-0087771, Understanding Math Classroom Affordances of Networked, Hand-Held Devices. Assertions and conclusions are those of the authors and not necessarily those of the Foundation.

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THE DEVELOPMENT OF STUDENTS' KNOWLEDGE FOR GENERATING, USING, AND EVALUATING ALGEBRAIC REPRESENTATIONS OF LINEAR MOTION

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Twelve pairs of 8th-grade students solved problems about a winch device that instantiated motions that can be modeled by pairs of linear functions. The paper examines how these students generated, evaluated, and refined initial algebraic representations to solve modeling problems. The findings extend previous research on algebra learning by providing further insights into ways that students must refine and reorganize understandings for constructing and using representations when moving from arithmetic to algebraic modeling and problem solving.

Understanding how students learn to represent and solve problems about "real world" situations is of practical and theoretical importance. Representing and solving problems about the surrounding world has always been central to mathematical thinking, and The National Council of Teachers of Mathematics (2000) has published standards that promote connections among mathematics, other disciplines, and the surrounding world. The *Standards* have led to curricula, including several funded by the National Science Foundation (e.g., Coxford et al., 1998; Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), that include modeling activities more varied than traditional word problems about boat trips up and down stream, mixtures of solutions with different concentrations, or other scenarios.

Although recent K-12 curricula place more emphasis on modeling, past research has documented numerous difficulties that students encounter when using algebraic, graphic, and other representations to model various contexts (e.g., Kieran, 1992; Leinhardt, Zaslavsky, & Stein, 1990). In summarizing past research, Kieran argued that learning to model with algebra is difficult because the representations are abstract and because the required operations conflict with those that students have learned to use through years of modeling with arithmetic. Kieran illustrated these differences using the following two problems.

> *Arithmetic Word Problem.* Daniel went to visit his grandmother, who gave him \$1.50. Then he bought a book costing \$3.20. If he has \$2.30 left, how much money did he have before visiting his grandmother? (p. 393)
>
> *Algebra Word Problem.* The Westmount Video Shop offers two rental plans. The first plan costs \$22.50 per year plus \$2.00 per video rented. The second plan offers a free membership for one year but charges \$3.25 per video rented. For what number of rental videos will these two plans cost exactly the same? (p. 393)

To solve the arithmetic problem, students first compute 2.30 + 3.20 = 5.50 to find how much money Daniel had after visiting his grandmother, and then extend this calculation by writing "2.30 + 3.20 = 5.50 - 1.50 = 4.00" to find how much money Daniel had before visiting his grandmother. Three features of this solution are common to elementary-school students' approaches to modeling problems. (a) Students start with the amount that Daniel had at the end and work backward through the sequence of described events. (b) Students do not think that 2.30 + 3.20 and 5.50 - 1.50 represent the same number when writing "2.30 + 3.20 = 5.50 - 1.50." Instead, they interpret the equal sign as a command to compute 2.30 + 3.20 and then continue their calculations. (c) Students describe amounts of money that Daniel had at two particular times with the expressions 2.30 + 3.20 and 5.50 - 1.50.

The following three features of a solution to the algebra problem contrast with those listed above for the solution to the arithmetic problem. (a') Students must work forward from the initial conditions to generate $22.50 + 2.00x = 3.25x$, which in turn requires students to reverse the mathematical operations that they previously used to model situations. (b') The equal sign in $22.50 + 2.00x = 3.25x$ no longer expresses results of calculations, but a relationship between two expressions that holds for a unique value of x. (c') The expressions $22.50 + 2.00x$ and $3.25x$ do not express costs of specific numbers of videos, but rather costs of arbitrary numbers of videos. Kieran's analysis demonstrates how students must modify their existing understandings about generating and using representations when moving from modeling with arithmetic to modeling with algebra. The present study focuses on further modifications that have not been widely reported.

Although students' difficulties modeling with algebra are well documented, cognitive accounts of mathematical learning have focused more often on the construction of concepts than on the construction of knowledge for generating representations and solving problems. For example, an oft cited family of explanations for how students construct core algebraic concepts focuses on *processes* being *reified*, or *encapsulated*, into *objects* (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Sfard & Linchevski, 1994). Recently, a few studies have begun to focus on students constructing knowledge for modeling situations and solving problems. For example, Doerr and Tripp (1999) and Izsák (2000) have focused on equilibration among students' internal conceptual understandings, uses of external representations to solve problems, and understandings of modeled situations. This work has suggested that mathematical concepts are necessary, but not sufficient, for solving modeling problems. The research reported here analyzes in further detail how students can coordinate knowledge of representations and represented situations to construct new knowledge for solving modeling problems with algebra.

This article is grounded in the following definition of modeling:

> *Mathematical modeling* consists of (a) examining various attributes of a particular mathematical, physical, or social context, (b) relating a subset of those

attributes through arithmetic operations, functions, or other mathematical structures, and (c) using resulting representations to solve problems.[1]

Parts (a) and (b) of the definition rely on the processes of *selecting attributes* and *relating attributes* (Izsák, 2000). The former refers to attending to a range of attributes and selecting a subset. The latter refers to combining attributes, for instance through addition or multiplication.[2] Part (c) refers to both internal and external representations. The example of imagined equations, graphs, or other artifacts can blur the distinction between internal and external representations because they are mentally held copies of external artifacts. The important distinction for this paper, however, is between artifacts, available either through imagination or perception, and knowledge for generating and using representations to solve problems. I reserve the word *representation* for external ones commonly used in mathematics or that students construct for themselves.

The Study

In the spring of 1996 and 1997, I interviewed 12 pairs of eighth-grade students attending middle school in a middle-class city in the San Francisco Bay area. By interviewing students in pairs, I hoped to get rich verbal data on what students were doing, and why, without having to interrupt the flow of their problem solving very often. Most pairs consisted of friends who had prior experience working together, but two pairs consisted of students who did not know each other well. Nine pairs consisted of students taking their first algebra course. The course emphasized formal steps for solving different types of equations and inequalities and included a limited number of word problems about linear phenomena. Three pairs consisted of students taking a pre-algebra course that contained some work with variables and functions. Five pairs were girls, five were boys, and two were mixed.

The purpose of the interviews was to examine how students can construct algebraic models without direct instruction from more experienced others. In such cases, students must regulate processes by which they examine problem situations and generate representations with which to accomplish problem-solving goals. Such self-regulation is particularly complex when students do not have established strategies at hand for representing and solving problems.

Each pair of students came after school for four or five, hour-long sessions during which they solved problems about a winch device (see Figure 1) that instantiated motions that can be modeled by pairs of linear functions, a core topic in introductory algebra courses. I adapted the winch from similar devices used by Greeno (1993, 1995), Meira (1998), and Piaget, Grize, Szeminska, and Bang (1968/1977). The device had weights attached to two spools, one 3 inches and one 5 inches in circumference. I refer to these as the 3- and 5-inch weights, respectively. Depending on how the winch was set up, the weights could rise, fall, or move in opposite directions as students turned the handle. A yardstick allowed students to measure heights of weights, changes in those heights, vertical distances between the weights, and changes in those distances.

The problems gave initial conditions for each weight and focused on three types of questions:

(1) Predict the distance between the two weights after an arbitrary number of cranks.

(2) Determine whether one weight will ever be twice as high as the other.

(3) Determine whether the two weights will meet at the same height.

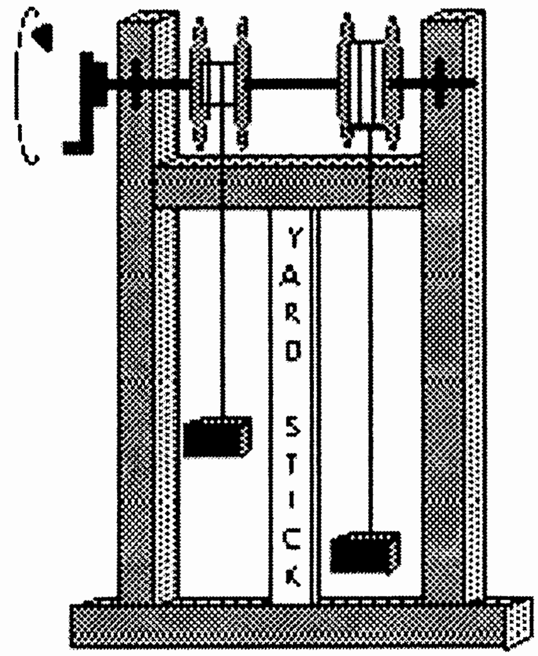

Figure 1. The winch.

Some questions asked students to imagine a larger 100-inch winch. I did not specify how to answer the questions because I was interested in when students introduced and used representations, including algebraic ones, to solve the problems.

A key difference between more traditional algebra word problems, such as the video shop problem discussed above, and problems about the actual winch is that the winch questions did not give any linguistic cues about particular quantities that could be used to generate representations for solving the problems. In contrast, the text of the video shop problem directs students' attention toward the cost of each rental plan individually. Students are less likely to focus on the difference between the two costs and to ask when the difference is zero dollars, although this is a reasonable strategy for solving the problem. Thus, in general, the winch problems provided less scaffolding for students than do more standard word problems because students have to examine the problem situation—the winch—directly and decide for themselves which are the key attributes and how they might be related. Questions about the 100-inch winch did describe in words the initial conditions for each weight. For example, some questions stated that the 5-inch weight started by the 88-inch mark and dropped as the crank turned. In these cases, however, students had already worked with similar motions when answering earlier questions about the actual winch and thus had already built some understandings of the described phenomena by working with the winch directly.

The students engaged in concentrated problem solving during which they discussed the winch, a variety of problem-solving strategies, and refinements of those strategies. When students were working in this way, I intervened only on occasion to clarify my instructions, to ask for further explanation of some comment, or to discuss possible strategies for making progress when students seemed stuck. All students eventually introduced drawings, tables, and/or equations. I used two video cameras—one to record students and the winch and one to record the students' scratch paper.

I performed microgenetic analyses (Schoenfeld, Smith, & Arcavi, 1993) on the interview transcripts, which included notes about students' representations and actions with the winch. In particular, I examined students' utterances, hand gestures, actions with the winch, and written work for evidence of what they were thinking moment-by-moment. More often than not, data in a particular segment of transcript were open to multiple interpretations. I built cases for particular interpretations of individual segments by knitting local accounts into consistent, global ones of the knowledge that students accessed and grappled with as they solved problems about the winch. The level of analysis represents an observer's perspective on what was initially problematic from the students' point of view and what understandings students subsequently constructed.

Analysis and Results

The theoretical framework summarized in Figure 2 emerged from analyses of the data. The frame is informed by theoretical perspectives on learning found in Smith, diSessa, and Roschelle (1993/1994) and Schoenfeld et al. (1993). These authors have described learning as the refinement and reorganization of prior knowledge useful in some contexts. The frame emphasizes the development of modeling knowledge as the emergence, refinement, and reorganization of knowledge for generating, using, and evaluating algebraic representations to accomplish problem-solving goals. The arrows represent reorganizations across the three categories of knowledge. This report focuses primarily on arrow 1, the arrow between *Evaluating Representations* and *Generating Representations*.

All 12 pairs of students answered questions of type 3—determine whether the two weights will meet at the same height—either by turning the handle or by generating equations smoothly. In contrast, students were unsure how to answer questions of types 1 and 2. Four pairs (all algebra students) generated, evaluated, and refined algebraic

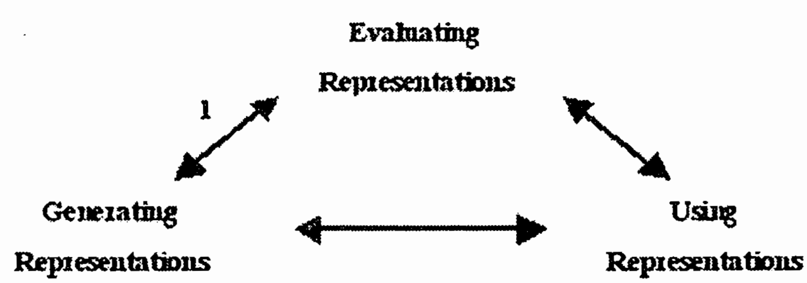

Figure 2. Developing knowledge for modeling with algebra.

representations as they worked on such questions. The analysis below emphasizes understandings that began to emerge as students evaluated and refined initial algebraic representations. The remaining pairs either generated no algebraic representations or generated equations and solved problems smoothly and quickly. Thus, these data did not provide access to students learning to model with algebra.

When the 3- and 5-inch weights started by the 14- and 0-inch marks, respectively, the 5-inch weight caught up to the 3-inch weight by the 35-inch mark after seven cranks. Pair 1 first represented the distance between the weights using $14 - 2n$ before the two weights met, and $2n$ after (See Figure 3. The subscripts in Figure 3 were added to clarify students' correspondences between symbols and winch attributes.). When using the first formula, the students thought of an initial 14-inch distance that decreased and counted cranks from the initial winch set-up. When using the second formula, the students thought of an initial 0-inch distance that increased and counted cranks from the place where the weights met. The students understood when to use each expression, and could substitute values to compute distances that matched measurements on the winch, but the students could not agree that the split approach was adequate for representing distances between the weights. A key understanding that

3-inch weight started by 14-inch mark and 5-inch weight started by 0-inch mark

	Generate: Equations	Generate: Winch Attributes	Use	Eval.
1	$14 - 2n_1$ $2n_2$	14: initial distance apart 2: distance change per crank n_1: # of cranks from initial set-up n_2: # cranks after weights meet	Sub. values	Single equation
	$\lvert(n_1 - 7)\,2\rvert = d$	n_1: # of cranks from initial set-up 7: total # of cranks to meet 2: distance change per crank d: distance apart	Sub. values	

3-inch weight started by 28-inch mark and 5-inch weight started by 0-inch mark

	Generate: Equations	Generate: Winch Attributes	Use	Eval.
2	$.5x = c$ $14 - q = c$	x: distance apart c: # cranks till meet 14: total # cranks to meet q: # of completed cranks	Sub. values	Single equation
	$14 - q = .5x$	14: total # cranks to meet q: # of completed cranks x: distance apart	Sub. values	Include all attr.

Figure 3. Pairs 1 and 2 generated, used, and evaluated algebraic representations for answering questions of type 1, predict the distance between the weights.

the students used for evaluating their split representation appeared to be *single equations*—the criterion that single equations are better than multiple ones. The students then reexamined the winch, focused on a new set of winch attributes, and generated the single $|(n-7)\,2| = d$ equation. Using *single equation* to evaluate and refine initial algebraic representations of distances between the weights is an example of arrow 1.

Pair 1 used a similar *single equation* criterion when working on questions of type 2 (see Figure 4). When the 3- and 5-inch weights started by the 28- and 0-inch marks, respectively, the students first introduced a set of three equations. One equation expressed the height of the lower 5-inch weight, one equation expressed the distance between the two weights, and one equation equated the height of the lower weight with the distance between the weights. The set of three equations expressed correct relationships among the heights of each weight and the distance between them when the 3-inch weight was twice the height of the 5-inch weight. The students questioned their set of equations, however, explicitly stating "we want one equation." They then generated, but also questioned, $0 + 5n = 28 - 2n$ because it was not true for arbitrary numbers of cranks. The students experienced conflict between their *single equation* criterion and *always true*—the criterion that algebraic representations should be true for any value of the independent variable. For example, the students knew that their equation was not true for $n = 1$.

Eventually, after solving their equation and confirming the four crank solution using the winch, the students dropped *always true* in this context. Refining those situations in which to apply *always true* is a second example of students coordinating knowledge for generating algebraic representations and evaluating algebraic representations, arrow 1. This coordination emerged in large part because students were able to use their representation (in this case solve their equation) to solve the problem. Thus the data illustrated the other two arrows in Figure 2 as well.

Pairs 2, 3, and 4 evidenced further criteria for algebraic representations. These students initially constructed algebraic representations by relating all attributes that they found salient and relevant. I will refer to this understanding as the *include all attributes* criterion. Pair 2 predicted distances when an imagined 100-inch winch was set up so that the 3- and 5-inch weights started by the 28- and 0-inch marks, respectively (see Figure 3). These students explicitly articulated the correct understanding that half the current distance between the weights was the same as the number of remaining cranks till the weights met. The students expressed this understanding with the equation $.5x = c$. When I asked if they could use the number of completed cranks instead, the students generated $14 - q = c$, which expressed the correct relationship between the total number of cranks for the weights to meet, the number of completed cranks, and the number of remaining cranks till the weights met. I asked Pair 2 if they could use just one equation—thereby introducing something like Pair 1's *single equation* criterion—and the students generated $14 - q = .5x$. They questioned their new equation, however, apparently thinking that c, the number of cranks till the weights met, should

3-inch weight started by 14-inch mark and 5-inch weight started by 0-inch mark

	Generate: Equations	Generate: Winch Attributes	Use	Eval.
3	$x - 14 + 2n = y$	x: height 3-inch weight 14: initial distance apart 2: distance change per crank n: # of completed cranks y: height 5-inch weight	Sub. values	Include all attr.
	$14 + 3n = (5n)2$	14: initial height 3-inch weight 3: height change per crank n: # of completed cranks 5: height change per crank	Tried to solve	
4	$(14 - 2x = 2b) = s$	14: initial distance apart 2: distance change per crank x: # of completed cranks b: height 5-inch weight s: height 3-inch weight	Sub. values	Include all attr.
	$14 + 3x = 2(0 + 5x)$	14: initial height 3-inch weight 3: height change per crank x: # of completed cranks 0: initial height 5-inch weight 5: height change per crank	Tried to solve	

3-inch weight started by 28-inch mark and 5-inch weight started by 0-inch mark

	Generate: Equations	Generate: Winch Attributes	Use	Eval.
1	$0 + 5n = h$ $28 - 2n = d$ $d = h$	0: initial height 5-inch weight 5: height change per crank n: # of completed cranks h: height 5-inch weight 28: initial distance apart 2: distance change per crank d: distance apart	Did not use	Single equation Always true
	$0 + 5n = 28 - 2n$	0: initial height 5-inch weight 5: height change per crank n: # of completed cranks 28: initial distance apart 2: distance change per crank	Solved	

Figure 4. Pairs 1, 3, and 4 generated, used, and evaluated algebraic representations for answering questions of type 2, will one weight ever be twice as high as the other.

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be represented explicitly because it was central to their understanding of the changing distances between the two weights. Only after substituting for q and calculating x did they realize that $14 - q = .5x$ combined a sufficient set of attributes to determine the distance between the weights. With new understandings about using their equation to determine distances, the students dropped *include all attributes* in this context (a third example of arrow 1 and an example of the other two arrows in Figure 2).

Pairs 3 and 4 evidenced a similar *include all attributes* criterion when working on questions of type 2 (see Figure 4). When the 3- and 5-inch weights started by the 14- and 0-inch marks, respectively, both pairs generated initial algebraic representations that expressed correct relationships among heights of both weights and distances between them. In the case of Pair 3, this is easier to see if one introduces parentheses: $x - (14 - 2n) = y$. These students began with the higher height and subtracted the distance to get the lower height. Pair 4 first equated the distance between the weights with the height of the lower weight and then doubled the lower height to get the higher height, generating in order $14 - 2x = b$, $14 - 2x = (2b = s)$, and then $(14 - 2x = 2b) = s$. The students in Pair 3 questioned the number of variables in their representation, and the students in Pair 4 questioned the number of equal signs in theirs. Although both pairs focused only on heights when introducing subsequent equations that did not have the same problems (see Figure 4), neither pair could solve their equations and so did not understand that their new approaches would work. In their work, Pair 1 may also have used an include all attributes understanding, but they did not express it as clearly. In contrast to Pair 1, the inability of Pairs 3 and 4 to use equations that they had generated may have blocked refinement of the contexts in which they applied criteria. Thus these data also pointed to important interactions between students' knowledge for generating, using, and evaluating algebraic representations as they tried to solve modeling problems.

Conclusion

The analyses above showed that students have knowledge for evaluating algebraic representations, and that they can use such knowledge to refine initial representations as they solve modeling problems. At the same time, the data did not provide access to the origins of criteria like *single equation*, *always true*, and *include all quantities*. A plausible hypothesis is that students construct such understandings through repeated experiences representing and solving addition and subtraction word problems, expressing linear patterns with "rules," and other experiences. For example, most addition and subtraction word problems require students to use all of the included information. The arithmetic word problem about Daniel discussed above is a prime example. Through solving such problems, students may develop an implicit, if not explicit, understandings that all relevant information should be included in representations of problem situations. Data from the present study suggests that students may continue to apply this understanding as they model situations in which subsets of salient attributes are sufficient for solving problems, and in which including all salient attributes, in fact, blocks solutions. Similarly, rules for linear patterns are usually expressed as single formulae that are true for all values of the independent variable. Through repeated experiences with such tasks, students may again develop and implicit, if not explicit, understanding about what makes a good algebraic representation for solving a problem. Students may carry this understanding inappropriately to new problem-solving situations where they need to generate equations that constrain the independent variable to a unique value. These hypotheses about the origins of students' criteria for algebraic representations extend Kieran's (1992) observation, cited above, that students have to refine and reorganize relationships they construct between representations and problem situations that afford algebraic solutions to problems.

I did not collect classroom data as part of this study, and so cannot say whether the interviewed students had had opportunities in class to discuss and compare different relationships between representations and modeled situations. Nevertheless, analysis of the data suggested that more explicit and extensive discussions focused on the ways that representations are generated and used to solve modeling problems could have helped these students to examine their existing understandings for modeling with algebra and to construct new understandings more quickly.

Notes

[1]Doerr and Tripp (1999) provided a similar definition of modeling, and English and Halford (1995) provided a compatible definition in the case where the problem situation is presented through text.

[2]Thompson's (1994) *quantitative operations* are similar to *relating attributes*.

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A STUDY OF THE CONVERSION FROM TABLE TO GRAPH: SELECTING THE TYPE OF GRAPH ACCORDING TO THE CONTEXT OF THE PROBLEM

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This paper presents the results of a research study which had as its main objective to investigate students' conversions from a table to a graph, where the underlying situation provides the meaning and therefore the suitability of the type of graph. We describe the responses of a group of nineteen students in a pedagogical university to a "constructing graph" and a "choosing graph" questionnaire, and to the interviews that followed. The analysis showed the students' difficulties to differentiate between continuous and discrete variables and how to represent them appropriately. We observed in general an overuse of bar charts and polygonal lines. The students also had difficulties distinguishing between the different types of graphs.

Introduction and Theoretical Framework

In many circumstances it is very important to choose the appropriate graph that represents a given real situation. A different type of graph might lead to a wrong interpretation of the information being conveyed. To start with, a continuous (discrete) situation should be represented by a continuous (discrete) type graph. But within continuous and discrete graphs, there are different forms which would be more suitable to a given situation. In a spreadsheet for example, when constructing a graph we need to choose among several types of graphs. Consequently, we advanced a research project which has as its main objective to find out what types of graphs students associate with different types of word situations.

There has been a large research effort studying students' conceptualizations of functional relationships and their difficulties in interpreting graphs (e.g., Chazan, 1994; Dugdale, 1993; Kaput, 1993; Kieran, 1993; Schwartz & Yerushalmy, 1995; Yerushalmy, 1994). Some of these authors have studied how students construct graphical representation from "mathematizing activities." Although in this project we were also interested in the construction of a graph from a given problematic situation, the ultimate purpose was different. We were interested in finding out what types of graphs were selected to represent different type situations and how well they are correlated. We also wanted to observe the difficulties in performing these tasks.

Although math functions (equations) usually correspond to smooth continuous graphs, in real type situations the corresponding graphs could be of many different types. To cite an important example, the SimCalc project, directed by Dr. James Kaput gives a graphical approach to the ideas of Calculus, through an educational software package called "MathWorlds." The basic graphing elements of this software are step and polygonal graphs. Thus, these are two additional types of graphs we considered in this study.

The topic of representations has been given widespread attention in the mathematical education research. As suggested by Duval (1993), the coordination of registers in different representations is crucial for the understanding of the related concepts and is by no means natural. Embedded in this conversion is the issue of congruence. The general problem we are interested in is the "congruence" between the tabular and graphic representations. However, we focus on a different aspect of congruency. Since a table is inherently discrete, it is seldom equivalent to a graphical representation. This implicit lack of congruency can be a source of students' difficulties in converting from one to the other. We need then a description of the situation to aid on the conversion from table to graph. In this research, we study the translation process from a table of values within a word problem to a graph and the effect of the type of situation in the graph selected.

Methodology

We considered in our research five types of problem situations (see the Appendix) according to five different types of graphs: two continuous (a smooth curve and a polygonal line); two discrete (a points graph and a bar chart) and one "hybrid" (a step graph).

We designed two questionnaires. The first one contained five construction type problems in which the most suitable graph was to be drawn according to the situation described (the appendix shows these construction problems). Each problem represented a different type of situation and therefore a different type of graph. The first problem corresponded to a polygonal line. The second problem related to a discrete situation and therefore corresponded either to a point graph or a bar chart. The third problem had intervals where the variable stayed constant and thus a step graph was the best choice. The fourth problem described a continuous situation. The fifth problem was a frequency distribution and thus was best described with a bar chart.

The second questionnaire contained five *election* type problems in which the best graph describing the situation had to be chosen from five graphs attached to the problem. All these five graphs, each one from the five types described above, were designed to represent the problem as best as possible (the questions on this second questionnaire were similar to the ones on the first questionnaire and therefore are not included in this paper).

In all the problems, the statement of the problem was accompanied by a table of values. The purpose of this was to observe if the students restricted themselves to the values given on the table, or completed the necessary information from the problem.

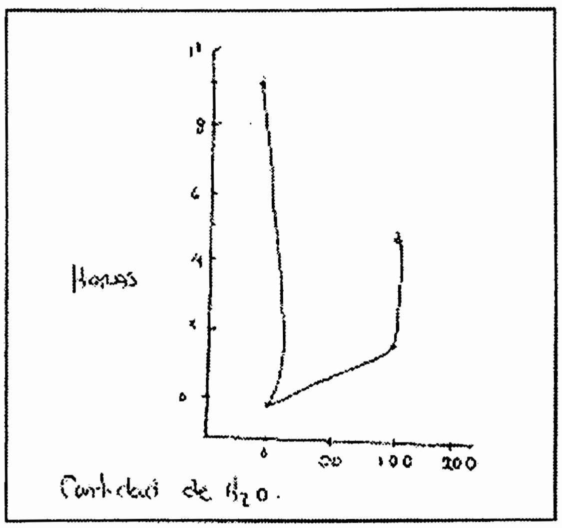
The two questionnaires were applied to 19 students of education at a pedagogical university (future secondary school math teachers) after their first year of study (completing their basic math training). After the answers to the questionnaires were analyzed, five subjects were selected (on the basis of representative and interesting work) for an interview to probe their ideas further.

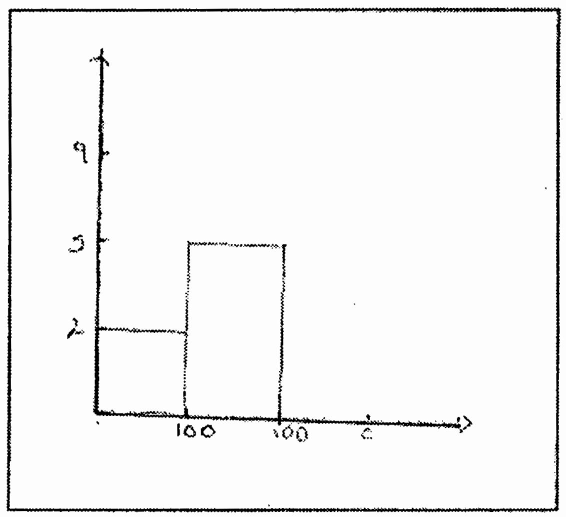
The analysis of the questionnaires and interviews focused on the type of graph chosen, with special attention to the students' difficulties in the conversion tasks.

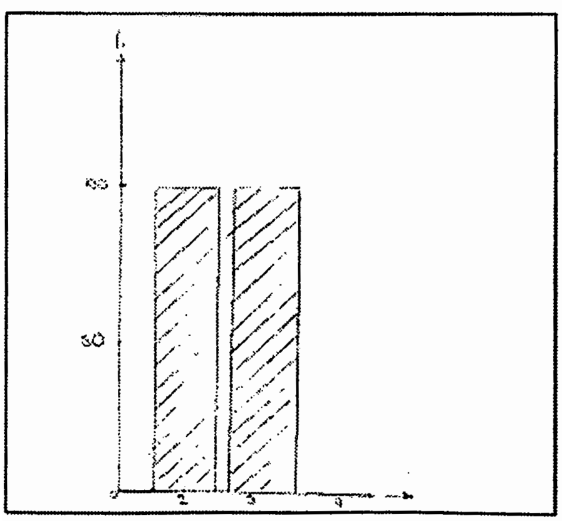
Results and Conclusions

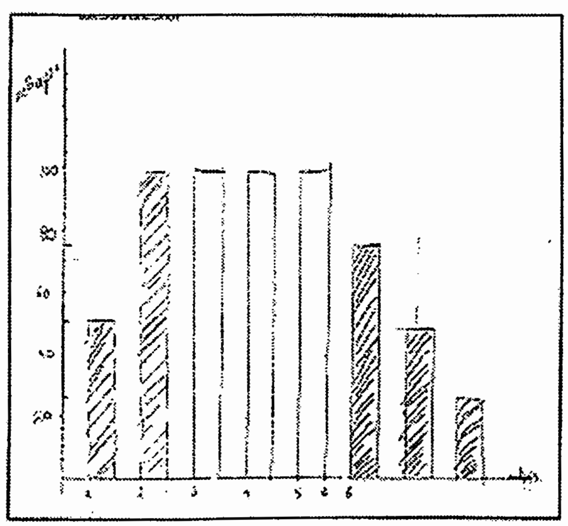
In the first problem of the construction questionnaire (see appendix) the dependent variable (amount of water) is a continuous quantity and the rates of filling and emptying are stated as constants, which means that a polygonal line would be the best choice to represent this situation.

To give an overall idea of the choice of graphs, they were classified into five groups, according to the five types described before. The graphs drawn by the students in this problem were: eight polygonal lines, two smooth curves, two point graphs and seven bar charts.







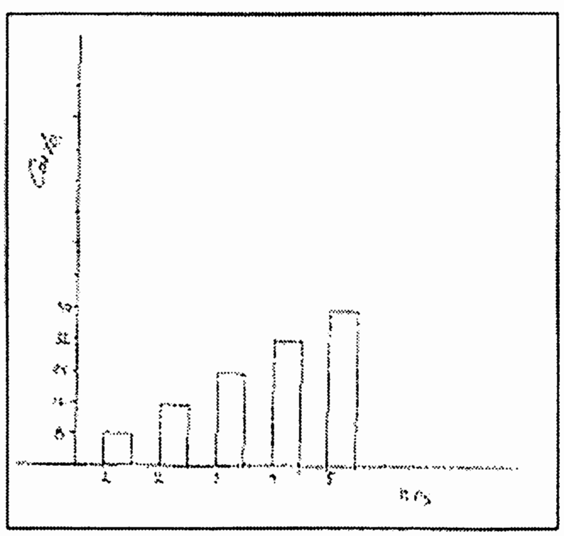


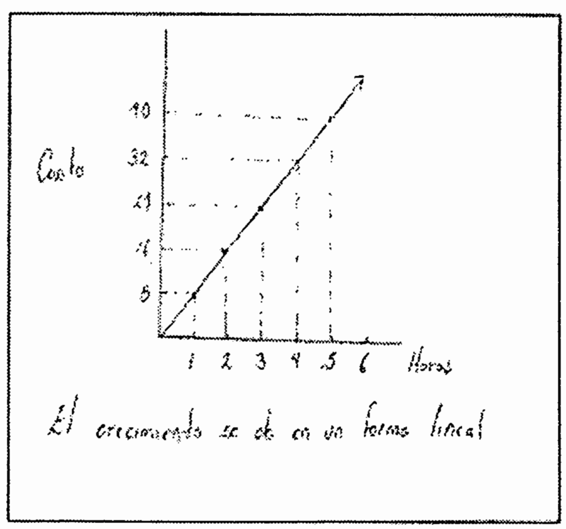
The four graphs shown on this page were drawn by students for this first problem. In the first graph (in the upper-left corner), the student tried to connect the points so that "only one y corresponds to each x" but because the axes were inverted, this was impossible. The second graph (in the upper-right corner) also has the axes inverted. The values of the amount of water were taken only as labels for the x-axis (0, 100, 100 and 0), so it is evident that the information from the statement was ignored and the graph was draw from the values of the table alone. The other two graphs can serve as contrast. The one in the bottom-left corner was drawn taking into account only the data in the table. The other one, in the bottom-right corner, was constructed by adding important information from the statement of the problem. It is evident that the first three graphs are far from representing the situation they originated from. In other words, it would be almost impossible to reproduce the problem situation from the graph. The fourth graph on the other hand, although not the "correct" type, conveys the situation much better.

In general, only very few of the graphs within the polygonal type (the correct choice for this situation) represented the problem posed in an adequate way.

The second problem has a characteristic discrete time and therefore either of the two discrete graphs (point graph or bar chart) would fit the problem. The graphs drawn by the students in this problem were: 12 bar charts, five polygonal and two with points. This is almost a "perfect" match since polygonal lines are also often used to represent discrete variations. We have to mention however that some of the bar charts are confusing to read since their labels were put in the boundary between the bars (like a real axis) and therefore it is not clear which of the two adjacent bars gives the correct value.

The third problem is better represented by a step graph because the time is considered as continuous but the charges of money made are constant through each whole hour. The graphs drawn by the students in this problem were: nine bar charts, seven polygonal and three with points. A complete mismatch. Two of the graphs are reproduced below.





The graph on the left is a bar chart and the one on the right a polygonal that turned out to be a straight line (the student even wrote, "The growth goes in a linear manner"). We emphasize again that these two graphs, as all the others drawn for this situation, were bad descriptions of the problem being represented.

The fourth problem refered to a situation in which the quantity of fertilizer can change continuously, so a smooth graph was the best representation. The graphs drawn by the students here were: seven polygonal, six bar charts, three smooth curves and two with points. Again a bad mismatch. This is very surprising because continuous graphs are used constantly in math and science instruction, so we would expect the students would prefer these type of graphs. It is also interesting to observe that a third of the graphs in this problem had inverted axes. This hints again that the students based their graphs from the table of values and not the statement of the problem, because only from the latter can be extracted which is the independent and dependent variables.

The fifth problem is a frequency distribution of a discrete variable, so either the bar chart or the point graph would be the best choice. The graphs drawn by the students here were: 10 bar charts, 7 polygonal and 2 with points. This is again almost a perfect match (considering the polygonals as acceptable).

Thus, we can observe in general an overuse of bar charts and polygonal lines. However, both of these graphs were used indistinctly in all types of problems and not with the expressed intention to match the behavior of the situation. Due to this, we can observe that the discrete problems had a much better proper matching than the others. The worst matching was done in the smooth and step type problems.

The following table summarizes the totals of the types of graphs selected in all problems combined. The first remark in the above paragraph (an overuse of bar charts and polygonal lines) can be seen clearly in the results of this table. As we mentioned before, surprisingly enough, there were very few smooth continuous curves. Step graphs were really not known by the students, although many real life situations behave in this fashion.

Type of graph selected (Totals):				
Smooth	Polygonal	Points	Bar chart	Step
5	**34**	**11**	**44**	**0**

The results of the second questionnaire, the election type problems, are summarized in the following table:

Type of graph elected (Totals):				
Smooth	Polygonal	Points	Bar chart	Step
19	**17**	**11**	**36**	**6**

It is interesting to contrast these two tables. When the students were shown the five types of graphs associated with each problem (second table), they elected in a higher proportion than before the smooth curves. Also, the step graphs were chosen some of the times. This obviously produced a drop on polygonal and bar charts selected, compared with the previous results. These numbers suggest that smooth curves are not drawn naturally, but, if they are provided as options, can be recognized as better descriptions of continuous problems (on the continuous type problem, half the students elected the continuous graph as the best choice).

The graphs drawn in the interviews followed the same patterns summarized above in the questionnaires. However, there were some interesting additional observations that are described below.

We noticed that there is confusion between discrete and continuous variables. In fact, when continuous graphs were read by the students, they only paid attention to their integer values. Surprisingly enough (because in math instruction, graphs are generally smooth), in continuous type problems, the students often used point graphs or polygonal lines. Two students remarked: "I don't use continuous curves" and "I don't like to draw curved graphs".

The students commented that they don't see a difference between the smooth and the polygonal graphs. Actually, they explained that they preferred the polygonal graphs because they could see the changes better. (One student commented: "I join the points to see if it moves back," and "in the curve, I don't find well the points.") Also, they did not distinguish between the polygonal and the points graphs. One student said: "... is like the polygonal but you don't join the points."

With respect to step graphs, the students have difficulty conceiving horizontal parts of graphs. They actually perceive them as bar charts (several students commented that "... it is very similar to a bar chart"). When they were given a different situation with a constant rate (like a velocity), they tended to graph instead its accumulation (the distance).

We studied here the connection between the statement of a problem, an attached table, and the corresponding graph. We observed throughout this study that there is confusion between discrete and continuous variables and how to represent them correctly (the discrete type graphs are clearly overused). This is due only in part to fact that part or most of the information about the variation of the quantities is contained in the statement of the problem and this is often ignored. But more importantly, even when taking into account the statement, the students did not know how to differentiate between the different types of graphs.

Although this research was exploratory in nature with subjects taken from a group of future teachers, we expect these same results for math teachers in general. Thus, it suggests already that in instruction, we don't pay enough attention to distinguish between the different types of graphs available and how these correspond to different types of situations. These are very important ideas that need further research with many kinds of subjects, because we constantly use graphs to convey information, and we should be able to reconstruct the situation entirely from the corresponding graph.

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Appendix

In this appendix, we reproduce the five problems used in the construction questionnaire (the tables in the real questionnaire were vertically oriented instead of horizontally as done below).

The five figures in the next page show the corresponding best fitting graph for each of these five problems.

1. A tank to store water is initially empty. During two hours, it is filled at a constant rate of 50 m^3 per hour. In the next three hours it is maintained closed. Right after this, it is emptied at a constant rate of 25 m^3 per hour. The next table registers the amount of water in the tank:

Time (hours):	0	2	5	9
Amount of water (m^3):	0	100	100	0

2. During seven days, there was an exposition of the work of a sculptor. The number of people attending the exhibition hall was registered every day. The following table shows this data:

Day:	1	2	3	4	5	6	7
Number of visitors:	531	689	203	100	50	45	200

3. In a parking lot, the charge is $8.00 for an hour or a fraction of it. Thus, if the car is there for less than an hour, the charge is $8.00; if the car is there up to two hours, the charge is $16.00 and so forth. This is shown in the following table:

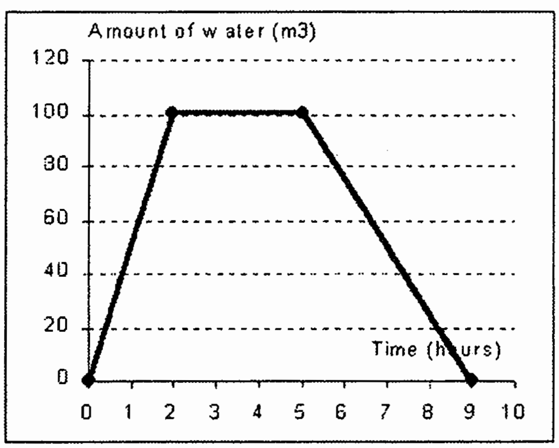
Complete hours:	1	2	3	4	5
Cost ($):	8	16	24	32	40

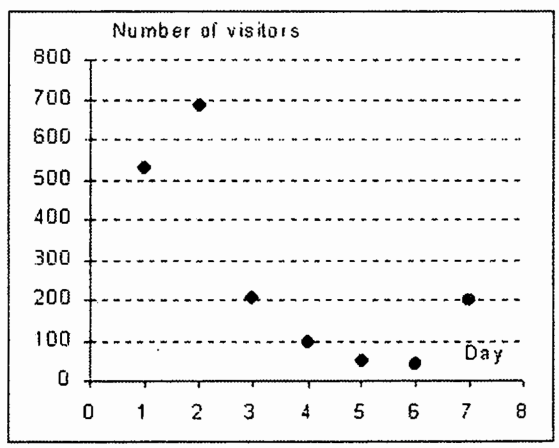
4. A farmer wants to know how much fertilizer is the optimum for his field of tomato plants. For this, he applies different amounts of fertilizer and counts the number of tomatoes produced. The results he obtained are shown in the table below:

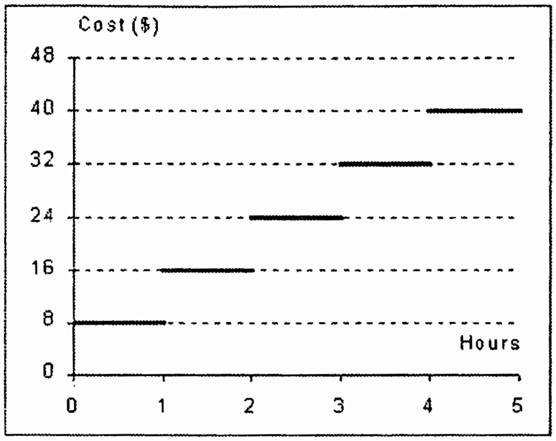
Amount of fertilizer:	0	100	150	250	450
Number of tomatoes:	300	425	600	800	650

5. In a demographic study, a poll was taken to 50 women about the number of children they bear. The data registered is given in the following table:

# of children:	0	1	2	3	4	5	6	7	8
# of cases:	1	3	6	7	9	7	8	4	5






Cost ($)
48
40
32
24
16
8
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Hours
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1
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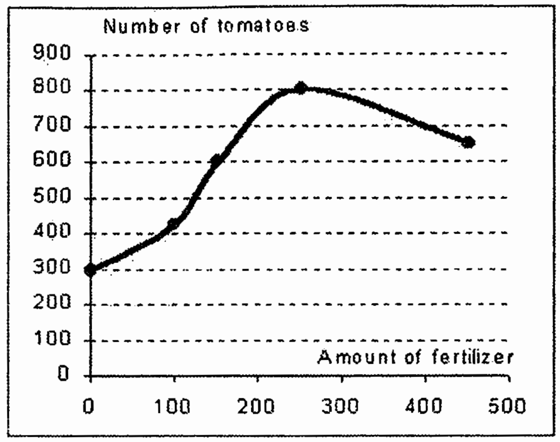
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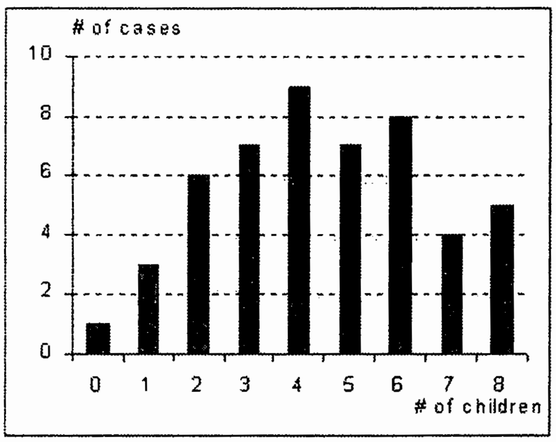
700
600
500
400
300
200
100
0
Amount of fertilizer
0
100
200
300
400
500


# of cases
10
8
6
4
2
0
0
1
2
3
4
5
6
7
8
# of children

SIMULATION OF AN ACTUAL CONTEXT AND ITS ALGEBRAIC REPRESENTATION

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The main purpose of the research study presented in this paper was to analyze how simulating an actual context and the coordinated work between the simulation and the tabular and graphic representations of a concept modify the meaning of the algebraic representation of that concept. For that purpose, we chose the concept of function, emphasizing the concept of linear function. We developed specific software for this research. This software simulates an actual context—filling containers—and allows the interaction with the graphic and tabular registers of representation. Duval's contributions to the study of representation registers of a mathematical object were a fundamental basis for designing the software and the research instruments. (Duval, 1998; 1999) One outstanding result was that most participants in this research study, through the activities of simulating the actual situation of filling containers, could give meaning to the slope of a linear function.

Theoretical Background

The characteristic feature of mathematical concepts, to be considered as such, is their high degree of abstractness. For instance, the mathematical concepts of variable and function can be considered as "the abstract generalization of concrete variables (such as time, distance, velocity, angle of rotation, and area of surface traced out) and of the interdependences among them (the distance depends on the time and so forth)" (Aleksandrov, Kolmogorov, & Lavrent'ev, 1956/1999, p. 43). Hence, the conceptual appropriation of a mathematical object is achieved by identifying its distinctive properties. However, we can only have access to a mathematical concept by means of its representations. These representations themselves reflect only part of the properties of the concept.

Duval called this contradiction a cognitive paradox of the mathematical thought. He stated his proposal to solve it as follows: "The coordination of various registers of semiotic representation appears itself as fundamental for a conceptual apprehension of objects: it is necessary that the object be not confused with its representations and that it be recognized in each of its possible representations." (Duval, 1998, p.175) The coordination of registers of representation refers to mobilizing several registers of semiotic representation, being able to choose one register instead of another, and switching from one to another representation during the same intellectual course. (Duval, 1999)

From the viewpoint of cognitive activities propitiated by representations, a semiotic system is a register of representation if it makes possible the following three activities:

> The formation of an identifiable representation based on a selection of features and data of the content to be represented.
>
> The treatment of a representation, which is the transformation of another representation, in the register itself where it has been formed.
>
> The conversion: transforming a representation of one register into the representation of another register keeping the whole or just a portion of the content of the initial representation. (Duval, 1998, pp. 177-178)

Incorporating computers, as an additional element, into the teaching and learning of mathematics processes opens new perspectives for students to achieve the conceptual apprehension of mathematical objects. Furthermore, the use of computers as a means to representing mathematical objects has made concepts to obtain a more diverse and deeper meaning. Access to representations has been widened (Tall, 1996). Computers have the capacity of bringing about treatments in representations, such as the graphic and tabular ones, to the level where such treatments acquire sense and meaning beyond the simple exemplification of static forms of algebraic representations. In a great variety of systems of representation, the action of the user changes the physical state of the representation (Kaput, 1994). 'Complex' activities of conversion with paper and pencil can also be carried out in a more accessible way (Mejía, 1996). From these considerations, the simulation of an actual context has been interpreted as a register of representation of one concept: the actual register.

The Simulation Tool

The actual context is implemented by the software called Recipientes [Spanish for Containers] (Monzoy, in preparation). The simulation of filling containers—a rectangular prism, a cylinder, and a truncated cone—is built into this software by establishing the geometric characteristics of the variables to be studied: filling volume, area of the transversal section, and filling level. Additionally, from the implicit mathematical model of the established functional relations in the simulation, the software implements the activities of identification, treatment, and conversion in the tabular and graphic systems. A typical window is shown in figure 1.

The activities are carried out in the registers by the intervention of the user. For instance: (a) in the actual register, it is possible to modify the dimensions of the containers, whether with the mouse or by assigning values to them; (b) in the tabular register, the menu option Tools—Analyze Table opens a window in which the quotient of increments can be computed, and the extreme values of the table and the increment in the column corresponding to the filling level can be modified; and (c) in the activities

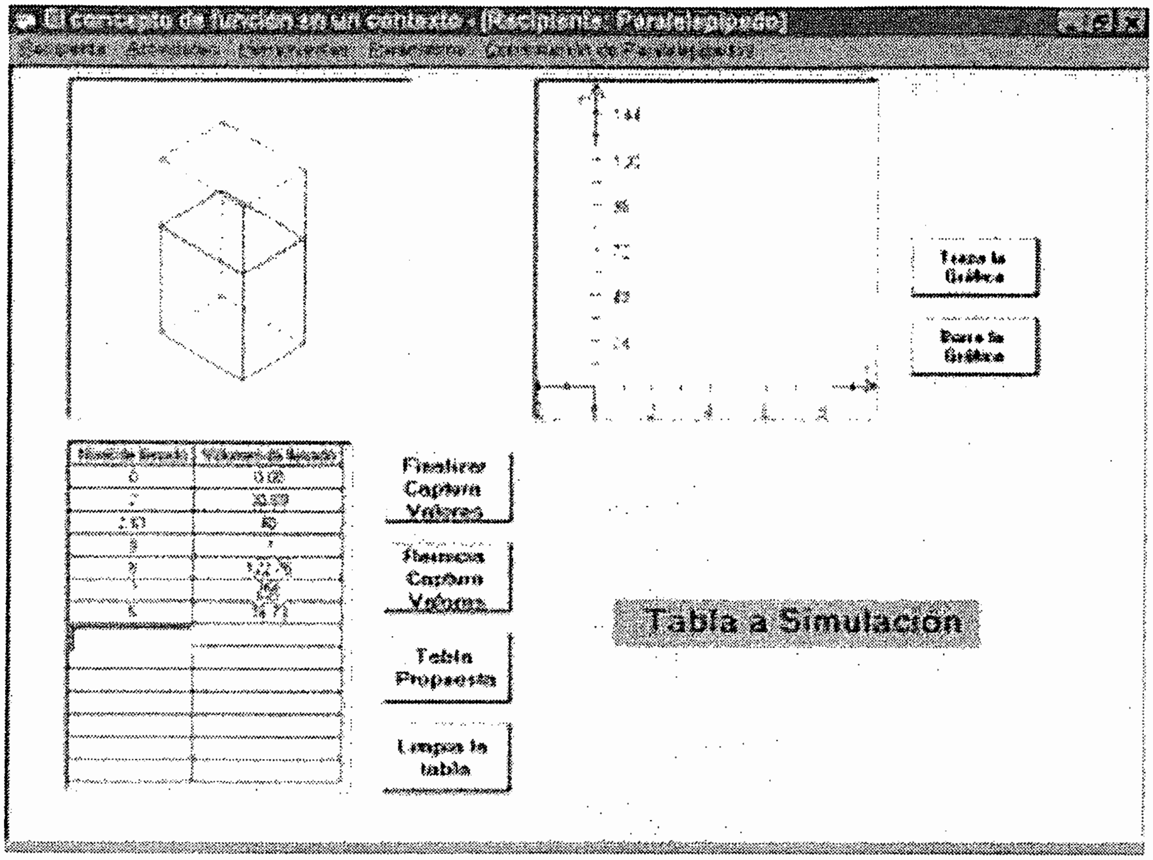

Figure 1. A window of the software Recipientes.

of conversion, the user can choose the two involved registers so that the activities in the initial register are manifested in the final register.

Methodology

The research study was carried out with 15 students (ages 15 to 17) who had completed the first and second grade courses of mathematics in higher secondary school (grades 10 and 11). They had studied the algebraic representation of the concept of linear function. Most of them had not passed in the second grade course of mathematics and, according to the teacher, “they were close to” passing in the course.

These were the stages of the research study:

1. Application of a test before the work with Recipientes. This pretest contained questions about identification in the tabular and graphic registers, and about situations such as obtaining the algebraic expression from a table, from the graph of the linear function, or from the representation of a physical context in a coordinate system (distance traveled by an automobile). For instance, two of the questions were:

(a) The values of a linear function, with independent variable x, dependent variable y, and domain the interval [-4,4] are given in the following table. Write the algebraic expression of the function.

Table 1

X	Y
-4	-7
-3	-5
-2	-3
-1	-1
0	1
1	3
2	5
3	7
4	9

(b) The two graphs plotted in the coordinate system represent the relation of distance to time, of the movement of two cars: car A and car B. The distance is measured in kilometers and the time, in hours.

- Obtain the algebraic expression for the straight line that shows the movement of car A.
- Obtain the algebraic expression for the straight line that shows the movement of car B.

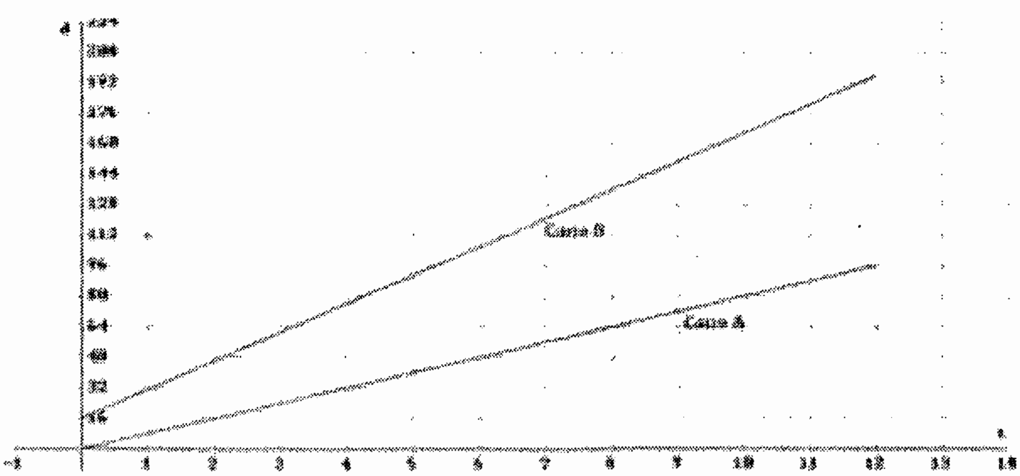

Figure 1

2. Students worked with the software *Recipientes* during four sessions of about 2 hours and 20 minutes each. The activities in these sessions were done using work sheets. During the first two sessions, the *training* tasks with the software were carried out. For example, participants had to obtain the dimensions of a parallelepiped, being given a table representing the relation between the filling level and the filling volume. During the third session, the instructor explained the terms 'independent' and 'dependent variable' and showed a procedure to obtaining the algebraic expression from a table and a graph. In the next work sheets, tasks that required a more open work with the software were indicated. That is, activities such as obtaining the algebraic expression of a linear function from the conditions of a container or from a table or a graph; or obtaining the dimensions of a parallelepiped from an algebraic expression which modeled the process of filling the container. An example of this type of questioning is the following:

   It is known that the algebraic expression that establishes the relation between the independent variable h and the dependent variable V of the filling process of a parallelepiped is

   $V = 35\ h$, with domain [0,10].

   Obtain the dimensions of a parallelepiped that satisfies the given algebraic relation of the filling process. Plot the corresponding graph and indicate the range of the function.

3. Application of a test after the work with Recipientes. This test was elaborated essentially the same way as the pretest.

4. Interviews with three participants. The purpose of the interviews was to confirm some results regarding the use of the software, and the responses given by the participants in the work sheets were the basis for formulating interview questions.

Analyses of Results

The activities of converting a table or a graph into an algebraic expression in the work sheets were done correctly by 60% of participants. Those who did not obtain the correct answer could obtain the value of the slope although they made mistakes in the algebraic operations.

The activities of obtaining the dimensions and the graph given the algebraic expression of the filling process of a container were done correctly by all participants. These results and the way in which participants obtained the answers suggest that the work with the simulation of filling a container did not provoke inconsistencies and that participants adopted it naturally for working with the concept of linear function.

Regarding results of the pretest and the posttest, there was a considerable increase in the number of participants who answered correctly the questions of the posttest referring to obtaining the algebraic expression of a linear function associated to condi-

Table 2. Results From the Pretest and the Posttest

Initial register	Final register	Pretest	Posttest
Tabular	Algebraic	0	7
Graphic	Algebraic	0	7
Graphic (context modeled by a graph that crosses the origin)	Algebraic	0	9
Graphic (context modeled by a graph that does not cross the origin)	Algebraic	0	6

tions given by a table, a graph, or a context. Results as to number of correct answers are shown in table 2. The situation that participants did not answer correctly the questions of the posttest but could compute the slope, making mistakes in algebraic manipulations, was different from the situation in the pretest, in which only two participants tried to answer the analogous questions.

The interviews with three participants were video-recorded for a better retrieval of information for this research. These interviews were done to precise how the actual register was used for obtaining the algebraic expression. Questions used for the interviews were based on results written in the work sheets. Interviewees had access to the software Recipientes during the interviews.

When we asked the first interviewee how she had obtained the algebraic expression from a table, her answer was not given immediately and, pointing to the monitor screen, she said: "The height and the filling level are used and they are divided."

Even though her assertion is not correct, she tried to base it on the actual register. Then, to the question "What data would you look for in the graph to find the quotient?", regarding how to obtain the algebraic expression from a graph, this first interviewee answered: "The value of x, the value of y."

Thus, she was asked to use the software to find the values she mentioned. She used the window of simulation to reflect it on the graph; she clicked on the filling level of the container and indicated that that is represented in the graph. She did not remember the procedure for obtaining the algebraic expression.

When she used the window of simulation, she tried to base her work on more concrete situations; she could obtain these from the actual register. Moreover, she did not restrict herself to working in a single register. That is, although she was asked to establish the algebraic expression from the graph, she used simulation.

The second interviewee was also asked about the procedure for obtaining the algebraic expression from the information given in a table. She answered: "I did not use the software because what was asked for was algebraic. Thus, I took two values… and so on." She pointed to the procedure used in the work sheet, which was correct. Then, she said: "I did use the software to see the quotient of differences, which is 20 in this case." Therefore, in this situation she used the software to check her results, doing activities of treatment in the table.

Then, when this interviewee was asked how to obtain the algebraic expression from a graph of the linear function, shown on the computer screen, and which corresponded to the work sheet, she answered: "It's the same but in another way. I took two values... First, I searched with the software... the dimensions of the container. I took the values from there. [She pointed to the table on the monitor screen.]

The behavior of this participant during the interview allowed to observe that the actual situation simulated with the software was an important basis for obtaining the data required to find the algebraic expression. Moreover, she showed that her activity with the software was not limited to a single register, but, on the contrary, that she was able to use the different registers available in the software. She used these different registers to check her assertions or to obtain the data she needed to use.

The third interviewee was nervous during the interview in spite that she knew well how to handle the software, in contrast with the other two interviewees. As to the question on how the algebraic expression is obtained from a table, she answered: "You take two values and make subtractions with those beside them." Then she explained the operations she had carried out to find the answer required in the corresponding work sheet—her answer was correct. Following, she was asked how to get from the graph the data needed to obtain the algebraic expression. She indicated: "The points with which the graph is plotted." The interview process was stopped because, on the one hand, her answer in the work sheet was correct and, on the other hand, the interviewee was getting increasingly nervous.

From the analyses of the interviews, we can conclude that participants exhibited a wider and more consistent interpretation of the algebraic expression from the filling of containers. Besides, when using the different registers built up with the software Recipientes, and selecting the register which was considered the most appropriate to solve the problem, they exhibited that they did coordinate the actual, tabular, graphic, and algebraic representations.

Discussion and Conclusions

No contradictory results were found in the analyses of the answers—right and wrong—to the pretest and the posttest, the responses in the work sheets, and the interviews. Hence, working with the software Recipientes—being one of its features the incorporation of the simulation as a register of representation—it is possible to obtain a wider and consistent meaning for the algebraic representation of the concept of linear function.

Results from the analyses of the work sheets and the answers to the posttest, which were verified through the interviews, also showed that the software Recipientes made it possible for participants to get a more concrete meaning of the slope of a linear function. Additionally, the interviews showed that the simulation of filling containers was an important support for the participants' assertions. This situation suggests that this physical referent supports the coordination of the actual, tabular, and graph registers of the concept of function and allows a favorable modification of the meaning of its algebraic representation. Besides, the relation between the variables and the dimensions of the containers did not provoke confusions.

This didactical experience afforded evidence that the incorporation of the simulation of an actual context as a register of representation is very helpful. Now, it is necessary to investigate whether the simulation of different actual contexts representing the same concept supports generalization aspects.

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REPRESENTATIONAL FLUENCY IN MIDDLE SCHOOL: A CLASSROOM STUDY

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This study investigated seventh- and eighth-grade students' ($N = 90$) abilities to solve problems using tabular, graphical, verbal, and symbolic representations and to translate among these representations before and after instruction. Students experienced more success solving problems using a given representation than translating among representations, with success strongly influenced by the particular representational formats involved. Performance improved with instruction, with the greatest gains occurring with an experimental curriculum (*Bridging Instruction*) that explicitly built on students' invented strategies and representations. Results suggest there are significant gaps between students' abilities to comprehend and to produce representations, that students may attain fluency with instance-based representations (tables and pointwise graphs) before holistic representations (symbolic equations and verbal expressions), and that instruction that explicitly bridges from students' intuitions about quantitative relations can enhance students' abilities to work within and translate among various representations.

One important aspect of mathematical competence is the ability to reason with and among multiple representations. The National Council of Teachers of Mathematics (2000) calls for an increased focus on a variety of mathematical representations, including graphs, tables, symbolic expressions, and verbal expressions, as well as the interconnections among them. These skills, which we call *representational fluency*, are growing in importance as the mathematics education community struggles to reform algebra instruction and curricula.

Despite the documented need for and benefits of representational fluency (e.g., Kaput, 1989), little is known about students' abilities to solve problems presented with different representations or to translate among different representations. This study investigated these issues as well as the impact of a theoretically guided approach for classroom instruction aimed at the development of algebraic reasoning in the middle grades. Specifically, we investigated 4 research questions:

1. How is initial student problem-solving performance influenced by representational format?

2. How is representation use during problem solving influenced by instruction?
3. How well can beginning algebra students initially translate among representations?
4. How is fluency among representations influenced by instruction?

Theoretical Framework

Motivated by the need for sustained, in-school research on students' representation use, this study was carried out in a classroom setting. It compared the influences of a theoretically guided experimental curriculum (described below) and *Connected Mathematics* (CM; Lappan, et al., 1998a, 1998b), a reform-based middle school mathematics curriculum shown to be effective (Hoover, Zawojewski, & Ridgway, 1997). CM emphasizes collaborative, discussion-oriented activities that use data gathering and representation as well as problem solving to make mathematics conceptually meaningful to students. Because it also addresses connections among mathematical representations, and because it was widely used at the school site under investigation, CM served as the control curriculum for this study.

The experimental curriculum implemented in this study, *Bridging Instruction* (BI), grew out of prior research on students' algebraic reasoning that showed the prevalence and power of students' informal and invented problem-solving strategies (Hall, Kibler, Wenger & Truxaw, 1989; Kieran, 1988, 1992; Koedinger & Nathan, 1999; Koedinger, Alibali, & Nathan, 2002; Nathan & Koedinger, 2000a, 2000b; Tabachneck et al., 1994) and their promise for supporting deep, conceptual understanding of more advanced mathematics (e.g., Nathan & Koedinger, 2000c). Like CM, BI takes a collaborative, problem-based approach to mathematics education. However, BI starts by eliciting students' invented strategies and representations and explicitly *bridges* from these to more formal and more efficient solution methods during classroom discussion. Thus, students' intuitive notions of how to organize data, how to depict them pictorially, and how to describe both linear and nonlinear relationships in words serve as precursors to their use of tables, graphs, and equations, respectively.

Method and Data Sources

Ninety students in four combined 7th/8th grade mathematics classrooms in a middle/upper-middle class school district in the Midwestern U.S. participated in this study for 9 weeks. Two of the classrooms were designated control classrooms and implemented CM, the school's standard curriculum, while the other two classrooms implemented the experimental curriculum, BI. The same, regular classroom teacher taught all four classes.

The CM curriculum is well described (Lappan, Fey, Fitzgerald, Friel & Phillips, 1998a, b) and commercially available. Students in the CM (control) condition worked through the seventh-grade units *Variables and Patterns* and *Moving Straight Ahead* during the time of this study.

The BI curriculum approached bridging in several ways. First, students' invented solution strategies and representations served as conceptual bridges to more formal procedures and representations. In prior studies (e.g., Nathan & Koedinger, 2000c) middle school students were observed using various invented solution methods to solve story problems and systems of equations. Guess-and-Test is one such method that students intuitively apply. It serves as a powerful inroad into formal algorithms because it highlights the structural aspects of the algebraic relations (Kieran, 1992) and procedurally grounds the concept of variable. Throughout the 9-week intervention, BI students were called upon to first decide for themselves, individually or in small groups, how they would represent data or solve for unknown values.

For example, in a variant of the "bridges and pennies" task (week 1; adapted from CM 8th grade series), students were asked to make a table to record the thickness of a paper bridge and the number of pennies that just breaks that bridge. They were then given wide latitude to graph the relationship between bridge thickness and number of pennies. Many insights emerged about students' understandings of graphical representations from the public displays of their graphs. Students did have a general understanding of the need for axes, labels, and plotted points, but did not all agree on how to label axes. Discussions of scale and interval size, and of the nature of dependent and independent variables naturally emerged. Students also did not all treat the independent variable (bridge thickness) as continuous. Many groups only included the values for which there were data collected, treating bridge thickness initially as nominal. This led to important discussions of the similarity of an axis to a number line. Finally, when students were asked to interpret their graphs, many read off relations like one would read values from a table. This led to the idea that one could use a line of best fit instead of the individual points to find an overall relationship between bridge thickness and bridge strength (measured in pennies).

In a second aspect of bridging instruction, concrete analogs served to ground more abstract representations. For example, in the "building squares" activity (week 3), students investigated the relationship between the side length and area of a square by building squares out of 1-inch square tiles. To see how area grows as a function of side-length, students used a method described by Kalchman (1998). Students decomposed the squares, stacked the tiles along grids of 1-inch graph paper, and marked off the values (heights) that showed the number of square-inches covered by each square. This provided a concrete way for students to both record the area of each square and to see how that area could be represented as the height along an axis of a graph.

In a third aspect of BI, linear and nonlinear relations served as contrasting cases for one another in order to make properties of each more salient to learners. In "building squares," students were explicitly asked to observe how both area and total tile perimeter (the sum of the side-lengths of all the tiles) grew as a function of the side-length of the squares, and to describe verbally and visually how these two growth patterns were similar and different. This idea was expanded further in "the cube problem" (weeks 8 and 9) in which the growth patterns for corner pieces (constant function), edge pieces (linear), face pieces (quadratic), and hidden pieces (cubic) were described and compared visually, verbally, and symbolically as functions of the side-length of the entire cube. One reason to use contrasting cases during instruction is to help learners form differentiated knowledge structures (e.g., D. Schwartz & Bransford, 1998). Linear functions are often taken as prototypical exemplars of functions. This can have detrimental effects on students' learning because linear functions have some unique attributes that do not generalize to all mathematical functions, such as (a) linear functions form a straight line when graphed against linear axes; (b) only two points are needed to determine a linear function; (c) the rate of change is constant; (d) missing values can be determined through linear interpolation and extrapolation (B. Schwartz & Hershkowitz, 1999). Most students, in fact, over-generalize the properties of linear functions and make erroneous assumptions on the basis of these over-generalizations, such as believing that only one function (a line) can pass between two given points. BI students directly confronted some of these misconceptions by simultaneously considering linear and non-linear functions throughout their algebra unit.

Classroom instruction and student interactions will be described in detail in future reports. Here we report on students' abilities to work within a given representation and to translate among representations, as assessed before and after the two different 9-week instructional interventions. The assessment instrument used a factorial design to allow systematic examination of the effects of problem linearity (linear or exponential), slope-sign (increasing or decreasing function), input representation (graph, symbolic, or word expression), and input-to-output translation (a graph, symbolic, or word expression input paired with a graph, table, symbolic, or word expression output) on student performance.

All problems given in the assessments were first introduced in words. Problems then presented the "input" representation (i.e., a graph, symbolic, or word expression) and asked students to respond to two problem-solving items and one translation item. The problem-solving items required students to work within the given representation to find a specific value of the dependent variable given a specific value of the independent variable or vice versa. The translation items required students to represent the presented functional relationship using a different output representation. Appendix 1 illustrates the multiple forms a linear problem could take given the variety of input and output representations involved.

Results and Discussion

Students' Initial Problem-Solving Performance

Pretest results indicate that linearity and mathematical representations do indeed influence students' problem-solving performance (see Figure 1). Problem-solving performance using graphical representations greatly exceeded that of all other representations. This graphical advantage held regardless of slope-sign (i.e., increasing or decreasing) for both linear ($M = 86.0\%$, $SD = 12.7\%$) and nonlinear ($M = 85.3\%$, $SD = 12.2\%$) functions.

In addition, students performed better on linear ($M = 58.3\%$, $SD = 32.1\%$) than nonlinear ($M = 39.7\%$, $SD = 36.0\%$) problems, $F(1, 68) = 47.50$, $p < 0.0001$. The interaction of linearity and representation was also significant, $F(2, 28) = 20.76$, $p < 0.0001$. Students experienced more success on linear problems when provided a verbal representation than when provided a symbolic one (linear word: $M = 65.1\%$, $SD = 22.3\%$; linear symbolic: $M = 24.0\%$, $SD = 21.0\%$), whereas symbolic representations led to greater success than verbal representations on nonlinear problems (nonlinear symbolic: $M = 20.8\%$, $SD = 19.8\%$; nonlinear word: $M = 13.0\%$, $SD = 12.3\%$; see Figure 1). This replicates the complexity-representation interaction reported by Koedinger et al. (2002) showing that verbal representations are most effective when solving lower complexity problems, whereas symbolic representations are more effective for higher complexity problems.

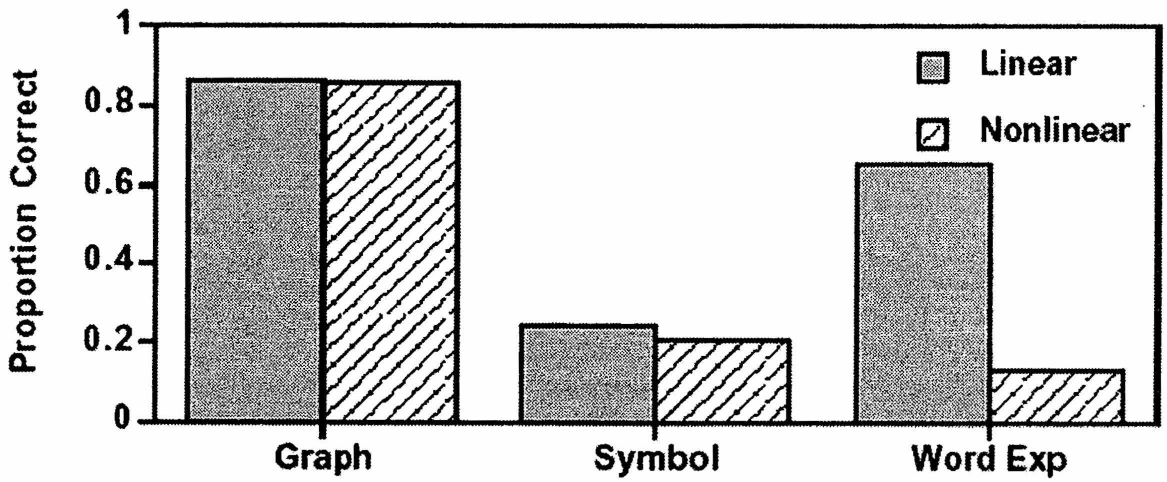

Figure 1. Proportiona of problem-solving pretest items solved correctly by problem representation and linearity.

Influences of Instruction on Problem Solving

BI students made greater gains in problem-solving performance from pretest to posttest than CM students (see Table 1), yielding a significant date by condition interaction, $F(1, 68) = 4.01$, $p = 0.05$, with CM students' improvements -limited to linear functions presented symbolically. Negative CM gains were not statistically different from zero. In contrast, BI gains were distributed across all representational formats and all levels of linearity.

Table 1. Proportion of Problem-Solving Pretest and Posttest Items Solved Correctly Along With Test Gains, Organized by Representational Input, Linearity, and Instructional Condition

Input Representation →		Graph		Symbol		Word	
Instruction ↓		Linear	Nonlinear	Linear	Nonlinear	Linear	Nonlinear
Control	Pretest	0.85	0.87	0.29	0.24	0.72	0.17
(CM)	Posttest	0.77	0.77	0.47	0.28	0.61	0.15
	Gain	-0.08	-0.10	0.18	0.04	-0.11	-0.02
Experimental	Pretest	0.87	0.84	0.19	0.18	0.58	0.09
(BI)	Posttest	0.88	0.90	0.39	0.25	0.67	0.32
	Gain	0.01	0.06	0.20	0.07	0.09	0.23

Students' Initial Translation Performance

Translation among representations appears to be a very advanced skill. At pretest, translation performance was essentially at zero for all but a few types of problems (see Table 2). Students could use pre-constructed input representations to solve problems (Table 1) far better than they could generate new representations as a part of a translation task. In the most dramatic example, pre-constructed graphs were correctly used for problem solving more than 80% of the time, but students across conditions could only correctly produce them about 6% of the time during translation tasks.

Table 2. Proportion of Translation Pretest and Posttest Items Solved Correctly Along With Test Gains, Organized by Representational Input and Instructional Condition

Output Representation →		Graph	Symbol	Table	Word
Instruction ↓					
Control	Pretest	0.11	0.19	0.11	0.02
(CM)	Posttest	0.10	0.10	0.34	0.05
	Gain	-0.01	-0.09	0.23	0.03
Experimental	Pretest	0.00	0.12	0.12	0.06
(BI)	Posttest	0.30	0.13	0.38	0.21
	Gain	0.30	0.01	0.26	0.15

Influences of Instruction on Representational Fluency

Overall, students were more successful on the translation items at posttest than at pretest, $F(1, 68) = 42.86$, $p < 0.0001$, though post-instruction performance was still low in both conditions (Table 2). BI students made greater translation gains than

CM students, $F(1, 68) = 5.94$, $p = 0.02$. The CM group experienced a 5.6% increase in correct responses, while the BI group experienced a 17.6% increase. Negative CM gains were not statistically different from zero. By posttest, CM students could translate from any input representation to a table of values at a rate significantly greater than zero while BI students could translate from any of the input representations to a table of values and to a graph, and could also translate between word expressions and symbolic equations.

Students understood and produced instance-based representations (i.e., tables and point-wise graphs) far better than more holistic representations (symbolic and verbal expressions, and line graphs). Graphs present an interesting case within the instance-based/holistic dimension, because they can be either instance-based (as with scatter plots and bar graphs) or holistic (as with line graphs). This duality does not apply to symbolic expressions, word expressions, or tables of values. To further understand students' performance with representations along this dimension, we compared experimental students' post-intervention abilities to produce accurate graphs when they were judged with instance-based versus holistic scoring criteria. An instance-based graph was defined as one including at least three correct data points. It was not necessary for the axes to be labeled with words or for the points to be connected with a line or exponential curve. Holistic graphs were ones in which the function (line or exponential curve) was drawn and the correct y-intercept was included. When evaluated from the point-wise perspective, students exhibited relatively high levels of performance (M = 29.5%). When they were evaluated using holistic criteria, performance was much lower (M = 4.3%). However, even when students constructed instance-based graphs, they often made other types of errors (see Figure 2).

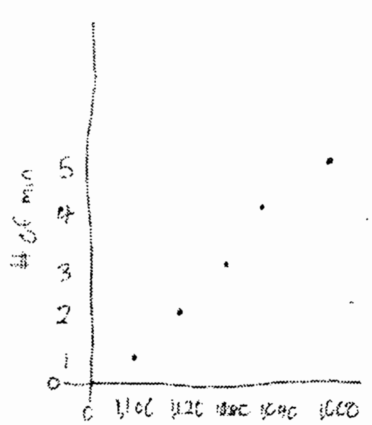

Figure 2. Point-wise graph of function d = 1200 - 40n

Conclusions

This study documents students' pre-instructional knowledge of mathematical representations and the effects of instruction on representational fluency. Students' pre- and post-instructional problem solving was heavily influenced by representational format, with students succeeding much more often with graphical representations than with symbolic or verbal representations, for both linear and nonlinear problems. On linear problems, students succeeded more often with verbal representations than with symbolic representations, and on nonlinear patterns, this pattern was reversed. This finding replicates earlier research showing a trade-off among representations, such that verbal representations are more effective for simpler problems, whereas symbolic representations are more effective for more complex problems (Koedinger, et al., 2002).

Our findings are consistent with Kalchman, Moss, and Case's (2001; Kalchman, 1998) psychological theory of the development of children's understanding of mathematical functions, which holds that procedurally based (i.e., computational) representations (e.g., tables of instances) and analogical representations (e.g., bar graphs) developmentally precede and form the basis for the more integrative, holistic representations (e.g., line graphs). In the translation tasks used in this study, students were most successful at generating tables of values. They were also more successful at generating instance-based graphs (e.g., scatterplots) than at generating holistic graphs (e.g., line graphs).

Pedagogically, one plausible hypothesis is that tables and instance-based graphs are a natural entry point into the mathematics of covariation, which serves as a central idea for the mathematics of functions. Because of their dual nature, graphs may be particularly effective in helping students to bridge from instance-based to more holistic representations. Ultimately, they may help students to learn the more holistic formalisms and to reap the rewards of such formalisms in the face of increasingly complex (e.g., nonlinear) relationships.

This study also demonstrated that *Bridging Instruction* was effective at facilitating both problem solving and translation. Why was *Bridging Instruction* so successful? One possibility is that the approach provides conceptual grounding for the meanings of various algebraic representations, by explicitly connecting them to students' informal reasoning and their intuitions about data organization and quantitative relations. This conceptual grounding may serve to support representational fluency. Another possibility is that it is pedagogically valuable to address linear and nonlinear functions within the same instructional unit. Each type of function can serve as a contrasting case for the other, reinforcing important common concepts while highlighting important distinctions. The present study does not allow for definitive conclusions about the individual or combined effects of bridging from students' informal reasoning or of teaching linear and non-linear functions in tandem. Future studies of each of these factors will be needed to tease out the basis for the effects we have observed. However, at a minimum, the present work documents that an instructional approach grounded in students' informal reasoning, and focused on both linear and nonlinear functions, can be highly effective and suggests that such instruction deserves greater consideration in the classroom.

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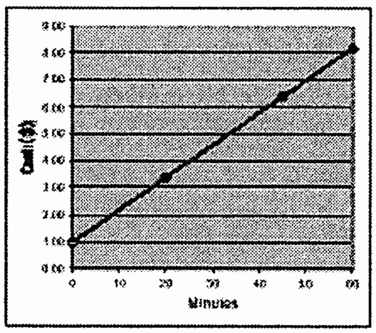
Appendix

Examples of Multiple Forms of a Linear Problem

Problem Section	Information Presented
Situation introduced	Cassandra sells phone cards to college students so they can make long distance calls for a good price. Each card has a base charge and a per-minute rate.
Input presented (For a given problem students get only one of these three input representations)	*Graph Input*: Below is a graph you could use to find the price of the card if you know the number of minutes on it.  *Symbol Input*: The expression below shows how to find the price of the card, p, of you know the number of minutes on it, n. $p = 0.99 + 0.12n$ *Word Expression Input*: The description below tells you how to find the price of the card if you know the number of minutes on it. To find the price of the card, you multiply the number of minutes by the per-minute rate of \$0.12, and then add the base charge of \$0.99.
Part a (problem solving)	What would be the price of a card with 30 minutes?
Part b (problem solving)	How many minutes would be on a card that cost \$6.99?
Part c (translation) (For a given problem students are asked to translate to only one of these three output representations)	*Graph Output*: Make a graph that you could use to find the price of the card if you know the number of minutes. (*Examination packets had graph paper included.*) *Symbol Output*: Write a mathematical expression that tells how to find the price of the card if you know the number of minutes. *Table Output*: Make a table of values that you could use to find the price of the card if you know the number of minutes. *Word Expression Output*: Describe in words how to find the price of the card if you know the number of minutes.

WHAT PROVIDES SUPPORT FOR STUDENTS' UNDERSTANDING OF SYSTEMS OF LINEAR EQUATIONS?

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In previous studies of students' understanding of systems of equations, researchers have speculated on how a particular instructional approach to graphs of linear equations affected students' understandings of systems of equations (cf., Goldenberg, 1988; Kieran, 1996; Sfard & Linchevski, 1994). To systematically explore this question, this investigation involved an 8-week classroom teaching experiment in a high school algebra classroom. The classroom teacher and I designed two technology-intensive instructional sequences. After the teacher taught each lesson, we met to discuss students' interpretations and revisions to plans for subsequent classes. The situated perspective was used to analyze the collective mathematical development of the class and to compare ways in which individuals participated in the collective development. From an analysis of the evolution of mathematical practices and students' conceptions as they participated in them, I have identified four key aspects of students' experiences with graphs of linear equations that supported their understanding of systems of equations.

Few studies specifically document students' understanding of systems of linear equations. On the one hand, Goldenberg (1988) and Sfard & Linchevski (1994) have hypothesized how students' prior experience with linear functions did not support students' understanding of a solution to a system as a single point, an infinite number of points, or no points. For example, students often think of a variable as representing only a fixed unknown value and they encounter difficulties when solving systems of equations where the system consists of coincident or parallel lines. On the other hand, Yerushalmy and Schwartz (1993) and Kieran (1996) have reported how a particular approach to linear functions supported students' understanding, specifically with regards to visualizing graphs of functions. Kieran (1996) reported that with a particular approach to linear functions variables were more readily interpreted as variables rather than a fixed unknown.

This study systematically explored how a particular instructional approach to graphs of linear equations affected students understanding of systems of equations. As part of an 8-week classroom teaching experiment, I worked with a high school teacher to design two instructional sequences--one for graphs of linear functions and one for solving systems of linear equations. We had several instructional goals. Among them was a goal of strengthening students' conception of a line as a set of points and its slope as a rate of change. Another goal was to support students' efforts to construct robust notions of the relationship between two quantities so that they could move freely between algebraic and graphical representations of this relationship. In an effort to support their understanding of systems of equations, we had a specific goal that students develop an awareness that a variable can be interpreted as more than a fixed unknown. Furthermore, with these goals in mind we inverted the traditional approach of an introduction to linear functions without a contextual base to one strongly situated in context. In addition to an analysis with regards to these instructional goals, the research described here attempts to account for social context, the role of students' personal experiences with various phenomenologies, and the role of language in the construction of socially-shared meaning.

Theoretical Perspective

In this study, I combined analyses of individual student learning with analyses of the social milieu in which this learning occurred. This investigation took individual students' reasoning as a unit of analysis while simultaneously viewing that reasoning as an act of participation in communal practices. This study used the situated perspective to analyze the collective mathematical development of the class and to compare the ways in which individuals participated in the collective development. In order to describe the collective development, the construct of a practice was used to examine the structure, responsibilities, and common activities of the mathematics classroom. Practices do not dictate a way to participate but are a description of the collective activity that arises from each student's interpretations of the social norms and socio-mathematical norms of a culture (Cobb & Bowers, 1999).

According to theorists of the situated perspective, knowing mathematics is considered an aspect of participation in social practices. Learning implies becoming a different person with respect to relationships in a community (Lave & Wenger, 1991). In particular, one can describe a student's learning of mathematics in terms of a trajectory of participation in the practices of mathematical discourse and thinking (Boaler & Greeno, 2000). I described four target students' individual growth in terms of their trajectory of participation in the classroom practices.

Setting

The participants attended a suburban high school yet about one-third of the students resided outside of the high school's attendance area. Twenty-seven percent of the students qualified for free or reduced lunch. The Algebra I class consisted of 34 students in ninth through twelfth grades. Approximately 20% of the class was classified as LEP (Limited English Proficient). Ten students were repeating the course as a result of having failed it on earlier attempts. The teacher had ten years experience teaching high school and was proficient with technology. She valued a classroom where students were expected to reason about mathematics and communicate their thinking to others in the classroom. The students met in a computer lab for just under half of the number of class periods.

Methodology

The methodology of this research was that of a classroom teaching experiment as described by Cobb (2000). The teacher and I developed a hypothetical learning trajectory based on how the collective mathematical activity might develop and by anticipating of the different ways in which students might participate in these activities (Simon, 1995). The high school teacher taught all the classes and I observed each lesson. After each day's lesson, we met to discuss students' interpretations and the implications for subsequent lessons. Furthermore, I interviewed four target students prior to and following each instructional unit, for a total of four times. There was a gap of two months between the two instructional interventions.

An interpretive framework developed by Cobb, Gravemeijer, Yackel, McClain, & Whitenack (1997) guided the analysis. This framework examines norms, socio-mathematical norms, and mathematical practices while simultaneously considering individual correlates. At the end of the teaching experiment, I identified phases that defined mathematical practices and shifts in socio-mathematical norms. This involved inferring taken-as-shared meanings of the collective and interpretations of individual students. I described how the classroom participants' contributions collectively constitute the evolving mathematical practices.

I also examined individual students' differing forms of participation in these practices. The four target students were the basis of case study analyses regarding three mathematical strands: their developing conceptions of speed as a rate, their evolving view of line graphs, and their representational activity. I further characterized their trajectory of participation in terms of their intellectual autonomy. Autonomy can be thought of as the degree to which students construct personal ways of judging (Yackel & Cobb, 1996). Increasing intellectual autonomy can be cast as students' evolution from reliance on a teacher's judgement of a mathematically acceptable answer with relatively little participation to increasing reliance on their own judgements (Lave & Wenger, 1991).

Data Sources

The data sources drew from classroom lessons, interviews with four target students, and planning and reflection meetings. The data corpus consisted of:

1) videotapes of whole class discussions,
2) videotapes of small groups in which target students participated,
3) fieldnotes regarding the small groups that did not include target students,
4) all the students' written work,
5) pre- and posttests given to the entire class prior to and following each instructional unit, and
6) interview data from the four target students.

The videotapes of whole class discussions, videotapes and fieldnotes of small group activity, and all the students' written work were used to identify practices. The target students' written work, interviews, and participation in whole class and small group discussions were used to examine their developing conceptions and to describe their trajectory of participation.

Overview of the Instructional Activities

One fundamental hypothesis in formulating our hypothetical learning trajectory was that we could build on students' pre-existing experience and understanding of motion to support their understanding of graphs of linear functions. We planned to use their experience with motion to help support several of our instructional goals.

Students began by trying to create target graphs using a motion detector hooked up to a computer. Students then wrote mathematical narratives for position graphs and predicted position graphs for given mathematical narratives. Students used SimCalc MathWorlds computer software to create position graphs that would simulate the motion for a character (Kaput, 1996). SimCalc MathWorlds is linked-representation software developed by a team of researchers to help students link formal representations with the experientially-real world (See Figure 1). Rather than focusing on a data set with a variety of symbolizations, SimCalc's focus is on the phenomenon, specifically motion (Kaput & Roschelle, 1997). SimCalc software enables students to explore motion with any combination of three graphs (position vs. time. velocity v. time, and acceleration v. time). Each of these can be linked to a character's motion, to each other, to a table of values, and to algebraic equations.

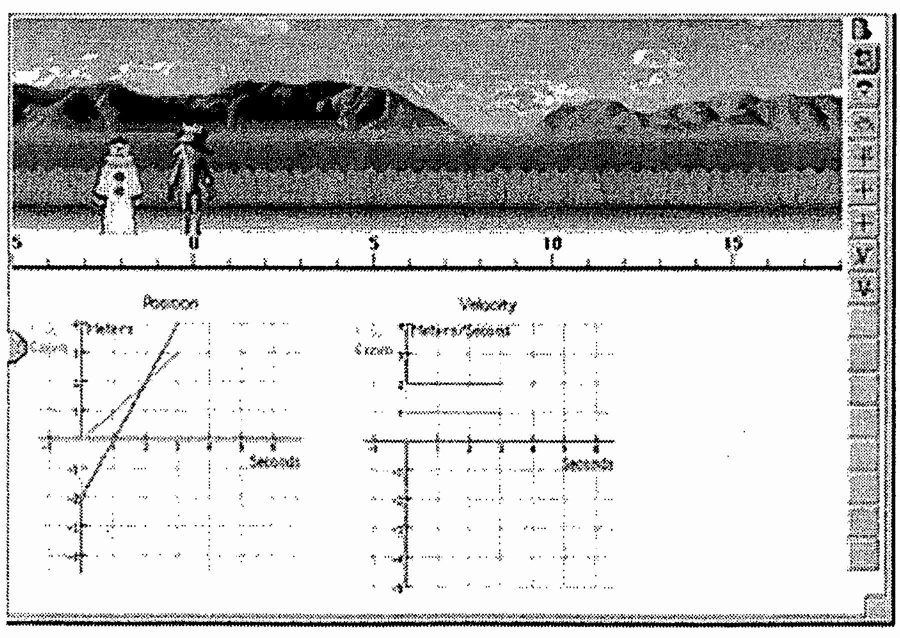

Figure 1. Interface of SimCalc MathWorlds software.

Over the course of the next couple weeks, students wrote equations for lines, explored the relationship among different representations (algebraic equations, position graphs, tables, and motion). They justified whether a given coordinate point was on a line. Students further gained experience interpreting rate of change and the y-intercept in contexts other than motion. They examined and created tables of values for determining a point of intersection, slope, and y-intercept. Finally, they solved systems by graphing, substitution, and by linear combination using other software developed for this purpose.

Results

These four mathematical practices emerged over the two months of the teaching experiment:

1) Describing a line graph in terms of a character's speed and direction
2) Describing speed as a pattern
3) Viewing slope as constant for a line
4) Using slope to determine if two lines meet

The first practice, describing a line graph in terms of a character's speed and direction, emerged as students described a character's motion by means of comparison. Initially, whole class discussions focused on negotiating what counted as an acceptable description of motion. Students wrote mathematical narratives for graphs of position. We then asked students what a position graph would look like for a character's simulated motion. These activities encouraged students to attend to specific points--a character's beginning and ending distances and time elapsed for a journey. SimCalc software's depiction of a journey with piecewise linear graphs probably provided support for students' tendency to focus on "completed journey" rather than an on-going one. As anticipated, students' narratives for position graphs were initially framed in terms of a walker moving forwards or backwards and fast or slow, but were not framed in terms of distances from the motion detector over time. Line graphs were interpreted comparatively and in terms of speed and direction. The taken-as-shared understanding of speed was with regards to the overall relationship between quantities (distance and time). Students described the effect of time and distance on speed. For example, "He ran back twice the speed because he went twice the distance in the same time. He'd be going twice as fast." Students did not initially have a perspective of a line graph as a set of points. Thompson & Thompson (1992) have noted that in order to develop a sophisticated understanding of speed as a rate, students first conceptualize distance and time as constituent quantities, which is an important first step to making a multiplicative comparison between two fixed quantities.

The second practice, describing speed as a pattern, evolved as students came to describe position graphs as a collection of points that revealed aspects of the story. One of our assumptions was that the learning of graphing entailed the refinement of visual, kinesthetic and narrative experiential domains (Nemirovsky, Tierney, & Wright, 1998). As students participated in the second practice, they indicated in the way they justified and in their symbolizing the graph of the line that they were thinking of speed in terms of an additive pattern. Justifications for speed consisted of language such as "You add three each time." When plotting points, students were able to identify patterns which they linked to writing the algebraic equation. Students connected patterns in a table of values with the algebraic equation. However, an analysis of practices indicated that students did not view slope as a rate of change.

The third practice, viewing slope as constant for a line, involved a shift to a view of slope as indicating a relationship between x and y coordinates. Students had been able to make a multiplicative comparison between the quantities of distance and time for a completed journey ("his speed is six meters in five seconds"). Upon realizing that students were having difficulty articulating a unit rate, the teacher posed a problem in which she asked students to examine a table of values and to determine where two characters met. The table recorded changes in x and y values that, like the former example of six meters in five seconds, involved non-integral slope. Students began to articulate a relationship between x and y values. A critical discussion involving a view of speed expressed as a unit factor initiated an evolution of practices. Furthermore, when graphing a line, students went from plotting multiple points to plotting two or three points. Students' language was suggestive of rate (e.g., "...for every one second, he moves 3 meters instead of 1.")

The fourth practice, using slope to determine if two lines meet, evolved from the practice of seeing slope as constant for a line. Once students saw rate as constant for the line. The taken-as-shared way of describing a line was uniquely determined by slope and intercept. Students looked at the equations of two lines and used information about slope and intercept to determine whether the two equations represented the same line, parallel lines, or intersecting lines. In the following discussion, as a follow-up to a student's comment, the teacher had asked, "What are all the possibilities for the graphs of two lines? What could happen if I graph two lines on the same set of axes?" Several students summarized for her the possibilities: parallel lines, same line, and intersecting lines. She asked for an example of two equations that represented the same line. One student, Cathy, suggested $3x + 2y = 6$ and $6x + 4y = 12$. Another student offered another example in which the coefficients and the constant term of the second equation were a constant multiple of the coefficients and constant term of the first equation. Knowing the difficulty students had with fractions, the teacher asked if they could write an equation that was the same as $y = (1/3)x + 5$. Evan suggested $y = (2/6)x + 10$. Walter said, "He's only doubled one thing." Much discussion ensued regarding slope. Finally the teacher brought them back to the original question.

T: So, for $y = (1/3)x + 5$?

M: $y = (2/6)x + 5$ is the same line?

T: Yes. Why is that the same line? Aibi, did I multiply anything? What has stayed the same? Let's talk about what has stayed the same.

W: The fractions are the same. [referring to the slope]

C: The starting point is the same.

T: Oh, starting position, which is the same as what?

Ss: y-intercept.

T: That's how we'll know, besides doubling and tripling it, if it is the same line for sure.

C: Like Anisha says. They are the same thing. One-third equals two-sixths. There's no difference there.

T: True, True.

M: $3y = x + 15$. That you get by multiplying.

T: Yes, Jesus. What's the slope [of the line $3y = x + 15$]?

J: One-third.

T: y-intercept?

J: Five.

[...]

T: Carlos, what do we need to determine if two lines are the same?

C: Get them like y equals something.

Students used information about slope and intercept to determine whether the two equations represented the same line, parallel lines, or intersecting lines. As the students solved systems of equations, the students indicated that they could estimate or compute the intersection of two lines based on rate of change. For example, when justifying how a target student knew two lines were parallel, he replied, "They'll never meet. The rate at which they grow is the same." Another target student would explicitly state how they each changed and stated the intersection in terms of "catching up."

Discussion and Conclusion

From an examination of mathematical practices, which describe the collective mathematical development of the class, I identified four key aspects of students' experiences with graphs of linear equations that appear to have supported their understanding of systems of equations and the nature of solutions to systems of equations:

a) Encountering line graphs situated in the context of motion

b) Engaging in point-wise interpretations of a line graph

c) Understanding that a line is uniquely determined by slope and intercept

d) Negotiating the nature of the socio-mathematical norms for justification

An analysis of the evolution of mathematical practices and students' conceptions as they participated in them indicated that students' work strongly situated in the context of motion did support their efforts to solve problems without a contextual base. Their prior activity of defining lines as sets of points in concert with work situated in the context of motion supported their efforts to construct meaning for the solution to a system as a single point, no points, or an infinite number of points. Their experiences enabled them to conceive of a solution as points that lie on both graphed lines and simultaneously satisfy both equations. The conception of a line as uniquely determined by slope and intercept supported their knowledge of equivalent equations and their interpretation of coincident lines as a solution to a system. Finally, the social norms and the nature of the whole-class ways of justifying can be linked to students' construction of meaning of systems of equations. The evolution of mathematical practices can be linked to particular social and socio-mathematical norms.

Encountering line graphs situated in the context of motion meant that students saw the solution to a system first in terms of two people meeting--in other words, two people being at the same place at the same time. Students who had a sophisticated understanding of rate used justifications that folded back to language about "catching up" as they described changing distance between graphs. Situating line graphs initially in the context of motion meant that students equated the x values with time and this supported awareness that a variable could represent an infinite number of values and not just a single unknown. This supported their construction of meaning for solutions to a system.

Engaging in point-wise interpretations of a line graph enabled students to construct a taken-as-shared meaning of a point of intersection in terms of a shared point. This enabled a view of lines as a set of points and they were able to conceive of a point of intersection that lies on both graphed lines and simultaneously satisfies both equations.

As described earlier, understanding that a line is uniquely determined by slope and intercept enabled students to determine whether two lines were equivalent, parallel, or intersecting at one point. Students used this knowledge of equivalent equations in solving systems by linear combination and in their interpretation of coincident lines as a solution to a system.

Finally, another key aspect of students' experiences with graphs of a linear equation involved the nature of what counted as an acceptable justification. The social norms and whole-class ways of justifying can be linked to students' construction of meaning of systems. The analysis revealed the importance of particular social norms, such as students explaining their answers and the expectation that they question an explanation if they did not understand. The negotiation of socio-mathematical norms was critical in the evolution of practices as the teacher helped the students formulate what counted as an acceptable explanation.

By situating individuals in the emerging practices of a mathematics classroom, I was able to contrast students who were participating at one level with other students who were becoming more central participants. Groups of students participated in the same practices but experienced whole-class conversations differently. The results portrayed how students' different forms of participation were affected, in part, by their conception of speed as a rate, by their participation in prior practices, and by their participation in the practices of other classes (e.g., concurrently taking a physics class). For example, as illustrated above, one target student with a less sophisticated image of rate experienced pivotal whole-class discussions about slope and intercept uniquely determining a line as being about how to arrange an equation in a format that would help identify critical elements.

With this work, we have a paradigmatic case of the situated perspective applied to a high school classroom. Through an examination of four individual students' trajectories of participation, I was able to situate students' changing interpretations as they participated in the evolving mathematical practices. This work also informs classroom teachers in that it embraces the complexity of the classroom. It accounts for what students and teachers contribute to the evolving community of practice. Analyses such as this can help teachers, curriculum developers and software designers anticipate ways in which other students will negotiate meaning.

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SUPPORTING LATINO STUDENTS' LEARNING IN EIGHTH GRADE ALGEBRA: USAGE OF LANGUAGE FOR UNDERSTANDING EQUIVALENCE

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This research report offers a framework for identifying emergent ideas about equivalence in beginning algebra students that is grounded in students' discourse. An exploration of the central question, "What are the conceptions that students have as they develop an understanding of equivalence?" is documented here. Latino students from six eighth grade classes were videotaped during group discussions and interviewed in pairs. Students' language on two tasks was used to develop and delineate a Reductive Model and an Equilibrium Model for equivalence.

The purpose of this study is to explore how students conceptualize equivalence as they move from arithmetic into algebra. National Council of Teachers of Mathematics (2000) calls for algebra throughout the K-12 curriculum and recommends that middle school students learn to generate equivalent expressions and solve linear equations. Equivalence is crucial for understanding algebraic expressions of quantitative relationships. Many state frameworks, such as the 1999 California Mathematics Framework (California Department of Education, 1999), call for every eighth grader to learn algebra, whereas previously most eighth grade algebra courses were designed with the highest achieving students in mind. There has been little research on algebra learning that shows how best to introduce formal algebra to younger students and specifically to a growing number of language minority students. This report discusses the first part of a larger study which links research and practice by investigating algebra classrooms as learning communities, student cognition in context, and how reform-based algebra curricula may have influenced students' understanding of equivalence.

Background

This research is based upon sociocultural theories that emphasize discussion and collaboration, within a community in order to explain children's learning (Ernest, 1996; O'Connor, 1998; Wertsch, 1985). Wertsch restates a Vygotskian view of the social aspects of learning in terms of a transformation of mental functions through social interactions with more experienced members of a culture. This theoretical orientation led me to choose a study site where the classroom community engaged in social interactions that allowed negotiation, agreement and dissent to take place in order to develop shared meanings. These broad sociocultural theories reflect my experiences in the classroom with student learning. As students build up their own unique set of knowledge structures from pre-existing structures through social interactions, students learning in the same classroom typically display different levels of understanding of a concept. The first research question in this yearlong study explores the variations in students' understanding of equivalence. I chose to explore student cognition in a situation that used social interaction by listening to and analyzing students' discourse interactions during specific tasks on algebraic equivalence. My research question lends itself to a qualitative methodology formulated by Moschkovich and Brenner (2000) called a Naturalistic Paradigm that integrates complimentary methods for naturalistic and cognitive studies and uses a recursive process of analysis allowing the researcher to follow leads and assumptions from the data. In this methodology, multiple points of view are considered, theory generation and theory verification are interconnected, and the importance of studying cognition in context is emphasized. Moschkovich and Brenner (2000) state " A naturalistic paradigm provides a road map for understanding learners in their own terms and for highlighting the potential in what they know" (p. 461). Using a methodology that incorporates multiple points of view allows me to incorporate diversity of students in a positive way and avoid a deficit model of learning that has been previously found in research with minority students (Khisty, 1995; Moschkovich, 2000). Rather than focus on barriers to learning and student misconceptions, I wished to hear from the students how they think about equivalence in their own words. It is from students' natural language that the framework for equivalence emerges.

For this study, I will use Gattegno's (1974) distinction between identity, equality, and equivalence, where equivalence is the least restrictive relationship wherein replacements are possible. In equality, at least one attribute remains the same, and in identity all attributes are the same. Equivalence is a wider set that includes identity and equality and permits substitutions in transformations of equations. Although the use of an equal sign is a necessary part of relating two quantities in an equation, few studies have explored algebra students' understanding of equivalence. Studies in the late seventies and early eighties showed that young children understand the equal sign to be a "do something" signal (i.e., operational) rather than a symbol showing "the same as" (i.e., relational) (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1976; Ginsburg, 1977). Kieran's study (1981) alluded to the possibility that large numbers of students in junior high, senior high, and college may take a simplistic view of equivalence. Kieran, in a short review of the research of equality from preschool through college, suggested that in the period of time between elementary school and high school, many students are confused about equivalence and hold a shaky grasp of the concept through high school. Kieran (1981) and Matz (1982) stress the particular importance of the transition period at approximately 13 years of age where students develop their concept of equivalence. Previous research suggests that equivalence is a stumbling block for students' success in algebra (Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Matz, 1982; Steinberg, Sleeman, & Ktorza, 1990) and thus a strategically important area of research to pursue. Yet, in a review of the literature, scant research exists on how 13-year-olds think about equivalence.

I present a framework for students' conceptions of equivalence that was formed from an analysis of eighth grade student discourse during interview tasks. (See Figure 1.) There are two main types of equivalence that can be discussed in algebraic thinking: that of equivalence between expressions and that of equivalence between transformed equations. Pimm (1995) and Kieran and Chalouh (1993) discussed one aspect of algebraic thinking that takes place between successive lines of transformed equations. To simplify the framework for this report, only equivalence between transformed equations is related here.

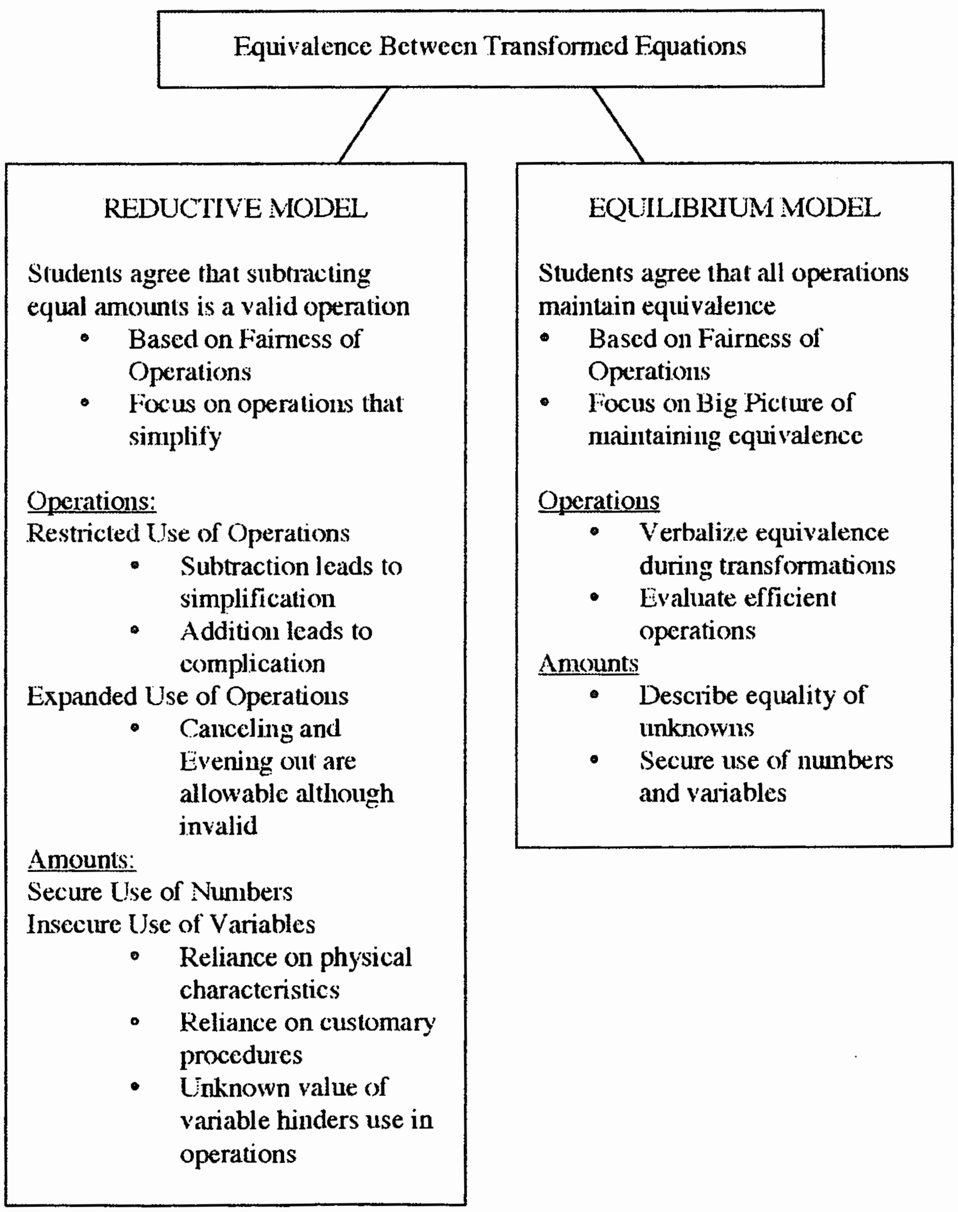

Figure 1. Framework for conceptions of algebraic equivalence in beginning algebra students.

Through student discourse, two major ways of thinking about equivalence emerged: what I call a Reductive Model and an Equilibrium Model for equivalence. In the Reductive Model, students fix their attention on subtracting equal amounts from both sides of an equation. Students focus on operations that reduce and simplify the equation and focus on the amounts being manipulated, not on the whole situation. In the Equilibrium Model, students conceive of equivalence as a balance where any operation that is applied evenly to both sides of the equation will maintain its equilibrium. In the Equilibrium Model students are focused, not on the amounts being manipulated, but on the maintenance of equivalence. I present data that support the models and further refine each after introducing the methods and two tasks used in student interviews.

Methods

A case study design was used for this classroom-based research with a student pair as the unit of analysis. After a yearlong participant observation of six eighth grade classrooms in a predominantly Latino school district, 37 pairs of students participated in semi-structured interviews near the end of the school year. The students in the six classes were taught by one junior high school teacher using mathematics reform-based algebra curricula that have been widely used across the U.S. As the school district and the teacher from this large California junior high school were committed to algebra instruction for all eighth graders, the students in this study were typical, ranging from high-achieving to low-achieving students, with both Spanish-dominant and English-dominant language backgrounds. Group discussions that took place during introductory units on solving equations were videotaped. During the analysis of the group discussions, several questions about how students conceived of algebraic equivalence arose and four tasks were created that focused on these questions. The tasks in the paired interviews maintained the usual classroom practices of collaboration and negotiation and allowed me to closely examine student language and cognition in context and in a natural setting. Two of the tasks will be discussed in this report.

Task 1: The Money Task

The first task in the paired student interview pertained to a realistic, hands-on situation. Foreign currency, in the form of bank notes, large coins, and small coins, which lacked Hindu-Arabic numerals for familiar numerical references, were used as manipulatives in the Money Task. I told each student to imagine that they are in a foreign country and they are each purchasing an identical item. Each student gave me their bank note and I gave them change in the form of large and small coins. To one student I gave 3 large coins and 4 small coins in change. To the other student I gave 1 large coin and 18 small coins in change. After students agreed that the value of the money that each received back in change was the same, students were asked to work together to find how many small coins were the same as one large coin.

The interviewer probed for student thinking about equivalence by referring to what the student pair did with the coins and asking the pair to justify their solution. The Money Task did not involve symbolic manipulation and provided a concrete situation for students to justify their solution. Students' physical movements of the manipulatives were captured on videotape as they worked through the task.

Task 2: Three Choices

The second task consisted of the initial steps in solving the linear equation $6x = x + 7$ and comments by three fictitious students about the step $-x = -x$. The interview pairs were shown a piece of paper that read as follows:

Three students were shown the following set of equations and were asked: "What can you tell me about the $-x = -x$ step?"

$$6x = -x + 7$$
$$-x = -x$$
$$5x = 7$$

1. The first student said, "You can't do that because you don't know what x is."
2. The second student said, "You can sometimes do that, but not all the time. It depends on x."
3. The third student said, "You can always do that, no matter what x is."

After student pairs selected the best response and discussed this problem, students were asked if the addition of unknowns was acceptable and the addition of unknowns with fractional coefficients was acceptable for certain equations.

The second task was designed to reveal what students perceive as acceptable steps for transforming equations. Several months before the interviews took place, all students had completed a four-week introductory unit on solving linear equations that included investigations into the addition and subtraction of unknowns.

Data Sources

Data from the transcriptions of the interviews were mapped (i.e., significant episodes were charted along with actions and statements) to provide the key features of the task. By mapping each interview, analysis of key elements of an interview and the language used by student pairs was simplified. A constant comparative method of analysis (Glaser and Strauss, 1967) was used to analyze the videotape data in order to find the natural language used in student discussions about equivalence. Descriptive matrices using direct quote raw data and key phrases were used to aid analyses (Miles & Huberman, 1984).

Results and Discussion

Task 1 Result

Use of equals

Students created their own usage for the word *equal* and other terms by stretching their usage to novel verb forms and phrases. In addition to using the word *equal* as a verb comparing two amounts in an arithmetic expression, students began to use the word *equal* as an action verb that involves the evening up of two entities. The following examples illustrate this usage.

> Using *equal* as a Comparison: "so that means these four coins are equal to this one" (Pair 30) and "which is equal to mine" (Pair 26)
>
> Using *equal* as an Action Verb: "we can equal the value to this" (Pair 15); "two of these large coins should equal up to seven of these small ones" (Pair 24); "We still just have to equal them out" (Pair 4); "so that equals out" (Pair 3); "we are supposed to equal two of like the same thing" (Pair 18)

Students used *balance* and *cancel* in much the same way (e.g., "we can balance them out") illustrating the transitional use of everyday language to describe algebraic processes. There was no standard way that students discussed algebraic equality, which may indicate that students created their own idiosyncratic forms to describe their actions for the interview tasks. Perhaps the new ways that students used the term *equal* shows transitional or novice entrance of the words into the mathematical register.

Patterns of use for terms of equivalence

Using a Naturalistic Paradigm, the researcher revisits the data with more focused questions and upon doing this, I found an interesting pattern. As I reconsidered the language of the student pairs who spoke of simplification using the terms *simple* or *less complicated*, I realized that most of these student pairs also described *taking away* in the same phrase. I looked specifically at the terms *take away, cancel, rid of, simple,* and *simpler*, and found a crossover of terms (i.e., students who used one term often used another of the terms *take away, cancel, rid of, simple,* or *simpler*). Originally, I had grouped the term *balance* with *cancel* and the other similar terms in my analysis, but I now saw that some student pairs used *balance* on its own without interchanging it with *take away, cancel, rid of, simple,* and *simpler*. I divided the students into groups that used the idea of simplification as its metaphor for equivalence and another group that used balance as the overarching metaphor for understanding equivalence. In the framework I characterized these two groups using the labels Reductive Model and Equilibrium Model.

Task 2 Result

Three general groups emerged from the responses of the second task, however, these groups do not respond to the three choices in the task. Table 1 illustrates the models students used for Task 2 based on the analysis of their language. The results appear to reinforce the findings from the first task.

Table 1. Two Models for Equivalence Used by Students in Task 2

Reductive Model		Equilibrium Model
X Value Matters Group (15 Pairs)	Subtract Only Group (7 Pairs)	Always Group (11 Pairs)
Choice 1 or 2	Choice 3	Choice 3
1) Concern with amounts in equation 2) Concern with operations – simplification means subtract 3) Customary procedures determine acceptability of next step 4) Physical shape of successive equations important	1) Hesitant to accept addition of unknowns 2) Prefer subtraction in order to simplify 3) Used Whatever You Do Slogan 4) Use invalid canceling operations 5) Call on higher authority rather than sense-making	1) Explicit in describing value of variables 2) Referred to Maintaining Equivalence in equations 3) Used Whatever You Do Slogan 4) Evaluated step for efficiency
Typical Responses for Each Group		
Pair 37: "You can't take *x* away because you don't know what *x* represents" Pair 10: "It has to depend on what *x* will be" Pair 18: "You need to solve for *x* first and then get the answer "	Pair 13: "So to find out what is *x* we have to take away X from here and from here so it can give you a number." Pair 15: "I think your supposed to take away a half." Pair 29: "So if we add one more x, … we're not going to be able to find the x at the bottom."	Pair 2: "and are equal, so it doesn't matter" Pair 9: "Since each is equal, you are allowed to take them away" Pair 12: "Yeah, if it's asking you to add it, because that's what you're trying to do, balance it."

Note. Four student pairs whose problems with coefficients made it difficult to determine conceptions of equivalence are not included in the table.

Reductive model- Excerpt 1

The most interesting results from Task 2 were the seven student pairs in the Subtract Only Group, who answered "You can always do that, no matter what x is," (Choice 3) but when questioned about an addition of x step (i.e., $+x = +x$) to both sides of an equation, student pairs rejected the addition step. The following interview excerpt from students A and J of Pair 23 illustrates a typical interchange between students in the Subtract Only Group and the interviewer.

1. A: because whatever you do to one side you have to do to the other.
2. Interviewer: Can you do that? [Can you add half an x?]
3. J: I don't think so, aren't you supposed to subtract all the time?
4. A: I think that you couldn't do that - you need to subtract it

[Later on in the interview, the students do a problem together with the interviewer that included an addition step. The transcript picks up from there.]

5. Interviewer: What about this step? You were thinking that you can only subtract, but we added.
6. J: Maybe you can add... [laughs]
7. Interviewer: Maybe!
8. J: It kinda seems wrong. We're used to subtracting it from both sides

Excerpt 1 captures the students' realization that their scheme for simplification may be widened to include operations other than subtraction (line 6). The students' explain that the customary procedure for solving equations is a routine that involves subtraction (lines 3 and 8). Although they stated, " whatever you do to one side, you do to the other," it was difficult for students to recognize operations other than subtraction were permissible. Six of the seven student pairs in this group used the *Whatever You Do* Slogan to justify the subtraction of x, but then decided that the addition of a variable did not hold under the rule that they had just stated.

Reductive model-Excerpt 2

The next excerpt with Student D of Pair 35 shows the unease that students in the *X* Value Matters Group felt when approached with the idea of adding unknowns. After asking the students in Pair 35 whether it is possible to add one half x to both sides of the equation, the students respond and illustrate in the transcription below that addition will not aid in simplifying an equation and finding a solution.

1. D: I don't think you can because they are both positive and that would just increase ... Increase it
2. Interviewer: And what's wrong with increasing it, tell me?
3. D: You have to find the value of x

4. Interviewer: And if you keep increasing it?
5. D: You won't find it.

Excerpt 2 shows a student verbalizing the idea that subtraction is the best way to simplify an equation and find the value of x; adding an unknown will complicate the equation and make it impossible to find the solution for x. The students in the X Value Matters Group did not readily see that operating with the variable x leads to "figuring out what x equals." The key issue in these students' explanations was the overriding concern to find the value of x and the indifference towards the steps leading to the solution. By focusing on the endpoint, customary procedures for solving, and the narrowing shape of the equation set, the students neglected the process of transformation and ignored equivalence.

Whatever you do slogan

I have termed the common phrase, "Whatever you do to one side, you do to the other" the *Whatever You Do Slogan.* I use the term *slogan* because it describes a catchy phrase whose meaning may vary from person to person, and thus, is ripe for misinterpretation. Because the terms *whatever* and *do* are vague, students developed their own meaning for the terms, and for some students in the Reductive Model the slogan came to mean: *use only subtraction in order to simplify.* For two student pairs in the Reductive Model, the slogan meant "cancel out" on both sides by adding a positive x to one side and a negative x to the other. For students using the Equilibrium Model, the slogan enabled them to use all operations and apply them commensurately to both sides of an equation. For 26 of 37 student pairs who used the slogan in the interviews, I learned to probe deeper in order to uncover the students' specific meaning behind the phrase.

Operational fairness

One constant and robust theme emerged from the data—that of students' ideas of fairness applied to operations. Students were often adamant about applying equal operations to both sides of equations and used ideas of fairness to justify equal operations. Out of the 37 interviews on each task, I saw only two cases where student pairs agreed that unequal operations were valid operations. Operational fairness is the firm belief that applying equal operations of equal amounts to both sides of an equation is a fair and valid operation. I make a distinction between agreeing that an operation is valid and realizing that a new equivalent equation has been created which continues to maintain equivalence to the original equation.

Reductive model overview

From the Money Task, a pattern emerged that showed two models for equivalence; one group that used balance and one that used equal subtractions as the means to simplification. With the use of concrete manipulatives in the Money Task, I wondered whether the task itself was influencing students to join the ideas of simplification and subtraction, however, the responses in the Second Task, which included both addition and subtraction operations of unknowns, also showed this pairing up of simplification and subtraction. The responses from the Second Task appeared to support the two models that emerged from the Money Task and led to a further refining of the models. In the Second Task, the Subtract Only Group was hesitant to accept addition of variables and preferred subtraction as the means to simplify. Students in this group had a restricted view of the operations that are used in transforming equations. Two student pairs in this group expanded their use of operations to include an invalid canceling step in order to simplify the equation. The X Value Matters Group was insecure in their use of variables although secure in their use of numbers in transforming equations. The student pairs in this group relied on physical characteristics and customary procedures in solving equations and did not tend to focus on overall goal of maintaining equivalence.

Equilibrium model overview

The results of both tasks appeared to show that some eighth grade students conceive of equivalence using a balance in equilibrium model that is based on principles of fair operations and is focused on the maintenance of equivalence as equations are transformed. Student pairs in the Always Group explicitly described the equality of the value of the unknowns and extended the idea of equivalence to variables. Most students also verbalized the maintenance of equivalence between equations during transformation and then evaluated the operations for efficiency. For example, Student Pair 17 suggested that a step was possible, but not productive when they stated, "I think you could, but why would you want to add half x because it would make it more complicated." As in this example, student language helped to refine the Equilibrium Model. The results of the second task, which used symbolic manipulation, helped to verify and delineate the characteristics of an Equilibrium Model that emerged in the first task, which provided a concrete manipulative situation without the use of variables.

Conclusion

This study reports that students express their ideas of equivalence in various and original ways and it is this language that grounds the framework for algebraic equivalence. Two models for eighth graders' conceptions of equivalence are documented: a Reductive Model in which students readily agree that subtraction operations are valid and where students focus on the amounts used in the operations; and an Equilibrium Model in which students agree that all operations maintain equivalence and where students focus on the overall maintenance of equivalence in transformed equations. Results from two interview tasks with 37 pairs of Latino eighth graders supports the emergence of two models for identifying ideas about equivalence in beginning algebra students. If students are to develop their ideas of equivalence by sharing meanings via discussions and interchange, then it is important that the curricular models and the vocabulary in the curricular models that are introduced to eighth grade students enable them to communicate their ideas to others and move them toward a better understanding of equivalence.

The goal of equity is embedded in mathematics reform curricula, and therefore it is important and imperative that the research community link research and practice in order to investigate how best to introduce algebra to young learners so that all students can succeed. This report is the first of a series of investigations designed to support Latino students in the learning of algebra by exploring how curricula influence students' understanding of equivalence and whether language minority students are benefiting equally from reform-based curricula.

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TEACHING ALGEBRA USING THE 3UV MODEL

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The 3UV (3 uses of variable) model involves a detailed decomposition of variable in its three main uses in elementary algebra: specific unknown, general number and related variables. This model is used as a guide to design teaching materials and a teaching strategy for elementary algebra through the differentiation and integration of the different uses of variable. Activities for an introductory elementary algebra course were developed and used with ten 12 to 13 years old students who were just being introduced in their approach to elementary algebra. We present the results obtained after 15 teaching sessions in which students worked following a spiral path of differentiation and integration of the different uses of variable. The results show, on the one hand, the appropriateness of the 3UV model for the design of teaching materials and of the teaching strategy. On the other hand, we show evidence of the possibility to introduce students to different uses of variable emphasizing their differentiation and integration. The capability of students to shift between the different uses of variable is illustrated by an excerpt of a group discussion.

Introduction

Several researchers state that the development of algebraic knowledge implies a solid understanding of the concept of variable (Bills, 2001; Philipp, 1992; Ursini & Trigueros, 2001; Warren, 1999). Supporting this view we consider that more emphasis should be made to help students develop an understanding of this multifaceted concept as a global entity. From our point of view students should be taught explicitly to differentiate between the different uses of variable and to integrate them in a single conceptual entity: the variable. In previous studies the basic capabilities underlying an understanding of variable have been summarized (Ursini & Trigueros, 2001). An elementary understanding of variable was described in terms of the following basic capabilities:

- to perform simple calculations and operations with literal symbols;
- to develop a comprehension of why these operations work;
- to foresee the consequences of using variables;
- to distinguish between the different uses of variable;
- to shift between the different uses of variable in a flexible way;
- to integrate the different uses of variable as facets of the same mathematical object.

Moreover, a theoretical framework, the three uses of variable model (3UV model), has been proposed as a basis to teach the concept of variable (Ursini & Trigueros, 2001). The 3UV model considers the three uses of variable that appear more frequently in elementary algebra: specific unknown, general number and variables in functional relationship. For each one of these uses of variable, the 3UV model stresses different aspects corresponding to different levels of abstraction at which it can be handled. The requirements for an elementary understanding of variable can be presented in a schematic way as follows:

The understanding of *variable as unknown* requires to:

U1. recognize and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem;

U2. interpret the symbols that appear in equation, as representing specific values;

U3. substitute to the variable the value or values that make the equation a true statement;

U4. determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations;

U5. symbolize the unknown quantities identified in a specific situation and use them to pose equations.

The understanding of *variable as a general number* implies to be able to:

G1. recognize patterns, perceive rules and methods in sequences and in families of problems;

G2. interpret a symbol as representing a general, indeterminate entity that can assume any value;

G3. deduce general rules and general methods in sequences and families of problems;

G4. manipulate (simplify, develop) the symbolic variable;

G5. symbolize general statements, rules or methods;

The understanding of *variables in functional relationships* (related variables) implies to be able to:

F1. recognize the correspondence between related variables independently of the representation used (tables, graphs, verbal problems, analytic expressions);

F2. determine the values of the dependent variable given the value of the independent one;

F3. determine the values of the independent variable given the value of the dependent one;

F4. recognize the joint variation of the variables involved in a relation independently of the representation used (tables, graphs, analytic expressions);

F5. determine the interval of variation of one variable given the interval of variation of the other one;

F6. symbolize a functional relationship based on the analysis of the data of a problem.

An understanding of variable implies the comprehension of all these aspects and the possibility to relate both, the aspects and the uses of variable in different problems. This implies developing the ability to *interpret* the variable in different ways depending on the specific problematic situation, acquiring in this way the flexibility in shifting between its different uses and the capability of integrating them as facets of the same mathematical object. It is necessary as well to develop the ability to *manipulate* symbolic variables in order to perform simple calculations. Both these abilities help students to develop a comprehension of why the operations work. The ability to *symbolize* rules and relationships in different problem situations leads to foresee the consequences of using variables. Starting algebra students should develop gradually these basic capabilities that are related to a deep understanding of elementary algebra.

Based on the 3UV model an approach to the teaching of algebra following a spiral path of differentiation and integration of the different uses and aspects of variable presented above has been suggested (Trigueros & Ursini, 2001). At each level of complexity all the three uses of variable are studied following a two phases pattern of instruction: first, situation are presented where the three uses of variable are introduced in a differentiated way; then students are faced with situations where it is necessary to integrate the three different uses of variable just studied.

In this paper we present an example of teaching materials for an introductory elementary algebra course, developed following the 3UV model and the suggested spiral path for teaching. We analyze 12-13 years old students' capability to differentiate between the different uses of variable when working with the proposed tasks. The teacher's knowledge about the concept of variable and of the 3UV model, her ability to conduct the activities, to interact with students and to generate enriching discussions are in the core of the success of the teaching strategy.

Methodology

Several activities of different levels of abstraction were designed in terms of the 3UV model. They were organized in three blocks. The first two blocks have the same structure: first there are a series of activities in which the different uses of variable are considered in a differentiated way; after that an activity involving the three uses of variable is presented. The third block originally contained only activities involving the three uses of variable, but it was slightly modified during the experiment. The following is a brief description of the content of each one of the three blocks:

Block 1. There are 12 activities in this block, all of them concerning numeric patterns. The first seven activities require working with general numbers. Through analyzing numeric patterns students are expected to approach the aspects G1, G2, G3 and G5. The same numeric patterns are after that used in the following two activities in order to work with specific unknowns. The aspects U1, U2 and U4 are involved. The activities 10 and 11 use as well numeric patterns as support, but students' attention is directed towards functional relationships. They are expected to approach F1, F2, F3, F4 and F5. The last activity requires working with the three uses of variable and the aspects involved are G1, G2, U1, U2, F1 and F4.

Block 2. Eleven activities integrate this block. All of them include areas of geometric figures. In the first five activities general numbers through the aspects G1, G2, G3 and G5 are present. The following four activities emphasize the aspects U1, U2, U4 and U5 of the specific unknown. The activity 10 involves variables in functional relationship with F1, F2, F3, F4, F5 and F6. As in block 1, the last activity requires working with the three uses of variable. The aspects involved are G1, G2, U1, U2, F1 and F4.

Block 3. In this block there are 7 activities. The first 3 activities involve related variables with F1, F2, F3, F4 and F5. These activities were added to the original design in order to equilibrate students' experience with the three uses of variable. In fact, during the first two blocks there were more activities with general numbers and specific unknowns than with variables in functional relationship. All the subsequent 4 activities require working with the three uses of variable and with all the aspects students worked with during blocks 1 and 2.

This organization of the activities aimed at following the spiral path suggested by the 3UV model for teaching.

A group of 10 students, aged 12 - 13 years old, attending the first grade of a public secondary school in Mexico, worked with these activities during 16 sessions. The 10 children were part of a larger group, but they were selected by their mathematics' teacher to attend the 16 sessions during the mathematics' regular hour. The researcher acted as the teacher of the 10 students' group. She presented the activities to the whole group and these were solved through a common discussion. She directed the group discussion by posing questions promoting the participation of all the 10 students in oral and/or written form. During the discussion a paper-board was used instead of a blackboard and students were invited to step forward and to write on it. Additionally they were asked to answer the posed questions individually on a paper sheet. All the sessions were videotaped, the paper-board pages and all the students' written productions were kept in order to be analyzed. At the end of the 16 sessions three students were interviewed in order to obtain more information about some of their written productions. Three researchers independently of each other analyzed the data. Their results were summarized, exchanged and negotiated until reaching a consensus. This method was used to validate the interpretation of the data.

Results

In order to present the results obtained we will discuss in detail the outcomes of the 16th session in which activity 5 of block 3 was developed. This activity requires working with the three uses of variable and with the aspects G1, G2, G3, G5, F1, F2, F3, F6, U1, U2, U4 and U5. Our purpose during this session was twofold: on the one hand we wanted to observe if the experience acquired during the previous activities pupils were able to shift between the different uses of variable approaching the different aspects; on the other hand we wanted to detect the possible difficulties in doing this.

A table describing the equivalence between pesos and dollars for 5, 8, 20, 35,and 100 dollars was presented. The column on the left represented the quantity of dollars and the one on the right its equivalence in Mexican pesos. The exchange rate used was 9.5 pesos per dollar. The teacher asked *What is this table about?* in order to introduce the activity and focus students' attention. The following dialog followed the introductory stages:

T (Teacher): What information can we get from this table?

In this way the teacher introduces the activity and calls pupils' attention on the data presented in the table.

M (Marcela): From dollars to pesos?

K (Karla): How much does it cost a given quantity of dollars.

M: How much do 5 dollars cost. (pointing with her pen to the table displayed on the paper-board)

T: Yes, how much do 5, 8, 20 dollars cost, any quantity of dollars. In that table, what quantities are we relating?

M: From dollars to pesos?

T: Although in this column it says "dollars", what are we actually representing here?

M: How many dollars.

T: How many dollars, that is the number of dollars. And in this one? (pointing to the other column)

K: The value of the dollars.

M: Yes, in this column we put the value of the dollars.

K: The equivalence from dollars to pesos. (moving her hand to indicate that she is going from one column to the other)

During this initial verbal interchange we can observe that Marcela and Karla immediately recognize that the quantities presented in the table are correlated. Karla stresses it using the word "equivalence" and Marcela by saying "from dollars to pesos" (F1). We can also see the teacher helping them to make explicit the quantities involved in the correspondence. The teacher comments help them to make their statements in a more precise form.

T: The equivalence from dollars to pesos. Then, in this table we are relating the number of dollars with the number of pesos. Marcela come to the board, please, now, what operation do we have to do to know how many pesos correspond to five dollars? And write the operation here. (showing the corresponding part on the table)

M: I have to multiply. Do I?

T: Yes, but, what is it that do you have to multiply?

M: ....

T: Yes, for example, if you want to know how many pesos correspond to five dollars.

K: 9.5 by 5.

T: Do you agree Marcela? (Marcela agrees by moving her head) Do it on the board (Marcela does the product using vertical notation on the board)

T: Then, what is the operation you do if you want to know how many pesos correspond to five dollars?

M: By five.

T: But what did you multiply? Write the operation in horizontal notation, here on the board. (pointing to a space on the board to the right of 47.5 Marcela writes9.5x5)

T: 9.5 by 5, well, what does the 9.5 represent?

M: the equivalence of 5 dollars.

Everybody: Noooo!

K: A dollar costs 9.5, a dollar is equivalent to 9.5pesos.

T: Very well, and what does the 5 represent?

M: The number of dollars.

T: The number of dollars. Now, what do we have to do to know how many pesos are equivalent to 8 dollars?

M: Multiply by 9.50.(Marcela writes 9.50 to the right of76. Marcela goes back to her place)

T: And what do we have to do to know how many pesos are equivalent to 35 dollars? Sandy?

S. (Sandy): 9.50 by 35.

In this passage the teacher wants to call the attention of the students about the rule that relates both columns of the table. She does it starting from particular cases and posing questions in a way that can help to fix the attention of the students in the mathematical form of the correspondence. In the group discussion more students intervene. From the responses given by Karla, Marcela and Sandy, we can observe that the students are capable of recognizing the appropriate method to determine the number of pesos that correspond to a given number of dollars (G1, F1), and also that they are able to state explicitly the operation needed to determine the number of pesos when the number of dollars is given (U4). We can see here that during the group discussion, the possibility of listening to different voices helps the members of the group to structure their thinking in a more formal way. This can be seen, for example when Marcela, who has difficulties in expressing verbally her point of view, is helped by the clear and precise interventions of Karla to formalize her reasoning.

T: Yes, and here what does the 9.50 represent?

S: The value of one dollar.

T: And the 35?

S: It is the number of dollars.

T: Well. We have seen that to determine the number of pesos that correspond to a given number of dollars, we have to multiply the number of dollars by the value of one dollar. Also, when we first looked at this table, what values did we say that we can write in this column? (pointing to the dollars column and inviting Jorge to come to the board)

The teacher summarizes, once more, the ideas that the students have just expressed, in order to generalize the procedure they have deduced taking as a starting point the previous examples. The teachers also introduces the idea of general number. By doing this she helps them to start the symbolization part of the activity.

J (Jorge): The value that we want.

T: Any value, and what symbol can we use to represent that value in general?

J: A letter.

T: A letter, which one would you use? Write it down on the table on the board (pointing one of the empty rows. J writes the letter h on the board)

T: Very well, and what does that h represents?

J: It represents any number.

T: And in this particular case, what does the h represent?

J: The number of dollars.

In this dialogue it is clearly shown that this student is capable of thinking about a general number as representing any value (G2), and that he feels the necessity of using a symbol to represent it, more specifically, a letter (G5). While he does this, the group follows attentively the discussion and assert when the letter h is chosen to symbolize the general number. These students are thus capable of symbolizing a general number by using any letter, h in this case, and they recognize in that letter a symbolic variable. They are not limited to using specific letters traditionally used by teachers and books as x or y. Another aspect that seems important to us is their capability to interpret the variable as the number of dollars and not as a label or abbreviation of the word dollars. The teacher's role in this dialog is as a guide to call the students' attention on the use of symbols to represent general numbers, and to make sure that the interpretation they do of the symbol is understood by the whole class.

T: Very well. And of we want to know how many pesos correspond to that number of dollars, what operation do we have t do?

J: We need the number of pesos, so that when I do a division I get the number of dollars.

T: Let's see again, what operation do we have to do to know the number of pesos that correspond to a number of dollars?

J: ah! Multiply.

T: What do you multiply? (Jorge writes on the board hx9.50 = 530)

The questions posed by the teacher are oriented to help the students verbalize the general rule and to verify if Jorge can write h x 9.5 as a general expression (G5) that represents the number of pesos. However, as can be noted, Jorge writes instead an equation (U5), where h has the role of an unknown number. This suggests that he has difficulties in symbolizing a general rule (G5) and that he needs to interpret h as an unknown (U2) to make sense of the question.

T: Well, but if I multiply the number h of dollars by9.50 what is the result of that operation?

S: Any number.

T: Yes, any number, we don't know which one. Then, how do we write that? (J: write h x 9.50 = z)

T: The result is another number that we have represented in this case, with a z.

T: How do we obtain the value of z?

V (Vanessa): The values of z is the result of a multiplication, and we represent it with a letter because we don't know its value yet. It is an unknown number.

The teacher goes back to the previous question trying to see if the students can generate the expression h x 9.5 (G5), and can interpret it as a general number which in this particular case represents the number of pesos. The students, however, do not produce the expected expression and in order to make sense of the operation on h they generate a functional relationship (F6). Jorge, who is still standing in front of the board, writes h x 9.5 = 530. From students' interventions we can also observe that they are interpreting the letter z as capable of taking any value, that is, letter z is interpreted as a general number (G2), but they are recognizing as well that its value depends on the given restriction h x 9.5, that is, that they are recognizing the correspondence (F1).

T: Very well, Jorge wrote h x 9.50 = 530, what is the value of z in this case?

V: It is 530.

T: Well. In this equation what is the value of h?

V: 530 divided by 9.5.

The teacher uses the expression previously written by Jorge to work on the aspects related to variable as specific unknown. From the dialog it can be observed that Vanessa recognizes that h is an unknown number that can be determined (U1), and she is able to indicate the operation needed to determine it (U4). During all this process, it can be seen how the students are shifting from one use of the variable to another without any difficulty. Observe as well how the teacher is coordinating the discussion following the path traced by the students' interventions.

T: Then, what is the value of h?

M: Any value.

Everybody: Noooo!

K: No, because the value of h depends on z and we already have z.

T: Well, because the value of h depends on a result that is already known, in this case it is 530, here. Then how many values can h take?

Everybody: One!

It is clear here that the students recognize that in this expression, an equation, the variable can take only one specific value (U2); and that this is due to the restrictions of the problem, in this case the value that has been given to z. Karla's answer shows that she is recognizing the correspondence between the variables h and z, that is, that the value of one of them depends on the value assumed by the other, and that for a given value of z there is a corresponding value for h (F1). Here again, the role of the group discussion and of the different interventions in guiding the reasoning process of the students, and in the way the teacher formalizes their findings, can be observed.

T: Only one and, what is that value?

Several students: It is 55.

The students determined the value of h doing the necessary arithmetic operations. This means that they could find the value of the independent variable h when the value of the dependent variable z was given (F2). At the same time, they were able to determine the value of the unknown considering the equation h x 9.5 = 530 (U4).

T: And if the value of h is 50, what is the equation that represents this situation?

J: It is represented like this… (he writes 50 x 9.5 =n)

T: And why n?

J: Because we don't know yet what is the result of 50 x9.5.

T: And if instead of writing 50 x 9.5 = n, we write50 x9.5 = z, are we changing the sense of what we are trying to say?

A (Alejandra): No it is the same. Because what is changing is the letter we are using to represent the variable, and the variable can have the same value even when it is represented using different letters.

In this part of the dialogue we can observe that Jorge does not have any difficulty to substitute the given value to h (F2). While doing this, however, he changes the representation of the dependent variable. This shows his ability to represent a specific unknown using any letter but, at the same time, puts forward the necessity to call students' attention on equivalent expressions. This propitiates a new question by the teacher in terms of the equivalence of the different written expressions. It is worth noting the clarity of the concise explanation given by Alejandra, which shows that she really understands the use of the variable. The teacher is also guiding the students in their understanding of the symbolization of the general rule by means of a functional relationship.

T: Well, in the expression 50 x 9.5= n, how many values can n take?

J: Any value.

Everybody: Nooo!

T: Why?

F (Fabiola): Because its value becomes dependent of 50.

T: Why?

F: It is dependent on h, but as we already know the value for h, which is 50, now n or z are depending on50. Now n and z have to be the same as the result of multiplying 50 x 9.5.

T: Then the expressions 50 x 9.5 = n and 50 x 9.5 are equivalent?

Everybody: Yes!

T: Now what is the value of n or z?

F: It is 475.

In this last part of the discussion it can be seen that the whole group agrees with Alejandra´s comment. Fabiola, for example emphasizes that the letter used to represent the variable is not important, and the whole group recognizes the equivalence between the two expressions. Here again it is shown that students are able to recognize that in an equation the variable can assume only one value (U2). They are capable of determining the value of the specific unknown that appears in an equation (U4). In doing this in the given example, they show as well their capability to determine the value of the dependent variable when the value of the independent variable is given (F3).

The analysis of students' interventions and productions show that after 15 sessions they had acquired the capability to recognize the different uses of variable and they could use them when working with simple algebraic problems. They were able to differentiate them and also to integrate them to solve specific problems during the global solution process. They showed a capability to shift between the different uses of variable in a flexible way. The excerpts we presented above show that students could approach the great majority of the different aspects involved in the given problem situation. There is evidence showing their capability to perceive rules (G1) and to deduce general rules (G3); to interpret a symbol as representing a general number (G2) and to differentiate it from the same symbol representing a specific unknown (U2). They could recognize the presence of a specific unknown (U1) and determine its value when it appeared in an equation by performing the required arithmetic operations (U4). Students recognized the correspondence between related variables (F1), and they could determine the value of the dependent variable given the value of the independent one (F2) or determine the value of the independent variable given the value of the dependent one (F3). They could as well express symbolically a functional relationship (F6) and write an equation (U5). However, some difficulties were detected as well. Students could not symbolize a general rule (G5) they had already recognized, although they were able to apply it to solve particular cases.

During this process the teacher's interventions were very important for guiding students' work and for calling their attention on the different uses of variable and the related aspects. Observe that all her interventions consisted in rephrasing students' answers to her questions and in posing new questions. This approach led to the participation of the whole group and to a common discussion of the ideas students were externalizing. On several occasions she asked students to make explicit the meaning of the literal symbols they were using. This helped them get aware of the role played by the variables in the different moments of the solution of the problem. Moreover, in spite of her schedule she was sensible to students' understanding and this led to a real dialog between students and with the teacher that led them to become completely involved in the proposed task.

Conclusions

This research shows that using the 3UV model as a basis for the design of a teaching strategy is feasible. The teaching approach followed a spiral path of differentiation and integration of the different uses and aspects of variable. In each block of activities students worked with all the three uses of variable following a two phases pattern of instruction: first, situation were presented where the three uses of variable were introduced in a differentiated way; then students were faced with situations where it was necessary to integrate the three different uses of variable just studied. The results obtained show that it is possible to introduce students to the study of the different uses of variable and work with its different aspects since their first courses of algebra. The episode shown in this paper makes it possible to appreciate students' strengths and weakness when they work with variables. The process of understanding the different uses of variables takes time. Some integrations and differentiations that have to be made are of a grater complexity than others. This implies that in some cases more attention has to be paid to the development of some notions. An example shown here is the difficulty students have to recognize an open expression as a conceptual unit, and not only as the result of the operation on a general number.

We want to stress that in an approach like this one, the role played by the teacher is fundamental. It is her responsibility to determine the degree of difficulty of the activities done in class, and to find a way to give enough opportunities to students to interpret, manipulate and symbolize the different uses of variables. She has also to be able to diagnose students' capabilities and difficulties during their interventions and to turn them into opportunities of conceptual growth in their learning of algebra. We want to stress as well, the fundamental role played by the explicit and implicit intervention of the group during discussions. Although these interventions can deviate the learning trajectory originally intended by the teacher into new directions, they become significant in the process of the students' understanding.

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DECONTEXTUALIZED RATIOS IN MIDDLE-SCHOOL STUDENTS' SOLUTIONS TO AN ASSOCIATED-SETS PROPORTION PROBLEM

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After noticing students' lack of success using fractional representations of ratios to solve an associated-sets proportion problem, we hypothesized that middle-school students who are successful at solving missing-value problems may not attach contextual meaning to fractional representations of ratios when necessary to solve other types of proportion-related problems. In our first step to gather evidence, we conducted a quantitative study and found significant associations between students' notation (fractional versus non-fractional) and whether students maintained the context of the problem in their solution, between students' notation and score on the problem, and for the students who used fractional representations, between students' score and whether they maintained the context of the problem in their solution.

Introduction

An understanding of proportion—the equivalence of two ratios—depends on a sufficient prerequisite understanding of ratio. According to Sowder et al. (1998), an understanding of ratio is essential to the transition from additive to multiplicative thinking; however, the topic of ratio receives little attention in elementary-school classrooms, and when it is covered the treatment often lacks depth (Post, Cramer, Behr, Lesh, & Harel, 1993).

While working on a larger study about how teachers understand and how textbooks present the relationship between ratios and fractions, we asked the following: Is there an association between students' notation for ratios—in particular, fractional versus non-fractional—and their success solving problems?

Conceptual Framework

We define proportional reasoning from the perspective of a relationship between two variables that describe a linear function (Karplus, Pulos, & Stage, 1983). The emergent perspective (Cobb & Yackel, 1996) is an appropriate lens for this investigation because of the conjectured reflexive relationship between classroom mathematical practices and individuals' mathematical conceptions and activity. For proportion-related problems, strategies and notation shown by teachers and in textbooks are adopted by many students who fail to connect them to their understanding of ratios and fractions and therefore have no conceptual foundation for using them in a meaningful way (Clark, Berenson, & Cavey, in review). The students who participated in this study attended 25 different middle schools; therefore, no single teacher or classroom environment had an impact on the results.

Methodology

Data Sources

We analyzed the written responses of 137 students to an associated-sets problem (Lamon, 1993), "the pizza problem," which is as follows: *There are 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys?* Responses were scored according to Allain's (2000) four-point rubric: 4 points for appropriate strategy and correct answer, 3 points for appropriate strategy but incorrect answer due to a computational error, 2 points for correct answer but no demonstration of strategy, and 1 point for inappropriate strategy and incorrect answer.

Coding Schemes

We coded the students' solutions to the pizza problem according to two dichotomous variables: *notation*, either fractional or non-fractional, and *context*, whether the student maintained the context of the problem in her solution by using labels or diagrams for the elements of the problem, such as boys, girls, pizzas, and slices of pizza (if she divided them). We also used another problem that we refer to as "the coffee problem," a standard missing-value problem, to narrow our sample for some statistical tests because we wanted to identify a subgroup of students who were successful at answering a missing-value problem. We used the coffee problem to separate the students into two groups—the 86 students (63%) who received a perfect score and the 51 students (37%) who did not.

Statistical Analysis

For statistical tests, we used SAS software and the Pearson chi-square test for association between notation and context and a cumulative logit model for multinomial ordinal data to compare groups' scores.

Results and Discussion

On the pizza problem, the students who used non-fractional notation were more likely than those who used fractional notation to maintain the context of the problem in their solution (see Table 1).

The Pearson chi-square value of 7.92 ($p < .01$) is sufficient to reject the null hypothesis of no association and would have been larger if we could have confirmed

Table 1. Frequency Counts for the Pizza Problem (N=137)

Notation	Context	No context
Fractional	34	40
Non-fractional	44	19

in interviews that some of the 19 whose solution was coded as non-fractional and no context simply guessed without working the problem and we therefore could have deleted them from this analysis.

While analyzing notation and context in the students' work, we began using the term *decontextualized ratio* to refer to students' written representations of ratios that were detached from the meaning of the problem. For example, a common error on the pizza problem was writing the relationships between the associated sets as 7/3 and 3/1 without any reference to what the numerators, denominators, or fractions stood for and to conclude that because 3/1 is the larger number that each boy got more pizza than each girl.

Of the 86 students who received a score of 4 on the coffee problem, those who used non-fractional notation on the pizza problem scored better than those who used fractional notation (see Table 2). The notation variable is significant ($p < .01$) in explaining the difference in scores.

Of the students in this subgroup who used fractional notation on the pizza problem, those who provided context in their solution scored better than those who did not (see Table 3). The context variable is significant ($p < .01$) in explaining the difference in scores.

Note

Data for this study was collected at a program supported by the National Science Foundation under Grant No. 9813902. The opinions expressed in this report are those of the authors and do not necessarily reflect the views of anyone at NSF.

Table 2. Frequency Counts for Notation and Score (N=86)

Notation	4	3	2	1
Fractional	25	2	14	16
Non-fractional	22	1	3	3

Table 3. Frequency Counts for Context and Score for Students Using Fractional Notation (N=57)

	4	3	2	1
Context	17	0	6	5
No context	8	2	8	11

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HOW 4-6 GRADE STUDENTS IDENTIFY ALGEBRAIC PROBLEMS THAT ARE ALIKE AND RANK THEM BY DIFFICULTY

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Problem setting has been researched as a factor that influences students' perception of problem likeness as well as success in solving problems (Krutetskii, 1976; Schoenfeld, 1985; Silver, 1981). The majority of these studies were conducted with high school or college students. Researchers discovered that expert problem solvers are able to recognize problem structure and disregard problem setting, as compared to novice and less successful problem solvers who attend to setting first. There is a paucity of research on young students' categorization of algebraic problems. The reported study promotes a better understanding of 4-6 grade students' perception of problem similarity and gains insight into the students' reasoning processes as they compare algebraic problems with several unknowns by difficulty and by likeness.

To find out whether young students categorize algebraic problems by setting or by structure, 25 students were interviewed prior to any formal instruction in algebra. An interview protocol was designed to provide consistency of the interview format. The interview included four parts. The first part of the interview helped students become comfortable talking aloud and describing their step-by-step procedures. The second part of the interview was designed to have students clarify methods they employed while solving the problems. They were asked to recall the solution methods they used and provide rationales for their solution steps (Dobrynina, 2001). The third part of the interview required students to order problems by difficulty. Students were asked to select the easiest and most difficult problem from among the two-variable problems and then repeat the procedure for the three-variable problems. With every selection, students were prompted to provide a rationale. The fourth part of the interview required students to consider all six problems, group those they thought were most alike, and provide a justification for each grouping. The results of the third and fourth parts of the interview are the focus of this paper.

The study used six problems: three problems in the study contained two variables and three problems contained three variables (unknowns) and relationships among variables were represented as systems of two or three linear equations. Symbolic representations of problem structures are shown below. In the equations, x, y, and z are unknowns and A, B, C, D, and E are natural numbers less than 30.

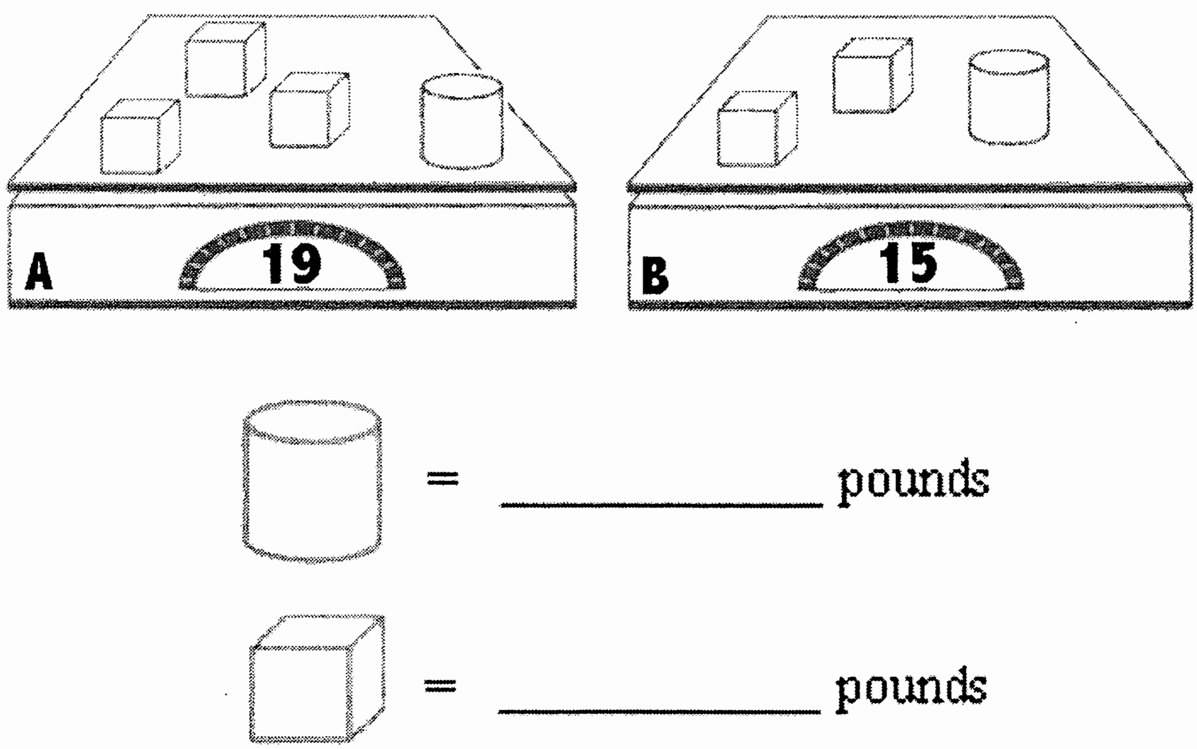
(1) $3x + y = \mathrm{A}$
$2x + y = \mathrm{B}$
$x = ?\quad y = ?$

(2) $2x + y + z = \mathrm{C}$
$x + y + z = \mathrm{D}$
$2x + y = \mathrm{E}$
$x = ?\quad y = ?\quad z = ?$

Each structure was represented in three settings. Three problem settings were used to present the two- and three-variable problems: weight scales, advertisements, and frame equations. Weight scales and advertisements utilized pictorial presentations with information in text. Frame equations, by contrast, employed an abstract presentation with the same text information. Below is an example of two-variable weight scale problem.

What is the weight of each block?

- Same blocks have same weights.
- Different blocks have different weights.



Results of the Study

Students were asked to compare six problems, group those that are "alike", present as many problem groupings as they desired, and provide reasons for their groupings. More than 50% of students stated that all six problems were alike because they could be solved by the same algebraic reasoning method, specifically, "You can always find the difference between pictures to find a missing thing (value), and then you can replace this found value in other pictures to figure out the rest of the object(s) (unknowns)."

Fifteen students stated that the six problems could be put into two groups: one group of two-variable problems and one group of three-variable problems. Reasons for this grouping included: "There is one less thing to consider in the first three problems than in the last three"; "There is one more step to take in the last three problems than in the first three"; "The number of pictures to look at is different, and number of things to figure out is different." Since students were asked to present as many groupings as they wanted, eight of the 15 students who put six problems into two groups, also grouped all six problems together. Fourteen students suggested that the six problems could be put into three groups as follows: weight scales, toys and geometric shapes. Reasons for this grouping involved descriptions of the nature of the unknowns as blocks, toys, or shapes. These fourteen students also stated that the six problems could be grouped differently. Thus problem setting grouping was not the only way the students viewed problem likeness. They grouped problems by solution method and by number of variables, as well.

The results of the study revealed that young students could group problems as "alike" based problem structure or nature of solution strategy. Furthermore, they were able to recognize similarities among problems despite differences in settings. This findings are in concurrence with Krutetskii (1976) who found that mathematically talented students of 7 to 10 year old were able to detect problem structure and ignore problem context; abilities that are crucial to obtaining mathematical information and solving problems (Schoenfeld, 1985).

When comparing the problems by difficulty, seven students selected the frame equation as the "easiest" problem, three students selected the weight problem, and one student selected the advertisement problem. When asked to provide rationales for selecting the frame equation as the easiest problem, students gave the following reasons: "I had practice with problems #1 and #2"; "The same strategy (comparison) can be used here as for the first two problems"; and "It is easy to find a difference here." For the "hardest" problem, six students named the weight problem, three students named the advertisement problem, and two students named the frame equation. When asked to provide justifications for their choices of the weight problem as the most difficult, students gave the following reasons: "It was a completely new problem for me"; "I had no clue how to start; and "I did not see a pattern at first."

Fourteen students declared that all two-variable problems were of equal difficulty. When those students were prompted to elaborate on reasons for their decisions, they said: "In all three problems you can compare A and B and find the missing object"; "The problems are set up differently but you can use the same strategy to solve them"; "You can use the same steps to find two things because one picture (A) has three of the same things and one of another, and the second picture (B) has two of the same things and one of another, so B has one less thing than A"; and "All problems have four objects in A and one object is taken away in B." Students' comparisons of three-variable problems and rationales for their ranking revealed similar results.

In summary, it was surprising that more than 50% of 4-6 grade students were not distracted by problem setting when they compared and ranked problems by difficulty and grouped them by likeness. Students recognized problems that had similar mathematical structures. They did not use algebraic terminology such as variable or unknown when they compared problems and provided reasons for their selections. Students did, however, use their own names to refer to unknowns, calling them "things" or "objects." The study revealed that elementary school students could solve and compare challenging problems with several unknowns without prior instruction. The results implied that algebraic instruction might begin earlier than upper middle grade school. To gain greater insight into this possibility and to determine the level of assistance students need, the "think aloud" algebraic problem solving might be accompanied by prompts, probes or solution hints.

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STUDENTS' CONCEPTUALIZATIONS IN A SETTING THAT INVOLVES COVARIATION BETWEEN FUNCTION AND RATE OF CHANGE

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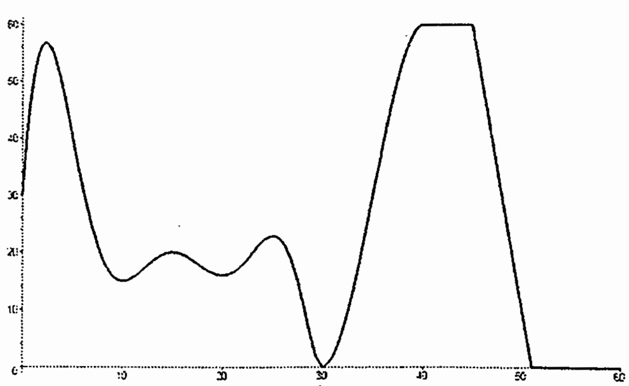
This study documents engineering students' conceptualization when they were requested to interpret a graph that represents the filling of a tank with a variable in-flow of water, but at the same time, there is a tap with a constant out-flow (10 liters/second). The only tap that has control is the in-flow of water. The focus of the study was to observe, how the students interpret relevant aspects of the graph, but in terms of the *manipulation* of the tap which controls the in-flow (e.g., What happens with the tap in the maximum, minimum or inflection points? Is it being opened or does it being shut or remain fixed?) That is, we want to see if the students can make any connection between the covaration of volume and the change of in-flow represented by the action of opening or shutting the tap.

Introduction

To assess student's understanding of calculus concepts, it is important that they are able to make significant connections between the mathematical ideas seen in a course and be able to apply them to interpret and describe e.g., the graph below. Thompson (1994) has pointed out that coordination of images of rate and rate of change are fundamental to reason about accumulation. In the same vein Carlson et al. (2001) states that covariational thinking is basic "in the development of students' understanding of the major conceptual strands of calculus" (p. 145).

Subjects, Methods and Procedures

The participants were twenty students that had taken a differential calculus course at a public university of Mexico. The graph (volume versus time) used in the study was:



The students were asked to make a description of the graph in terms of the actions of the tap that controls the in-flow of water, e.g., "at what time-intervals is the tap being opened, being shut or does it remain fixed or it is completely shut?"

Features of the Task

Analyzing what happen with the handling of the tap that controls the in-flow of the water in the tank based on the graph is not an evident issue e.g.. Is it enough (sufficient) to focus on the volume's increase or decrease (represented in the graph) to determine what is occurring to the tap in terms of being opened or shut? Is it necessary to attend simultaneously the fluctuations of *net* consumptions (first derivative = inflow *minus* outflow, with constant out-flow) and associate these ideas with dynamic geometrical view point (enact or imagine the behavior of tangent line over the curve as the time varying in the functions' domain, and therefore observe the changes in the value of the slope) The establishment of these connections are crucial in order to analyze what happens with the handling of the tap, e.g., if the first derivative is positive it means that the in-flow is larger than the out-flow, but this data is not sufficient to know if the tap is being opened or being shut. We need the information of the changes of the *net* consumption. This gives the information about how the curve is bending. Both data allow to determine if the tap is being opened or being shut. In order to, help the students to analyze the situation, the first derivative (a concept) can be *manipulated* through the actions to open or shut the tap. We assume that this tangible support can promote the student's development of connections (rate of change and changing rate of change)

The data was gathered from a questionnaire and an interview. The former contained fifteen questions. The answers were categorized in highlight tendencies. The concepts involved were: increase or decrease function, first derivative, maximum and minimum, change of concavity and inflexion points. As an illustration, some questions of the questionnaire were:

"At what time-intervals does the change of in-flow increase or decrease?"

"At what instant is the in-flow the same as the out-flow?" (maximum and minimum)

"At what instant does the tap which was being shut begin to be opened or vice versa?" (inflexion points)

Presentation of Results

The most recurrent conceptualization was the students noting the increase or decrease of volume represented in the graph, as the key aspect to determine if the tap which controls the in-flow was being opened or being shut that is if the volume increases or decreases, then the tap is being opened or being shut ("we know when the tap is opened because the graph shows an increase of the volume and being shut because it shows a decrease of the volume") Noted that the students left aside or didn't see the change of net consumption (first derivative) as relevant information for the analysis of what happens with the tap. This misconception leads the students to another misunderstanding about what happens at maximum and minimum points e.g., "the tap is shut in the maximum point and is open in the minimum point".

Most of the students were not able to link the handling of the tap going from opening to shutting or vice versa with the change of concavity or inflexion points. These actions were associated with what happened at the maximum and minimum points, e.g., "the tap is being shut and being opened in the maximum or minimum".

Conclusions

The framework of Carlson, et al. (2001), permits to explain the above misconceptions: the students showed a lack or loose coordination of how the variables are changing with respect to each other [e.g., volume (function) and first derivative (net consumption)]. These findings, in addition to the fact that the students confused the volume with the change of the in-flow, caused an ill interpretation of the graph. In spite of these difficulties, we believe that this type of tasks has potential to asses the mathematical ideas in calculus and promote student to connect ideas, thus to achieve a comprehensive learning.

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UNDERSTANDING STUDENTS' COGNITIVE TENDENCIES IN SOLVING ALGEBRAIC PROBLEMS: AN EXPLORATORY STUDY

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The main purpose of this paper is to construct an understanding of low achievement students' cognitive tendencies in solving algebraic problems. In terms of cognitive development, it is an interesting exploratory to go through some processes of algebraic expressions and analyze the abstract levels. The common cognitive tendencies in processing the algebraic expressions between the more concrete and the more abstract were highlighted in this paper. Forty-four students, enrolled in my 1st-year Junior College Mathematics core course, participated in this study during the 2001-2002 school year. It was found that their cognitive tendencies were trapped in a lower and/or more intuitive level of algebraic expressions. This research proposes another picture of students' algebraic thinking. The recognition of students' algebraic thinking may serve as a basis to improve school algebra curriculum and instruction.

The foundation for the research is a proposed theoretical framework that originated from the perspectives of historical development and cognitive development in mathematics. From historical perspectives, the development of algebraic concepts has been classified into two constructs: expression and abstraction (Boyer, 1991). These two constructs serve as a base of cognitive framework for this study. In school mathematics the learner is expected to translate algebraic expressions between the more concrete and the more abstract and vice versa (Filloy & Sutherland, 1996). Learning to operate fluently in algebraic expressions, to organize them and to represent them in alternative forms is the key ability to find exact solutions for equations. The process of finding solutions for equations has always been conceived essentially as the heart of algebra curriculum. In this study, by observing how these participating students find solutions for equations or/and inequalities, I identified these participating students' cognitive tendencies in dealing with algebraic expressions. On the other hand, the process of abstraction leads to the formation of mathematics structures of higher order and broader generalization (Skemp, 1987). It is interesting to know these students' cognitive tendencies in formatting new algebraic expressions with respect to each abstract order and/or generalization.

As a teacher-researcher, empirical data were collected through my own mathematics classroom in a natural setting. Forty-four students, who were considered to have low achievements in mathematics, participated in this study during the 2001-2002 school year. All participating students were invited to complete two problem-solving tasks. Each problem-solving task consisted of twenty algebraic problems. These algebraic problems were constructed on the basis of junior high school algebra curriculum. They were carefully designed to reflect these students' cognitive tendencies of algebraic expressions at different levels of abstraction. For example, how these students solved a first-degree equation and a second-degree equation respectively was assumed to show their tendencies of algebraic expressions at successive abstraction. The writing samples of two problem-solving tasks from these students were mainly used to interpret these students' cognitive tendencies in algebra. Semi-structured interviews, which conducted to these participants, were utilized to support the interpretations of their writing samples. The predetermined theoretical framework was applied to explain the collected data. It was found that the cognitive processes of these students were trapped in a lower and/or more intuitive level of algebraic expressions. These students tended to (a) recognize a given structure of algebraic expressions to an elementary one, (b) avoid the manipulation of algebraic symbols in expressions, (c) decompose all the quantities, (d) prefer more intuitive and natural numbers as answers, or/and (e) represent a quadratic function as a linear graph. This study shows that students with low achievement in mathematics tend to over simplify the algebraic expressions and abstraction because they have difficulties with higher order structures and formalism.

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USES OF CURRICULUM MATERIALS IN HIGH-SCHOOL ALGEBRA: THE CASE OF AN "OLD-FASHIONED" TEACHER

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This paper reports preliminary results of a larger study conducted to explore the nature of algebra teachers' pedagogical content knowledge, by looking at how teachers use curriculum materials in high-school algebra courses. Research in the learning of school algebra has shown that students apply the rules of symbolic manipulation without understanding the reasons behind them and that many of the recurrent mistakes are a consequence of such lack of understanding (Bednarz, Kieran, & Lee, 1996). The quality of teaching, more than curriculum (e.g., textbooks) has been signaled as a source for such outcomes, but so far little has been done to understand the relationship between teachers and textbooks, and in general, to understand the nature of the knowledge that is needed to teach algebra. I am interested in learning how and why high-school teachers select, modify, and use textbook content in teaching algebra and in using this knowledge to understand the impact of reform in the teaching algebra.

Theoretical Perspectives

I used Artigue's and colleagues' framework (Artigue, Assude, Grugeon, & Lenfant, 2001) which describes three interrelated dimensions of knowledge: Epistemological (ED), Cognitive (CD), and Didactical (DD) that influence different "professional gestures" and the competencies that teachers exhibit in their teaching of algebra. The framework is compatible with and complementary to Shulman's (1986) categorization of knowledge and it is useful because it articulates characteristics of teachers' knowledge particular to algebra. The ED encompasses notions about the historical development of algebra, the distinction between object and tool (Douady, 1984), and notions about the mathematical structure of algebra. The CD includes knowledge about general learning theories, difficulties and misunderstandings with algebra, problems of the relation between algebra and arithmetic, and knowledge of meaning of different semiotic representations (graphs, symbols). The DD includes knowledge of the curriculum organization (e.g., goals of algebraic teaching by grade or progression of tasks), textbooks, reform materials, tests, motivation, and so forth (Artigue et al., pp. 26-27). In this study, the emphasis is on teachers' curricular knowledge within the didactical dimension. In particular I posed the following question: how do teachers use textbooks and other materials to define an algebra curriculum? I report here results of a pilot case study that explored this question.

Methods and Data Sources

I conducted a case study (Stake, 1995) of an algebra teacher who teaches algebra 1 in a small (450 students) magnet high school in the Midwest. The teacher had the freedom to select the textbooks for his classes. After working with an integrated curriculum for a year, he chose an "old-fashioned" algebra textbook (Dressler, 2000) which offers a sharp contrast with the standards-based curricula students have encountered in all their previous schooling. With this case study I wanted to single out issues that would be pursued as hypothesis in future studies about the nature of knowledge needed for teaching algebra. Data came from weekly class observations, notes, and debriefing; videotapes of selected classes; collection of textbook materials, worksheets, tests, and other written activities developed by the teacher; and three types of teacher interviews: linked to particular lessons, general interviews in which particular problems and questions were posed, and video-tape interviews, in which the teacher and the researcher observed segments of his teaching and discussed its potential for creating discussions for future secondary teachers. Data were collected in the academic year 2001-2002 and were codified into broad categories according to the three dimensions of knowledge with further data collection used to refine and provide descriptions for sub-themes, a method consistent with the constant comparative analysis (Corbin & Strauss, 1990). The teacher read documents produced by the researcher to test for validity of the assertions.

Results

Derek is an engineer by training and became a teacher after 5+ years in business for personal reasons; he holds a master's in mathematics and has been teaching in this high school for 9 years "just when the [NCTM] Standards were published." He selected the textbook (Dressler, 2000) over other "integrated" because he feels "very comfortable with everything [it is] doing" he thinks the material is learnable and consistent whereas the other [integrated] books "I needed to jump around, I either liked the material or the problems and I felt that there was always something new and no continuity." He sensed a lack of focus and was unable to "cover all the material; I felt that it was difficult to develop the ideas... This book is simple, clear for teaching and learning; and has plenty of examples and problems." He follows the textbook, uses most of the examples in instruction, and assigns almost all (either the even or the odd) problems in each section, occasionally skipping some lessons (about 4 in all). He supplements with two types of worksheets, puzzle-type (Marcy, 1983), because with these "students can check by themselves whether their work is correct" and teacher-created type, which usually contain problems taken from examples drawn from Dressler (2000) or from other textbooks. Derek describes this class as a "normal group of kids" and frequently highlights that one of his goals is for students to be autonomous in their learning. One of the manifestations of this goal is his decision of not using class-time for checking homework—which consists of about 15 problems taken from the book, assigned weekly—because "some kids do learn without doing every [assigned] problem" and because "they are old enough" to be responsible for their own learning and assume the consequences of not practicing. Students receive extra points in tests when they do their homework. On rare occasions Derek has brought ideas not present in the textbook (e.g., big Xs to find two integers that added or multiplied give two given numbers) but had not attempted to link them to the content studied. The class has a relaxed atmosphere, with students freely raising questions and volunteering answers, and a special sense of togetherness. Most of the time students work individually but in groups helping each other and much of their talk is about personal issues. Students have acknowledged publicly their appreciation for the class and their sense that they are learning a "lot of stuff" and that they feel " very capable of doing mathematics, more than anytime before" some for the first time. During class Derek frequently remarks the importance of working hard and learning the materials and in all the interviews, he has talked about how the reform has fallen short in providing answers to pressing questions of teaching and learning. I have observed a continuous and uniform progress in students' ability in solving the problems posed in the textbook.

Discussion

The question that I sought to answer-how a teacher's use of curriculum materials defines an Algebra 1 curriculum-falls within the realm of the DD of the framework but the other dimensions (ED, CD) were present and relevant for the analysis. Two themes helped in explaining this teacher's professional gestures with respect to his selection, use, and modification of curriculum materials in algebra: "Algebra is a tool for future work" (ED, DD) and "Ambivalence about reform" (DD, CD). Derek's view of algebra as a tool is pervasive in his talk and work. Epistemologically algebra consists of "algebraic manipulations, modeling world problems using variables and solving equations" a knowledge that makes a "well rounded person" and enables one to "pursue any career." Didactically this view of algebra justifies his selection of textbook and other materials because that is knowledge "you can put your hands on," meaning you know when it is new and when it is not. Derek's ambivalence about the reform is also present in his talk and work. His view of the reform curricula as treating too many topics which do not do justice to any of them has implications in his didactical choice of the textbook; a choice that is also supported by his cognitive knowledge: the reform curricula do not take advantage of students' capabilities of making abstractions because there is "so much emphasis in concrete work, with problems that are not real for the students" and so little room for abstraction that it makes the "students think that they are just doing the same thing over and over again."

Derek has a reasonable mathematical preparation and experience and has welcomed the reform; but he has used reform textbooks that have done a poor job in maintaining the purpose and abstract complexity of algebra. As a person with autonomy to make a choice, he consciously rejects alternatives that researchers see as fostering a greater and deeper engagement with the subject because from *his* point of view, students' engagement comes from having a sense of control of what they are learning. A critical textbook analysis of reformed curricula could illuminate whether indeed students can not experience such sense of control and a study on how teachers cope with engagement with the subject matter given in the curriculum is then urgently needed.

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THE EFFECT OF A MULTIPLE REPRESENTATION-BASED ALGEBRA CURRICULUM ON A STUDENTS CONCEPTUAL UNDERSTANDING OF FUNCTIONS

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This study focused on examining the students' ability to make connections between the different representations and the students' ability to use the various forms to solve quadratic equations. The students showed a significant increase in their level of understanding through the use of a multiple representation-based curriculum.

There is an agreement among mathematics educators about the importance of the function concept in high school mathematics (NCTM, 2000). Lloyd and Wilson (1998) state that it is important to remember that one representation of a function may have "varying limitations or strengths in different contexts" (p. 253); therefore the ability to move among the representations is necessary to be able to interpret various problem situations.

Theoretical Framework

For the purposes of this paper, multiple representations refers to the presentation of a concept or process using numerical, graphical, and symbolic viewpoints (Noble, Nemirovsky, Wright, & Tierney, 2001; Porzio 1999). Skemp (1979) described conceptual representations as schemas or structures that are constructed by relating other concepts or objects to one another.

Objectives

This study was focused on examining the students' ability to make connections between the different representations and the students' ability to use this knowledge to categorize the cards into groups that represent the same functions using different representations. The purpose of the research is: 1) to determine whether students are able to make the connections among the various representations of functions, and 2) to determine whether students' ability to make the connections among the representations can be influenced by teaching that stresses connections between representations.

Methods Of Inquiry And Data Sources

The setting for the study was a large southeastern university. The students were enrolled in two different sections of an intermediate algebra course. The students in the classes participated in lectures, used graphing calculators for class work, homework, quizzes, and tests, classroom discussions, group activities, and submitted their homework online. The study was conducted during a 15-week constructivist teaching experiment, including pre and post task-based interviews. The subjects were taught in a class that emphasized and encouraged the use of multiple representations. Pre and post function sorts were conducted with individual students. These function sorts were videotaped and questioning was employed to determine student reasoning behind a particular sort. The teacher-researcher and a non-participant observer analyzed these videotapes. The type of sort they perform will be analyzed by the teacher-researcher and an observer to determine the level of thinking behind the sort. The levels of thinking will be classified according to the following table:

Level of Sort	Type of sort	Example	Level of Mathematical Connections
1	Surface sort	Student sorts graphs in one pile, tables in one pile and algebraic representations in a pile	No connections between representations
2	Surface sort with some additional sorting factor	Student puts the graphs together, tables together and algebraic representations together; use some mathematical scheme to separate further, e.g. put all quadratic graphs together, all linear graphs together, all linear equations together, etc.	No connections between representations, but some mathematical understanding about the type of representation
3	Partial Connection Sort	Student puts the graph of one type of function with either the equations or table of values of the same type of function; the graphs of linear functions together with one of the other types of the same function	Connections between representations are being made, though not between all three representations
4	Full Connection Sort	Student places the tables, graphs and algebraic representations of the same function together	Connections are being made between all three representations.

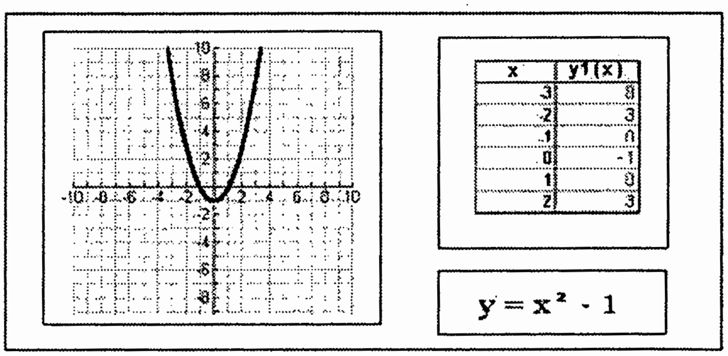

Figure 1. Sample of 3 types of cards used in card sort.

The number of students moving from a level with no connections to a level in which connections were made and analyzed using a distribution-free signed rank test (Wilcoxon) to determine if the number students making connections between the representations has increased.

Results

The a Wilcoxon Signed Rank Test indicated that there was enough evidence to state that there was a difference in conceptual levels of students between the pre and post-card sort when teaching using a multiple representation-based curriculum. The sort level of the students' increased 1-2 levels after they had been taught function concepts using a multiple representation-based curriculum.

Subject	Pre-Sort	Post-Sort	Difference
1	2	4	2
2	1	2	1
3	1	2	1
4	1	3	2
5	1	3	2

$n = 15$
Wilcoxon Statistic = 15
$p = .03$ (significant)
Achieved Confidence 96%
Confidence Interval is (1.00, 2.00)

Conclusions

The a Wilcoxon Signed Rank Test indicated a significant strengthening in students connections between representations which would allow students to be more flexible with the representation they use to solve a problem and to move more freely between the representations. Therefore, multiple representation-based curriculum should be used in the teaching of function concepts. As educators, we need to help students make the connections between the various representations.

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SYSTEM OF EQUATIONS AND ALGEBRAIC SUBSTITUTION: SYNTAX AND SIGNIFICANCE EXTENSION

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It is of interest to describe both the way in which children apply a learned syntax in order to solve linear equations with an unknown (using a geometric model) to a *system of linear equations with two unknowns*, as well as the new significances obtained through this extension process. In this presentation we analyze the way in which 12 to 13 years old children begin to solve a system of linear equations both within the context of word problems solution and that of pure syntactic solution.

Background

Following the research program proposed by Filloy, Puig and Rojano (Rojano, 1994), which aims to clarify the details of algebraic language acquisition, we accept the *Local theoretical models* (Filloy, 1990) as our theoretical framework. From this perspective, the use of *algebraic substitution* implies an abstraction level --both in the interpretation and in the use of algebraic symbols-- higher than that needed to operate with a single unknown in the solution of one linear equation. Specially regarding the solution of certain system of equations such as (S):

$$4x - 3 = y$$

$$6x = y - 7$$

The *algebraic substitution* entails putting into practice the idea that *an unknown* which represents an unknown quantity *can be expressed in terms of another unknown.* So, a problem with two unknowns can be reduced to a problem with just one unknown quantity and then it is possible to apply the syntactic skills previously learned so as to solve linear equations involving one unknown.

When dealing with algebraic substitution the idea of the unknown becomes more complex, since algebraic expressions are used to represent unknowns. Moreover, algebraic relation and algebraic transformation do change. For example, the idea of equivalent equations and transformations that can be applied to obtain equivalent equations is extended (e.g., see Freudenthal, 1983; Kieran, 1998; Matz, 1980).

The theoretical framework of *Local theoretical models* provides the semiotic tools to analyze these different levels of complexity, permitting both the study of the mathematical formal meaning of these concepts and of the meaning assigned by their use for solving problems: their *pragmatic meaning* (Filloy & Kieran, 1989). This theoretical framework allows, then, the study of strategies and *intermediate codes* produced by the learner as he or she becomes more competent in the use of mathematical knowledge.

Analysis Results

In this presentation, Manuel and Mariana's cases are analyzed (P and Mr, respectively). These cases belong to the study *La adquisición del lenguaje algebraico: la operación de la incógnita* developed by E. Filloy and T. Rojano (Rojano, 1984; Filloy & Rojano, 1989, 2001). Here the analysis is carried out coming from the view of the research project *School algebra as a language: the first algebraic syntax and semantics elements* (Solares, 2001). Within our project a clinical study with children from 12 to 14 years old is being carried out at a school center in Mexico, and the results and questions that arose from the afore mentioned previous project are being reformulated:

> What are the difficulties derived by the learning of *algebraic substitution* when solving verbal and syntax problems with two unknowns? How do learners extend their already learned syntax in order to solve equations with just one unknown?

The analysis of the interviews with P and Mr offer evidence that allow us to say that *algebraic substitution* is essential for manipulating the unknown.

When a competent subject makes the *algebraic substitution* he or she uses the equivalence of two algebraic expressions. For example, in the system (S) a competent user considers as equivalent the expressions "$4x - 3$" and "y", for him or her these are names of the same object: the corresponding algebraic function. In contrast with this, we have that spontaneous reading by children can be centered upon the operations necessary to calculate the value of "y". This difference on meanings attributed to the algebraic expressions is recurrent in algebra teaching, in which case the teacher is a competent user and the pupils are learners, generating thus difficulties not clarified yet by research.

Also, the analysis of these interviews shows that those children that have become competent users in the *manipulation of the unknown* (Filloy & Rojano, 1989) generate solution-strategies for verbal and syntactic problems with two unknowns. These strategies can either become obstacles for the learning of equation systems, or general methods for their resolution –thus the necessity for their study.

For example, when solving problems, children can express an unknown in terms of the other by using the context. And when solving equation systems they can develop strategies based upon the extension of the arithmetical use of the sign 'equal', as P did. He constructs the transformation consisting in the addition of a same amount to both members of an equation. This transformation maintains both the equivalence of members and the value of the operated unknowns. Thus, besides numbers or identical expressions, P is able to add complete algebraic expressions as long as they are equivalent (in order to solve systems like (S)):

P. "...now I will combine these two (both equations), I have "*10x - 3 = 2x - 7*" (he added both equations), and I have to eliminate this "*-3*" here and as a result I have "*10x = 2y - 4*"; now I will use the ***criss-cross method*** which gives me as a result..."*4x + y - 10 = 6x + y*", (he adds the left member of one of the equations to the right one of the other), now I will simplify as much as possible, I eliminate "*4x* ", the result is "*2x*" here I eliminate the "*y's*" and the result is "*2x = -10*" from which I may deduce that "*x = -5*", then ..."

Lastly, this analysis suggests the existence of the risk of generating syntactical errors during the extension process of the notion of the unknown and its syntactical manipulation, providing a perspective for the analysis of certain algebra syntax errors different from those presented by Matz (1980) and Booth (1984) (refer to Filloy, Rojano, & Solares, 2002).

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COGNITIVE TENDENCIES IN THE SOLVING WORD ALGEBRAIC PROBLEMS USING SYMBOLIC MANIPULATORS: CLINICAL STUDY ON THE ALGEBRAIC SYNTAX-SEMANTICS RELATIONSHIP

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The objective of this investigation is:

1. To carry out lessons in the symbolic manipulator's screen, making the student to meditate on what has and what brings in her again (formulas, algorithms, etc.) so that he generates and use algebraic expressions with the purpose of exploring, to discover and to manipulate things that he ignored. Beginning this way the observation stage.
2. To investigate the way as the symbolic manipulators using for represent and solve word algebraic problems by students, relating them with their previously arithmetic experiences and the progressive use of the Mathematical System of Signs algebraic.
3. To characterize cognitive tendencies, product of solving word algebraic problems to take the part to the students, using the symbolic manipulators.

The fellows that begin in the learning of the school algebra face the necessity of developing an operability it has more than about symbols that not very before corresponded to relating and meaning sources characteristic of the Arithmetic, but that now they have, completely different senses. These phenomena can be studied from a point of view that considers to this school content as a Mathematical System of Signs in use. With foundation in the above mentioned and with the use of symbolic manipulators, as a programmable calculators, it is sought to create an atmosphere for modeling and solving word algebraic problems, in which think about interesting queries in the algebraic syntax-semantics relationship. Interesting results in the solving arithmetic-algebraic problems, were given with didactic models that put emphasis in the necessity of being competent in more and more abstract and general uses of the logical-mental representations that are required to par excellence reach a full competition in the algebraic method, the Cartesian method (MC), this necessity, is the one that contrasts this method with the competitions that required other two: the method of successive analytic inferences (MIAS) and the analytic method of successive explorations (MAES). The analysis of these models guides and they tune the formulation of verbal algebraic problems that they generate and propitiate the use of algebraic expressions, with the purpose that the students explore, discover, meditate, manipulate, check, operate and validate the things that they ignore and this appear in the symbolic manipulator (formulas, algorithms, etc.) and that they can move, to cut, to erase, and to invoke when it is wanted.

In the investigation utilizing the theoretical and methodological proposal of the Theoretical Models Local, and the notion of Mathematical System of Signs. The Theoretical Models Local doesn't privilege some component grammar, logic, mathematics, cognitive, pragmatic, of teaching or of communication, rather it builds Models Theoretical Local only adapted for specific phenomena but able of taking into account their four components; Teaching models, models of Processes Cognitive, models of Formal Competition and Communication models.

It is necessary also to use a notion of sufficiently wide of Mathematical System of Signs, so that it serves as tool of analysis of the texts that the students take place –these texts are conceived as the result of processes of sense production- since the Mathematical System of Signs is the product of a process of progressive abstraction, either in the History of the Mathematics or in the personal history of the student. The stratification of the Mathematical System of Signs is a result of this process. They are studied then, Mathematical System of Signs intermissions, together with their corresponding personal codes –that the students invents- to discover the obstructions that take place to try with different available Mathematical System of Signs, while he is trying to be competent in the use of a new Mathematical System of Signs and of having a good performance in terms of the pragmatic meaning certain socially.

CAN STUDENTS' USE OF REPRESENTATIONS BE INFLUENCED BY TEACHING?

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It is the purpose of this research to determine if teaching with multiple representations will enable the college level remedial math student to broaden their network of mathematical knowledge and allow them to see familiar concepts in a new light so that they can construct meaning for the concepts and claim ownership of them. The hypothesis is that this expanded network of related concepts will allow them to choose to a representation other than symbolic manipulation to solve a problem when convenient and applicable.

Studying the effect of teaching on student use of different representations to solve problems will promote deeper and better understanding of the psychological aspects of learning mathematics. If a link is established between a method of teaching encouraging the use of multiple representations and a students' willingness to change from traditional algebraic manipulations to using an alternate representation to solve problems, classroom teaching can be modified to help students move freely between representations. This will enhance their mathematical understanding and increasing the structure and hierarchies of their networks of mathematical knowledge.

Experience strongly influences what representation a student will choose to use to solve a problem (Keller & Hirsch, 1998). "Because each representational format has varying limitations or strengths in different contexts, it is beneficial to have the choice of which representations to employ and the knowledge needed to make such a choice" (Lloyd & Wilson, 1998, p. 253). For students to have this choice and this knowledge, they must have experience with each different type of representation. "Mathematics teachers' help students learn to use representations flexibly and appropriately by encouraging them as they create and use representations to support their thinking and communication" (National Council of Teachers of Mathematics, 2000, p. 284).

This study examined students' choice of representation (algebraic, graphical or tabular supplied on the testing instrument) to solve five basic algebra problems. The 13 students who participated in the study took a pretest and a posttest and the type of representation they used to solve the problem was recorded. On the pretest if the student did not know how to solve the problem, he or she was asked to select the representation that was most familiar or with which they were most likely to use to solve the problem. Because of the small sample size and unknown distribution of the data, a nonparametric Wilcoxon Signed Rank Test was used to evaluate the significance of the differences between the number of problems the students solved algebraically and the number of problems the students solved either graphically or with a table on the pretest. The same was done on the posttest. The pretest showed no significant difference between using algebraic and using a graph or a table (testing a two-sided alternative for a difference obtained a p-value of .485). The posttest showed a significant change in method of solving the problem. The difference between the number of times a students chose to solve a problem algebraically was significantly less than alternative methods (testing a one-sided alternative for a difference between number of times a student used algebraic methods vs. the number of times a student used an alternative method was less than 0 obtained a p-value of .002).

This study is a preliminary study examining the effects of teaching with multiple representations on student choices in how to solve a basic algebra problem. The study found that students can be influenced to choose an alternative way to solve a problem when taught several different ways to interpret a problem. There was no indication in the study that students abandoned symbolic manipulation altogether, rather the evidence seems to suggest that students incorporate other methods of solving problems in their network of knowledge of algebra, and move from one representation to another with equal consistency.

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Volume 2: Research Reports, Short Oral Reports, Poster Presentations

PME-NA XXIV

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History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implementation thereof.

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Preface

It is with great pleasure that we at the University of Georgia host the 24th annual meeting of PME-NA as we also hosted the 4th annual PME-NA meeting exactly 20 years ago in 1982. A quick glance at the differences between then and now shows how the field has grown and changed: The proceedings that year were about 250 pages long, and there were 34 papers presented in 7 topic areas. This year the proceedings will be nearly 2000 pages long with over 200 presentations in 15 topic areas. Despite the growth of the field and the organization, the hallmarks of collegiality and open intellectual exchange remain.

The theme of this year's conference is *Linking Research and Practice*. The theme is intended to highlight the interplay between the ways that research is used in practice and the ways that research grows out of practice. The invited plenary speakers were asked to address the theme in their areas of expertise by challenging the audience to think critically about the research we do–the questions we ask, the methods we use, the contexts in which we do research, the people with whom we do research, how we communicate the results of our research, etc. We hope that those in attendance as well as those who will read these papers in years to come will be stimulated to think deeply about our roles as researchers and consumers of research.

We received over 250 proposals for sessions at PME-NA and are grateful for the work of the many reviewers who helped shape the program. We undertook only structural editing (format and references) on the final papers so as to leave intact the integrity of the authors' work.

We wish to express our appreciation to the many people at the University of Georgia who have made these volumes and this conference a possibility, including, but not limited to, Patricia S. Wilson, Margaret Caufield, Elizabeth Platt, Salli Park, Bernice Peters, Teresa Banker, Nancy Williams, Joseph Allen, Brian Wynne, and all of the faculty and graduate students in the Department of Mathematics Education.

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Assessment

A CONTENT ANALYSIS OF EXIT LEVEL MATHEMATICS ON THE TEXAS ASSESSMENT OF ACADEMIC SKILLS: ADDRESSING THE ISSUE OF INSTRUCTIONAL DECISION-MAKING IN TEXAS

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High stakes tests are increasingly used to monitor systemic improvements in mathematics and teachers are expected to rely on the results of such tests to adapt their instructional practices. We examine the Texas Assessment of Academic Skills (TAAS) over a period of three years to examine to what extent its results can be used to guide instructional decision-making. We present the results of an expert content analysis of the 10^{th} grade TAAS mathematics test for 1999, 2000, and 2001 which reveal that problem solving objectives mask significant content emphases. We further examine the variation in raw scores by objective across grades and years to show this information is not reliable enough to guide changes in instruction. We examine the sampling of topics within objective to gauge the distribution across topics. Finally, we attempt unsuccessfully to account for changes in difficulty using a combination of changes in sampling, item characteristics and composition of distractors. This leads us to question the utility of providing teachers raw data by objective and points to the urgency of developing better methods to link content analyses and psychometric methods of scoring.

Since 1995, the state of Texas relied on the Texas Assessment of Academic Skills (TAAS) to drive the current accountability system for Texas schools. Along with attendance and course credits, passage of he TAAS was required for a student to graduate. A new assessment, the Texas Assessment of Knowledge and Skills (TAKS) will be implemented in 2003, and although better aligned with secondary topics, it will produce similar data artifacts for teacher use. TAAS results, in the form of raw scores by test objective, scaled scores known as Texas Learning Index (TLI) scores, and item analysis percentages have been provided to teachers and administrators to guide them in improving school performance and providing quality instruction in mathematics. Our interest was in examining these data to determine if they could validly be used for these purposes. Our interest in the question was piqued by data showing discrepant results as student scores on TLI increased while raw scores have declined, typically explained by the state as differences in test difficulty (Confrey & Carrejo, 2002). We sought to understand at the level of classroom practice, if the data provided by raw score for each of thirteen objectives could be used to make instructional changes, as it is widely believed.

In the case of the TAAS test, we propose that even within a single test and its analyses, drawing data-driven conclusions reveals how psychometric traditions for creating scaled scores and equated tests seem to produce contrasting results from content analyses. The key issue of content validity (Heubert & Hauser, 1999) "whether a test measures what it purports to measure and what conclusions can be drawn from the results – and whether the conclusions or inferences drawn from the test results are appropriate" (p. 71), requires one to link these two arenas. Thus our goal is to link psychometric and content analytical traditions by building protocols to help teachers refine their analysis and increase their statistical capacity to interpret and critique data and create plans of action (Confrey and Makar, 2002). In addition, we wish to argue for the role of outside content experts when conducting analyses of the TAAS. We argue for more discipline-based protocols for conducting and interpreting item analyses, and point out that these analyses are likely to be neglected or unpublished under the current accountability system.

In this paper, we provide an account of our content analysis of the mathematics portion of the TAAS test for the 10th grade level over a period of three years (1999, 2000, and 2001). We address, based on available information, the question, "Can data provided to teachers in the form of raw scores by objective provide an accurate description of student performance and support instructional decision making?" In addressing this question, we relied on answers to three sub-questions:

1. Using an independent content protocol, do the constructed TAAS objectives provide an adequate description of the content tested?
2. How much variation is there in students' mean performance by these objectives across the grades over time?
3. Can we identify the possible factors that result in changes in performance by objective through a content sampling and item analysis?

As development of the new assessment, TAKS, is underway, examination of the TAAS, its construction, format, and content, provides important experimental groundwork for future analysis of the new test, designed to be closely aligned with TAAS.

TAAS Construction and Format

According to the test makers, the Texas Education Agency (TEA, 2001), construction of the TAAS first involves a review of state standards for mathematics, the Texas Essential Knowledge and Skills (TEKS), to determine appropriate learning objectives by grade level. Following this review, educator committees develop drafts of test objectives, linked to the TEKS, to be reviewed by teachers and other specialists. The objectives are then refined based on feedback, and sample test items are written. Educator committees develop guidelines for assessing each objective which include eligible test content, test item formats, and sample items. A test blueprint is then developed by item writers, some of whom are identified as former teachers. TEA curriculum specialists then review the items. During this process, item review committees judge the content, difficulty, and bias of each item. Further item revision occurs and then they are field tested. Field-test data is analyzed for reliability, validity, and possible bias. Data review committees then use statistical analyses to determine if items are worthy of being used. The final blueprint is then developed. Field-test items are placed in an item bank and the final tests are built from the bank and are designed to be equivalent in difficulty from one administration to the next.

TEA outlines thirteen objectives grouped in three categories: concepts, operations, and problem solving. Within concepts there are five objectives: (1) Number Concepts, (2) Relations and Functions, (3) Geometric Properties, (4) Measurement, and (5) Probability and Statistics. Within operations are (6) Addition, (7) Subtraction, (8) Multiplication, and (9) Division. Within problem solving are the four remaining objectives: (10) Estimation, (11) Solution Strategies, (12) Representation, and (13) Reasonableness. TAAS questions are clustered in groups of four under each objective with the exception of Solution Strategies and Representations which have eight items each at the exit level. Therefore, sixty questions comprise the final multiple choice exit test with each question having a possible 4-5 answers. Furthermore, TEA outlines which state mathematics standard(s), i.e. TEKS standard(s), is tested by each aforementioned TAAS objective.

Examining TAAS Content with an Expert Protocol

Our approach to the analysis began by:

1. Creating a protocol based on our selection of topics relevant to K-12 mathematics education. The topics comprising the protocol are indicative of the breadth of subject matter teachers and researchers find most relevant to many, if not most, implemented curricula. Topics were also chosen based on current research in mathematics education.
2. Obtaining copies of the tests (available from the TEA website) and categorizing each item according to the constructed protocol without referring to their respective TAAS designation. If a general topic was missing for an item, it was identified and the categories were adapted until we could account for all items on the test.
3. Comparing our categorization with that of TEA's for all three years.

Our protocol includes the following topics (followed by subtopics): a) numeration (scientific notation, sequences), b) geometry (angle, congruency, coordinate plots, formula, spatial reasoning, similarity, symmetry, and vertices, edges, and faces), c) measurement (linear/perimeter, area, volume, weight, and the Pythagorean Theorem), d) operations (addition, subtraction, multiplication, division), e) rate (ratio and proportion), f) probability (combination, experiment outcomes, and mean, median, mode), g) data and statistics, and h) equations (literal, equation with variable, inequality). For d) operations, we constructed the following table to indicate specific number types used in the items we placed under this topic (see Figure 1).

Likewise for g) data & statistics, we constructed the following table to indicate the representation format involved in the question and the representation involved in the answer choice (see Figure 2).

Our categorization separated the topics of ratio and data analysis as separate categories (see Figure 3).

Our categorization separated the topics of ratio and data analysis as separate categories (see Figure 3). We used it to examine and display content distribution on the test over the three years. The following chart shows the number of items per topic. It is in marked contrast with the test specifications which specify four items per objective for most objectives and eight for solution strategies and representation.

One can see the emphasis on certain topics such as addition and subtraction remain relatively unchanged from year to year. However, other topics such as mul-

	Whole	Integers	Fractions	Decimal	Percent	Exponents
Addition						
Subtraction						
Multiplication						
Division						
Combined ops.						
Comparison						
Representation						
Ordering/sequence						

Figure 1.

		Answer form					
		Read value	Table	Line	Bar	Verbal	Circle
Question form	Table						
	Line						
	Bar						
	Verbal						
	Circle						

Figure 2.

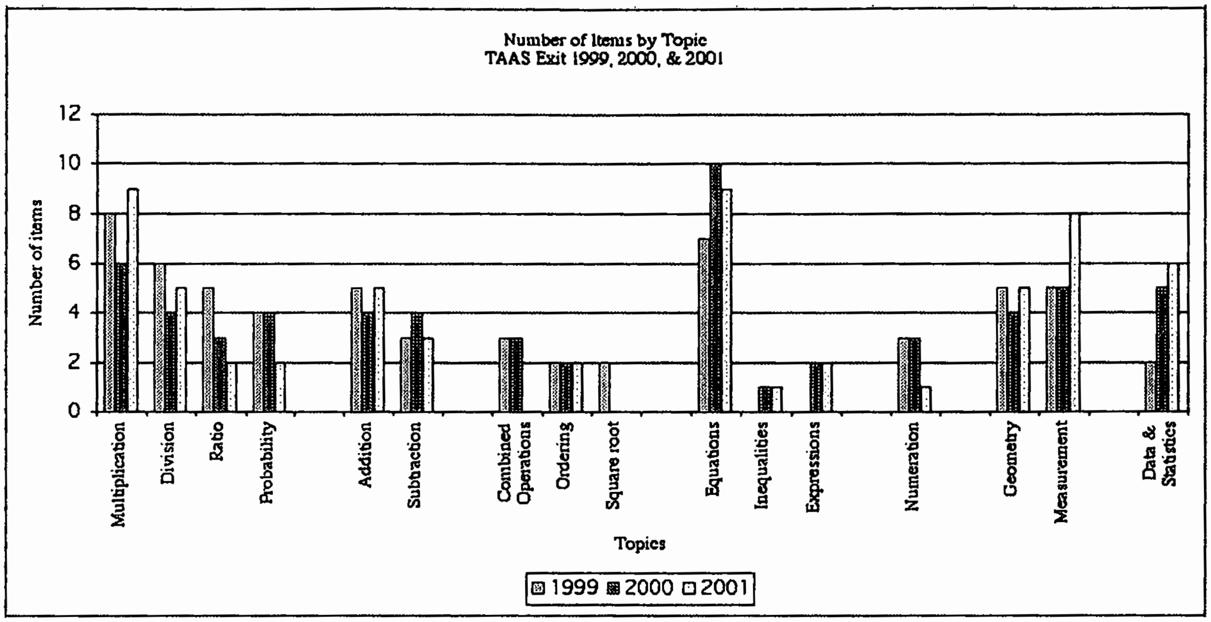

Figure 3. Distribution of items by topic, 1999, 2000, 2001

tiplication, division, ratio, equations, and data receive different emphasis from year to year. (Notably, measurement received a heavier emphasis in 2001 compared to the previous two years.) The major difference lies in our elimination of the Problem Solving objectives (Estimation, Solution Strategies, Representations, and Reasonableness) and categorization of their items into content categories. The following table lists the item numbers in each of these Objectives and the heading under which we placed them on our protocol (see Figure 4). Also note the variation in topics year to year.

The total number of items in the 1999 test related to multiplicative structures (multiplication, division, ratio, rate) is eighteen out of the sixty items. The total number of items in 2000 is eleven out of the sixty and the total number for 2001 is fifteen out of sixty. We found multiplicative structures far more heavily represented than teachers recognized, and hence its importance in passing TAAS could have been easily neglected. Furthermore, TAAS Objective 5, Probability and Statistics, contains four items, yet on the overall tests, the total number of items related to data, statistics, and probability under our protocol is six out of sixty for 1999, nine out of sixty for 2000, and eight out of sixty for 2001. This area is also underrepresented in the test specifications relative to the actual test. We conclude that constructs in multiplicative structures and data and statistics (including probability) comprise 24 out of 60 or 40% of the items for 1999, 20 out of 60 or 33% of the items for 2000, and 23 out of 60 or 38% of the items for 2001. This analysis indicates how the Problem Solving category with its four Objectives mask content that teachers should emphasize in their instruction. We understand why the Problem Solving objectives were included but point out that they

Objective 10 – Estimation					
1999		2000		2001	
Item	Protocol Topic	Item	Protocol Topic	Item	Protocol Topic
23	Mult/whole	21	Comb.Op./whole	21	Mult/whole
24	Mult/whole	31	Equation/literal	23	Mult/dec.
36	Division/whole	42	Comb.Op./dec.	26	Data/line, read
37	Division/%	50	Mult/%	27	Mult/%
Objective 11 – Solution Strategies					
1999		2000		2001	
Item	Protocol Topic	Item	Protocol Topic	Item	Protocol Topic
22	Numer/sequence	22	Numer/sequence	25	Equation/literal
25	Rate/ratio	24	Meas/perimeter	28	Meas/area
26	Equation/literal	26	Rate/ratio	29	Equation/literal
30	Rate/ratio	28	Meas/perimeter	30	Equation/literal
31	Rate/ratio	30	Equation/literal	32	Meas/volume
34	Numer/sequence	34	Equation/literal	35	Meas/perimeter
39	Numer/sequence	35	Equation/literal	36	Meas/volume
43	Comb.Op./whole	43	Comb.Op./whole	38	Mult/%
Objective 12 – Representations					
1999		2000		2001	
Item	Protocol Topic	Item	Protocol Topic	Item	Protocol Topic
29	Equation/literal	25	Equation/inequal	22	Equation/vari.
32	Equation/literal	29	Equation/literal	24	Geom/coord
33	Equation/literal	32	Data/line, table	31	Division/%
35	Comb.Op./dec.	33	Data/line, table	33	Equation/inequal
40	Equation/literal	36	Equation/literal	34	Data/line, table
41	Comb.Op./%	37	Equation/vari.	39	Equation/vari.
42	Equation/literal	38	Data/table, line	42	Equation/vari.
44	Data/Venn	39	Equation/vari.	43	Equation/vari.
Objective 13 – Reasonableness					
1999		2000		2001	
Item	Protocol Topic	Item	Protocol Topic	Item	Protocol Topic
21	Data/bar, verbal	23	Data/table, verbal	37	Data/table, verbal
27	Mult/dec.	27	Data/bar, verbal	40	Data/bar, verbal
28	Mult/dec.	40	Equation/literal	41	Mult/whole
38	Mult/whole	41	Mult/dec.	44	Mult/dec.

Figure 4. Content protocol for objectives 10-13.

should be crossed dimensionally with the Content objectives rather than designed as their own categories.

Variation in Students' Mean Performance by Objective

Our second investigation concerned the variation in student performance for each objective. Teachers were using declines or gains in performance by objective as evidence of instructional need or success. We decided to examine the variation in students' mean scores in two ways. First, we examined student performance for TAAS assessments 1999, 2000, and 2001 disaggregated by objective. Then we examined performance over three years by grade level to see if any trend lines suggested patterns of improvement or decline (see Figure 5).

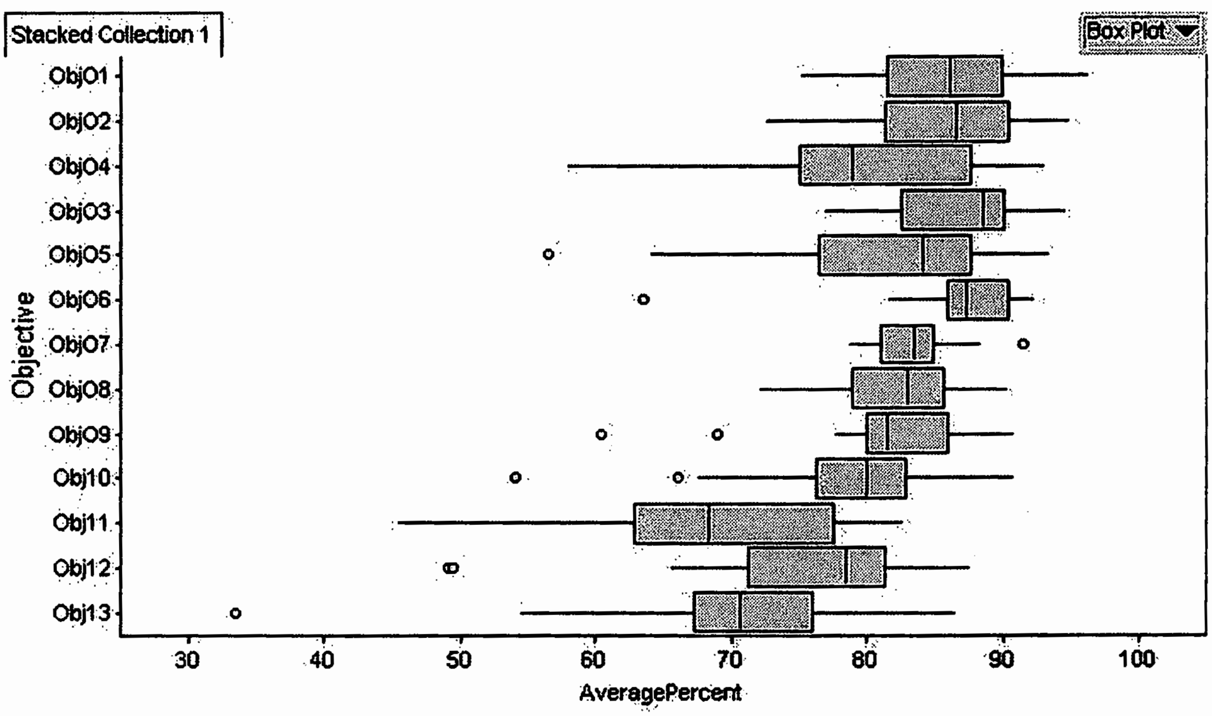

Figure 5. Variation in mean scores for all grades.

Note that two of the four Objectives related to operations; namely, Objectives 6 & 7 show little variation for all grades. However, Objectives, 4, 5, and 11, Measurement, Probability & Statistics, and Solution Strategies respectively, show considerably more variation. These data raise questions whether raw scores reported by objective are relatively stable enough over time to guide instructional decisions.

One possible explanation for the variation is that students are consistently gaining ground on particular objectives as teachers implement revised strategies for instruction. To examine this question, we examined student performance on the objectives individually by grade over three years. Our analyses revealed little consistency in results. From 1999-2000, over student performance measures on six Objectives increased, while performance decreased on the remaining seven Objectives. From 2000-2001, student performance increased on two Objectives and decreased on the remaining eleven. These data suggest that variation in student performance by Objective was due to improvements over time. Below, we provide the trend charts for Objectives 4, 5, and 11 for all grades as illustrative of the variation found.

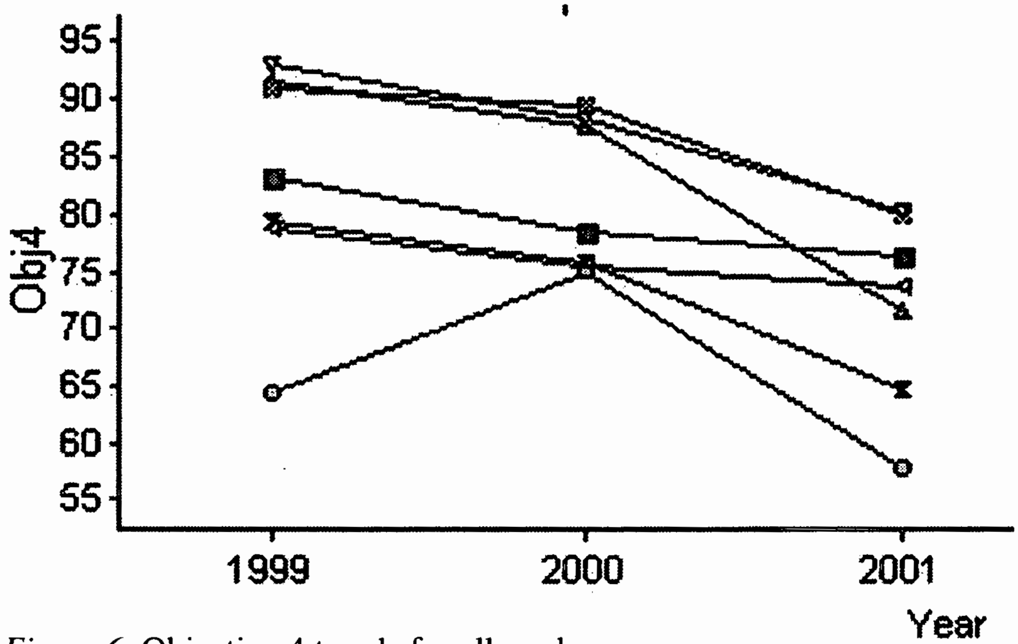

Figure 6. Objective 4 trends for all grades.

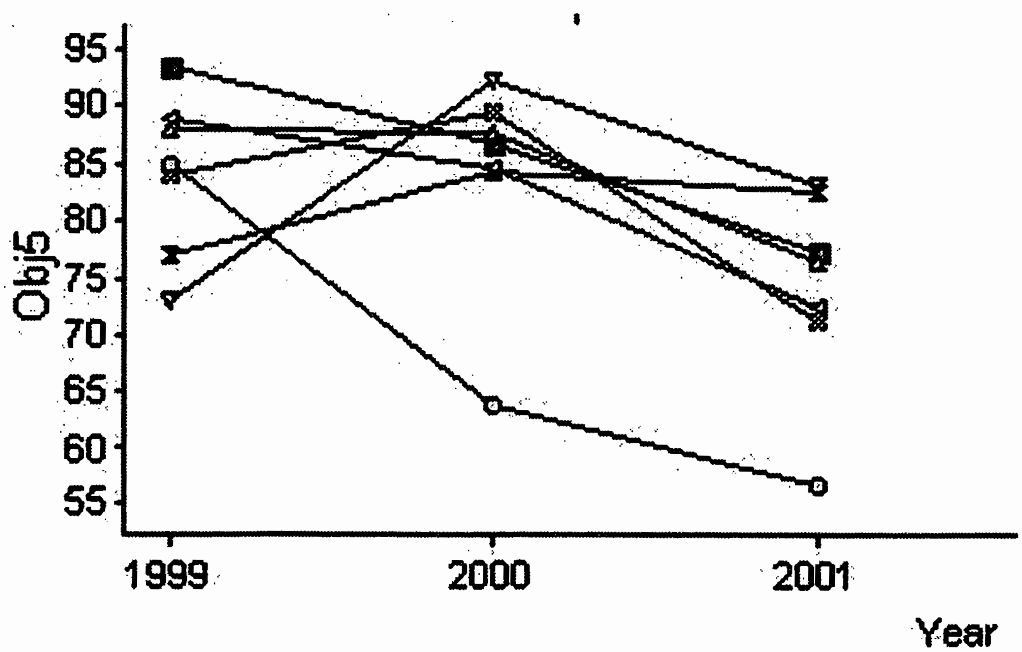

Figure 7. Objective 5 trends for all grades.

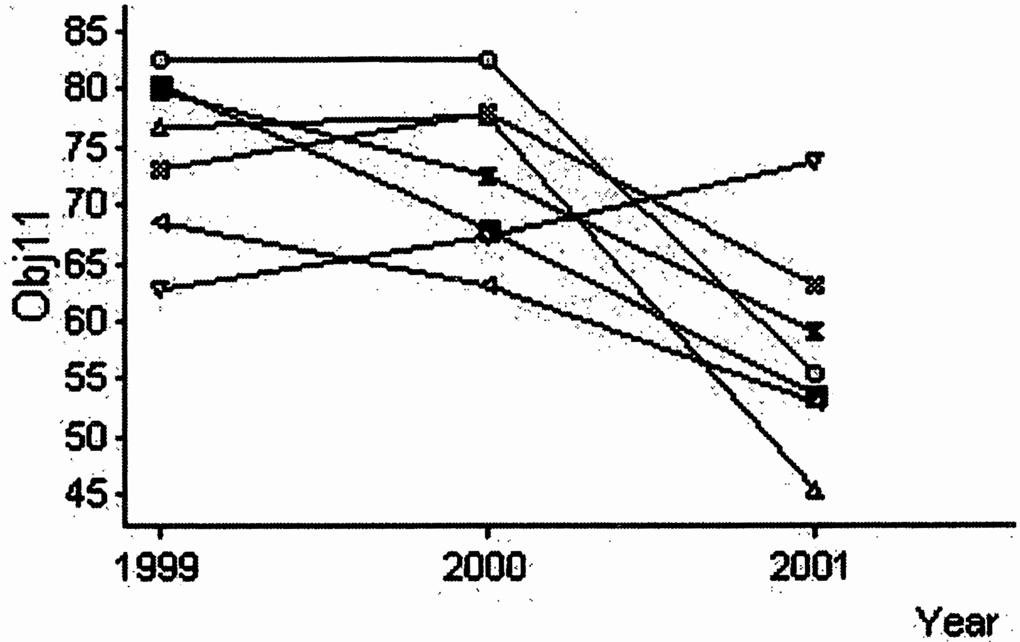

Figure 8. Objective 7 trends for all grades.

This analysis demonstrates that teachers cannot simply use changes in their mean student performance by objective to guide their planning. This seems obvious as one recognizes that the raw scores by objective are the product of not only the content of the objective but also the sampling of topics and the difficulty of the items and distractors. This led us to our next protocol for analysis.

Possible Content Factors Affecting Student Performance by Sampling and Difficulty

Returning to content analytical methods, we tapped into another possible source of information for teachers that could be useful in examining the variation in sampling of subtopics in each objective. TEA outlines which TEKS standards are being aligned with each TAAS objective. We created a table outlining the items clustered under each TAAS Objective. Within each TAAS objective, we identified which associated TEKS standard(s) each Objective tested (TEA, 2000). Each clustered item within a TAAS Objective was aligned with an associated TEKS standard (for example, see Figure 9). Our conjecture was that too little variability would permit the test to lose validity as teachers could virtually drill students on likely topics for inclusion. We expected that a valid test over time would sample proportionately across subtopics. We also expected that if teachers were assuming consistency in certain items, changes in the sampling of those objectives would lead to drops in performance.

Continuing this process for all the TAAS Objectives, we quantified the amount of variability in item sampling between 1999, 2000 and 2001. For example, since two items in 2000 are sampled from a new topic while two items remained from the old topic, we coded this as a 50% change. Eight item objectives can produce changes such

TEKS standards aligned with TAAS Objective 1: Number Concepts	1999	2000	2001
(1)(A) compare and order rational numbers	2, 15	2, 5	8, 20
(1)(C) approximate the value of irrational numbers	8, 17		
(1)(D) express numbers in scientific notation, including negative exponents		10, 13	3
(d)(3)(A) use patterns to generate the laws of exponents and applies them			16

Figure 9. TEKS to TAAS alignment for objective 1.

as 37.5%. Likewise, one item in 2001 differs in alignment with the 2000 test. This constitutes a one out of four or 25% shift in content emphasis. Overall, we calculated the percent change in content alignment from 1999 to 2000 and then from 2000 to 2001 (see Figure 10).

Obj	1		2		3		4		5		6	
Years	99/00	00/01	99/00	00/01	99/00	00/01	99/00	00/01	99/00	00/01	99/00	00/01
% change	50	25	75	25	100	50	100	50	0	50	0	0

Obj	7		8		9		10		11		12		13	
Years	99/00	00/01	99/00	00/01	99/00	00/01	99/00	00/01	99/00	00/01	99/00	00/01	99/00	00/01
% change	25	0	0	0	0	0	50	25	37.5	12.5	62.5	37.5	0	0

Figure 10. Percent change in content alignment.

Our initial analysis showed that from 1999 to 2000 and 2000 to 2001, five of thirteen objectives were unchanged. For 1999-2000, an additional four objectives and for 2000-2001, an additional seven objectives showed under fifty percent changes, showing only four objectives in the first year and no objectives in the second year were altered more than fifty percent. We examined the variation in student performance to see if there was a simple relationship between student performance and variation in sampling (see Figure 11). There was not. This is an area that requires a more sophisticated form of analysis.

Our final analysis included an attempt to quantify difficulty in three ways: task consistency, task characteristics, and distractors. Within task consistency, we considered whether the content being tested was comparable between years. When the content topic was the same, we noted changes in task characteristics including number type, language use, single-step versus multi-step procedures, and the use of the money context. Distractor analysis was performed at the level of identifying answer choices that could easily be disregarded an/or common misconceptions. We identified the Objectives that a) appeared to have task consistency, were amenable to test preparation, and should produce relatively stable performance; b) objectives with varied difficulty in the items (harder or easier) across years and c) objectives with varied items

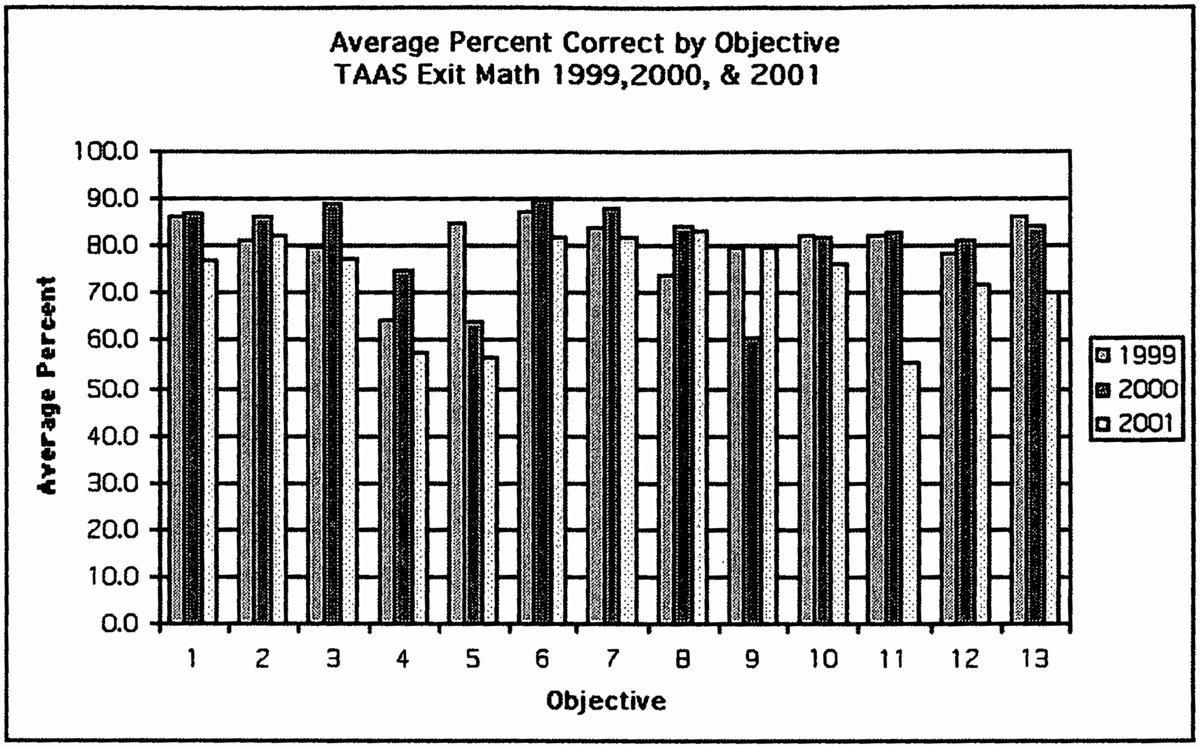

Figure 11. 10th grade student performance by objective.

and incommensurability in assessing difficulty. We compared our prediction and the item analysis provided by TEA of actual results of the percentage of students passing.

The results of our analyses showed the unreliability of making predictions based on our measures of item difficulty. We predicted that the 2000 test would be more difficult on all objectives based on our measures of difficulty except for Objectives 6 and 7 that we predicted would remain the same. According to the chart, we were correct for Objectives 5 and 9. Other results showed we were incorrect or only minimally correct for Objectives 10 and 13. Between 2000 and 2001 we had similar results. This is not surprising as test equating on TAAS is actually done using a Item Response Theory approach based in Rasch analysis. The difficulty levels are not publicly released.

Based on our analysis of TAAS for tenth graders, we found it difficult to draw any conclusions about student performance and improvement based on objective level analysis. These results suggest serious doubts about how teachers, after quantifying such results, are supposed to use this information to make a judgment about performance and thereby influence their instructional decision-making.

Conclusions

Our research was focused on examining whether the data provided to teachers for instructional decision-making were valid for this purpose. We worked to extend the analysis of the data in relation to the published test structure, and found that there were a number of problems in reconciling the results. The problem solving objectives obscured the underlying content dimensions making it difficult to judge the relative needs of students as regards topics. The variability in the student raw scores by objectives makes it unlikely that variations in student performance year to year represent real changes in student knowledge. A lack of trend data by objective suggests this variability is unlikely to represent systematic improvements in instruction. Finally, it did not appear that one could easily construct a content valid analysis of changes in difficulty in terms of topic selection, item characteristics or distractors.

In future work, we plan to continue to work to develop protocols that can validly guide teachers in undertaking content analyses of test results that can inform instructional decision-making. We hope to be able to link such analyses with the methodologies of test equating to determine whether there is a way to resolve the competing influences of psychometric test analysis and score preparation and content analyses. We encourage our colleagues in mathematics education to become similarly involved in close analysis of content dimensions of testing, to ensure that reform efforts at the curricular and instructional level are consistent with the messages given teachers from high stakes tests.

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CAN HIGH STAKES TESTING IN TEXAS INFORM INSTRUCTIONAL DECISION-MAKING?

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This paper reports on a study of high stakes testing in Texas as viewed from a ruling on the legality of the test for graduation. A summary of the Court ruling is analyzed as it relates to professional standards, empirical data and consequential systemic validity. An argument for increased empirical study of these accountability systems by content specialists is made if they are to be used as effectively as possible to improve classroom instruction.

High stakes testing is being used widely as a means to link standards to student outcomes within an accountability system designed to hold students, teachers and administrators responsible for systemic improvements in mathematics education. With the new Elementary and Secondary Education Act (ESEA) legislation and the Mathematics and Science Partnerships (MSP), this method of school reform is being extended nationally as a basis for receiving federal funds. It is essential for researchers in mathematics education to be observing and studying these evolving systems of education and ensuring that they are working to the benefit of students. This paper, two companion pieces (Confrey & Carrejo, 2002; Confrey & Makar, 2002) and a book, *Systemic Crossfire* (Confrey, in progress), present the results of two years of investigation of the Texas Assessment of Academic Skills (TAAS), its structure as a test, its content and its impact on classroom instruction. Our research is focused on the question, "can the results of high stakes testing inform instructional decisions?" rather than on the question "are the State scores going up?" Focusing on improvement in scores and the impact of tests on instruction are essential to ensure the accountability system is functioning as intended.

In previous work (Confrey, Bell & Carrejo, 2001), we argued that conflicting meanings of the terms standards and accountability can lead to "systemic crossfire" at the classroom level. We identified conflicts in the way accountability programs sought the improvement of standards (largely by monitoring rates of student passage on high stakes exams) while organizations of science and mathematics education scholars employed Standards as expressions of consensual conceptual targets for instruction. We examine how these tensions play out at the level of the instructional core as we conduct a form of research we call "implementation research" (see Confrey, Castro-Filhó & Wilhelm, 2000). Implementation research involves partnering with practitioners to improve student achievement and studying and documenting the catalysts and impediments to change using a model of systemic reform. Our cyclic model links student outcomes to professional development to teacher knowledge and community to implementation of standards-based curriculum back to student outcomes (see Confrey & Makar, 2002).

In this set of papers, we concentrate on unpacking the use of the high stakes test first as they are embedded in the legal frameworks of the state where graduation is dependent on passing the tenth grade test; then as seen through content analyses protocols (Confrey & Carrejo, 2002), and finally, as viewed by teachers undertaking inquiry studies on their students' performances on the test (Confrey & Makar, 2002). To set the legal context, we present a summary of the case: G. I. Forum vs. Texas Education Agency (TEA). We examine the ruling in relation to its interpretation of professional standards, empirical analysis, and its consequential systemic validity. Our findings offer lessons to other states as well as to Texas as they revise their examination system.

In 1999, nine students challenged the legality of the TAAS test as a requirement for graduation. Most had failed to pass the mathematics portion of the test, a sixty item multiple-choice test covering thirteen objectives. The TAAS exit test was given in 10th grade with up to six subsequent opportunities for students to pass parallel versions. The TAAS test is currently being replaced by the Texas Assessment of Knowledge and Skills (TAKS) which is designed to more fully measure the State's standards (Texas Essential Knowledge and Skills or TEKS). TAAS was heavily focused on arithmetic proficiency, and though touted as a higher cognitive test than its predecessor, TEAMS, it required little mathematics more advanced than single or double step solutions to word problems and interpretations of patterns in graphs and charts. More difficult items were added such as simple probability and a few literal equations. No calculators were permitted. The new test, TAKS, will however, be given in 9th, 10th and 11th grade with objectives covering algebra, geometry and problem solving and permit the use of the graphing calculator. Passing the 11th grade test will be required for graduation. Understanding TAAS, which has an eight year history and a court battle behind it is valuable in preparing the system what to expect from TAKS and in anticipating possible grounds for Court challenge. It can also inform accountability system development in other states.

The accountability system behind these tests remains quite stable. The raw score is converted into a scaled score, known currently as the Texas Learning Index (TLI) using a Rasch test equating methodology. Originally, in 1994, a raw score of 70% was set for passing by individual students, and subsequent tests have been equated to permit the Texas Education Agency to claim that one's learning index is an indicator of one's likelihood to be on track to pass the exit exam. To avoid low performing status, fifty percent of each subgroup greater than thirty students must pass the test, dropout rates must be lower than a given percent and the data must be of good quality across the district. This fifty percent rate has increased from 30 percent in 1996 as it is used a means to drive the system towards consistent improvement. However one must keep in mind that student graduation remains consistently dependent on obtaining the TLI score of 70, so that while the system (teachers and administrators) is given relief early on, individual students are not.

After reviewing the one month of conflicting testimony in the fall of 1999, Judge Edward C. Prado of the Western Division of Texas (San Antonio) ruled, on January 7, 2000 that "The Court has determined that the use of the TAAS examination does not have an impermissible adverse impact on Texas's minority students and does not violate their right to due process of law" (p. 2, opinion). First of all, it is important to recognize that while the Judge ruled against the plaintiffs, he did acknowledge that from the results of the test, there is a clear indication of disparate impact. Disparate impact discrimination "outlaws unjustifiable practices that are facially race neutral but that fall more harshly on certain minorities." (p 4). His determination of disparate impact was based on data showing that "The evidence is undisputed that the Defendants knew, based on early projections, that a larger number of minorities would fail the TAAS than would white students. The Defendants projected that at a cutoff score of 70 percent; at least, 73percent of African Americans and 54 percent of Hispanic takers would fail the math portion of the test." (p. 5)

How can a judge rule that there is disparate impact but not rule against the test? This can only occur if the ruling limits the rights of the Courts to intercede, or the Judge determines that the test is not part of the problem that produces disparate impact but part of the solution to address it. Judge Prado did both. Drawing on three areas of law: desegregation, employment, and testing law, the Judge set the boundary conditions for the case.

Education is the particular responsibility of state governments. Moreover courts do not have the expertise, or the mandate of the electorate, that would justify unwarranted intrusion in curricular decisions. On the other hand, these considerations cannot be used to tie a court's hands when a state uses its considerate power impermissibly to disadvantage minority students (p. 2).

Judge Prado decided that the Courts could intercede to a limited extent because for the students the high school degree was deemed as a property right that was being denied to them. However, he also limited the Courts influence over curricular and instructional mandate in relation to setting objectives and monitoring instructional practices, and determined that on the whole, the test was soundly designed and seemed to be contributing to improvement rather than further discrimination. His ruling was based on a set of related findings that form his argument:

1. There was sufficient *notice* of the curricular objectives in question through widespread distribution of the Texas Essential Knowledge and Skills and the plans for implementing a graduation test.
2. The TAAS test was described as "solely an objective measurement of mastery"

and it "accomplishes what it sets out to accomplish, which is to provide an objective assessment of whether students have mastered a discrete set of skills and knowledge" (p. 24). In making this ruling, Judge Prado determined that the testing was necessary to overcome subjective teacher evaluations and inflated grades. By doing so, he *rejected correlations with grades as* necessary evidence of test validity. Further, he eliminated the need *demonstrating predictive validity* that is, correlating the test with future student success in college or work.

3. The process of *test construction as consistent with professional standards.* This process, summarized in Smisko, Twing, and Denny (2000) involves the identification of a set of curricular objectives to be tested, test objectives, writing items by experts (including teachers, test writers), reviewing items, field-testing and further review. He wrote, "... there was no testimony demonstrating that Texas has rejected current academic standards in designing its educational system. Educators and test-designers testified that the design and use of the test was within accepted norms"(p. 29).
4. The pass rate was viewed as set in a manner consistent with professional standards and constituted the right of the state to demand that students' demonstrate *70% test mastery of the declared material.* Furthermore, he accepted the Rasch model of test equating which focuses on item difficulty parameters but not on item weighting as an appropriate way to adjust scores from year to year... Thus, the pass rate could vary relative to raw scores as indexed by Texas Learning Index score transformations.
5. The multiple administrations of the exit test were considered to be a valid way to avoid the stipulation that graduation should not be dependent on a single test score. The conjunctive requirements of passing TAAS, and accumulating course credits were also viewed by the Judge as *avoiding reliance on a single test score.*
6. The system of *remediation* was considered adequate as a means to provide students sufficient assistance in overcoming test failure. Judge Prado explicitly *validated test driven instruction* as a legitimate professional practice. On this, he wrote, "The Plaintiffs introduced evidence that, in attempting to ensure that minority students passed the TAAS test, the TEA was limiting their education to the barest elements. The Court finds that the question of whether the education of minority students is being limited by TAAS- directed instruction is not a proper subject for its review. The State of Texas has determined that a set of knowledge and skills must be taught and learned in State schools. The State mandates no more than these "essential" items. Test-driven instruction undeniably helps to accomplish this goal. It is not within the Court's power to alter or broaden the curricular decisions made by the State" (p. 26).

7. "The State of Texas... has designed the TAAS *accountability system* to reflect an insistence on standards and educational policies that are uniform from school to school" (p. 5). This uniformity in relation to essential skills and knowledge required for graduation helped the Court decide the test was fair, despite disparities.

8. He viewed the process of notice and test construction as a form of *curricular validity* and a surrogate to *instructional validity.* On this, he wrote, "in reviewing items, educators are instructed to consider the following issues: relevancy of the item, difficulty range, clarity of the item, correctness of the keyed answer choice, and plausibility of distracters. ... Reviewers are asked to assess whether or not each item on the TAAS exam covers information that was sufficiently taught in the classroom by the time of the test administration"(p. 9). This ruling relies heavily on the assumption that the reviewers of the tests know what is actually taught in the various schools around the state.

9. He accepted the position that *items showing diverse impact* could be retained on the test if it is a "fair measure of its corresponding state objectives *for all students*, and is free of offensive language and concepts that may differentially disadvantage minority students" from the testimony of S. Philips, cited p. 10.

10. He was convinced that the *empirical evidence supported progress* and rejected testimony concerning the increasing dropout rates particularly at grade nine. The evidence showed decreasing gaps between minorities and Whites on TAAS scores, showed statewide progress on the National Assessment of Educational Progress, and showed increases in enrollments in higher level courses and Advanced Placement exams. He wrote, "... the Court has had to weigh what appears to be a significant discrepancy in pass scores on the TAAS test with the overwhelming evidence that the discrepancy is rapidly improving and that the lot of Texas's minority students, at least as demonstrated by academic achievement, while far from perfect, is better than that of minority students in other parts of the country and appears to be getting better" (p. 4).

11. His ruling was also based on his claim that the plaintiffs had provided *no better alternative.* One almost sees in his ruling a lament that more effective systems of education are not in place. He wrote, "Ultimately, the resolution of this case turns not on the relative validity of the parties' views on education but on the State's right to pursue educational policies that it legitimately believes are in the best interests of Texas students. The Plaintiffs were able to show that the policies are debated and debatable among learned people. The Plaintiffs demonstrated that the policies have had an initial and substantial adverse impact on minority students. The Plaintiffs demonstrated that the policies are not perfect. However, the Plaintiffs failed to prove that the policies are unconstitutional, that the adverse

impact is avoidable or more significant than the concomitant *positive* impact, or that other approaches would meet the State's articulated legitimate goals. In the absence of such proof, the State must be allowed to design an educational system that it believes best meets the need of its citizens" (p. 7).

The impact of the case on the testing climate in the state has been to increase the adversarial relationships between those who support and oppose the tests, rather than to encourage the development of improvements to bridge that gap. For example, perhaps due to the success of the legal case, changes have been made to the TAKS testing system abandoning plans to use subject specific End of Course exams. We believe the action is designed to create a single comprehensive exam as a graduation requirement, thus aligning it more closely with the TAAS system and lessening its vulnerability to legal action. Having survived a legal challenge, TAAS has become a testing blueprint. It is essential that researchers in mathematics education are aware of the legal conditions surrounding high stakes testing and that we consider both theoretically and empirically the impact of these conditions on classroom practice, especially in relation to the quality of mathematical content. Our discussion of these issues is placed in the context of a critique of the ruling as it is related to three categories of use of high stakes data. These are:

1. Questioning whether the ruling is consistent with the professional standards of testing
2. Reporting on the contested data trends that were used to establish overall system improvement, and
3. Raising the neglect of consequential systemic validity as a weakness in the ruling.

The first category permits us to address more deeply the adequacy of the Prado ruling as regards points 1, 5 and 8 of his argument. The second focuses further discussion on point 10 and the third category of critique permits reexamination of points 6, 7 and 9 in the ruling. The second paper builds on the second type of critique and relates it to the limited adequacy of the data to guide instructional decision-making. The third further explores what is possible in relation to consequential systemic validity. We hope to propose a revised system of accountability and its links to the instructional core in subsequent work.

Professional Standards of Testing

In a relatively recent publication, *High Stakes: Testing for Tracking, Promotion and Graduation* (Heubert & Hauser, 1999), the Board on Testing and Assessment of the National Research Council discussed their interpretation of professional standards for testing. Using the primary constructs of reliability, validity and fairness, they emphasized the need for these powerful instruments of policy and accountability to be used carefully and wisely. They warn, "When tests are used in ways that meet technical, legal and educational standards, students' scores provide important information that, combined with information from other sources, can promote both learning and equal opportunity. On the other hand, they can reinforce and legitimate biases and inequalities that persist in American society and its schools. Used improperly, tests can have serious negative consequences—for individuals, particular groups, and society, as a whole" (p. 26).

A major message of the book is that " concern is also reflected in current psychometric standards, which recommend that a decision that will have a major impact on a test taker should not be made solely or automatically on the basis of a single test score, and that other relevant information about the student's knowledge and skills should also be taken into account (American Educational Research Association, American Psychological Association, & National Council on Measurement in Education, 1985; American Educational Research Association, et al., 1998, Standard 8.12)" The question is "does the opportunity to retake the TAAS test up to six times constitute an acceptable means to address this requirement?" According to Judge Prado, it does. Our reading of the testing standards indicates that multiple opportunities to take a test designed under identical specifications do not meet this standard. While repeated administrations might improve students' chances of eliminating the results of random error in their score, we interpret the standard to be addressing the need for qualitatively different forms of measurement. The legal ruling accepted empirical evidence of subsequent passes as sufficient evidence that adequate remediation had been provided and hence that multiple testing opportunities met the professional standard of non-reliance on a single score. Future statements of professional standards will need to address this question more explicitly.

The second professional standard compromised by the ruling involves the decision to rely on curricular validity rather than on instructional validity. The authors of *High Stakes: Testing for Tracking, Promotion and Graduation* state that "It is neither straightforward nor inexpensive to measure the content of actual instruction (Popham and Lindheim, 1981). As a result, there is little evidence to suggest that exit exams in current use have been validated properly against the defined curriculum and actual instruction; rather it appears that many states have not taken adequate steps to validate their assessment instruments and that proper studies would reveal important weaknesses (Stake, 1998). Our classroom research suggests that many urban students have been inadequately prepared on multiplication, division and ratio reasoning, in part, perhaps due to over reliance on calculators which were prohibited in TAAS testing. A failure to consider instructional validity compromises standards of fairness and leads to deficit thinking about certain student groups.

Challenges to Empirical Data

The second challenge to the ruling has been in relation to empirical data. Below, one sees the state of Texas's data that the students in Texas are improving in part as a result of the accountability system when the percent of students equaling or exceeding a TLI of 70 is graphed over time (see Figure 1).

In this graph, one notes that both the overall scores on the test are improving and the gap among Whites, African-Americans and Hispanics is decreasing. The graphs constant improvement and narrowing of disaggregated trend lines are cause for optimism. The fact that they are not linked to drop out rates forces one to ask whether the students are being lost in the process. Contention over drop out rates is common in the state (House Research Organization, 1999). References to improvements in NAEP and AP enrollments can be found elsewhere.

However, it is instructive to also examine same data by the mean score on the TLI rather than the percent passing (see Figure 2). This score will be affected by the performance of not just those students who were successful but also by those who scored poorly on the test.

These data are somewhat less promising with lower slopes and less narrowing of the gap. These results suggest that either the means vary over time less than the percentage pass as a function of the test equating methodology or that the improved successes of the students' passing the test are not accompanied by improved performance by students at the other end of the spectrum. Another possible implication of these data is that there is an excessive amount of effort by schools in improving their passing rates by working disproportionately with students who are near the cut-point known as the "bubble kids."

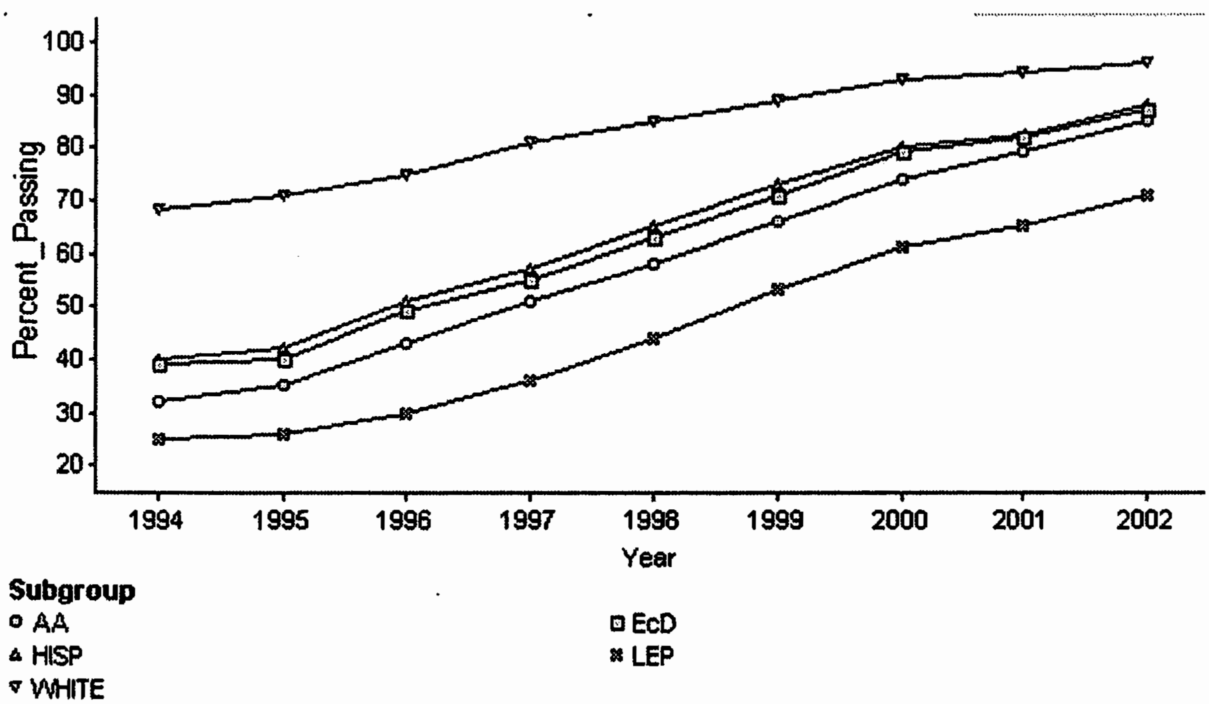

Figure 1. Percent passing over time.

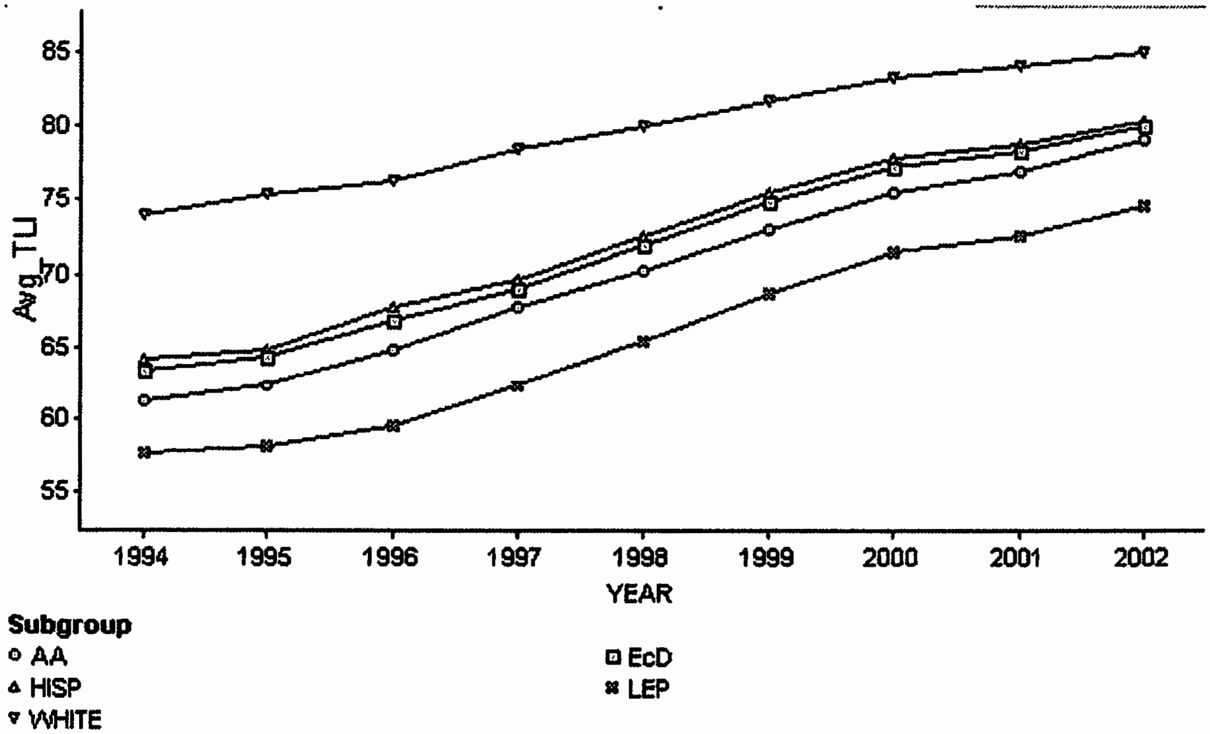

Figure 2. Average TLI over time.

A third set of data over time relates to the raw scores of students on the test. As shown by the next graph, the raw scores of students on TAAS exit test have dropped precipitously between 2000 and 2001 (see Figure 3). For the last two years, the average score on TAAS by students is 43.2 of 60 items. Keep in mind on a multiple choice test with four or five distracters; one can expect a score of 12-15 correct by chance alone. One immediately wonders if the loss in the mean in the raw scores is experienced by all students or only those near the bottom.

By examining a graph of the raw score corresponding to a TLI of 70, or a pass on the test for an individual, one can see that the students passing the test are scoring lower (see Figure 4).

We present a listing of the means and standard deviations by the whole group and subgroups to permit one to examine trends in the distributions as well as the means (see Figure 5).

We recommend an emphasis on the examination of distributions of scores rather than solely percent pass as a fundamental tenet of our research protocols. Only by doing so can one begin to determine if students in all performance levels are adequately served.

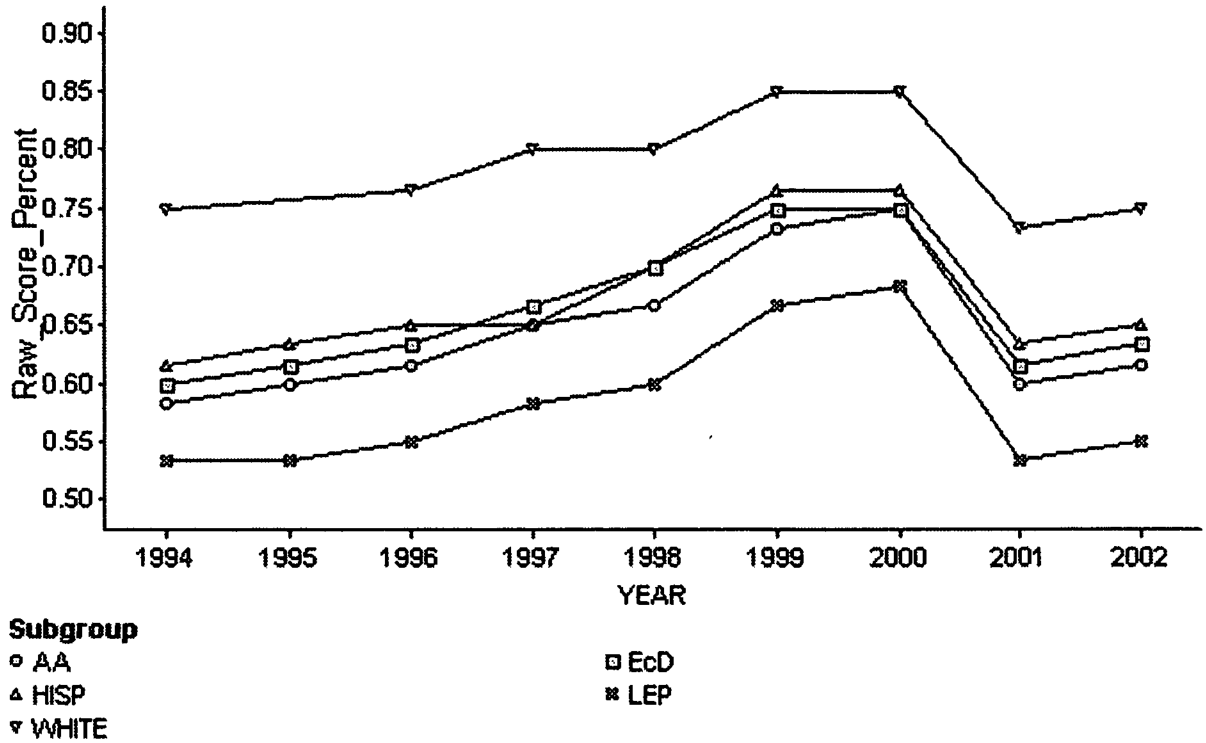

Figure 3. Average raw score over time.

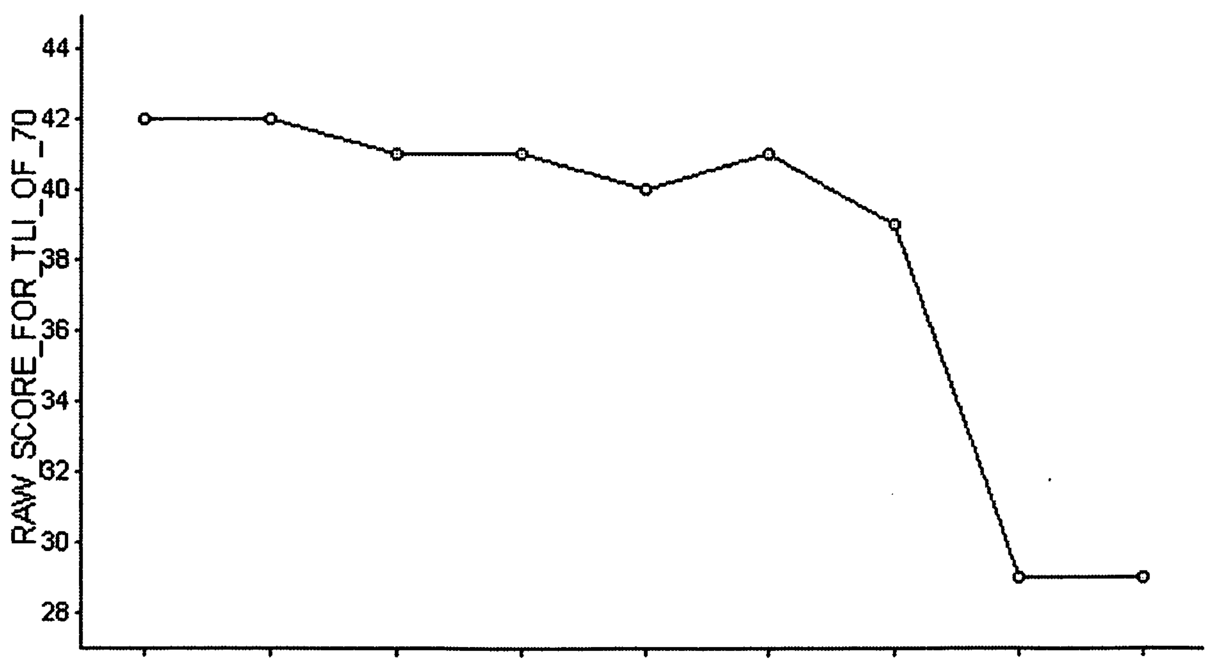

Figure 4. Raw score corresponding to a TLI of 70 over time.

	Year							
Ethnicity	1994	1995	1996	1997	1998	1999	2000	2001
All-Mean	41.9	42.9	43.6	45.4	46.2	48.9	49.2	43.5
All-SD	12.3	12.4	11.8	11.6	10.9	9.8	9	10.6
AA-Mean	35.3	36.3	37.7	40	41.2	44.4	44.9	38.4
AA-SD	12.2	12.5	12.4	12.5	11.6	10.9	10.1	10.7
HISP-Mean	37.7	38.4	40	41.7	43.1	46.4	46.9	40.4
HISP-SD	12.4	12.5	12.2	12.3	11.5	10.6	9.6	10.7
WHITE-Mean	45.5	46.7	46.8	48.8	49.2	51.4	51.4	46.6
WHITE-SD	10.9	10.7	10.3	9.7	9.3	8.1	7.4	9.4

Figure 5. Means and standard deviations by group and subgroups.

Reconciling the data between raw scores and TLI demands empirical tests. According to the Texas Education

Agency, the test items are becoming more difficult as the test is increasingly aligned with the state standards, TEKS. We have tried to examine this question using content analyses as reported in the second paper. We have also tried to examine the difficulty levels by Rasch analysis but these data are not public. Regardless of whether the test equating is done correctly, it is of critical importance that teachers are provided their classes data in the form of the raw scores disaggregated by objective, and an item analysis of the test. This is the data that teachers as a group use to adjust their classroom practices. In our second paper, we use content analysis and an examination of the objectives over time to see if these data can be useful in informing classroom practice. In subsequent work, we plan to undertake statistical analysis of the test equating process in order to link it to an examination of the test by objective. We are concerned to understand better how the psychometrics of test equating may be affecting the construct validity of the links between TEKS and test objectives and items.

Consequential Systemic Validity

The final analysis we offer is that the Court ruling seems to be very naïve as regards the impact on the testing system on the system itself. This is due to a tendency to view the education system as a production system rather than as a feedback system. We point to four ways in which this can lead to serious problems in school level practices.

1. The results of testing can be used to short circuit the curriculum and lead to teaching to the test As stated in *High Stakes: Testing for Tracking, Promotion and Graduation,* "A comparison of low- and high-stakes state testing programs found that, as the stakes of testing increase, "there is a point at which district strategies take on the flavor of a single-minded devotion to specific, almost 'game-like' ways to increase the test scores" (Wilson & Corbett, 1991:36 p. 173). Finding the line between reasonable test familiarity and excessive test preparation requires one to examine the question empirically.

2. The accountability system does not hold teachers accountable for all students, and hence over time, students in the bottom 30% of the achievement group may fail to be adequately served. A longitudinal empirical study of the consequential validity of the test for this group would permit one to ascertain this impact, but difficult to do as the privacy features of the data sets tend to make it difficult to examine. A similar argument can be made for dropouts who may leave the system anticipating or fearing failure (Haney, 2000). A closed model linking test performance to dropouts as a population model needs to be devised to monitor this issue.
3. Teachers are not provided with sufficient training in statistics nor are they provided with enough opportunities to access the data for study. Our third paper reports on efforts to use the test results in this inquiry based manner.
4. There is a perception that legal safety is increased by minimizing empirical investigation. Rather by accepting that consequential systemic validity must be investigated empirically and scientifically, we can increase the likelihood that the system will mature rather than polarize.

Conclusions

This paper and its companion pieces raise questions as to how well the accountability system in Texas is currently acting as effectively as possible as a useful guide to improving classroom practice. It argues that mathematics education researchers must become more deeply involved in examining the ways in which test construction and analysis can be informed by a deeper knowledge of content. Further, it suggests that serious questions of fairness may be neglected without more informed and empirical study of the links between testing and classroom practice.

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THE IMPACT OF AUTHORITY AND REFLECTION ON TEACHERS' USE OF OPEN-ENDED ASSESSMENT ITEMS

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Research has shown that mathematics teachers' current assessment practices are not consistent with recommendations in the mathematics education reform literature. One recommended strategy is the use of open-ended assessment items. This paper reports the findings of two studies designed to investigate factors that facilitated or impeded teachers' use of such items. Two middle grades and three secondary mathematics teachers who had participated in professional development projects designed to expand their understanding of the purposes and uses of assessment, as well as to change their instructional and assessment practices, were participants in the studies. Findings from the studies include that teachers' orientations to authority affected their use of open-ended assessment items, as did the focus and extent of the teachers' reflective thinking.

Introduction

The teachers in the studies participated in professional development projects designed to help them meet mandated student assessment requirements. The mandate was that at least 20% of the items on all classroom tests must require higher-order thinking on the part of students. County policymakers expected teachers to change their instructional methods to incorporate more reform-oriented instruction when faced with a requirement to do so with their classroom assessments. The professional development projects provided an analysis of the rationale for and characteristics of reform-oriented assessment practices, as well as the development of strategies for writing open-ended assessment items and practice in scoring student responses to such items. The projects resulted in open-ended item banks created by the teachers which they could use to meet the assessment mandate. The findings of prior research have pointed to such factors as resources, time, and knowledge as impediments to teachers' success with implementing alternative assessment strategies (e.g., Nash, 1993; Senk, Beckmann, & Thompson, 1997). The professional development projects addressed these factors in numerous ways, which included providing an open-ended item bank and practice in scoring student responses to open-ended items. Once attempts were made to control for the impeding factors, it was possible to investigate the more cognitive factors that affected the teachers' use of open-ended assessment items.

Theoretical Perspectives

Prior literature indicates a relationship between teachers' orientation to authority and their instructional practice (Thompson, 1992). Therefore it was reasonable to expect that orientation to authority would also affect teachers' assessment practices. The teachers' orientation to authority was investigated using epistemological development models (Baxter Magolda, 1992; Belenky, Clinchy, Goldberger, & Tarule, 1986; King & Kitchener, 1994; Perry, 1970/1999). These models imply a continuum from a dependence on authority for knowledge to a development and evaluation of knowledge in context.

During the studies it became evident that the teachers differed in the focus and extent of their reflection. Thus, a theoretical perspective on reflective thinking was necessary. Reflection was viewed from Dewey's perspective. Dewey (1933) defined *reflective thinking*, as the "active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (p. 9). The process of reflective thinking occurs in two phases. First, a person enters "a state of doubt, hesitation, perplexity, mental difficulty, in which thinking originates" (p. 12). The person recognizes something as problematic. What follows is "an act of searching, hunting, inquiring, to find material that will resolve the doubt, settle and dispose of the perplexity" (p. 12). The demand for a conclusion to a problem is the definitive characteristic of reflective thinking. Dewey stated that there are occasions in which something is considered problematic and a suggestion for resolution emerges, and yet reflective thinking does not occur. In these instances the person is not "sufficiently *critical* about the ideas that occur to him" (p. 16). The person takes the first answer or solution that occurs to her or him, and thus does not become completely immersed in the careful search for a resolution.

Methodology

The participants in the studies were two middle grades mathematics teachers (Sue and Leah) and three secondary mathematics teachers (Todd, Robin, and Keith), all five of whom participated in the professional development projects previously described. The qualitative data were collected during extensive classroom observations and from interviews, surveys, and artifacts. Sue and Leah were each observed teaching in an eighth-grade Algebra I class. Todd was observed teaching Algebra Concepts (a traditional Algebra I course taught over two years) and Algebra II, Robin teaching Algebra I and Calculus, and Keith teaching Algebra II and Pre-calculus.

There were two phases of inductive data analysis (Patton, 1990). The first phase consisted of a descriptive analysis in order to answer basic questions about the teachers' practices. The second phase consisted of analyzing the data with an emphasis toward interpretation using the theoretical orientations.

Findings

Beliefs, authority, reflection, knowledge, and constraints emerged from the two studies as factors that influenced teachers' use of open-ended assessment items. The present paper addresses two of these factors, authority and reflection. The five teachers varied in what they used as sources of authority for their teaching practices. The teachers also varied in the focus and extent of their reflective thinking. The focus of the teachers' reflective thinking refers to the issues that they found problematic. The extent of their reflection refers to the degree to which they were sufficiently critical in their search for resolution with respect to the issues they found problematic. Both authority and reflection are discussed below for each participant along with their impact on the teachers' use of open-ended items.

Sue

Sue's main source of authority for her teaching was the textbook. She progressed through the textbook, page by page, section by section, chapter by chapter. She presented the textbook examples and then assigned textbook exercises. She told the students that "The most important thing you are going to learn is in this textbook" (Ice, Observation). Sue's dependence on the textbook for the structure and content of her teaching was obvious when she questioned the sequencing of topics in the textbook but followed the given order anyway. The textbook included the topic of the mean of a set of numbers in the section of chapter 2 about dividing rational numbers. Sue said that it was "a weird place to put mean" (Comment to Ice, Observation), but she still taught the lesson according to the order and examples given in the textbook. During instruction, Sue focused on the procedures that were presented in the textbook. Since most of the text's items were procedural, that is what Sue's instruction reflected. Thus the authority of the textbook impeded Sue's use of open-ended items during instruction. Other sources of authority for Sue included high school mathematics teachers and courses and the county school system. These sources of authority facilitated her use of open-ended items because she believed her students would encounter those types of items in high school and because she wanted to meet the county assessment mandate. However, Sue only used open-ended items on quizzes and tests and only for the purpose of assigning grades. When asked if she would make any changes in her teaching as a result of using the items, Sue replied, "No" (Ice, Interview 3).

Issues that Sue found problematic were student attendance, student course level, and lack of student learning. She believed that absences caused students to miss a large amount of mathematical content and that they were unwilling to make the effort to come in for extra instruction. To Sue, students' absences accounted for much of their misunderstanding. Sue determined that the students needed to be more diligent in their efforts to attend class. She also believed that it was problematic that too many students were taking Algebra I in the seventh grade and Geometry in the eighth grade. In addition, she thought too many students were taking Algebra I in eighth grade. She believed that the students' level of maturity was not sufficient for them to be successful in those classes at those grade levels. She believed that even though those students could pass Algebra I and Geometry in the middle grades, their lack of maturity would keep them from learning the material as well as they should, and thus the students would encounter considerable difficulty in the mathematics classes in high school. As a result of her belief and her position as department head, Sue decided to make an effort to reduce the number of seventh- and eighth-graders taking Algebra I and Geometry. Finally, lack of student learning, as evidenced by low grades on quizzes and tests, was also problematic for Sue. She thought that maturity was the main factor that contributed to a lack of student learning. She believed that the students were just not ready to make the kind of effort that was required to be successful in Algebra I. Also, she believed that they did not want to spend time interpreting both traditional and open-ended items correctly, did not have the patience to read them thoroughly, and did not take the class seriously. Sue also believed that the students were not motivated, that their parents pushed them into Algebra I when they were not really ready to handle the course, and that the students' prior instruction in Pre-Algebra did not sufficiently instill the basic skills such as operations with integers and solving simple equations. Sue said she would continue not being easy on the students and demanding that they raise their level of effort in her class. Because Sue's most problematic issues were students' attendance and level of coursework, factors that were not under her control, they did not appear to have the ability to impact her use of open-ended items in any significant way. The issue of lack of student learning did appear to have the ability to impact Sue's instructional and assessment practices, however, her ability to explain the students' lack of learning in terms of student shortcomings inhibited her reflection.

Todd

Todd's sources of authority included the textbook, the leaders of the professional development projects, the county school system, and other teachers in his school, especially his mentor teacher who also participated in the professional development projects. Todd was relatively new to the teaching profession and therefore did not consider himself to be an authority on teaching. He used the textbook as a prescription for teaching. He used a lot of "they" language when he talked about how and what he taught. When asked if he planned to address a particular topic, he would respond with something such as "They don't get into that," they meaning the authors of the textbook. When asked how he would teach logarithms (a topic he had not yet taught), he responded, "I don't know necessarily what the book or the curriculum wants me to teach" (Sanchez, Interview 7). Todd felt a responsibility to follow the county assessment mandate. Because Todd used open-ended items on his tests and it was discussed during the professional development that assessment should be aligned with instruction, he also used open-ended items during instruction. He also claimed to use open-ended items because they require students to think on a higher-level. Todd used open-ended items often, both during instruction and on quizzes and tests. However, the way Todd used open-ended items minimized the potential for fostering higher-level thinking in his students. He often gave open-ended items as group work and assigned a completion grade. He was not willing "to waste more than a couple minutes on a problem" (Sanchez, Interview 3). Although he assigned open-ended items often, during class he did not spend much time on those items and often "glossed over" them. Although the textbook did not facilitate Todd's use of open-ended items, his other sources for authority did support his use of those items, albeit in a superficial way.

Issues that Todd found problematic were students' prerequisite mathematical knowledge, teacher expectations, students' perception of his mathematical knowledge, and lack of student motivation. Todd was concerned that students did not have a mastery of the basic mathematical skills that they should have prior to entering his class. In order to address this issue, Todd focused his teaching on basic skills and procedures. Although he assigned open-ended items, his concern about basic skills limited the time he was willing to spend addressing students' thinking about those items. Todd found it problematic that many teachers accepted student mediocrity and did not have high expectations for student learning. Thus, he felt it was important to ask questions that challenged students, and the use of open-ended items helped him meet that goal. Todd wanted to be a mathematical authority in his classroom and found it problematic if his students perceived that he was mathematically unsure. Open-ended items, by nature, introduce unpredictability in the classroom. Although this unpredictability did not inhibit Todd from assigning open-ended items, it did limit his willingness to explore students' thinking about the items and caused him to quickly escape situations that made him feel mathematically uncertain. Finally, Todd found students' lack of motivation to be problematic. He used open-ended items to address this issue by assigning them as a warm-up or group activity for which he gave either a completion grade or bonus points. Thus, Todd's use of open-ended items was facilitated by his desire to address motivation, but again, the use was superficial.

Leah

Leah made instructional decisions based on her teaching experience and what felt appropriate to her. She had confidence in her expertise as a teacher and relied on that expertise to make decisions. She decided on her own explanations and examples to use during instruction and sometimes did not present material in the same order as the textbook. She used the textbook for student homework assignments. Leah's belief in herself as an authority was obvious from the decisions she made regarding her teaching. For example, when she felt it was best for her students to spend additional time covering basic skills, she did so, even though she commented that other teachers covered the same material more quickly. Also, she chose not to use the open-ended item scoring rubric developed during the professional development project because it did not fit into her grading scheme. Instead, she developed her own way of scoring student responses that was more appropriate for her situation. Leah used open-ended items on tests, as was mandated by the county. She did not, however, integrate open-ended items into her instructional routine even though she considered herself an authority for instructional decisions. Perhaps since Leah's main source for authority for instructional decisions, her prior teaching experiences, did not include the use of open-ended items during instruction, she did not include open-ended items as part of her instructional approach. Additional sources of authority for Leah were the NCTM and the county school system. The NCTM facilitated Leah's use of open-ended items on her tests because she believed they helped her approach NCTM's vision for assessment in mathematics. Similar to Sue, Leah used the items to meet the county assessment mandate.

Leah found the ability to change her teaching practice problematic, as well as low student test grades and student responses to open-ended items. Leah seemed to be searching for something to help improve her teaching. She kept adding discrete items to her repertoire of instructional and assessment techniques. Even though she considered herself a learner, Leah claimed it was difficult to change her teaching practices:

> I think it takes a long time and a lot of hard work. It cannot be done in a very short period of time. I think we have to try one or two things a year and then make a change maybe over a 10-year period. I think it takes a long time. (Ice, Interview 1)

Leah's motivation for change in her teaching was grounded in her ability to "step outside [herself] and see some things that are not the best" (Ice, Interview 2) and resulted from her analysis of her teaching and her efforts to find better ways to help students learn mathematics. Leah found change difficult and the way she dealt with the difficulty was to add small changes in over extended periods of time. Thus, the changes she incorporated with respect to assessment were limited to adding open-ended items to her tests.

Leah searched for reasons to determine why students made low grades on her tests. She concluded that there were several actions that she and the students could take to help them improve their grades. With respect to open-ended items, Leah maintained that the items were new to the students and that they would become better at responding to open-ended items with practice. Also, the day before the chapter 2 test, she reviewed with her students how to answer four open-ended items similar to ones that were on the test. In order to learn their mathematics better, Leah suggested to the students that they acquire better study habits as well as habits that would improve their ability to learn from her instruction (such as taking notes better, less talking in class), that they put forth more effort to learn, and that they stop being so careless and making mistakes in their work. These suggestions, along with encouraging the students to find someone outside of class (a parent, sibling, friend, or tutor) to help them learn their algebra, were ways in which Leah believed the students could improve their grades. Her conclusions led her to reduce the number of open-ended items on her tests because she was going beyond the 20% mandate.

Leah was surprised, and thus found it problematic, that students could not respond to open-ended items as well as she thought they should be able to. One reason for their poor performance was that, as mentioned above, she felt that the students did not know their mathematics well enough to appropriately respond to the items. Further, Leah believed that the students struggled with the details of open-ended items and that they did not have the ability to pay attention to so many details in one item. The last reason that she believed caused the students to respond poorly to open-ended items was that they were not clear about the directions, wording, or intent of some of the items, and thus interpreted them incorrectly. Along with the suggestions to help students improve their grades, Leah tried to help students respond better to open-ended items by sometimes giving hints, and she made an effort to select open-ended items that she believed were worded more clearly.

Robin

Robin made decisions about teaching based on her own teaching goals and experiences. One of those goals was that she should finish the county curriculum. This goal may seem to be related to a reliance on external authority, but more likely it was related to the fact that Robin helped create the curriculum and felt ownership of it. The fact that Robin did not always follow the assessment mandate illustrated her willingness to deviate from the authority of the county school system. She said,

> I feel a responsibility to myself more than I do to any of those other outside groups or people or anything. Because I'm...gonna do what I need to do in the classroom because of why I think I'm here. Not because [the] county tells me to cover XYZ....But because, you know, I'm a serious committed person to teaching, and I'm going to do it the way that I feel it needs to be done. (Sanchez, Interview 4).

Robin's commitment to the curriculum sometimes overrode her commitment to open-ended assessment, but when she did use open-ended items, she used them to support higher-level thinking on the part of her students and to inform her teaching. She did not use open-ended items because she was supposed to; she used them because they helped her reach her broader educational goals.

Robin found the issue of time to be problematic. She also viewed her instructional and assessment practices as problematic. Robin's commitment to finishing the county curriculum inhibited her use of open-ended items because she was hesitant to use valuable class time to investigate student thinking. She also recognized that it took students longer to respond to open-ended items than more skill-based items. She was only occasionally willing to take the time to incorporate open-ended items into her instruction. Robin also found her instructional practices problematic in the sense that she was always searching for better ways to increase student mathematical understanding. If a student approached a problem in a way that made more conceptual sense to Robin than the way she had presented it, then she said she would adopt the student's approach the next time she taught the topic. She viewed her teaching as open to scrutiny from herself and others and accepted that other methods could be better than the one she originally used. While Robin used open-ended items sparingly during instruction, and more often on tests and quizzes, she used the student responses to those items to meaningfully inform her instruction. Finally, Robin viewed assessment as problematic because she realized that it was possible for students to have a conceptual understanding of mathematics while lacking precision in carrying out computations. These students, according to Robin, are unfairly assessed by traditional means in which students are required to perform rote procedures. Thus, Robin used open-ended items on quizzes and tests to provide opportunities for students to demonstrate their conceptual understanding.

Keith

Keith's teaching was mostly unaffected by the authority of the county school system and the textbook. He based his teaching decisions on a variety of sources. He used the county curriculum "as just kind of an outline.... It's my responsibility to decide what it is that the students learn" (Sanchez, Interview 4). When asked if he used the textbook to decide how to teach a particular topic, Keith responded, "Yes, because it's just another point of view from mine. If I know of better ways to present the topic, I usually present them my own way" (Sanchez, Interview 4). Keith used open-ended items during instruction as a way to find out what students knew. In his Algebra II class, Keith presented an open-ended item about systems of linear equations, to which students were unable to provide satisfactory responses. What followed was a discussion that led to increased student understanding of solving systems of linear equations. Because Keith used this item, he was able to determine that the students lacked some understanding of solving systems of linear equations and as a result used the information to make an appropriate instructional decision.

Keith found the issues of student mathematical thinking, instruction, and assessment as problematic. According to Keith, students think about and learn mathematics differently. He recognized that students can know and understand mathematics differently than he does. He valued student approaches to solving problems in his classroom by having students explain their reasoning or by presenting more than one method if students were not comfortable with his original method. Open-ended items allowed students to approach problem-solving in a way that was consistent with how they understood the mathematics. Keith therefore embraced the use of open-ended items as a window into his students' thinking. Because he believed that students learn mathematics in different ways, Keith found instruction particularly problematic because one approach would not necessarily reach all students. Therefore, Keith used open-ended items as part of his instructional practice so he could identify student thinking and modify his instruction accordingly. Keith found that traditional assessment, in which students are asked to generate specific responses, does not allow students to demonstrate their own mathematical knowledge. Keith's use of open-ended assessment items helped alleviate this aspect of assessment that he viewed as problematic.

Discussion

The teachers' sources of authority impacted their use of open-ended items in the sense that if the teacher relied on an authority who supported the use of those items then the teachers tended to use them. Some sources of authority promoted the use of open-ended items (NCTM, the county school system, the professional development leaders) while others hindered the use of those items (the textbook). We did not find, however, that relying on a source of authority which supported the use of open-ended assessment resulted in a use of the items that is consistent with the reform efforts in mathematics education. In other words, authority might play a role in getting teachers to use open-ended items but it might not be able to affect how the teachers use them.

Sue, Todd, and Leah each identified aspects of their teaching practice as problematic and searched for and determined suggestions for resolution. However, it appeared that, as Dewey (1933) described, the teachers were not sufficiently critical about the issues and seemed to accept somewhat superficial solutions to their problems. Thus, according to Dewey, these teachers did not engage in reflective thinking. Since the teachers did not appear ready or willing to engage in reflective thinking about their problematic issues, they did not reach the stage of carefully searching for a resolution. Thus, open-ended items were not considered helpful to the resolution process. In contrast, Robin and Keith engaged in reflective thinking as a way to address the issues they found problematic. Keith explained,

> Before I got involved with alternative assessment, it was like, I don't know what to do—something is missing. And now I have something I can grab onto. I've always been somebody who likes to ask questions [while teaching]. It is just that I didn't always have, I don't think, the framework that I needed to ask a good question, thinking about possible responses and thinking about how you would determine if those were good responses. Now I'm thinking about that all of the time. Not only do I just reflect about what I have taught, but I think more about it before I teach. (Sanchez, Interview 7)

Both Robin and Keith used open-ended items to better understand their students' thinking and used the information they gleaned about students' thinking to inform their instruction.

Not only was there a difference between the extent of reflection among the teachers, but the focus of their reflection—the problematic issues—was fundamentally different. Sue and Todd found issues such as attendance, motivation, and prior knowledge to be problematic. Although these issues can affect classroom dynamics in significant ways, they are largely out of the teachers' control. These three teachers indicated that poor student performance is due to characteristics of the students. In contrast, Leah, Robin, and Keith found issues such as student responses to open-ended items, low test grades, instruction, assessment, and students' mathematical thinking as problematic.

Not only are these issues more in the domain of issues that teachers can control or impact, but they are also the constructs that are epistemologically and pedagogically central to teaching, namely instruction, learning, and assessment. While Leah did not engage in reflective thinking as defined by Dewey, Robin and Keith did engage in such thinking. These findings suggest that teachers who engage in reflective thinking about constructs that are central to teaching tended to use open-ended items in a way that was consistent with the intended purpose of those items—to inform teachers about students' mathematical thinking in order to guide instruction and assess understanding.

Conclusions and Implications

Teachers' orientations to authority impacted the use of open-ended items. Some sources of authority hindered teachers' use of open-ended items while other sources facilitated such use. However, a reliance on sources of authority that facilitated item use did not necessarily promote teachers' meaningful use of those items. Thus, while authority can lead teachers to use open-ended assessment items, it does not guarantee that teachers can use such items in ways to promote students' deeper understanding of mathematics.

The focus and extent of teachers' reflective thinking also impacted the use of open-ended items. Teachers who engaged in reflective thinking about issues that were within their realm of control and were central to the practice of teaching were more likely to use open-ended items in a way that helped promote students' mathematical understanding.

The findings from these studies suggest that preservice education and professional development programs should help teachers move beyond just using alternative assessment strategies, such as open-ended items, and help them use student responses to make instructional decisions that will promote students' deeper understanding of mathematics. Programs should provide teachers with opportunities to develop specific strategies for using information gained from alternative assessment techniques to inform their teaching. If professional development programs aim to help teachers move more towards reform-oriented practice, we suggest that those programs go beyond teaching specific techniques for instruction and assessment and challenge the fundamental assumptions that underlie teaching. These programs need to provide teachers a context for reflection in order to enable them to view teaching as problematic. Viewing teaching as problematic is the precursor to engaging in reflective thinking about teaching. Teachers need to be provided with opportunities to frame teaching as a problematic activity as well as to engage in the process of searching for resolution. After these programs are implemented, future research could investigate their ability to impact teachers' practice.

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TEACHING AND TESTING AT THE FOURTH GRADE LEVEL: AN ANALYSIS OF A TWO-YEAR STUDY IN NEW JERSEY

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This research, briefly summarized, was designed to document the teaching practices of a sample of fourth grade mathematics teachers from across the state of New Jersey, over the course of a two-year period. A major part of the study investigated the teachers' reactions to a new statewide testing program. We were particularly interested in how they perceived the new test to be impacting their teaching, in the context of what was actually happening in their everyday instruction. To this end, the research methodology incorporated both a sample survey and interviews, and direct observation with post-observation interviews. This particular report focuses on the classroom observation and interview components of the study. The codes used to assess classroom practice were based upon the work of other researchers who have developed coding systems for use in analyzing classroom practice. Briefly, the data suggests that the teachers have adopted new strategies in response to the state test but are not changing their basic approach to teaching mathematics.

Introduction

In this research[1], we were interested in learning more about how teachers responded to a new state test that was designed to encourage them to implement more student-centered instructional practices as embodied in state and national standards (c.f. NCTM, 2000). More specifically, we were interested in gaining a deeper understanding of how teachers feel that the fourth grade test is impacting their instruction, and also in learning more about their actual mathematics teaching. When we began our study, several questions came into focus. They included the following: How did the teachers view the new test? What, if any, changes did they believe that they were making in their actual teaching, and why? As we considered their responses, we became increasingly interested in their actual classroom practices. Some additional questions came into focus. They included the following: Would the teachers build instruction based upon close attention to student thinking? (as cited in, for example, Davis, 1984; Cobb, Wood, Yackel & McNeal, 1993; NCTM, 2000; Klein and Tirosh, 2000; and Schorr, 2000). Would they implement more open-ended types of tasks in their classrooms? Would they encourage students to defend and justify their solutions (as cited, for example in Hanna & Jahnke, 1996; Harel & Sowder, 1998; Maher, 1998; Maher & Martino, 1996)? Would they adopt specific strategies, like using more manipulatives or small group instruction (as cited in Spillane & Zeuli, 1999)? Would they simply ignore the test and not make any changes?

This paper presents two years of data regarding the actual practices of these teachers and their perceptions of how the test has influenced their teaching. Our approach in this research was to observe a sample of 4th grade mathematics teachers from across the state, and interview them about their practice and their reactions to the test, including any changes that the test has prompted in their teaching. The test that we focus on is New Jersey's fourth grade Elementary School Performance Assessment (ESPA) that has been in place since 1999. This paper describes the results.

Theoretical Framework

In the last two decades, most states have either introduced or extended their testing of children (Editorial Projects in Education, 2001). As of the 2000-2001 academic year, all 50 states and the District of Columbia have a large-scale testing program in place (Education Week on the Web, 2002). This trend, however, extends beyond the United States (Niss, 1996; Keitel, and Kilpatrick, 1998 as cited in Abrantes, 2001; Firestone and Mayrowetz, 2000; Abrantes, 2001). Some advocates of tests consider these tests to be part of a broader effort to raise educational standards and to make educators accountable for reaching them. They see testing as a way to use the authority of the state to improve teaching and learning and enhance equity by holding all children accountable to the same high standards (O'Day & Smith, 1993). Some maintain that a test that is well designed can prompt teachers to revise their practices because teachers will inevitably "teach to the test", and that can be good if the test is well-designed. Others maintain that content that is emphasized on tests gets emphasized in class, and that untested content either falls out of the curriculum or gets put off until the latter part of the school year (Corbett & Wilson, 1991; McNeil, 2000). The types of items that are placed on the test are also purported to ostensibly influence the types of problems teachers use in class. The argument is that by including items that require students to solve more complex types of problems, teachers will be more likely to provide students with the opportunity to do the same in class. One reason given for the great interest in various forms of performance assessment and portfolios in the 1990s was the hope that tasks requiring students to show their work and explain their answers would promote inquiry-oriented instructional practices (Resnick & Resnick, 1992; Rothman, 1995). Currently many tests combine conventional, multiple-choice formats with other formats intended to measure higher order thinking and problem-solving abilities. However, even when tests employ formats where students construct responses, some of the same risks that are typical of the more traditional tests have been found to occur (Smith, 1996; Stecher & Barron, 1999).

Some opponents of testing believe that extensive testing will encourage measurement of less relevant skills, and reinforce traditional approaches to teaching (McNeil, 2000). There are also those who believe that the effects of state tests have been overstated and that any modest changes in teaching exist alongside what has been conventional practice (Wilson & Floden, 2001). Regardless of the format, the evidence that testing promotes instructional change remains unconvincing or inconclusive at best (Newmann, Bryk, & Nagaoka, 2001; Smith, 1996).

While the research literature on testing remains divided, research in mathematics education (c.f. Schorr and Firestone, 2001, Simon and Tzur, 1999) has consistently shown that changing actual classroom practices is not easily accomplished. Mathematics teaching has a long tradition of focusing on student acquisition of rules and procedures, often with little or no understanding of them. Helping teachers to change their instructional practices toward those recommended in the NCTM Standards requires more than the acquisition of a new strategy or technique. Simon and Tzur (1999) suggest that teachers often have a limited understanding of what the mathematics reforms are about, and mathematics educators and researchers caution that "some of the pedagogical ideas from the NCTM Standards...have been enacted without sufficient attention to students' understanding of mathematics content" (NCTM, 2000, pp. 5-6). Teachers who employ particular methods often insert them into their basic paradigm of teaching without moving towards the more student-centered approach into which they are intended to fit. Thus, specific instructional techniques may not involve students in more compelling and challenging mathematical problem-solving experiences in which they have the opportunity to build and understand different representational systems, defend and justify their solutions, and ultimately develop a deeper understanding of the content.

Based upon the above research in testing, combined with the research on teacher development in mathematics education, our guiding questions were: How do teachers believe that the test is impacting their teaching? and, How are the changes that are being reported manifesting themselves in the actual classrooms?

Methods and Procedures

Sample

Year 1: The observation study focuses on 63 teachers drawn from two different sources. The first came from a statewide survey of 4th grade teachers of mathematics designed to be representative in terms of both district wealth and geographic spread (see Firestone et al., 2001). Ultimately, 22 teachers were selected. The remainder came from a second sample consisting of teachers involved in professional development provided by institutions of higher education.

Year 2: The second year's sample consisted of 28 fourth grade teachers from the first year's professional development districts.

Actual Observations

Each teacher was observed twice. The classroom researcher kept a running record of the events in the classroom, focusing on the activities of the teacher as well as capturing the activities of students. The field notes recorded all problem activities and explorations, the materials used, the questions that were posed, the responses that were given—whether by students or teachers, the overall atmosphere of the classroom environment, and any other aspects of the class that they were able to gather.

Interviews

At the conclusion of each lesson, the teachers were asked to respond to a series of open-ended questions about the observed lesson. These included questions relating to the concepts and ideas that the teacher intended to elicit; what changes the teacher might make in future lessons; and how state testing and professional development affected their teaching.

We also asked teachers to describe any professional development or other learning experiences that they might have had and their reactions to the new state test (see Schorr, Firestone and Monfils, 2001; Schorr and Firestone, 2001 for analysis of the first year of study).[2] Briefly, the data suggests that the teachers have adopted new strategies in response to the state test but are not changing their basic approach to teaching mathematics.

Coding

While observations were underway, researchers conducted detailed analyses of records of classroom observations, and adapted several pre-existing coding schemes to be used for coding the classroom data. These were based on the works of Stein, Smith, Henningsen, and Silver (2000); Stigler & Hiebert (1997, 1999); Stein and Smith (1998); and Davis, Wagner, and Shafer, (1997). These codes were selected because they reflected ideas about effective mathematics instruction as indicated in national and state standards. They included attention to the mathematical discourse that emerged, the opportunity for conceptual understanding to take place, the nature of student conjectures, the opportunities students had to share ideas and defend and justify solutions, etc. They were also chosen because we felt that they would supply information on the nature and use of reported strategies (i.e., manipulatives, small group instruction, use of different types of problems and activities, questioning strategies, etc.).

Coding of the classroom observations was performed by six members of the research team who collectively represent a wealth of educational experience including elementary and mathematics classroom teaching, supervision, teacher training, and mathematics education research. Two individuals independently coded each observation—at least one coder was an experienced mathematics education researcher. The other coder also had extensive experience in elementary education.

Interview data were transcribed and entered into a qualitative data analysis software package. Interviews were sorted by question. Responses were analyzed in clusters, as there was considerable overlap in responses given to individual questions.

Within each cluster, responses to specific questions on test preparation practices were reviewed and coded according to emergent themes. For each returning teacher, year 1 and year 2 responses were compared, and trends across teachers were noted. Interviews from 58 of the 63 first year teachers and all 28 of the second year teachers were available for analysis.

Results

Consistent with our earlier findings (c.f. Schorr and Firestone, 2001; Schorr, Firestone & Monfils, 2001), teachers reported that both the test and their professional development experiences surrounding the state test and implementation of state standards had influenced their teaching. For example, they said that they were using more manipulatives, small group instruction, and open-ended reality based problem activities. Indeed, we found that manipulatives were used in about 60% of all observed lessons. Similarly, students worked in groups for at least a portion of the time, in almost 65% of all observed lessons and in almost half of all cases, teachers made an effort to connect the lessons to the students' real life experiences. Again, consistent with prior results, the adoption of these strategies was not necessarily accompanied by a change in overall approach to teaching mathematics. For example, while manipulatives were used extensively, they were used in a non-algorithmic manner in less than 19% of all observed lessons. This means that the manipulatives were used in ways that did not foster the development of conceptual understanding. In fact, in almost two-thirds of the lessons where manipulatives were used, they were used in a very procedural manner, where the teacher generally told the students exactly what to do with the materials, and the students did it as best they could to mimic what the teacher had done or directed. In many lessons, there was little or no opportunity for the students to develop their own solutions to the problem or to see the relationship between the problem activity and the concrete (or alternative) representations.

The following example will serve as an illustration. In this observation, the teacher supplied the students with pretzels to be used as concrete manipulatives. She then said:

> T: All right, any questions? All right, Mrs. T is going to give you a bag of stick pretzels. What I want you to do is take your [pretzels] and polygon and move it to the side. I'm going to ask you to make many things with your stick pretzels. You can eat them later but you might have to take the stick pretzels and break them in half for some of the activities I want you to do.

In her directions, the teacher specifically told the students, in effect, that although they had been given a concrete material to work with, they would be expected to follow specific procedures. Ms. T. continued as follows:

> T: First thing I want you to do is when I give you the pretzels I want you to form three different pairs of parallel lines. I want parallel lines that

> are horizontal, parallel lines that are vertical and parallel lines that are diagonal. That's your first chore. Ok, and you put 'em right on your desk.

Ms. T's demonstration left little to the imagination. Indeed, all they had to do was to take the pretzels and form the shapes that Ms. T. described. In this example, the students had a concrete material (the pretzels), but they were only allowed to do what the teacher wanted them to do. In essence, their role was simply to follow her instructions. Other examples indicate that even when children were given more freedom, the teacher often stopped them without ever asking what they had built, how it connected to the problem or some other representation, or how it compared to what others had built. By telling students exactly how to use the materials, the teacher may have undermined the opportunity for students to better understand the mathematical idea.

A main area that the teachers repeatedly noted as being influenced by the test involved the nature and type of mathematical tasks given to students. They reported that they were now emphasizing more open-ended types of tasks, because of the presence of such items on the test. When we examined the actual mathematical tasks that students were asked to perform, we found that most (77%) involved memorization or doing procedures in an algorithmic manner. Only 3% of all observed lessons involved situations where students were required to do non-algorithmic thinking.

Many teachers also reported that they were interested in having students explain their reasoning. They also said they were interested in having students find and understand multiple strategies for solving problems. Yet despite the fact that many teachers said that they wanted students to explain their own answers, and consider multiple strategies for solving problems, they rarely insisted on such activity. For instance, one code documented whether or not the teacher encouraged students to reflect on the reasonableness of their responses. In almost 80% of all cases the teacher never posed questions that would prompt students to consider whether or not their answers were reasonable. If a student gave an incorrect response, another student was asked to provide a correct answer, but there was little discussion of an appropriate strategy to solve the problem. In the following excerpt, in a discussion involving probability and the fairness of a particular spinner, the teacher asks a potentially good question, but then when the student responds with an observation that could have led to a more mathematically sophisticated discussion, she states the conclusion for the class, thereby eliminating the possibility of extending the knowledge and probing for deeper understanding.

T: Was one of the spinners unfair?

Boy: Yes.

T: Which one was unfair? Rob.

Rob: The second.

T: Why?

Rob: Because one part was bigger than the others.

T: I don't think that we want to play with that spinner. Right?

Class: No.

T: Of course, because many times you will finish in blue. Right?

Of interest in the context of state testing, is the degree to which test preparation directly influences classroom instruction. Our data provide evidence to support claims made by both advocates and critics of state testing. We observed lessons in which teachers openly state that they prepare students for ESPA in the course of their daily instruction. For example, in one such lesson the teacher had the students use pattern blocks to consider the concept of symmetry. To begin, the teacher asked students to classify five plane figures placed on the overhead. After whole class discussion about different methods students had used to group the objects, they moved into a discussion of how some students categorize the objects by the number of lines of symmetry. Students broke into groups and worked on an activity in which they used pattern blocks to "build a mirror image below a dotted line". Then they created similar problems and worked on each other's problems. Students were required to defend their answers by explaining their reasoning. Incorrect responses were tested by students moving objects on the overhead or by paper folding. This lesson, according to the teacher, was one that would prepare students to answer test questions similar to one open ended ESPA item released on the state website in which students are required to use manipulatives to trace the reflection of a plane figure drawn by connecting points plotted in the Cartesian plane, and then discuss the relationship of the area of two figures.

We also observed lessons in which students were drilled on multiple-choice questions taken from district provided ESPA practice books. We found that the students were repeatedly reminded that their answers to open ended practice items would not gain full credit unless they wrote more. In essence, the teachers encouraged the students to write more, as the more they wrote, the more likely they we be to get credit. In addition, we heard from teachers, particularly in the low-wealth districts, that selected students were being pulled out of math class for individualized test preparation in order to help them boost their scores. We also heard that again, particularly in low-wealth districts, in the month prior to the test, test review delivered by commercial providers replaced teacher-facilitated math lessons.

We found that many teachers reported that they had participated in professional development designed to improve their practice, their ability to teach in a more "standards-based" manner, and/or their test scores. They also reported that they were under pressure to raise test scores and were subject to surprise district level classroom visits to verify their adoption of strategies such as small group work, and rubric scoring of written responses to open-ended math problems. The teachers reported that many of the demands of this type of district accountability motivating commercial test preparation and classroom monitoring often impacted their efforts in positive as well as negative ways.

Most teachers referred to the state test as a catalyst for change ("the test is fueling change") and cited ways in which their practice has changed in response to the test. For example, one teacher stated, "I like the way the questions are challenging and make them think. I think it certainly has affected the way that I've taught, I teach, and that I'm very, I'm always looking for opportunities to have an open-ended question somewhere and that's good."

The following quote from a teacher in a progressive, high-wealth district summarizes much of what teachers across the state told us about the test and about differential impact across districts:

> The benefits of the test are that I think it is making district sit up and take notice that you know what, we can't teach the way we used to teach because the kids are not getting the skills they need for real life. ...the main thrust of the test is to change the way teachers are teaching in the classroom to match the skills that are needed for the students of today. We can't teach knowledge. We have to teach skills for getting knowledge. That was what the test was supposed to drive was this kind of teaching in the classroom. ...I do think that the test is reflecting more of the methods we were already using in the classroom. I think that change had taken place before we ever got to the test in many districts. I'm sure there are districts that still aren't up to that point yet but they are evolving towards that. Especially now the ESPA is in place. They know that their students aren't prepared. I felt like our students were pretty prepared before we even got the ESPA and saw it for the first time. Much more so. Again, we have made those changes in our curriculum to reflect the ESPA. So, is the test pushing us in a right direction or are we pushing ourselves to match these standardized tests for fourth and eighth grade? That is the big question. It has always been the big question. I will say that I think it is difficult to teach to the test. It probably must be easier to do with other standardized tests like the Iowa and the CAT. Those type of tests lend themselves to teachers teaching to the test. Whereas, this was much more difficult to do that. If you don't have it in place all year long and in previous years they are not going to be able to do it.

When teachers talked about how test preparation has affected what they are doing in their classrooms, most teachers referred to a more embedded approach to preparing their students for the state test, using phrases like "teaching life skills", "teaching that reflects the standards", "critical thinking", "yearlong preparation", and stating that by covering the curriculum they were preparing their students for the test. For example, one teacher said, "My classroom is set up to work with ESPA, not separate that I'm gonna stop and train for. My writing folders, right from the beginning my math portfolios--everything is open-ended questions, rubrics are used from the beginning, language is used. It's just part of our way of life in here."

The interview data also suggest that teachers in the poorest districts are responding differently to the state assessment than teachers in the wealthier districts in terms of change in content and test preparation. For example, when asked about subjects that might have been ignored in the past, one teacher stated that her district did not teach science prior to implementation of the state test. "I think because it wasn't measured on standardized tests. And unfortunately, in an inner-city district, the number one means of deciding what's important in the classroom is what are we gonna be tested on? Reading and math were it, and that's where the focus is."

Many teachers in the disadvantaged districts perceive the test to be a force for improving instruction, while others express doubts about fairness. One teacher in a special needs district expressed praise for the state test, but also talked of the constraints faced by teachers in her district

> And in my classroom the ESPA, and the accountability issues of the ESPA for myself and for my students, have made me a better teacher.... I was very pleased with the fact that I was allowed to include both math and technology aspects in my PIP [Professional Improvement Plan] because I was able to indicate that I wanted to improve student growth in math. Not through text study and worksheet completion but through problem-solving activities that was sometimes not finished in a day.

Conclusions

The teachers involved in this research have indicated that they have been motivated to change their styles of teaching as a result of the ESPA test. Indeed, our observations confirm that they do incorporate many of the strategies and techniques that they reported in our interviews (such as small group instruction and the use of manipulatives). This research does not and cannot document just when these strategies first became part of their teaching practice; we can only note that the teachers attribute the implementation of many of them to the test. This study provides evidence that the teaching practices that we noted in our observations, however, are not focused on the more conceptually oriented aspects of instruction. In fact our data, overall, suggests that the teachers we interviewed and observed are adopting new procedures as part of their instructional practice but not changing their basic approach to teaching mathematics. For example, they tend to continue to assign tasks that reinforce a procedural view of mathematics, and classroom discourse does not tend to encourage students to defend and justify solutions. In sum, while many teachers report incorporating standards-based practices in response to the state test, their practice suggests a more surface level adoption of specific strategies rather than a fundamental change toward meaningful mathematics instruction. Those teachers who mentioned professional development in their interviews claimed that it has influenced their practice and their thoughts about meaningful instruction. Perhaps with appropriate support, teachers who are ready and willing to make changes in their teaching will be able to incorporate practices that will enable children to have access to mathematical instruction that fosters the growth of mathematical thinking.

Notes

[1]A portion of this paper was presented, in a preliminary format, at the annual meeting of the American Educational Research Association, April 2001 and at PME – NA, October 2001. Neither presentation focused on both the test data and the observation data. This paper was prepared with supports from grants # 9804925 and 9980458 from the National Science Foundation. The opinions stated here are those of the authors and not the National Science Foundation, Rutgers University or Rider University.

[2]Earlier reports (PME-NA, 2001and AERA, 2001) that relate to this research did not report on the survey data and/or the observational data. In addition, those reports did not focus on the final year of the study as the results had just been coded and were not fully analyzed.

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IS STATE ASSESSMENT A VIABLE TOOL FOR REFLECTION ON CLASSROOM ASSESSMENT?

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North Carolina uses *Dimensions of Thinking* (Marzano et al., 1988) as a framework to incorporate higher order thinking in their End-of-Course exams. The goal of this research was to explore whether this effort could also help facilitate the inclusion of higher order thinking in the classroom. Based on reviewing state EOC-related documents and interviews with state officials, a modified teacher-friendly version of the state's assessment framework was created for classroom use. In Spring 2002, over 60 teachers "recreated" the EOC test item development process and analyzed their current classroom assessment practices using the framework. Most teachers found the framework useful; however, a variety of problems including lack of time, lack of "fit," a restricted context, and low state expectations all worked against using the framework effectively in the classroom.

Introduction

State testing has grown significantly in the US over the past decade. Many critics argue that external, state-mandated testing is detrimental to the quality of instruction and assessment in the classroom by (a) narrowing the curriculum, (b) taking flexibility away from teachers to meet the needs of their students, and (c) forcing teachers to "teach to the test." (Kohn, 2000; McNeil, 2000) However, others propose that if state tests focused more on higher order thinking skills, then these tests might actually help teachers improve classroom instruction and assessment by encouraging teachers to include these thinking skills in the classroom. (Yeh, 2001)

North Carolina (NC) is one of many states with yearly testing, and a goal of promoting higher-order thinking skills. The North Carolina Department of Public Instruction (DPI) officially uses *Principle and Standards for School Mathematics* (NCTM, 2000) and *Dimensions of Thinking* (Marzano et al., 1988) as a theoretical framework to develop test items for their End-of-Course (EOC) exams (DPI, 1999; Munk, 2001). The goal of this research was to explore whether a state's effort to assess higher order thinking skills could help facilitate the inclusion of higher order thinking in the classroom. And in particular, to ascertain if NC's assessment framework was a viable tool for improving higher order thinking in classroom assessment.

NC's Use of *Dimensions of Thinking*

Dimensions of Thinking includes: (1) metacognition; (2) critical and creative thinking; (3) thinking processes; (4) core thinking skills; and (5) the relationship of content-area knowledge to thinking. Although NC DPI encourages teachers to incorporate all five dimensions in their teaching, only the "core thinking skills" are used as basic organizers in the development of EOC test items. The eight core thinking skills listed in *Dimensions of Thinking* (*DOT*) include: (1) focusing; (2) information-gathering; (3) remembering; (4) organizing; (5) analyzing; (6) generating; (7) integrating; and (8) evaluating.

The seven thinking skills used by NC DPI to construct EOC test items include knowledge, organizing, applying, analyzing, generating, integrating, and evaluating. In NC DPI's framework, *focusing*, *information-gathering*, and *remembering* from *DOT* are combined into a single thinking skill called "knowledge." NC DPI also added the additional thinking skill of *applying* not found in *DOT*. Knowledge and applying appears to originate from Bloom's Taxonomy which includes knowledge, comprehension, application, analysis, synthesis, and evaluation. Despite combining the thinking skills of *focusing*, *information-gathering*, and *remembering* into the category "knowledge," *remembering* is the only aspect of this thinking skill used in labeling test items as *knowledge* in NC's EOC exams. In this case, the actual use of this term is more consistent with Bloom's Taxonomy than *DOT*.

Methodology

To frame the professional development experiences for this project, we utilized the ideas of Bryant and Driscoll (1998) to design and facilitate workshops as teachers explored the potential relationship between state and classroom assessment. The "guiding principles for design" included: (a) *purposefulness*: this project was defined around a specific goal / objective with an assessment activity framework specifically created for this project, (b) *connection to the concerns of participants*: this project grew out of our discussions with teachers and their desire to learn more about state testing; (c) *opportunities for exercising judgment*: the project incorporated opportunities for teachers to exercise, share, and discuss their professional judgment through the development and evaluation of state test items; and (d) *opportunities for reflection*: the project was designed to encourage participants to reflect individually and in groups during and after the workshop and relate what they've learned to the classroom.

In Fall 2001, the authors met with DPI officials to discuss the process and theoretical framework (*Dimensions of Thinking*) used for constructing mathematics test items for EOC exams. Shortly thereafter, several teachers agreed to recreate the test item development process in a trial workshop. Using feedback from this process and sharing this result with NC DPI officials, the state's framework was modified to make it more teacher-friendly by grouping the seven thinking skills into three broad categories. In January and February 2002, over 60 teachers participated in two different workshops which involved both analyzing and writing EOC test items. In March 2002, the state assessment framework was modified once again by adding a narrative section to the category descriptors (a more detailed description of the evolution of the assessment framework is provided below).

In creating the workshops the main objectives were to: (a) share the state's process and theoretical framework for creating EOC test items; (b) engage teachers in re-creating the test item development process themselves; and (c) help teachers analyze and compare their current classroom assessment practices with the EOC exams. Following an overview of the process used by DPI to develop test items, each workshop had three small-group activities each followed by whole group discussions.

Activity I: Teachers were given "released" state test items in order to categorize them by thinking skill and level of difficulty. They compared this to the state's categorization for these test items.

Activity II: Teachers wrote their own test items covering a wide range of thinking skills and levels of difficulty. Each group presented their test items for whole group discussion and feedback.

Activity III: Teachers brought copies of their own classroom tests to the workshop. They analyzed their tests by categorizing the questions by thinking skill and level of difficulty and compared this to the proportion of questions found on the EOC exams.

Of these sixty teachers, twenty agreed to continue participating in additional follow-up discussions as they used the assessment framework in their classrooms. For the remainder of the spring semester, they submitted copies of their exams as well as reflections on their experiences in using the assessment framework.

Modifying the State's Assessment Framework for Use by Teachers

Teachers who participate in the development and review of EOC test items describe the process as a rewarding professional development experience. The purpose of this research project was to provide opportunities for teachers to recreate the test item development process to see if it helped them incorporate these higher order thinking skills into the classroom. Our original effort to use the framework "as is" in a "trial workshop" met with little success. Teachers found it too awkward and impractical to use so many different types of thinking skills. In particular, there were large disagreements among teachers when asked to categorize test items by thinking skill. Despite these disagreements, teacher categorizations often fell along similar lines. When reviewing test items, teachers generally grouped "knowledge" items with simple or "one-step" application" items. Similarly, "organizing" was often combined with more complex or "multi-step" application items. And finally, "analyzing," "generating," "integrating," and "evaluating" (referred to as the higher order thinking skills) were often used to classify the same test item.

As a result, we modified the state's assessment framework by grouping the seven thinking skills into three broad categories: *Category I*: Knowledge – Applying (simple or "one-step"); *Category II*: Organizing - Applying (more complex or "multi-step");

and *Category III*: Analyzing, Generating, Integrating, and Evaluating. After two further workshops with teachers and discussions with DPI officials, each thinking skill category was provided with a narrative based on how NC DPI used the thinking skills when developing EOC test items and how teachers often interpreted the thinking skills categories. These descriptions are based on the familiarity a student has to a question and whether or not a student is likely to have been taught an algorithm or procedure to solve the problem. The narrative for each of the categories is as follows:

Category I: Knowledge – Applying ("one-step"). These are recall questions and simple applications. The problems are very familiar to the student and the student should already know an algorithm or procedure to solve the problem.

Category II: Organizing - Applying ("multi-step"). These are more complicated questions and applications, but the student should already know an algorithm or procedure to solve them. The student may also be asked to apply a known procedure to a new situation or the student may need to first organize the information in order to apply a known procedure.

Category III:Analyzing, Generating, Integrating, & Evaluating (Higher Order Thinking Skills). How to solve these questions should *not* be immediately apparent. Students will have to think through the problem before choosing a procedure or will have to create their own procedure / method to solve the problem.

The following are examples of released test items and accompanying thinking skills from the NC DPI website.

Category I

- Simplify: $(6x^3y^3 - 12x^3y^2 + 8x^6y^5) \div 2xy$
- Solve: $2x + 1 = 5$
- Which of the following is a characteristic of a chord?

Category II

- Express, in terms of x, the mean of $(4x^2 - 6)$, $(2x^2 - 3x)$, $(^{-}13x + 3)$
- The US balance of trade is calculated by subtracting imports from exports. If $0.71x^2 + 2.15x + 67.53$ models exports and $0.82x^2 + 6.42x + 55.07$ models imports for the period 1970 – 1998 ($x = 0$ for 1970), find the algebraic expression for the balance of trade.

Category III

- A soup company decides to increase the height of its cans by 30% but to keep the volume the same. Approximately how much must the radius of the can be decreased to keep the volume constant?

- For the line $y = 3x + 5$: If the y-intercept moves to 7 and the slop remains unchanged, how does the x-intercept change?

In sharing this categorization scheme with DPI officials, they stated that 40% of EOC test items fall in Category I; 20% in Category II, and 20% in Category III. Level of difficulty is another aspect of test item analysis conducted by NC DPI. The level of difficulty of a test item is identified quantitatively through field-testing and qualitatively by teachers. In general, DPI defines easy questions as those 70% of the students would answer correctly; medium questions are those 50% to 60% of the students would answer correctly; and hard test questions are those 20 or 30% of the students would answer correctly. On NC EOC exams, approximately 25% of the items are written at the easy level; 50% at the medium level; and 25% at the hard level.

Thinking skill and level of difficulty are clearly related, but not synonymous. If students rarely have the opportunity to experience Category 3 questions, then all of these questions will be difficult for students to answer. To help teachers differentiate between the thinking skill and the level of difficulty of a test item, an additional framework was created. This framework (in table form) included all possible combinations of thinking skill vs. level of difficulty including the state's percentage goals for each of the nine possible thinking skill–level of difficulty problem types. Teachers used this framework to classify test items from released EOC items or for items they wrote themselves and to place the items into one of the nine categories.

Teachers Use of the Framework in the Classroom

Twenty teachers agreed to use the framework in their classrooms by revising a previous exam and to write a new exam for a future test. Below are examples of Category III questions teachers included on exams that they previously had not. Some teachers took existing questions and extended / modified them to a Category III question. For example,

(Original): "Find the x-intercepts of the function $f(x) = x^3 + 3x^2 + 2x$."

(Revised): "Explain the connections between the x-intercepts of the function $f(x) = x^3 + 3x^2 + 2x$ and the solution to the equation $x^3 + 3x^2 + 2x = 0$."

(Original): "Angela walked along the diagonal of a 30m by 16m rectangular field. How far did she walk?

(Revised): "Angela took a shortcut by walking along the diagonal of a 30m by 16m rectangular field. How much farther would she have had to walk if she had walked along the edges of the field?"

Another common strategy teachers used to write Category 3 questions involved students explaining their answers. For example:

- "Use the discriminant to predict the number and type of solutions of the equation: $9x^2 + 6x = {}^-1$. Use a graphing calculator to check your solution and explain/describe how the graph verifies your answer."

- "In which of the following triangles can the Pythagorean Theorem be used to find the missing length of a side? Tell why or why not."
- "Can a right triangle be isosceles? Explain."
- "Solve $x^2 + 6x + 2 = 0$. What solution method did you use? Was this the only choice that seems reasonable? Explain."
- "Explain how natural logarithms are different from and similar to common logarithms."
- "Explain why a vertical line has no slope."

Another strategy used by teachers to create Category III questions was reversing problems. That is, give the students the answer and ask them to generate possible questions. For example:

- "Create a word problem that could be solved using the equation $x - 6 = 14$."
- "Find two different equations with roots of -2, 3."

Additional strategies used by teachers to create Category III questions included asking students to

- find an error in a worked problem, then explaining and correcting the error.
- classify equations by type, or comparing/contrasting equations.
- solve problems in more than one way then comparing/contrasting these solution techniques.

Difficulties in Teachers Using the Framework

Most teachers found our modified version of the state's assessment framework a useful tool to reflect on tasks and thinking skills used to solve these tasks. In interviews with teachers, most explained that they previously only used two criteria when creating test items: content coverage and degree of difficulty. However, this framework provided them with a new way to analyze/select/generate assessment items. Nonetheless, there were four problem areas that made it difficult for teachers to use the framework in their classroom: lack of time; lack of "fit;" restricted context; and low state expectations.

Lack of Time

The first area of difficulty was simply a lack of time. Teachers expressed concern over how long it took to write and grade tests which included higher order thinking skills as well as the amount of time it took to prepare students to be able to answer these types of questions on an exam. In the context of the problems discussed below, many of the teachers expressed concern over whether it was time worth spending.

Lack of Fit

The state's identification of test items occasionally did not match the *Dimensions of Thinking* framework nor was the state's own classification scheme consistently applied. This made it difficult for teachers to identify questions by thinking skill. Although many of the questions matched thinking skills from *Dimensions of Thinking*, others did not. Our first two examples[1] are mis-categorized by NC DPI as *evaluation* (category 3) questions:

Example 1: Which expression is equal to $\sqrt{169}$?

a. $\sqrt{121} + \sqrt{9}$ b. $\sqrt{196} - \sqrt{1}$ c. $\sqrt{256} - \sqrt{36}$ d. $\sqrt{144} + \sqrt{16}$

Example 2: Given $y = {}^1/_3x + 2$ and $y = 2x - 13$, which ordered pair below is valid for both equations?

a. (-2 , -1) b. (9, 5) c. $(-{}^4/_3, 0)$ d. (3 , 16)

In these cases, the term "evaluation" as commonly used in mathematics to "find the value of" is incorrectly used as synonymous with "evaluation" as a thinking skill defined in *Dimensions of Thinking* (evaluation: assessing the reasonableness and quality of ideas).

In *Dimensions of Thinking*, integrating refers to connecting and combining information efficiently into a cohesive statement; changing existing knowledge structures to incorporate new information. Nonetheless, the following question was mis-categorized as a Category 3 "integration" question: The length of a barn is 58 feet and the width is 32 feet. Find the area of the barn. (choices not included here).

Sometimes, test item classification was inconsistently used by NC DPI. On one Algebra I EOC exam in 1998, the question "Which of the following is a rational number?" was classified as a "knowledge" (category 1) question while the exact question was listed as an "organizing" (category 2) question on another Algebra I EOC exam. Similarly, some questions identified as "knowledge" (category 1) on one test were categorized as "analyzing" (category 3) on another despite being the exact same question.

This lack of fit often led teachers to believe they were already teaching for higher order thinking even though "routine" textbook problems continued to dominate their exams ... including those they labeled as Category 3. For example, teachers often categorized question such as "Graph $y = 2x - 3$" and "Find an equation with slope 2 and y-intercept 3" as category 3 (Generating) questions. Although these problems have routine solutions, it was actually consistent with NC DPI's use of the term "generating." An example EOC test item classified as generating included: The sum of a number and two is five. What is the equation for this statement? Answer among four choices was $x + 2 = 5$. However, it is not clearly apparent that this fits with the definition of *generating* in *DOT* where it is described as producing new information, meaning, or ideas; going beyond available information to identify what reasonably may be true; anticipating next events, or the outcome of a situation; explaining details, examples, or other relevant information.

Restricted Context

NC EOC tests are limited to multiple choice questions only. This makes it very difficult to use *Dimensions of Thinking* as originally intended. For example, it can be difficult to construct higher order thinking questions that require students to *evaluate* or *generate* using a multiple choice format. As a result, in the workshops, we asked teachers not to limit themselves to multiple choice questions and encouraged them to broaden its use for all types of classroom assessment/activities.

In addition, teachers recognized that due to the restricted context of the exam, many of the questions could be solved using different cognitive skills ... depending on the knowledge and experience of the student. Teachers felt there was no way to tell if a student was required to use a certain thinking skill. Problems initially non-routine when learned in class may become routine on a test. Although a student may have used higher-order thinking skills to learn the mathematics, eventually the problem may become "routine" and higher-order thinking skills no longer needed on the EOC exam. For example, the following question could be reduced to "find the vertex" by many students; yet, it was classified as an "integrating" question by NC DPI:

> The height in meters of a ball thrown vertically upward is given by the function $h = 20t - 4t^2$. At what time will the ball reach its maximum height and what will be the height at that time?
>
> a. 4 seconds; 25 meters b. 4 seconds; 16 meters
>
> c. 3 seconds; 45 meters d. 3 seconds; 40 meters.

In addition, in the problem above, students are given possible answers. Therefore, students can also substitute the time values to find the corresponding heights. Although this problem does require knowledge of functions, its potential to elicit "higher order" thinking is limited.

Low Expectations

The vast majority of teachers in our workshops stated that they were unaware of the "thinking skills" framework used to construct EOC items (in particular, none of the teachers had heard of *Dimensions of Thinking*). However, most teachers felt their students were doing fine on the Algebra I EOC exam despite their lack of knowledge of the higher order thinking skills. In discussions with DPI officials regarding the Algebra I EOC exam, students often are not required to answer correctly more than 50% of the questions to pass -- implying a student can miss all "category three" questions and still pass the exam. Thus, the low score needed to pass the exam does not "require" students to learn category 3 questions nor do teachers have to prepare their students to answer these types of questions. (Relatedly, as discussed above, many category 3 questions were mislabeled as "higher – order" thinking questions further lowering the higher order thinking required on the Algebra I EOC exam). [Of note, deeper investigation into the EOC Geometry exam shows higher expectations and a better fit / consistency with *Dimensions of Thinking*. Not surprising, Geometry EOC scores in NC are lower than Algebra I EOC scores.]

Teachers Comments on the Usefulness of the State's Assessment Framework

Despite the diverse interpretations and uses of the NC DPI assessment framework, the general consensus among teachers was that they felt it was useful to have a better understanding of how EOC test questions are created and that they felt that they would make at least a few modifications to how they taught and test so their students would be better prepared for the EOC exam. Several teachers commented that after using the state's assessment framework, they realized they needed to make changes not only to classroom assessment, but instruction as well. As one teacher commented, "When I categorized my test using the framework, I found that 95% of my questions were knowledge [Category I] questions. My new (revised) test is much better because it has a variety of thinking skill levels." She later continued, "To help my students become successful on this test, I would need to implement more high level thinking as part of my daily instruction so that my students could gain more experience in solving problems of this type."

Another teacher, after commenting that her students did very poorly on the "revised" test, explained, "But I did gain some valuable insights from this test. I realized that my style of teaching really does sell short my students' abilities as problem solvers. I try to use discovery type activities, but that is where I seem to stop. Everything after that becomes traditional and rote, especially in my assessments." In general, although the framework was used and interpreted in diverse manners, teachers stated that the framework was a useful starting point to help them select and create tasks using "thinking skills" as a criterion.

Summary

The goal of this research was to explore whether a state's effort to assess higher order thinking skills could help facilitate the inclusion of higher order thinking in the classroom. And in particular, to ascertain if NC's assessment framework was a viable tool for improving higher order thinking in classroom assessment. Our goal was to develop a framework that teachers would find practical in analyzing their classroom assessment practices by helping them introduce "cognitive goals" into their teaching. This project shows that a state's assessment framework can be a viable tool for reflection on classroom assessment. However, the impact on teachers was diverse, and its long-term impact is unclear. There is little evidence to suggest that the state's effort to encourage higher order thinking skills on their exams has had much effect in the classroom without additional training on using the state's assessment framework. Additional research is needed to continue to follow these teachers to ascertain the impact of their knowledge and use of the state's assessment framework on their teaching and eventually, on the success of their students.

Note

[1]Some of the examples used in this section are similar (but *not* the same) to those found on the NC Algebra I 1998 EOC exam. The 1998 exam questions are secured by the state and used for EOC practice at the district level.

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CLASSROOM ASSESSMENT AS A BASIS FOR TEACHER CHANGE (CATCH)

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The objective of the CATCH project[1] is to develop and test a program of professional development to bring about fundamental changes in teachers' instruction by supporting changes in teachers' classroom assessment practices. Classroom observations of middle grades mathematics teachers prior to participation in the CATCH project revealed that most teachers retained conventional assessment practices, even though they were using various reform mathematics curricula. After one year of participation in CATCH, teacher interviews and classroom observations revealed that teachers who initially enacted design principles for classroom assessment sought to apply similar principles to their selection of instructional activities, interpretation of student class work, and their use of instructional assessment. Findings suggest that student-centered instruction is supported by professional development activities that focus on design principles for classroom assessment.

The professional development program we are studying is a product of the Research in Assessment Practices (RAP) project of the National Center for Improving Student Learning and Achievement in Mathematics and Science (NCISLA). From 1996 to 2000, as we studied reform and teachers' approaches to assessing their students in a variety of mathematics and science classrooms, a strategy to develop teachers' formative assessment practices evolved. The CATCH project is studying what is necessary for this strategy to "travel" to new sites. We have found that a key to change is getting teachers to shift their assessment practices toward assessing for understanding. When greater attention is given to developing teachers' expertise with classroom assessment, teachers are better prepared to make principled decisions about the selection of assessment tasks and the interpretation of student work. Then, as teachers broaden their conceptions of classroom assessment and make greater use of instructionally embedded assessment, they are better prepared to make instructional decisions based on the student thinking they listen to and observe. Our belief is that improving the alignment between teachers' instructional decisions and student thinking results in learner-centered instructional environments that are more likely to result in improvements in student achievement (Bransford, Brown, & Cocking, 1999).

Theoretical Framework

There are three theoretical components to our change strategy: (a) a domain view of mathematical content, (b) learning with understanding, and (c) formative assessment.

Mathematical Domains

Epistemologically organizing mathematical content by domains rather than by a large collection of concepts and skills is not new. Two well-known publications—*On the Shoulders of Giants: New Approaches to Numeracy* (Steen, 1990) and *Mathematics: The Science of Patterns* (Devlin, 1994)—have described mathematics in this manner. To describe each domain, the key features and resources of the domain that are important for students to find, discover, use, or even invent for themselves are identified. The domain view of mathematics differs from conventional perspectives in at least two important ways: First, the emphasis is not on the parts of which things are made but rather on the whole of which things are part (i.e., how concepts and skills in a domain are related) and, in turn, with the relationship of those parts to other parts, other domains, and ideas in other disciplines. Second, the domain view rests on the meaning of signs, symbols, terms, and other mathematical representations—the language that humans have invented to communicate with each other about the ideas in the domain. In fact, students should experience the need for the elements of the language of mathematics, and as a consequence, teachers must introduce and negotiate with the students the meanings and use of those elements.

Learning with Understanding

In any mathematical domain, learning with understanding occurs on a widespread basis in classrooms that provide students with opportunities to (a) develop appropriate mathematical relationships, (b) extend and apply their mathematical knowledge, (c) reflect on their own mathematical experiences, (d) articulate what they know, and (e) make mathematical knowledge their own (Carpenter & Lehrer, 1999). To accomplish this, the ideas in any domain need to be structured according to *hypothetical learning trajectories* (Simon, 1995) to help teachers plan and carry out instruction so that students progress from informal, to preformal, and then to formal understanding of the domain.

Formative Assessment

From an assessment perspective, a hypothetical learning trajectory can be viewed as a loosely sequenced subset of benchmark evidence for student learning with understanding in a particular domain. The quantity and quality of evidence of student learning are bounded by the practical means by which teachers can reasonably assess individual and collective learning within a classroom setting through formal, informal, and instructional assessment methods. In this sense, *hypothetical assessment trajectories* are organizational frameworks to locate student thinking on a network of developmental paths that are accessible and achievable through appropriate sequencing of learning opportunities.

Although research confirms the contention that formative assessment supports student learning (Black & William, 1998) and can be used to facilitate learning with understanding (Shepard, 2000), many mathematics teachers show limited understanding of the ways formative assessment can be fully incorporated into their classroom practices (Fennema & Nelson, 1997; Fennema & Romberg, 1999). As a result, teachers often find it difficult to make didactic decisions based on their students' work and therefore leave their students with incomplete information about their progress (Romberg, 1999). Students in such classrooms can find themselves at a loss in assessing what they know, or don't know, reinforcing flawed procedures and mathematical misconceptions. To teach for understanding, teachers must first assess students' prior knowledge, skills, and understanding, and then use this information as a point of departure for instruction. As students progress, teachers must continue to monitor their progress, not in terms of correct or incorrect answers on some percentage scale, but in the broader and deeper sense of their conceptions of mathematical content and their growing ability to adapt what they understand to solve unfamiliar problems embedded in new contexts.

To operationalize assessment trajectories, teachers need to value students' informal knowledge as instructional anchors. As argued by Kilpatrick, Swafford, and Findell (Kilpatrick, Swafford, & Findell, 2001) in *Adding It Up*, "learning with understanding involves connecting and organizing knowledge, learning builds on what children already know, and formal school instruction should take advantage of children's informal everyday knowledge of mathematics" (p. 342). Whereas conventional classroom assessment focuses primarily on student outcomes and student recall of formal knowledge and procedures, assessing for student understanding requires that teachers attend to students' incoming knowledge and the progression in student knowledge across a wider range of informal, preformal, and formal representations. Even experienced teachers with degrees in mathematics struggle with how to make use of students' informal knowledge and emergent student strategies. Teachers who value students' informal knowledge assess and expand on their thinking in ways that transcend the objectives of a particular activity, thereby bridging content across lessons and providing opportunities for students to apply knowledge in new ways.

Building from the results of a series of in-depth studies with middle grades mathematics teachers that demonstrated the value of developing teacher expertise with classroom assessment (Romberg, in press), our goal in the CATCH project was to study the design, implementation, and impact on teacher practice of a professional development program to support teacher assessment of student understanding. Over the duration of the study we will also analyze how changes in teachers' assessment practices impact student achievement results on compulsory state and district assessments. The focus of this report is to report a subset of findings from this study in response to the following question: How do teachers' assessment practices change as a result of their participation in this professional development program?

Approach and Methods

To have an impact on teacher practice, professional development activities must leverage mutually supportive principles that promote teacher learning. King and Newmann (2001) concluded that teacher learning is most likely to occur when:

- teachers can concentrate on instruction and student outcomes in the specific contexts in which they teach;
- teachers have sustained opportunities to study, to experiment with, and to receive helpful feedback on specific innovations;
- teachers have opportunities to collaborate with professional peers, both within and outside of their schools, along with access to the expertise of external researchers and program developers; and
- teachers have influence over the substance and process of professional development (p. 86).

Each of the principles described above is consistent with the design of professional development activities for our work with teachers in CATCH.

Our aim was to train professional development cadres at each of the participating sites who would then conduct professional development programs on classroom assessment for teachers in their respective districts. The objectives of the institute were to articulate the goals of the project, present examples and activities to communicate classroom assessment principles (e.g., making one's assessment goals explicit, considering features of assessment task design, scoring and grading student work), and provide opportunities for cross-district collaboration and team building. Participants were involved in adapting and constructing formative assessment instruments, which they later used in classroom experiments. Materials developed at the institute were consistent with the principles of mathematical literacy described in OECD's (1999) *Measuring Student Knowledge and Skills: A New Framework for Assessment.* Participants were also provided with a number of existing assessment resources they could adapt for their own use — e.g., *The Great Assessment Picture Book* (Dekker & Querelle, 2002) and *AssessMath!* (Cappo, de Lange, & Romberg, 1999). Attention was also given to critique of existing assessments and how to select instruments to meet their own goals. In general, the seminar emphasized more constructive ways to assess student understanding and gave teachers the opportunity to discuss how such ideas could be incorporated into their classroom practice and supported through ongoing professional development. The initial CATCH seminar served as an example for the summer seminars that were conducted by lead teachers.

Our support efforts have since shifted to providing technical support to teachers as they design assessment tasks, participating in monthly meetings (when possible), and supporting teachers and administrators in the design of local professional development for new cohorts of teachers. Local professional development activities facilitated by the research team and lead teachers were designed to promote the following developmental model:

- During the initial CATCH summer institute, lead teachers and administrators critique and develop greater understanding of existing assessment instruments;
- Teachers select and adapt assessment instruments for their own use with students and report about results during monthly and ad hoc meetings with colleagues;
- In considering and using these instruments, teachers examine the role and function of assessment instruments versus the desired learning outcomes and the potential for positive feedback;
- Scoring and grading of student work is used to provide insight into student (mis)conceptions that guide instruction;
- Teachers implement design principles for classroom assessment and learn how a series of items can be constructed to design balanced assessments that reflect a hypothetical assessment trajectory. This provides students with the opportunity to demonstrate the full range of mathematical competencies including making mathematical arguments, non-routine problem solving, developing their own models and inventing new strategies;
- Teachers explore assessment opportunities embedded within instructional contexts, learn how to balance the use of formal and instructional assessment, and examine the relationship between classroom assessment and student achievement on external assessments;
- Teachers inform their colleagues during successive summer institutes, thus helping ideas and outcomes of the CATCH project to "travel" to new classrooms, schools and districts.

Participants and Data Collection

The participants in this study include professional development cadres (lead teachers, staff developers and district administrators) from two school districts. District A is a small urban/suburban district serving over 3,000 students in which lead teachers work with a student population that is predominantly European American (85%), with approximately 30% receiving free or reduced-cost meals. District B is a large urban district in Eastern United States in which lead teachers work in middle schools with a school population of predominantly African American and Hispanic students, with more than 75% receiving free or reduced-cost meals. In March 2001, fifteen teachers and administrators from Districts A and B participated in the initial CATCH institute. After one year, just over 100 teachers across both districts had participated in CATCH professional development activities.

Data collection during the first year of the study focused on the emerging classroom assessment practices of lead teachers. These practices were documented through an initial survey, classroom observation protocols, and teacher interviews. Two interviews were administered every semester and classroom observations were completed every eight weeks. Changes in teachers' assessment practices were also documented via an assessment portfolio, which included all formal assessments administered by each teacher, and samples of student work.

Results

Overall, teachers became more aware of what they were doing (and not doing) to assess student understanding. In monthly meetings and summer institutes held during the 1st year of the CATCH project, both district teams chose to emphasize the selection and design of tasks according to a tri-level Dutch assessment pyramid (see Dekker & Querelle, 2002). As lead teachers articulated classroom assessment principles as presenters during their respective summer institutes they revealed the assessment aspects from the initial CATCH institute they found relevant and demonstrated their degree of ownership of assessment theory and practice. Teacher interviews and classroom observations revealed emergent assessment practices and increased attention to assessment opportunities that occurred during instruction. Classroom observations and teachers' assessment portfolios from both research sites indicated greater teacher use of tasks that provided students opportunities to demonstrate a range of mathematical understanding. For this research report interviews will be used to highlight teachers' justification for using more cognitively demanding tasks.

Selection and Design of Tasks to Assess Student Understanding

Teachers at both research sites chose to apply the Dutch assessment pyramid in their own classrooms and shared these design principles with colleagues in local CATCH institutes. Lead teachers realized that most assessment instruments used at their schools consisted of items that assessed only basic skills, facts, and routine procedures (*Level I questions*), often in short-answer or multiple-choice format. Teachers found that they rarely used problems designed to deepen student knowledge and understanding as students use basic skills and routine procedures in unfamiliar problem contexts and choose appropriate mathematical tools to solve problems (*Level II questions*). Nonexistent in teachers' classroom assessments was the use of questions to encourage generalization, mathematical reasoning, and argumentation (*Level III questions*). Teachers concluded that they were not giving students opportunities to gain ownership of the mathematical content and were only asking students to reproduce what they had been practicing. In a process described as "leveling," teachers analyzed their own assessments to discern the reasoning elicited by different assessment tasks. As one teacher noted, the most helpful aspect of CATCH was "looking at questions on levels and leveling and really having the opportunity to re-examine myself through that information that I have gotten [from students]."

The Dutch assessment pyramid was often used by teachers to justify greater use of projects, extension problems and prompts requiring written explanations.

- One of the things that I am doing is [with] a group of four girls who really are together and getting it. I try to get over to them ... and push them over the edge a little bit more with trying to get them to think a little, because they can. (Ms. Pitkin, District B, 01/28/02)
- I like to put a question on each exam that allows them to think about and maybe apply some other information they already know to what they, to what we are doing. ... They really have been doing well with them and really understood why they get these [problems]. Explaining why this formula is so and not just saying this is how it is [written in] the math guide. (Ms. Graves, District B, 01/31/02)

In District B, the Dutch assessment pyramid was presented at the local CATCH institute as a model for classroom assessment that was consistent with compulsory district and state assessments, which included problems requiring students to provide mathematical explanations and engage in non-routine problem solving.

- I think it differs from last year because I am looking more to see if kids are writing more. Last year they didn't do a lot of writing. This year I am trying to focus on the things that are outlined in their testing, for standardized tests. A lot of the things that they have to do require that the children explain their answers in detail. Not always with numbers, but with words. So it is a little bit different. I am not looking for just computation this year. It is computation plus an explanation. (Ms. St. Pierre, District B, 01/30/02).

Reduced Emphasis on Formal Assessment

As teachers gave greater attention to assessment design they began to view assessment opportunities in other aspects of classroom practice. The attention to task design allowed teachers to apply the same principles to tasks used in particular phases of the teaching-learning cycle. Tasks used for instructional activities, class work and homework were now potential assessment opportunities. As a result, several teachers reduced their reliance on tests and quizzes as assessment instruments.

- I think I am getting away from [using] as many tests, as many quizzes. It seems to be more the observation and discussion. Show me an example of how your work is done in class or something like that. I don't feel the need to test as much. It seems like a waste of time. I know what they are going to do pretty much [and] it seems to be just a waste of time now, the process of formal testing. So I try to cut that back. We were in the past giving two tests per chapter and have cut it back to one. (Mr. Lemke, District A, 03/05/02)
- I think the subtler the assessment probably the better it is.... I walk around all the time. I can get more out of that assessment of just looking and seeing what the

kids are doing. And for one thing it is a lot faster than having to put it on a piece of paper. And it is immediate and it is very quick and it doesn't have to necessarily break the flow of the class. (Ms. Edwards, District A, 02/15/02)

- I do more observation of classwork and homework as a tool than I do my test. And that is the truth! (Ms. Carter, District B, 01/31/02)
- Observation in the classroom. Just watching them as they work. Listening. So keeping a better record, a more accurate record of who is answering what. Before it was kind of like a general sense of things. And now I kind of have a more detailed recording of who answers, how often they answer correctly, who doesn't answer, who will answer if I call on them and things like that. So that is a little different from last year. (Ms. Kruger, District B, 01/28/02)

However, as teachers gave greater attention to observational assessment some struggled with how to document such information. Teachers were not interested in using evidence gathered through observational assessment for reporting or grading. Rather, teachers viewed record-keeping as a way to bring greater clarity to the "hodge-podge of information in their head"—a tool to support further reflection on student thinking.

Several teachers described changes to their practice of scoring and grading student work. With regard to student performance, they grew to value qualitative over quantitative information. Student models and explanations were of greater interest than aggregate scores. In order to devote more time to interpreting student work, teachers were more selective in the student work they collected and scored. Becky, a teacher with over 30 years experience teaching middle grades students explained,

- I assess formally less often. ... and I tell the kids I expect them to monitor their progress to get their needs met. Major shift. Major shift. I talk to teachers in my building who have hundreds of grades per quarter. 'Hey, guys! We've got to get past this!' ... You don't have to do that. You have to give the kids feedback.... You have to discuss it. Answer their questions. But you don't have to grade it. You don't even have to collect it. And it seems like any person with two ounces of brains could figure out that they can't possibly grade 120 papers every night and stay sane. But somebody had to tell me that! (Ms. Chamberlain, District A, 02/15/02)

Impact on Classroom Instruction

In addition to demonstrating greater use of Level II and III problems in their formal assessments, teachers began to interpret instructional activities and problems in terms of the level of reasoning required. Several participating teachers used principles of assessment design to evaluate instructional activities and motivate questioning techniques to extend student reasoning. This entrée to instructionally embedded assessment began to emerge in the second round of post-observation interviews, in which teachers began describing instructional activities, class work and homework problems in terms of Levels I, II, and III reasoning. As explained by one teacher,

- Not just in the design of the test, but in the actual problem work that they are doing, students need to be able to take their concept or their knowledge of the concept and reapply it in a slightly different way. Maybe related, but different. (Ms. Petersen, District A, 03/05/02)

Ms. Graves, a District B lead teacher, observed that the curriculum she used did not appear to include enough Level II and III problems. Therefore, she began to supplement student class work with problems requiring further application and generalization of concepts and adapted her curriculum to extend student thinking.

As teachers gained confidence in the design of assessment tasks, they began to apply the same design principles to the questions they used during instruction. During the first round of observations, the trajectory of the lesson followed the outline of the curriculum and teachers restricted their exploration of student thinking to questions as written in the textbook. By the third round of classroom observations, several teachers in District A demonstrated greater use of questions to probe and encourage elaboration of student responses which emerged during the activity. Teachers were more responsive to student thinking and used "their own questions" to facilitate classroom discourse.

The emergence of student-centered pedagogy was observed in Mr. Lemke's class. In a post-observation interview, Mr. Lemke identified the questions he developed "on-the-fly" as giving greater insight into student thinking and attributed his improved instructional decisions and use of questions to a shift in the locus of instructional assessment, from "as a class" to "to the kid."

- I guess I am moving toward a better assessment. I still think I have a long way to go. I guess I should be able to predict better what is coming for kids and I think I am getting better at making decisions as I go along. I think I can read the kids better than I could two years ago. 'To the kid' instead of 'as a class.' [Assessment] 'as a class' was not as helpful. I think now I can pretty much 'to the kid' know what they are doing. (Mr. Lemke, District A, 03/05/02).

Conclusion

Teachers who teach for understanding find the need to design tasks to assess Level II and III goals in order to assess for understanding. As teachers operationalize this assessment design model, they begin to rethink the learning objectives of their curricula and the questions they use during instructional activities. To make sense of students' mathematical representations elicited by Level II and III questions, teachers need to have the opportunity to construct explanatory models that relate students' verbal and written artifacts to hypothetical assessment trajectories and models of progressive formalization.

Our experience shows that teachers, and subsequently their students, benefit greatly from exploring features of classroom assessment, such as the design of assessment tasks, the interpretation of students' written and verbal responses, and strategies for eliciting or responding to student ideas during instruction. But teachers have to feel the need for change themselves: they must recognize the limited information their current assessment practices provide, they must realize the necessity of using tasks and practices that can reveal student understanding and they must view teaching for understanding as an important goal. By integrating assessment and instruction, teachers will develop an informed basis for making instructional decisions. These developments lead to greater student understanding of mathematics and improve student achievement.

Notes

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DEVELOPING A MULTI-TIER ASSESSMENT DESIGN IN MATHEMATICS EDUCATION: THE EFIT AND EMAT PROJECT

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When coordinating a complex and rapidly expanding project in Mathematics Education that involves developmental, research, and implementation components all intrinsically changing, how does one go in evaluating and assessing? We were obliged to answer this question when the implementation of the EFIT-EMAT (Teaching Physics and Mathematics with Technology) Projects were supported by the Ministry of Education in Mexico. In this presentation we will justify the use of a multi-tier assessment design by giving examples of how this design encourages interaction and helps document and give opportunistic feedback to all the participants, and thus, leads to the development of the EFIT-EMAT Project. We will also describe thought-revealing activities for students and teachers, and how these tools have been used to document and evaluate students' mathematical understanding and teachers' mathematical practice.

Context: The EFIT and EMAT Projects

The EFIT (Enseñanza de la Física con Tecnología, or Teaching Physics with Technology) and EMAT (Enseñanza de las Matemáticas con Tecnología, or Teaching Mathematics with Technology) Projects for public middle schools were supported by the Ministry of Education in Mexico as a response to the Educational Reform (SEP, 1994). This reform's goals are to (a) increase quality in the teaching and learning of Science and Mathematics (including teachers, students, curriculum, theoretical perspective), (b) increase coverage of education, and (c) enhance professional development of teachers in Science and Mathematics. Thus, the EFIT and EMAT Projects were supported at a pilot stage from 1997 to 2000 in 32 schools countrywide to assess feasibility for further expansion. Characteristics of these projects included the implementation of a pedagogical model, aligned to the new curriculum, and using pieces of technology to create a classroom environment that would allow a learning approach that wouldn't be possible to achieve without these components.

The pedagogical model integrates innovative elements, like a transformation of the role of the traditional teacher to a more active guide and facilitator; the encouragement of collaborative learning to discuss and solve scientific and mathematical projects; the awareness of a closer and more meaningful relationship between students' every day life events and math and science in the classroom; the use of the technology as a mediated tool; and the development of activities that incorporate the technology, the pedagogical approach, and the reformed curriculum.

Classrooms are equipped 10-15 student computers; a teacher computer, local access network connections with the Internet; a printer; and distance communication via Red Escolar (Educational Network). For the EFIT Project, software included Interactive Physics, Micro-computer based lab (sensors), and LXR Test. For the EMAT Project, software included Cabri-Géomètre, Spread Sheet, Stella, SimCalc Mathworlds, and Calculator TI-92.

After three years, the Ministry of Education assessed the results (Rojano, Moreno, Bonilla, & Perrusquía, 1999) and decided to support an expansion of this project to other states and schools, including Telesecundarias (middle schools offered through satellite system devices, mainly aiming at rural areas) and Normales (Normal Schools). This expansion, occurring in the next 5 years (SEP, 2001), urges us to rethink the evaluation and assessment strategies used in the pilot stages, and to formulate a feasible plan for such an up-scaled venture. A plan that in addition to evaluation and assessment, will also promote implementation, developmental, and research components.

The main goals to consider for the evaluation and assessment of the EFIT and EMAT projects are: 1) To document the development, including conditions that enhance or hinder growth, 2) To help in the decision-making instances of different participants, and 3) To give access of documentation to all the Project participants as feedback.

Research Design: Multi-tier Design Experiment

The need for an evaluation and assessment plan that is dynamic, iterative, adaptive, and adoptive, leads us to consider design experiments (Brown, 1992) as an appropriate framework. Lesh and Kelly (2000) describe a multi-tier design experiment (MTDE) as a complex system that allows researchers to document and analyze the development in which students, teachers, schools, parents, and researchers interact. MTDE's involve at least three levels of investigators: students, teachers, and researchers, all of whom interact and interweave through their experiences and roles to develop towards common goals. For example, Tier 1 may focus on the nature of students' developing knowledge and abilities; Tier 2 may be aimed at investigating teachers' developing assumptions about the nature of students' mathematical knowledge and abilities; and Tier 3 may center on the researchers' conceptions about the nature of students' and teachers' developing knowledge and abilities. Each tier can be thought of as a longitudinal development study – all of the tiers interact and affect the others. Thus, a MTDE provides a complex methodology to collect and analyze documentation on student/teacher development, and their mathematical ideas.

Methodologies for Documentation

Aspects we anticipate should be documented include: 1. Educational community profile, 2. Technology, 3. Curriculum content, 4. Motivation, and 5. Pedagogy. For the design, we'll consider these methodologies:

(a) Standardized Testing: Including diagnostic computer and paper-based student testing and pre-testing to evaluate initial conceptions on mathematics and scientific content. Testing at the end of the school year will allow us to do a pre and post-test analysis. The methodologies used to design, collect, and analyze these tests involve item response theory and other statistical methods. Software that will help us achieve these goals is LXR Test.

(b) Qualitative Tools: Aimed for teachers, instructors, parents, school principals, and administrators, these include the use of questionnaires, interviews, observation guides, and journals. The methodologies used to design, collect, and analyze these tools are qualitative, and involve different approaches (Miles & Huberman, 1994).

(c) Thought-revealing Activities (TRA): Designed to engage students in developing math and scientific concepts, these can also be used to document participant development within the project. They reveal the way students think about a given real-life situation which is modeled through mathematics (Lesh, Hoover, Hole, Kelly, & Post, 2000). The problem is solved by groups of 3-5 students who create a mathematical model. An identified client requires the students' solution in order to solve the problem. To implement the model adequately, the students must clearly describe their thinking processes and justify the solution. In this way, students are forced into a cognitive situation where they can refine their ideas iteratively until they develop a construct that is useful and meaningful for them and the client. Ultimately, they need to describe, explain, manipulate, or predict the behavior of the real world system to support their solution as the best option. As in real life, there is not a single solution, but there are optimal ways to solve the problem. Because of their characteristics, TRA's blur the distinction between instruction and assessment. They can be used for student assessment because (i) students' learning is being assessed WHILE students are learning, and (ii) since these problem-solving situations promote learning, they are not taking time away from learning to assess their thinking. TRA's allow students to document their own thinking and learning development locally (in a single solving episode) and longitudinally (solving a collection of activities during the school year). Thus, they allow teachers and researchers to access a trail of documentation for ways of thinking. When used in a classroom context, teachers find themselves in a situation where their role is shifted to become facilitators, observers, mentors, and learners – all at the same time. In addition, the teacher must also adjust to collaborative learning in the classroom. Therefore, when teachers begin using TRA's for students, they gradually shift towards new ways of teaching that are more consistent with the EFIT-EMAT Project. Analogously, TRA's can be designed for teachers, providing trails of documentation of their development within the multi-tier design experiment (Carmona, 2000).

In the presentation, an example of one TRA for students and one for teachers will be shown, and how these can document students' and teachers' development (locally and longitudinally) and how these activities give opportunistic feedback for teachers, parents, administrators, and researchers.

Note

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TIMSS 1999 DATA: ONE SINGAPOREAN'S PERSPECTIVE

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Singapore ranked first among 38 participating countries in the 1999 Mathematics Test in the Third International Mathematics and Science Study (TIMSS). Their achievement will be considered from two perspectives: (1) the correspondence between the TIMSS Mathematics Test and Singapore's high- stakes mathematics exam for sixth graders, and (2) a personal reflection on the roles of schooling and testing in Singapore's society.

Singapore's Secondary 2 (equivalent to U.S. eighth grade) students performed significantly better than U.S. eighth graders in each of the five content areas covered by the 1999 Mathematics Test. Using the data and categories in TIMSS 1999, this paper identifies the test items on which the Singapore students outperformed the U.S. students. These items were compared with test preparation materials used widely by Singapore students for the national examination, *Primary School Leaving Examination* (PSLE).

These analyses were based on two data sources: (1) the 82 questions in *TIMSS 1999 Mathematics: Released Set for Eighth Grade* and (2) the weighted summary statistics for each participating country on each mathematics item as listed in *Main Survey TIMSS Report* (available: http://www.timss.com/timss1999i/timss_test.html). U.S. students performed better than Singapore students in only three out of the 82 released items. All three were multiple-choice items, two in *probability* and one in *number sense*, two in *knowing* and one in *investigating and solving problems*.

Among the five content areas, Singapore students performed significantly better than U.S. students in *fractions and number sense* and *measurement*; among the performance expectation categories, they performed better in *investigating and solving problems*, and *communicating and reasoning*. Many of the *investigating and solving problems* items are considered routine procedures for Singapore students as similar practice problems in PSLE preparation materials can be easily found to match 17 of these 25 items. Other than probability, the items in the released sets are well within the national syllabus for the *Primary Mathematics Programme* (available: http://www1.moe.edu.sg/syllabuses/doc/Maths_Pri.pdf). Singapore students will generally find the TIMSS Mathematics Test easy because they took the PSLE Mathematics two years prior.

Singapore's Educational Climate

The mathematics curriculum in Singapore is examination driven. Primary education is dedicated to preparing students for the PSLE, in which results are used to assign Singapore students to either the *Special*, *Express* or *Normal streams* for their secondary education. Secondary 2 exam results are used to track them into either the science, technical or arts streams. Students must pass the General Certificate of Education (GCE) "O" Level examination to pursue "pre-university" education; each student's grades will determine his or her admission into either junior colleges (two-year program) or pre-university centers (three-year).

The educational environment in Singapore is highly competitive. Not only do students compete among themselves, schools compete to attract good students. Students must perform well in the PSLE to gain admission into higher-ranking secondary schools, in the GCE "O" Level exam to get into top junior colleges, and in the GCE "A" Level exam to be admitted into their national universities. The pressure to perform well on exams may help explain why Singapore students reported spending more out-of-school time doing homework in mathematics than U.S. students (Mullis et al., 2000).

Having experienced the education system in both Singapore and the U.S., I am aware of many social and cultural differences affecting mathematics education. For example, citizens in Singapore generally place greater importance on education and emphasis on effort. Singapore students are generally more competitive and motivated to excel academically, and they face more social pressure of having to meet their parents' and teachers' expectations.

To benefit from comparative studies, differences in cultural traditions must be considered (Graf & Leung, 2000). Based on my reflections on my experience in both countries, I wish to share my perspectives on (1) how culture affects education, (2) how the Singapore education system is designed to meet the needs of the country rather than those of the individuals, and (3) why care must be taken when we import educational ideas (e.g., Singapore textbooks) from another country. There is more to the story than can be told solely by test scores.

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MIDDLE SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF CLASSROOM ASSESSMENT

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Despite the reformed vision of assessment in mathematics classrooms, vast differences exist in how teachers use and view classroom assessment (Stiggins, 1999). A pervasive focus on summative assessment and the use of lower level recall permeates classrooms (Black & Wiliam, 1998). This perspective raises important questions about mathematics teachers' understanding of classroom assessment. The purpose of this study was to examine a cadre of middle school mathematics teachers' understanding of their assessment practices as they began to implement the reform-based middle school curriculum *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). Classroom assessment of student work is construed in a broad manner and is intended to include all traditional methods such as quizzes and tests as well as homework, class assignments, observation, group work and questioning techniques. Related research questions included: How do the teachers use classroom assessment? What types of assessments do they use? How does the curriculum influence their assessment practice? How do teachers negotiate classroom assessment within the broader context of mandatory state assessments in eighth grade?

Six female teachers in two middle schools in an urban area in the northeastern United States were recruited from schools implementing the reform-based middle school mathematics curriculum *Connected Mathematics.* Teachers in both buildings met regularly and at least weekly to discuss the curriculum. Teaching experience ranged from 6 to 28 years. The intent was not to follow individuals per se; rather, the intent was to see how the group as a whole understood their classroom assessment practice. The goal was to describe how teachers understand their classroom assessment practices through semi-structured interviews, observation and assessment artifact collection. The data collected in this study consisted of three semi-structured interviews, conducted at the beginning of the school year, at the end of the first marking period in early November, and at the end of the second marking period at the end of January; classroom observations; and collection of assessment artifacts when they deviated from the curriculum.

The results of this study indicate a multifaceted picture of the teachers' understanding of classroom assessment. How these teachers understand classroom assessment is inextricably linked to the looming state exams given at the end of eighth grade. However, the teachers understood classroom assessment to be much broader than "teaching to the test" and engaged in reformed practices. Analysis showed how the teachers knew what students had learned. The results show great promise that these teachers, involved in ongoing professional development, and utilizing a reformed middle school mathematics curriculum understand assessment to be far beyond formal quizzes and tests. These teachers' perception of assessment includes a formative component that is broader than traditionally summative and formal modes.

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SPATIAL VISUALIZATION:DO TESTS ASSESS WHAT WE THINK THEY ASSESS?

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Results from the sixth National Mathematics Assessment recommended more experience with concrete models to visualize and explore concepts (Strutchens & Blume, 1997). Studies have enumerated the strategies and errors that middle grade students utilize in counting the total cubes in a 3-D arrangement of cubes (Battista & Clements, 1996; Battista, 1999). The purpose of this study was to determine if spatial visualization instruction at Van Hiele level 0 would improve learning in volume and surface area (Van Hiele level 1 and 2) with eighth graders. To extend past research, visualization errors were analyzed for a surface area open-ended problem.

The teaching experiment was conducted in regular eighth grade class settings over two weeks in a rural middle school. With the experimental group (n = 14), the teacher introduced volume and surface area with visualization activities using linking cubes then moved to the formulas for the calculations. The comparison group (n = 15) studied volume and surface area from traditional pictures on the board and in the textbook. Each group was given a pretest in which students calculated volume for two prisms, two cylinders, one pyramid, and one cone and surface area for two prisms and two cylinders. For the posttest, ten problems similar to the pretest were given on a 20 item unit test. In addition, the posttest included a surface area problem in which students counted the surface area visualized in squares rather than using a formula. Standardized state criterion referenced tests given at the end of the 7th and 8th grade were reported.

Table 1. Experimental and Comparison Means

	7th Grade State Test	8th Grade State Test	Pretest % Correct	Posttest % Correct
Experimental (n = 14)	167.8	174.4	5.7	60.9
Comparison (n = 15)	168.3	173.9	3.6	61.2

Quantitative results did not show improvement in volume and surface area learning by using linking cubes for visualization. However, in an interview, the teacher explained, "Yes, the manipulatives did help!" She noted, "In class, comparison students experienced difficulty in comprehending the figures and determining measurements for the formulas. She added, "The experimental group was able to transfer the measurements from the figures to the formulas".

With the visualization surface area problem, only 4 students in the comparison group attempted the problem while 9 students in the experimental group attempted it. Only one student from each group gave the correct answer and explanation. However, the 9 students in the experimental group gave explanations using the concept of surface area. Many student errors were in miscounting the surfaces.

All students scored at Level III (170) on the state test, "mastery of grade-level subject matter and skills and well prepared for the next grade level". These results reveal a discrepancy between student understanding shown in the open-ended question and posttest scores and also the achievement measured by the state test.

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THE EFFECTS OF EXTERNAL TESTING ON CLASSROOM PRACTICES

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The centralization of curriculum via the creation and measurement of standards offers a promise of accountability for the work that is done in schools. Accordingly, the Ontario government introduced a ninth grade assessment in mathematics aligned with the Ministry of Education curriculum guidelines. The expectation is that teachers will align their classroom practices with the policy to a greater degree as a result of the pressure of the impending test. The premise, which has limited empirical evidence, assumes that external testing can alter classroom practices. The results of this comparative case study indicate that external testing has a limited influence on teacher classroom practices.

Background and Purpose

The Ontario government introduced a Grade 9 Mathematics assessment designed to reflect the new curriculum which values student understanding and concept building via investigation, experimentation, and their ability to reason using multiple strategies and representations (Ontario-Ministry-of-Education, 1999). Consequently, we should observe teaching strategies that foster constructivist mathematical learning behaviors. The Ministry's expectation in aligning the assessment to the curriculum guidelines (EQAO, 2001) is that teachers will adjust their classroom practices to enhance the match to the documents because of the pressure of the impending tests. The supporting premise is that assessments have great potential to alter classroom practices. However, the limited studies that have empirically investigated the effects of testing on classroom practices by collaborating observational data with teacher interviews (see Cimbricz, 2002) have contradictory claims. Consequently, irrespective of the prevalent use of testing as an agent of change in schools, the collective claims from relevant studies offer inconclusive results. This study attempts to illuminate the relationship between changing classroom practices and testing through a comparative case study of two mathematics teachers.

Method and Results

The teachers were teaching both ninth and tenth grade mathematics courses in the same large urban secondary school in Toronto. Both courses received a new curriculum within the last three years, but only the ninth grade had an external assessment introduced last year. Hence, comparing the classroom practices in the two courses offers an opportunity to discern if classroom practices change when implementing a new curriculum in the presence of an external test. Government documents, student work, six guided classroom observations followed by unstructured post-observational interviews for each class, and three one-hour semi-structured interviews comprised the data. The data was analyzed using an opportunity to learn (Wang 1998) framework whose specifications were established via the document analysis, and validity was established through triangulating the results and performing a member's check.

The most influential factor on classroom practices was departmental policy, which promoted the belief that if the entire content was delivered, the students would succeed on the external assessment. Accordingly, teacher efforts were focused on a timely delivery of the syllabus, which highlighted a perceived time constraint. Teacher reaction to this time restriction was to consciously employ conventional instruction that does not match the exploratory teaching strategies promoted in the Ministry guides. However, the test's effect on course pacing may be limited, as the scheduled lessons for the assessment dates was a series of review classes to prepare for the internal departmental exam. Hence, there is no evidence demonstrating a particular impact on classroom practices from the mandated external assessment.

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Beliefs

TEACHERS' GOALS: SPRINGBOARDS OR CONSTRAINTS

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Although Piaget's definition of learning includes the mention of goals, a participant's goals are often overlooked in teacher education research efforts. In response to research findings that have alluded to teachers' goals as influential factors in their professional development, I have extended the study of teachers' beliefs and orientations toward knowing to include the study of teachers' goals. I designed this study of secondary novice mathematics teachers' goals to determine whether or not relationships among a teacher's goals might lead to identification of overarching goals, from which, along with core beliefs and orientations toward knowing, I could represent a coherent model of a teacher's actions in relation to her beliefs. Three participants participated throughout their teacher education program and first year of teaching. The emergence of overarching goals related to respect, the politics of schooling, and mathematical inquiry provided the fabric that linked these novice teachers' beliefs and actions.

The view of learning as a response to perturbation that an individual generates in relation to a *goal* (Piaget, 1970; von Glasersfeld, 1995) set the stage for this study of three novice teachers' goals in relation to their process of becoming secondary mathematics teachers. Erffmeyer & Martray (1988) noted the contextual importance of goals as well as the perceived difficulty of achieving goals within contexts. Research findings related to the professional development of secondary mathematics teachers include discussion of teachers' goals (Helms, 1989; McGalliard, 1983; Owens, 1987; Thompson, 1982) and suggest further research in this area. Former research on teachers' beliefs and their orientations toward knowing have shed light on influential forces in the professional growth process of teachers (Cooney, Shealy, & Arvold, 1998), and this research adds yet another dimension to further our understanding.

By investigating the goals that are core to one's identity, namely those that are part of one's embedded tradition (MacIntyre, 1981, 1988), I learned that a teacher's goals may also serve to safeguard identity and provide refuge from conflicting messages (Arvold, 1998). As the foundation upon which one attends to, interprets, and interacts with the rich social structure that defines one's world, embedded tradition is unique to each individual. Consideration of goals as integral components of embedded tradition and the exploration of the relationships among goals, beliefs, and orientations toward knowing in light of philosophies of mathematics teaching (Ernest, 1991) may help explain the challenges that are associated with reform-oriented mathematics teacher education programs. I designed this study of novice teachers' goals to determine whether or not relationships among a teacher's goals might lead to identification of overarching goals, which, along with core beliefs and orientations toward knowing, would help represent a coherent representation of a teachers' actions in relation to her beliefs. Skott (1999) explained perceived inconsistencies between a teacher's beliefs and actions through the multiple motives of a teacher, whereas I am investigating the possibility of an underlying structure that supports the coherent nature of a teacher's beliefs and actions. A matrix of overarching goals, beliefs, orientations, and responsive actions captures dominant features that contributed to a coherent representation of beliefs and actions for each of the participants and augments the discussion in this report. (See Appendix A.)

The expert/novice research revealed profound differences among teachers. Leinhardt (1987) identified a transparent system of goals that supported the fluidity of interconnections within and among lessons of experts but only ambiguous systems of goals and fragmented lesson structures associated with the lessons of novices. In several other studies the process or product nature of goals was highlighted. As students learned to read, the "read to learn" or process-oriented groups produced higher reading skill scores than the "read to answer question" or product-oriented groups (Schunk and Rice, 1989). Students with a goal of learning how to solve problems rather than a goal of completion, experienced higher self-efficacy ratings as well (Schunk, 1996). These findings led me to Ames' work (1992). According to Ames, a mastery goal orientation is based upon goals that motivate a search for personal understanding or a process, whereas a performance goal orientation is based upon goals that motivate achievement and recognition or a product. I, like Kloosterman (1996), used this framework in my analysis of goals. Although Kloosterman noted that students often express both performance and mastery goal orientations, he also found that students' overall performance goal orientations most often related to their concern about their teachers' views of them as mathematics learners. His research led me to believe that overall goal orientations were related to orientations toward knowing and the deeply rooted embedded traditions that constrain or serve as springboards for growth (MacIntyre, 1981, 1988).

I assumed that the nature of most goals was dynamic and responsive to many influences including personal interests, social interactions, and context. I was most interested in identifying the goals that were less dynamic, that related to other deeply set embedded traditions, and that acted as a foundation through which the process of becoming a mathematics teacher could be interpreted and illuminated. I focused on the research question: What are the goals that are part of the embedded traditions of the participants and how are they related to their beliefs and ways of knowing and to the process of becoming a mathematics teacher?

Research Design

This case study was an outgrowth of a project entitled Research and Development Initiatives Applied to Teacher Education (See endnote.). The primary purpose was to reason toward hypotheses (Glaser & Strauss, 1967; Patton, 1980) and illuminate the process of becoming a mathematics teacher through the study of teachers' goals and goal orientations (Ames, 1992) in relation to embedded tradition (MacIntyre, 1988).

An interactionist perspective (Blumer, 1969) grounded the study and guided data collection methods. Theories of belief structures (Green, 1971), orientations toward knowing (Belenky, Clinchy, Goldberger and Tarule, 1986), and mathematics teaching philosophies (Ernest, 1991) guided analyses. The interdisciplinary theoretical foundation of the study supported my search for connections and relationships among goals, beliefs, and orientations toward knowing.

The interpretive mode of inquiry relied primarily upon patterns of discussion initiated by the participants (Lee & Yarger, 1996) and their introduction to sensitizing rather than definitive concepts (Blumer, 1969). Data were collected almost daily throughout the 12-month preservice teacher education program and monthly during the first year of teaching. Primary data sources included focus surveys with follow-up interviews, individual and group interviews, pre- and post- observation interviews, and observation fieldnotes. Triangulation with respect to responses to specific questions, unsolicited mention of goals, choice of projects and foci of independent personal pursuits, and researcher observations of and participant explanation of instruction assisted in determining each participant's goals. Constant comparative analysis (Glaser & Strauss, 1967) supported the reiterative coding process. Predetermined coding of beliefs, orientations toward knowing, and goals, led to beliefs about the nature of mathematics and orientations toward knowing mathematics. Emergent goal-oriented themes included tradition, respect, policy issues, and mathematical inquiry.

Three of the seven volunteers who participated in the first year of the study secured teaching positions immediately after graduation and thus were able to participate throughout their first year of teaching. Novice teachers Monica, Cheryl, and Alice (pseudonyms) were among a cohort of 15 preservice teachers who entered the secondary mathematics teacher education program at the start of their final year of full time university undergraduate study at a large southern university. Each was dedicated to a teaching career but with somewhat different goals in mind. The study of their goals in relation to their family backgrounds, experiences immediately prior to entry, in the program, and while student teaching, synthesizing, and teaching provide stories which illuminate the embedded traditions of each and the coherence among goals, beliefs, and actions. In this brief report, I include greater depth into Monica's story, and a glimpse into the stories of Cheryl and Alice.

Monica's Story

Monica was the middle child of a small rural farming community in the south. She planned to return to her home community as a teacher and active member of her church community. Immediately prior to entry in the program, she received a full scholarship from the state and the top mathematics award from her school. Both were based upon outstanding performance in her studies.

Monica described her ultimate goal as being a respected member of her home community. She described how she wanted her students to walk away from the school at the end of a year and look back to the school and comment to each other, "She really made a difference in my life." She offered no explanation of how she might come to make such a difference. She often mentioned how she strove to be a good student and reveled in being known as "the A student." She consistently described a good student as one who follows the directives of the teacher and models the teacher's behavior, even so far as to "know the mathematics just like the professor does."

Upon entry into the teacher education program, she continued her excellent performance in her concurrent mathematics classes. Because of her religious views she stayed away from social activities that involved alcohol and instead spent most evenings studying mathematics. She practiced being a good role model for others. She had not contemplated what the teacher education program would entail. She responded to the unfamiliar discussion and activity-oriented environment in her special way. She remained silent unless called upon by a professor. She seldom entered into discussion even during small group investigations. Instead she worked independently and shared her results and solution strategy only when asked to do so. She preferred highly structured directives and she followed them precisely. Ideas presented in her written and oral responses mirrored the ideas her professors had shared or the literature she had read. The multiplistic ideas presented in her teacher education classes were interpreted through her personal lenses. For example, when asked to use an alternative method to solve a system of equations, she obeyed and then explained that it was good to know alternative methods in case a student inquired, but then added that she preferred to use the correct method.

During Monica's earliest field experiences, she focused on the positive teacher-student relationship that enabled the teacher to help students individually and encouraged the respect of students. In subsequent experiences, she was overwhelmed by students' disrespect for teachers and their lack of interest in learning. In a team teaching field experience she chose to study her segment on reflections and plan the day-2 lesson independently. After the activity-filled first day introduction to transformations, her lecture on finding the coordinates of reflective images confused the high school geometry students as well as her team members. Ignorant of what had gone wrong she opted to take an observer role during the remainder of the two weeks.

Monica entered student teaching apprehensively, was unaccustomed to the non-directive approach of her cooperating teacher and still amazed by the disrespectful nature of the students. Monica associated disrespectful behavior with poor family backgrounds and former life experiences, rather than relating it to the present environment or the teacher-student relationship. Monica chose to focus on acquiring record-keeping skills and on tutoring students who cared enough to come for help after school. When university-related observation was scheduled, she attempted to engage the students in a reform activity. Post-observation interviews revealed that she lacked an understanding of the purpose of the reform activities she introduced. In one case for instance, an activity she found in the margins of her textbook had great potential for promoting mathematical communication, if only she had facilitated student discussion instead of quietly sitting at her desk. She was disappointed that she had not learned how to teach these unmotivated students during her experience and panicked after learning that she would be teaching a number of general mathematics and pre-algebra classes during her first year as teacher in her home community.

During the following synthesis class, Monica was determined to gain knowledge of how to teach these lower level students. The absence of a professor as an authority figure and her growing knowledge of classmates' rich student teaching experiences motivated her to turn to classmates for expert advice. She seemed transformed. She chose readings pertaining to motivation, joined her classmates in critiquing various methods, and shared her personal analyses openly. She formed friendly relationships with several classmates and socialized with them. She completed two full weeks of plans for each of her three classes. She felt well prepared for teaching and was excited to begin. Her renewed confidence was bolstered by departmental recognition as the program graduate with the most potential.

Monica struggled during her first year of teaching. During the first week of teaching, Monica introduced angles by using various golf clubs to demonstrate the angle that determined the flight of a golf ball. She became flustered when students responded with questions about the angle's effect on the hook of the ball. In their excitement students had scurried to her desk to see the golf clubs. The noise level rose significantly and Monica grew anxious and directed students to their desks and instructed them to focus on the pictures of angles in their textbooks and complete the exercises in the book. She needed to be in control. She tried to facilitate another planned activity the next week but again grew anxious and abruptly stopped the activity. She reverted to teaching in the way she was taught, namely teacher lecture followed by student seatwork. She rationalized that this conventional method had worked for ages. When students begged for more activities, she told them that they had not shown proper appreciation for all the time and effort that she had put forth to design the activities. She insisted upon their respect first but she was never satisfied. Worksheets and textbook exercises filled the remainder of the block schedule's semester long course and the students provoked her constantly. Teachers who recognized her struggle offered assistance but Monica refused their help. The majority of the students failed the course but a fellow teacher assured Monica that her failure rate was typical. Monica seemed unconcerned. Since teaching had drained her energy and enthusiasm, she finally accepted the advice of her fiancé, also a first year teacher, and decided to put a strict behavior policy into effect the second semester but student behavior remained the same. Only Monica's second semester calculus class of 5 male students provided her with a sense of satisfaction as a teacher during her first year of teaching.

Obviously, this overview provides only a glimpse into the process through which Monica was becoming a teacher but it highlights how her goal for gaining respect through being the good student did not transfer into a reform-minded notion of being a good teacher. Only in the absence of an authority figure had she opened herself to the support of colleagues and dare to share her unique voice. She may not have realized that through her actions in the synthesis class, she had gained respect and that she would continue to be respected if she accepted assistance from others. She never understood becoming a teacher as a process and instead expected to know all she should know to be a teacher during her first year. She reverted to isolation from the mathematics education community only two weeks into her teaching career. Monica's goal was easy to identify because her mention of being respected, especially by students, permeated her dialogue from the first interview through the last. She only initiated discussion of mathematics once during the two-year study and seldom mentioned key aspects of teaching and learning. Her orientation toward received knowing transferred to her expectations of her students. Through her lens of embedded tradition she interpreted the teacher education focus on empowering students by creating opportunities for higher order thinking quite differently than intended. Her focus on responses that reiterated what an authority had told her or what she had told her students was connected to her notion of respect. Classroom observations provided evidence that she connected extraneous comments or questions with disrespect. Unfortunately the study's interpretive mode of inquiry allowed Monica to avoid discussions of mathematics, learning, assessment, and teaching methods. When I asked questions about these topics within the context of her teaching, she responded by diverting the discussion to her students' lack of respect or to comments from other teachers about these students and their unfortunate circumstances. This diversion was consistent throughout the study as was her self-centered orientation.

Cheryl's Story

Cheryl grew up in a small community, and as a teenager she began to question the blind faith of her religious family. She was intrigued by the nature of communities and their influence. Her childhood desire to become a teacher transformed into the desire to not only teach in a classroom but also promote quality education for *all*. She entered college immediately after high school, but her journey toward certification continued for 17 years. She married after her first year of college study and became a mother two years and again four years afterward. She responded to the annual or biannual transfers of her husband by becoming active in her respective communities, especially school communities. She continued part time study until she became a full time student and entered this study. Although her college mathematics performance was average, she enjoyed the structure within and the mystique surrounding mathematics. She deemed skill mastery, inquiry, and applications as fundamental components of all mathematics curricula. Her work experiences as a teacher's aid in elementary and special education classrooms and her children's public school experiences enhanced her view of teaching and education.

Upon entry into the program, Cheryl's stated goal was to prepare for a career in teaching. She expected to share what she had learned through experience to better prepare herself and others for teaching. She expected to learn how to work with and within a system as well as a classroom, and she believed that her study of mathematics was complete. Disappointment reigned as her years of mathematics study proved to be insufficient preparation for the mathematics investigations that were integrated into the first teacher education class. Her enthusiasm plummeted during the 5-week study of functions, yet she continued to interject related stories. She felt alienated from her professor and classmates when their interest in her stories waned, but she continued to share her perspectives and the experiential evidence that supported them. Her classmates referred to her as the voice of experience.

Although Cheryl embraced the vision of NCTM standards, a healthy skepticism accompanied her study of reform-oriented teaching methods. She felt confident in her ability to teach individuals and small groups of students and was anxious to learn how to facilitate learning for large groups of students. She showed a great interest in moving beyond the design of activities and a desire to explore elements of classroom and school environments that support or deny access to learning for all. Her early field experiences provided her with opportunities for personal exploration but constraining factors overwhelmed her as she tried to satisfy the expectations of others. She felt constrained by a system that failed to provide the space, technology, and support necessary for quality teaching.

She set clear goals for student teaching. She planned to provide the students with the best instruction she could while she learned to work within the system and possibly convince her cooperating teacher of the benefits of technology. Although she experienced tension working within the system throughout her student teaching experience, she believed that she expanded her teaching expertise to teaching large groups of students. Fieldnotes verify her effectiveness as she facilitated probability experiments and exploration of graphical representations.

Experiences during her first year of teaching were varied. She was highly respected by teachers in her department and worked with them for the professional growth of all. Her goal was to seek a balance of school and home life and begin to affect positive change in education in general. Her school system awarded her time and funding to pursue her interest by attending conferences and workshops in the Spring of her first year of teaching. She also experienced obstacles that threatened her commitment to teaching. A student threatened her with bodily harm and although the student was suspended from school, the administration would not remove the student from her class and the return of the student caused her profound anxiety. The year ended on a sad note as well. An administrator asked her to change a grade of a student in response to the parents' request. She had entered into contracts with the student several times during the school year and the student had not honored the contract. After many meetings and with an unclear conscience, she reluctantly acquiesced. Although individual learning was of great importance to her throughout the study, Cheryl most often initiated discussion about how to interpret the actions of, manage, and teach groups of students. She also initiated discussion of the inner working of school systems. She enjoyed coming to understand the dynamics of classrooms and school communities. She did focus on student learning and teaching but the politics of the classroom and the school were of major importance.

Alice's Story

Like many of her teacher education classmates, Alice was from a suburban community in which both parents worked. Unlike classmates who openly shared their Christian philosophies, she lacked religious commitment and was torn between the Jewish and Christian beliefs of her parents. While in high school, she experienced the divorce of her parents. In response, she sought the companionship of a new group of friends and although she did not abandon her commitment to learning in school, she encountered a drug and music culture that was not supportive of formal education. As she assisted her new boyfriend through difficult times, she gained a new perspective of education. She became determined to provide all students the opportunity for empowerment

Alice's goal upon entry to the program was to learn how to help students learn to think for themselves. She focused on mathematical thinking. Although the message promoted in the teacher education program emphasized individual sense making, she perceived the climate in her education class as counter to the message. She quietly studied students in her class who simply followed the path set out for them. She also focused attention on her overarching goal of promoting inquiry and used her creativity to design motivational activities that would encourage mathematical thinking. During early field experiences she facilitated mathematical inquiry and students responded quite enthusiastically. During her student teaching, she began to follow the mathematical tangents initiated by students and their multiple ways of sense making.

During Alice's first year of teaching in a rather affluent suburban school, she expanded upon her methods and became quite adept at co-creating a learning environment in which problem solving and sharing of multiple strategies were the norm. Unfortunately parents again caused turmoil in her life. Monica had responded to a student's question about an alternative solution method with, "I don't know if this strategy will work. Let me study it tonight and we'll discuss it tomorrow." I had recorded a few student expressions of "gotcha" in my field notes and was alerted to a possible underlying attempt to discredit the first year teacher. Indeed, the parents contacted the head of the mathematics department and stated their displeasure with having their children in a classroom with a first year teacher who obviously did not know her mathematics.

Fortunately Alice had already gained the respect of her peers and department head for her in-depth knowledge of mathematics. She had asked her fellow teachers about how they introduced functions and after learning that they simple stated the definition, she shared a few of the ides she had learned in her teacher education class. The head of the department subsequently asked her to conduct a more in-depth workshop for all the teachers. Nonetheless, the lack of parental support for her passionate attempts to encourage students to become mathematical inquirers was disheartening for her. With the support of her colleagues, she became more resolute in her goal of teaching students to think for themselves. During most research interviews, she initiated mathematical problem solving activities. She discussed the different student approaches to the problems only after I had an opportunity to engage fully in the mathematics. She was particularly enamored with questions students brought to her. Her favorite was determining the appropriate size and placement of audio speakers in the trunk of a car. Her student-centered and inquiry-directed orientation dominated her experiences throughout the study.

Synthesis and Implications

The embedded goals of three preservice teaches led them to attend to different aspects of their teacher education program and use their embedded traditions as constraints or springboards. Monica was able to work outside her embedded traditions for a short time but her embedded tradition simple could not support activity-oriented teaching in the classroom. Performance goals and isolationist tendencies (Cooney et al., 1998) constrained her professional growth as a mathematics teacher. On the other hand, Cheryl and Alice's mastery goal orientations and commitment to the life-long nature of the process of becoming a teacher more often served as springboards from embedded tradition despite obstacles. The goals of learning the politics of schooling and promoting mathematical inquiry resembled the fluid and interconnected goals of expert teachers whereas the fragmentation associated with novice lessons was more aligned with the goal of respect that lacked a pathway toward it.

This research informs research and teacher education practices in many ways. The inconsistencies reported in the research may not be inconsistencies at all. We, as teacher educators, need to practice what we preach and connect to student interests that are much more profound that simply becoming a mathematics teacher. A program that recognizes goals within embedded tradition might have offered Cheryl and Alice greater opportunity for professional growth and offered Monica a pathway to use her embedded tradition as a springboard. Goal setting activities might not only inform instructors but also assist teachers in building pathways toward their goals. My current research findings support this conjecture as well as the empowering nature of action research. Novice teacher need not spend their first year in survival mode. They have much to offer their students, colleagues, and themselves if they learn how to springboard from their traditions to grow professionally.

Note

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Appendix A

Embedded Goals	Goal Orientations	Primary Foci	Orientations toward Authority	Mathematics Instruction	Teacher Image/ Role Model	Responses To Lack of Movement toward Goal
Respect	Performance/ Product	Behavior Management Classroom Management	Self as authority Received knowing Dualism Narrow lens	Textbook driven Dissemination Reward correctness	Teacher as totally in charge	Distance self Busy work Community service
Institutional Change	Blend of Performance/ Product & Mastery/ Process	Curriculum and policy Group dynamics Planning Creativity Flexibility	Institution as authority Self as learner/ facilitator/leader Subjective & procedural knowing Wide lens	Curriculum driven Interactive Lecture Reward effort	Teacher as part of working group of teachers and school community	Share attempts Professional educational programs
Learning and Inquiry	Mastery/ Process	Instruction Student inquiry and multiplistic thinking	Individual as authority Self as learner/catalyst/ leader Constructivist knowing Medium lens	Concept driven Joint inquiry Reward novel Negotiate meaning Explore	Teacher as partner in life-long learning of students and colleagues	Share concerns Seek advice Assume leadership

Relationships between Overarching Goals and Beliefs, Orientations toward Knowing, and Practices

IMPLEMENTING STANDARDS-BASED MATHEMATICS CURRICULA IN THE MIDDLE GRADES

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A series of case studies was undertaken to better understand the process of implementing Standards-based middle school mathematics curricula. Based on data collected from 4 school districts over 3 years, the findings of this study indicate that there are identifiable factors that can support curriculum implementation or, in their absence, can impede the implementation process. Five broad factors were identified across the 4 sites: (a) a plan for implementation of the materials; (b) district- and building-level leadership and support; (c) ongoing professional development aimed at the changing needs of teachers; (d) teachers' philosophical agreement or compatibility with the curriculum; and (e) curriculum alignment with state standards and tests. At the end of 3 years, we found stable implementation in only one district; in each of the other three districts conditions threatened the success of the implementation.

In response to the need for curriculum materials that reflect and support the philosophy and pedagogy of the NCTM Curriculum and Evaluation Standards (1989), the National Science Foundation funded a series of curriculum development projects that subsequently published what have come to be known as Standards-based curricula. At the middle school level, five core curriculum programs are currently on the market: *Connected Mathematics Project*, *Mathematics in Context*, *MathScape*, *MATH Thematics*, and *Pathways to Algebra and Geometry*. These programs have become recognized as legitimate alternatives to traditional mathematics textbooks and are being adopted in school districts across the country.

There are vast differences in the content, and to an even greater extent the pedagogy reflected in Standards-based materials, and that of traditional mathematics textbooks. According to St. John, Heenan, Houghton, and Tambe (2001), these innovative curricula "demand new modes of instruction, require more expertise, resources, time, and effort to implement"(p. 3). Such demands on instructional practice can be overwhelming to teachers and change should be cultivated and supported through professional development (Manouchehri & Goodman, 2000). The research literature has begun to identify factors that contribute to the successful implementation of Standards-based curricula, as well as factors that present obstacles to implementation. Factors that support the implementation process include: the establishment of collaborative relationships between school districts, institutes of higher education, and business and scientific communities (Bay, 1999; Batchelder, 1998; St. John & Pratt, 1997); strong alignment between the curriculum and state policies (Batchelder, 1998; St. John & Pratt); and district allocation of resources (Batchelder, 1998; St. John &

Pratt, 1997). Obstacles to implementation are often created by a teachers' superficial understanding of mathematics reform (Batchelder, 1998; Haug, 2000), negative student reaction and parental perception (Bay, 1999; Reys, Reys, Barnes, Beem, & Papick, 1998), and emphasis on high stakes state and district assessments (Batchelder, 1998; Bay, 1999; Haug, 2000).

Our study builds upon and extends this research by systematically examining the process of district-wide implementation of Standards-based middle school mathematics curricula. More specifically, three research questions were investigated: (a) To what extent are the curricula being implemented? (b) What are the supports and barriers to implementing the curricula? and (c) What is the impact of implementing the curricula on students and teachers?

Method

When this study was initiated in the fall of 1998, only four of the middle school curricula were published commercially: *Connected Mathematics Project*, *Mathematics in Context*, *MathScape*, and *MATH Thematics*. Although *Pathways to Algebra and Geometry* was later published and included in our study, this paper reports only on the findings related to the other four curriculum projects.

Case Study Sites

Four school districts, each in the process of implementing one of the NSF-funded curriculum programs, were identified for case study analysis. In selecting case study districts, we attempted to include sites from different parts of the country and to represent a variety of demographics. Only sites that reported district-wide adoption and implementation were selected for participation in the study. For each curriculum project, one case study site was selected from a list of school districts provided by the curriculum developer. A brief description of each site is presented below.

Arnold Community Schools.

Arnold is a predominately middle-class Midwestern community of about 60,000 people. The population is about 85% white, 14% African American, and 1% other ethnic groups. The school district serves approximately 11,000 students in 22 schools: 16 elementary, 3 middle, and 3 high schools. Until recently, the economy in Arnold was dominated by industry and accommodated an unskilled or semi-skilled workforce. However, the economy has dramatically changed and the need for unskilled laborers has diminished. The community has realized the need to prepare students for an information economy and has focused on revitalizing and improving the educational system. The school district began full implementation of *MathThematics* during the 1998-99 academic year in grades 6 through 8, with the exception of the 8th-grade Algebra 1 class. Standards-based curricula were not adopted at the elementary or secondary levels.

Harrison County Public Schools.

Harrison County surrounds a large city located on the east coast. Its communities include affluent suburbs in the western part of the county to rural areas in the east. There are also urban neighborhoods closer to the city limits, in the center of the county. The school district serves 40,000 students in 39 elementary, 9 middle, and 8 high schools. The student population is 62% white, 33% African American, 3% Asian, 1% Hispanic, with the remainder of students from other ethnic backgrounds. During the 1998-99 academic year, the *Mathematics in Context* curriculum was being implemented in all middle schools except the alternative school. We came to learn, however, that use of the materials was optional in all but the advanced learner classes. Hence, rather than a full adoption and implementation, *Mathematics in Context* was one of two adopted texts; the other being a traditional middle school series. Standards-based curricula were not being used at either the elementary or secondary levels.

Montgomery Public Schools.

Montgomery is a large urban city in the Midwest. The school district serves about 102,000 students in 114 elementary schools (10 of which are K-8 buildings), 22 middle schools, and 18 high schools. The student population is 61% African American, 20% white, 13% Hispanic, 4% Asian, and 2% other ethnic groups. At the middle school level, 76% of the student qualify for free lunch. As participants in an urban systemic initiative, the district was encouraged to adopt a Standards-based mathematics curriculum. During the 1998-99 school year, the district undertook full implementation of the *Connected Mathematics Project* curriculum in grades 6 – 8 and the Standards-based *Investigations in Number, Data, and Space* in the elementary grades.

Valley Springs Unified Schools.

Valley Springs is a bedroom community for a major metropolitan area on the west coast and is also a small city in its own right with a population of 120,000. The socio-economic status of the community ranges from urban poor to suburban middle class. The school district serves 20,000 students in 17 elementary, 4 middle, and 3 high schools. The school population is 25% African American, 20% Filipino, 20% white, 16% Hispanic and 6% Asian and Pacific Islander. Valley Springs began full implementation of *MathScape* during the 1998-99 academic year in all classes in grades 6 through 8 except the 8^{th}-grade Algebra 1 class. The impetus for reform in the middle grades came from the high schools' use of an integrated reform curriculum and piloting one of the Standards-based secondary programs. Also, a Standards-based elementary program, *Trail Blazers*, was adopted at the elementary level.

Procedure

Beginning in the academic year 1998-99 and continuing through May 2002, a series of site visits were conducted in each of the case study districts. This paper will report on the first three years of that study. During Year 1 of the study, three of the sites were visited on two occasions, while the fourth site received only one visit. In Year 2, all sites were visited once. In Year 3, one site was dropped from the study and the remaining three sites were visited once. Because the purpose of this study was to examine the implementation process within a school district, a different school was targeted for each site visit. Typically, 1.5 days were required to complete a site visit.

During each site visit, interviews were conducted with classroom mathematics teachers, the principal, the lead teacher or building-level math coordinator, and district personnel. The district contact person in each site was interviewed during each visit, thus providing a stable perspective over the three years of the study. Approximately 70 teachers, 11 principals, and 15 different building-level and district personnel were interviewed.

Most interviews were conducted individually, but some were conducted in small groups, usually as grade-level teams. An interview protocol was developed to address the following issues: (a) the extent to which the curriculum materials were being used at each grade level; (b) perceived barriers and supporting structures for implementation as related to the district, principal, teachers, and parents; and (c) the impact of the curriculum on student's achievement, students' disposition, teachers' knowledge, and classroom practices. These issues remained the focus of each interview throughout the three years of the study. All interviews were conducted according to the protocol, but specific follow-up questions were tailored to probe or clarify a particular response.

Most interviews were audio taped, although there were situations where the physical setting was not conducive to taping or teachers were uncomfortable being taped. For all but a few interviews, two researchers were present and they both took written notes throughout the interview. As possible, pertinent artifacts such as locally generated pacing guides, schedules of professional development sessions, and summaries of student achievement data were collected.

Data Analysis

Data analysis was an on-going process that occurred in two phases. In both phases, a data reduction approach (Miles & Huberman, 1994) was used to discern patterns or trends in the data. Given the self-report nature of the data, interviews with principals and other building-level and district personnel were used to triangulate the teachers' perceptions of the implementation process. During the first phase of analysis, interview data from each site visit were analyzed in terms of the three research foci: extent of implementation, supports and barriers to implementation, and impact of implementation on students and teachers. Following each site visit, one researcher reviewed her written notes and the interview tapes to organize and synthesize the data around the three research foci. Data tables and narrative summaries were constructed to present the findings. These tables and summaries were then examined by one of the other researchers, using her notes (and if necessary, further review of the tapes) as a check. Any discrepancies in interpretation were discussed and resolved. This process was followed for each of the site visits and resulted in the identification of *within-case* trends pertaining to the implementation process as perceived and enacted at each school.

In the second phase of analysis, trends *across* the four case study sites were examined. For each year of site visits the data tables and narrative summaries for each site were analyzed and patterns across the four sites were identified in terms of the three research foci. Cross-case summaries were developed to characterize the process of implementing Standards-based curricula at each point of implementation (i.e., year 1, year 2, year 3). These yearly cross-case summaries were also analyzed to discern generalities across the three years of implementation.

Findings

This section is organized according to the three research foci. General findings pertaining to the extent of implementation, support structures and barriers to implementation, and the impact on students and teachers are presented across the four case study sites.

Extent of Implementation

Given our criteria for selecting sites that had initiated district-wide adoption and implementation, it is not surprising that the majority of teachers we interviewed reported using their respective curriculum materials as the basis of their instruction. That is, these teachers reported using the materials at least 40% of the time, and most teachers indicated major use at more than 60% of the time. The fidelity of implementation, however, varied greatly within districts by schools, teachers, and grade levels.

The most common finding across sites was that almost all teachers, even those who reported using the materials "fully," supplemented the curriculum with extra review and practice activities. For some teachers the supplementing was extensive, for others it was only minimal. Many teachers commented that students entered their classrooms without the prerequisite skills and concepts assumed by the curriculum. These teachers believed it was necessary to develop students' skills prior to using the curriculum materials and thus drew upon activities they had used previously or found in other resources. Some teachers did not believe the curriculum materials devoted enough time to allow students to fully understand a particular topic (e.g., fractions, percents, integers) and supplemented units or modules with extra activities and worksheets. Similarly, teachers reported that the curriculum materials did not provide enough practice for a particular skill or concept within a lesson. At all sites, teachers provided extra practice and even specific instruction beyond the scope of the curriculum materials in preparation for state and standardized tests.

In all districts, there were some teachers who failed to implement the adopted curriculum or used it only rarely. Often, these were teachers who were new to the district and had not received professional development related to the curriculum or who were long-term substitutes. This was especially evident in Valley Springs where there was a shortage of certified teachers, particularly at grade 6. In Arnold, the dissenters were typically grade 8 teachers who did not believe the materials were appropriate for their students. According to one teacher, "These students need fundamental drill work." In Montgomery, the implementation was weakest in the two magnet schools where teachers openly refused to use the materials because they believed their gifted students needed a "more challenging" curriculum.

By the third year of implementation there was evidence that the Standards-based curricula in three of these school districts were struggling for survival. In Arnold, there was only one 8th-grade teacher fully implementing the curriculum. Most of the other grade 8 teachers openly opposed to the reform philosophy and actively pursued a skills development curriculum using worksheets and other materials. This was beginning to create a negative influence on teachers at the other grade levels. The Harrison district was dropped from the study in Year 3 because we learned that the materials were rarely being used, even in the advanced learner classes, and that the district no longer planned to extend the use of the materials to non-advanced classes. Although the district personnel in Valley Springs perceived the implementation as going well, there was concern about the fate of the curriculum because of a state mandate that only approved textbooks could be purchased and *MathScape* was not on the state textbook adoption list. Montgomery was the only district in which the curriculum implementation appeared to be relatively stable. The district mathematics coordinator reported that about 90-95% of the teachers were actively using the curriculum, albeit at different levels of implementation.

Supportive Structures and Barriers to Implementation

A variety of factors that either supported implementation, or in their absence created barriers, were identified for each case study site. Five broad factors were identified across the four sites: (a) a plan for implementation of the materials; (b) ongoing professional development aimed at the changing needs of teachers; (c) district- and building-level leadership and support; (d) teachers' philosophical agreement or compatibility with the curriculum; and (e) curriculum alignment with state standards and tests. These factors, some of which were mentioned above with regard to the extent of implementation, are discussed individually although many of the issues are closely related.

Implementation plan.

To facilitate the implementation process, two districts developed plans for gradually implementing the curriculum at each grade level and one district reorganized the sequence of topics to better match the state standards. In Montgomery, core units for each grade level were identified and a sequencing and pacing guide was developed to facilitate the phase in of the curriculum over three years. In Year 1 the 6^{th}-grade classes used the grade 6 materials, the 7^{th}-grade classes used some of both grade 6 and 7 materials, and 8^{th}-grade used the grade 7 materials. In Year 2 the 7^{th}-grade classes used the grade 7 materials while the 8^{th}-grade used some modules from both of grades 7 and 8. In Year 1 the goal at each grade level was to complete 5 units, and by Year 3 it was expected that teachers would complete all of the core units targeted for their grade level. In Valley Springs, a customized sequencing and pacing plan was developed for Year 1 only, on the expectation that teachers would be unable to complete all the units in their first year using the materials. Specific units and lessons at each grade level that closely matched the district's mathematics framework were identified for implementation in Year 1. It was expected that in subsequent years the teachers would increase the number of units covered to include the entire curriculum. Although Harrison County did not develop a plan for the gradual implementation of the curriculum within each grade level, they did reorganize the order of topics within the materials to more closely match the scope and sequence of the state standards.

Almost all of the teachers we interviewed, including those in districts with specified sequencing and pacing plans, reported that they had difficulty completing all of the curriculum modules or units expected by the district. Moreover, this concern continued to be voiced in Year 3, after many of the teachers had been using the materials for two years. In each district few teachers, if any, were able to address all of the required curriculum modules or units. However, teachers in districts with sequencing and pacing plans had an idea of which units could be skipped and which ones should be emphasized. In districts without such guides, individual teachers addressed the materials quite differently, picking and choosing units or lessons within units. In these cases the coherence of the curriculum within and across grade levels was potentially compromised.

Professional development.

All of the teachers we interviewed indicated that professional development was needed beyond the first year of implementation and that it should address the changing needs of teachers, as they became more familiar with the materials. Teachers reported that the most valuable professional development activities were ones conducted by facilitators who had actually taught the materials and who could "walk through" a module and highlight specific lessons, investigations, and problems that should be emphasized and ones that could be skipped.

Teachers also believed it was important for all of their mathematics colleagues to be able to attend workshops. In some schools, principals sent only lead teachers or one teacher per grade level to workshops offered during the school day. The expectation that these teachers would bring information back to share with the rest of the faculty was rarely fulfilled and teachers did not find this approach acceptable. In three of the four districts, teachers expressed concern that attendance at professional development sessions was voluntary rather than required. Hence, the perception was that teachers who most needed professional development did not take advantage of the opportunities provided. Some teachers, as well as district personnel, felt this had a negative impact on the implementation.

Leadership and support.

The importance of strong administrative leadership and support were considered important factors contributing to the success of the implementation. At the district level, the implementation process was usually spearheaded and managed by key district personnel. These were also the people who coordinated, and often conducted, professional development activities. Principals also provided support in a variety of important ways. They provided the resources and materials needed to implement the curriculum, opportunities for common grade-level planning or department meetings, and release time for teachers to engage in professional development activities.

Philosophical agreement.

The teachers in this study who never, or only rarely, used the Standards-based curricula espoused traditional beliefs about the teaching and learning of mathematics. Essentially, their beliefs about what it means to know and learn mathematics did not match the philosophical perspective of the curricula. The most common of these beliefs was that students should be taught specific concepts and skills prior to engaging in investigations. For these teachers, the mastery of computational procedures was seen as a prerequisite to problem solving. Thus, in their view, the curriculum materials did not provide enough examples and practice problems for students to become proficient in "the basics."

Alignment with state standards and tests.

In each district, a significant factor in choosing to adopt a Standards-based curriculum was the belief that it was closely aligned with state and national standards. More particularly, all of these districts were involved in some aspect of mathematics reform through participation in national or state projects (Arnold and Montgomery) or through the initiatives of the district (Harrison and Valley Springs). District personnel believed that reform efforts would be supported through the use of a Standards-based curriculum.

Over the course of this study, state standards or testing procedures changed in three of the four districts to the extent that the success of the curriculum implementation was threatened. In Valley Springs, as mentioned above, the *MathScape* curriculum was excluded from the state textbook adoption list. There was also a move by the state to require all 8th-grade students to take Algebra 1. This mandate prompted the district to consider merging the three-year mathematics program into two. In Harrison County, performance on state assessments and proficiency in meeting the state standards overshadowed the focus on mathematics reform. This was prompted by the fact that the district led the state in percent of schools meeting state standards. The Superintendent was anxious to maintain that status, thus, extraordinary emphasis was placed on maintaining and increasing test scores. In Arnold, the state standards were significantly revised to increase the number of concepts and skills required at each grade level. For many topics, the organization and sequencing of the curriculum materials no longer matched the standards. Montgomery was the only site where state and district requirements continued, over the course of this study, to be supported through use of the curriculum materials. In fact, district proficiency tests for grades 6 through 8 remained very closely aligned with the *Connected Mathematics Project* curriculum. In many cases assessment items were directly modeled after problems or activities in the curriculum.

Impact on Teachers and Students

Students' disposition toward mathematics.

Students' reactions to these curricula were mixed. For many, their first experiences were frustrating because of the amount and level of reading required. Also, many students were not accustomed to having to explain their thinking and justify their responses. Nevertheless, teachers reported that most students' attitudes improved as they became more familiar with the materials and knew what to expect. Teachers also reported that students enjoyed the hands-on explorations and discovery activities. Some students showed a greater appreciation for mathematics and expressed more interest in the subject. Teachers also noted an increase in students' abilities to think mathematically to solve problems.

Students' achievement.

Although most teachers and district personnel indicated that they expected an increase in achievement scores following the implementation, we were unable to document any significant overall growth in these districts. In the third year of implementation, state test scores in Arnold had declined district-wide. In Valley Springs, standardized test scores had remained relatively flat at all grade levels over the three years of this study. In Montgomery the percentage of students meeting grade-level proficiency requirements increased steadily over the three years, but state test scores had not changed significantly overall.

Teachers' knowledge and classroom practice.

Many of the teachers we interviewed reported that using these curriculum materials had prompted them to think differently about mathematics and sometimes deepened their understanding at a conceptual level. Most teachers, however, reported that the materials had impacted their practice more than their knowledge of mathematics.

For some, the materials complemented and supported the pedagogical approaches they already used. Others recognized changes in their pedagogy such as incorporating more group work, use of manipulatives and calculators, asking for student explanations and justifications, having students share solutions at the overhead, and using journal writing.

Discussion

The findings of this study support claims in the literature regarding the difficulty of implementing a Standards-based curriculum. In examining the challenges encountered by the teachers in our study, we have identified critical support structures and potential barriers that are pertinent to the implementation of any of the middle school curricula. In essence, teachers need ongoing technical assistance as well as intellectual and emotional support in order to implement these materials successfully. We also found that over time students adapt to the pedagogy of these curricula, and many develop a more positive disposition toward mathematics.

One conclusion that can be drawn from our work is that ultimately, the most significant factor in determining the long-term success of these Standards-based mathematics curricula could be their impact on student achievement in terms of state standards and assessments. The implementation process in each of the districts in this study was influenced in some way by such measures. In three of the districts specific conditions that threatened the success of the implementation were directly connected to new state standards and issues related to student achievement. Even in the one district where the implementation did not appear to be threatened, district personnel were concerned that there had not been greater improvement in state test scores.

There is some evidence in the research literature (Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1997; Hoover, Zawojewski & Ridgway, 1997; Tetley & DuBose, 1997) that suggests that students using these Standards-based curricula perform as well as students using traditional curricula on traditional achievement measures and even better on measures that account for problem solving and higher-level thinking. Unfortunately, overall achievement gains could not be identified in any of the districts in our study. One reason is that it is difficult to gauge the impact of a curriculum when the faithfulness of implementation varies so greatly from teacher to teacher. The achievement of students from classes with strong implementation could be averaged out by the achievement of students from classes where the implementation is weak. The uneven implementation of curriculum materials highlights how important it is to analyze achievement data, not only by grade level, but also by individual students and teachers. Across a district, student achievement data should be disaggregated by the number of years in high implementation classrooms. This type of analysis could better document the impact of a Standards-based curriculum on student learning. Although Briars and Resnick (2000) have presented a good model for this type of research, access to the required data is not easily obtained. Yet this is exactly the kind of data school districts need to garner support for these curricula. Over the three years of this study we were unable to obtain the kind of data to produce such an analysis. This is an area of research we will continue to explore and one that should be given top priority by other researchers as well.

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ENTAILMENTS OF THE PROFESSED-ATTRIBUTED DICHOTOMY FOR RESEARCH ON TEACHERS' BELIEFS AND PRACTICES

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In research on teachers' beliefs, a distinction is often made between what teachers state ("professed beliefs") and what is reflected in teachers' practices ("attributed beliefs"). Researchers claim to have found both consistencies and inconsistencies between professed and attributed beliefs. In this paper, I examine methods and research designs typically used in studies of teachers' beliefs. I assert that, in some cases, the perceived discrepancy between professed and attributed beliefs may actually be an artifact of the methods used to collect and analyze relevant data. In particular, I contend that the apparent dichotomy can be the result of a lack of *shared understanding* between teachers and researchers of the meaning of terms used to describe beliefs and practices. Traditional and novel methods are described, a data example is provided to illustrate my claims, and implications for future research are discussed.

Introduction

Theories, methods, and findings in research are inextricably intertwined. A researcher's theoretical perspective influences and shapes methods used to conduct research. A researcher's methods encapsulate and embody theories. Methods, however, only allow researchers to "see" what those methods provide access to (Schoenfeld, 1992). Theoretical perspectives may also constrain what researchers see in phenomena and what they seek to see, hence shaping findings and developments in theory. These are inherent characteristics of the research enterprise. It seems productive, however, to examine theory, findings, and methods collectively–especially in a developing field such as mathematics education research–to see whether one or more may be potentially shaping how phenomena are examined.

In this paper I examine the use of the dichotomy of professed versus attributed beliefs as it is found in research on teacher's beliefs and practices. Beliefs about mathematics, teaching, learning, and students are often described as either "professed" or "attributed." Professed beliefs are defined as those stated by participants and attributed beliefs are those that researchers infer based on observational or other data. I assert that in some cases differences between professed and attributed beliefs may in fact be the result of a methodological artifact and not an accurate reflection of the nature of the phenomena researchers seek to understand. The use of this construct has influenced the methods used in studies of beliefs and practices and has also shaped the nature of research findings and resulting theories.

This paper begins with a very brief survey of literature on teachers' beliefs and practices. I then describe methods typically used in this area of research (questionnaires/interviews and observations, respectively). In the third section, I describe the professed versus attributed dichotomy and discuss how, in some cases, perceived inconsistencies may be the result of a lack of *shared understanding* among researchers and teachers about what descriptive terms mean. In the subsequent section, I present alternative data collection methods, based in part on "videoclubs," designed to address this issue. In the fifth section, I present a data example to illustrate my points and then conclude with a general discussion and potential implications.

What Is Known About Teacher Beliefs And Practices

Overview

In the 1960s and early 1970s, research on teachers was strongly influenced by behaviorist views about teaching and learning. The objective of much of the work was to "describe teaching in terms of sequences of behavior, and then to investigate the relationship of that behavior to children's learning" (Calderhead, 1996, p. 709) The development of information processing theory and other aspects of cognitive science shifted studies of teaching away from behaviorist perspectives. The process-product paradigm that characterized the earlier behaviorist work was superseded by other approaches that focused on teachers' thinking and decision making (Thompson, 1992). More recent research focuses more extensively on cognition and contexts of learning. Instead of examining only teachers' behaviors, this recent work addresses teachers' perceptions, thinking, judgments, reflections, evaluations, and routines (Calderhead, 1996). This research agenda has developed to include investigations of the knowledge and beliefs behind teachers' practices.

From findings in teacher cognition and other sub-fields of educational research, we know there are many factors that influence the decisions and practices of teachers (Borko & Putnam, 1996; Clark & Peterson, 1986). A teacher's subject matter knowledge, pedagogical knowledge, and pedagogical content knowledge are all important influences on teaching practices–so too are the curriculum in use, the teacher's goals, and myriad other social and contextual factors. Research suggests, however, that beliefs are one of the significant forces affecting teaching (Calderhead, 1996; Richardson, 1996; Thompson, 1992).

Beliefs have been described as a "messy construct" with different interpretations and meanings (Nespor, 1987; Pajares, 1992). Current definitions of beliefs found in the mathematics education literature, however, focus primarily on how teachers think about the nature of mathematics, teaching, learning, and students. In this context, beliefs are defined as conceptions, personal ideologies, worldviews, and values that shape practice and orient knowledge (Ernest, 1989; Pajares, 1992; Thompson, 1984; Thompson, 1985; Thompson, 1992).

Influenced by their own experiences as students as well as more formal teacher preparation, teachers form beliefs about what it means to teach, how it is one learns, what the nature of math is, and what students are and should be like. Researchers contend that "beliefs are instrumental in defining tasks and selecting the cognitive tools with which to interpret, plan, and make decisions regarding such tasks; hence they play a critical role in defining behavior and organizing knowledge and information" (Pajares, 1992, p. 325). Beliefs appear to be, in essence, factors that shape decisions teachers make about what knowledge is relevant, what teaching routines are appropriate, what goals should be accomplished, and what the important features are of the social context of the classroom. Beliefs shape teachers' practices and hence the learning experiences of students. Teachers' beliefs are therefore vital determinants of the type of understanding of mathematics that students have the opportunity to acquire. As a consequence, examinations into the description, nature, and role of beliefs in teachers' practices are an important and rich arena for research.

Belief Categorization and Classification

Researchers have delineated categories of beliefs (Ernest, 1985; 1988; 1989; Kuhs & Ball, 1986; Lerman, 1990; Prawat, 1992; Skemp, 1978). Often-sited categories include beliefs about mathematics, teaching, learning, students, and related subcategories. For beliefs about mathematics, some researchers have appropriated categorization schemes from philosophy of mathematics. For example, Ernest (1989) proposed different views of mathematics: problem solving, Platonist, and instrumentalist. The problem-solving view takes mathematics to be a "continually expanding field of human inquiry," where as in the "Platonist" view mathematics is a unified, static body of knowledge that is discovered, not created. In the "instrumentalist" view, mathematics is a collection of useful unrelated facts, rules, and skills. Other categorizations are derived from different views of the nature of mathematical knowledge. Lerman (1990), for example, sorts beliefs into two categories: "absolutist" and "fallibilist" views of mathematics. Skemp (1978) says there is "relational understanding" of mathematics (knowing both what to do and why) and "instrumental understanding," which he describes as "rules without reasons."

Beliefs about teaching and learning have also been categorized. For example, Kuhs and Ball (1986) summarized four views of teaching and learning. One is described as "learner-focused," while another is characterized as "classroom-focused." Two others are closely tied to content, with one emphasizing conceptual understanding and the other centering on performance of skills.

Relationships Between Teacher's Beliefs and Practices

Once categorized by researchers, teacher's beliefs are sometimes compared with observations of practices. Research has documented that the beliefs teachers hold related to mathematics, teaching, students, and learning play out in interesting ways in their practices. In some cases, researchers found that professed beliefs were consistent with what was reflected in observations of classroom practice. For example, Thompson (1985) described a teacher named Kay who viewed mathematics as a "subject of ideas and mental processes rather than a subject of facts" (p. 288). She also viewed the study of mathematics as one of "discovery and verification of ideas." Kay's views about mathematics and teaching were said to be consistent with Thompson's observations:

> She frequently encouraged the students, in a rather persuasive tone, to guess, conjecture, and reason on their own, explaining to them the importance of these processes in the acquisition of mathematical knowledge. (p. 289)

Inconsistencies between beliefs and instructional practice have also been documented. Thompson (1984) described Lynn who believed mathematics instruction should encourage students to ask questions and participate actively in class discussions. Thompson observed, however, that Lynn's practice consisted primarily of lectures followed by routinized seatwork. These practices, which severely restricted student participation, interaction, and the opportunity to ask questions, were deemed inconsistent with Lynn's beliefs. Cohen (1990) provided a similar case analysis of a teacher named Ms. Oublier. Ms. Oublier believed she was implementing reform mathematics practices such as cooperative learning. Cohen, however, claimed that she had in fact maintained very traditional teaching practices that did not reflect the sense-making envisioned by the authors of the reforms.

There are many possible explanations for these perceived inconsistencies. It is certainly plausible for teachers to state beliefs and behave in a manner inconsistent with those beliefs. There are, however, other potential explanations tied to the methods and research designs used in this work that warrant consideration. In the following section, I provide a brief survey of methods typically used in this work to help explain how this interpretation may be an artifact of the methods used.

Typical Methods and Research Designs

Beliefs: Questionnaires, Interviews, and Observations

Much of the work on beliefs is based on self-reports from teachers. Through questionnaires and/or interviews, researchers build descriptions of beliefs expressed by teachers. The methods employed range from large-scale surveys to in-depth case studies. From this research, a great deal is known about the beliefs teachers typically state. Beliefs are often categorized based on schemes such as the ones described above. These categorization schemes are sometimes derived empirically from the data and other times predefined categorization schemes are used in the collection and analysis of the verbal data.

In some instances, beliefs teachers state are augmented with ones inferred from observations. Some researchers contend that for an accurate portrayal of beliefs, "investigations of teachers' mathematical beliefs should examine teachers' verbal data along with observational data of their instructional practice or mathematical behavior, it will not suffice to rely solely on verbal data" (Thompson, 1992, p. 135). Typically, researchers match their characterization of particular beliefs (e.g., mastery of skills is important) to behaviors observed that the researcher would consider consistent with holding that belief (e.g., devoting considerable class time to drill work on those skills). The methods used to attribute beliefs to teachers from observational data are not usually specified in much detail. In addition, rarely are definitions of descriptive terms presented—it is assumed that the teachers, researcher, *and the reader* understand the terms to mean the same things. Consider for a moment what saying "I believe group work is important" means to you. Would everyone share that meaning?

Practices: Observations and Teacher Self-Reports

Data on teaching practices typically come from two sources: observations and teacher self-reports. Classroom observations are sometimes documented on video or audiotape. Fieldnotes or more structured systems for recording observations are common among written data-gathering methods. In most cases, these studies provide little detail about the methods used to analyze the observational data, and descriptions are often portrayed as if they have universally agreed upon definitions.

In some cases, data about instructional practices come not from observations but instead from teacher self-reports in interviews or on questionnaires. In these instances, teachers are asked to describe what their classes are like or queried about their use of specific teaching practices. Again, specifics about the methods and definitions are not usually provided. What does "The teacher led a discussion about problem solving" mean to you? What might it mean to others?

Connecting Beliefs and Practices

Using data collected in the manner described above, some researchers examine the relationship between teachers' beliefs and their in-class teaching practices. Typically, researchers look for correlations among beliefs and observed or reported practices. As with some of the methods described above, how these analyses are conducted is often described only in very general terms.

Professed and Attributed Beliefs and *Shared Understanding*

As noted earlier, researchers have found both consistencies and inconsistencies between beliefs and practices. This has led to separate classifications of what teachers say (professed beliefs) and what is reflected in their practice or inferred from other data (attributed beliefs) (Calderhead, 1996; Thompson, 1984). These classifications have influenced the research designs and methods used in studies of teachers' beliefs and practices.

On the surface, using both professed and attributed beliefs seems like a very sensible way to ensure data are more representative of teachers' beliefs. It is reasonable to presume that one needs information in addition to what is stated on a questionnaire or in an interview to fully understand teachers' beliefs and practices, especially when teachers' actions appear to contradict what teachers claim to believe.

As noted above, primary sources for information about teachers' beliefs include questionnaires and interviews. On a questionnaire or in an interview, terms used by the researcher to describe beliefs and practices (for example, "problem solving," "cooperative learning," "sense-making," etc.) may not carry the same meaning for the teacher as they do for the researcher. When a teacher says she believes she should foster "sense-making" of mathematical ideas in her classroom, can researchers be certain they know what the teacher means? When researchers ask teachers what they believe the role of problem-solving should be in their classes, how certain can they be that they and the teachers have similar understandings of what problem solving is? These disconnects mean that the resulting data may not accurately represent the teacher's beliefs or practices. I refer to this potential disconnect as a lack of *shared understanding* between the participants and researcher about the meaning of terms used.

This issue of shared understanding is also problematic when researchers make attributions of belief based on observations of practice. A typical example begins when a teacher states a belief (for example, "Problem solving is important"). Next, the researcher observes a class where the teacher has students work a series of procedural exercises–a practice the researcher does not see as reflecting a belief in the importance of problem solving. The researcher concludes that there is an inconsistency between what the teacher claims to believe and what she must really believe. The problem lies again in the presumed shared understanding. What the teacher did in the class may fit *her* conception of problem solving, and the teacher and the researcher may merely hold very different definitions. Saying her teaching does not reflect the *researcher's* conception of problem solving is not the same as saying that her beliefs and practices are inconsistent. (An example of this phenomenon can be found in Cooney's 1985 study of "Fred.") Unfortunately, the methods typically used to study beliefs and practices usually do not make it possible to formulate such a distinction.

I contend that the distinction between professed and attributed beliefs is a false dichotomy and an example of a particular consequence of the lack of shared understanding discussed above. It is quite plausible that there are situations in which teachers state one belief and behave in ways that are inconsistent with that belief. There are likely to be instances where teachers make statements that are (intentionally or unintentionally) inconsistent with what they carry out in their classrooms. I think, however, that it is possible that perceived discrepancies are sometimes the result of incomplete or inaccurate understanding of the terms and descriptions used by teachers and researchers. The apparent dichotomy may be a consequence of the methods used to study beliefs and not a true reflection of the nature of teachers' beliefs.

I also contend that it is inappropriate to classify any belief as "professed." When a teacher makes a statement of belief, the researcher shapes the way the data are portrayed. Even in cases for which transcript or other teacher-generated statements are used as data, the researcher has framed the presentation of the excerpt and provided surrounding prose that color how the teacher's belief is portrayed to an audience. In these ways, all beliefs reported are to some extent attributed to the teacher by the researcher. Instead of designing methods that just access both attributed and professed beliefs, it seems more productive to devise methods that allow for the most accurate attributions possible.

Description of Novel Methods

Videoclip Interviews

Work on beliefs is highly dependent on the descriptions and terms used by researchers and teachers. To improve the accuracy of the attributions of belief I wanted to increase the chances that teachers and I had shared understanding of terms used to describe beliefs and practices. To build shared understanding, videotape data of classroom practices was collected and used as the context for "video clip" interview discussions. For a more extensive description and discussion of these methods see Speer (2001).

The video clip interview technique was based in part on other researchers' use of "videoclubs" as a professional development activity and as a source of data on teacher cognition (Frederiksen, Sipusic, Gamoran, & Wolfe, 1992; Nathan, Knuth, & Elliot, 1998; Sherin, 1996). The teacher's class was videotaped on multiple occasions. After each class, I viewed the tape and selected clips for discussion. During each interview, I played clips and asked the teacher to explain what he or she was doing and why.

By using video clips of the teacher's class in the interviews, I hoped to obtain information beyond what is possible to obtain in traditional, de-contextualized interviews or in a combination of interviews and observations. As discussed below, I believe the use of video clips can help build shared understanding by allowing vocabulary to emerge and by providing a meaningful context for the discussions.

Emergent definitions and vocabulary. In a traditional interview, a researcher might ask, "What role does problem solving play in your teaching?" Terms such as "problem solving" do not have precise, universally agreed upon definitions. Researchers cannot assume teachers (or other researchers or readers) share definitions. I wanted the descriptive vocabulary and definitions used in the interviews to emerge in the context of the conversations and be inductively derived from the discussions.

If video clip interviews are used, a teacher can describe what was going on in the classroom in his or her own terms. Instead of introducing words into the discussion with potentially ambiguous meanings, the words come from teachers. This comes with the obvious caveat that the researcher must seek to understand what the teachers mean by the words they use–a task made easier with the use of actual examples of classroom practices as discussed below.

Shared artifacts and increased connections to practice. Video clip interviews improve shared understanding by utilizing a shared artifact that grounds the conversation in examples of the teacher's practice. Instead of discussing beliefs and teaching practices in the abstract, a video clip provides a concrete context around which the discussion can be focused.

In a de-contextualized interview, the researcher might ask about questioning practices. Interpreting the teacher's responses in this situation might be challenging because of the multiple layers of inference needed. For example, the teacher had to interpret what the researcher meant by questioning practices and then the researcher had to interpret the teacher's response to the questions. A more complete understanding of what teachers and researchers mean when using particular words is possible if those words are tied to actual examples of practice. Although there is certainly still inference necessary in a video clip interview, the video clip provides a concrete example to build questions around and to make reference to in the discussion. The researcher can then match terms as understood by the teacher to illustrative examples and create a more accurate and complete understanding of what he or she means when using particular terms.

Data Example

In this section I illustrate my claims about professed and attributed beliefs and the importance of shared understanding with a data example. The teacher was Karl, a doctoral student in mathematics. He was teaching a discussion section in a second semester university calculus course where substantial class time was devoted to collaborative group work. I first describe Karl's beliefs and practices as they might be reported using traditional methods. Viewed in this manner, there were inconsistencies between what Karl professed to believe and the beliefs that might be attributed to him based on observations. From video clip interview data, however, it was possible to develop a more thorough profile of Karl's beliefs and the apparent inconsistencies were resolved. I focus here on illustrating my claims and only discuss a few of Karl's beliefs. This example is drawn from a larger study of teachers' beliefs and practices (Speer, 2001).

Traditional Methods

Analysis of interview data revealed that Karl believed he should promote "independence" in his students. He believed it was important that students learn to solve problems in their groups without relying on extensive assistance from him. He said, "there's some sense of independence you're trying to get them, I'm trying to get them." He said this was "because they're not always going to be in a nice, structured class where there's somebody who knows more than them. I mean at some point they're going to be in a situation where nobody can tell them what the answer is, nobody can tell them what the reason is."

In support of "independence" Karl believed students should figure things out for themselves and that his primary mode of interaction with them should be to ask questions to help them solve problems on their own. Karl described his role in class as, "Socratic. I mean, it's, it's, they know that I have the answers. It's like everybody, you know, I've got the answers for it, I'm not going to tell them. But I'm going to try to ask them and see if I can get them to figure it out by asking the right questions." Karl also believed mathematics was about ideas and relationships and felt that in addition to developing skills, students needed to understand mathematical concepts.

Inferences from observations, however, might lead one to conclude that there were inconsistencies between Karl's professed and attributed beliefs. Karl's manner of interacting with students did not reflect the idea of independence in the ways one might have expected. He did have students work on problems, but he did not provide very substantial scaffolding or problem solving support. His professed beliefs would lead one to expect a questioning style that drew out student ideas and helped them build on their existing understanding of the material. Yet, although he did ask many questions, he did not probe for student understanding or illuminate the mathematical ideas and relationships he professed to value. For example, he frequently asked very general questions such as, "What does the problem ask?" and "What should you do next?" When students were unable to answer, he neither asked more specific questions to bring out what the students did not know nor directed them to relevant features of the problem. His use of questions rarely helped students make progress and were not what most would call "Socratic."

Videoclip-based Methods

Videoclip interviews revealed what Karl meant by terms he used and the perceived inconsistencies between his professed and attributed beliefs were resolved. The shared artifact of videoclips of Karl's actual practice permitted me to develop a richer understanding of Karl's descriptions of both his beliefs and teaching practices. To Karl, asking the kinds of questions he asked and interacting with students in the manner he did were both consistent with his beliefs about promoting independence.

Karl believed that to promote independence, students needed to learn to ask themselves a set of "stock" questions. He believed the main obstacle preventing students from solving problems was their failure to ask themselves these questions. He said that "often when I asked them these questions, I try to ask the same questions enough times that they start getting it in their heads that these are good questions to ask themselves." He went on to say, "I try to always have kind of a stock set of questions so that after awhile they start hearing the questions in their own head." He said that the questions were the main thing that would help students learn to be independent and he added, "Especially, especially the more general questions." Although as an outside observer, one might not infer Karl's belief in "independence" from his use of questions, Karl believed he was helping students learn to be self-reliant and solve problems on their own. To Karl, the general questions he used were completely aligned with his conception of "Socratic."

From these and other discussions with Karl, I came to understand what he meant by the terms he used. Not only did "promoting independence" have a different sense than one might have expected, but so did his concept of "Socratic questions," and other descriptive terms he used. Once I had this understanding, there was no longer any inconsistency between his professed beliefs and what I would attribute to him from observing him teach.

Conclusions and Implications

Why is it that researchers have found a dichotomy between professed and attributed beliefs? Current theory does not explain *why* consistencies or inconsistencies might exist. I think the answer may be in part because the methods used have influenced the data that have been collected. I contend that some differences may be in part due to a methodological artifact. In particular, the dichotomy may arise when researchers and teachers lack a shared understanding of the words and terms used to describe beliefs and practices.

When researchers discuss "professed" beliefs, most often they are referring to words teachers have said. Although data may be actual statements made by teachers, researchers introduce layers of interpretation simply by selecting particular data and conducting analyses. In addition, the discussion and descriptions are not linked to specific referents. Therefore, although beliefs may be classified as professed because the data source is statements made by the teacher, in reality, the researcher has made attributions. Using videoclip interviews to illicit teachers' thinking and reasoning about their teaching practices provides access to information that it not possible to obtain using typical interviews and/or observations. The methods I used provide one way of building shared understanding and generating more accurate attributions of beliefs.

Whether researchers use videoclip interviews or other methods for obtaining data on teacher's beliefs, it seems likely that substantial progress in this area of research will occur if we concentrate on developing and using methods that will enable the most accurate attribution of beliefs possible instead of focusing on distinctions between professed and attributed beliefs. As attention to beliefs becomes an increasingly common component of teacher preparation and professional development programs, it is important that the research community strives to ensure that representations of teacher's beliefs are as accurate as possible.

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BELIEFS ABOUT STUDENTS' MATHEMATICAL ABILITIES: HURDLE OR SPRINGBOARD TO GRAPHING CALCULATOR USE?

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The research study, on which this paper is based, focused on preservice teachers' beliefs and practice regarding the use of graphing calculators in mathematics instruction. In particular, the study looked at how the graphing calculator was used and with what mathematics the calculator was used. The paper describes the teachers' beliefs about student learning and beliefs about students' mathematical abilities because the study showed that teachers' beliefs about students' mathematical abilities influenced how the graphing calculator was used and influenced the range of mathematics taught in the classroom.

This report is based on a 15-week study of two secondary mathematics preservice teachers, and it describes their belief systems about students' mathematical abilities in terms of Green's metaphor (1971). The teachers' belief systems are consistent with the structure proposed by Green, and the uses of the graphing calculator by these teachers during student teaching paralleled the uses found by Simmt (1997).

Perspectives or Theoretical Framework

Green (1971) developed a thorough description of the structure and properties of belief systems. This description posits how beliefs are held rather than what those beliefs are. He explained that belief structures have three aspects of which they are comprised: the relationship between beliefs, the strength with which they are held, and the cluster effect of beliefs. The organization of a person's beliefs is a logical order consisting of primary beliefs and derivative beliefs. Primary beliefs are those beliefs that are very basic to a person's actions; usually the person cannot justify this belief because it is self-evident to her. Derivative beliefs are those beliefs that develop logically out of other beliefs.

Psychological strength is another descriptive term about how beliefs are held. Beliefs that are held passionately are called core beliefs and live at the very center of the person's belief structure. These beliefs are the person's foundation, and, thus, are very difficult to change. On the other hand, beliefs that are held with less strength are called peripheral beliefs and can be more easily changed.

A person's belief structure contains isolated clusters of beliefs, and within these clusters, the beliefs are consistent. However, from cluster to cluster, the beliefs may be inconsistent. Such inconsistencies may cause turmoil only when the clusters are compared side-by-side; thus, a person can hold inconsistent beliefs as long as the beliefs remain in isolated clusters.

Evidence for holding beliefs is another important aspect that Green described. Some beliefs are held evidentially and some are held non-evidentially. Beliefs that are held evidentially are open to challenge because the reasons for the belief may be disputed by other evidence. Non-evidential beliefs are highly resistant to change because they are held without evidence or reason.

Simmt (1997) provided categories of use for the graphing calculator that described how inservice teachers used the tool when teaching quadratic transformations. Those categories of use were as a tool for 1) checking work, 2) plotting graphs of functions, 3) finding graphical solutions, 4) understanding word problems, 5) exploring beyond the concept being taught, and 6) providing a picture. Green's description of core beliefs best relates to the preservice secondary mathematics teachers' beliefs about students' mathematical abilities and how those beliefs influenced the teachers' uses of the graphing calculator.

Methods and Data Sources

Two preservice teachers, Fiona and Hope, were observed and interviewed about graphing calculator use during student teaching. The observations and resultant interviews occurred both at the beginning and the end of student teaching to provide a comparison of how the graphing calculator was used in instruction. Preliminary essays and surveys with an interview were used to determine existing beliefs about mathematics and mathematics education at the beginning of student teaching, and two essays were written after the conclusion of student teaching. The constant comparative method of data analysis (Creswell, 1998; Merriam, 1998) was used to shape subsequent instruments for data collection.

The data for this study were collected during student teaching and the seminar for teaching mathematics that follows student teaching. The data consist of preliminary essays, surveys (one essay and two surveys), and one individual interview during the first week of student teaching; two individual interviews, five classroom observations, and three e-mail prompts-responses during the remaining weeks, and two final essays written during the seminar—all for each preservice teacher.

Results and Conclusions

During student teaching, both preservice teachers worked in situations that involved skill-challenged mathematics students in classes of pre-algebra and applied algebra. The researcher examined with what mathematics content the graphing calculator was used and found two content areas: computation and algebra. Within each of these areas were several arenas of that content. Thus, the researcher labeled five arenas for computation (i.e., arithmetic operations, statistical, probabilistic, evaluative, and trigonometric) and six arenas for algebra (i.e., solving one-and-two-step equations and inequalities, lines—including linear regression problems, and creating T-tables).

Fiona's students had low arithmetic skills, evidenced by previous failure in the class she was teaching and in the skills assessments she did during the first week of student teaching. She saw this evidence as an indication of what was possible in her classroom. In most, if not all, of the classroom observations for Fiona, the researcher noted Fiona's statement that students could get correct answers if they knew how to use the graphing calculator. Her core belief was that students' low arithmetic skills limit graphing calculator use to verifying answers. As a result of this belief, she planned her lessons around the use of verifying answers, the lowest level of graphing calculator use. More evidence to support this core belief presents itself for Fiona's class in the limited content arenas. For computation, only two arenas were evident: arithmetic operations and evaluative. For algebra, the only arenas evident were solving one-and-two-step equations and inequalities and creating T-tables for lines. Fiona confirmed the researcher's conclusions with this statement: "I was wrong. I thought their low arithmetic skills meant they could only verify answers, but they were using the calculator to generate alternate solutions and to explore other concepts. Next time I'll know what to do."

Hope's students also had low arithmetic skills, evidenced by the level of the mathematics curriculum in which they were enrolled. However, Hope's core belief stemmed from her confidence that all students could do many levels of mathematics given appropriate tools. Hope believed that the graphing calculator was an appropriate tool to help students do all levels of mathematics, and she used the graphing calculator accordingly. She said, "Look at the linear regression. This would not even be available to my students without the calculators. If I tried to teach them how to do linear regression, I'd lose them, and they wouldn't see the line. The calculator is amazing!" Hope's core belief about her students' mathematical abilities and graphing calculator use opened a broader range of mathematics. For Hope's students, it appears that the teaching of much deeper and richer mathematics was possible. Evidence for this conclusion also comes from the content arenas used in Hope's teaching: computation content involved the arenas of arithmetic operations, statistical, probabilistic, evaluative, and trigonometric; algebra content involved the arenas of solving one-and-two-step equations and inequalities, lines—including linear regression problems, and creating T-tables.

The study showed that teachers' beliefs about students' mathematical abilities do influence graphing calculator use. For Fiona, her belief that low arithmetic skills implied verifying answers with the calculator was a hurdle for graphing calculator use. The limited content arenas in both computation and algebra suggested this problem, also. For Hope, her belief in the students' ability to learn many levels of mathematics given an appropriate tool, represented by the calculator, served as a springboard for graphing calculator use that provided opportunities for teaching deeper and richer mathematics in her classroom.

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EXAMINING THE ROLE TEACHERS' BELIEFS ABOUT MATHEMATICS AND EXPECTATIONS OF STUDENTS PLAY IN SELECTING AND IMPLEMENTING CURRICULA

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Low achievement and failure in mathematics classes are far too familiar experiences for too many students. Consequently, mathematics teachers and mathematics departments are challenged to respond to these experiences in ways that enhance learning and facilitate success. This research study examines two high school mathematics teachers' efforts to address student failure. The teachers collaborate to design and "co-teach" a mathematics class for students who had failed the first semester of their mathematics class. I refer to this newly formed class as Math Skills. The teachers selected a skills-based curriculum in an attempt to construct a mathematics class that enabled students to acquire or fortify their proficiency in pre-algebra and algebra topics. The teachers' beliefs about the nature of mathematics were integral to the selection of the curriculum and to the goals and standards the teachers set for the course and student performance. The research questions that guided my inquiry were: (1) What are the teachers' beliefs regarding the nature of mathematics? (2) What are the relationships among the teachers' beliefs about the nature of mathematics, their beliefs about students' enrolled in Math Skills capacity to do mathematics, and their goals for the class and the students in the class? and (3) What are the results of the implementation of the curriculum, especially as it relates to the impact this implementation had on the teachers' aforementioned beliefs.

This research was conducted in a mathematics class that was taught in a high school with an enrollment of approximately1500 students. The school, Prairie High School (a pseudonym), is located in a small, Midwestern city. The mathematics department and the district's Board of Education authorized the addition of the class in order to provide a mathematics learning experience to students who had failed the first semester of their mathematics class and who would otherwise be placed in a non-mathematics course for the second semester. Two primary goals of the course were: (1) the students would learn mathematics skills that would promote their success when they reentered the mathematics class that they had previously failed and (2) the class would present an opportunity for continuous mathematics learning which would not have been afforded if they were placed in a study hall or other content area classes.

This research considers the teachers' efforts and how their views about mathematics and their expectations of their students influenced those efforts and vice versa. Research supports the claim that teachers' views of mathematics influence instruction and curricular-related decisions (Dossey, 1992). Moreover, different views of the nature of mathematics (e.g. a body of absolute truths or fallibilist) promote varied foci during the selection of curricula and instruction (Lerman 1998). Stodolsky & Grossman (2000) argued that teachers' perspectives on the nature of mathematics influence their views on what content to teach. Thus teachers' views on the nature of mathematics are significant when considering the selected curriculum and the methods of instruction utilized in the implementation of that curriculum.

Additionally, teachers' beliefs and expectations about their students are significant. Research suggests that students perform at higher levels when teachers hold high expectations of them (Gutierrez, 1996; Silver & Stein, 1996). Talbert, McLaughlin, & Rowan (1993) asserted that teachers' decisions about what should be taught were influenced by their expectations of their students. Therefore, the examination of teachers' expectations of their students can shed light on both the selection of particular content to teach and the methods of instruction in which the teachers would engage in the implementation of selected curriculum.

I collected data in the form of field notes from weekly classroom observations, audio-taped, semi-structured interviews with the teachers, and journaling. Collaborating with a research team that consisted of one professor of mathematics education and two other graduate students, I developed a coding system to analyze the transcripts of the interviews and field notes. Our inquiry was framed by research on teachers' beliefs and how those beliefs are challenged (Cooney, Shealy, & Arvold, 1998; Green, 1971). Predetermined codes on the nature of mathematics (procedural, static, dynamic, sequential, or well-defined) and teachers' expectations of students (high, low, and/or related to innate ability) were defined and later revised throughout analysis. Codes regarding how mathematics is learned included drill and practice, sense making, applications, and rote/memorization. I triangulated across data sources and participated in weekly research team meetings in which data were presented and analyzed and alternative analytic perspectives of inferences were examined. Prevalent themes that I identified through the implementation of constant comparison analysis (Strauss, 1987) guided the focus of subsequent observations and interviews. Emergent themes included a procedural view of mathematics, low student expectations, and skill-acquisition as the goal for mathematics learning.

This study reaffirms connections between teachers' beliefs about the nature of mathematics and how mathematics is learned and the curriculum they implemented. The teachers held a procedural view of mathematics and believed that drill and practice would achieve mathematics learning for students in Math Skills. The selection and implementation of the curriculum were influenced by the teachers' low expectations for the students. While the aforementioned findings are not surprising, the students' poor performance in Math Skills appeared to have opened a door for the teachers' reconsideration of how mathematics is learned and the efficacy of a low-level, skills-based curriculum in meeting their students' needs. Both teachers' stated that assessing the success of the course would be partially dependent on the students' performance when they reentered the previously failed class. However, they both suggested that the drill and practice approach had not been successful for promoting engaged learning or skill retention as they previously thought. One of the teachers taught a similar class the following year and incorporated field trips, projects, and connections with college mathematics students in an effort to promote more engagement and learning. However, a focus on drill and practice remained. Yet, teaching Math Skills provided the teachers with firsthand evidence that brought the efficacy of a low-level, skills-based curriculum in meeting the needs of previously unsuccessful students into question. Findings suggest that efforts to enhance mathematics instruction that allow opportunities for teachers to engage in the practices they believe will work coupled with critical discussions and reflective interchanges on students' performance can promote a reconsideration of beliefs about how mathematics is learned.

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MATHEMATICS TEACHERS' PERCEPTIONS OF COMPUTERS USE IN MATHEMATICS INSTRUCTION: A CASE STUDY FROM TURKEY

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The purpose of this study was to explore Turkish middle school mathematics teachers' perceptions of a mathematics learning environment when computers were used for drill and practice purposes. Data for this study were collected through interviews. The results of the study were presented with a focus on the perceptions of the roles of teachers and students, environment, instruction and interaction when computers were used.

Introduction

Research has shown that a vast majority of teachers in Turkey had positive beliefs about the use of computers in education, but had limited knowledge and experience on how to use this technology in their teaching (Cakiroglu, Cagiltay, Cakiroglu, & Cagiltay, 2001). A major obstacle for not using computers in teaching was the lack of hardware and lack of teachers' knowledge in computers (Cakiroglu et al., 2001). International experience on this issue suggests that, even if the computers are available and accessible, mathematics teachers tend to not use computers in their classrooms (Rosen & Weil, 1995). Especially, when mathematics teachers hold negative beliefs about the computer use in mathematics instruction, this may be a reason for not using the computers in their teaching (Norton, McRobbie, & Copper, 2000). In this sense, how mathematics teachers perceive the use of computers in the mathematics classrooms becomes an important issue for understanding the issue of integrating the computer technology in mathematics curriculum.

Research suggests that when used in mathematics classes, computers may have an important impact on classroom culture, which could result in different roles for both teachers and students (Fraser, Burkhardt, Coupland, Philips, Pimm, & Ridgway, 1987; Heid, Sheets, & Matras, 1990). A major concern of teachers in such a learning environment was related to classroom management (Jensen & Williams, 1992). Teachers' perceptions of computers in the mathematics classes seemed to be affected by various factors related to their background. Individual teachers' resistance to the use of computers in the class was found to be connected to their beliefs about mathematics teaching and learning and their existing pedagogies, including their perceptions about examinations, concerns about time constraints, and preferences for particular text resources. When teaching was considered as transmission of knowledge with a subject-centered approach, individual teachers had a restricted image of the potential of the computers in teaching and learning. However, when teaching was viewed as a social constructivist process with a more student-centered approach, an individual teacher had a broader view of the potentials of the computers in mathematics teaching (Norton et al., 2000).

The purpose of this study was to explore five Turkish middle school mathematics teachers' perceptions of a mathematics learning environment when computers are used. There was a focus on the differences that may likely to appear in their regular mathematics lessons in terms of roles, environment, instruction and interaction when computers were used.

Method

Setting

The research site was a private middle school in one of the major cities of Turkey. The school had two computer laboratories, and no computers were available in the classrooms. Available software for mathematics instruction was determined by the collaborative work of the research and development department and the mathematics department of the school. One of the major criteria in the purchase of an instructional software was its consistency with the national curriculum (In Turkey, there is a strict centralized curriculum that schools are expected to follow). Available software for mathematics classes was mostly based on drill and practice, where students usually perform the mathematical skills covered in the regular lessons, or solve questions related to that week's topics.

Data Source

Among nine mathematics teachers in the school, the ones who had conducted lessons in the computer laboratory were selected as participants in this study. In total, five mathematics teachers were selected as participants. These teachers were interviewed about their perceptions of the use of computers in teaching mathematics. Interviews were recorded on the basis of teachers' approval. They were transcribed and analyzed initially by content analysis and then by phenomenological analysis as described by Bogdan and Biklen (1998).

Results and Discussion

Findings suggest that, among the participants, views of using computers in teaching mathematics seemed to be related to their interest in computers. For instance, one of the teachers, who was quite competent in computers and expressed the importance of computers in her daily life, was interested in mathematics education software. She provided clear ideas on the use of computers in the mathematics classes, and how the software should be designed and used. She also described how she would use computers in her lesson, if they were available in her classroom instead of computer laboratories. Another teacher who had limited experience in using computers was not very interested in the use of computers in mathematics instruction. She claimed the uselessness of the computers in the mathematics classes almost on any condition.

Teachers' expectations from their students had similarities in terms of the behaviors they expect to see in their classes. Respect was one of the most emphasized issues. In terms of mathematics instruction, teachers claimed that the students should contribute to their own learning. In this respect, doing homework on time was clearly emphasized by most of the participating teachers.

A common concern of the teachers during the mathematics lessons in the computer laboratories was related to the management of the classroom. They stated to have difficulty on keeping track of student progress, and maintaining the orderliness of the lesson. In this respect, teachers tended to believe that students loved dealing with computers during mathematics lessons conducted in the computer laboratory. As one of the teachers stated, to some extent, this may be due to the fact that students are "freer and more relaxed in computer sessions." She stated that in her regular mathematics lessons, she has the control of what is going on during the lesson. According to her, the case is quite different in computer sessions, where "students can sit anywhere they want and chat with classmates" which are "disorderly" and inconsistent with her regular mathematics classes. In this sense, she preferred to have the computers in her classroom, instead of computer laboratory. Unlike their regular mathematics lessons, she felt to have a more "passive" role in the computer laboratory sessions, which she did not prefer. She thought that students are more "serious" in her mathematics lessons without computers.

Although teachers appreciated that students enjoy mathematical work they do on computer sessions, they complained that they lost control of the flow of the lesson. Such concerns of teachers related to the classroom management were consistent with the arguments of Jensen and Williams (1992).

Willingness to use various instructional methods also seemed to be related to their perceptions of using computers in the classroom. Among the participating teachers, the ones who seemed to be open to changes, such as trying to use different teaching methods, were also open to the use of the computers.

A common belief among the teachers was that during the mathematics lessons in the computer laboratories, male students were mostly interested in computers just because they like to use them, while female students are interested more on the mathematical tasks presented by the computers. During these sessions, teachers believed that female students were more "careful and spend more time on the mathematical tasks" presented by the computer, compared to male students.

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ONE & DONE: "VERY LITTLE PREPARATION FOR TEACHING WITH TECHNOLOGY"

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This paper highlights the results of an investigation of pre-service elementary teachers' perceptions of their ability to teach via technology and of how their degree program prepared them to so teach. Results indicated that participants tended to perceive themselves as capable of teaching via technology. However, they had difficulty identifying specific technological resources for teaching. Further, participants tended to be less positive in their perceptions of both the extent to which their degree program prepared them to teach via technology and the extent to which their instructors modeled the effective use of technology than in their perceptions of their own ability to teach via technology. Accordingly, modifications to teacher education programs may be warranted and some suggested alterations are provided.

Goals and Participants

In the spring of 2000, I (M. H.) began an investigation of pre-service elementary teachers' perceptions of their preparedness to teach via technology and of how their degree program prepared or failed to prepare them to so use technology. The participants were 43 pre-service elementary teachers within 2 weeks of completing their undergraduate degrees. Cohorts of students took fixed blocks of courses during their senior year, which for most participants was a precursor to a fifth year of study and an internship leading to an MAT in Elementary Education. However, no participant had completed a student teaching experience.

Of the 43 participants, 27 were my (M. H.'s) pupils. Of these, 7 were enrolled in one integrated instructional methods course 20 in another. I taught methods of teaching mathematics to both sets of students, and a colleague taught the language arts components for both courses. An instructor I shall refer to as Mr. K taught the social studies and science components of the course in which 20 of my students were enrolled. Another instructor taught the corresponding components for the rest of my pupils, and the remaining 16 participants were students of three different educators.

Methods and Tentative Hypotheses

Data was collected through a survey that contained both 5-point Likert scale and open-response items. Frequency counts, percentages and chi-square tests were used to identify patterns in Likert scale data and coding was used to identify patterns in the open-response items. As a check on the participants' assessment of their preparedness to teach via technology, I (M. H.) asked them to list 3 specific technological resources for teaching each of the disciplines of mathematics, science, language arts and social studies.

I began the study with the tentative hypothesis that the pre-service teachers would not perceive themselves as ready to teach via technology. I also suspected that the participants would tend to disagree with the statements that their teacher preparation program prepared them well for teaching via technology and that their instructors modeled the effective use of instructional technology (Topp, 1996).

Results

The pre-service teachers generally perceived themselves as capable of teaching through technology (83% A or SA & 100% N to SA). When asked to list 5 technological resources with which they had become familiar through their education courses, participants sited resources such as the Internet, e-mail, Power Point, spreadsheets, software and videos. When asked to identify 3 specific technological resources for teaching each of the disciplines of mathematics, science, language arts and social studies, most participants again noted 2 or 3 broad categories of technological resources such as the Internet, videos and email. The Internet, software, videos and laser discs were noted on the questionnaire, so those and other broad responses were not considered specific tools. Only 8 of 43 or 18.6% were able to list at least one title. However, 16 more participants for a total of 24 or 55.8% of the participants provided a response that indicated some knowledge of specific technological resources or where to locate them.

We find these percentages disconcerting. It is possible that the participants simply had difficulty remembering titles, but the percentages could be indicative of insufficient knowledge of educational technology. Accordingly, it is possible that the participants' perceptions of their ability to teach via technological resources are unrealistic. Some support for the latter possibility is found in the fact that the participating teachers tended not to be as positive in their evaluation of the extent to which the university prepared them to use technology as an instructional tool (SA 7 %, A 2 8%, N 37%, D 21%, SD 7%). Likewise, the participants tended not to agree as strongly when asked if their instructors had modeled the effective use of technology (SA 7 %, A 39.5%, N 41.9%, D 1 1.6%) (Topp, 1996). Moreover, very intriguing patterns existed in the responses to the questions requiring participants to list 3 specific technological resources for each of 4 disciplines. As noted above, 24 participants gave responses indicating some knowledge of specific technological resources or sites at which such resources could be found. Eight of the 24 were able to identify at least one specific technological tool. However, 62.5% of those noting specific resources and 75% of those displaying some awareness of such resources were in Mr. K's class, which comprised only 46.5% of the participants.

In light of the above, we separated the participants into two groups, those in and those not in Mr. K's class and re-examined their ability to identify a specific technological resource. We found that 5 of Mr. K's 20 pupils were able to highlight a specific technological tool, while 3 of the remaining 23 participants were able to do likewise.

Further, 16 of Mr. K's 20 students displayed at least some awareness of specific technological resources, while only 8 of 23 remaining participants displayed such awareness. A ^{2} test for significant differences in frequencies revealed no statistically significant difference in the frequencies of identification of a specific technological tool across the two groups. However, a ^{2} test did reveal a significant difference ($p = 0.003$) in the frequencies with which Mr. K's students and the rest of the participants displayed at least some knowledge of specific technological resources. Thus, there is a relationship between being a pupil of Mr. K and demonstrating awareness of specific technological resources. This does not mean that Mr. K. was the cause of the awareness, but it is certainly a possibility and an issue that warrants further investigation.

The previously noted lack of support for the claim that the participants' instructors modeled the effective use of instructional technology was also highlighted in 60% of the responses to an open-ended question concerning how the university had failed to prepare the participants to teach with technology. Moreover, 83.7% of the participants indicated that they did not receive enough instruction regarding methods of teaching through technological resources. Similarly, 44.2% of the pre-service teachers highlighted having only one course dealing with technological resources as a weakness of their degree program (Topp, 1996). Even so, 46.5% of the participants also noted that they were given assignments such as developing Power Point presentations, which would certainly contribute to the participants' preparation to use technology as an instructional tool. Additionally, 39.5% of the pre-service teachers noted that some instructors did incorporate technology in their teaching.

Implications and Potential Solutions

Data indicates that educational technology courses are preparing pre-service teachers to download lesson plans, find useful websites and to create spreadsheets and Power Point presentations. However, the difficulty participants had in identifying specific technological resources and the pervading perception (83.7%) of a need for further instruction concerning such resources and methods of teaching through them, suggests that teacher educators (at least at the participating university) need to reconsider the nature and scope of teacher preparation programs. Additional support for this claim is found in the pre-service teachers' assertions that their instructors did not model the effective use of technology (39.5%) and that having only one course focusing on technology was a weakness of their degree program (44.2%).

However, due to the wide range of majors served by educational technology courses, it is impossible for one course to adequately prepare pre-service teachers to teach via technology. Even so, we assert that pre-service teachers should be exposed to some technological resources for teaching mathematics, science, language arts and social studies during their educational technology course(s). Likewise, such courses should address criteria for evaluating technological resources and some general guidelines for teaching with technology. Nevertheless, methods courses should also address these issues. Further, pre-service teachers need to extensively explore technological resources and to experience the use of such resources from the perspectives of both learner and teacher (Benson, 1997; Roblyer & Erlanger, 1999).

We propose that one method of helping pre-service teachers to become familiar with technological resources is to have them review and critique such resources. Assignments of this nature afford the bonus of requiring pupils to consider guidelines for evaluating technological resources. Pre-service teachers should also use technology to resolve problems, practice skills and learn or apply concepts and procedures. Finally, pre-service instructors' knowledge of technology and methods of teaching through it could be broadened by planning and teaching lessons in which technology was used as an instructional tool.

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CHANGE IN URBAN TEACHERS' BELIEFS BETWEEN THE END OF STUDENT TEACHING AND THE END OF THEIR FIRST YEAR OF TEACHING

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Rationale and Theoretical Perspective

The role of beliefs in teacher change is well documented. A substantial body of literature exists which suggests that teachers' beliefs drive their teaching of mathematics (Pajares, 1992; Richardson, 1996). In order to change teachers' practices, we need to consider teachers' beliefs. However, we know that beliefs are difficult to change (Lerman, 1997; Pajares, 1992), that the beliefs teachers' espouse are not always consistent with the way they teach (Cooney, 1985), and that changing teacher's beliefs takes time (Richardson, 1996). Moreover, Pajares (1992) tells us that beliefs about teaching are well established by the time a student enters college. They are developed during what Lortie (1975) calls *the apprenticeship of observation* that occurs over their years as a student. They include ideas about what it takes to be an effective teacher and are brought to their teacher preparation program. Given this, it seems imperative that teacher education programs assess their effectiveness, at least in part, on how well they nurture beliefs that are consistent with the program's philosophy of learning and teaching. Also, it is important to study how consistent the beliefs teachers espouse are with their teaching practices, i.e., can teachers do more than *talk the talk.*

The Intervention

The Urban Alternative Preparation Program (UAPP) is housed in the Department of Early Childhood Education at a large, urban university. Students enter as a cohort taking all coursework together. They hold non-education undergraduate degrees and are preparing to become certified in a K-5 *urban* setting. The program is grounded in constructivist learning theory and promotes practices that support that perspective.

In Phase I of the program, students obtain certification after four semesters. Unique aspects of this teacher preparation program include coursework and field experiences in the urban classroom and an integrated four-course sequence in mathematics-related content: two in mathematics and two in mathematics education. One faculty member teaches the twelve semester hours of content and methods as an integrated, seamless course over three consecutive semesters. Instruction is consistent with a constructivist philosophy. Students learn the mathematics from the mathematics book using many of the methods suggested in the methods book. They use oral reflection, videotaping and written logs to examine the methods used to deliver the content. The same mathematics faculty member who delivers the content in their coursework mentors them in their field placements. (For a more complete description of Phase I see Hart 2002.)

To continue on into Phase II students must have accepted jobs teaching elementary school in an urban setting (defined for our purposes as schools with at least 75% free or reduced lunch). During this time students pursue a masters degree. Their mathematics coursework during Phase II includes a course that provides monthly coaching in mathematics (content and methods) in their individual classrooms and a university course on Issues in Teaching Mathematics in the Elementary School. In addition to delivering content, the course provides a mechanism for these novice teachers to collaborate and reflect on their practice. The faculty member who teaches the mathematics in Phase I provides the coursework and mentoring in Phase II. (For a more complete description of Phase II see Hart 2001.)

The Study

This study reports on the first cohort of students who entered the UAPP in May, 1999. In particular, the results of an item analysis of the Mathematics Beliefs Instrument (MBI) used at the end of Phase I and at the end of Phase II of the project will be discussed.

Fourteen of the original cohort of 20 teachers completed Phase I and obtained certification. Eight teachers took a teaching position in an *urban* school and are part of this study. They ranged in age from 25 to 41. There were two African-American females, one Asian male and five Caucasian females. Two teachers were in a first grade classroom, five teachers were in second grade and one was in third grade.

Results from the study comparing beliefs before entering UAPP to beliefs at the end of Phase I suggested that teachers had in fact changed in a direction more consistent with reform practices (Hart 2002). However, that research was only the first step in assessing change in their beliefs. As first year teachers in Phase II, I had the opportunity to look at their practice and assess how consistent it was with Phase I. Results from that study (Hart 2001) found that teachers had generally maintained their reform perspective that their teaching behaviors were moderately consistent with their espoused beliefs, but that they were greatly influenced by the culture of the schools within which they worked.

In reporting this work at PME last year it was suggested that an item analysis of the responses from the comparison of Phase I and Phase II instruments might provide a richer picture of the nature of change for these teachers. This short oral presentation will report on those findings. Because of the small number of participants (n=8), only descriptive statistics are used.

Of the 224 possible response changes in self-reported beliefs on the MBI (28 items x 8), 80.4% (180 items) remained unchanged, 8.9% (20 items) changed in a direction more consistent with reform philosophy, and 10.7% (24 items) changed in a negative direction away from reform philosophy.

In analyzing the items I ordered items by the total *amount* of change in either direction and reviewed the qualitative data sources for possible explanations for the change. I will give one example here. Item #17 (*some people are good at mathematics and some aren't)* had the largest amount of change with 6 teachers varying from their original response. Five teachers moved away from a reform perspective, accounting for 29% of the total amount of change on the survey. One teacher shifted toward reform. After analyzing the logs possible explanations emerged.

- All five teachers who had changed away from reform had, as first-year teachers, been given classrooms with the lowest performing students at their grade level; perhaps skewing their perceptions.
- Results from the MBIs indicated these teachers had changed their beliefs about what it means to be good at mathematics. However, the teachers were all in a culture of traditional mathematics practice. Being *good* at mathematics may have meant something different.
- The teachers reported that their students resisted engaging in conversations about mathematics and in sharing their thinking. One teacher commented that the students "just want me to tell them the answer." The students did not fit the teachers' new definition of being good at mathematics.
- In one class racial issues may have played a role in the teacher's perceptions of her students and the students willingness to demonstrate their real ability. A white female teacher reported that "[one child] would tell me that she didn't care what I said and that she wasn't going to do the work because she hated white people." This unwillingness to perform may have been perceived by some as lack of ability.

Discussion

What have we learned from these results? For the most part, at this point in their professional development, these teachers appear to have generally maintained beliefs that are consistent with reform practices. However, a few teachers on a few items demonstrated some interesting change. Using the example of item #17, there appear to be many possible explanations for the teachers given the situation these teachers were working in, it is not unreasonable that after teaching for a year in a low-income setting with a traditional view of mathematics that they would believe that not all students are good at mathematics. From a programmatic point of view that information can inform our work with teachers during their pre-service experience so that we can do a better job of preparing them for the urban classroom.

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NAVIGATING THE WORLDS OF CONCEPTUALLY-ORIENTED MATHEMATICS AND STANDARDIZED TEST PREPARATION

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This paper examines how a third-grade teacher navigated the potentially conflicting aims of developing students' conceptual understanding and preparing them for standardized tests. We situate our analyses of the teacher's understanding of mathematics and of herself as a mathematics teacher within the broader context of the school within which she practices and the educational culture in the United States. Like many elementary teachers, Gina Tyro found herself struggling to overcome her own negative experiences as a mathematics student and develop practices with which she was unfamiliar (Ball, 1994). We argue that it was a combination of the teacher's beliefs and experiences surrounding mathematics that prompted her to embrace a conceptually-oriented curriculum. However, the same beliefs and experiences initially limited the connections she was able to draw between the curriculum and the test-preparation lessons. The pressures she faced to both prepare students to do well on standardized test while fostering deep mathematical understanding is typical of many teachers in urban schools. How teachers understand and respond to these often competing goals can provide insights into current classroom practices and can guide efforts to support teachers in this work.

Theoretical Framework

Our analyses were informed by research that examines classroom practices—in particular the participation structures established in the classroom in relation to teachers' beliefs about mathematics, teaching, learning, and educational goals. Typical classroom practices reflect views that mathematics consists of rules and procedures to be memorized and practiced. Thus mathematical learning can be measured by the extent to which a student can use these procedures accurately and efficiently. These beliefs about mathematics and what it means to know, which permeate mainstream culture, are acquired, "through years of watching, listening and practicing" (Lampert, 1990, p. 32) and influence the ways in which a teacher sets up her classroom. Lampert argued that teachers must work to initiate alternative classroom norms which "build a *participation structure* that redefines the roles and responsibilities of both the teacher and students in relation to learning and knowing" (p. 32).

Establishing such norms, however, often involves working against one's own deeply rooted beliefs about mathematics teaching. Thompson, (1992) used the term *integratedness* of belief structures to refer to the extent to which one's "beliefs and views in a given domain form a coherent conceptual system, as opposed to each belief existing in isolation from others" (p. 122). Lack of integratedness becomes apparent when teachers try to develop practices that go against their established classroom norms.

Teachers' beliefs are not the only factors at play in shaping classroom practice. Other local goals, meanings, and events can mediate the classroom practices, leading to further inconsistencies between stated teacher beliefs and classroom practices (Thompson, 1992). In other words, knowledge and practices are contextualized socially and historically (Engeström, 1999), making change in practices considerably more complex than many might perceive.

Methods & Data Sources

Our analyses involved a year-long ethnographic study of Gina, a third-grade teacher in an urban elementary school. Six teachers at the school had been involved in a project that combined professional support and research for two years. During that time, they used the reform-oriented curriculum, *Investigations in Numbers, Data, and Space* (TERC, 1998). Gina, a third year teacher in her first year at the school, used *Investigations* for the first time.

The data consisted of classroom observations, semi-structured interviews with the teacher, and informal conversations and classroom observations. We used QSR NVIVO to code and analyze the data. The initial coding was based upon a set of analytical questions about the observations, but emergent codes were also added.

Results & Conclusions

Gina's recollections of her mathematics instruction were primarily negative and notably different from the practices she experimented with when using the *Investigations* curriculum. Her willingness to use the curriculum was impelled by her desire to offer her students a mathematics experience that differed from her own. From September to January, Gina drew upon the curriculum to encourage mathematical thinking, understanding, and the generation of computational strategies and explanations. Gina's confidence in and assessment of the curriculum were bolstered by her students' successful participation in these activities.

Even though they used the *Investigations* curriculum, many teachers at the school questioned the extent to which it prepared students to perform as expected on standardized tests, specifically the SAT-9. Thus, in consultation with a veteran third-grade teacher, Gina decided to devote two 2-week sessions to test preparation; she taught one in January and one in April. Her goal was to familiarize students with what she believed they need to know: the mechanics and rules of test taking and standard algorithms for multi-digit computation.

In January, she treated the test-preparation sessions as disconnected from her previous lessons. This shift was evident in the expectations she held for students and the norms she established in the classroom. She introduced the unit as follows: "This kind of math that I'm gonna show you today, it really doesn't involve a lot of thinking. And that's why schools, decided to stop teaching it that way" (Observation 01/14/02). The students worked primarily on timed computation worksheets. Gina taught the students the conventional algorithms for addition, subtraction, and multiplication and expected the students to use those to solve the problems. The substantial change in participation structure observed during this instruction and the absence of any connections to their prior conceptual work was striking, particularly since much of the instruction we observed between September and January involved student-generated approaches to computing with 2- and 3- digit numbers.

Between January and April, Gina returned to the *Investigations* curriculum. In April, she began a second test-preparation unit. This time she made explicit references to work they had done from the *Investigations* curriculum. For example, she encouraged students to solve problems using whatever strategy they found to be quickest or easiest. While Gina encouraged students to connect these two approaches to learning math, she maintained that the two were incompatible and even irreconcilable. And she shared this view with her students: "SAT-9s use this weird math language we don't use. . . This is not math, this is SAT-9 practice" (Observation, 04/16/02).

The shift we found in Gina's inclination to encourage students to draw on unconventional computation strategies seems related to her observations of students using these strategies over time. In an interview in April, she explained with some surprise that many student-generated strategies were more efficient than conventional algorithms. In other words, Gina's teaching practices changed in response to changes she observed in students' mathematical practices. The interdependency of teachers' and students' practices illustrates the ways that classroom norms and practices are co-constructed and constantly shifting.

The pedagogical approaches Gina used in January suggest that at that time her beliefs about conceptual understanding and traditional algorithms were discrete as opposed to integrated (Thompson, 1992). Thompson (1992) (among others) argued for the importance of teacher reflection on the coherence between their beliefs and practices as a critical way for teachers to develop integrated systems. We are struck that Gina's reflection focused on the practices she and her students jointly constructed.

Implications

The case of Gina Tyro raises several issues related to reform in mathematics education. First, Gina Tyro illustrates the possibility for students' mathematical practices to stimulate teacher learning. It is important to point out, however that the practices students developed were co-constructed by the teacher, the students, and the curriculum. Teacher learning and classroom practice do not follow a linear arrangement, but are intertwined and recursive. Teachers' beliefs are complex and deeply rooted and thus mediate practices and interpretations of new ideas. But they are also mediated by practice. Second, she illustrates the enormity of demands, sometimes conflicting, currently placed on teachers. As demands for accountability intensify, it is important to understand how teachers navigated their associated conflicts. The field could benefit from more clarity on the relationship between computational mastery and conceptual learning. In recent years, the mathematics education community has been understandably driven by a pursuit of conceptual development. It is increasingly evident, however, that such emphases alone, can result in underserving many mathematics learners, particularly those in low-income communities. The field is in need of more studies of students' mastery of both computational and conceptual understanding in relation to different teaching practices (e.g., Boaler, 1997).

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A CASE STUDY OF A NOVICE AND AN EXPERIENCED TEACHER IN THE IMPLEMENTATION OF A MULTIPLE REPRESENTATIONS CURRICULUM

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The mathematics education community, in its call for reform, underscores the importance of mathematics instruction emphasizing the use of multiple representations in the presentation of concepts. One focus of the study was how teacher beliefs affect their ability to implement a multiple (algebraic, graphical, tabular) representations curriculum. The novice instructor's attitude remained neutral while the experienced instructor's attitude remained somewhat positive. The teachers had a great impact on the students' preferences for a particular representation and ease of technology use. Implications for further research were that teacher training is essential if reform curricula are to be properly implemented.

Introduction

As mathematics educators explore ways to strengthen and energize their pedagogy, and as technology becomes more common in the classroom, the use of multiple representations is gaining more attention in the mathematics curriculum. The phrase *multiple representations*, in this study, refers to the presentation of a concept or process using numerical (table), graphical (Cartesian graph) and symbolic (equation) viewpoints (Noble, Nemirovsky, Wright, & Tierney, 2001; Porzio, 1999). Numerous researchers have investigated the effects of instruction using multiple representations of concepts (Dreyfus & Eisenberg, 1982; Harvey, 1991; Keller & Hirsch, 1998; LaLomia, Coovert, & Salas, 1988; Lloyd & Wilson, 1998; Noble et al., 2001; Porzio, 1999; Stein, Baxter, & Leinhardt, 1990). The conceptions teachers have about mathematical content can affect their ability to implement reform curricula (Fennema & Franke, 1992; Kidder, 1989; Lloyd & Wilson, 1998; Millsaps, 2000; Stein et al., 1990; Thompson, 1992; Wilson, 1994). The purpose of the study was to determine if and how teachers' beliefs about multiple representations and technology use will affect their implementation of the new curriculum. The research was guided by the following questions:

1) How will the teachers' beliefs about technology affect their ability to implement a multiple representations curriculum?
1) How will the teachers' beliefs about multiple representations affect their ability to implement a multiple representations curriculum?

Methodology

The participants were instructors of two different sections of Intermediate Algebra at a large southeastern university. One instructor, "Joe", was identified as the novice teacher and the other instructor, "Mary", was identified as the experienced teacher based on teaching experience, educational background and calculator proficiency. The teachers each selected four students from their respective classes to participate in the study. The students ranged in age from 17 to 19 and were selected to reflect diversity with respect to race, gender, and academic performance in class.

The data collection methods included interviews, classroom observations, audiotape recordings, videotape recordings, and artifact reviews. All interviews were videotaped and transcribed. All classroom observations were audiotaped and transcribed. Field notes taken during the observations focused on any teacher-student interactions that indicated the uses of multiple representations during classroom instruction. Written artifacts included the pretest, posttest, technology surveys, and observation sheets.

Results and Discussion

Research shows that teaching with multiple representations leads to a better understanding of concepts (Hiebert & Carpenter, 1992; Kaput, 1989, 1992; Porzio, 1999). Contrary to the research, however, many educators are reluctant to embrace a multiple representations curriculum. Eisenberg and Dreyfus (Eisenberg & Dreyfus, 1991) have found that one of the reasons that teachers avoid visual explanations of mathematics is a sociological one; visual is harder to teach. The novice instructor, Joe, revealed in his interview that he was reluctant to teach the visual representations because he was not comfortable teaching them. He found it difficult to communicate to his students about mathematical relationships. He often missed the chance to promote meaningful connections between the concepts and the different representations. Joe also hesitated to use the calculator in class because he found it difficult to use during instruction. His hesitance was reflected in his teaching and affected his students' choices for representation use.

In contrast, Mary incorporated the graphing calculator into her instruction throughout the class period. She used each of the three representations almost equally (algebraic – 25.9%, graphical – 29.0 %, tabular – 16.1%, and a combination of the representations 29%). Similarly, her students also opted to use the various representations to solve problems and did not totally rely on the algebraic method. Mary's positive attitude, strong calculator proficiency, extensive teaching experience and beliefs played a crucial role in her implementation of the reform curriculum. Mary's conceptions contributed to a classroom environment in which she encouraged students to employ a variety of representations and make connections among them.

Conclusion

It is not reasonable to expect Joe and Mary to approach the multiple representations curriculum in the same manner based on their dissimilar levels of experience and teaching styles. It is, however, reasonable to expect that Joe can effectively implement this reform curriculum with the proper guidance and preparation. This training must include how to make connections between the various representations, and how to integrate technology into the classroom.

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BELIEFS AND PRACTICES OF TEACHERS REGARDING FAILURE IN ALGEBRA

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This qualitative case study sought to understand the beliefs and practices of four Algebra I teachers who had high failure rates. The research used ethnographic techniques to examine teachers' beliefs about the nature of algebra, teachers' attributions for student failure, teaching efficacy as it related to student failure, and the ways in which teachers' attributions influence their practices. Strong connections were observed between the beliefs of the teachers regarding the nature of mathematics, their attributions for student failure in algebra, their self-efficacy, and the modifications they made in instructional practices. The study has implications for teachers, administrators, and teacher educators.

Introduction

Currently there is a national trend to require algebra for all students. Yet historically many students have been unable to achieve this goal; the National Center for Educational Statistics reports a typical failure rate of 40-50% (Mullis, Dossey, Owen, & Phillips, 1991). The purpose of this research was to describe and understand the beliefs and practices of some Algebra I teachers who have high failure rates.

Theoretical Framework

According to Pajares (1992), the broad construct of educational beliefs includes beliefs about the teacher's ability to affect student learning (teaching efficacy), about confidence in oneself to perform certain tasks (self-efficacy), about the nature of mathematics (ontology) and mathematical knowledge (epistemology), and about causes of teachers' or students' success or failure (attributions, locus of control, motivation). Beliefs play a critical role in selecting tasks and making decisions regarding such tasks, and therefore are instrumental in defining behavior (Pajares, 1992). A growing body of research provides evidence that teachers' beliefs about mathematics and mathematics teaching affect instructional practice (Brown & Baird, 1993; Thompson, 1992). Gibson and Dembo (1984) described several studies that found a positive correlation between teacher efficacy and student achievement.

Research Methodology

The research project was a qualitative case study using ethnographic techniques. Participants were four teachers of beginning algebra at a metropolitan high school. Data were collected primarily through semi-structured interviews, supported by data from classroom observations and examination of artifacts. A teacher efficacy scale created by Gibson and Dembo (1984) was also administered. The researcher developed a conceptual framework classifying factors in student failure as related to teacher (pedagogy, assessment, classroom management), student (ability, motivation, prerequisite skills), or system (curriculum, class size, resources). Attributions for failure were also classified according to whether they were in the teacher's locus of control. Data were analyzed to ascertain teachers' beliefs about mathematics and mathematics teaching, attributions for student failure, and the way these beliefs impacted teaching practices.

Findings

All four teachers in the study had been taught reformed pedagogy and were familiar with the National Council of Teachers of Mathematics Standards (2000). However, they held an Instrumentalist view of the nature of mathematics, seeing it as a set of rules and procedures that must be mastered in order to advance to the next level of mathematics (Thompson, 1992). Their role was to demonstrate procedures, structure the classroom environment so that students could listen to explanations and practice the skills, and encourage students to work hard. Strong connections were observed between the beliefs of the teachers regarding the nature of mathematics and mathematics teaching, their attributions for student success and failure in algebra, their self-efficacy, and the modifications they made in instructional practices.

Beth

Beth (all names are pseudonyms) believed that the purpose of Algebra I was to prepare students for upper level courses, and was concerned about her accountability for their performance. Most of her attributions for student failure were related to student factors: their lack of motivation and self-discipline, and their inability to listen, understand and make connections, and retain knowledge. The locus was external to Beth and uncontrollable by her. These attributions had a strong influence on her self-efficacy and practice. Beth said she tried to explain things to students over and over again, "on the simplest level possible and use as many analogies as possible." She also talked about "making sure that they understand, making sure they do enough examples," though she acknowledged these strategies were generally ineffective. Beth had a lack of general and personal teaching efficacy, even stating at one point, "I'm just depressed about the future of mankind." She had trouble relating to students who found math difficult, and had difficulty with classroom management. By the end of the study Beth revealed that she was leaving education.

Chad

A strong proponent of the NCTM Standards, Chad believed that in Algebra I students should develop conceptual understanding to provide a good foundation for later mathematics. Unlike Beth, Chad stated that intellectual ability was a strong factor in determining success; students must be able to see patterns and organize schema efficiently. His judgment of students' achievement levels was based on a combination of their cognitive ability, maturity and behavior, motivation and work ethic, and the influence of home environment. Although he attributed failure to many student-related factors, Chad believed that as a good teacher he could overcome most of these. He reflected on the pedagogy he employed, asking himself, "What is it I can do that... helps them to connect" concepts, and asserting that "the pedagogy really matters." This sense of control contributed to Chad's strong sense of efficacy.

Melissa

Melissa focused on the negative aspects of the school's alternating day block schedule and large class sizes, both system factors out of her control. Another of her strong concerns was the fact that many of the students in Algebra I lacked the necessary prerequisite skills. Melissa attempted to take control by maintaining a highly structured classroom and, like Chad, she attributed much of her students' success to the teacher. Melissa modified instructional practices to accommodate her attributions for student success and failure, which in turn enhanced her teaching efficacy.

Ron

Ron, the only one who did not hold a master's degree, was quick to point out that he was not a "math geek," asserting that his ability lay in explaining mathematics to at-risk students. Most of Ron's attributions for failure were student factors, particularly lack of ability and effort. Ron sometimes used projects to engage students and encourage them to put forth more effort, thus asserting his control and enhancing his efficacy. Ron had the highest failure rate of all informants (57%); however since these same students were failing many other classes, he determined that the failure was not attributable to him. As he told me, "I do my thing and I try to do it the best I can. And, you know, I think that in a lot of ways, see, I really care about the lower kids...And I can handle them."

Conclusions

The teachers in this study believed mathematics is a set of facts, rules and skills to be learned and remembered, and the primary purpose of Algebra I is to teach students the skills and the organizational strategies needed for success in subsequent classes. Each teacher modified instructional practices in ways that were strongly linked to attributions for success or failure, to methods that enhanced personal teaching efficacy, and to strategies that had helped the teacher to learn.

This study has implications for teachers, educational administrators and policymakers, and teacher educators. Teachers need to hold and communicate high expectations for all, building understanding through tasks that require higher order thinking skills. Administrators and policymakers must clearly define expectations for algebra, and provide teachers with the necessary curricula, training and resources. Teacher education programs must provide an opportunity for prospective teachers to learn content from instructors who model Standards-based pedagogy, and lead them to reflect deeply on their experiences in order to effect a change in their beliefs and practices. Teacher educators also need to help teacher candidates develop new sources of efficacy based on their ability to help all students gain mathematical power.

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GEOMETRY

THE ROLE OF A DYNAMIC SOFTWARE PROGRAM FOR GEOMETRY IN HIGH SCHOOL STUDENTS' UNDERSTANDINGS OF GEOMETRIC TRANSFORMATIONS

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This study investigated the nature of students' understandings of geometric transformations, which included translations, reflections, rotations, and dilations, as they used the technological tool, *The Geometer's Sketchpad*, within a high school Honors Geometry class. The researcher identified four purposes for which students used dragging and measures and analyzed the ways in which students applied their mathematical understandings to interpret and utilize technological results.

Purposes

There has been a resounding call made by mathematics educators to incorporate the use of technology in high school mathematics classrooms (NCTM, 1989, 2000). In its most recent document, *Principles and Standards for School Mathematics*, NCTM 2000 states, "technology is essential in teaching and learning mathematics; it influences what is taught and enhances students' learning" (p. 24). Many believe the use of technology not only enhances students' learning but also provides students access to mathematical topics that were once reserved for college mathematics courses. One such topic is geometric transformations. The study of geometric transformations is an important topic to include in high school mathematics because it provides opportunities for students to think about important concepts (e.g., functions, symmetry), provides a context from which students can view mathematics as an interconnected discipline, and engages students in higher-level reasoning activities. The use of computer software, such as *The Geometer's Sketchpad* (GSP, version 3, Jackiw, 1991), offers students tools to explore geometrical relationships and may allow more opportunities for students to reflect on their activities. It is possible that the technology, because it makes visible students' actions and provides feedback that is based on geometrical relations and theorems, will facilitate students' development of deep understandings of geometric transformations. This study investigated the nature of students' understandings of geometric transformations, which included translations, reflections, rotations, and dilations, as they used the technological tool, *GSP* and were encouraged to create and explore conjectures and develop explanations. In particular, this paper addresses the following research question: In what ways does the technology support and constrain the understandings that high school students develop when learning geometric transformations with *GSP*?

Perspectives

With such strong encouragement for using technology in mathematics classrooms, it is important to examine how the use of technological tools supports or constrains students' development of mathematical understandings. The role of such tools in shaping students' understandings is often described in terms of the affordances offered by and constraints imposed by tools. The affordances of the tool, as described Wertsch (1998), refer to how tools can enable and empower actions or thoughts. One can also examine the ways in which tools may constrain the actions that are allowed and the understandings that develop. In thinking about *GSP* as a tool, one can study the ways it may support students' thinking about geometrical objects, properties, and relations because of its dynamic nature; this type of reasoning might be more powerful than simply thinking statically about the material object present on paper or on the computer screen. The former type of reasoning might be thought of in terms of reasoning about the figure whereas the latter type of reasoning might be thought of in terms of reasoning about the drawing (Laborde, 1993). Several researchers have described how different modalities inherent in a dynamic geometry tool, such as dragging, may offer affordances and impose constraints on students' reasoning about drawing versus figure. For example, Arzarello et al (1998) described three different dragging modalities exhibited by students: 1) wandering dragging, 2) dragging test, and 3) lieu muet dragging. Wandering dragging refers to dragging in a somewhat random fashion. The student may not have a particular goal in mind for the dragging activity. Dragging test refers to dragging to check the construction to determine if the construction resists dragging. If a geometrical object has been constructed that its properties should remain invariant under the drag mode. Lieu muet dragging refers to dragging in a way that preserves some regularities in the geometrical object. The researchers reported that analyzing these different types of dragging modalities provides insights into how students are switching between reasoning empirically and theoretically. This connection between empirical evidence and theory was also analyzed by Chazan (1993) with a focus on how students made use of measures. Chazan, who conducted research with students using a non-dynamic geometric construction program, found that students believed that empirical evidence such as measures, computer-generated drawings or pencil and paper drawings were sufficient to prove a statement true for all cases. Students believed that empirical evidence was proof. The case study analysis for the current study examined students' interpretation and use of technology including students' use of capacities of the dynamic tool, such as dragging and measures as they were learning about geometric transformations

Methods

The participants for the study were selected from a class of seventeen high school Honors Geometry students. Six of the students, who reflected a range of mathematical abilities and technological familiarity, were selected to serve as participants. The van

Hiele Geometry test (Usiskin, 1982) was administered at the beginning of the study and it revealed that the six selected participants were at or above level 3. The test was administered only at the beginning of the study to provide additional data for selecting participants. Case studies were conducted on four of the six participants, Jennifer, Kevin, Ben, and Sarah, for whom data seemed to hold the most promise in addressing the research questions.

During the 1999-2000 school year the researcher taught a seven-week unit on geometric transformations to all students in the Honors Geometry class. The instructional unit was based upon curriculum materials written by a team of mathematics educators at The Pennsylvania State University and University of Iowa through the CAS-Intensive Mathematics Project (Heid, Zbiek, Blume & Choate, 1999). The materials are written with the assumption that students have constant access to dynamic geometry tools and computer algebra systems.

Data Sources and Analysis

Major sources of data were verbatim transcripts of whole class discussions, the six participants working in small groups, task-based interviews, photocopies of written work, and field notes. Data were collected during 28 classes and during 3 task-based interviews conducted with each participant prior to, in the middle, and at the completion of the instructional unit. The activities in which students engaged while working on the computers during class and interviews were recorded directly using a ProScan converter connected from the computer to a videocassette-recording device.

Subsequent to collection of all data, the researcher engaged in an in-depth case analysis, which involved developing characterizations of individual students' understandings of geometric transformations and their use of technology. An analysis across individuals was also conducted to identify cross-cutting themes related to the research question. The case study method (Stake, 1995) and constant comparison method (Glaser & Strauss, 1967; Strauss & Corbin, 1990) were used. Results from the cross-case analysis will be presented.

Results

The ways in which the computer mediated students' understandings of geometric transformations was analyzed by focusing on students' purposes for using the technology, when they were allowed to make choices about its use, and determining how such uses contributed to or limited their problem solving efforts. Three themes related to the role of technology in students' understandings of geometric transformations are reported. They include students' understandings and use of dragging, students' uses and interpretations of measures, and students' uses and interpretations of dynamic sketches.

Students' Understandings and Uses of Dragging

The purposes for which students used the dragging modality were to test a construction by applying the drag test, to verify a conjecture, to observe behaviors of points under the drag mode, and to search for invariances. The purposes were sometimes explicitly stated by the student and sometimes inferred by the researcher from the actions and statements of the student. An example of students' use of the drag test will be presented.

What is dragging?

When one drags a point in a dynamic sketch he or she is, in essence, sampling points in the plane. From this perspective, a point is like a variable. As it is dragged in the plane it is assigned different coordinates. The label of the point is similar to the name of a variable. The name of the point does not have significance. Sarah's understandings of the effects of dragging on points appeared to be related to how she viewed the appearance of the point. Because the point retained its label and looked the same, she stated that she believed the point had not changed even though the coordinates of the point had changed.

During a whole class discussion that took place in class 9 the researcher asked students to think about what was happening when they were dragging various objects within a dynamic sketch. In particular, the researcher asked students to consider the reflection of points A, B, and C, over line DE. The image points, A', B', and C' were drawn on the board and the researcher asked the students to think about the reflection in terms of input, output, and parameter. Students were asked to consider what was being varied (input, output, or parameter) if line DE was dragged. Because the labels of the two points on the line (D and E) remained the same, students often thought they were considering the same object and had a difficult time thinking about moving the line of reflection as varying the parameter. Later in this same class, students were asked to describe if dragging a pre-image point was most similar to changing the input, output, or parameter. Sarah's partner said that she was unsure how dragging this point allowed her to examine different input output pairs.

Sarah's partner: Well, how do you change it though? Because all you're doing is moving it.

Researcher: Right. So, by moving...

Sarah: But that means it's in a different place than it originally was. You didn't like change the point name or anything. [Class 9, 302-308]

The fact that the name of the point remained the same seemed to suggest to Sarah and her partner that it was, in fact, the same point. They seemed focus on the material representation of the point (the drawing) rather than reasoning about the theoretical definition of point. The students seemed to believe that simply changing the point's location did not cause it to vary. However, when the researcher suggested students consider the coordinates of the points, something that would vary as the point was dragged, they seemed to begin reasoning about dragging a point as varying the input (or output). The use of coordinates seemed to support their reasoning about points as variables. Researchers have reported that the concept of variable is difficult for algebra students to understand (Boers-van Oosterum, 1990). It is not surprising that Sarah and her partner had difficulty attributing characteristics of algebraic variables to geometrical points.

The Use of the Drag Test

Although all four students were at first hesitant about dragging, during the course of the study they developed different purposes for dragging. One purpose involved dragging to check whether the construction they created maintained the properties of the figure they were constructing. Hoyles and Noss (1994) introduced this type of drag test to students to determine whether they had created a figure that is messable (does not maintain its properties when dragged). All four students in the current study used the drag mode for this purpose. However, the way in which students interpreted the results of the drag test varied. Some students used the drag test to determine which properties needed to be included in their sketch and used this information to modify their construction. Other students simply described the results of the drag test (e.g., "it is no longer a rectangle") without focusing on properties and experienced more difficulties in creating a correctly constructed figure. An example from the second interview will be used to contrast the two different interpretations that were made from the drag test. During the second interview, the researcher asked students to construct a rectangle using at least one reflection and other construction menu commands as needed. An example of the first interpretation of the drag test occurred with Kevin. He drew (rather than constructed) a vertical line CD and a vertical segment AB. He reflected the segment over the line which he marked as a mirror line and connected pre-image/image points as shown in Figure 1.

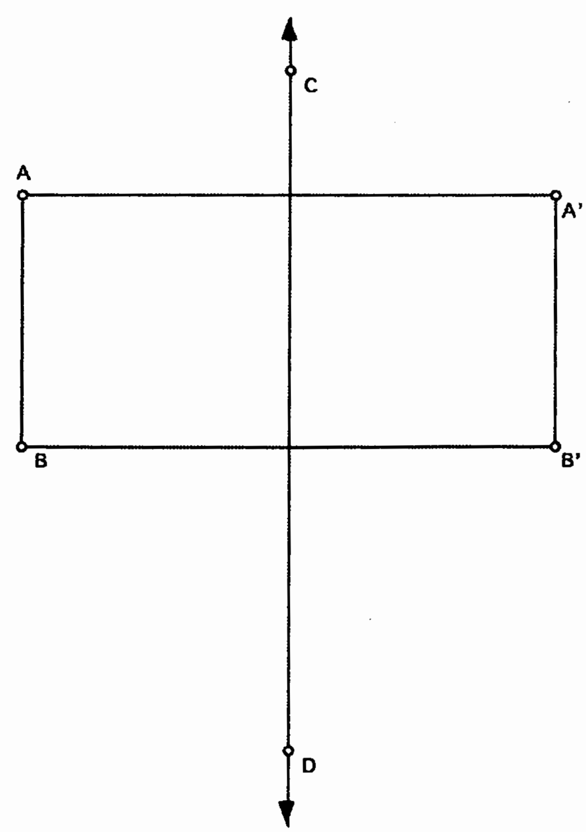

Figure 1. The "rectangle" Kevin produced during the second interview.

Kevin dragged one of the endpoints and noted that because the object was no longer a rectangle his construction method would not work. Kevin explained that in order for the construction to work, segment AB needed to always remain parallel to the line of reflection (line CD). The fact that the drag test failed seemed to prompt Kevin to step back and think about the properties of a rectangle. After he considered what was needed he explained the condition that the segment be parallel to the line of reflection was needed. This is contrasted with Sarah who constructed a rectangle using the same method. However, when she applied the drag test, she explained that her construction did not work because the rectangle became a trapezoid. Rather than focus on the properties that were needed to construct a rectangle (opposite parallel sides), as Kevin did, Sarah simply described what happened in her drawing when she dragged. Students' interpretations and uses of the dragging feature of the technology seemed to be related to the extent to which they made use of their mathematical understandings as a filter through which the irrelevant aspects of the sketches could be removed allowing them to focus on the more important features. The use of mathematical understandings was also critical to their use and interpretations of measures.

Students' Uses and Interpretations of Measures

GSP allows students to measure geometric objects (e.g., line segments, angles) and updates those measures dynamically as objects are changed by dragging. The purposes for which students used measures included exploring relationships, creating and verifying conjectures, and checking the correctness of a construction. An example that illustrates the first three purposes and another example to illustrate the last purpose will be provided.

Use of Measures to Explore Relationships, Create and Verify Conjectures

An example of different ways in which students used measures to explore relationships within a sketch and to create conjectures occurred when students were asked to determine to what single transformation the composition of two given translations is equivalent (See Figure 2.). Jennifer and Ben created triangle ABC, translated it by vector DE (which Ben had drawn using the ray tool) and then translated its image by vector FG (Jennifer suggested to Ben that he draw this using the segment tool). They conjectured that the composition of two translations was equivalent to a single translation using vector AA''. The researcher challenged them to determine how vector AA" was related to the original two vectors. To investigate this, Ben seemed focused on finding a relationship among measures without focusing on what the measures represented and whether they made sense in terms of the relationship they were exploring. He measured the magnitudes and slopes of the vectors and after determining that it was not possible to write one as a linear combination of the other two he began to look for other objects to measure.

Ben: Could the distance between, here and right here, could (they) be equal [Ben drew segments AF and AD to measure the distance from A to F and A to D]? Could they be equal maybe?

Jennifer: I'm not sure of that. See, I'm not sure that that would matter. Hmm. And it's not like it matters where these [vectors] are. So that distance [A to F and A to D] shouldn't matter. You can get rid of those. [Ben deleted segments AF and AD] Okay, well, let's figure out what counts here. [Class 12, 333-340]

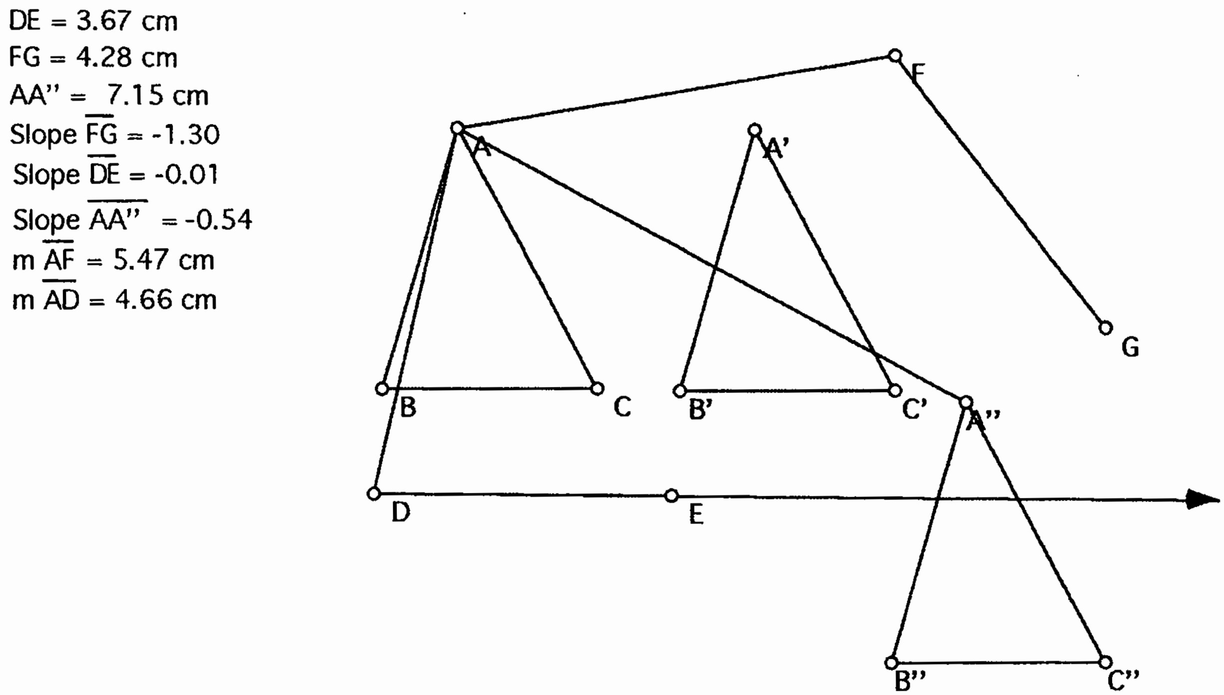

Figure 2. The dynamic sketch Ben and Jennifer created while exploring the composition of translations

Jennifer suggested to Ben that he drag segment FG so that point F coincided with point E. She then directed Ben to create segment DG and measure its slope. She immediately noticed that its slope was the same as the slope of segment AA". Although these students had not been taught how to add two vectors, Jennifer was able to create this on her own by thinking about the composition of two translations. Perhaps, because Jennifer was able to step away from the problem and think about what information was relevant she was able to notice this relationship among the vectors. Her focus on the relevant feathers in the sketch and her purposeful use of measures and dragging which was guided by her understandings of transformations allowed her to create and test her conjecture about the composition of two translations. However, the use of measures and Ben's focus on the quantities rather than the objects to which they referred seemed to constrain his thinking about the sum of two vectors.

Use of Measures to Test the Correctness of a Construction

Several students used measures to test whether or not their construction, which they often created by reasoning about properties, was correct. To illustrate this approach an example from Jennifer's work will be provided. Jennifer was asked by the researcher to explain how she could construct a square using rotations on GSP, during the second interview. Jennifer explained that she would start with segment AB mark A as the center and rotate point B 90 degrees about A. She explained that then she would mark B as the center of rotation and she would rotate point A 270 degrees about point B. She said then she would "connect the dots" and the resulting figure would be a square. She explained that all the angles would measure 90 and all the sides would have equal lengths because a rotation preserves the lengths of segments. However, after she executed this construction she measured the sides to make sure the lengths were equal, as shown in Figure 3. She then explained that if she measured opposite angles and they were both 90 then the other two angles would also measure 90.

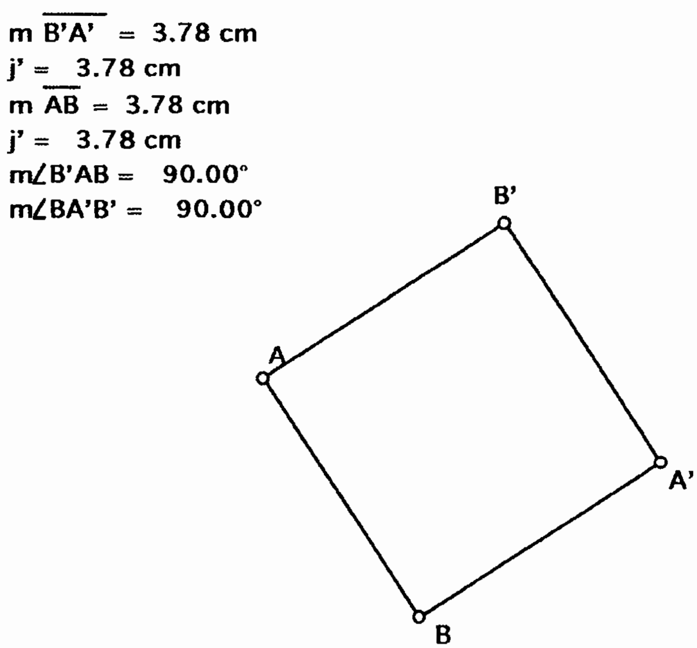

Figure 3. The square Jennifer constructed using *Geometer's Sketchpad*

Jennifer's decision to measure only two angles seemed to be based on her assumption that the quadrilateral was a rhombus because the measures she found were all equal. She seemed to take the empirical evidence as given information from which she deduced what other measures would be sufficient to gather. There seemed to be an interplay between her reasoning from empirical results and reasoning about properties of squares. Her reasoning was not deductive and not solely empirical. She seemed to employ a type of pseudo-deductive reasoning.

Jennifer appeared to use measurements to check the construction she created even though she explained why the construction should result in a square based on the properties of rotations. The measures might have been used to provide further corroborating evidence to assure her that she had constructed a square. In this way their availability within the tool seemed to support her problem solving. The measures also might have been taken to illustrate to the researcher that she had constructed a square.

Interpretations and Uses of Dynamic Sketches

Students developed a variety of ways of interpreting and using dynamic sketches. One student treated the dynamic environment as an electronic drawing board (e.g., MacDraw). All four initially interpreted the dynamic sketches statically and all four used strategies that were described by the researcher as reactive or proactive. Whether students were using premade sketches or sketches from "scratch" did not seem to influence their use of reactive or proactive strategy. Rather, their focus on figure or drawing seemed to be connected to the type of strategy employed.

Reactive and Proactive Strategies

The analysis revealed that students appeared to employ two different types of strategies when working with the computer: reactive and proactive. Reactive strategies were strategies that involved performing an action and then determining the next action based on what appeared on the computer screen. The ways in which the student determined the next action was not based on their understandings of mathematics but rather the next action seemed to be determined by using a trial and error approach. Reactive strategies seemed to be based in reasoning about the visual appearances offered by the computer (drawings). Both Kevin and Sarah, when asked to construct an equilateral triangle using rotations appeared to reason from the visual feedback that was generated by the computer. Their approach to these problems seemed to be characterized as a process that involved marking a center, rotating an object through some number of degrees, and then viewing the screen to determine if the angle measure or center needed to be adjusted. Rather than reason about the properties of the object they were trying to construct and relations to the rotation they were applying they seemed to employ a guess and test strategy.

Proactive strategies were characterized by a student having a plan in mind about how they were going to use the computer and were guided by students' understandings of geometrical properties and relations (figures). For example, the researcher presented Jennifer with a pre-constructed sketch (See Figure 4.) and told Jennifer that a single transformation was applied to all points in the plane.

Jennifer was asked to determine what transformation was applied. To do so, Jennifer carefully dragged one of the two corresponding pre-image/image points until each pair coincided (See Figure 5.).

She used the pattern of the coinciding pre-image/image pairs to create a line segment that she said represented the line of reflection. She stated that the transformation that was applied was a reflection. Jennifer's understandings about fixed points appeared to guide her dragging activities that were afforded to her by the technology.

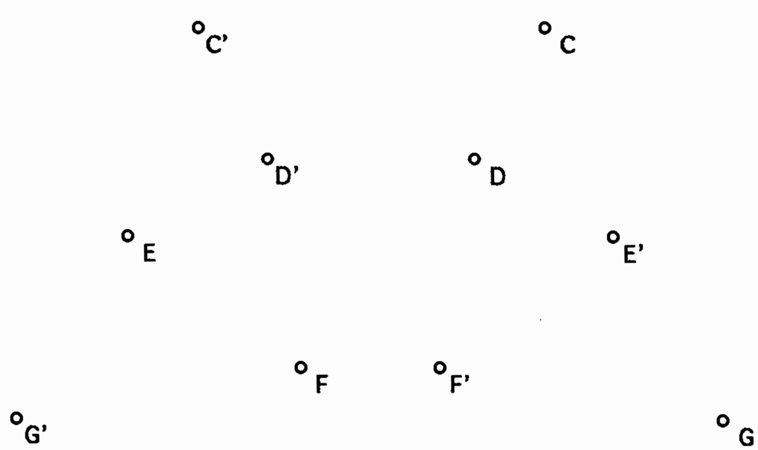

Figure 4. A picture of the matching task.

Discussion

A characteristic of students' reasoning and problem-solving activities as they came to understand transfor-

Figure 5. The diagram after Jennifer dragged pre-image (image) points until they coincided with their corresponding image point (pre-image)

mations in a technological environment included the ways in which they used different feature of the technology including the use of dragging and measures. Students' uses of these features were related to and influenced by their mathematical understandings. Their understandings were also influenced by students' engagement in anticipating the results of their actions and reflecting on what was produced by those actions using their mathematical understandings. The lack of anticipation or reflection was demonstrated by students who often relied on visual information to provide cues that allowed them to determine their next action. The availability of this visual information offered by the technology seemed to constrain students' reasoning about transformations; students tended to reason about the material drawing rather than the geometrical properties. On the other hand, students who engaged in anticipation and reflection and drew upon their mathematical understandings to interpret visual information generated by the computer seemed to reason about the figure. The technology seemed to support students' developing understandings of transformations by allowing students to use proactive strategies employing measures and dragging. Many researchers, teachers, and reform documents suggest that the use of technology promotes students' understandings of mathematics and recommend that technology be incorporated into the teaching and learning of mathematics.

Relatively little is known about how students use technology and what understandings they develop when it is used to learn new mathematical concepts. This study provides evidence to support that technology can promote students' understandings of mathematics and it provides insights into the ways in which students interpreted technological results and used technology as they were learning. There is still much more that needs to be understood about the complexities that are involved when technology is used by teachers and by students in the learning and teaching of mathematics.

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COORDINATING MEDIATION OF ACTIVITY IN THE LEARNING OF GEOMETRICAL TRANSFORMATIONS

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This is an attempt to contribute in a research perspective related to the use of computerized scenarios and the introduction of historical contexts (Bartolini, 1998; Bartolini & Boero, 1998; Boero, Pedemonte & Robotti, 1997) in the search of significant tools (Confrey, 1993) for mathematical learning. Particularly we are presenting evidence, referring to the manner in which students acquired coordinating experiences, obtained through activities in connection with the scenarios and some set up tasks. Collective and group activities designed for eighteen students aged fourteen carried out within a teaching experiment based on the implementation of two learning scenarios focused on elementary geometrical transformations. In these scenarios, activity mediation through cultural devices became useful to promote intuition and objectification of geometrical properties in play; both processes become essential for developing discourse from a communicational approach to mathematical learning (Sfard, 2001).

Theoretical Framework

From an educational perspective derived from the psychological school of Vygotsky, a great number of arguments have resulted (Bartolini & Boero, 1998; Boero et al., 1997; Mariotti et al., 1997) in favor of the introduction or the use of certain kind of mechanisms to increase the understanding of mathematical notions and ideas in the classroom, as the manipulation of large pantographs that permit the tracing of drawings or exact designs by the students.

Besides, microworlds and computer science have allowed to glimpse improvement in diverse educational problems, like succeeding in the control of the execution of solving algorithmic procedures, or the students' possibility to assess used solving methods (Hoyos, Capponi & Geneves, 1998; Hoyos, 1999); or "real" implementation of novel work proposals (Jörgensen, 1999).

The fundamental role of the tools as mediation for educational purposes can be recovered to a large extent by the socio-cultural perspective of the learning research. In fact, one of the basis of the investigation framework from these standpoint, consists of the assumption that notion and mediation are inherently related (Werstch, Del Rio & Alvarez, 1995, p. 19), so that:

> "Experience does not go on simple inside a person... in a word we live from birth to death in a world of persons and things which is in a large measure what it is because what has been done and transmitted from previous human activities. When this fact is ignored, experience is treated as if it were something, which goes on exclusively inside an individual body and mind. It ought not to be necessary to say that experience does not occur in a vacuum. There are sources

outside an individual which give rise to experience". (John Dewey, 1938/1933, p. 39, quoted in Werstch et al., 1995, p.190)

Finally, in the implementation of the learning experiment that is presented in this paper, the relationship was basically aimed –in the context of tasks designed for that effect, within the manipulation of significant cultural artifacts and the promotion of mathematical speech in the classroom. According to Sfard (2001), "communication should be viewed not as a mere aid to thinking, but as almost tantamount to the thinking itself" (Sfard, 2001, p.23).

So, a mathematical learning investigation based on communication taking place in the classroom, should investigate the development of a mathematical discourse focusing three main aspects: a) Vocabulary used by students during work sessions; b) Mediation artifacts (even though Sfard only mentions explicitly visual mediation); c) Meta-discursive rules. (Sfard, 2001, p. 28).

Aims, Methodology and Results

In this work, it is interesting to point out the way in which students modify their discursive actions. In order to do that, we have observed how pupils approach new mathematical terms, using scripts for the activity, the manipulation (and support) of certain cultural artifacts, and describing their accomplished requested tasks. Particularly, it was interesting to analyze the different descriptions that are carried out at the end of the performed activity. From our point of view, these descriptions reflect how pupils coordinate their obtained experiences.

With this purpose in mind we set up two learning scenarios, which involved, in the first one, an introductory exploration of the geometrical transformation topic using the menu comprised in Cabri II.

The second scenario consisted basically in the manipulation of large pantographs, which allowed students to trace drawings guided by specific geometrical configurations underlying in the articulation machines.

In total, 10 practical, 50 minute long work sessions were carried out in April 2001. In them participated 18 students, 14 years old, attending junior high school (9th level).

Actually, in the first scenario, the script used in order to introduce the topic of geometrical transformations in Cabri-II, allow students to a first encounter with the terms in use. It is important to mention here that the "*Help*" command in Cabri-II was activated at all times. When the *Help* command is activated, Cabri-II shows on the screen a scientific description of all necessary requirements involved in a certain application or construction, and the order that some choices or constructions might be done.

It was extremely important to provide students with scripts that indicated to appeal to the *Help* command so, they would be able to access to a first perception of the objects and relationships involved in a geometrical transformation, to a visual configuration of properties and relations between them.

Templates and Mediation Artifacts

Next, we will show the first prototypical student's answer. It was obtained when a description of the geometrical configuration was required and resulted from activating the Cabri-II *Dilation* command:

> Abraham: Let's say if we are reflecting... Well, we are drawing a line going through this point [the dilation centre point], it is going to pass through the dilated, by the reflected figure. As it is divided into pieces [i.e. tracing the segments going from the dilation centre point to a point P on an initial object, and from the point P to point P', P' the dilated point, or image from P] then, the traced segments are going to be on a straight line.

We can observe that even as Abraham is using the mathematical terms in a very diffuse manner, it is possible to show up what was used when answering –because he is pointing at the figure produced. So, to understand that Abraham is expressing the alignment property of dilation between corresponding points P and P'.

A first analysis carried out at students of standard level performance tells us the following. In general, even the students after working within Cabri-II could visualize some relationship and properties, these were enunciated in a cut way, formulated without a description of the situation between the designed figure in question and its image under the dilation.

Similarly, relating to the other first requested student's answers, they were very vague too. For example, when they were asked to observe the effect of moving any of the points P in initial triangle and describe what was happening to the P' point on the reflection image. Most of the students could not determine or characterize in any way (by the orientation, for example) the different trajectories perceived. In general, the answers were of the type "a point moves with the other". However, that answer does not characterize any of the transformations because it is valid for all of them.

What actually happens, for example, comparing the reflection and translation cases, is that in a triangle drawn figure we set a point P and, when we move it clockwise, then, in the reflected figure the movement of the P' point is against-clockwise. While the effect of moving P under the translation geometrical transformation is always in the same orientation. (See Fig. 1, and Fig. 2 at the end of this paper.)

In fact, the students "*have just entered the template-driven phase*" (Sfard, 2001, p.33). They were trying a new way of speech about certain concepts or notions, using words and phrases supplied by the script and through the Cabri-II *Help* command. They were trying their first use of a template at hand, and their later elaboration of some phrases was, in fact, restraint.

It is worth to mentioning here, one hypothesis working there. It has been applied in both learning scenarios –and in the type of didactic situations instrumented there. The hypothesis is that a crucial point in order to produce learning results it is constituted by enunciation of the underlying mathematical properties in the drawing (Laborde, 1993). Such enunciation could be established through arithmetical relations between drawing's properties, or textual expression of general properties noticed in the tracings –question facilitated by the software dynamism (in case of computer support), or from geometrical configurations of the material support provided to students; or finding algebraic terms or equations.

In general, any of the required enunciation will depend on the background and academic level of the student, and on the comprehension of the properties underlying the drawing carried out.

In reference to the second scenario, pupils were required to manipulate some pantographs brought into the classroom in order to make students aware of geometrical properties involved in their drawing. In this scenario, the whole script used consisted only in general instructions to explore and describe the perceived relationships or geometrical properties between tracings carried out.

Students were asked to draw a figure manually, guiding one end of the pantograph. Therefore, in this way, they simultaneously obtained two figures, one draw by themselves and another one made by the machine articulations. Next, they were asked to describe the relationship between the object drawn by them and the reproduction effected by the machine. They were also asked to express what was the transformation implied.

Next, we will show the type of descriptions accomplished by the students at the end of the work sequence in the second scenario –following the manipulation of the pantographs, when they were asked to describe what they had just obtained by the accomplishment of the given task.

Interviewer (I): What did you do?

Dulce: This is the original figure, this point is the one that traces the original figure [being point A, the one that is nearer to the dilation centre point O], this is the resulting figure; these figures…

Interviewer: Ok. And how did you draw them? Show me, [Coral moves a ruler, bringing it near the drawn figure and point A]

Interviewer: No, but how did you draw it with the machine?

Dulce: [Dulce moves the machine and explains] Following this point [A], for example, drawing, and this one follows [point B]

Coral: And it forms a bigger figure.

Interviewer: Ok.

…

Interviewer: Ok, all right. But, could we know, let's say, what I'm looking at here is that this figure is smaller than the other one, isn't? But could we know how small is it? Let's say it is what part? Is it half, or one fourth, or what part is it?

Dulce:	(Dulce measures the sides of the drawn squares with a ruler) Half.
Interviewer:	It's half. How do you know?
Coral:	It's the fourth part.
Dulce:	Because we measured it.
Interviewer:	Because you measured... And then what?
Dulce:	This one is 3.5 and this one is 7 [Dulce is placing the ruler along the sides of the drawn squares].
...	
Interviewer:	The side, the side measures half.
Coral:	It measures half; and on this side [on one of the perpendicular sides of the resulting square or image], again it measures half.
Interviewer:	And then, I would have...
Dulce:	The figure
Coral:	The figure is formed, and it is about the fourth part of the original square...Of the resulting square [she is referring to the dilated square].

As in the case of Abraham the students Coral and Dulce visualized the linear relationship existing between O (the dilation centre point), P and P' (initial and dilated point, respectively). However, in this case we can observe a great improvement in the mathematical use of terms. Because the students perceive a relationship between both figures, the initial and the dilated (remember that Coral almost immediately said "And it forms a bigger figure"), and they arrive to establish a numerical relationship between the measurements and the area of the drawn figures.

It is probable that these students were near to establishing all the numerical interrelationship of proportionality implicit in the drawn figures.

In doing that, dilation might become represented by certain new mathematical terms, those numerical interrelationships emerged from their activities.

Practical Working Sessions at School

The teaching of school mathematics in junior high school in Mexico has, among its aims, more than learning rules and procedures, to promote understanding and problem solving. In the same way, intends that classroom activity let to emerge conjectural processes, communication and validating (cf. Alarcón, et al., 1994; National Council of Teachers of Mathematics [NCTM], 2000).

While from the declarative point of view, validating knowledge or negotiation of meaning has always had a privileged place within the framework in the constructivist knowledge. In general, within the scholar practice in Mexico this has been limited to a formal institutionalized knowledge performed mainly by the teacher.

Besides, that most teaching experiments have been pursued is to report mathematical subjectivities or conceptualizations reached through small groups, or pupil-teacher interaction.

However, a collective negotiation of such intersubjectivities needs to plan a long-term individual and collective learning. It is also needed, a collaborative and supportive substructure of research which is more complex than the one centreed on individual or in small groups learning.

Little or none are the efforts in our system aimed to set up a collective negotiation of meaning. Nonetheless, the validation or such collective negotiation postulates are the cornerstone of the current curriculum.

Now days, in the context of the everyday educational practice, Mexican pupils attend, five days a week, 50 minutes daily, to mathematics lessons. These courses are usually taken in the same classroom, with the same teacher and, with no other tools but pencil and paper.

Thus, one of the goals of the project on which this paper is based (CONACYT, # 30430-S) was to look for grounds favoring to alternate practical and normal lessons for the quality improvement in the learning of mathematics in Mexico.

In particular, the empirical results presented in this paper hold that practical working sessions similarly to those presented here must take place at school everyday, as it is noticeable the behavior of the girls who participated in the experiment on equal terms as boys. Remember that the performance of the pupils Coral and Dulce is prototypical of the standard reached in class. (See Fig. 3 at the end of this paper).

> Regarding the participation on equal terms Szendrei (1996) says that the role of the use of concrete materials (common tools and educative materials) by pupils of both gender is that of facilitate equal development of mathematical skills in both, boys and girls of the same age at school. Skills that are not equally well developed through out-of-class experience (Szendrei, 1996, p.427).

Supporting Szendrei's view, there are careful analysis of the data collected during the Second International Mathematics Study (SIMS), carried out by the International Association for the Evaluation of Educational Achievement between 1982-83. These analysis bring out that there are not important sex differences regarding the mathematical advances among 13 year olds:

> "... It appears that girls and boys assimilate the subject matter taught in class equally well in Arithmetic, Algebra and Statistics. Thus it seems reasonable to hypothesize that sex differences at that age, when they exist, are due to out-of-class experiences and psychosocial processes rather than to biological differences". (Hanna, 1989, quoted by Szendrei, 1996, p.427)

As a matter of fact, in various societies girls (Szendrei, 1996, p.429) are forced to play only girls games. Therefore, activities in the mathematics lesson using concrete materials contribute to give both girls and boys an equal chance for development.

It is also important to point out again, the fact that in order to carry out practical working sessions on mathematics at school, different concrete materials, as well as new technologies, like computers, internet access, calculators, etc. must be used. In the same way, it is of great importance to have scripts and or working prototype sequences for pupils, similar to those here mentioned, which should be prepared to accomplish complex tasks on mathematics and to be used as templates for developing discourse in the classroom.

Summarizing, a didactical hypothesis emerged from this work is that it is probable that the learning scenarios based on the mentioned elements will allow the advance and set up of mathematical communication within the classroom. These principal elements were the following: (a) The use of scripts –as templates, to guide activities; (b) [In order to] Manipulate cultural materials or artifacts (concrete and virtual); (c) Working on mathematical complex domains.

Finally, by means of this teaching experiment, it was possible to insert in school, in a successful way both previously mentioned learning scenarios during a normal mathematics course in a government school. These grounds the fact that practical lessons conformed as scenarios mentioned here might be very welcome for school actors. Such practical lessons would be complementary to the traditional mathematical courses in our country, which are frequently the only ones carried out all the way in our educational system.

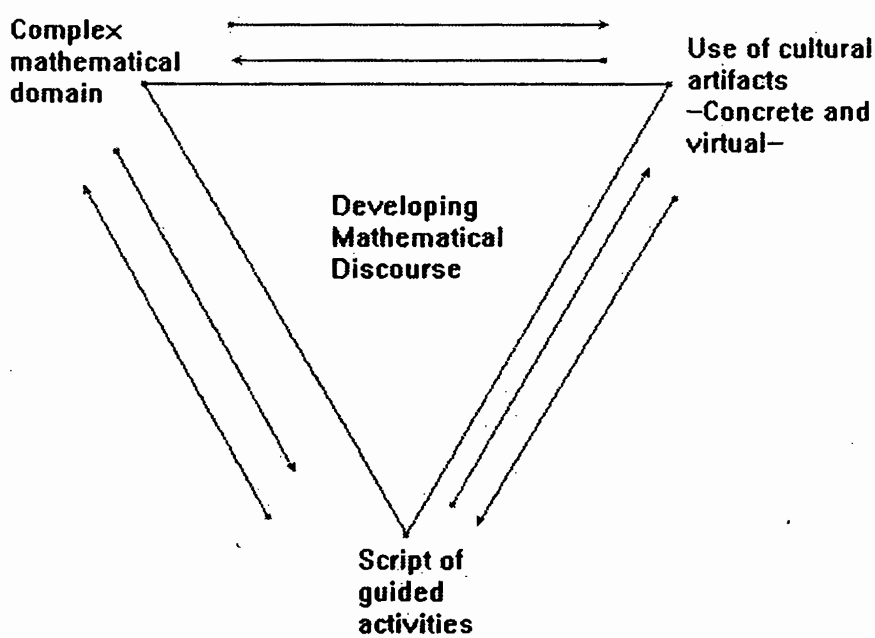

Figure 1. Didactical elements to promote mathematical discourse within the classroom.

Conclusions

The discourse or utilization forms of mathematical terms in use by Abraham allowed taking on account an intuition process related to mathematical notions and properties in use. Such intuition was characterized by a diffused or weak use of new terms, which in our point of view refers to one of the characteristics of phase one of the developing process of discourse in the classroom (Sfard, 2001, p.32). This characteristic is that of the use of signifiers by templates, in this case those provided by the scenario script.

As a complement, the derived objectification from the manipulation of articulated machines (or pantographs) conducted to a greater comprehension relating to a mathematical use of the terms. The evidences of the establishment of a numerical relation between the drawn figures might be given account of how the terms or symbols in use would become appropriate representations of drawing.

Finally, we only want to underline the didactical possibilities emerged from these empirical findings, as a elemental base for generating mathematical thinking and discourse. That is to say, it is probable that a base to a suitable mathematical meaning of a topic in question could be to establish trough a sequence of actions consisting of the following steps: a) To start with the emergence of intuitions by means of activities within the dynamical software. b) To continue to the objectification of the geometrical properties at stake, by means of activity through cultural artifacts. c) To communicate (with the support offered by the software dynamism and the use of new technologies) strategies and solutions to possible proposed problems.

Acknowledgements

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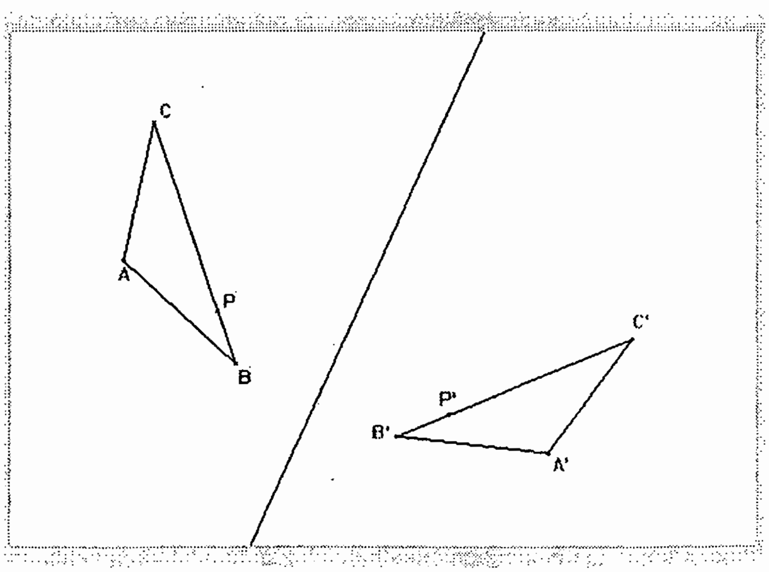

Figure 2. A prototypical situation of reflection within Cabri-II

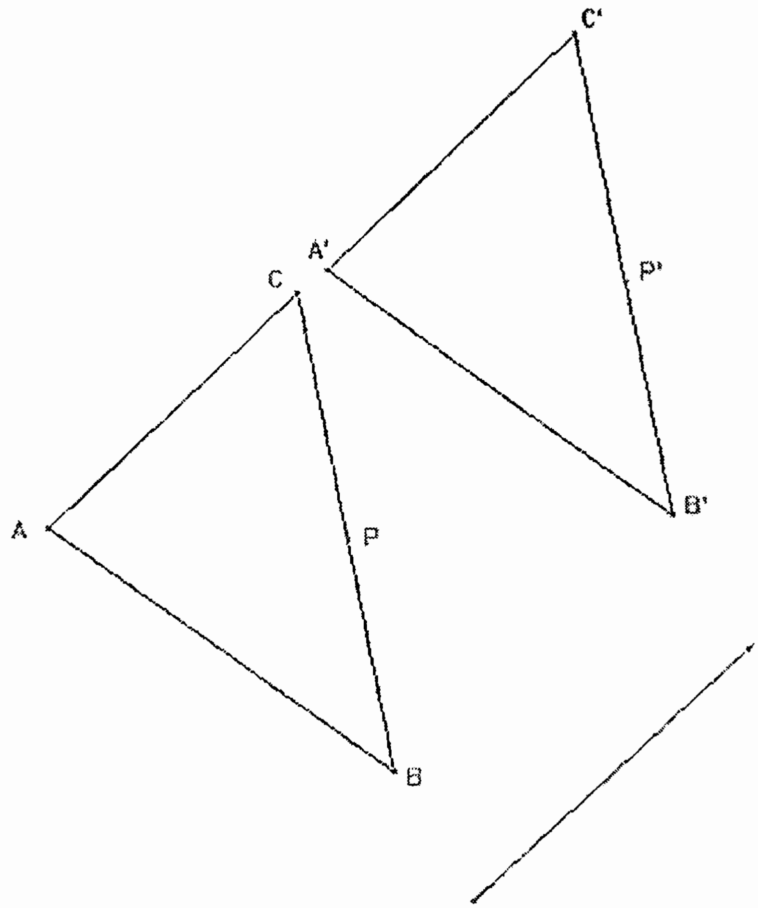

Figure 3. A prototypical situation of translation within Cabri-II

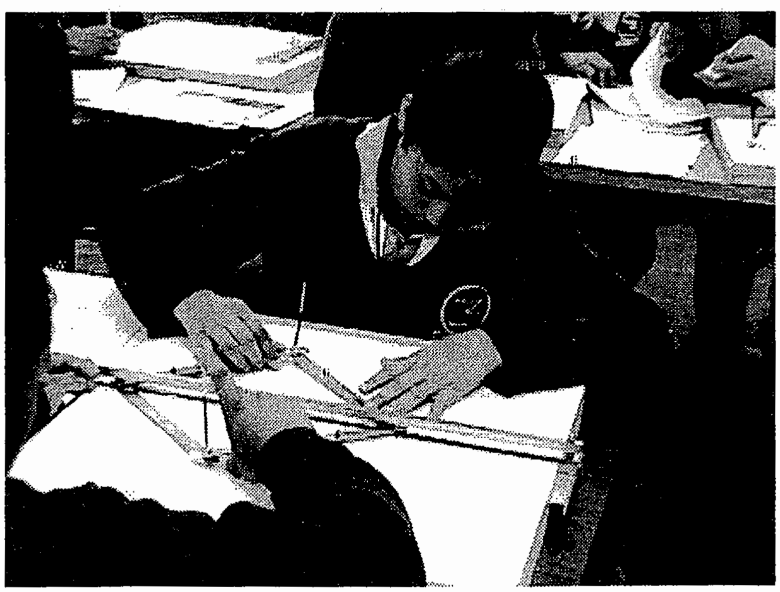

Figure 4. Dulce and Coral are manipulating the pantograph of geometrical reflection

DEVELOPING PRESERVICE TEACHERS' MATHEMATICAL REASONING AND PROOF ABILITIES IN THE GEOMETER'S SKETCHPAD ENVIRONMENT

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This study investigated the actual learning processes of two preservice teachers as they explored geometry problems with dynamic geometry software and the effects of using the software on developing their mathematical reasoning and proof abilities. A constructivist teaching experiment was designed and implemented for the investigation. Through participating in the study, the subjects developed a new learning style—exploring problem situations through a learning process characterized by initial conjecture – investigation – more thoughtful conjecture – verification (or proof) – proof (or verification). They changed their conceptions of mathematics and mathematics teaching as well. The Geometer's Sketchpad (a dynamic geometry software package) was found to be an excellent teaching and learning tool that can enhance students' mathematical reasoning and proof abilities.

A research study was conducted in the Fall semester 2001 within the course "Learning Mathematics with Technology" (MAE 3651) for secondary school mathematics preservice teachers at Florida International University. The goal of the course in which the subjects were enrolled during this study is to help the preservice teachers experience learning mathematics with technology, determine important content areas for the secondary school mathematics curriculum and develop competency in these content areas. Using dynamic geometry software to revisit high school geometry is a major focus of the course.

The Purpose of the Research

The purpose of the research was to investigate the actual learning processes of the preservice teachers as they explored geometry problems with dynamic geometry software and to find the effects of the use of dynamic geometry software on developing their mathematical reasoning and proof abilities.

Conceptual Framework

The conceptual framework of this study came out of the constructivist perspective that knowledge is not passively received from the teacher but actively constructed by the learners themselves, the van Hiele model of geometric thinking, and the research on mathematical reasoning and proof. According to van Hiele (1986), students progress in geometric thinking through a taxonomy of levels, and progress from one level to the next higher level is dependent on the nature of the instruction provided to students. The van Hiele levels are numbered differently in various sources. Based on the numbering system used by Battista & Clements (1995), the levels are: Level 1 - visual, Level 2 - descriptive / analytic, Level 3 - abstract / relational, Level 4 - formal deduction, and Level 5 – rigor / metamathematical. de Villiers (1987) suggests that deductive reasoning in geometry first occurs at level 3 when the network of logical relations between properties of concepts is established. He claims that because students at levels 1 or 2 do not doubt the validity of their empirical observations, formal proof is meaningless to them - they see it as justifying the obvious (Battista & Clements, 1995). Other researchers (Senk, 1987; Van Dormolen, 1977) give similar suggestions.

Design and Procedure

A constructivist teaching experiment was designed and implemented for the investigation. During the teaching experiment, I attempted to provide opportunities for the students (preservice teachers, the same hereafter with only a few exceptions) to learn by judiciously selecting tasks, posing questions, and encouraging active explorations. The study lasted for ten weeks when the course concentrated on geometry topics using the Geometer's Sketchpad (GSP), a dynamic geometry software package. Since the students' detailed changes and growth in their learning processes were the focus of the investigation, two students were selected from the MAE 3651 class to participate in the study. They did not join the big class during the ten weeks except for the first class session in which a pre-test was administered. Teaching interviews (two times a week with 75 minutes each time) were conducted for them, sometimes individually, and sometimes in a pair (for cooperative learning purposes). The time for the subjects to learn geometry and the problems for them to explore were basically the same as those for the big class. The difference between the two settings was that there was much more one-on-one interaction between me as the instructor and the subjects in the teaching interviews. The two subjects were selected based on both their willingness of participation and their different thinking levels in the van Hiele model.

The study consisted of three phases. Phase 1 was the pre-test. Phase 2 consisted of nine-week teaching interviews. Phase 3 was the post-test.

The Pre-test

For the pre-test, I used 30 questions - 20 questions selected from Mayberry's (1981) instrument and 10 questions that I had devised - to assess the student's van Hiele levels of geometric thinking. All questions dealt with triangles and quadrilaterals. The 10 questions that I had devised looked more on whether a student can do simple proofs. The 10 questions and the criteria for scoring were developed with the same structure as that of Mayberry's instrument. From among the students who were interested in participating in the study, a male student (Fred) and a female student (Lisa) were chosen. It was found that Lisa was at level 3 and Fred was at level 2 in the van Hiele model.

The Teaching Interview

The teaching interviews for each of the two subjects explored the detailed processes in which the students' problem solving, mathematical reasoning, and proof abilities were enhanced. The subjects worked on the assigned activities using GSP as a tool. I interacted with them by probing with questions, and giving hints only when they needed help.

The problem situations used for the interviews were designed based on the following considerations: 1) The tasks assigned to the subjects should "focus on important mathematics" (NCTM, 2000, p.15); 2) These tasks should be both challenging and consistent with the subjects' van Hiele levels of geometric thinking; 3) The sequence of tasks should help the subjects gradually progress from lower levels of geometric thinking to higher levels; 4) The tasks should emphasize the subjects' active hands-on activities with the dynamic geometry software to assist their mathematical thinking; and 5) The tasks should allow the subjects to generate feedback from which they can judge the efficacy of their methods of thinking. Nine sets of problems were designed and assigned to the subjects during the interview sessions. Solving these problems required the grasp of important concepts and relationships related to the following three topics: Triangles, Quadrilaterals/Polygons, and Circles. Since a proof-oriented geometry course requires thinking at least at level 3 in the van Hiele hierarchy (Battista & Clements, 1995), the problems (especially the problems at the early stage of the interviews) given to Fred didn't require formal proofs but informal arguments, while all problems given to Lisa required both intensive investigations and formal proofs. The problems were sequenced so that the level of sophistication and difficulty of problem contexts increased gradually. Taking full advantage of the dynamic, multiple, linked representations given by GSP was encouraged in explorations of the problems.

The Post-test

At the end of the ten weeks, the subjects took a written examination (with the other students in class) and an additional post-test. The post-test given to Fred was the same as the pre-test except for three items, which were new activities for him and went beyond any of the tasks he encountered during the interview sessions. The post-test given to Lisa consisted of 10 proof-oriented questions or problems.

Data Collection and Analysis

Each teaching interview was videotaped. All significant parts of the videotapes were transcribed. Careful notes were taken for all interview sessions and related after-session discussions. The answer sheets of the pre-test and the post-test, the computer files on the disks, as well as the written work (assignments and tests/exams) and comments completed by the students in each interview session, were also collected for analysis

A constant comparison approach (Glaser & Strauss, 1967) was used in this study for the data analysis. This method of analysis is inductive - it moves from data to tentative theory, to new data, to refined theory. The transcripts of the videotapes and other data were analyzed during interview sessions and after all sessions were completed. These analyses took the form of editing data with commentaries. The commentaries depicted the story lines of the corresponding students' progression through the sessions and revealed the critical developmental points of the students.

Results

A New Learning Style

In the teaching interviews, the subjects were encouraged to explore each problem situation through a learning process characterized by initial conjecture - investigation – more thoughtful conjecture – verification (or proof) – proof (or verification). This learning process effectively facilitated the subjects' development of mathematical reasoning and proof abilities.

During the third week of Phase 2, the subjects were assigned to solve the problem below:

> A road is proposed that will connect two towns A and B on opposite sides of a river. The road will cross the river in a bridge that is perpendicular to the riverbanks. Where should the bridge be placed so as to minimize the total length of the road?

First, the subjects were encouraged to give their initial conjectures. Lisa used her knowledge from a previous problem in which the river was simply a line (river width ignored). In that case, the shortest path was the straight line segment from point A to point B. So, she automatically assumed the same approach for this problem. She drew a straight line connecting points A and B. This time, however, since the river had a width, the line intersected the river at two points. She decided to find the midpoint of the segment determined by these two intersection points. She then went on to conjecture that it was the segment through this midpoint perpendicular to the riverbanks that would give the shortest path from A to B. Fred had a different conjecture. He believed that the bridge on the shortest path should lie on the line passing through the midpoint of segment AB and perpendicular to the riverbanks.

The subjects then tested their conjectures through experimentation with GSP. Using the dynamic feature of GSP, Lisa was able to construct an arbitrary path and move it along the river until she found the shortest path (path AMNB in Figure 1) by using the Measure tool. This path disproved her initial conjecture. Fred "located" the shortest path in a similar way. His conjecture was also found to be incorrect.

However, the shortest path that was "located" was not the real path that must be constructed. The subjects were led to carefully observe the "located" shortest path, and try to find its characteristics. Through observation and checking by doing necessary measurements, Lisa found that the most important characteristic of the path was segment AM // segment NB. Because of this, if AM was moved down by the width of the river, then AM and NB lied on the same line. In addition, point N that located the bridge on the shortest path was the intersection point of this line and the bottom riverbank. This analysis revealed an idea to construct the shortest path - Translate point A by vector XX' (Segment XX' in Figure 1 represents the river width.) to locate point

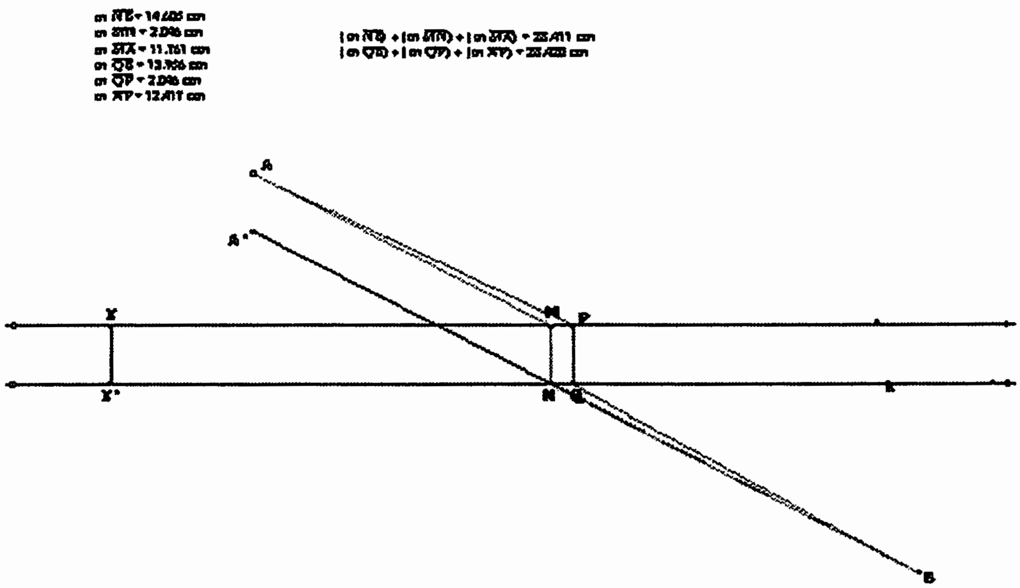

Figure 1

A', connect points A' and B using a segment, and construct the intersection point of segment A'B and the bottom riverbank. Then it was easy to complete the rest of the steps of the construction. Fred had difficulty understanding the process. I arranged the interview session so that Lisa and Fred worked together as a group. Lisa was happy to answer questions that Fred asked. Both subjects completed the construction of the shortest path - in other words, made a thoughtful conjecture.

(In order to save space by not introducing another figure, let's now consider path AMNB in Figure 1 to be the constructed shortest path.) To verify whether the shortest path was constructed correctly, both subjects created an arbitrary path APQB where Q was a free point on riverbank k and segment PQ represented the bridge (see Figure 1). By continuously dragging point Q and comparing the length of path APQB with that of path AMNB, both subjects observed that path APQB was always longer than path AMNB. Therefore, AMNB was indeed the shortest path.

Since verification itself is not a proof, the problem was not solved yet. Lisa continued her GSP exploration to construct a proof. She began dragging different points around to see how it would affect the distance of the paths, hoping that she would find a clue for proving the conjecture. As she was doing this, she came across exactly what she needed. She found that if she dragged one of the riverbanks close enough to the other so as to make the river width approach 0, a triangle was formed with sides AP, PB, and AB. This meant that when the river width was 0, the constructed path became side AB, and the arbitrary path became sides AP and PB. By the Triangle Inequality Theorem, AB AP + PB, confirming that AMNB is the shortest path. Since the river did have a width, the situation was different. However, the only difference was the river width, and the width was included in both paths. When the river width was subtracted from both paths, the situation would be similar to that when the river width was 0. To remove the river width from both paths, the only thing needed to do was to translate segment AM by vector XX' (see Figure 1). After the translation, the parallel segments AM and NB would become one segment, and the Triangle Inequality Theorem would be used to complete the proof.

An Excellent Teaching and Learning Tool That Can Enhance Students' Mathematical Reasoning and Proof Abilities

GSP has been widely used by teachers, mathematics educators, and students at both school and college levels as an effective teaching and learning tool. However, most of the people use GSP dynamics to help discover properties and relationships, make and test conjectures, and construct geometric objects. In this study, I not only emphasized these aspects of the use of GSP, but also went one step further. I also emphasized that after conjectures were made and tested, the subjects continued GSP explorations to come up with insight for reasoning and proofs. By analyzing the data that I collected in the study, my finding supported that of Battista & Clements (1995) - Sketchpad explorations can not only encourage students to make conjectures, they can foster insight for constructing proofs.

A good example of this aspect has been seen in the Shortest Path problem: Dragging one of the riverbanks close enough to the other and observing the resulting situation fostered insight for proof. Another example in which GSP helped the subjects both make and prove their conjecture was the problem solving process below:

> ABC is an arbitrary triangle. Points D, E, and F are respectively on sides BC, CA, and AB. BD= (1/3)BC, CE = (1/3)CA, and AF= (1/3)AB. PQR is formed by the construction of line segments AD, BE, and CF. What is the relationship between PQR and ABC? (see Figure 2)

When asked to make an initial conjecture, Lisa observed the figure for a while, and then said, "If you ask for the area relationship, it's 1/3, ... no, 1/9, ... no, I am not sure. Less than 1/3." Fred chose 1/9 without being sure either. Through GSP investigations, both subjects found that area(PQR) = (1/7)area(ABC). Because this problem was quite challenging, the task to prove the finding/conjecture was only assigned to Lisa. She spent quite a while thinking about a proof, but all efforts "led to no where" (her own words). In this case, I asked her to continue her investigation with GSP. Through further work with GSP measurements, she found that Area(BDP) = Area(CEQ) = Area(AFR) = (1/3)Area(PQR) = (1/21)Area(ABC); Area(Quadrilateral DCQP) = Area(Quadrilateral EARQ) = Area(Quadrilateral FBPR) = (5/3)Area(PQR) = (5/21)Area(ABC); BP : PQ : QE = 3 : 3 : 1; CQ : QR : RF = 3 : 3 : 1; and AR : RP : PD = 3 : 3 : 1. She was very interested in these new findings. I suggested that instead of proving the original conjecture directly, she might try to prove a new finding first - for instance, prove the relationship between the smallest triangles (such as BDP) and ABC. She agreed, and continued her active thinking. With minimal help from me (I asked a question: Let x be the area of BDP and y be the area of CEP. What are the areas of DCP and EAP?), she finally came up with a proof similar to the one shown in Figure 2:

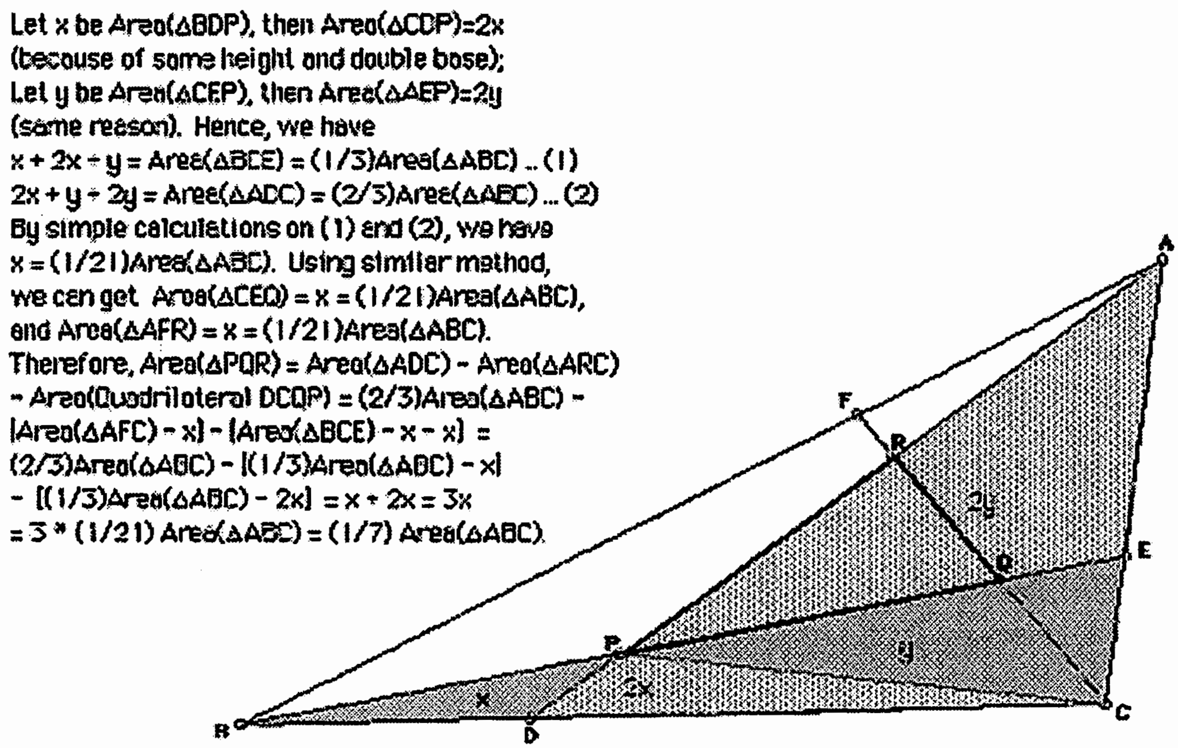

Figure 2

She deeply felt that GSP can foster insight for constructing proofs and hence is an excellent teaching and learning tool that can enhance students' mathematical reasoning and proof abilities.

Different Roles in Helping Students at Different Levels of Geometric Thinking

Research has demonstrated the effectiveness of using GSP in geometry teaching and learning (Dixon, 1997; Choi-koh, 1999). Based on the data collected in this study, I found that GSP plays different roles in helping students at different levels of geometric thinking. The following problem and the subjects' exploration processes provide an example.

STREET PARKING. You are on the planning commission for Algebraville, and plans are being made for the downtown shopping district revitalization. The streets are 60 feet wide, and an allowance must be made for both on-street parking and two-way traffic. Fifteen feet of roadway is needed for each lane of traffic. Parking spaces are to be 16 feet long and 10 feet wide, including the lines. You job is to determine which method of parking – parallel or angle – will allow the most room for the parking of cars and still allow a two-way traffic flow. (You may design parking for one city block (0.1 mile) and use that design for the entire shopping district.)

Both Lisa and Fred gave the same initial conjecture: Angle parking would allow more cars to be parked than parallel parking. Then they began their investigations.

Lisa used paper and pencil to draw diagrams to help thinking. She quickly determined how many cars could be parked in one city block in the parallel parking situation by saying, "The length of the block (528 ft) would be divided by 16 ft, the length of each parking space. This would allow 528/16 = 33 parking spaces on each side of the street, giving a total of 66 available parking spaces."

For angle parking, Lisa's investigation process was not so smooth as that for parallel parking. Her diagram is shown in Figure 3. She stated, "Using the Pythagorean Theorem to find x, we get $x = \sqrt{16^2 - 15^2} = \sqrt{31} \approx 5.567$ (feet). Now we are left with 528 – 5.567 = 522.433 (feet). If we divide this length by the width of the parking space, 522.433/10 = 52.24 (feet). Then, *at most* 52 cars would be able to park on each side of the road. Thus, giving 104 available parking spaces."

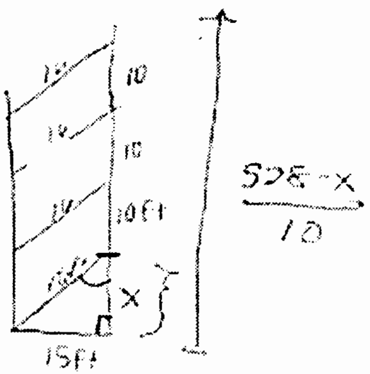

Figure 3

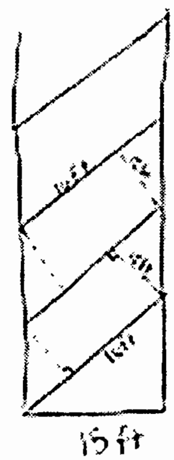

Figure 4

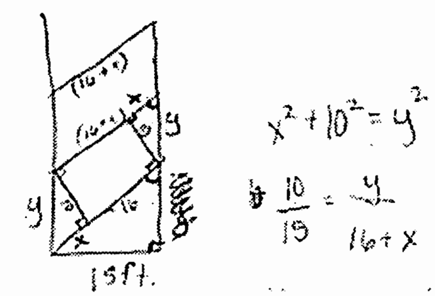

Figure 5

While indicating that it was good for her to understand the length of the curb space (10 ft in her thinking) was the key to determine how many cars could be parked, I had the following dialogue with her (I stands for the investigator and L stands for Lisa):

I: If a car is exactly 16 feet long and 10 feet wide, can the car be parked in your parking space? (I drew a rectangle on paper to represent a car.)

L: (after thinking for a while) No. The parking space is too small in comparison to the car.

I: What does this tell you?

L: My drawing is wrong.

I: Do you still think angle parking is better?

L: I don't know yet. Need more work.

I: How will you correct the mistake in your diagram?

L: The rectangle should be inside the parking space. (She immediately drew a new diagram shown in Figure 4.)

Even though the new diagram was far from accurate in terms of ratios between segments, the structure was correct. Lisa continued active thinking, and in about ten minutes, she came up with a system of two equations shown in Figure 5. Solving the system of equations would give the length of the curb space, and then the number of cars that could be parked would be easily calculated.

Lisa did not stop at this point, but instead constructed the drawings shown in Figures 4 and 5 in GSP and used GSP measurements and calculations to verify the solution she had obtained. This became her habit later – using GSP as a tool to verify her work after she had developed mathematical solutions or formal proofs.

Fred experienced a much more difficult time in formulating a mathematical solution for the angle-parking situation although he also correctly calculated the number of cars that could be parked in the parallel parking situation, In order to help Fred, I opened a pre-constructed GSP sketch for him (Figure 6).

Below is the dialogue between Fred and me:

I: Look at the current position of the car parking. Is this situation OK?

F: No. It blocks the traffic.

I: How will you change the situation so that the parking is acceptable with the regulations?

F: (Looked at the sketch seriously, and then dragged the point "Drag" down until the yellow rectangle only "touches" the traffic lane [see Figure 7].)

I: What do you observe?

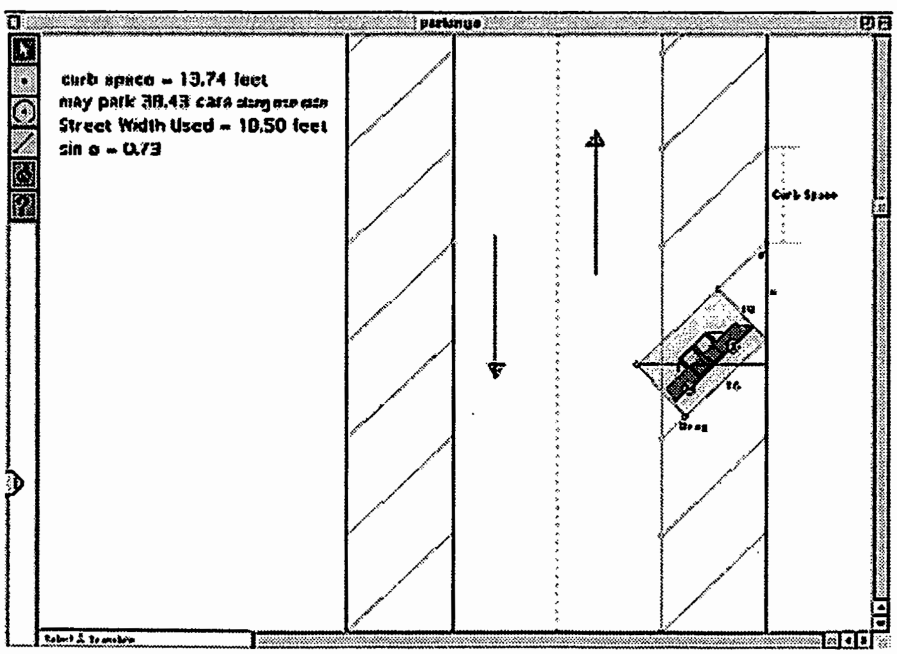

Figure 6

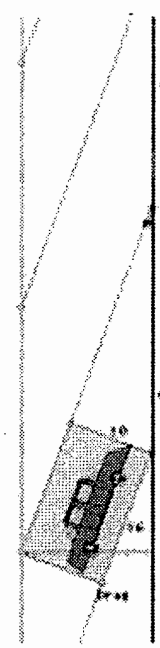

Figure 7

F: (Thought for quite a while.) Now the traffic flows, but space is wasted a lot.

I: Did you see the numerical values displayed on the left-hand side?

F: Yes. Now the curb space is very long. Only 18 cars can be parked on one side of the road.

I: What is your conclusion?

F: Parallel parking is a better choice.

It seems that without GSP as an investigation tool, it would have been difficult for Fred to come up with the more thoughtful conjecture that parallel parking was a better choice. More help was provided to Fred when he tried to explain the new conjecture. This was not the case for Lisa. Sufficient GSP experience had helped her achieve a thinking level at which she was able to make thoughtful conjectures and even develop formal solutions or proofs (for many problems) with or without using GSP. In this case, GSP was more a verification tool than an investigation tool for her.

Improvement of the Subjects' van Hiele Levels of Geometric Thinking

At the beginning of the study, it was found via the pre-test that Lisa was able to perceive relationships between properties and understand logical implications and class inclusions. She was able to do some simple proofs using a two-column format, but not beyond that. For instance, she constantly used the Triangle Angle Sum Theorem, but did not know how to prove it. Therefore she was considered to be at van Hiele level 3 of geometric thinking. During the ten-week teaching interviews, she gradually progressed to level 4 at which she was able to self-initiate efforts to reason deductively and construct proofs. For the ten proof-oriented questions or problems in the post-test, Lisa correctly completed eight and partially completed two. The two problems that Lisa only partially completed had a certain degree of difficulty, and it was understandable that she was not able to complete them in limited time. Among the eight problems that she correctly completed, one was: *Two circles intersect at Points A and B. A line passing through A intersects the two circles respectively at C and D. A line passing through B intersects the two circles respectively at E and F. What is the relationship between lines (or segments) CE and DF? Can you prove your conjecture?* Lisa quickly conjectured that line segments CE and DF were parallel through visual observation. She also quickly verified her conjecture by dragging points on the circles to see if the lines would still look parallel, and they did. To prove the conjecture, however, was a challenge. It required the understanding of the Cyclic Quadrilateral Theorem and the theorems to prove two lines are parallel. Furthermore, the key for developing a proof was the flexibility to create cyclic quadrilaterals that did not exist explicitly in the first place. After active thinking, Lisa showed this flexibility by constructing segment AB to form two cyclic quadrilaterals. Then the rest of the proof became straightforward. It was evident that Lisa had progressed to solid van Hiele level 4 of geometric thinking.

Fred also demonstrated the development of his geometric thinking during the ten-week teaching interviews. His performance in the post-test showed that he had progressed from his original level (van Hiele level 2) to a thinking level at which he was able to identify relationships and categorize figures in a hierarchical order (van Hiele level 3). For example, he knew that a square is a rectangle, but a rectangle is not necessarily a square. Similarly, a rhombus is a parallelogram, but not vice-versa. He recognized that the set of all rectangles is bigger than the set of all squares. Furthermore, he was able to understand and sometimes even present logical arguments. For example, Fred had difficulty to do the deductive proof in the parking problem. However, when he discussed the problem with Lisa, he was able to understand Lisa's reasoning. After the discussion, he was able to explain why the parallel parking was better on his own.

Changes in the Subjects' Conceptions of Mathematics and Mathematics Teaching

The subjects' conceptions of mathematics and mathematics teaching significantly changed as a result of their experience in the interview sessions.

The interactive, dynamic, and innovative features of GSP allow students who might not be interested in mathematics to become active participants in exploring mathematics. Previously, teachers lectured and were the ones doing all the work on the board. Students had little determination so that they became unproductive and uninterested in the subject matter. However, with GSP, students now have the opportunity to get involved in the teaching and learning processes by making, verifying, and proving their own conjectures. Before participating in the teaching interviews, Fred didn't like geometry, and fell asleep in geometry classrooms. Now, he enjoys geometric learning. He indicated, "One of the reasons that I am so interested in this class is because we use computers. I am like a little kid and my attention is easily caught by the use of a computer."

Before the teaching interviews, Lisa agreed to the following statement, "To understand mathematics, students must solve many problems following examples provided." She explained that this was the way she had been taught mathematics her entire life. The teacher provides examples and the students imitate the procedures. She wouldn't know any other way of learning and teaching mathematics. However, after participating in the teaching interviews, she realized that students can understand mathematics, perhaps even better if they discover/explore solutions to mathematics problems themselves than if they simply mimic the teacher's examples. She indicated that, in the MAE 3651 class, the instructor hardly provided examples for students to follow. Rather, the students were given problem situations and had to find solutions themselves. This teaching method got all students involved and interested. In the process of finding a solution, students learned why the solution worked. Thus, they understood the related mathematics without following a given example.

Before the teaching interviews, Lisa agreed to the statement, "Students should have opportunities to experience manipulating materials in the mathematics classroom before teachers introduce mathematics vocabulary," but did not think it was *necessary*. She stated, "Yes, students should have the opportunity to use manipulatives now and then, but it is not essential. Additionally, I hesitated to say that I strongly agree because I also found it a little difficult to imagine that students could work with materials without knowing the terms of what it was they were looking for." However, after participating in the teaching interviews, she realized that knowing the vocabulary is not critical when trying to find a solution or an explanation/proof. What is important is the process of problem solving and conceptual understanding.

Implications for Preparing Mathematics Teachers

This study was conducted within the course "Learning Mathematics with Technology" (MAE 3651). The results of the study seem to suggest that a course like MAE 3651 be necessary for preservice mathematics teachers (at least currently). First, preservice teachers should understand the mathematics they will be teaching at and beyond the level expected of their future students (MAA, 1991). To the study of geometry, judging, constructing, and communicating mathematically appropriate arguments remain central (NCTM, 2000). Therefore, preservice teachers should be provided with opportunities to develop their mathematical reasoning and proof abilities. In terms of the van Hiele model of geometric thinking, they should reach at least level 4 (ideally level 5). A course like MAE 3651 can assist them towards that goal. Second, geometry has always been a neglected area in mathematics education (NCTM, 1991). A majority of the preservice teachers did not get chances to seriously learn geometry in high school where the focus was on procedures and finding answers. Students like Lisa have stronger geometric background than most of their peers. Even so, however, they lack the necessary knowledge and skills in developing proofs. Students like Fred lack both geometric intuition and conceptual understanding. Therefore, most of the preservice teachers are not ready for *College Geometry*, which is a required course of most teacher preparation programs in the nation. In that course, formal proofs are a norm and students reason about mathematical systems. Students at a thinking level lower than van Hiele level 4 are not able to make sense of the formal aspects of deduction. This explains why it was difficult for most preservice teachers to even pass that course. With a course like MAE 3651 as its preparation, the situation can be fundamentally improved. Third, preservice teachers often have little exposure to the type of investigative mathematics that lies at the heart of NCTM Standards (2000). In a course like MAE 3651, they are required to learn mathematics through hands-on and minds-on investigations and explorations. Finally, dynamic geometry in general, and GSP in particular, is one of the best examples of technology that can be used to support innovative learning and teaching of mathematics. Teacher preparation programs should recognize the need to increase future teachers' knowledge of geometry, help them develop their own mathematical power, and expose them to innovative methods of mathematics learning and teaching through the use of dynamic geometry software, especially GSP.

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ENGAGING TEACHERS IN RESEARCH THROUGH PROBLEM SOLVING AND PROBLEM POSING WITH DYNAMIC GEOMETRY SOFTWARE

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The main research objective of this study is to engage future and current secondary mathematics teachers in mathematical research by conjecturing a geometry result using dynamic software. A college geometry course for future and current secondary mathematics teachers taught in the spring was selected as the setting of this study. We report here the effects of incorporating problem solving and problem posing activities in this course. Students were asked, as a final project, to find a "new" geometry theorem (i.e., a result that they did not know or was not covered in class) using dynamic geometry software. A total of 46 conjectures were produced. Additionally, students were asked to react to this activity in a 1-2 paragraph. Although students felt initially intimidated by the assignment, conjectures produced suggest, on one side, that future and current mathematics teachers can generate interesting and significant geometry conjectures, and on the other side, that all the students found the assignment engaging, interesting and satisfying. The data reported correspond to two sections of this course (2001 and 2002).

Objectives or Purposes

This study is part of an ongoing project that seeks to involve future and current secondary mathematics teachers in mathematical research. This paper reports the effects of incorporating problem solving and problem posing activities in a college geometry course for secondary mathematics teachers. First, we wanted to investigate students' abilities to pose problems using dynamic software with a semester-long exposition to problem solving and problem posing activities. Second, we wanted to study students' affective responses to investigating a mathematical situation without a specific task. To this end, teachers in a college geometry course were asked to find an "original" conjecture in geometry with the aid of dynamic geometry software. Original was defined as a result that had not been covered in class or they had not seen in a book. Students also wrote a 1-2 paragraph about how they felt about this task.

Perspective(s) or Theoretical Framework

A central goal in mathematics education (NCTM, 2000) is that students become mathematical problem solvers. Teachers play an essential role in this enterprise (NCTM, 2000). Teachers are expected to model and emphasize "aspects of problem solving, including formulating and posing problems, solving problems using different strategies, verifying and interpreting results, and generalizing solutions" (NCTM, 1991, p. 95).

Posing problem or problem formulation (Silver, 1994) is increasingly receiving attention from both a curricular and pedagogical perspective (NCTM, 2000). For instance the *Principles and Standards for School Mathematics* (NCTM, 2000) states that "students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking" (p. 52).

Unfortunately teachers lack abilities to pose problems. For instance, the authors describe that future teachers both elementary (2000, 2001) and secondary (1999) teachers have limited abilities to pose mathematical problems. In the particular case of secondary mathematics teachers, the authors (1999) identify the need of future teachers to learn how to pose problems in systematic ways. A systematic approach to pose problem has been built using Contreras (1998) and Moses, Bjork & Goldenberg (1990). This systematic approach to pose problems is referred as a problem-posing model and consists of seven strategies: Variation of unknowns, variations of knowns or givens, variation of restrictions, reversing knowns and unknowns (converse-type problems), generalizing, thinking of patterns, and proving.

"The college geometry course is especially important for prospective secondary mathematics teachers. [...] Not only does the college geometry course need to lay a strong foundation for the content they will eventually teach, but it is also one of the few courses that might develop the preservice teachers' ability to create and present proofs" (Gover & Connor, 2000, p. 48). Furthermore, we propose that the college geometry course supported with interactive geometry software provides an environment in which secondary mathematics teaching majors can engage in problem posing as well.

Methods or Modes of Inquiry

The *Principles and Standards for School Mathematics* (NCTM, 2000) recommends that students should "examine patterns and structures to detect regularities [and] formulate generalizations and conjectures about observed regularities" (p. 262). Geometry, in particular, offers many opportunities for students to examine visual regularities and hidden patterns. Interactive geometry software such as the Geometer's Sketchpad (Jackiw, 1995) facilitates searching for geometric patterns and the formulation of conjectures because it can construct geometric configurations with precision. It also enables the learner to test the observed patterns and the formulated conjectures or generalizations by dynamically and quickly generating a large number of instances.

A college geometry class taught by one of the authors during the spring was chosen as the setting for this study. The study spans over a two-year period and includes two section of the college geometry course, one section in Spring 2001 and one section in Spring 2002. The class is an undergraduate course but it is also open to graduate students. During the spring 2001, eight males and 13 females completed the course. There were 12 undergraduate students, six graduate students and three post-degree students. During the spring 2002, 13 females and 12 males completed the course. There were 19 undergraduate students, two graduate students, and four post-degree students.

Students were asked to produce an original conjecture as a final project for the class. This assignment was included in the syllabus and students were notified of it the first day of class. Students were also told that they were supposed to "discover" the conjecture with the aid of geometry software. They were told as well that the class would provide instruction on the use of the software and ample opportunities to solve and pose problems with its aid. Since the authors did not know the level of difficulty of the conjectures that the students would pose, to release some of the stress from the task, students were asked not to prove the result, but at least to try to prove it. Also, students were told that an original result meant a result that had not been covered in class or that they had not seen in a book.

The class met twice every week, Mondays and Wednesdays from 4-5:15pm. All the meetings took place in the computer lab. Hence each student had access to a computer. We used Geometer's Sketchpad, GSP (Jackiw, 1995) version 3 during Spring 2001 and version 4 during spring 2002. Some students purchased the student version of this software.

The second author was invited to illustrate the use of a framework to pose problems (Contreras, 1998) in systematic ways during the second week of Spring 2001. The presentation centered around geometry tasks aided with dynamic geometry software (GSP; Jackiw, 1995). This class then continued covering the material and emphasizing the application of the framework when relevant. Towards the 11th week of classes the instructor provided students with a list of suggestions for problems. The purpose of this list was for the students to formulate or pose new problems by varying or modifying the conditions of the problem. In contrast, students in the spring 2002 did not meet the second author or received a list of problems. However, the instructor invited a former student, Ron, to present his conjecture during class. Ron found a conjecture working on the diagram associated with the Pythagorean Theorem (McAllister, Martínez-Cruz & Gannon, in preparation). Also, as the class in spring 2001, the spring 2002 section experienced the application of the problem-posing framework when relevant. To illustrate, one of the problems in homework was the following: "Given a square, construct the centroid of each of the triangles formed with three consecutive vertices of the square. Next join pairs of consecutive centroids counterclockwise. What kind of quadrilateral is formed? Prove it!" (See Figure 1).

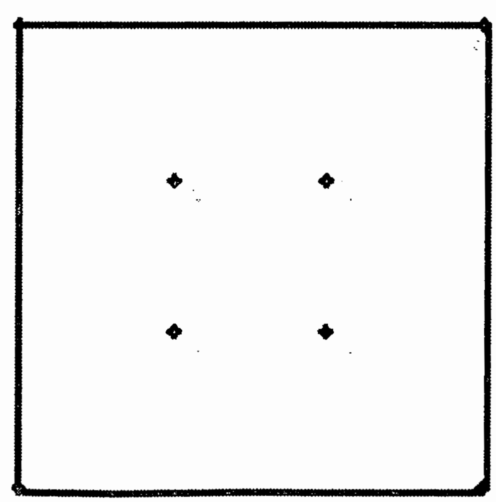

Figure 1

After, the homework was graded and the problem was discussed in class, the instructor discussed the possibility of generating new problems from this problem. In particular, we looked at varying the knowns, in this case a square—which is a regular quadrilateral. So we looked at the situation when we have other quadrilaterals such as a rectangle or a parallelogram. We constructed the newly generated situation for the rectangle with GSP and discussed in more detail than when working with a parallelogram. We also suggested how to prove the result observed. No further comment was done on what other figures to try, however it was underlined how we had generated another problem by varying a known, and conjectured a result with the aid of geometry dynamic software.

Data Sources or Evidence

Students presented their projects in class during the last two weeks of each semester. A total of twenty-one projects were collected in Spring 2001, and twenty-five projects in Spring 2002. Some projects were presented as conjectures and others as problems. We provide three samples of students' work from Spring 2001. All three contain a sketch. We have included only the sketches for the last two projects. This presentation does not imply any ranking.

(1) Conjecture: Given a golden rectangle, the sum of the areas of the infinite number of circles that can be made using the sides of the inscribed golden rectangles as radii is to the largest circles as the longer side of the original rectangle is to it's (sic) shorter side.

(2) Theorem: In any ellipse, the triangle inscribed in a semi-ellipse with the base as the minor axis is acute. (<90). Likewise, the triangle inscribed in a semi-ellipse with the base as the major axis is obtuse. (>90). See Figure 2.

(3) Properties of ΔKEN in comparison with ΔUSC. ΔKEN is constructed by taking any interior point "P" of ΔUSC, connecting point "P" with each of the vertices to make 3 Δ's and then constructing the centroids of each of the 3 Δ's to form ΔKEN.

Theorem: The triangle (ΔKEN) formed by connecting the three centroids of three triangles formed from an interior point of a given triangle (ΔCSU) are similar. In other words, $\Delta KEN \sim \Delta CSU$. See Figure 3.

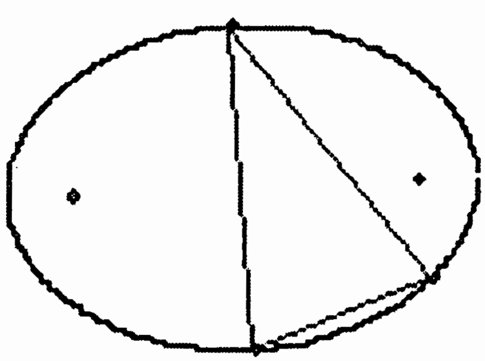

Figure 2

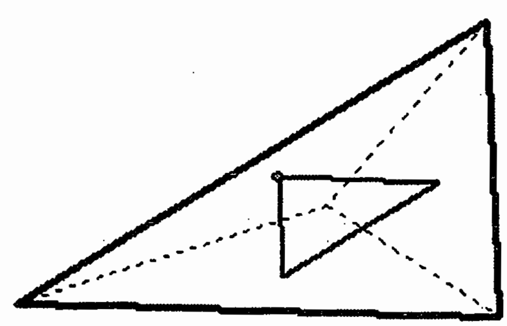

Figure 3

Additionally, as a last assignment, students were asked to "write 1-2 paragraphs on how they felt about the assignment 'discover a new theorem' using GSP". During the conference we will add to the discussion the projects from Spring 2002.

Results and/or Conclusions/Points of View

The results discussed in this section refer to data collected during Spring 2001. During the conference we will combine and discuss the data collected in both sections.

Students' Abilities to Pose Problems Using Dynamic Software

We used the same framework (Contreras, 1998) to analyze the problems. During the presentation, we will illustrate how this was done. A first analysis of the 21 problems students proposed shows that the most common strategies to pose problems were: (a) variations of knowns or givens, variation of restrictions, and generalizing. For instance, the second problem mentioned above came out of the Theorem "An angle inscribed in a semicircle is right." The student posed the problem for an ellipse (instead of a circle) and varied the situation accordingly (in the case of a circle, it is necessary to use a diameter, but in the case of the ellipse, the student worked with major and minor axis).

Unfortunately, this framework does not address the quality of the problem. In other words, what constitutes an interesting, nice, or beautiful problem? Certainly, not all the problems the students came out with have all of these features. However, the authors agree that each of the three problems presented above are interesting, nice and beautiful to some extent. When we bring into the discussion these qualities, it is hard to assess them and separate them from students' affective responses to the assignment. This is addressed below and adds information to the students' abilities to pose problems.

Students' Affective Responses to Investigating a Mathematics Situation Without a Specific Task

All students (in both sections) mentioned that the activity had been "enjoyable" with the majority at the beginning of the semester feeling it would be "impossible" to complete. As one student said "I never figured I would find anything and if I did we would already have learned about it in class."

Some students mentioned that the task integrated geometry patterns, exploration and learning. The use of the software had facilitated "the exploration of geometric relationships from the investigate point of view," and "exploration is...one of the best ways to learn something. When you explore, you seem to better understand that which you have 'discovered', and you learn more along the way than you with most other activities".

The disappointment of not finding a result, maybe right away, emerged in one comment: "The only dissapointment (sic), however, was encountered when I would explore some hypothesis I had expecting to find some geometric constant. Most of the time, to my dissapointment (sic), I could not find any truths." However, another student pointed to this situation in a constructive way. "I thought a particular relation might hold for all cases, but I was able to find a counterexample without too much effort."

Several students mentioned about the ownership of the material through discovery: "Discovering something new, even if it is only new to yourself, is wonderful". Finally, several students mentioned about how the task made them "feel like they were actually contributing to the field of mathematics".

One of the situations that we noticed after students had observed a pattern was the quick tendency to claim the conjecture as a fact. We overemphasized that to be fact the students needed to find a proof of the conjecture and stressed the importance of writing their conjectures starting with the phrase "It seems that ...." However, still during the presentations, several students started their presentation as if the result were a fact. So, it seems that more can be done for the importance and need of proofs as opposed to a large number of evidence.

We believe that we instilled in some students the idea of being curious in mathematics. This is important since we believe it gets to the core of the teacher-researcher issue. To what extent it is an open question. However, we are pleased to report that the first author worked with three students on independent studies during Fall 2001 to pursue and deepen the investigation of their conjectures. A presentation at MAA (Martínez-Cruz & Ohgi, 2001) addressed the work of one of these students.

Relationship to Goals of PME-NA

This study builds on previous research reported at PME-NA meetings. In particular, it seeks a better understanding of the psychological aspects of teaching and learning geometry and the implications thereof.

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STUDENTS' DEVELOPMENT OF THREE-DIMENSIONAL VISUALIZATION IN THE GEOMETER'S SKETCHPAD ENVIRONMENT

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This article reports a series of four studies carried out over a period of four years. These related studies were clinical and qualitative, as they investigated middle and high school students' development of geometric thought, particularly as it related to three-dimensional visualization. The studies were carried out in the constructivist pedagogical style. The Geometer's Sketchpad was instrumental in all of these studies and it was this environment in which the subjects learned to increasingly develop their visualization abilities as they solved challenging geometry problems. Platonic solids, Archimedian solids, other geometric solids, and the mathematical relationships related to these solids were the objects of study. Students' van Hiele levels of geometric thinking were monitored as they constructed dynamic GSP representations of these solids to conduct geometric investigations. A fundamental finding was that GSP provided opportunities to have a distinct positive affect on students' learning of three-dimensional geometry.

According to the NCTM Standards documents (1989, 2000), in high school geometry, all students should have opportunities to visualize and work with three-dimensional figures in order to develop spatial skills fundamental to everyday life and to many careers. Geometry instruction should focus increased attention on the analysis of three-dimensional figures. It should focus on the continued development of students' skills in visualization and pictorial representation of three-dimensional figures.

Research (Ben-Chaim, Lappan, & Hoaung, 1988) suggests that spatial strategies can be taught successfully to middle and high school students. Appropriate use of three-dimensional representation and computer software is of particular value in development of three-dimensional visualization. Particularly, the Geometer's Sketchpad (GSP) developed by Jackiw (1995), a dynamic geometry construction and exploration tool has become widely introduced into classrooms of mathematics and science, especially of geometry. Its effects on different fields have been topics of research. Dixon (1997) investigated its effects on eighth grade students' spatial visualization. Her research results suggest that the GSP instructional environment was more effective than the traditional instructional environment at improving students' two-dimensional visualization, but was not more effective than the traditional instructional environment at improving students' three-dimensional visualization.

Bishop (1983), in describing his previous works in spatial thinking, presents two different but similar spatial abilities:

1) The ability for interpreting figural information (IFI). This ability involves understanding the visual representations and spatial vocabulary used in

> geometric work, graphs, charts, and diagrams of all types. Mathematics abounds with such forms and IFI concerns the reading, understanding, and interpreting of such information. It is an ability of content and of context, and relates particularly to the form of the stimulus material. 2) The ability for visual processing (VP). This ability involves visualization and the translation of abstract relationships and nonfigural information into visual terms. It also includes the manipulation and transformation of visual representations and visual imagery. It is an ability of process, and does not relate to the form of stimulus material presented. (p. 185)

According to Bishop, these definitions refine and extend the definitions given by McGee (1979). IFI extends the spatial orientation of McGee to include geometric and graphical conventions and refines it by emphasizing "the interpretation demanded by those representations" (p. 185). VP as well is refined and extended by emphasizing "the process aspect rather than the form of the stimulus" (p. 185). Thus, three-dimensional spatial ability includes both the ability of interpreting figural information in the context of three-dimensional objects and the ability for visual processing of these objects.

The Purpose of the Research

The purpose of the research was to investigate the effects of the GSP dynamic instructional environment on middle and high school students' three-dimensional visualization. The following questions were investigated in the studies:

1) What role can the GSP dynamic instructional environment play in the development of students' three-dimensional visualization? Is there evidence to indicate improvement in students' three-dimensional spatial ability when GSP is used to teach three-dimensional geometry?
2) What role can the GSP dynamic instructional environment play in the development of students' geometric thinking as defined by the van Hiele theory? Is there evidence to indicate improvement in students' geometric thinking when GSP is used to teach three-dimensional geometry?

Significance

This article describes a series of research studies that focus on the use of GSP in improving students' three-dimensional visualization, one of the important aspects in developing a reform-oriented geometry curriculum. More and more educators believe that the use of technology can effectively facilitate the teaching and learning of mathematics. The NCTM Standards documents (1989, 2000) emphasize the effective use of technology as one of the main feature of the reform curriculum. However, technology-related research on students' three-dimensional visualization is very limited. Increasingly, mathematics educators and teachers need quality technology-related research to "inform us of the conceptual linkages among new and old ideas and orientations and how these might be influenced by various instructional strategies and materials" (Kaput & Thompson, 1994, p. 680). It is this concern that the research studies took into consideration. They explored evidence supporting the belief that students could benefit from the use of technology. The research results will contribute to our current limited understanding of the students' development of three-dimensional visualization and geometric thought within a technology-rich environment.

Conceptual Framework

The conceptual framework of the research was the constructivist perspective that knowledge is not passively received from the teacher but actively constructed by the learners themselves, the van Hiele model of students' geometric thought, and the previous research on three-dimensional visualization. According to van Hiele (1986), students progress in geometric thinking through taxonomy of levels, and progress from one level to the next higher level is dependent on the nature of the instruction provided to students. The five van Hiele levels are Level 1: Visual (figures recognized by appearance alone), Level 2: Analysis of Properties (properties perceived, but isolated and unrelated), Level 3: Ordering/Hierarchy (relationships, implications, & class inclusions), Level 4: Deduction/Proof (deduction is meaningful, can construct proofs), and Level 5: Rigor (formal aspects of deductions, symbols manipulated). Within this framework, the van Hiele model was examined in light of the use of technology. van Hiele's research and particularly the view of levels of thought and the lack of communication among levels may need to be updated for an educational system in which technology is an integral component.

Design and Procedure

Constructivist teaching experiments were used for these investigations. Over a four-year period, twenty-four students were provided increasingly more challenging geometry units with GSP as a central feature throughout. The class as a whole was engaged in problem solving in geometry, which included GSP constructions, investigations, and dynamic experimentation. However, since we were interested in the students' detailed changes and growth in their thinking processes, we focused on looking at individuals and used a clinical approach to assess students' growth in geometric thought.

The class of twenty-four students was a part of the Partnership Academic Community (PAC) program, a collaboration between Florida International University and Miami-Dade County Public Schools. This program aims to help at-risk minority students of metro Miami area to improve their academic and behavioral achievement by placing them in a reform-oriented, technology-rich learning environment. The PAC students are from three middle schools and a high school of Miami-Dade School District. Most of them are African-Americans and Hispanics and from lower socio-

Table 1. Research Phases 1 and 2

	Phase 1	Phase 2
Grade Level	Grade 7	Grade 8
Length of Study	20 weeks at 2 hours per week	10 weeks at 3 hours per week
Subjects	Whole class of students	Whole class of students
Students' Learning Activities	Constructions of basic geometric figures, and explorations on geometric transformations, projective concepts, perspectives, and dynamic representations of geometric solids	Further explorations on dynamic representations of geometric solids, and simple reasoning challenges.

Table 2. Research Phases 3 and 4

	Phase 3	Phase 4
Grade Level	Grade 9	Grade 10
Length of Study	15 weeks at 1 hour per week	10 weeks at 7 hours per week
Subjects	Four subjects (two boys and two girls)	Four subjects (two boys and two girls)
Students' Learning Activities	Problem-solving sessions and task-based interviews	Problem-solving sessions and constructivist teaching interviews
Research Focus	Subjects' ideas and reasoning processes concerning their solutions to the problems, and their van Hiele level of thinking	Students' conceptual understanding , their ability to solve problems by modeling, and their van Hiele level of thinking

economic families. The subjects involved in the clinical studies were selected for their willingness and also for their different mathematics thinking levels.

The teaching experiments progressed through four phases, which are summarized in the following two tables:

A part of the progression from study to study is the extent to which the envelope is being pushed. For example, in studies dating back to 1997, the inquiry was on tasks directly related to the studies of the van Hieles, of basically plane geometry concepts with an easy entry into solid geometry concepts such as perspective views of solid figures. In more recent studies, the investigation of plane geometric concepts embedded in three- dimensional objects was undertaken. Another dimension of growth in inquiry was providing subjects with physical geometric objects and requesting their construction in GSP as dynamic representations on which transformations such as rotations and translations and specific components such as cross sections could be investigated.

The exploration activities/problems used in the studies were devised in ways that (a) the subjects were required to use hands-on explorations with GSP to learn new geometric ideas; (b) they were required to make conjectures based on their explorations; (c) they were required to make meaningful explanations for whatever conjectures they made; and (d) they were asked to look back to check if they had reached a complete understanding. Our goal was to use these activities to help each subject develop three-dimensional visualization and make orderly progress in the movement from the van Hiele level he or she was currently at to a higher level.

Data Collection

Detailed notes were taken when observing the teaching sessions in all four phases. Some of the sessions and all teaching interviews were videotaped. Significant parts of the videotapes were transcribed. The computer files on the disks, and the written tasks completed by the students throughout the four phases, were also collected.

Data Analysis

A constant comparative approach was used to analyze the collected data. The constant comparative method of qualitative data analysis (Glaser & Strauss, 1967; Grove, 1988) serves as a model demonstrating how systematic analysis can improve the validity of a qualitative research study. Constant comparison involves analyzing and interpreting data during and after data collection. Using this method, the transcripts and other data were analyzed during teaching sessions and after all sessions were completed. These analyses took the form of editing a set of transcripts and other data with commentaries. The commentaries depicted the story lines of the corresponding students' progression through the sessions.

Results

The results of the research suggest that GSP and the associated activities were effective in helping the students develop three-dimensional visualization and achieve conceptual understanding of geometry content.

GSP helped the students generate and sustain insight and enthusiasm in learning three-dimensional geometry.

Starting in grade 7, the subjects of these studies had positive attitudes toward the use of GSP. They found it interesting and dynamic. They found it a friendly environment in which they could play around and experiment. Using GSP to learn geometry in an experimental fashion has contributed to the subjects' enthusiasm for geometric exploration. In some sense, GSP was not considered a geometric environment so much as an artistic and creative environment by these subjects.

Periodically, the students in these studies were asked their views of using computers, particularly their views related to GSP. Consistently, they expressed their continuing enjoyment and valuing of GSP as a tool for learning. The ways that the subjects used GSP changed form over the years. Initially subjects made simple creative uses as they learned the capabilities of GSP. Then they began to make sense of problem situations; for example, making operational the definition of geometric objects (e.g., a dodecahedron). The three-dimensional uses of GSP emerged early (during the 7th grade) when they turned to the study of perspective drawing. Problems such as finding a hexagonal cross section of a cube and showing it visually moved the use of GSP to focused challenges.

The dynamic possibilities of GSP emerged during the 8th grade year, as students used hints and challenges to do transformations of three-dimensional objects and their cross sections. They expressed the positive attitudes that emerged in the new ways of experiencing the GSP environment. One of our subjects expressed it well, as follows:

> I like the ones that move...the animation...it's different from the object just staying still on the screen...It becomes easier because that knowledge from GSP is in your head and your head is working like GSP, so you can think about it, rotate it. GSP helps you really think about it, you use the program in your head, as a screen in your mind so that if you construct a figure...say a perpendicular line, you can see it in the screen (in your mind).

The dynamics held continuing opportunities for geometric exploration far beyond what we had done before, and the subjects were able to create solids that could be transformed. The solids could be rotated, and the cross section slices on it could be translated and rotated as well. Our subjects found these uses of GSP brought new challenges. They realized the implications for a deeper level of study of geometric objects. In the 9th and 10th grade years, Platonic solids and Archimedian solids became objects of study. The animation of GSP helped students to analyze the structure of these objects in an in-depth way. Some tasks were provided that challenged the subjects of these studies to make sense of the meaning of duality while finding the dual of each Platonic solid. Other tasks challenged our subjects to use slicing planes to create Archimedian solids from given Platonic solids. These tasks brought a new dimension to the GSP exploration. Likewise, tasks that requested our subjects to construct in GSP an Archimedian solid from the net of such a solid expected higher levels of visualization. Through these types of tasks, our subjects built GSP planar representations of the related polyhedra. The use of these dynamic GSP representations permitted our subjects to study the features of each solid, and for some the result was an understanding of the structure of the solid. However, possibly the greater value of these tasks was that they enticed the subjects into a greater mental involvement in the study of three-dimensional geometry. Their enjoyment of this type of study was a key, we believe, to the development of spatial thinking of our subjects. The evidence that we have, beyond what we could observe the students doing and saying was that about 50% of them reported in post-study surveys the influence that GSP had upon their spatial understanding. Some of the students' comments were:

> "I like working on the computer more because it is fun, and GSP helps me to learn geometry but at a higher level of understanding because I have all the visual I need right in front of me."

> "I learned a lot about things like the properties of solids, regularity, shapes, and 2-D dynamic constructions. The computers really helped me to see and know these things because sometimes I couldn't visualize these things that were so easily shown on GSP… The GSP program would be a great learning tool to use because the student in order to construct a figure or a shape must take the properties of that shape into consideration."

> "GSP helped us look at the objects and shapes from different points of view and in a way that helped record the information into our minds."

Thus the GSP dynamic environment effectively facilitated the affective learning, and the sense of enjoyment and ownership of our subjects. We believe this affective learning indirectly contributes to the development of spatial visualization abilities.

GSP, by providing immediate feedback, helped students efficiently confirm or correct their conjectures.

In the teaching experiments, the investigators followed the constructivist approach where the subjects were requested to learn new ideas by exploring given task situations. As the first step of the exploration, the investigator always encouraged the subjects to make initial conjectures. Looking back over the many initial conjectures the subjects made we found that a large portion of those conjectures were incorrect. Making incorrect initial conjectures was not a bad thing, and seemed to be normal in students' learning process. From the constructivist point of view, substantive learning takes place over a long period of time and occurs during periods of confusion and conflict. In the teaching sessions designed for these studies, as soon as the subjects made their initial conjectures, they were allowed (and at the same time they were very eager) to manipulate the GSP dynamic models to check their ideas. The immediate visual feedback provided by these models stimulated the subjects' strong desire to struggle against and clear their mental confusion and conflict, thus achieving improved understanding. The interview with a subject (Laura) in Phase 3 gives a good example of this aspect.

The interview was designed to compare the GSP dynamic planar representation with a static paper-and-pencil test in facilitating the subject's three-dimensional visualization. Laura was given a two-dimensional figure of a regular icosahedron. The given figure only showed its front elements (vertices, edges, and faces), with its back elements missing (Figure 1). The task was to add all missing elements to complete the figure.

Laura took a pencil-and-paper test first. To add the missing elements, she tried hard to image the spatial structure of a regular icosahedron. Even though she added some elements (e.g., one missing vertex on the upper pentagon cross section and two missing vertices on the lower pentagon cross section) correctly, she was not able to visualize the complete structure of the regular icosahedron. Figure 2 shows her drawing - an apparently incorrect conjecture.

Soon after the paper-and-pencil version of the task, Laura began the GSP test. In this case, she was given a GSP sketch of dynamic planar representation of the regular icosahedron. She rotated the planar representation by dragging the free point (on the small circle) around (see Figure 3). In the course of rotation, she had opportunity to observe the continuous change of images of the icosahedron. In other words, she experienced different orientations of the planar representation of the solid. This effectively enhanced her visualization on the solid. She quickly said, "Oh, my drawing was wrong." To add the missing elements correctly, she continued to drag the free point and moved it around, sometime fast, sometime slowly.

The investigator observed that when the subject rotated the figure, she approximated the positions of the missing elements and checked her approximation by slow-

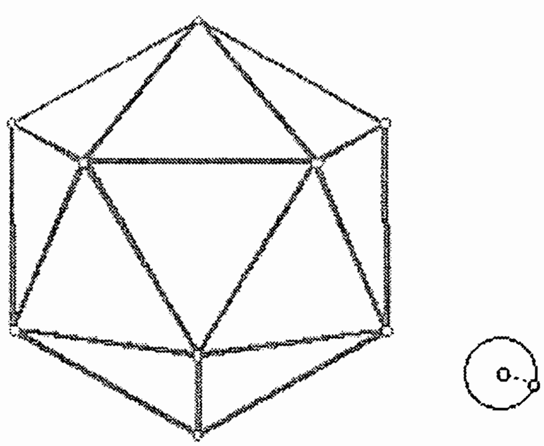


Figure 1. A figure of an icosahedron with missing elements.

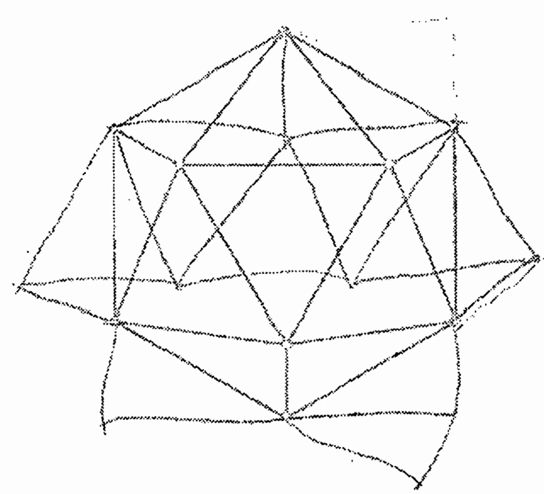


Figure 2. Laura's drawing for the missing elements.

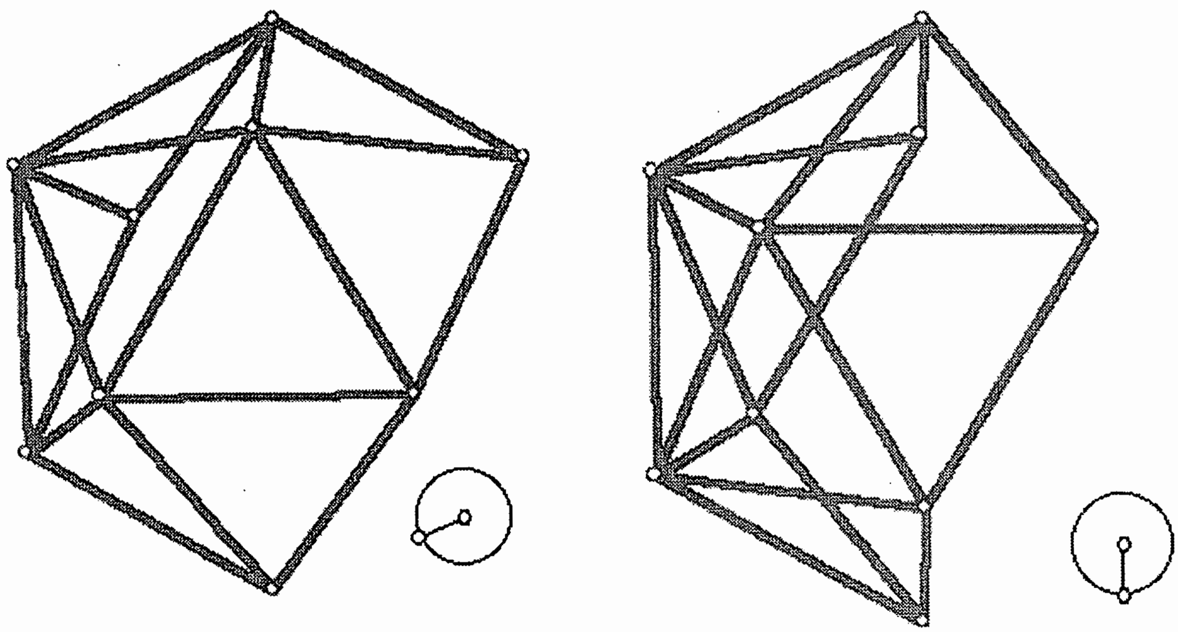


Figure 3. Two specific positions in the course of rotation.

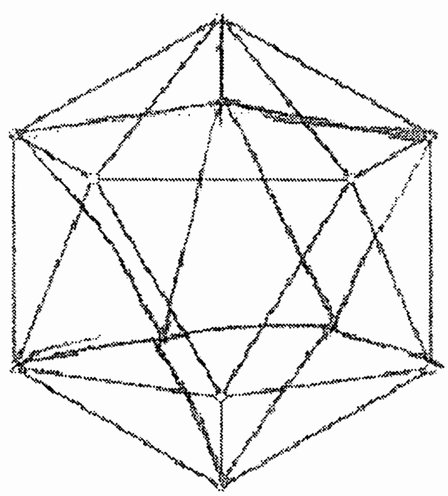


Figure 4. Laura's drawing in the GSP test.

ing down her rotation and moving the free point back and forth. In about three minutes, she finished her drawing (See Figure 4). This time, she figured out the positions of all missing vertices accurately and connected all missing edges correctly. The drawing suggests that with the feedback that GSP animation provided, Laura corrected her misconceptions and achieved a better understanding of the spatial structure of the regular icosahedron.

In the paper-and-pencil test Laura was struggling to visualize the complete spatial structure of the object, but she failed based on the limited information that the static planar figure provided. She was only able to develop an incomplete internal representation of the solid. After she finished the GSP test, the investigator had a conversation with her.

I: Look at the picture you drew on your paper-and-pencil test.

L: I know there was something wrong.

I: I would like to understand how you drew the picture like this.

L: Well, I was thinking there should be ten faces in the middle.

I: Could you try counting the faces? How would that help?

Then she began to count the faces in the middle. "Here we go. When I counted the faces in the middle, I missed these two triangles," she said, while pointing at the two most "outside" triangular faces (see Figure 2). Thus she was able to clear up her own confusion. The conversation allowed us to see the clear picture of her mental process. At first, she had some understanding of the spatial structure of the object, and so she knew there were five faces in the upper part and ten faces in the middle. In fact, she drew the missing vertices in the upper cross section correctly. However, because of her confused spatial visualization on the static planar figure she was not able to draw the missing vertices in the lower part correctly. In the GSP test, through manipulating the dynamic planar representation of the solid, she reached a better understanding of the structural properties, made progress on the logical connections, and found out what was wrong with her previous drawing. Thus, she developed a more complete mental representation of the solid.

Another example is the following task given to the subjects in the interview sessions in Phase 4. They were requested to produce and study the polyhedron represented by the net shown in Figure 5. Typical misconceptions that arose in this setting included interpreting the adjacent equilateral triangles as a parallelogram. Two of the subjects then concluded that the resulting polyhedron was a rectangular prism, and two concluded that the solid could not be formed. To help the subjects visualize the solid they were studying, they were introduced to a pre-constructed GSP sketch that allowed them to fold the net into a solid and unfold the solid back to the net by clicking the related buttons. This experience allowed the subjects to see that the triangles could fold into different planes (i.e., their conjectures were incorrect) and led them to

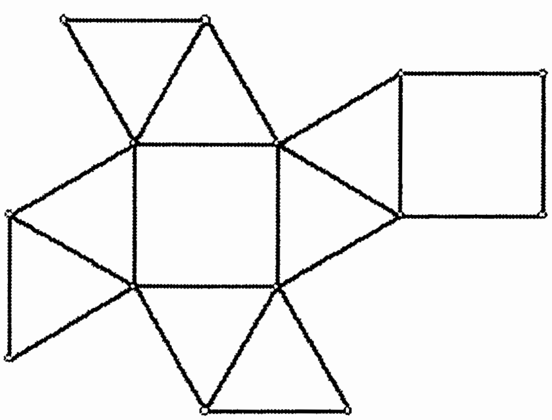


Figure 5. Task: Construct the polyhedron that this net represents.

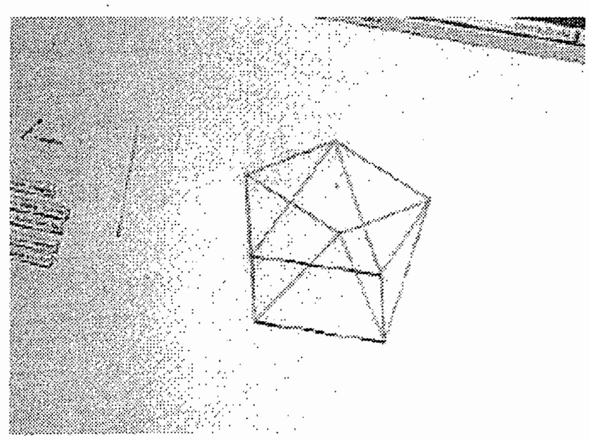


Figure 6. Manuel 's GSP construction of the polyhedron from the net.

re-analyze the net. This type of feedback mechanism became a pattern of inquiry that aided visualization. With minimal study, the subjects were able to determine the polyhedron that the net represented. One of the subjects (Manuel) was able to construct a dynamic GSP model of the solid within a few minutes because he was able to visualize the polyhedron. He rotated his GSP model and quickly indicated that this was it. The rotations enabled him to test out this construction and see if it fitted his image. This use of GSP is instructive and provides avenues for constructing meaning.

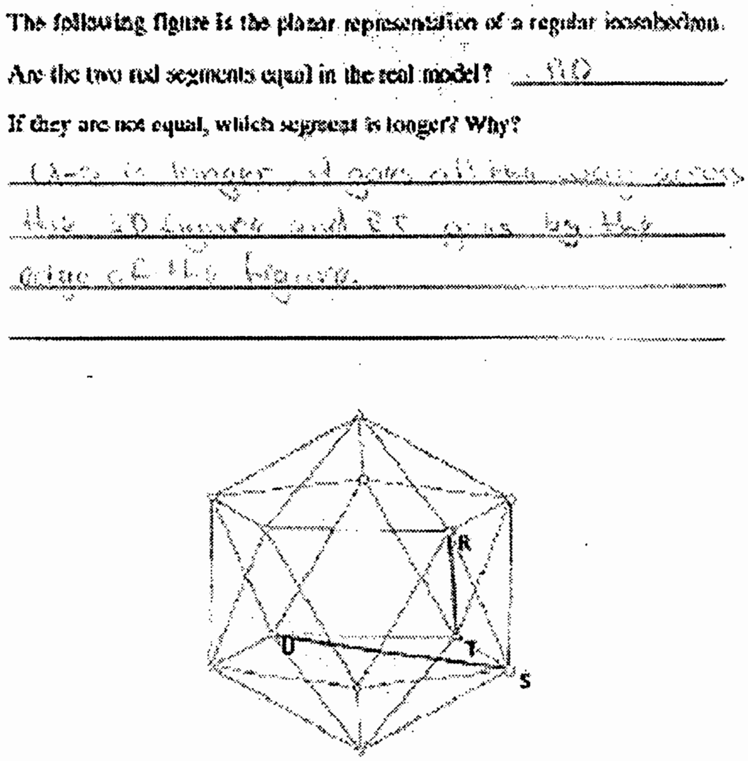
Along with the subjects' progression in the teaching experiments, we did find the subjects gradually made fewer incorrect conjectures.

The GSP dynamic environment helped the students concentrate on the logical (rather than visual) properties of three-dimensional objects.

For a two-dimensional object, if we construct its external representation which preserve all of its logical properties, then the visual properties of the representation are consistent with the logical properties of the geometric object. In this case, students can guess some properties of a geometric object based on the visual properties (using observation and measurement tools) due to the consistency. For a three-dimensional object, however, even though we construct its two-dimensional representation using the perspective approach, the visual properties of the two-dimensional representation are usually not consistent with the logical properties of the geometric object. For example, parallel perspective planar representation maintains some properties of three-dimensional objects such as parallelism and the ratio of collinear segments, but the representation does not keep some other properties of the objects such as the perpendicularity or congruency of segments and angles. Due to the influence of the approaches they applied to exploring two-dimensional objects and the limitation of their logical reasoning ability, students tend to draw conclusions based on the visual properties of the static planar representation rather than the logical properties of the three-dimensional objects. GSP can provide students with dynamic two-dimensional representations of three-dimensional objects, and adding the dynamic dimension can help students overcome the obstacle. That is, by manipulating the dynamic planar representations such as rotating them, students are able to induce logical properties of the geometric objects, because the visual properties of the external representations move to the background. In addition, students can control the motion. In this way the GSP dynamic instructional environment is valuable to develop students' internal representation (mental image) of three-dimensional objects.

As an illustration of how GSP dynamics helped our young subjects user logical properties to make and verify conjectures, we offer the following example - the problem situation shown in Figure 7, as it was presented to a subject (Mary) in Phase 3:

The problem asked for a comparison between the two segments highlighted in the two dimensional-representation of a regular icosahedron. In the paper-and-pencil test that was given to her first, Mary thought that segment US was not congruent to but longer than segment RT. She was led to a faulty inference because the static view was interpreted so that RT was seen as an edge of the icosahedron. Her explanation was that segment US went all the way across the three-dimensional figure (solid) but segment RT went by the edge of the solid. The explanation revealed that her incorrect judgement came from the wrong visualization of the planar representation of the solid. Mary was at a higher thinking level than many students in her class, but she still visualized RT as an edge of the figure before she was able to relate the visual and logical structures. We can see from here the sharp difference between the static two-dimensional representation of a solid and the solid itself, and the strong influence of the difference on how students interpret the visual.


The following figure is the planar representation of a regular icosahedron.

Are the two red segments equal in the real model? no

If they are not equal, which segment is longer? Why?

Figure 7. Mary's answer sheet for the paper-and-pencil test.

After Mary completed the paper-and-pencil test, the investigator showed her a GSP dynamic two-dimensional representation displayed in Figure 8. Mary clicked the Rotation button, and found immediately a quite different visualization from that in the paper-and-pencil situation. In order to see all possible orientations given by the dynamic representation, she chose to drag the free point on the small circle to observe the rotation in a controlled speed. The feedback that the dynamic

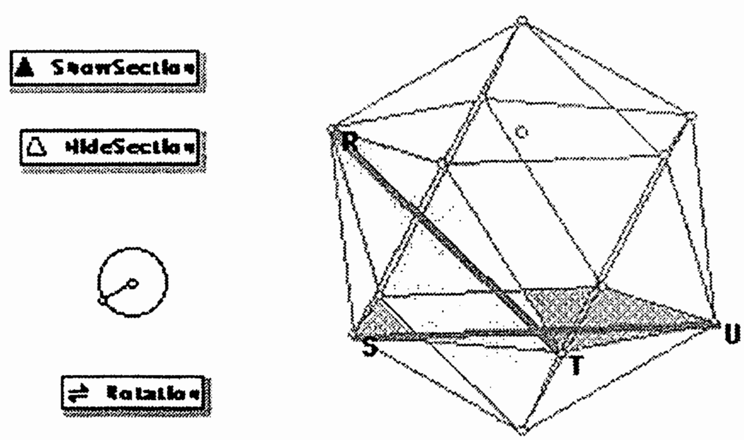


Figure 8. GSP dynamic two-dimensional representation of the problem situation.

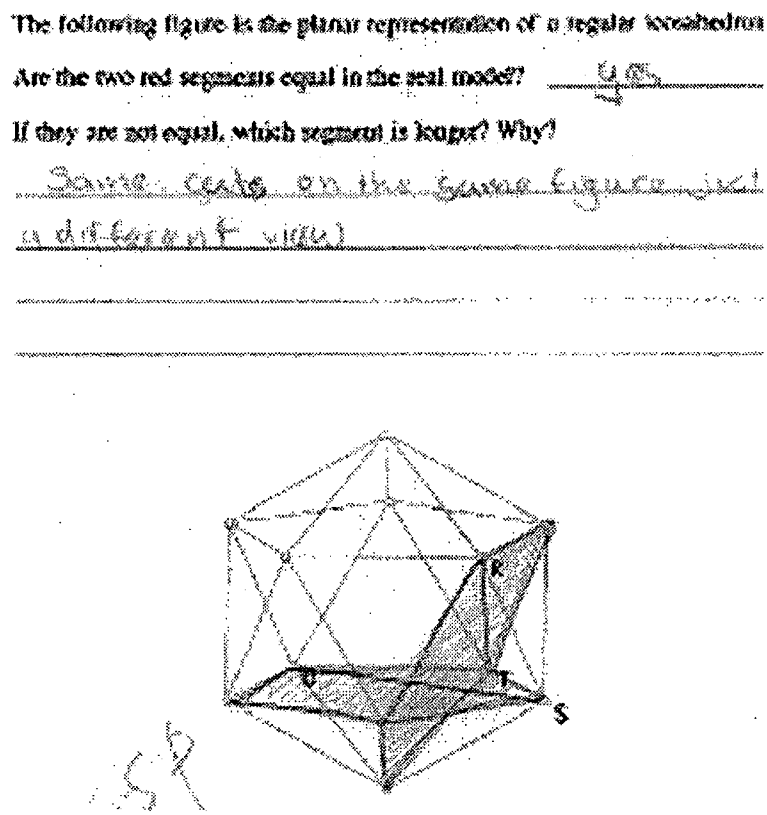


Figure 9. Mary's answer sheet in the GSP version of the test.

representation provided allowed her to realize the congruence of segments US and RT. This time, she was able to visualize segments US and RT as diagonals of two pentagonal cross sections of the solid. Even though she did not mention this explicitly in her explanation, she did indicate the new visualization by shading these two pentagonal cross sections in the figure (Figure 9). Her idea was clearly that the two pentagons were congruent, and hence the two corresponding diagonals were congruent. This is an indicator that she used logic properties to explain her new finding and achieved conceptual understanding of the relevant spatial relationships.

The goal of enhancing the subjects' geometric thought by at least one van Hiele level was basically achieved.

Through the teaching experiments of the research, all subjects made significant progress. We used modified versions of the van Hiele level test developed by Mayberry (1983) to measure the subjects' levels of geometric thought at the beginning of the studies and at the end of each phase. Of these tests, the questions on two-dimensional shapes were from Mayberry's instrument, while those on three-dimensional objects and the criteria for scoring were created by us following the structure of her instrument. By the end of Phase 2, all subjects' geometric thought was found to be improved with the increase of at least one van Hiele level. By the end of the research, all four subjects who participated in the teaching interviews in Phase 3 and Phase 4 improved their geometric thought by at least two van Hiele levels. Two of them (Manuel and Pedro) were able to do logical reasoning for the three-dimensional tasks, reaching the Deduction level (van Hiele level 4). For example, Manuel showed what we consider van Hiele level 4 thought in a variety of tasks. In the task shown in Figure 10, the front of an Archimedian solid was provided on paper. This solid is characterized as being composed of a pair of squares and a pair of equilateral triangles at each vertex. Manuel was asked to draw (on paper) a new solid from which this solid could be sliced. This task was quite challenging because it required considerable knowledge of Platonic and Archimedian solids, as well as strong three-dimensional visualization. Manuel easily completed the diagram (of the Archimedian solid) by drawing the hidden parts of the solid. He was then able to complete drawings of both a cube from which this solid could be sliced and an octahedron from which it could be sliced.

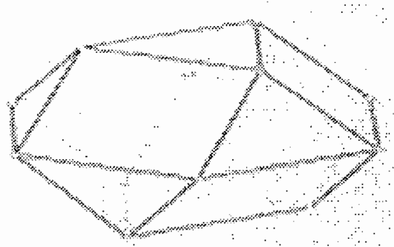


Figure 10. A sketch of an Archimedian solid characterized by the completion of the backside outlines.

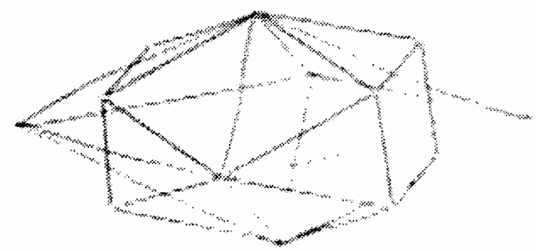


Figure 11. The sketch of one of two Platonic solids from which the Archimedian solid is "sliced."

Figure 11shows the octahedron that he drew . This indicates that Manuel was easily able to use the two-dimensional representations of the octahedron and the cube to produce the Archimedian solid. This task represents an example of complex problems that require mental processing equivalent in thought to that of a geometric proof. Sufficient GSP experience effectively helped him develop solid conceptual understanding so that he was able to solve challenging problems with or without referring to GSP representations. We believe that with consistent performance on such tasks, the subject was performing at level 4 in van Hiele's scheme.

Another example is the following task assigned to Pedro: Construct an Archimedian solid that is defined by the characteristic of having three squares and one equilateral triangle at each vertex. Pedro responded to this challenge by first drawing what he conjectured the solid would look like (see Figure 12). He then proceeded to create

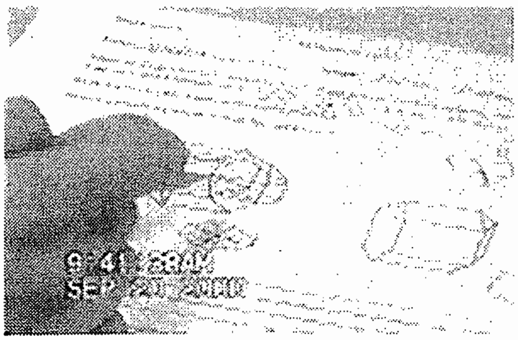


Figure 12. A drawing of the Archimedian solid that has three squares and an equilateral triangle at each vertex.

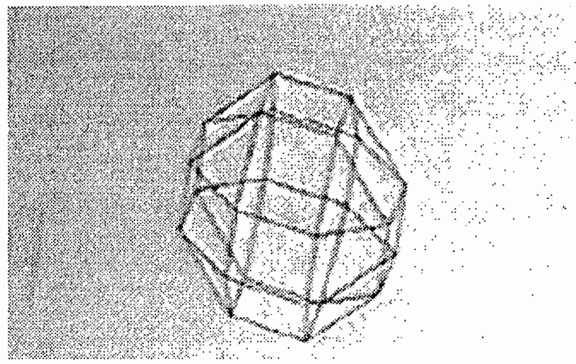


Figure 13. A dynamic GSP version of the Archimedian solid.

a dynamic GSP version of the solid shown in Figure 13. The complexity of the task was substantial. Pedro started with a "local definition" of the solid and proceeded to a paper and pencil drawing. The drawing carried with it analysis and detail so that Pedro was able to discern the structure of the solid. He was then able to construct a dynamic GSP sketch with capabilities of rotation and creation of cross sections. This seems to be indicative of the nature of deductive reasoning, and leads us to believe that this student was functioning at level 4.

Dixon (1997) found in her study that GSP instructional environment was not more effective than the traditional environment at improving students' three-dimensional visualization. However she suggested that the (test) result might have been different if the three-dimensional visualization had been measured using manipulation of blocks or the simulation of three-dimensional behavior using the computer. The results of these studies supported her conjecture.

Conclusion

Three-dimensional visualization is a complicated cognitive procedure in which the subjects need to process the information perceived from external representation and the information retrieved from internal representation in an interwoven, integrative, and dynamic manner. During the studies reported by this article, the subjects actively participated in the explorations of three-dimensional visualization in the GSP dynamic instructional environment. Through interacting with the dynamic environment the subjects had many opportunities to construct their understandings of spatial structures and spatial relationships of three-dimensional objects. The way that the GSP environment differs from the traditional environment is that the GSP environment can provide students with dynamic two-dimensional representations of three-dimensional objects, powerful transformation tools, and some other useful features such as animation and the use of buttons. Specifically, the GSP environment helps students concentrate on the logical rather than visual properties of the three-dimensional objects. The immediate feedback provided by the dynamic environment allows students to verify or change their conjectures. At the end of the studies, the subjects had progressed significantly in terms of their van Hiele levels of geometric thought. In summary, GSP and the associated activities had a distinct positive affect on the subjects when they were developing three-dimensional visualization and pursuing conceptual understanding of geometry content.

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COGNITIVE ELEMENTS OF STUDENTS' USE OF METAPHORS IN A COLLEGE GEOMETRY CLASS

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The aim of the research, of which this paper reports a part, was to obtain a deeper understanding of factors that enhance or inhibit the learning of mathematics in a university-level geometry course. In particular, the study investigated what metaphors and other forms of analogy were constructed by students in the course. These analogies provided a window into the developing geometrical conceptions of the students. In order to attain these goals, this research implemented a qualitative case study methodology using action research. This paper reports on a preliminary analysis of the metaphorical ideas of point, line, and plane of 16 students, with an in-depth analysis of the responses of two of these students.

Theoretical Framework

In the mathematics education community there has been a recent interest in the metaphorical basis for mathematical thinking and reasoning, not only as an aid to learning, but as the very substance of mathematical thought itself (Lakoff & Núñez, 2000; Núñez, 2000). Theoretical formulations have argued persuasively that metaphor and other forms of analogy are important components of mathematical cognition, and thus should be taken into account in its teaching and learning (Sfard, 1997). However, there are very few empirical studies that have investigated the role of metaphors in high school mathematics education (e.g., Presmeg, 1992, 1997), and even fewer at the collegiate level (e.g., Sfard, 1994). Thus we aim to fill a lacuna in the literature on metaphor use in mathematics education by investigating how students in a university course, College Geometry, use metaphor and other forms of analogy to make sense of geometrical concepts. Our theoretical framework, grounded in the literature cited in the foregoing, includes the linking of target and source domains through metaphors in which the source consists of familiar elements of everyday life (for example, containers) and the target domain is a mathematical construct (for example, sets, as used in set theory). In addition to the basic grounding and linking metaphors described by Lakoff and Núñez (2000), we acknowledge that there are idiosyncratic metaphors designated by these authors as "extraneous", which may nevertheless play a vital role in the learning of mathematics by individual students (Presmeg, 1992). Thus our framework is wide enough to encompass all of the spontaneous links between source and target domains constructed by the students in the study during their learning in the geometry course. Because of the exploratory nature of the study, we did not make a fine distinction between use of simile and use of the corresponding metaphor (e.g., "a point is like a period on the end of a sentence", vs. "a point is a period...").

In the broad study, because we are interested in the personal constructions of individual students and how these might be reflexively implicated in and influenced by social interactions during the geometry course, our theoretical framework resonates with the emergent perspective of Cobb & Yackel (1995), in which social and psychological lenses are used to interpret the varied phenomena of learning, in this case in one university geometry class. In the present research we focused on the psychological dimensions as a starting point. There is one further strand in the theoretical formulation of the research issues that we wanted to be sensitive to, although it was not a focus in this study. Some metaphors constructed by individual students may be highly personal, charged with affective meaning. Thus it would be valuable to take into account both cognitive and affective elements of the use of metaphor in geometry, for instance, using Goldin's (2000) model of affective pathways, which involve "not global attitudes or traits, but local changing states of feeling that the solver experiences and can utilize during problem solving" (p. 209). There were hints of the emotionally charged nature of some uses of metaphor in the research reported here.

Methods of Inquiry

Because this research involved one university class during one semester (spring, 2002), it may be conceptualized as a bounded study. A qualitative case study (Merriam, 1998) is appropriate to the objectives. Thus the methodology is qualitative and interpretive, and the methods of inquiry include surveys (initial and at the end of the course), clinical interviews with selected students, and observations during whole class and small group sessions throughout the semester, as well as collection of students' work. Both researchers were present at most of the class sessions, one as observer, and the other as teacher of the course – thus introducing elements of action research into the methodology (Kelly & Lesh, 2000). Quality of the research will be monitored through triangulation of data sources, through respondent validation, and through coordination of the interpretations of the two researchers and the students involved.

All of the 17 students in the course gave their consent to use of the survey results in this research. We have complete journal entries from 16 of the students. Thirteen students agreed to participate actively through interviews and detailed classroom observations. Six students (three women and three men to balance the genders) were selected from the thirteen students for clinical interviews. Selection was based on the initial visualization survey results to include visualizers and nonvisualizers (see next section), because previous research findings had suggested that preference for visual imagery might be a factor in the students' construction of metaphor (Presmeg, 1992). In the following section, the data sources are described.

Data Sources

An attitude survey was administered to all students during the second week of the course. In addition to initial questions relating to their previous experiences of geometry, students responded on a five-point scale to 38 questions that addressed their conceptions of the nature of geometry and how it should be taught, followed by a free-response section entailing related questions (e.g., question 1, "Please describe how your previous teachers presented and explained geometric concepts. How did this make you feel about geometry?" 5, "What do you consider geometry to be?") A modified form of the same survey was administered again at the end. The purpose of the survey was to provide baseline data and to monitor changes through the course in a global fashion. More detailed information was provided from other data sources.

A "preference for visualization" instrument was administered on the first day of the course (Presmeg, 1985). Students completed 12 everyday problems using whatever method was comfortable for them (all problems could be solved using either visual or nonvisual means). Then they completed a questionnaire that asked them to identify the methods of solution they had used. Visual solutions (written or imagined) yielded a score of two points, nonvisual solutions zero, giving a possible total of 24 points on a "mathematical visuality" scale. The purpose of this instrument was to identify the "visualizers" in the class, that is, those students who prefer to use visual imagery or diagrams in solving problems that may be solved with or without these methods, in case preference for use of imagery was a salient factor in the representation they chose for geometric analogies.

Students submitted reflective email journal entries each week of the course. In the first journal entry, they were asked to expand on the two questions, "What is geometry to you?" and "How would you like to learn geometry?" In subsequent entries they reflected on experiences in the class, and on geometry learned that week and how they felt about it. The journals had the twofold purpose of establishing a line of ongoing communication, and of providing data relating to cognitive and affective aspects of students' learning. Students were also asked in specific journal entries to address their notions of proof, and their metaphors for point, line, and plane.

The six students who were selected from the 13 who agreed to full participation discussed cognitive and affective aspects of their mathematical concepts during the interviews. These interviews were video- and audio-recorded because body-language might have been important (not addressed in this paper).

Preliminary Results

The survey results are in the process of being analyzed. The "preference for visuality" instrument yielded scores ranging from 2 (nonvisual) to 22 (highly visual) out of a possible 24, with a class median of 12. Students with extreme scores were included amongst those interviewed. The scores of the six students chosen for interviews ranged from 6 (nonvisual) to 22 (highly visual). It was mainly in the interviews that metaphors were investigated in depth, but students' journal entries and classroom observations also yielded metaphors. Metaphors and analogical reasoning abounded while class participants were talking about solids, non-Euclidian geometry, tessellations, and defining concepts such as point, line, and plane. No use of analogies was explicit throughout the middle of the course while students were tackling the Hilbertian style of proof and working through proofs involving standard geometric theorems.

Some of the richness of the spontaneous idiosyncratic metaphors generated by various students can be gleaned from the following list:

- a truncated pyramid is a sliced chunk of cheese;
- the x-axis on a coordinate plane is water level;
- an icosahedron is the interlocking jaws of a monster;
- a slide (translation) is a raindrop going down a window;
- the discovery of "tessellations" is like the discovery of America: they didn't really invent it but they named it "America".

These analogies arose in the interviews, or were generated in class discussion. Some of these were explored in greater depth with the students but these are not a focus in this paper. In the next section, after an in-depth description of two students' use of analogies for point, line, and plane, we provide a general overview of 16 students' metaphors for these concepts.

Conceptions of the Concepts of Point, Line, and Plane

The constructs of point, line, and plane are usually taken as undefined terms in the axiomatic system of incidence geometry (Rainich & Dowdy, 1968). They can be viewed as building blocks upon which the edifice of a geometry course can be laid. The text used in their class, College Geometry (Kay, 2001), adopted the formal system found in Moise's Elementary Geometry from an Advanced Standpoint, an adaptation of Hilbert's original axiomatic development of elementary geometry. Point, line, plane, and space were taken as undefined terms. However, in class, the instructor stressed that the visual representations of points, lines, and planes were not the theoretical constructs themselves. We now take a closer look at the development and maturation of these ideas in the eyes of two college geometry students to whom we have given the pseudonyms Jane and Mary.

"A point is that which has no part" – Euclid

Initially, both conceived that a point was tiny. Mary believed that any dot on a piece of paper was a point and expressed amazement that until this course, she did not realize that a point was actually just represented by that dot. She implied during the second interview that she was shocked a point was not a dot because her high school teachers had always told her it was. She added, "Those who are going to major in math can think about this so that when they get to college they are not so shocked when someone says that to them." Even though she felt it was important for her and other mathematically inclined students to understand the theoretical conception of a point, she made sure to state, "It's not that our high school teachers were lying to us, they just did not think we needed to know that." Jane appeared to struggle less with reconciling the current definition of point with her previous notions, stating, "I think of the points you are looking at when you look at a piece of paper...I have a theoretical image of a point: that it's as small as you can possibly get. There is nothing actually there, but there is." Jane appeared to be realizing that there were two ontological entities involved, namely, the representation of a point and the point itself.

Mathematically, a point is dimensionless. It has no size, even though our bodily experience suggests to us that if a point is a fundamental component of lines, planes, and solids, then it must have size, else how else could it exist? The students struggled with this idea of a nothing that is, just as Euclid and his contemporaries did. This is best seen by considering that Aristotle adopted the definition given by the Pythagoreans: "that which is indivisible in every way in respect of magnitude and without magnitude but has position" is a point. Plato apparently objected to this definition because it implied that points were a "geometrical fiction". Euclid, himself a Platonist, omits that a point must have position (Heath, 1956). Trying to reconcile a something that has no dimension was implicated in Mary's statement, "It's there, it exists but you can't see it. It's hard for me to understand but I guess a point is there but you don't actually see and it's like microscopic- it's not there- it's teeny, teeny, tiny," and Jane's statement "There is nothing actually there, but there is. I think of it in terms of an atom. I can't really look at an atom in a microscope or anything like that but I know what it is." Interestingly, the concept of a point being an indicator of position did not explicitly appear in any discussion with or journal entries of Jane and Mary, although it might have been implicit in Mary's journal-entry metaphor, "a point is a star in the sky".

As indicated by the journal entries detailing students' personal metaphors for point, submitted in the last weeks of the course, Jane and Mary decided that a point was smaller than they could conceive. As another student (who has not been interviewed at this time) pointed out, a point is a grain of salt in a beaker of water just before it dissolves. In these journal entries they appeared to be reconciling this notion of no size versus existence by providing metaphors for point that likened a point to something that was really small compared to its surroundings. Comparably to Mary's macroscopic metaphor, "A point is a star in the sky," Jane likened a point microscopically to "an atom in one of the paper fiber molecules used to make a thin piece of paper."

During the final interview, the students were asked once again to discuss point, line, and plane. They also were presented with a table that had point, line, and plane paired with the descriptions their classmates had provided in their last journal entries. The students were asked which pairs seemed alike or not alike and to explain their reasoning. Jane and Mary still focused on the size. In discussing their choices they both emphasized the importance of choosing objects that were small compared to their surroundings. Jane claimed to have used her chemistry background to adjust her final journal entry metaphor to become a point "is a fraction of a proton." Mary's initial conception that any dot was a point (which she always drew as a small circle) seemed to manifest itself in her justifications for her metaphor selections. Her favorite, a point is a pinprick, was selected because a pinprick is "precise and is circular".

This leads to the question of whether Mary's use of metaphor to describe point is actually restricting her to a concrete understanding of the notion of point. Or is it the nature of undefined concepts to be difficult? Historically, there has been much discussion on how to describe a point. In modern times, geometers have tried to circumvent the problem by actually using objects from the material world as illustrations from which the concept is abstracted (Heath, 1956).

"A line is breadthless width" – Euclid

Both Jane and Mary, when initially asked to define line, immediately defined straight line. By the end of the course Jane seemed to become aware that the word line triggered the concept of straightness for her, but it was unclear whether Mary had this self-knowledge. Mary initially defined line as a "sequence of points that are connected" and "it's just a group of points lined up — together, one on top of the other all touching." This original definition manifested itself at the end of the course in her personal metaphor "a line is a string of beads", which was chosen because "the beads are all strung up like the points are on a line."

The attribute of straightness was important to her throughout the course. When asked how she would describe a line to a child she initially sketched a line and then remarked that it was not straight enough and proceeded to fold a corner of the paper and point to the crease as a good example of a straight line. This also seemed to indicate that the attribute of infinite length was not as important to her. In the final interview, some of her preferred metaphors, a line is "a dividing line on a road because it is thin", or a line is "a straight edge of a ruler" indicate that she still focused on the straightness of a line and also on her perceived thinness of a line. It appears she saw a line as having some kind of thickness if we take these together with her comment that length and distance are not good metaphors for line because "they cannot be seen and a line can be seen." None of the discussion in the third interview indicated a need for a line to have infinite length. A slight indication that Mary saw a distinction between line and the concept of straightness appeared when she explained why a piece of rope was a good metaphor: "this one is good because you can bend a rope around into different shapes like a line." However in the same interview, when asked what she thought when she heard the word line in geometry class, she stated, "Shapes, what shapes are made of — you *bend* a line into the different shapes."

Jane appears to have an early conception that a sketch of a line is a representation and that a line must have infinite length: "the line would be something that has a distance but you can't ever tell how thick it is and because in order for you to draw it you would have to give it thickness you can't ever really draw a line." During the last interview, Jane acknowledged that when asked about a line she immediately thought "straight, in the Euclidean sense", until their exposure to taxicab geometry. In her early descriptions she also used the Archimedean property of a straight line, "the shortest distance between two points and the fastest way to get between those two points in space." By the last interview, Jane stated that any representation of a line should involve an infinite set of points, have no start or end, and was straight in the "Euclidean sense". When discussing which metaphors she preferred, she primarily focused on the last two properties. The metaphor, "a line is an arrow with no surrounding forces (like gravity) acting on it", was rejected because it had a visible starting point. Another rejected metaphor, a line is a road, did not work because "I have seen too many curved roads."

In these metaphors, the issues that arose were straightness as an attribute of a line, and dimensional attributes of length and width in the representations. The metaphors appeared to bring these issues to the fore. The discussion in the interviews facilitated reflecting by the students, without pre-empting by the interviewer. This was also the case in the discussion of metaphors for a plane.

"A plane surface is a surface which lies evenly with the straight lines on itself" – Euclid

Both described a plane as a flat surface that contained points and lines throughout the course. Jane stated that she still thought of a plane as it was initially described to her: "a sheet of paper that keeps on going forever but its so thin that you can't even see it because its only the thickness of a point." She also included the condition that you can't curve the paper because a plane must be flat.

Mary also claimed that a sheet of paper would be a plane and justified this by saying a plane is "just where you could draw points and lines." She also stated that a plane could have any size, paper was just a plane in "minimized area" and that "you could extend it as far as you wanted to. Mary implied that her notion of the size of a plane had changed by stating that the faces of a tetrahedron would be "parts of planes" and that a circle is a "cut-out" of a plane. In the last interview, both Jane and Mary found the metaphor "a plane is a chalkboard" the best because it was a flat surface that would contain lines and points. Jane also mentioned that this was a good metaphor for a classroom because it was convenient.

Jane was very careful to always include a disclaimer of infinite size in all of her descriptions of plane. With the chalkboard she stated, "it goes on thru earth and its sides extend forever", and she liked the metaphor "plane is a wall" because she stated that with a wall you could think "longer than the great wall of China and however tall." Her choice of metaphors for plane also seemed to indicate that the concept of zero thickness was important to her. She rejected a pane of glass because she imagined seeing it from the side and substituted a plane is "a sheet of infinitely long saran wrap." When reminded that a chalkboard also has thickness she remarked that you do not look at it from the side.

In these metaphors, Jane and Mary both seemed to realize that the plane extends infinitely, even if the piece of paper or the chalkboard do not. This is a dimensional issue. The other dimensional issue of thickness was addressed by Jane very explicitly ("paper one atom thick"), but not by Mary, who predominantly focused on the flatness attribute of a plane.

General Overview

From the final interview discussions with Jane and Mary, the text's discussion, Heath's translation and commentary of Euclid's Elements, and the researchers' understandings of the concepts of point, line, and plane, the two categories, dimension and structure, for the attributes of the three concepts emerged. Dimension contained the attributes of size (length, width, height) and structure included attributes of the nature of the object such as its position, shape, or relationship to other objects. In the following we discuss the metaphors that 16 students (including Jane and Mary) provided for point, line, and plane in their metaphor journal entries according to these two categories.

As discussed earlier, a point is dimensionless. Although asked to create a metaphor for a point, one student chose to provide defining attributes of the point such as "a point on a plane has no space but has a definite position." One student provided a simile "a point is like the period on the end of the sentence," and the remaining 14 provided 21 metaphors for point. Most of the metaphors (16 of 21) likened the point to an object that was small: "a point is a microchip" or to an object that was small compared to its surroundings: "a point is a baseball stopped in space." Three of the 21 analogies for point had no size and were also the only metaphors that referred to the position aspect of point, a point is "an imaginary number", "a place but not a spot", "a designation." Four of the students did use the metaphor a point is dot; one called it invisible, another mentioned it was a dot with no length or width, and two put no qualifications on the point. It is not known if the last two realized that a dot is the physical representation of a point or if like Mary they thought that a point was the dot. Some of the 23 responses are shown in table 1.

A line is infinite in length but with no thickness. Of the 22 responses provided by 16 students, 19 were metaphors, three were similes. 13 of the responses provided for the attribute of infinite length including such adjectives as never ending, infinite, forever. Four of the responses explicitly addressed the fact that a line should have no breadth or width. Six of the responses appeared to address the breadth issue, "a line is a never-ending straight path to any destination you wish..." The other responses included such items as a line is "a never ending string", "the horizon" and could be interpreted as referring to the width issue. Some of the 22 responses are shown in table 2.

Overwhelmingly, the 15 students likened a plane to a flat surface such a chalkboard, sheet of paper, wall, or pane of glass. Two students provided mathematical descriptions for a plane such as "a two dimensional region of space." Of the 26 responses, seven addressed the infinite size attribute explicitly by using such words

Table 1. A Selection of Students' Metaphors for a Point

A point is ...	**Dimension**	**Structure**
a dot (two students)	small	circular
the size of a pea in space, only infinitely smaller	small compared to surroundings	
a star in the sky	small compared to surroundings	
a period at the end of a sentence	small compared to surroundings	circular
atom	small	component
grain of salt dropped in a beaker of water right before it dissolves	small	position?
a spot within an area	small	position
a space, but not a spot	no size	
an atom of one of the fiber molecules used to make thin paper.	small compared to surroundings	component
microchip	small	
imaginary number	no size	

as never ending or infinite. Two of the metaphorical responses commented on a plane having the mere thickness of a point and only extending in two dimensions. Some of the 26 responses are shown in table 3.

This paper could address only a small part of the study, and the analysis is ongoing. The categories of dimension and structure were grounded in the metaphors generated by the students. The eliciting and discussion of the students' metaphors turned out to be a powerful research tool that allowed for deeper probing of their understandings of point, line and plane. Because many of the students' metaphors were new to the researchers (although they were very familiar with concepts of point, line, and plane), these novel ways of thinking enabled the researchers to search more deeply for what the students meant, and to probe further by questioning. Instead of using a geometrical task or problem in the interviews, the discussion of the metaphors generated by these

Table 2. A Selection of Students' Metaphors for a Line

A line is ...	**Dimension**	**Structure**
a ray of light	infinite length	straight and thin
a straight road		straight
a stretched piece of rope		thin
a piece of straw	finite length	straight
an infinitely long marker that separates two lanes of traffic on an infinitely flat road	infinite length	thin
a never-ending string	infinite length	thin
the horizon	infinite length	straight boundary
an arrow with no surrounding forces (like gravity) acting on it	infinite length	straight and thin
a string		thin
a never-ending telephone wire	infinite length	thin
a path that never ends	infinite length	
a hockey puck path (with no friction to stop it) in an arena with no walls	infinite length	straight and thin

students and others provided a window for starting to understand their notions of the nature of point, line, and plane, and the connections between these concepts and their physical representations.

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Table 3. A Selection of Students' Metaphors for a Plane

A plane is ...	**Dimension**	**Structure**
a table top	finite	flat surface
a chalkboard	finite	flat surface
a wall (two students)		flat surface
a pane of glass that extends infinitely on all sides	infinite	flat surface
a sheet of paper	finite	flat surface
the song that doesn't end	infinite	
a roof	finite	
a never-ending chalkboard	infinite	flat surface
a floor	finite	flat surface
a football field, tessellated to go to eternity	infinite	flat surface
a smooth piece of wood		flat surface
a piece of paper that keeps extending forever	infinite	flat surface
a lake that will never end	infinite	flat surface

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WRITING CONJECTURES IN GEOMETRICAL ACTIVITIES WITH CABRI-GÉOMÈTRE

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We present here some advances concerning a project whose aim is to explore the relationships existing between the geometrical activities resulting from some students' activities in an environment of dynamic geometry (using Cabri-Géomètre software, in this case) and the production of written formulation of conjectures and proofs. The study was carried out with eight senior high school volunteer students, 16-17 year-olds, who had previously attended a Euclidean Geometry course of one semester in a public school.

The Cabri-Géomètre scenarios help the student to accede to the knowledge of geometrical facts. The validation of these facts is based on the "dragging" technique. This is an exploration tool that helps the student to discover the invariant relationships that are present in the geometrical constructs (Hölzl, 1996).

In the production of this knowledge some input texts (instructions), the student's interaction with software, and the geometrical representations produced and perceived on the screen are all involved. However, according to standard mathematical criteria, the scientific consolidation of these geometrical facts demands that they be constituted in a system of results that needs to be expressed in written argumentation and in a language adapted to the demands of deductive proofs. Hence, the importance arises that conjectures and propositions are formulated in writing.

Furthermore, writing, usually, and also oral language, gives shape and precise meaning to thoughts or ideas, which previously lacked exact contours. When this happens, not only the existing thought is transferred into a written, or oral formulation, but also, in doing so, it is modified and it is given a precision of which it lacked, making it remarkably clearer.

In the case of students' geometrical ideas, their activity with dynamic geometry allows them, even before the writing phase, to have a representation which is free of ambiguity and imprecision, and also gives them a conviction of the validity of such ideas; and this representation should help them to make a written formulation, since their linguistic competence is now acting on a less unshaped material. A question, however, arises. How does the representation of data processing of geometrical facts help the students to accede to the possibility of writing the representation of such geometrical ideas?

Background

Some researchers have recognized the importance of the role played by oral and written language in understanding mathematics (Duval, 1995; Laborde, 1990, 1992; Pimm, 1987); and including *communication* (oral and written) as a standard process in

NCTM (NCTM, 2000) is a demonstration of that assertion. On the other hand, studies about mathematical proof have highlighted features that are making clear the complex field of mathematical proof: the role played by empirical evidence in contrast with deductive arguments (Chazan, 1993; Martin & Harel, 1989); the difference between argumentative reasoning and deductive reasoning (Duval, 1991); and the students' proof schemata (Harel, 1996). Particularly, with the arrival of computers, the problem of the influence of activities with dynamic geometry on learning mathematical proof has been posed (Hoyles & Jones, 1998; de Villiers, 1998). Balacheff (1987) emphasizes the role of language in the passage from pragmatic proofs to intellectual proofs. In spite of his remarks about the importance of formulating situations, the results about conjecture processes (Kakihana, Shimizu, & Nohda, 1996; Boero, Garuti & Mariotti, 1996) do not consider the difficulties imposed by language, which enrichment and broader usage could open an access to proof. In our study, we consider that possibilities provided by dynamic geometries could allow us to reconsider the relationships between the language and mathematical proof since the very moment of discovery and formulation of conjectures.

Experience Design

In the summer of 1999 we organized a geometry workshop with eight senior high school students (16–17 years old) who were volunteers from a public school (IPN) in Mexico City. They had attended a geometry course. In eight four-hour sessions, distributed along two weeks, they were taught to use the software (Cabri–Géomètre). Then, they were asked to do some activities in order to remember basic facts of geometry and other activities so that they could discover relationships between the medians and the areas of a triangle. The activities were taken from a document from the teamwork of Cabri (Équipe EIAH, 1996).

Each student worked individually. We counted on the support of four observers and each one of them was assigned to two students. The observers had the permission to intervene in order to help students only with the related difficulties of software use, without giving their opinions about questions of mathematical content. Moreover, each instructor wrote down the most relevant facts observed from the students' activities. We have considered in this report only the instructors' remarks that plainly show whether each student arrived to the correct conclusions of each activity.

In order to analyze the written productions, we considered the correct syntax of the statements (i. e., we observed whether the sentences obeyed elemental grammatical rules of language (Spanish) and whether the expressions in natural language were coordinated adequately with the geometric symbols and with the figure).

Another criterion more specific consisted in the comparison of the statements with the conditional structure ($P \Rightarrow Q$) of the corresponding proposition. In particular, we observed whether the statements each student wrote had a complete description of the antecedent. This feature would tell us if a conjecture was conceived as a complete proposition, in which the formulation of the antecedent is fundamental.

We observed in a punctual way whether a student formulated the propositions without making reference to the figure or if he/she used it to facilitate its expression. The use of symbolic notation tied to the figure mitigated considerably the difficulty in expressing the geometric ideas.

Summing up, the following three aspects help us to get closer to the texts produced by students: A) Correct syntax; B) Complete description of the antecedent; C) The support of the text on the figure and the use of symbols.

The Conjectures

The propositions to which the activities we posed to the students could lead, constitute a system in the sense that a proposition can be deduced from other (or others) previous proposition(s). The propositions related medians to areas of triangles. We set them forth as they are below in a conditional form.

Theorem 1: If in the triangle ΔABC, AM is a median, where M is the midpoint of BC, then Area (ΔAMB)=Area (ΔACM)

Theorem 2: Consider a triangle ΔABC and a median AM, where M is midpoint of BC. The height of triangle ΔABM that departs from the vertex B is congruent with the height of triangle ΔAMC that departs from the vertex C.

Theorem 3: If in the triangle ΔABC, M is an inner point so that Area (ΔAMB)=Area (ΔACM), then M is upon the median AX, where X is the midpoint of BC.

Theorem 4: Let □ABCD be a trapezoid, with AB||DC. Let S be the point of intersection of the extension of the sides that aren't parallel (i.e., S = AD BC) and let I be the midpoint of AB. Then, segment SI is a median of ΔABS. Let X be the point of intersection of the beam SI with segment DC. Then segment SX is a median of ΔDSC.

Theorem 5: Consider a triangle ΔABC and let A' be the symmetrical of A with regard to C; B' the symmetrical of B with regard to A and C' the symmetrical of C with regard to B; then Area (ΔA'B'C') = 7 × Area (ΔABC).

Theorem 6: The medians of a triangle divide it in six triangles with equal areas.

In the following, we present activities 1 and 3 that were posed to the students in the second phase of the workshop.

Activity 1

- Construct a triangle; label the vertexes with A, B, C.
- Mark the midpoint of segment BC, calling it I.
- Define triangles ΔAIB and ΔACI
- Obtain the areas of triangles ΔAIB and ΔACI.

- Grab any vertex A, B, or C and drag it to a new position. Observe what occurs with the obtained areas.
- Write a conjecture about your observation and write the corresponding test.

Activity 3

- Draw a triangle, label the vertexes with A, B, C and elect a point M in the interior of the triangle; create triangles ΔABM and ΔACM and include their areas; construct semi–straight line AM, name X the intersection AM with BC. Include the measurements of segments BX and XC.
- Move point M until you have achieved that the areas of triangles ΔABM and ΔACM are equal or almost equal. What can you now say about the position of X?
- Can other positions of M be found for which the areas are equal? What conjecture can you formulate for the position of X? Write it down.

Upon carrying out the previous activities, students would have formulated theorems 1 and 3 enunciated above.

Observations and Analysis

In order to refer to the students we have numbered them from 1 to 8. According to the observers' reports, activity number 3 was one of the most problematic ones. Although all students made the construction they were asked and they related the pertinent aspects, that is, the equal areas with the inner point placed upon the median, they were not able to distinguish exactly what they were asked. They enunciated the inverse proposition. However, they were not aware of their confusion. Only the observer perceived they were confused.

In the rest of the activities, except for two other cases, all the students made the construction adequately and "saw" what the activity was about. Student number 4 could not organize the activity number 4 very well and student number 8 got confused with activity number 5. So, in general terms, we can affirm that the students noticed the reason why these activities were conducted.

On the other hand, their achievement on writing the corresponding propositions was very poor. If they were evaluated with certain rigor, almost all the statements had mistakes. In spite of them, the authors unified the criteria to decide which statements could be considered correct in the sense that they expressed the asked idea; in the following frame we indicate with the ones that were evaluated as correct and with number 0 the ones that were considered as incorrect (See Table 1).

Based on the pointed out categories, we would like to summarize the students' observations in the following chart 'Student by Activity' (See Table 2). In the columns with the heading "A", numbers are assigned in relation to the correct syntax of the

Table 1

Student	1	2	3	4	5	6	7	8
Activity 1					½	½	0	
Activity 2	0			0		½	0	0
Activity 3	0	0	0	0	0	0	0	0
Activity 4	0	0	0	0		½	0	0
Activity 5	0	½	0	0	0	0	0	0
Activity 6	0	0	0	0	0	0	0	0

Table 2

Student	1			2			3			4			5			6			7			8		
Activity	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
1	1	1	0	1	1	0	1	1	0	1	1	0	½	½	0	½	1	0	0	0	0	1	1	0
2	0	1	0	1	1	½	½	1	1	0	0	1	0	1	1	½	1	0	0	0	0	0	1	0
3	0	0	1	0	0	1	½	1	0	1	0	0	0	0	1	0	0	0	0	0	1	½	0	1
4	0	1	0	0	0	1	0	0	1	½	0	0	½	0	0	1	0	0	0	0	1	0	½	1
5	1	0	0	1	½	0	1	½	0	½	0	1	½	0	0	1	0	1	1	0	1	0	½	0
6	1	-	1	0	-	0	½	-	1	1	-	0	0	-	½	½	½	0	0	-	0	½	-	0

written propositions from the corresponding student. Number 0 has been assigned to the enunciations that violate several syntactical rules in a way that makes the enunciation very confusing. We assigned the fraction 1/2 to the statements that, although they are defective, they were able to transmit the meaning of the proposition. Number 1 was assigned to the statements that were judged to be totally correct. Naturally, the beforehand statements and the following ones reflect the point of view of the researcher and his collaborators; so, the assignations are somewhat subjective.

In the columns marked "B", we grade the complete description of the antecedent with 1 if the enunciation is present and with 0 when it is absent; we have placed 1/2 when it is difficult to decide between both extremes. Finally, with the category "support of the figure" the columns with the heading "C", we assign 0 if the enunciation is not supported by the figures nor utilizes symbolic notation, with 1 when elements of the figures are expressed trough symbols such as P, PQ, AB, etc.

In the first two activities almost all students described the antecedent in their statements, but in the activities 3, 4, and 5 only in three occasions there was a complete description of the antecedent (in activity 6 it was not required to formulate the conjecture).

In several occasions inadequate words were used to refer to objects that students had already understood in the screen. For example, one of them uses *perpendicular* *bisector* instead of *median* (in Spanish: *mediatriz* and *mediana*, respectively), another one uses *vector* trying to say *vertex*, another one writes *heights* of the triangle to designate the *sides* of the triangle others than the base.

We observed that the corresponding statements to activities 2, 3 and 4 display a bigger difficulty in the syntax. This could be happening because the complexity of the statements is greater and they demand specialized phrases that become complicated if the figure and the symbolic notation are not used, for example: "the perpendicular to the straight line... that crosses the point...", "the areas of the triangle formed by the interior point to the triangle, the superior vertex and every vertex of the base segment are equal...", "the triangle formed by prolonging the non-parallel sides of the trapeze...".

The lack of vocabulary and of control in the formulation of appropriated phrases to describe the geometric objects, and the reluctance of the students to use symbols, make it harder for them to do the task. In effect, only in few occasions they utilize graphics and symbols. We had to consider that in some cases they needed to use symbolic notation.

We now present some observations regarding the students' productions as answers to *Activity 1*. We noticed that there were two ways in which the question was understood. Four students properly understood that they were being asked to write the statement of the proposition on the areas of the triangles that are formed by means of the median. The other four gave more ample answers while trying to produce a proof. Only two students of the first group produced statements that, although not quite precise, they were syntactically and mathematically correct:

Student 2: The triangles formed when drawing the median of a triangle have equal areas.

Student 3: When dividing a side in half and then joining this point with the opposite vertex, the area(s) of the two triangles thus formed are always equal.

The other two students established syntactically incorrect statements, although we were able to ascertain that they were thinking in terms of the correct idea:

Student 5: The area of two triangles that are inside another one, whose side is the perpendicular bisector of one side of the first triangle, is equal for both.

Student 6: If in any triangle the middle point of a segment is obtained and when joining it with the opposite vertex, the area of such triangles thus formed are equal. If you obtain the midpoint of the base segment and this, when joined with the third vertex, the areas of the two triangles are equal.

Of those offering an argument, only one achieved an acceptable formulation, and another one almost succeeded:

Student 1: When drawing the midpoint in any side of any triangle, and when joining it with the vertex opposite to that side, the two triangles thus formed will always have the same area because the height is the same for both. And they will be compensated with each other.

Student 4: A median divides the area of the triangle into two equal areas. For, since they have the same common base, with the same dimension, in these two triangles their height will be the same.

The other two students formulated only redundant statements, without succeeding in arriving to an argument:

Student 7: They have the same area because when obtaining the midpoint, that is the median; and therefore, they have the same area.

Student 8: When in a triangle one of its sides is divided with a median, two triangles with the same area are formed, because such a median divides them in two exactly.

We can observe that, in spite of the fact that the students can show that they know the result, and that it furthermore convinces them, they find great difficulties when trying to formulate it correctly.

Conclusions

Balacheff suggests that in order to accede to what is called mathematical proof, it is necessary that the student accede to a functional language that would not be only a way of describing the actions. Such language is characterized by the introduction of certain quantity of symbolism (Balacheff, 1987, pp. 58-59). Our observations show students' difficulty to accede to that functional language. Furthermore, we can say that it is necessary for the students to learn how to use the symbols. It is also important for them to acquire and use the vocabulary of geometry and learn the syntax of phrases that describe geometric objects.

The frame that a student requires in order to organize information with the purpose of carrying out a deductive geometric demonstration is different from the frame, which is required in order to have a certain control of geometric objects on the screen and observe behavior. Nonetheless, in the written activity a complexity is revealed which the software yields and hides at the same time. But beyond all doubt, different from the past, the activities with dynamic geometry can permit the explorations of the properties of the objects. Otherwise, as a consequence of this study, we would advocate that parallel activities of writing in all levels of education not be abandoned.

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SHAPES, ACTIONS AND RELATIONSHIPS: A SEMIOITIC INVESTIGATION OF STUDENT DISCOURSE IN A DYNAMIC GEOMETRIC ENVIRONMENT

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It has been argued that trying to describe ways in which students interact in dynamic computer environments may extend beyond Piagetian developmental levels and stage theories of learning (Connell, 2001; Doerfler, 1991). Therefore, the purpose of this study was to use a semiotic perspective to describe the discursive activity of students learning about quadrilaterals in a dynamic geometric environment. During a two-week lesson study, the onscreen work and dialogue of two students was captured on a video recorder. An analysis of the students' discourse gave evidence that students concurrently construct their understanding of properties and relationships between different quadrilaterals. Furthermore, the students seem to treat the on-screen-geometric representations as mathematical objects within themselves and not as geometric tools.

Within the last decade, rapid developments in computer technology and mathematical software have given students of mathematics new means by which mathematical objects and relationships can be visualized (Battista, 2001; Connell, 2001; Sfard, 2000). For example, dynamic geometric software allow students access to mathematical shapes that may be manipulated and morphed while preserving the defining properties that make up that shape. Due to this action on object capability that dynamic geometric software provide, it has been argued that the trying to describe ways in which students interact in these computer environment may extend beyond Piagetian developmental levels and stage theories of learning (Connell, 2001; Doerfler, 1991). In order to investigate this claim, this teaching experiment examined the nature and development of student discourse in a computer based dynamic geometric environment using a semiotic framework of inquiry. The emphasis of the study was to document and describe how fifth grade students visualize and conceptualize the dynamic on-screen representations of quadrilaterals in an interactive hypermediated environment. In particular, in what ways do the students' discursive activity about these on-screen representations differ from more traditional discourse about quadrilaterals.

Actual Reality, Virtual Reality, and Prototypes

The initial context for this investigation lies in what Sfard (2000) calls actual reality (AR) discourse and virtual reality (VR) discourse. AR discourse is a form of communication in which the objects being discussed may be perceptually mediated by the senses. For example, the statement, "*I moved* the chair *to the other side of* the room," is an AR statement in that one can perceive the actual objects being moved. In contrast, VR discourse is the form of communication in which the perceptual mediation is limited if not impossible, and may only be understood through the use of symbols representing the objects being discussed. The statement, "*I moved* the x *to the other side of* the equal sign," is an example of a VR statement. In this case, x, is a variable, a varying quantity of unknown value. The equal sign is a symbol to denote some type of relationship between two mathematical quantities. Note that in both cases, the symbols of x and = refer to abstract mathematical objects that are not immediately perceivable to the human senses. However, in spite of the differences between the symbol/object relationship between AR and VR discourse, note that both types of discourse are structurally similar and use symbols mediated by objects, be they perceived or not (Sfard, 2000). While most of mathematical discourse lies under the category of VR discourse, it is not always the case for geometry.

One major difference between geometry and other areas of mathematics (algebra, arithmetic, or probability) is the perceivable, AR, connection between geometric concepts and actual objects (Doerfler, 1996). From a conceptual standpoint, a triangle, being a 2-dimensional figure, does not have a real word object that represents the defining qualities of a triangle. A triangle in its purest mathematical form is a two-dimensional object having no depth, that is made up of the union of three one-dimensional line segments at their endpoints, each having no thickness. Such an object does not exist in the physical world. However, what does exist is the ability to construct a drawing of a triangle that accurately reflects the essence of the definition. Furthermore this object, for all practical purposes, is a widely agreed upon as the representation of a triangle which allows for the AR discourse of triangles.

Despite the AR nature of geometric figures, there are many relational concepts in geometry that are not immediately perceivable to the senses and are mediated through the process of deduction on known properties and symbols of geometric objects. The statement, "A square is a parallelogram," is a relationship that is traditionally constructed not based on the objects themselves, but rather through a deductive mental process. A parallelogram is a quadrilateral with two sets of parallel sides. A square has two sets of parallel sides, therefore a square is type of parallelogram. This differs from the statement, "A collie is a dog." In this AR statement there are perceivable characteristics, namely that a collie looks like other dogs that allow one to construct a meaningful relationship.

What has changed as result of the advent of dynamic geometric software is a new class of geometric objects that may be manipulated in a manner to allow for AR discourse on traditionally VR concepts. In the program Shape Makers (Battista, 1998), one example is a dynamic on screen object known as a parallelogram maker. This on-screen object may be dragged, turned, and morphed into various forms while preserving the defining qualities of a parallelogram. We use this notion of AR and VR discourse to talk about the ways in which children construct mathematical relationships through perceptual mediation that previous to dynamic geometric software could only be constructed through abstract deductive processes.

Another important concept in this investigation of student discourse in a dynamic geometry environment is Doerfler's (2000) notion of prototypes. These processes involve the perceptive, discursive and cognitive interaction with, and the manipulation of, some kind of mathematical model. This model, known as a concrete carrier, may be an object, a drawing, or a mathematical expression be it physical or imagined. In this case, the carrier is a computerized on-screen representation of geometric objects. It is important to realize that the carrier themselves do not represent the respective concept. For example, a common carrier used in understanding rectangles is a pencil line drawing of a rectangle. The concept of a rectangle that an individual brings to the drawing is restricted to their prototype of a rectangle. To a young child, the prototype may be limited to a number of real life examples. However, to a mathematician, the prototype may allow for the visualization of non-Euclidean representations of rectangles. In this way, the same drawing is a carrier for very different prototypes.

Doerfler (2000) describes three types of prototypes that are applicable to this investigation into student discourse and thinking in a dynamic geometry environment. They are figurative prototypes, operative prototypes, and relational prototypes. In constructing figurative prototypes, the learner's interaction with the concrete carrier leads to a perceptive understanding focusing on the properties, distinctive features and internal relations of the concept. A student's understanding of a square's defining properties, congruent sides and congruent angles, would be considered a figurative prototype for that shape. One powerful feature of a dynamic geometric representation of a square via the square maker is that these figurative prototypes may be developed saliently in that while the square maker is morphed and manipulated, it preserves its defining characteristics. Essentially, a figurative prototype emphasizes the qualities of an object in the foreground, with the operations on the objects in the background.

The second type of prototype, operative, focuses on the operations on the concrete carrier. In geometry, possible operations would be isometric and non-isometric transformations. For example, a shape made with a trapezoid maker may be isometrically translated, rotated or reflected allowing for the visualization of congruent trapezoids in different orientations. These concepts of isometric transformational geometry, which is often illustrated by teachers with paper cut out of shapes that are rotated and flipped on the blackboard or overhead are easily visualized via dynamic geometric representations. A particularly distinctive feature allowed by dynamic geometric representations is the capacity to morph and reshape certain objects to build relationships with other objects. For example, a trapezoid maker can be reshaped to make a figure that is a square. This morphing capability leads us to the last type of prototype, relational.

Relational prototypes refer to the construction of specific relations constructed at the concrete carrier. In a paper and pencil curriculum, unlike figurative and operational prototypes, these relationships are not immediately perceivable, but must be mediated and constructed cognitively. For example, the fact that a square is a parallelogram would represent a relational prototype between a square and parallelogram. Such a conclusion transcends both figurative and operative understanding of the concept of a square in that a relationship with another figure has been established. In a traditional geometry curriculum, this relationship can only be meaningfully constructed through deductive reasoning. However, the use of dynamic geometric software allows for the construction of this relationship through visual inductive reasoning. In a sense, these types of hierarchical relationships once a product of VR discourse become products of AR discourse through perceptual mediation using the on-screen carriers for the relational prototype.

In general the relationship between concrete carrier and prototypes is particularly important in a dynamic geometry environment. Furthermore, although each type of prototype consists of certain characteristics with varying degrees of complexity, there is not a hierarchical or sequential relationship between them. In other words the acquisition of one type of prototype is not a prerequisite, nor does it necessarily lead to the construction of another. Rather they are networked in a non-linear manner to give meaning to more complex concepts. This framework diverges from more typical frameworks like the Van Hiele levels for geometric understanding (Bell, 1998; Frerking, 1994) and the SOLO taxonomy (Olive, 1991; Pegg & Davey, 1998) used in many of the previous research studies on geometry. Given the need to describe the dynamic nature of the geometric representations we chose to depart from these frameworks to explore the nonlinear development of meaning and prototypes. According to Doerfler (1991), the development of meaning has a holistic aspect corresponding to the prototype for the respective concept in which development is not divided up into more elementary particles or ideas to be progressed though sequentially. Rather, to attain meaning, suitable concrete carriers must be developed and presented to the students in a manner and context to promote the construction of the appropriate prototype.

Methodology

This study took place in a fifth grade classroom (25 students) in a large Midwestern elementary school (approximately 700 students) at the beginning of the second semester. The study began with careful observations of one teacher's current pedagogical practice in a technology-rich fifth grade classroom. One important characteristic of this classroom was the emphasis on technology in all curricular areas. Students had access to a classroom lab of desktop computers, a classroom set of laptops with wireless Internet connections, and digital video cameras with digital video editing software. Students used some form of digital technology on a daily basis through the creation and maintenance of their own personal web page portfolios. It was common for the teacher and students to video tape their own classroom activities, edit the video clips, and post them on their web page. Furthermore, all the video taping and digital editing were done by the students. This technology-rich classroom provided a context that was conducive to this projects use of a computer-based geometry curriculum and video technology for the collection of data.

A total of eleven 45-minute lessons were co-taught by one of the researchers and the classroom teacher. Their role was to act as a facilitator for learning between a dynamic geometry curriculum called Shape Makers (Battista, 1998) and the students. Each class session consisted of two parts. For the first 30 minutes, students were given a set of tasks that they were to complete either individually or in pairs using the Shape Maker curriculum. During the last 15 minutes, a whole class discussion, facilitated by one of the researchers was used to help summarize and synthesize the concepts from the day's activities. After one week of whole class instruction and discussion, two students from the classroom were then asked to participate in case-study observations during the second week of instruction. The two students worked on the program together while one member of the research team member interacted and ask occasional questions for clarifying how the students were interpreting, following the instruction or working on some task.

As the students worked on the computer, all on screen work and dialogue was captured on a digital video recorder to document the on screen actions with the discourse between the two students (Hall, 2000; Lesh & Lehrer, 2000). The purpose of videotaping individual students was to identify any idiosyncratic interactions with the computer based lessons. The following results and discussion are based on two days worth of video taped material taken during the second week of instruction.

Results

In the first videotaped session, after exploring the properties of the sides and angles in square maker and rectangle maker, the students moved on to the parallelogram maker. When they first opened up the parallelogram maker, their initial observation was the unequal angles in the parallelogram. However, their first action on the parallelogram maker was to make it into a square. This important action, making the parallelogram maker into a square or rectangle, was done repeatedly with the other shape makers throughout their investigation. The choice of square and rectangle as referent objects makes sense in that the students seemed to have the most formal and informal knowledge about both shapes that was reinforced by their exploration with the square and rectangle makers. During the second videotaped session, the students' construction of figurative and relational prototypes was extended through their investigation with the trapezoid maker. While working with the trapezoid maker, the students observed that all the sides were not congruent. The two students begin to discuss properties of parallel sides, non-parallel sides, and side length. In the midst of this discussion, Student J then proceeded to turn the trapezoid maker into a parallelogram by making the two non-parallels sides parallel.

Student J: [At this moment, two sides of the trapezoid are not parallel]...they won't be the same, here, ok, ready...they intersect on D and C... right...there [Now, she drags one vertex of the trapezoid, so that all the sides become parallel], they're the same...they don't intersect no more

Student A: Oh, A and B (length AB) and C and D (length CD) they are 79, and B and C (length BC) and A and D (length AD) are 84. So that means the opposite sides are the same on B and C and A and D...so we gotta say that the opposite sides on...sides on B and C and A and D are parallel...

They began by extending their figurative prototype of a trapezoid. They developed a relationship between the opposite sides being parallel and congruent to each other, prompting Student J to connect all of the shapes together in a hierarchical manner. This was an extension of Student J's relational prototypes. To make her point, she closed the trapezoid maker and then opened the rectangle maker.

Student J: Oh, do you get it now? ...you know how he said the squirrel [Earlier that day, their teacher discussed the hierarchical relationship between a squirrel and a mammal]...the rectangle can make into a square, it can make into a kite...some kind of kite. Trapezoid...so that means the rectangle is a square, and the square is the kite...

Student A: But we're not on rectangle.

Student J: Let's see what the trapezoid can make, since we're on trapezoid. [Student A closes the rectangle maker and then reopens the trapezoid maker.]

Student A: So we're on trapezoid.

Student J: Yeah, so let's see what this thing can make...can it make...are they (sides) even? No none of them are, Ok, see if you can make a square, then we can say the trapezoid makes a square, and the square can make a trapezoid, or the trapezoid can make a rectangle, the rectangle can make a square, and the square can make a kite.

This excerpt suggests that student began to informally explore the various relationships between square, trapezoid, rectangle and kite by morphing and manipulating the trapezoid maker. This illustrates the early stages of an action-based relational prototype of the square, trapezoid, rectangle and kite. Again, the students used the square and rectangle as referent object to help organize their relational understanding of the less familiar trapezoid and kite.

The students then consulted with one of the researchers to share their ideas. Notice the tension between the researcher using the term trapezoid maker and the students using the term trapezoid to describe the on screen geometric figure.

Student J: [To the researcher] We said the trapezoid can make a square, the square can make a rectangle, the rectangle can make a square, and the square can make a kind of kite.

Researcher: ...Let's use (trapezoid). Ok when you say this trapezoid, you're talking about this program right? OK, so that's the trapezoid maker.

Student J: Yeah, were talking about the trapezoid can make a square and a kind of kite out of the square. Like if you make a....

Researcher: ...Show me what you mean.

Student J: She's going to make a square....

Researcher: So the trapezoid maker can make a square, good.

Student J: The square can make a rectangle.

Researcher: Oh, I see what you are saying.

Student J: Out of the square we can make a rectangle, so that means what he was talking about the squirrel ...is these things can make different, this one can make more and more shapes, like squares.

Researcher: ... what you're saying is that the trapezoid maker can make a square...

Student J: The square can make a rectangle

Researcher: Ok...just so you know, you're still playing with the trapezoid maker...

Student J: Yeah.

Researcher: ...so the thing making the rectangle is actually the trapezoid maker.

Student J: Yeah, the trapezoid is the.... A square.

On the one hand the students wanted to share their findings. On the other hand, the researcher was trying to help the students distinguish between the shapes they were making and the tool they were using to make the shapes (i.e. - the trapezoid maker).

Student A: So probably what she is trying to get at is that the trapezoid and I think the kite they're like bendable...kind of like some people are double jointed and they can do more stuff kind of like that...

Student J: The trapezoid is a rectangle and a square and a kite

Researcher: Ok, it can make all those things...so what you are saying then is that a trapezoid maker can make a square...

Student J: So it is a square.

Researcher: Ah, not quite, it's the other way around, if a trapezoid maker can make square, then a square is a type of trapezoid

Student J: Yeah. That.

Researcher: That's what you want to say.

Student J: Cause a square can't make a trapezoid.

First, it appears that the students seemed to treat the shape makers as mathematical objects within themselves. This was in contrast to the researcher's view in which he treated them as on screen mathematical tools. It appears that the students were unable to make the distinction between trapezoid and trapezoid maker. However, this did not seem to limit their ability to construct meaningful relationships. In fact, their discursive practice gives evidence that their constructed notion of a trapezoid is very dynamic. Second, these dynamic geometric representations of the quadrilaterals allowed the students to discuss the hierarchical relationships in a form of actual reality (AR) discourse. Throughout the discussion, the two students characterize the trapezoid through the use of terminology like, "makes," "bendable," "double jointed," and "do more stuff." Of particular interest is the notion of "do more stuff" when referring to the kite and trapezoid maker. In the hierarchy, both shapes are right below the quadrilateral because their respective definitions are the most general and therefore, from a virtual reality (VR) point of view, encompass a greater variety of shapes. However, the students' characterization of doing "more stuff" in many ways is an AR description of the same hierarchical principle. The discourse is AR because the relational concept, a square is a type of trapezoid, is being mediated perceptually through the on screen figures. Furthermore, it appears that the students recognize the trapezoid's versatility over the square supported by student J's statement, "Cause a square can't make a trapezoid." The use of the square and rectangle as referent objects allowed students to begin a primitive construction of the quadrilateral hierarchy, supporting their understanding of trapezoids and kites.

Discussion

Two important themes that emerged out of the data were the concurrent construction of figurative, operative and relational prototypes and the treatment of the shape makers as mathematical objects to construct hierarchical relationships between quadrilaterals. This was done without knowing the defining properties of each shape. In the video taped activity, the lessons first had the student explore shapes most familiar to themselves (squares and rectangles) and then progressed to less familiar shapes (trapezoid and kite). The purpose of the activity was to help the students learn the individual properties of each quadrilateral. In the first videotaped session, both students initially were focused on the specific properties of the quadrilaterals as they manipulated, morphed and dragged their dynamic shape makers focusing on the parallel sides, equal side lengths, and equal angle measures. In a sense they were constructing both figurative and operative prototypes. However, many of the figurative prototypes developed were general in nature. For example, when the students began their investigation of the parallelogram, they concluded that, unlike the rectangle, all the angles on a parallelogram were not equal to each other. The students never discussed specific angle relations like adjacent angles summing to 180 and opposite angles being congruent. This supports the holistic nature of prototypes in the development of a concept.

However, the lack of knowledge or understanding of specific quadrilateral properties did not prevent the students from concurrently constructing basic relational prototypes for these shapes.

Throughout the lessons, both students seemed to grasp onto the square and rectangle as referent shapes in which all other shapes were compared and evaluated. However, the students' comparison was done through their attempts to make one shape maker into several other shapes. This morphing activity with referent objects allowed the students to compare the properties of the various shapes. The students were in a sense, discursively constructing an understanding of each figure's side and angle properties (figurative prototypes) through this morphing process (operative and relational prototype). Our findings illustrate the concurrent construction of the students' figurative, operative and relational prototypes.

Conclusion

Many traditional geometry curriculums see instruction as a construction of an axiomatic system through the use of definitions, postulates and theorems. In contrast, the dynamic geometric figures in Shape Makers (Battista, 1998) represent complex objects that embody a multitude of elements that make up that system. As a result, rather than a deductive construction of a geometric system, instruction takes the form of an inductive deconstruction of these very complex computer models. These semiotic theories by Sfard (2000) and Doerfler (2000) helped us to explain the non-linear construction of abstract relational geometric concepts, and their operative and figurative counterparts.

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HIGH SCHOOL MATHEMATICS STUDENTS AND GEOMETRIC REASONING

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This study compared the developmental reasoning stages of beginning and experienced geometry students with reference to a theoretical framework developed by Pierre van Hiele and Dina van Hiele-Geldof. The van Heile model has had an impact on the design of geometry instruction and curriculum as a way of overcoming difficulties teachers encounter in teaching high school geometry. Several results from the study are significant for mathematics education. We detected no statistically significant difference between the beginning and experienced geometry students' mean scores. There was a statistically significant gender difference among the experienced geometry students in the van Hiele levels. We were able to determine a difference in the van Hiele levels among the beginning geometry students and among experienced geometry students. The students' distribution seems not to be optimal for an efficient learning and teaching, since the properties of the levels suggest that two students who attempt to reason with each other but at different levels will experience difficulty in understanding each other.

This study compared the developmental reasoning stages of beginning and experienced geometry students with reference to a theoretical framework developed by van Hiele (e.g., Fuys, Geddes, & Tischler, 1988; Usiskin, 1982). In particular, the study attempted to answer the following questions: (1) What differences exist according to the van Hiele levels between beginning geometry students and experienced geometry students? (2) What is the significance of gender among the experienced geometry students in the van Hiele levels? (3) What generalizations can be made among low van Hiele level students, high van Hiele level students, beginning students, and experienced students?

The study was based on two theoretical positions. The first is a pedagogical orientation that acknowledges the importance of teachers' knowledge of student cognitions (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Shulman, 1989). The second is a cognitive framework (cf. Usiskin, 1982) developed by Pierre van Hiele and Dina van Hiele-Geldof that captures the manifold nature of students' geometric reasoning and, in particular, identifies five developmental levels. The cognitive framework was used to generate assessment protocols and evaluation tasks and to evaluate the impact of instruction on development stages of students' geometric thinking. The van Heile model has had an impact on the design of geometry instruction and curriculum as a way of overcoming difficulties teachers encounter in teaching high school geometry (e.g., Hoffer, 1981; Usiskin, 1982). Pierre van Hiele and Dina van

Hiele-Geldof proposed their model in 1957 and refined it between 1957 and 1986. The model delineates five development stages experienced by all learners of geometry: visualization or recognition (level-0), analysis (level-I), abstraction (informal deduction) (level-II), deduction (level-III), and rigor or axiomatic (level-IV).

One of the first lines of research on the model was reported by Wirszup in 1976 (as cited in Mayberry, 1983), and eventually attracted educators' attention. Usiskin (1982) worked on the validity of the van Hiele levels. He confirmed the validity of the first four levels in geometry learning. In 1986, Burger and Shaughnessy focused on descriptions and characteristics of the van Hiele levels. Fuys, Geddes and Tischler (1988) examined the effects of instruction on a student's predominant van Hiele level. Some of these researchers confirmed the validity of levels and investigated students' behavior on tasks while others studied evaluation and assessment of the geometric ability of students as a function of the van Hiele levels. Mayberry (1983) reported that her findings with 19 undergraduate elementary teachers supported the hierarchical aspect of the theory, but she did not support the idea that an individual showed the same level of thinking in all areas of geometry. Gutirrez, Jaime & Fortnuy (1991) replicated these results with preservice teachers in Spain for levels I-IV.

Methods of Inquiry

We defined beginning geometry students as those who have just begun their study of high school geometry. Experienced geometry students are algebra-II students who had already completed the study of geometry during the past year. The criterion for high van Hiele level was determined to be level III (deduction) or level-IV (rigor or axiomatic); otherwise the student was considered to be at a low van Hiele level. The subjects for this study were four classes of secondary school mathematics students. Two classes were taking high school geometry (beginning geometry students) and two classes were taking algebra-II (experienced geometry students). Those who were enrolled in algebra-II at the time of the study took geometry the previous year. Algebra-II is the next course following geometry in the high school mathematics sequence. The total number of subjects for the study was 100 students, 53 from geometry classes and 47 from algebra-II classes.

We used two types of test instruments, the Entering Geometry Test (EGT) and the van Hiele Geometry Test (VHGT), taken from Usiskin's Cognitive Development and Achievement in Secondary School Geometry Project (1982) used by his permission. The EGT is a beginning test consisting of 20 multiple-choice geometry questions for 25 minutes. The VHGT is an advanced test containing 25 multiple-choice geometry questions for 45 minutes. It is designed for testing and evaluating the students' geometric reasoning ability rather than computation. In particular, the VHGT is designed to identify an individual's van Hiele level of geometric thought and to compare differences among low and high van Heile levels of students and beginning and experienced students.

Both tests – first the EGT and next the VHGT – were administrated by the researcher during two arranged sessions. After the second test, the researcher selected twelve volunteers for individual interviews. Interview items, from the written tests, were decided and selected after the administration of the tests: two questions from EGT and three questions from VHGT, depending on the students' responses to the questions. The aim of the interviews was to see how students perform orally in an attempt to confirm their level of performance on the written test and to validate written test results. Two characteristics, teachers' personal judgment of students' potential and the results of the VHGT, played an important role in the selection of volunteer students for the 30-minute individual interviews.

The sessions were audiotaped to create a transcript of each interviewee's responses. The data sets are the responses from the students' answer sheets and the interview transcripts. The responses of students and the interview transcripts were evaluated in light of the van Hiele theory. In the analysis of scores from the EGT and the VHGT, a one-way and a two-way analysis of variance were used to confirm significance of differences found.

Results

The data from the study indicated several results that are significant for mathematics education. Although the beginning geometry groups' mean scores appeared numerically higher than the experienced geometry groups' mean, there was no statistically significant difference between the two. There was a statistically significant gender difference among the experienced geometry students in the van Hiele levels, and gender had an effect on scores. The male students' scores were significantly statistically higher than the female students' scores on the VHGT test. We were able to determine a difference in the van Hiele levels among the beginning geometry students and a difference in the van Hiele levels among experienced geometry students. The students' distribution seems not to be optimal for an efficient learning and teaching, since the properties of the levels suggest that two students who attempt to reason with each other but at different levels will experience difficulty in understanding each other.

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PROSPECTIVE MATHEMATICS TEACHERS´ LEARNING IN GEOMETRY – THE BEGINNING OF A LONGITUDINAL STUDY ON TEACHER EDUCATION

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The way mathematics teachers understand geometrical concepts influences their teaching and consequently what their pupils might learn. Every semester since the spring of 1999, prospective mathematics teachers at Göteborg University have responded to a questionnaire that address questions regarding how the students conceive concepts in geometry. The written answers are analyzed with respect to concept images and the way the students make use of their concept images and the mathematical definition of certain concepts. As a base for an extended analysis some other qualitative aspects are identified.

Introduction

When discussing teacher qualifications in mathematics and education of future teachers in mathematics, the branch of geometry is of course only a small part of the science mathematics. Nevertheless, there exist reasons and arguments for why we should try to observe and educate competencies for teachers in mathematics with geometry in focus (Niss, 1998). This will include teachers' views and knowledge of geometry as a mathematical topic as well as the kinds of knowledge and views that they should have of the processes of learning geometry.

In Sweden, as well as in other countries, geometry was an essential part of school mathematics during the period from the middle of the 1800-century to the end of the 1950's. One interpretation of this fact is that geometry and especially the Euclidean plane geometry was removed by the introduction of the "new mathematics". Later on, there have been discussions (Kapadia, 1985) whether the loss of geometry in the school mathematics may have caused a drawback regarding student's understanding of general concepts or not. As a result, geometry and measuring has a more prominent position in the mathematics syllabuses for the Swedish school today (Skolverket, 2000a, 2000b). The programs for teacher education seem to have followed the same patterns, only with a slight displacement in time. Despite this, teachers of today are expected to fulfill the national curriculum and ensure that their students reach proficiency in these areas.

The Study

At Göteborg University, students preparing to become teachers of mathematics and natural science for Grades 4 to 9 or for the gymnasium (Grades 10 to 12) take courses in mathematics that are offered by either the department of mathematics or by the department of mathematics education. One such 5-credit point (5 weeks full-time study) course is called *Geometry and Mathematical Modeling,* in which there is a strong emphasis on the use of graphing calculators and a variety of software, both from a mathematical and a didactical perspective. The students come to the course after having taken different courses in mathematics where the content vary over time. At present time the students come with courses in number theory, Euclidian geometry, linear algebra, and real analysis, which approximately correspond to 30 weeks of full time studies. Beginning in the spring of 1999, the students who came to the course were given a questionnaire, which address questions regarding how the students conceive concepts in geometry.

This study is based on written responses that have been collected from all the entrance questionnaires on geometry administered during the last 7 semesters, including the Spring 2002 course. The students were given the questionnaire the first day of the course and there was no time limit for them working on it, although up to now no student has used more than 80 minutes. At present, 197 students have responded to the questionnaire. The issue in focus is the identification of the prospective teachers' concept images and the way they make use of their concept images and the mathematical definition of certain concepts, geometrical concepts that they will find central when they begin their professional life as mathematics teachers. The two elements, concept image and concept definition relates to the work done by Hershkowitz and Vinner (Hershkowitz, 1990) and are used as a framework for the analysis of the students' written responses. The starting-point for the analysis is the interpretations of what the students´ have described in their written answers to the questionnaire (Patton, 1990). The way the questions in the questionnaire are formulated is indicating that the geometrical meaning of the concepts is in focus.

The selection of questions in the questionnaire are closely related to the national curriculum for the Swedish comprehensive school and for the Swedish Gymnasium (Skolverket, 2000a, 2000b), to analysis of common text books that are used in Swedish schools, and to the kind of geometrical concepts that are used in national tests. The students were asked to give an explanation to the following five concepts: *diagonal*, *congruence*, *parabola*, *rhomb*, and *cycloid.* Beside this, the students responded to two different questions regarding the differences of construction and drawing in plane geometry. Different responses were classified according to the critical attributes of the actual concept used by the students, but the reasons for the differences were not analyzed.

Findings and Conclusions

Most of the findings correspond to these described by Hershkowitz and Vinner (Hershkowitz, 1990). The concept image expressed by a student is often only part of or may not coincide at all with the definition of the corresponding mathematical concept. The prototype phenomenon was also observed frequently in the students' descriptions, which means that each concept has one ore more prototypical examples that are attained first and therefore exist in the concept image of most students in the study. Besides this there are results that could form the basis of an extension of the model used for analysis. There are reasons to separate the students´ answers in a qualitative aspect, not only in 'correct' or 'non correct'. This will mean if a description of a certain concept is bounding its meaning to a specific situation (see the response from *student 1* below) or if the description will allow generalizations (see the response from *student 2* below).

> Explain in short terms the following geometrical concepts; please use both pictures and words. Even if your belief about the concept is vague, we appreciate if you as clear as possible will try to give an explanation.
>
> *Diagonal*
>
> Student 1: A diagonal is dividing all rectangles, squares, rhombus, parallelogram in two equal parts.
>
> Student 2: A straight line between two corners in an object with n corners, $n \geq 4$, $n \in$ N. Note. Not between two adjacent corners.

This aspect is of particular importance when a teacher use his or hers explanations to meet and support students in their learning of geometrical concepts. With this in mind, another key aspect when analyzing the data is the students' use of different languages, a more formal or an everyday language. They also use metaphors, like a description of what to do to get an impression of the meaning of a certain concept.

Despite the fact that several students demonstrate serious lack of concepts in the questionnaire, quite a few of them manage to illustrate acceptable mathematical reasoning during the final examination. One important tool for teaching and learning geometrical concepts in the course was The Geometer's Sketchpad (Jackiw, 1995). The students' work with the software was complemented with seminars and discussions regarding computer supported learning in geometry reported by for example Laborde (2001).

The results emerging from the described study will form the basis for deeper studies of what these prospective mathematics teachers learned in geometry and how their understanding of the theoretical mathematical content later on was expressed in their teacher practice.

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THE BENJAMIN BANNEKER PROJECT: A STUDY OF AFRICAN AMERICAN CHILDREN'S PROBLEM SOLVING

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The Benjamin Banneker Project responds to the achievement gap in mathematics and science (Irvine & Armento, 2001) and the digital divide in technology (Wenglinskey, 1998) that continue to exist between mainstream and African American students. The purpose of this one-year project is to develop and field-test computer software that uses intelligent computer-assisted instruction (ICAI) to help third-grade African American children solve culturally relevant mathematics and science problems. However, developing software that is culturally relevant is exploratory and somewhat risky because such software is initially very expensive to develop, and it targets only a specific group of students. The rationale for selecting third graders is that differences in minority students' motivation and achievement in mathematics begin to occur around fourth grade (Ladson-Billings, 1997); the rationale for selecting African American students is that they have the lowest scores in science and mathematics (Tate, 1997). Yet, if the needs of these children are met by intelligent tutoring, then ICAI has the potential to impact on the learning of all children.

The goals of this project are to influence the classroom environment as well as the instruction of African American students as teachers are encouraged to use technology as an emerging curriculum rather than as a reward or only to supplement traditional instruction. Furthermore, rather than merely being receivers of digital information, the students would be able to interact with the computer, which will respond with cues to help students solve culturally relevant mathematics and science problems. Our study takes place in two charter schools in Philadelphia: Alliance for Progress Charter School and Harambee Institute of Science and Technology Charter School.

Integrating technology into the school curriculum remains a daunting challenge, despite the fact that high-tech computer workstations are commonplace in most teachers' classrooms (Ertmer, Addison, Lane, Ross, & Woods, 1999). However, teachers participating in this project have learned to use the software and how to become more culturally responsive to their students. Thus, teachers have learned how to become cultural brokers in the classroom (Aikenhead, 1996) as well as how to run the software program and monitor student progress. In addition, the teachers have attended ongoing inquiry group sessions to learn how to integrate computers more fully in the classroom and how to help children develop their own word problems. Teacher activities focused on using technology and pedagogy to help African American children to become proficient in solving culturally relevant word problems in order to increase their persistence and repertoire of problem solving strategies for use in other contexts (cultural brokering).

During this poster presentation the authors will present the software that has been developed to date. The software is unique in that mathematics and science problems will be embedded within the theme of the Underground Railroad. The main character, Moses, travels from a plantation in Bucks County, Maryland, to freedom in Philadelphia, Pennsylvania, along the Underground Railroad. His journey involves both mathematical and scientific decision-making as he encounters obstacles such as poison ivy, wild animals, and general survival issues along the way. Moses uses the stars to guide him North and has to travel an average of 8 miles per day to get to Philadelphia before his rations run out. This software is patterned after the Oregon Trail and is being pilot tested with third-grade African American students. Data from teacher professional development workshops and student interactions with the media as well as the problems and problem solving strategies will be presented along.

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INITIATING PROBLEM POSING THROUGH EXPLORATION WITH SKETCHPAD

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Problem posing can be seen as a necessary, or at least desirable, precursor to problem solving. As a creative, personal form of mathematical thinking, problem posing is an activity that is often difficult for teachers and students to engage in—where to start? Where to go? Hot to get there? Based on both theoretical and empirical inquiry, this poster presentation proposes that "mathematical play" is an essential precursor to problem posing and can be stimulated in learning environments that explicitly enable aesthetic engagement and exploration.

Problem posing is an essential component of the professional mathematician's practice and has received increasing interest among mathematics educators. However, despite the many strategies offered by Brown and Walter (1983), researchers have found it difficult to elicit problem posing behavior in the classroom (Silver, 1994). This may be due to students' limited exposure to varied problem situations in the classroom, as well as to the disconnect between students informal, intuitive mathematical knowledge and their formal "school math" knowledge. Furthermore, in order to pose a problem, one must perceive a situation as being problematic, that is, as surprising, strange, or unexpected enough to initiate wondering and questioning (Dewey, 1938). Therefore, researchers such as Hawkins (2000) have suggested that problem posing can only occur after a period of exploration during which an inquirer gets acquainted with a mathematical situation. Often, the pedagogical challenge is to find ways in which to engage students in meaningful (to them) and relevant (to the curriculum) exploration.

This poster presentation reports on three examples of mathematical situations given to a small group of middle school students using The Geometer's Sketchpad (Jackiw, 1991) that succeeded in fostering exploration and, in turn, problem posing. I compare these three examples with the types of tasks more commonly proposed to students using DG software, such as "discovery problems" or "guided inquiry problems" and highlight the often implicit pedagogical theories and intentions that influence students' attitude and behavior.

The first example was inspired by Walter's (2001) description of her own process of exploring and mathematizing Theo van Doesburg's *Arithmetic Composition I.* The second example involves Archimedean tessellations and the third star polygons. The two latter examples are common topics in textbooks and other resource materials, but were designed and presented to students as "guided inquiry" as opposed to "discovery problems." Based on my observations of students' interactions with Sketchpad, on analyses of their activities, and on follow-up interviews, I identify some of the mathematical ideas, Sketchpad abilities, and habits of mind that were initiated and developed throughout their work. I pay particular attention to: the motivational characteristics of the situations, that is, those that elicited the attention of the students and thus provoked subsequent constructions, actions and conjectures; the way in which Sketchpad supported student exploration and problem posing; the evaluative characteristics of the situations, that is, those that initiated the types of reflections and judgements that lead to student evaluation of the significance or interest of a particular problem or solution.

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Learning

CONCEPTUAL UNDERSTANDING AND COMPUTATIONAL EFFICIENCY: CHILDREN'S STRATEGIES FOR MULTIPLYING MULTIDIGIT NUMBERS

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This paper aims to examine children's conceptual understanding and computational efficiency in the domain of multidigit multiplication. In this study, children's invented strategies for multiplying multidigit numbers were used as a window to explore their understanding of multiplication and its underpinning properties. Fifty-eight children in grades 4 and 5 were individually interviewed with multidigit multiplication problems in equal grouping context. Based on the use of groups, knowledge of place value, and use of the properties of multiplication, children's invented strategies were classified into several categories and children were clustered into strategy groups based on their primary choice of strategies. The analyses of invented strategies for multiplication demonstrate sophistication of children's understanding of the operation and the distributive and associative properties and limited understanding of the commutative property. In addition, the findings of this study provide empirical evidence of the interconnectedness of children's conceptual understanding of multiplication and computational proficiency.

Recent reform efforts in mathematics education hypothesized the interdependency of the conceptual understanding and computational proficiency, and research in cognitive science provides a general support for the interdependency (Kilpatrick, Swafford, & Findell, 2001; Wu, 1999). However, there has been little empirical evidence that showed how these two strands played a role in children's problem solving activities. This paper aims to examine children's conceptual understanding and computational efficiency in the domain of multidigit multiplication. In this study, children's invented strategies for multiplying multidigit numbers are used as a window to explore their conceptions of multiplication and understanding of underpinning concepts and properties. This window allows investigation of what the invention may afford or limit with the respect of children's understanding and computational skills.

Theoretical Background

Invented Strategies and Conceptual Understanding

A significant body of research has shown that children can construct solution strategies for addition and subtraction without explicit instruction in specific procedures (Carpenter, Ansell, Franke, Fennema, & Weisbeck,1993; Carraher, Carraher, & Schliemann, 1987; Cobb & Wheatley, 1988; Hiebert & Wearne, 1996; Olivier, Murray, & Human, 1990; Saxe, 1988). Researchers focused on children's invented strategies based on a hypothesis that invention can play a central role in helping children develop conceptual understanding of numbers and operations. Carpenter and Lehrer (1999) provided a theoretical framework to define what it means to understand mathematics and discussed how the process of inventing alternative strategies and articulating underlying reasoning could enhance children's conceptual understanding. They also discussed the pitfalls of learning mathematical procedures without understanding and when children learn skills in the relation to developing a conceptual understanding from the beginning of schooling, mastery of skills is facilitated as well as understanding.

A longitudinal study by Carpenter and his colleagues (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998) provided empirical evidence that supports the proposition of the relationship of invented strategies and children's development of conceptual understanding. They traced the development of children's solution procedures for addition and subtraction for three years and found that children in grades 1-3 who used invented strategies before they learned standard algorithms demonstrated better understanding of place value concept and were more successful in transferring their knowledge to new situations than were students who initially learned standard algorithms. These results are consistent with the theoretical perspective that the development of conceptual understanding before mastery of skills enhances children's mathematical learning (Hibert & Carpenter, 1992).

In this paper, I extend this line of research to the domain of multidigit multiplication and examine conceptual understanding that children develop through the process of constructing alternative strategies in a more complex domain of multidigit multiplication. Furthermore, I investigate children's development of computational procedures and its connection to their conceptual understanding of multidigit multiplication.

Children's Invented Strategies for Multiplication

An understanding of multiplication plays a central role to children's reasoning of multiplicative structure. Sound understanding of multiplication enables children better understanding of division, ratio, proportion, and rational numbers. In spite of the importance of the concept and calls for more research in this domain, most research on children's learning of multiplication has focused on students' misconceptions of multiplication (Fischbein, Deri, Nello, & Marino, 1985; Greer, 1988; Mangan, 1986) and research on children's understanding has been relatively limited (Kilpatrick, Swafford, & Findell, 2001).

Several studies investigated children's invented strategies for single-digit multiplication problems (Anghileri, 1989; Carpenter et al., 1993; Heege, 1985; Kouba, 1989). Anghileri, Carpenter et al., and Kouba illustrated strategies constructed by children as young as five or six and the strategies in the studies showed that young children's invented strategies develop from direct representation of the problems with physical materials to several forms of counting strategies. Heege identified six different strategies that children in grades 2-6 constructed: doubling, halving, increasing a familiar product by adding the multiplicand once, and decreasing a familiar fact by subtracting the multiplicand once.

Recent research showed that children can construct a wide range of sophisticated strategies for multiplying multidigit numbers (Ambrose, Baek, & Carpenter, in press; Baek, 1998; Kamii, 1994; Lampert, 1986). Teaching experiments by Kamii and Lampert in a third- and a fourth-grade classes illustrated that children can develop partitioning strategies using intuitive understanding of the distributive or associative properties. In my previous studies (Ambrose, Baek, & Carpenter, in press; Baek, 1998), I attempted to characterize the nature of children's invented strategies by analyzing the strategies based on underpinning mathematical concepts and properties of the strategies. I constructed a comprehensive taxonomy by reconstructing the classification schemes of Heege (1985) and Kamii (1994). However, the data was not sufficient to generate firm conclusions about the development of children's understanding of multidigit multiplication (Kilpatrick, Swafford, & Findell, 2001). In this study, the classification scheme is further refined by comprehensive analyses of underpinning mathematical concepts and properties of multiplication, and the relationship of the level of sophistication of invented strategies and the level of computational efficiency is investigated.

Methods

Subjects

The students in this study consisted of 58 students in 3 fourth- and fifth- multi-graded classes at one school. Thirty-two of the 58 students were in fourth grade and the other 32 were in fifth grade. All 32 fourth graders and 7 of the 26 fifth graders were in traditional classroom in the previous year. About 70% of the students who participated in the study were White and about 15% of the students were on free or reduced lunch.

Classroom Instruction

Classroom instruction is potentially an influential factor on children's inclination and ability to generate invented strategies. The three teachers whose children were interviewed in this study had participated in a research program designed to focus on helping teachers understand children's mathematical thinking in primary grades (Carpenter, Fennema, & Franke, 1996). At the time of the study all three teachers had participated in the program for more than 7 years and shared the perspective that children could invent procedures to solve problems and instructional decisions should be made based on children's thinking, so that teaching can support children's understanding of mathematics more effectively. The teachers routinely provided problems in real world context and built a classroom norm that children construct their own strategies and discuss them. The standard algorithms were not taught in the classes. When a child who learned them at home or previous year used the algorithm, it was not accepted as an appropriate solution unless the child could explain why the algorithm worked.

Interviews

Clinical individual interviews with fifth graders were conducted in November and December, and the interviews with fourth graders were conducted in the following March through early May. I interviewed all students and each interview lasted between 30 to 90 minutes. To encourage children to use abstract strategies, only paper and pencil were readily available during the interview. After the child gave an answer, the interviewer asked the child how he or she solved the problem. I recorded detailed notes on each child's solution strategies. All the interviews were audiotaped, and child's written work was collected.

The interview tasks assess (a) students' knowledge of place value concepts, (b) their invented strategies for multidigit multiplication problems, and (c) their abilities to use specific invented strategies. The problems for place value assessment consisted of two word problems involving multiplication and division problems with 10 objects in each group. In the second set of the tasks, eight multiplication problems in equal grouping context were asked (for different types of multiplication problems, see Greer, 1992). Children were classified as demonstrating knowledge of base-10 number concepts if both of their responses were coded as demonstrating place value knowledge. The multiplication problems to assess children's invented strategies were chosen to investigate how the size and role of the multiplier and multiplicand influence children's choice of strategies. Two problems constructed with one- and two-digit numbers, four problems with two-digit numbers, and two problems with two- and three-digit numbers. The third set of the tasks assessed children's understanding of three specific strategies: (a) partitioning both numbers into decade numbers strategy, (b) compensating strategy, and (c) commutative strategy. Children were shown how a hypothetical child used each strategy, and were asked to explain and replicate the strategy for a different problem.

Analyses

Children's responses were initially coded whether the answer was correct or not. Strategies that lead to an incorrect answer were analyzed if the strategy would have resulted in a correct answer if the solution was carried out without any calculation errors. If the calculation errors were simple counting, adding, or recall fact errors, and two or fewer errors were made for a single problem, the strategy was coded valid. Strategies for multiplication problems initially coded in predetermined categories (Ambrose, Baek, & Carpenter, in press; Baek, 1998). On the basis of interview notes and audiotaped protocols of each child's explanations, new categories were constructed or the existing categories were modified to capture mathematical differences between strategies. Descriptions of each of these strategy types are provided in results.

After the strategies for all eight multiplication problems were classified, each child's most prevalent strategy was identified and children were clustered into strategy groups. Because children used different kinds of strategies for smaller numbers, strategies for problems with two-digit numbers or larger were given more focus for clustering. A child who solved 5 or more of 6 problems with larger numbers correctly was classified in a specific strategy group if he or she used the type of strategy for four or more problems. A child who solved 4 of 6 problems correctly had to solve more than three problems using a specific strategy to be classified in that strategy group. A child who solved three problems or fewer had to use a single strategy for all problems.

Results

Children in this study invented a wide range of strategies for multiplication problems. The invented strategies showed different levels of children's understanding of units, place value, and properties of multiplication. The problem context seemed to help children understand the operation of multiplication and construct meaningful computational procedures. Many children constructed reasonably efficient strategies using their intuitive understanding of distributive and associative properties, and were able to justify the solution procedures using the problem contexts. By the same token, the problem situations seemed to influence children's strategies in that they assigned different roles to the multiplier and multiplicand. Most children did not use the commutative property even when switching the roles of the two factors would have made the calculation procedure easier.

Children's Invented Strategies for Multidigit Multiplication

I classified children's strategies into four main categories: (a) direct modeling, (b) adding/ doubling, (c) partitioning, and (d) compensating. The strategies in the adding/ doubling category and the partitioning category were further classified into a number of subcategories. Children who used direct modeling strategies represented the number of groups and the number of objects in each group. The adding and doubling strategies were more abstract than direct modeling. Those strategies are based on repeated addition, which some children shortened by doubling. When children used the partitioning strategies, many split the multiplier into smaller numbers and created multiple sub-problems that were easier to deal with. This procedure allowed the children to reduce the complexity of the problem and to use multiplication facts that they already knew. Children who used compensating strategies adjusted the multiplicand or/ and multiplier based on special characteristics of the number combination to make the calculation easier. Children then made corresponding adjustments later if necessary. Only a few students used the direct modeling strategy. Students who used the direct modeling strategy represented each of the groups with tally marks or other drawings and counted the total number of objects. In following sections, each of the invented strategies is described and illustrated.

Adding and Doubling

Repeated adding

The most basic abstract strategy that children invented in this study was repeated addition. This strategy involved a simple addition of one of the numbers in the problem and in most cases children added the multiplicand, following the problem context.

Doubling

Children in the study used doubling strategies to shorten tedious addition procedures. Most children who used a doubling strategy wrote down the multiplicand as many times as the multiplier, made as many pairs of the multiplicand possible, and figured a sum of every pair, pairs of pairs, and so on (Figure 1).

Complex doubling

The complex doubling strategy involved a more powerful efficient form of doubling and depended upon more advanced conceptual understanding of properties of arithmetic. Consider Lauren's strategy for the problem, 47 children paying $34 per child. Figure 2 illustrates how Lauren doubled the number of dollars as she kept track of the number of children in each running total. Lauren's strategy shows that she did not list each of the 47 children's 34 dollar cost. Instead she kept doubling to create a new unit involving a given number of children and the associated cost for that number of children. Her notation allowed her to keep track of number of children as well as the dollars associated with each as they were doubled and to make necessary adjustments.

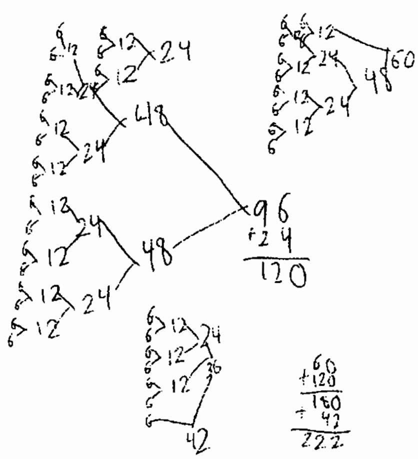

Figure 1. Chris' Doubling strategy for the problem, 37 ants with 6 legs each.

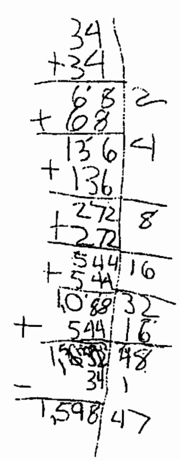

Figure 2. Lauren's Complex Doubling strategy for the problem, 47 children pay $34 each.

Partitioning

Partitioning a single number into non-decade numbers

Most children in the study used partitioning strategies to solve problems with two-digit numbers or larger. One of ways children partitioned numbers was non-decade number partitioning. In solving the problem, 35 sets with 23 blocks per set, for example, Jennie partitioned the multiplier 35 into 7 groups of 5s, using her knowledge of factors of 35 (Figure 3). Her strategy shows that Jennie implicitly used the associative and distributive properties, even though the calculations involved in this strategy were only simple addition.

Partitioning a single number into decade numbers

Children in the study constructed several different strategies that utilized their understanding of place value at different levels and had a potential to take a full advantage of the base-10 structure of our number system. One of the least sophisticated strategies that used grouping of ten was to partition a number into 10s and 1s and to use addition to figure out the sub-product of 10 groups. As Figure 4 illustrates, John listed ten 6s and added them solving the problem, 34 ants with 6 legs. Other children who used decade number partitioning strategies took advantage of knowledge of multiplies of 10. As Figure 5 shows, Josh partitioned the multiplier into tens and ones just like John did but he knew that 10 × 32 was 320.

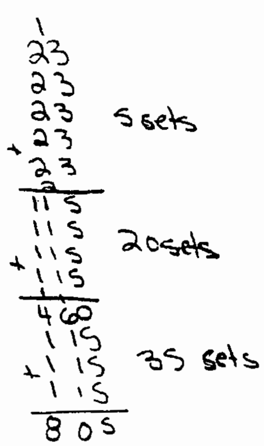

Figure 3. Jennie's Non-Decade Number Partitioning strategy for the problem, 35 sets with 23 blocks per set.

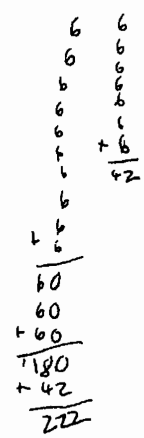

Figure 4. John's Decade Number Partitioning and Used Adding strategy for the problem, 37 ants with 6 legs each.

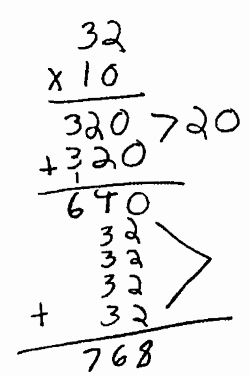

Figure 5. Josh's Decade Number Partitioning and Used Place Value Knowledge strategy for the problem, 24 classes with 32. children per class.

Partitioning both numbers into decade numbers

More sophisticated strategies entailed partitioning both the multiplier and multiplicand. In the following example, Sean partitioned both numbers to solve the problem, 43 boxes with 24 cards per box (Figure 6). Sean set up a table with 4 and 3 on the top and said that he kept 20 and 4 at the side of the table in his head. He multiplied 20 by 40 and by 3, and 4 by 40 and by 3. It was an efficient in a way that did not involve any addition procedures to generate four partial products by taking advantage of Sean's knowledge of number facts. Whereas children who partitioned only the multiplier often explained their solution procedures using references in the word problem context, children who partitioned both number never specifically related solution procedures to the problem context.

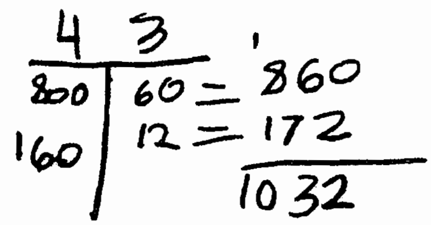

Figure 6. Sean's Both Number Partitioning strategy for the problem, 43 boxes with 24 cards per box.

Compensating

A few children in the study adjusted one or both of the factors in the problem on the basis of the special characteristics of the number combinations. Then they made corresponding adjustments later to figure out the final product. Most children who used compensating strategies rounded one of the factors up to the nearest decade number in order to simplify their calculation by taking advantage of their knowledge of place value. In the following example (Figure 7), Sarah used the strategy, saying that she knew 47 was very close to 50 and a half of 100 × 34 would get her 50 × 34. She knew 100 × 34 was 3400 and calculated a half of it, saying "I know a half of 3000 is 1500 and a half of 400 is 200. 1500 and 200 is 1700." Then Sarah calculated 34 × 3 at the bottom, saying "I know 30 × 3 is 90 and 4 × 3 is 12. So 90 plus 12 is 102." She figured out the final answer by subtracting the three groups of 34s from 50 groups of 34s. Sarah's compensating strategy demonstrated her flexible number and operation sense that capitalized on her knowledge of place value. Sarah used her implicit knowledge of the associative and distributive properties to round up the multiplier to the decade number and then to the hundred.

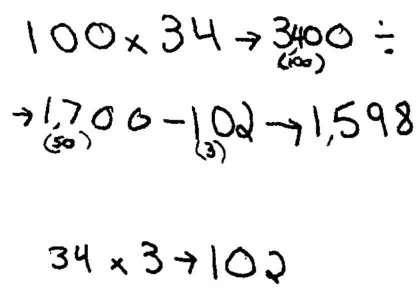

Figure 7. Sarah's *Compensating* strategy for the problem, 47 children pay $34 each.

Clustering Children into Primary Strategy Groups

Next I describe students' strategy groups constructed based on the dominant invented strategies that students' used for multiplication problems. The clustering process revealed that children's use of strategies was remarkably consistent; out of 54 children, 35 children used a single strategy for the problems involving two-digit numbers or larger; 15 children used two strategies; only 4 children used more than 3 strategies.

Table 1 shows the number of children in each of strategy groups, the number of children who demonstrated place value understanding, the average number of prob-

Table 1. Number of Children in Each Strategy Group and Mean of Three Sets of Tasks of Each Strategy Group (N=58)

Strategy Group	Number of Students	Number of Students with Place Value Knowledge	Mean Correct of Multiplication Problems (max=8)	Mean of Replication Tasks[a] (max=3)
Direct modeling/ Adding/Doubling	7	2	1.72	0.14
Complex doubling	3	3	8.00	1.67
Single number partitioning into non-decade numbers	6	2	5.33	1.00
	2[b]	1	7.50	0.00
Single number partitioning into decade numbers: adding	6	5	6.83	0.67
Single number partitioning into decade numbers: base-10	15	15	7.20	2.27
	1[b]	1	7.00	2.00
Both number partitioning	2	2	5.00	2.00
Buggy algorithm	3	3	2.67	0.50
Formal algorithm	2	0	4.50	0.00
Others				
Not solved	4	1	0.00	0.00
Multiple Strategies	3	3	6.67	1.67
Incomplete	2	2	1.00	0.00
Not in sequence	2	2	7.50	0.00

a. Correct replication and valid justification.

b. Numbers which fall between two strategy groups represent children who used both two strategies.

lems solved by children in each group, and the average number of specific strategies replicated by children in each group. The construction of clusters reveals that there are tight relationships between the type of primary strategy, the number of problems that children solved successfully, their place value knowledge, and the number of specific strategies that children could replicate.

With an exception of the complex doubling group, the clusters show that the level of abstraction of the strategies is closely related to the number of problems that children could solve. Children in the direct modeling, adding, and doubling group could not solve any problems beyond one-digit times two-digit numbers. Children in the non-decade number group found that the multiplication problems involving two-digit and three-digit numbers were too difficult for them to solve. It seemed that the size of units that children in the non-decade number partitioning group created was relatively small (less than 6) and it required many steps to solve a problem involving large number of groups. Children who partitioned one of the factors into tens and ones and used addition procedures to figure out the partial product were very successful solving all problems involving two-digit numbers and a problem involving a two-digit number times a three-digit number. However, they could not solve a problem involving a three-digit number times a two-digit number. Children who consistently partitioned one of the factors into decade numbers and utilized the place value structure in our number system were highly successful solving all the problems. In this group, children who alternatively partitioned the multiplier and multiplicand appeared to be more successful solving the multiplication problems than the children who always partitioned the multiplier only. In spite the fact that a both number partitioning strategy requires more an abstract translation of problem situations, two children who primarily used this kind of strategy was not as successful as children who mainly used single number partitioning strategies. They could not figure out the product of decade numbers or generated only two partial products when four partial products were necessary (*buggy algorithm*).

Children in the groups of direct modeling/adding/doubling and non-decade number partitioning did not demonstrate the place value knowledge in the interview tasks. Children in the group of decade number partitioning with adding showed that most of them knew the product of b groups of 10 but they needed to go through an addition procedure to figure out 10 groups of b. It shows that knowledge of *b* groups of 10 objects does not easily translate to 10 groups of *b* objects for children.

Children's understanding of specific strategies was also closely related to their primary strategy. Children in the decade number partitioning strategy with place value knowledge group could replicate more specific strategies than children in other groups. It showed that these children possessed highly fluent computational skills to solve the problems as well as conceptual understanding to make sense of strategies that they had not constructed. Also these replication tasks shed a light on a couple of unexpected findings. First, a number of students who could explain the both number partitioning strategy failed to replicate it because they could not correctly multiply two decade numbers, such as 30 × 40. Many children answered that 30 × 40 was 120 or 112. Second, a couple of children in the non-decade number partitioning group understood the mechanism of the compensating strategy but failed to replicate the strategy because their computational skills could not take advantage of the multiple of a decade number.

Discussion

Invented Strategies and Understanding of Multidigit Multiplication

The results of this study support the hypothesis that invention plays a significant role in children's development of conceptual understanding. Children employed fundamental principles of multiplication, underlying concepts of base-ten numbers, and their implicit understanding of the distributive and associative properties to derive invented strategies. Children with a high level of partitioning strategies showed that they are not only fluent at decomposing, keeping track of the number of groups that they were calculating, and recomposing the partial products to generate the final answer, but also flexible in applying and extending their knowledge to make sense of new strategies and to use them for themselves.

Computational Efficiency

Although children's invented strategies indicate their deep understanding of multiplication, they revealed children's limited knowledge in two specific areas. First, a significant number of children knew the product of multiples of 10 but used addition procedures to calculate the product of 10 groups of objects. The difficulty with this task is also related to their understanding of commutativity.

Second, a small number of children failed to carry out both number partitioning because their knowledge was limited in calculating a product of two decade numbers, such as 30 × 40. It was unexpected that children with this level of abstraction say 30 × 40 = 120 because you have to add a zero. One of the teachers of the students speculated that children might have had difficulty because they thought that there would be easy rule that they could apply in order to deal with zeroes and they were less inclined to use partitioning strategies.

These difficulties seem to be due lack of explicit experiences of the tasks. Children in the study spent a great deal of time constructing meaningful strategies for word problems. The specific knowledge to carry out particular strategies was not necessarily a focus of the instruction. The results of the study suggest that computation efficiency of these children would greatly improve with specific knowledge of multiplying 10 and decade numbers.

In conclusion, the findings of this research support the hypothesis of interdependency of conceptual understanding and computational skills and call for more research on children's strategies to inform research and practice about related concepts and skills.

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FACTORS AFFECTING NINTH GRADE TURKISH STUDENTS' SELF-CONFIDENCE IN LEARNING MATHEMATICS

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In this study, we will present the findings of a research study on factors affecting ninth grade Turkish students' self-confidence in learning mathematics. 841 ninth- grade students (374 females and 467 males) participated in the study. The data was collected through conducting the Mathematics Achievement Test (MAch) developed by Bulut and Tag (2000). We also used the scales that Bulut and Tag (2000) adopted from Fennema and Sherman scales: Self-Confidence in Learning Mathematics (Con) Scale, Attitude Toward Success in Mathematics (AtSc) Scale, Mathematics Anxiety (Anx) Scale, Father (FcSt) Scale, Mother (McSt) Scale, and Teacher (TcSt) Scale. We can predict the student's self-confidence in learning mathematics by using above equation included MAch, AtSc, Anx, FcSt, McSt and TcSt. In other words, students' self-confidence in learning mathematics is affected by their mathematics achievement, attitude toward success in mathematics, mathematic anxiety and perceived attitude of father, mother, and teacher toward one as a learner of mathematics.

The main goal of this study is to examine the relationship between 9th grade Turkish students' mathematics achievement, attitude toward success in mathematics, mathematics anxiety, gender, perceived parent and teacher characteristics related to student, and their self-confidence in learning mathematics by using correlational research methods.

Perspectives/Theoretical Framework

Confidence in learning mathematics refers to confidence in one's ability to learn and perform well on mathematical tasks (Fennema & Sherman, 1976). Armstrong (1981) stated a relatively strong relationship between confidence in learning mathematics and achievement in mathematics. In particular, students who are sure of their ability in mathematics will probably choose tasks involving mathematics more often and persist longer than those who are not sure they will succeed. Confidence in learning mathematics is also reflected in course taking and career aspirations in quantitative fields. Consequently, research studies about confidence in learning mathematics indicate the importance of this variable in relation to student achievement.

According to a meta-analysis, there is a significant gender difference in mathematics self-confidence (Hyde, Fennema, Frost & Hopp, 1990). Researchers have identified parental and societal attitudes as being influential in making girls internalize the feeling that they are inferior to boys in mathematics (Parsons, Kaczala & Meece, 1982). Since previous research provides evidences that gender differences account for self-confidence in learning mathematics, in this proposal, gender is considered as a predictor variable.

Teacher's attitudes and effectiveness in teaching mathematics are viewed as prime determiners of students' attitudes and performance in subject matter (Carter & Norwood, 1997). The students' family background is also influential even in learning mathematics, which may appear to be learned exclusively in school (Wang, Wildman & Calhoun, 1996). Thus, in the present study, perceived parent and teacher characteristics related to student are considered as key variables to predict students' self-confidence in learning mathematics. These variables refer to students' perception of their parents' and teacher's attitudes toward them as learners of mathematics. They also include parents' and, teachers' interest, encouragement, and confidence in the student's ability (Fennema & Sherman, 1976).

Fennema and Sherman (1976) stated that attitude toward success in mathematics refers to students' anticipation about positive or negative consequences as a result of success in mathematics. They also explained that mathematics anxiety refers to feelings of anxiety, dread, nervousness, and associated bodily symptoms related to doing mathematics. As seen in the definitions, there could be relationships between mathematics students' attitude toward success, mathematics anxiety, and self-confidence in learning mathematics. Thus, in the present study these variables are taken as predictor variables to explain self-confidence in learning mathematics.

Data Sources

In this study 841 ninth- grade students (374 females and 467 males) participated in the 1999-2000 academic year. The study was conducted with a conveniently selected sample of students from both private and public high schools in Ankara- Turkey.

Instruments

In the present study, data were collected through using the Mathematics Achievement Test (MAch) developed by Bulut and Tag (2000). It consists of 20 multiple-choice type questions. Item discrimination powers and item difficulty powers were computed. The validity of the test was confirmed by a mathematics educator and a mathematics teacher. Its alpha reliability coefficient was found as 0.81.

We also used the scales that Bulut and Tag (2000) adopted from Fennema and Sherman scales: Self-Confidence in Learning Mathematics (Con) Scale, Attitude Toward Success in Mathematics (AtSc) Scale, Mathematics Anxiety (Anx) Scale, Father (FcSt) Scale, Mother (McSt) Scale, and Teacher (TcSt) Scale. One of the item, "My mother/father hates mathematics" was removed from the Father and Mother Scales because they are not related to mothers'/fathers' opinion about their children. The other scales consist of 12 Likert-type items. The items in the scales had 5 response alternatives: strongly agree, agree, undecided, disagree and strongly disagree. While positive items were scored from 5 to 1, negative items were scored from 1 to 5. The alpha reliability coefficients for each scale range from 0.79 to 0.93.

Results

Correlational Analyses

We present in Table 1 pearson correlations, means, and standard deviations for study variables [Criterion variables: self-confidence in learning mathematics (Con); Predictor variables:mathematics achievement (MAch), attitudes toward success in mathematics (AtSc), mathematics anxiety (Anx), gender, perceived father characteristics related to students (FcSt), mother characteristics related to students (McSt)and teacher characteristics related to students(TcSt)]. Low correlations between predictor variables in Table 1 indicate that the multicollinearity was not a problem.

Table 1. Correlations, Means, Standard Deviations for Seven Variables

Variables	Con	MAch	AtSc	Anx	FcSt	McSt	TcSt
Con	-						
MAch	.34*	-					
AtSc	.39*	.22*	-				
Anx	.71*	.22*	.28*	-			
FcSt	.42*	.25*	.44*	.30*	-		
McSt	.40*	.22*	.41*	.24*	.59*	-	
TcSt	.47*	.18*	.35*	.46*	-.40*	.38*	-
Gender	.06	.01	-.06	.07*	.02	-.02	-.06
Mean	42,328	10.845	52.408	41.451	46.214	41.342	42.328
SD	8.793	4.813	6.575	10.414	7.084	5.903	7.685

Note: Gender is coded 0=girls and 1=boys.
* $p<.05$

Regression Analysis

Multiple Regression Coefficient (MRC) Analysis is conducted in order to test the relationship between mathematics achievement, attitudes toward success in mathematics, mathematics anxiety, gender, perceived father, mother and teacher characteristics related to students, and the self-confidence in learning mathematics. The data analysis indicates that gender is not a significant predictor variable for criterion variable ($p>0.05$). So, this variable was removed and MRC is accomplished one more time. The variables related to the results of this analysis are given below:

As seen in Table 2 there is a significant relationship between self-confidence in learning mathematics and linear combination of the predictor variables: Mach, AtSc, Anx, FcSt, McSt and TcSt ($F=212.963$, $p<0.05$). Approximately 61% of the variation in the self-confidence in learning mathematics are explained by the variation of the combination of predictor variables (R square= 0.61).

Table 2. Summary of MRC Results for Combined Effect of Six Predictor Variables

Regression Statistics			Values		
Multiple R			0.78		
R Square			0.61		
Adjusted R Square			0.60		
Standard Error of the Estimate			5.545		
Number of Cases			841		
	df	SS	MS	F	Sig F
Regression	7	39294.263	6549.044	212.963	0.000*
Residual	834	25647.159	30.752		
Total	840	64941.422			

* $p < 0.05$

Table 3. Results of Multiple Regression Analysis for Six Predictor Variables

	B	Standard Error	t-ratio	Sig. of t
Constant	-2.357	1.781	-1.323	0.186
MAch (X1)	0.248	0.042	5.921	0.000*
AtSc (X2)	0.134	0.034	3.946	0.000*
Anx (X3)	0.481	0.021	22.763	0.000*
FcSt (X4)	0.009	0.036	2.458	0.011*
McSt (X5)	0.169	0.042	4.038	0.000*
TcSt (X6)	0.009	0.030	3.039	0.002*

*$p < 0.05$

As indicated in Table 3, mathematics achievement, attitude toward success in mathematics, mathematics anxiety, students' perception of their parents' and teachers' characteristics related to students are significant factors affecting self-confidence in learning mathematics. The empirical model constructed by the significant factors is: $Y= -2.357+0.248X1+0.134X2+0.481X3+0.009X4+0.169X5+0.009X6$; where Y is "self-confidence in learning mathematics" and Xi's are variables identified in Table 3.

Conclusions and Implications

We can predict the student's self-confidence in learning mathematics by using above equation included MAch, AtSc, Anx, FcSt, McSt and TcSt. In other words, students' self-confidence in learning mathematics is affected by their mathematics achievement, attitude toward success inmathematics, mathematic anxiety and perceived attitude of father, mother, and teacher toward one as a learner of mathematics. The results of the present study are consistent with the study of Armstrong (1981) who found a strong relationship between mathematics achievement and self-confidence in learning mathematics. Like Wang, et al. (1996), we revealed that students' family background influences their self-confidence in learning mathematics. On the other hand, the findings of the present study, unlike Hyde, et al. (1990), could not show that gender is a significant factor determining self-confidence.

The effect of the combination of these variables on self-confidence in learning mathematics is very important because the variation of this combination on self-confidence in learning mathematics as 61 percent. There is a statistically significant relationship between criterion variable and predictor variables. Consequently, we should improve student's self-confidence in learning mathematics and organize our mathematics instruction by taking into account all study variables stated in the present study.

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EXPLORING THE TEACHER'S ROLE IN DEVELOPING AUTONOMY

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Understanding students' mathematical thinking and efforts to improve instruction have focused on classroom discourse "designed to illuminate what is entailed in a 'discussion' and to probe the specific moves that teachers and students engage in that lead to productive rather than unproductive discussions" (National Research Council, 2001, p. 359). This paper investigates the interactions of a group of fourth grade students and their teacher by providing a detailed description and analysis of a classroom episode of group problem solving. In particular we explore how interactions within the classroom's micro-culture impact the development of individual and group autonomy. The purpose of this paper is to deepen our understanding of the ways in which teachers and the social contexts of their classrooms can promote or restrict the development of autonomy through discourse.

Theoretical Considerations

In Piaget's (1973/1948) view, the central purpose of education is the development of autonomy. This view is consistent with current reform movements in education particularly in mathematics (Kamii & Dominick, 1998; Yackel & Cobb, 1996). Autonomy also plays a key role in complex adaptive systems theories, where autonomy is defined as the capacity for self-regulation. In both of these cases, autonomy does not imply "freedom" as the term is generally used; rather it is an individual's capacity to act in accordance with core knowledge and values (Deci, 1995; Kamii, 1994), responding to environmental factors in intelligent and adaptive ways. According to Piaget, unequal power or authority relations tend to constrain the development of autonomy while cooperation encourages reciprocal interaction and the growth of individual autonomy. For Piaget cooperation meant, "striving to attain a common goal while coordinating one's own feelings and perspectives with a consciousness of another's feelings and perspectives" (DeVries, 1997, p. 5). It is characterized by mutual respect in which adults seek to minimize their use of authority while providing opportunities for children to practice governing their own behavior.

The development of individual autonomy is determined by interactions and the social contexts in which they occur. By engaging in the process of self-determination through autonomous action individuals learn how to make better decisions and thereby develop their own creative potential. Thus the growth of individual autonomy is supported in a social context of mutual respect, cooperation, and freedom of choice. Likewise it is constrained in a context of coercion, competition and control. Since it is apparent that social context plays a key role in the development of individual autonomy the idea of socio-autonomy (Fleener & Rodgers, 1999) is introduced to provide a theoretical lens for examining the social dynamics associated with its development.

Although socio-autonomy shares many of the characteristics associated with cooperative learning, it is defined as an aspect of the dynamics within a social context rather than a set of actions or organizational structure. Socio-autonomy describes how groups develop their own identities and autonomous characteristics that may support the development of individual autonomy. Some key features of socio-autonomy are: shared meanings; communication; active engagement in common purposes; opportunities for meaningful choice making; and "dialogical encounters" in which "one begins with the assumption that the other has something to say to us and to contribute to our understanding" (Bernstein, 1992, p. 337).

In analyzing classroom interactions we use Wittgenstein's (1953) view of language games as communication networks within social contexts and Sfard's (2000) conception of thinking as "communication with one's self" (p. 297). For the researcher, Sfard (2000) contends, we need to (re)conceptualize what we may mean by communication and how we can define communication as a process rather than as a bartering, exchange, or conduit of ideas. She shifts attention to Wittgenstein's approach to the construction of meaning. Rather than looking for underlying structures or meanings, Wittgenstein explored the emergence of meaning as language-games. To employ this method in educational research, Sfard suggests exploring not the meanings of discourse, per se, but the discourse itself from which meanings, intentions, understandings, and feelings can be examined as a way of reading the language-game.

Wittgenstein's language-games conception of communication considers the emergence of meaning as evolving through use and social context rather than correspondence. Language games are the enactment of social dynamics and thus are the existing meaning structures in specific social settings. At the same time, language games reveal the dynamics among authority, autonomy and the construction of shared meanings. Language games get played out in a social context, and thus, meaning is understood from a social perspective. Rather than language as the origin of meaning, Wittgenstein remarks, "to imagine a language means to imagine a form of life" (Wittgenstein, 1953, section 19), implying that who we are and how we live are fundamentally related to the language we use. Meaning is understood not as something outside of language, but emergent as language in use. "Meaning is never going to be satisfactorily explained by pointing to the mental, to things like intention" (Genova, 1995, p. 120). The difficulty with denying that words have meaning, however, is not the problem of meaning as reference but shared meanings. How is communication possible at all? How do we use language to derive meaning and share understandings? From a Wittgensteinian perspective communications and thus shared meanings are possible because we engage in language games. This understanding of language games is consistent with social systems as meaning structures and Sfard's conception of thinking as a form of communication. The language-games being played are integral, inseparable and indistinguishable from the meanings that are being constructed during discourse. We will follow Sfard's approach by observing student and teacher actions and interactions in analyzing communication and the construction of shared meanings. We will look to the language games being played in a brief classroom episode in order to better understand these classroom interactions and the implications they have for teaching.

Social Context

The Benjamin Banneker Elementary School is a school-within-a-school. It is a part of the larger Adams Public School, which is located on the far south side of Chicago. It is also a part of a loosely organized network of more than one hundred "small schools" within the Chicago Public Schools. The entire Adams School had an enrollment of 550 students in the 1996-97 academic year when the data described here were collected. Eighty-nine percent of these students are from low-income families and 98.8% are African-American. The Benjamin Banneker School first opened its doors to students in the fall of 1996. The school's origin can be traced to two years earlier when a group of Adams teachers began to meet and organize a new school based upon a shared vision of educational reform. In the "Benjamin Banneker School Vision Statement" they describe their goal as creating a school where "all children can succeed and feel good about themselves and where learning is a joy due to creative leadership and innovative teaching." One of the early decisions made by these teachers was the adoption of *Math Trailblazers*, a "Standards based" K-5 mathematics curriculum developed with NSF funding at the University of Illinois at Chicago. The lesson presented here is taken from a third grade *Math Trailblazers* unit that focuses on fractions.

The interactions presented here came from a single lesson taken from a series of videos made during regular classroom visits occurring over a one-year period. Although the interactions described here were common for this classroom, this particular episode was not selected as a general representation of the class as a whole. Rather, this episode was chosen for analysis because of the variety of discursive patterns found in an extended interaction involving the students and their teacher. The group consists of four girls, Tiffany, Diane, Mary, and Tina. Tiffany describes the specific problem task at the beginning of the transcribed dialogue.

The Fair Shares Lesson

The class begins with a student reading the instructions out loud for the problems and students cutting out paper pizzas to assist them in their work. After about ten minutes the teacher stops the class and asks another student to read the directions again. She then asks the class to explain what the instructions mean by the phrase "dividing fairly." The first student responds by describing a situation in which she has experienced sharing. Another student explains: "everyone gets a piece." The teacher follows up by asking: "How many pieces?" to which the student responds "an equal number of pieces." Other students respond that to share fairly means that "everyone receives the same amount", "the pizzas are divided equally", and that "they had a congruent amount." The teacher summarizes the discussion by repeating one student's response "to share fairly means everyone receives the same amount." She then instructs the students begin working on the problems in their small groups.

Small Group Interactions

For about the next 45 minutes all seven of the student groups appear to be actively engaged in solving the problems as the teacher moves around the room providing assistance to various groups. The interactions, which are described here occurred near the beginning of the group work and lasted for about ten minutes. Our observations begin as the group is completing the initial task of cutting out the "pizzas" and starting to work on the assigned problems. The group is working together cooperatively and has assumed an independent identity characteristic of socio-autonomy. Each of the girls appears to be busily engaged in a task related to the problem and over-all group focus. Tiffany is quietly reading the problem to the group while Mary carefully counts out 10 paper pizzas to each of the girls and Diane is cutting out the last of the pizzas. Tina, who has been providing some leadership for the group, appears to be thinking as she writes something down on her paper. A few moments earlier she has initiated a creative problem solving approach by enthusiastically asking Tiffany to play the role of Amber from the problem. Now, after further thought, she says to Tiffany:

Tina: Tiffany what's the question?

Tiffany: OK. (Tiffany reads the question from the textbook louder this time as Tina listens attentively and the other two girls continue their work.) Amber, Mario, Denise and Jake have a pizza they want to share fairly. How much pizza will each one get?

Tina is writing on a piece of paper while Tiffany reads the question. Just as Tiffany finishes reading the teacher walks up behind Tina and stands quietly listening to the group.

Diane: How many pieces?

Tina: (Counting the number of names in the problem) 1, 2, 3, 4, so each person would get two pieces of pizza.

Teacher: (this unexpected response of "two pieces" catches the teacher's attention and she enters the discussion) O.K., how many? (She holds a paper pizza up in her hand.) They have one pizza they're working with right?

Girls: Yes—

Teacher: O.K. now what does that question say?

All the girls together: How much will each one get?

Teacher: So how many people are getting a pizza?

Two of the girls: 4

Mary: (who appears to be in deep thought and also ignoring the teachers question, jumps ahead with her answer and begins speaking at the same time as the other two girls) 1 ½ , they're getting 1 ½ .

Teacher: (directed to Mary who has caught the teacher by surprise with her unexpected answer) So if you have one pizza and each gets a fair share ...

Mary: 1 ½ (Mary interrupts)

Teacher: Baby its 1 pizza not 4.

Here the teacher is clearly trying to help Mary understand the problem, but she also fails to make any attempt to understand Mary's answer of 1 ½. Mary responds with this same answer, "1 ½"at least five times during the episode. It seems that, for Mary, "½" is her fraction word and her answer of "1 ½" is equivalent to "Each got one piece--a fractional piece. Even though Mary's answer did not fit into the language being used in her mathematics classroom to describe the fractions, she was clearly trying to engage in the language games of mathematics being played in her group. But since no one attempts to find out what she means by "1 ½," an opportunity for her to receive valuable feedback and possibly improve her understanding of fractions is missed.

Responding to Mary's apparent confusion, Tina continues her leadership role by attempting to explain her answer to Mary. Tina's attempt to help Mary understand might have been an indication of Tina's interest in having fun with the mathematics, making it relevant, or helping the group come to a shared understanding of her answer. It's not clear, however, whether the others, including the teacher, were open to playing along with Tina's new language game. And before Mary has time to respond to Tina, the teacher tries to clarify Tina's explanation saying, "Oh, you're going to cut it into 8 slices?"

From this point forward, an important shift can be observed in the group's interactions. Almost every verbal exchange in this small group is initiated by the teacher and occurs between the teacher and one or more of the students in the group. The cooperative work and dialogical interactions characteristic of emerging socio-autonomy found at the beginning of our observations have now shifted and the teacher has become the focal point of the group's interactions.

After a few more questions from the teacher, Tiffany confidently provides the desired answer of 1/4 and the teacher repeats the answer while giving Tiffany an approving nod. Tiffany smiles triumphantly while Diane and Mary appear to be waiting for the teacher's next question. Although the teacher and Tiffany have "solved" the problem by playing in the teacher's language game, Tina has continued working independently on her eight-slice scenario and soon provides this response to a question posed earlier by the teacher:

Tina: 2/8

Teacher: O.K., O.K. now what can we say, you said 2/8 and you have 2/8 and she said 1/4. (pause) O.K. you said 1/4 and you said 2/8 now that's two different things, so what does that mean, so what can we say, you're only going to get 1/4 and you're only going to get 2/8? So what does that mean?

Diane: Uhh, (excited, stands up, searching for the correct word) Uhhh, well it's a mixed uhm.. and uhmm...uh

Teacher: You said each person is going to get ¼ and you said each person is going to get 2/8. What can we say about 1/4 and 2/8? (There is some excitement as the girls search for the correct word.)

Diane: They are ...

Mary: They are...

Tiffany: There are 4 people; hang-on it's like this...

Teacher: O.K. one at a time

Here the teacher is attempting to take advantage of Tina's scenario to illustrate the idea of equivalent fractions. The game she is playing is to get her students to recognize that 2/8 is equivalent to 1/4. The effort of the group to make sense of her questions about the relationship between sharing a pizza cut into fourths and sharing one that is cut in eighths is confused by an inability of some of the students to understand the language of "amount" versus "pieces."

Teacher: Let's listen to Tina.

Tina: On my birthday I was having a party and I invited 7 people plus me and...

Teacher: O.K. you know what dear, now is it 7 people you're inviting or don't forget we're still on this one here (pointing to the problem in the book), we're still on number 6, think of inviting those 4 people.

Tina: So instead of inviting 7 people I invite 4 people and I had a pizza, I had a pizza that was cut into 8 and I wanted to see how many slices everyone would get including me if my pizza was cut into 8.

Teacher: So if it is divided into 8 slices, so how many slices would each person get?

Mary: (lost in thought, then suddenly speaking up) 4 ½

Mary's response appears to be consistent with her previous responses of "1 ½". In her language, she is indicating that each person would get a fractional piece. There are four people, so there will be four fair shares of a fractional amount. Once again though, no one pursues Mary's answer or tries to make sense of it. Instead Tina continues trying to develop a scenario acceptable to her teacher.

Tina: But, but I said 4 people plus me.

Teacher: No you have to be counted as one of those people. But remember; let's not get away from the problem.

Tina: O.K. O.K. so I would be one of them. So everyone at my party would get 2 slices. *Teacher:* O.K. how can we say 2 slices as a fraction?

Tina: 2/8

Teacher: So now my question--what can we say about 2/8 and 1/4?

Tina: (Finally acquiescing to the game the teacher is playing) They are the same.

Teacher: O.K. do you know another word for the same?

The teacher has not received the answer she wants, which is "2/8 is *equivalent* to 1/4." She is trying to help her students adopt a more standard use of mathematical language in answering a specific question about the relationship between 1/4 and 2/8. When, after further questioning it becomes apparent that the group is not following her reasoning she tries shifting to a more accessible representation of the idea of equivalent fractions.

Teacher: Four-eighths equals? —

Diane: One-half of the pizza.

Teacher: 4/8 is *equivalent* to one-half. O.K. so you've answered number 6. What does number 6 say again?

Mary: (tries to begin reading the problem but struggles and Tina takes over)

Tina: Amber....how many slices of pizza will each person get?

Mary: 1 ½

Mary's return to her original answer of 1 ½ is significant but not surprising. It provides compelling evidence that Mary's interactions during these exchanges have done little to alter her understanding of fractions or to change her mind about the correct answer. Her use of "fraction language" is different from that of her teacher. Without trying to make sense of the fractional language being used by the other, neither has improved their understanding of what the other is trying to say. Confounding communications about fractions is the conflated use of fractional language to describe, "how many pieces" rather than "how much pizza" each person receives in a particular scenario. These students, it seems, are still relying primarily on counting strategies rather than working with fractions on a deeper conceptual level as equal parts of a whole. The teacher tries to make the transition from discrete quantities to fractional relationships, but within the current language game, the students are unable to follow the new script. While the teacher is trying to get her students to play in the game by using the standard fractional words, fractional concepts as negotiated meanings are not part of the game.

Discussion

Our observations began with a group of four girls working together cooperatively. In the first observed interaction between Tina and Tiffany, we see Tina excitedly asking Tiffany to play the role of "Amber." With these actions she appears to be exercising leadership of her group in at least two directions. On the one hand she is focusing her efforts on the assigned problem while at the same time trying to make the process more interesting by asking Tiffany to role-play a person from the problem. Tina continues her leadership by asking Tiffany "What is the question?" Tiffany reads the question and Tina quickly solves it with the creative and unexpected solution of "two pieces." In so doing, she has bypassed the most simple and direct solution to the problem and introduced a new avenue of questioning. All of these actions are carried out in a playful and energetic way as Tina uses her enthusiasm and leadership to pull the others in her group into the language game she has initiated.

Although Tina appears to be giving the group strong leadership, each of the other girls is also engaged in a task related to solving the problem. Not only is the group working together in a self-sustaining way to solve the assigned problem, they are also generating new and potentially more interesting problems as a result of their interactions. Since these students have chosen to work together cooperatively towards a common goal, the group appears to have taken on a life of its own and we see the emergence of socio-autonomy. It is in social settings such as this, where students have the opportunity and motivation to make and learn from their decisions that the development of autonomy is most likely to occur. Creating a social environment where cooperative work like this can occur is a complex and challenging task for any teacher. To have brought her class to this point is a significant accomplishment, which requires both thoughtful preparation by the teacher and a curriculum like *Math Trailblazers* that emphasizes cooperative problem solving.

Shortly after Tina gives her unexpected answer of "two pieces," before we have an opportunity to see how the group will respond to Tina, the teacher enters the discussion. Her intentions are to facilitate the group's discussion by steering it in an appropriate and pre-determined direction. But the group dynamics from this point forward are significantly altered and become dominated by the teacher's understanding of her role as a teacher of mathematics. This understanding of her role is clearly illustrated in her interactions with Mary. Even though Mary sticks with her answer of 1 ½ throughout the discussion the teacher never stops to probe what Mary may have meant by this answer. Meaning in the sense of a medium of communication does not occur once the language game focuses on students' adopting particular understandings rather than engaging in discourse that allows meanings to emerge through the discursive process. The teacher tries to correct Mary's mistake, a realistic understanding of her role as the teacher, rather than hermeneutically listening (Davis, 1997) to Mary's response. Although she uses a variety of well-intentioned approaches in trying to help Mary get the "correct" answer, Mary's answer remains unchanged. Their actions, as seen from a language games perspective, indicate communication, as a medium for meaning evolving through language games, has not occurred. There is no evidence of the types of negotiation necessary for the construction of shared meanings as communications.

It may seem disappointing that a competent, well prepared teacher using a curriculum designed to emphasize student participation in problem solving, appears to see her role as (1) getting her students to adopt the standard language of mathematics and (2) directing her efforts toward telling students what they need to know rather than listening to determine what they understand. An alternative interpretation of these findings is to recognize the complex challenges that occur when teachers who are accustomed to traditional pedagogical practices seek to bring about educational transformation. Teachers engaged in such attempts are inevitably faced with the conflicting demands of teaching for understanding versus covering content; supporting the development of autonomy versus maintaining classroom control; and the social negotiation of meaning versus the efficient transfer of specific information. All of these conflicts and the constant lack of resources and support add to the daunting challenges faced by teachers such as the one described here who seek to bring about educational reform.

Changing our Vision of Teaching

In his development of the idea of "vision" Wittgenstein distinguishes between having a world-view and having a world-picture. For him, a world-view implies a vision of the world as a static unchanging reality waiting to be discovered, capable of transmission from one individual to another (Wittgenstein, 1969). Unfortunately, schooling, with a focus on the teaching-learning relationship, at least according to Wittgenstein, takes on an air of indoctrination by measuring learning against the yardstick of a *Weltanschauung*.

Traditional mathematics instruction engages the teaching-learning dialectic to advance a particular *Weltanschauung* implicating instruction as well defined concepts and procedures that are to be passed from teacher to student. In this model of teaching the social negotiation of meaning and dialogical interactions are of little value since knowledge is pre-determined and always passed from teacher to student. Also the development of autonomy is thwarted since discourse fails to facilitate communications as the medium of meanings and rather promotes a particular world-view.

While *Weltanschauung is* static, *Weltbild* is dynamic and evolving as we add to and alter it with each new experience. Wittgenstein associates *Weltbild* with "seeing-as"; a way of seeing that is more about making connections than identification and naming (Genova, 1995). An aspect of experience and revealed through language games, *Weltbild* evolves as seeing-as changes.

> I contemplate a face, and then suddenly notice its likeness to another. I *see* that is has not changed; and yet I see it differently. I call this experience "noticing an aspect."...The expression of a change of aspect is the expression of a new perception and at the same time of the perception's being unchanged....'See-as'...is not part of perception...it is related to the experience (and)...it can also be called the expression of thought...you are also *thinking* of what you see. (Wittgenstein, 1953, pp. 193-197)

From an instructional perspective, adopting Wittgenstein's ideas of a *Weltbild* and "seeing-as" offers important advantages. First if teachers adopt this view then mathematics is seen as an ever changing, evolving construction shaped both by social interactions and individual thought. The focal point of instruction then becomes the conceptual development of mathematical ideas through dialogical interactions, social negotiations and the construction of shared meanings. Secondly, understanding the importance of language games as on-going mediums for meaning, the teacher's role shifts from getting students to use pre-determined language and concepts to listening and engaging students in knowledge-in-context (Applebee, 1996). The discourse domain of mathematics then becomes a *Weltbild* and not the *Weltanschauung* found in traditional mathematics instruction. Within the social context of the classroom, mathematics curriculum as conversation, engaging in the discourse domain of mathematics, and seeing the world as mathematical contribute to both emergence of socio-autonomy and the development of individual autonomy.

The significance of this study for changing our ideas about curriculum include developing a model of interpretation for classroom language dynamics and deepening our understanding of the social contexts which support the development of autonomy. The perspective of language games *as* meaning rather than as ways of understanding some underlying meaning structures is crucial to understanding an emergent curriculum founded on the principles of self-organization and social meaning structures. In order to move toward the integration of these curricular ideas into instructional practice we must first find effective ways of altering our vision of mathematics. We must move from the widely held view of mathematics as an unchanging set of concepts and procedures to a dynamic vision of mathematics as an ever-changing socially negotiated construction. While we may still find value in the mastery a specific *Weltanschauung* of mathematics, how powerful and gratifying it would be for teachers and students to see the world as mathematical relationships, and to participate together in the creation of a *Weltbild* of mathematics.

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THE COMPLEXITY SCIENCES AND THE TEACHING OF MATHEMATICS

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We report on a year-long teaching experiment in which principles of complex emergence were used to frame activities and interactions within a junior high school mathematics class. We begin with a brief introduction to complexity science, endeavoring to situate the discourse alongside contemporary theories of learning, knowledge, and culture. This discussion is used to frame the report of the teaching experiment, which is in turn used to illustrate an examination of some pragmatic aspects of complexity science research for mathematics teaching practice.

Complexity science is interested in questions of emergence—that is, in those instances when coherent collectives arise through the ongoing effort of individuals to maintain their fit within evolving circumstances. These sorts of phenomena might be further described as 'bottom-up', as emergent macrobehaviors arise through localized rules and behaviors of individual agents, not through the imposition of top-down instructions.

On these counts, complexity science is more a metadiscourse than a new field. It has arisen in the coalescence of relatively isolated research efforts and insights across economics, sociology, psychology, neurology, political science, and other academic domains. (See Capra, 1996 and Johnson, 2001 for extended lists.)

Weaver (1948), in an article that is regarded by many as seminal to complexity science, was among the first to formally categorize those phenomena that have become the subjects of complexity science research. In fact, he distinguished among three sorts of phenomena. The first category, *simple systems*, includes those dynamic events that involve only a few interacting parts or variables. Framed by the insights of Galileo, Descartes, Newton, and their contemporaries, such systems include simple trajectories, billiard ball collisions, orbits, and the like.

As was recognized early on, the analytic tools developed to describe and predict such simple interactions can give rise to intractable calculations when the number of interacting components increases only slightly. In the 19th century, as scholars met up with more and more such phenomena, new analytic tools based on probability and statistics were developed. These methods were useful for the examination of phenomena that might involve millions of parts or variables—in Weaver's terms, *disorganized complex systems*. The development of statistical and probabilistic methods was more a resignation than a shift in thinking. The driving assumption was still deterministic, consistent with Laplace's (1795/1951) often-cited pronouncement:

> Given for one instant an intelligence which could comprehend all forces by which nature is animated and the respective situations of the beings which compose it—an intelligence sufficiently vast to submit these data to analyses—it would embrace in the same formula the movements of the greatest bodies and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. (p. 4)

The move to probability and statistics, then, was an acknowledgement that no flesh-based intelligence was sufficiently vast. The more intricate and complicated the phenomenon, the more one was compelled to rely on descriptions of gross patterns rather than on analyses of interacting factors.

Weaver was among the first to note that these two categories—simple systems and disorganized complex systems—did not cover the full range of possibility. Specifically, they omit those phenomena in which the interacting parts themselves are not inert or mechanical, but fluid and co-specifying. Such phenomena as societal coherence, cultural evolution, and brain function, for instance, do not readily submit to analytic and statistical tools. The essential quality of phenomena in this category is that they maintain some level of coherent organization.

Organized complex systems, as Weaver named them, represent a rejection of determinism and a demotion of analytic reduction. A different sensibility is required for their study, in large part because a comprehensive knowledge of the parts that comprise such systems is not adequate for the prediction of the qualities or behaviors of the collectives. The problem here is not a matter of inadequate calculation power; it is a matter of the sorts of transcendent possibilities that emerge when complex unities cluster into more complex unities.

The adjectives 'disorganized' and 'organized' have since been dropped in discussions of complexity, in favor of a more fundamental distinction between 'complicated' and 'complex'. As Waldrop (1992) develops, complicated is used by complexity scientists to refer to deterministic, mechanical events, and thus includes both simple and disorganized complex systems. Complex, in contemporary parlance, is used to refer to what Weaver described as organized complexity.

Over the past 30 years, the emergence of complexity science has paralleled the embrace of non-representationist epistemologies among mathematics education researchers. The two trends are compatible on several counts, including shared emphases on post-Darwinian evolutionary dynamics and a greater use of biology-based metaphors to describe knowing, knowledge, and other complex phenomena. The movements, however, are not coterminous, as becomes clear through comparisons of their objects of study and the utility of their conclusions. Regarding the objects of study, contemporary non-representationist discourses tend to be focused on particular, well delineated phenomena. Radical constructivism, for instance, follows Piaget in its focus on the individual's construals of the world (von Glasersfeld, 1995), whereas particular social constructivist discourses might focus on the emergence of a particular body of knowledge (e.g., Ernest, 1991), cultural sensibilities (e.g., Skovsmose, 1994), or some other collective phenomenon. Complexity science, by contrast, collects together such phenomena as individual sense-making, collective knowledge, and social habitus as a class of phenomena.

At the same time, however, complexity science explicitly rejects any attempt to collapse such phenomena into instances, variations, or elaborations of the same thing. A core tenet of the discourse is that complex unities must be studied at the levels of their emergence, as new possibilities arise and new rules apply in each case. A perhaps surprising implication of this assertion is that discourses that are concerned with different phenomena (such as radical and social constructivisms) can be simultaneously valid and incompatible. For instance, the implicit dynamics of individual sense-making and collective knowledge (or neuronal interaction, stock market behavior, etc.) may be similar and co-implicated, but the events themselves cannot be collapsed into or mapped onto one another.

In its first few decades of development, complexity science consisted mainly of descriptions of various complex systems. More recently, Johnson (2001) notes, the emphasis has shifted toward more deliberate efforts to trigger complex emergence. Much of the current work in complexity science is concerned with the identification and manipulation of the qualities that are common to all complex systems—that is, the conditions that must be present for agents to come together into collectives that might supersede the possibilities of those agents on their own. On this count, complexity science suggests that the *possibility* exists for transcendent collectivity wherever self-specifying agents are allowed to interact. However, as has been demonstrated in many mathematics classrooms, something more is needed for complex emergence than proximity, common goals, and common tasks.

Unfortunately, because complexity is a-deterministic, the "something more" that is needed cannot be specified. A shift in interpretive focus is required here, away from what *must* or *should* happen toward what *might* or *can* happen. Pragmatically speaking, efforts to occasion complexity revolve more around the specification of boundaries and conditions than around efforts to prespecify outcomes and means—or, as Varela (1987) frames it, on proscription rather than prescription. Proscription is a matter of what is not forbidden is allowed, versus the more prescriptive attitude of what is not allowed is forbidden. Proscriptive situations include, for example, most games, social interactions, business dealings, and artistic endeavors. In formal and explicit terms, these sorts of activities are framed more by what one must not do than by what one must do.

At first glance, applied to discussions of school mathematics, this prescription/proscription distinction might seem to map onto the extremes of popular debate. At one end of the spectrum, the caricature of a strictly rote-based, 'traditional' classroom seems the epitome of a prescriptive situation. On the other end, an exploration-oriented, student-centered, 'progressive' classroom appears to embody a proscriptive sensibility. Such a mapping, however, is not entirely appropriate.

The difficulty arises in the extreme interpretation of the progressive classroom, in terms of an anything-goes abandonment of rigid structures and imposed expectations. A proscriptive attitude does not necessarily support such moves. In fact, proscription is not an abandonment of constraints, but a shift in thinking about the sorts of constraints that are necessary for generative possibility. As Johnson (2001) explains, complex emergence occurs in rule-governed systems:

> their capacity for learning and growth and experimentation derives from their adherence to low level rules.... Emergent behaviors, like games, are all about living within boundaries defined by the rules, but also using that space to create something greater than the sum of its parts. (p. 181)

Occasioning Complexity in a Mathematics Classroom

We turn now to an account of a classroom event that was explicitly informed by principles of complex emergence. Specifically, the lesson (taught by Simmt) incorporated the necessary but not sufficient conditions of internal diversity, redundancy, decentralized control, organized randomness, and neighbor interaction. The meanings and significance of these notions will be elaborated in the discussion that follows our report of the event.

The episode unfolded in a Grade 7 classroom, near the end of November, two weeks into the study of integers. The initial focus of the unit was to render explicit the sorts of applications and images that students brought to the topic—including number lines, thermometer scales, profit/loss figures, and altitudes, as well as more primitive (and more tacit) images for counting and other binary operations such as set/container schemas and path schemas (see Lakoff & Núñez, 2000).

In this lesson, students were asked to consider the statement, "3 x $^{-}4$ = ?" They were paired off (by the teacher) and each pair was given ten minutes to agree on a product and to "Show how you know"—that is, to prepare an explanation on chart paper for presentation to their classmates.

In our analysis, students' responses fell into five categories:

- *grouping of objects*, whereby 3 x $^{-}4$ = $^{-}12$ was described and explained as three groups of four negative chips;
- *number line movement*, where 3 x $^{-}4$ = $^{-}12$ was interpreted as three hops of length four, starting from zero, moving in a negative direction (which was not always presented as the conventional leftward or downward);
- *repeated addition*, where 3 x $^{-}4$ = $^{-}12$ was rewritten as $^{-}4$ + $^{-}4$ + $^{-}4$, and the already established rules of addition invoked;

- *appeal to external authority*, of either a previously learned rule or the answer generated with a calculator;
- *extending a rule*, in which one pair determined the magnitude of the product by ignoring the signs (i.e., $3 \times 4 = 12$), and assigned the product a negative value "because the minus was on the bigger number".

A feature of these presentations (which took up the rest of the 50-minute block) was that the students closely attended to one another's responses and explanations, without prompting or nagging from the teacher. Their engagements with one another's work were notable around two explanations in particular: the case of extending the rule and one version of repeated addition.

Unfortunately, the case of extending the rule came too late in the session, and there was no opportunity to pursue the challenge, raised by the teacher, that while this strategy gave the same response as the other to $3 \times {}^{-}4$, it would not for $4 \times {}^{-}3$. It did, however, fulfill a sort of cliffhanger function as some students began to engage in rather heated debate as the bell rang and as they filed out of the room.

In the other noted case, one of the pairs that had written on their poster that "$0 + {}^{-}4 + {}^{-}4 + {}^{-}4 = {}^{-}12$". The zero had clearly been squeezed into the left margin of the page, apparently added after the rest of the statement had been written. Noticing that point of difference from the other groups, Simmt asked about the zero. As a member of the pair pointed to a classmate in a different group, he explained, "He said that we have to start somewhere". In other words, although the explicit representation offered by this pair was a case of multiplication by repeated addition, that interpretation was rooted in the image of movement along a number line.

The pair's response about their inclusion of a zero in the sum prompted some rumbling in the room, amid which another student, Tim, announced that "It actually doesn't matter where you start from. It will be the same". As neither the original explanation nor Tim's elaboration had yet been explicitly linked to the number line image, the teacher asked Tim to say more. In response he offered, "Well, negative four, three times, is always negative twelve, even if you start somewhere else".

Interpreting Complexity

We focus our analysis on just the two interpretations developed in this already very partial account. We have organized this section as five intertwined discussions of some of the conditions necessary for complex emergence, as mentioned above.

Internal Diversity

In complexity terms, intelligence is about the capacity to respond not just appropriately, but innovatively to novel circumstances. The key to intelligent response is the diversity represented among a system's agents, as a system's range of possibility is dependent on the variation among and mutability of its parts. Pools of internal diversity enable a system to respond in flexible ways, in response to shifts in circumstance, whether internal (among agents) or external (with the context) to the system.

An orienting principle of the lesson as taught was the importance of diverse contribution. As such, the task, $3 \times {}^{-}4$, was not heard by most as a challenge to home in on a definitive rule or explanation, but as a place to articulate and compare different ideas. Even the pairs that seemed to frame their activities in terms of rule-generation were very much part of this diversity game. A member of the pair that extended the rule was quite proud of the fact that their response differed from all of the others.

The importance of diversity was perhaps most evident in Tim's contribution—in particular, his point that it doesn't matter "if you start somewhere else". In formal terms, we interpret Tim's thinking to be consistent with a vector-based interpretation of multiplication, where direction and magnitude matter but location does not. Tim's insight is an instance of conceptual blending, in which a property of 'portability' was added to the mix of multiplication as set grouping, repeated addition, and number-line hopping. As Lakoff and Núñez develop, these sorts of metaphor mergers have often occurred in moments of significant advances in the history of mathematics.

Tim's contribution was clearly hinged to ideas that had already been represented. This juxtaposition of diverse interpretations, that is, satisfied the condition of internal diversity.

Redundancy

A system's capacity to maintain coherence is tied to the redundancy expressed among its agents. Just as internal diversity enables flexible response, internal redundancy is essential to the co-specification of agents. Educators interested in complex interactivity must thus attend to the common ground of participants. In the mathematics classroom, such redundancy involves more than shared vocabularies and resources. Commonalities of experience, expectation, and purpose are also important.

The redundancies that underlie a system's robustness can be difficult to discern, as they tend to serve as the ground of activity, not the figure. Arguably, then, this part of our analysis should be the lengthiest, and had we the luxury of a more full-bodied description of the setting, it would be. In the months preceding the reported episode, Simmt had devoted considerable attention to the development of standards around acceptable explanation and engagement, in addition to an explicit pedagogical emphasis on the sorts of experiences, images, tools, and metaphors that infuse interpretations of various concepts.

The intention of these efforts was to establish a necessary redundancy among participants—that is, to ensure that agents' *individual understandings* were sufficiently compatible to enable the emergence of *collective understandings*. The redundancy among participants was not framed in terms of master of facts and procedures, but on shared standards. Such groundwork, although not the central focus of this analysis, is arguably the main work of the mathematics teacher.

Decentralized Control

Mentioned in a discussion of teaching, the idea of decentralized control might be heard as an endorsement of student-centered instruction and a condemnation of teacher-centered classrooms. It actually is intended to point to quite a different sensibility.

To appreciate this point, one must be clear on the nature of the complex unities that are desired in the mathematics classroom—which, for us, are mathematical ideas, insights, concepts, and understandings. This point prompts our attentions away from matters of teacher- versus learner-centeredness and toward disciplinary knowledge. In this case, the issue is not *who* is at the center, but *what* is at the center.

To be more explicit, traditionalists and progressivists alike have tended to regard the mathematics classroom as a collection of individuals. A complex theoretic presses our attentions to the possibilities of interpreting the classroom as a collective, in terms consistent with Varela's (1999) description of a complex unity:

> The whole does behave as a unit and as if there were a coordinating agent present at its center.... [A coherent global pattern] emerges from the activity of simple local components, which seems to be centrally located, but is nowhere to be found, and yet is essential as a level of interaction for the behavior of the whole. (p. 53)

This description, we believe, is an apt one of the emergent understandings in the episode described above. The 'knowledge' in this setting clearly did not reside in any person in particular. Nor was the authority exclusively lodged in one character, one argument, or one resource. The control of the knowledge, that is, was truly decentralized, and that decentralization was necessary for some of the ideas to arise.

This is not to say that the teacher did not exercise a particular authority. Simmt was, in fact, very directive on certain aspects. In particular, the collective rules of engagements were explicit in the manner in which she set the task and embodied in the accumulated history of her responses within the collective. And while such aspects of teaching can be analyzed in terms of power and authority, an interpretation based on such monological metaphors would have to ignore the complex, co-specifying activities during the session. It would also have to focus on the activities of the individuals, rather than on the emergent knowledge of the collective.

Organized Randomness

Complex systems must exist in disequilibrium, maintaining a delicate balance between sufficient organization (to structure the agents' interactivities) and sufficient randomness (to allow for flexible response). Such balances are matters of neither "every agent does the same thing" nor "every agent does something different", but of all agents participating in a joint project.

We prefer to use another apparent oxymoron—*liberating constraints*—in our efforts to make sense of the implications of this condition of complex emergence for the mathematics classroom. Liberating constraints are those proscriptions that are placed on classroom activity that, it is hoped, will open up rich spaces of interpretive possibility.

Consider, for example, the decision to structure an entire 50-minute block around the item, "$3 \times {}^{-}4 =$ ". On the surface, this might seem a rather severe constraint. But, of course, the lesson was not really about determining the product of 3 and $^{-}4$, but about the sorts of images and metaphors that enframe and inform discussions and applications of integers and binary operations. (At least, this is how Simmt understood her activity. It is not at all clear that any of the students regarded their engagements in such terms.)

Nor was that particular item the only constraint. Students were also expected to "Show how you know", a demand that effectively opened the door to a diversity of possibility. Unlike some classroom settings in which such randomness might be ignored or not noticed, in this case it was deliberately represented, displayed, and organized.

Importantly, the point here is not that one must seek to prompt randomness in the mathematics classroom. That quality is always and already present by virtue of the complex natures of human thought and sociality. The challenge for the teacher is to find means of tapping into that wealth of possibility. The posters idea was developed in specific response to this concern, and the resulting artifacts helped to establish a balance between sufficient organization to orient students' actions and sufficient randomness to allow for flexible and varied response. Phrased differently, the structure of the activity served to mediate individual insights and the collective project of mathematizing.

For us, the deliberate attention to the mediation of individual and collective projects represents a rejection of linearized curricula—and, with that, an elaboration of efforts toward 'task analysis'. While we believe it important for teachers to have well developed understandings of how mathematical concepts fit together, complexity science prompts us to assert that such analytic understandings must be part of broader appreciations of concepts' experiential requisites, metaphoric roots, and so on.

Neighbor Interactions

On the surface, it might seem almost unnecessary to develop the point that there must be neighbor interactions in order for complex possibilities to emerge. What is not so obvious, however, is what might constitute a 'neighbor' in the context of a mathematics community.

In the rarified and abstracting epistemic body of mathematics, the neighbors that must bump against one another are not physical bodies, but ideas, hunches, and queries—a point that Rotman (2000) underscores in his critique of the popular image of the mathematician as a loner:

> If one observes [mathematicians], then it would be perverse not to infer that for large stretches of time they are engaged in a process of communicating with themselves and one another; an inference prompted by the constant presence of standardly presented formal written texts (notes, textbooks, blackboard lectures, articles, digests, reviews, and the like) being read, written, and exchanged, and all informal signifying activities that occur when they talk, gesticulate, expound, make guesses, disagree, draw pictures, and so on. (pp. 7–8)

Taken to the context of the mathematics classroom, an upshot here is that group work, pod seating, and class projects may be no more effective at occasioning complex interactivity than straight rows and textbook exercises—if there is no emphases on the display, juxtaposition, and interpretation of diverse, emergent ideas. Concepts and understandings must be made to stumble across one another. Without these neighbor interactions, the mathematics classroom cannot become a mathematics community.

Such was the sort of thinking that lay behind the paired interactions, the poster presentations, and the surrounding discussions in the episode reported. These conditions enabled the emergence of an idea not previously considered by anyone in the room, at least not in the terms presented: Tim's merging of the *location-specific* suggestion that "we have to start somewhere" (if multiplication is understood as number-line hopping) with the *location-independent* idea of multiplication as repeated addition. The consequent blending gave rise to the vector-like notion that multiplication is a directed movement of specified magnitude.

Complexity Science and Mathematics Teaching

For us, the main attraction of complexity science is that it provides means of embracing radical, social, and critical constructivist insights into knowing and knowledge while doing something that these discourses are often unable or reluctant to do: speak to the particular, multileveled, deliberate, and practical concerns of formal education.

To that end, the five conditions discussed in this report are not merely useful tools for after-the-fact analyses of classroom events. They can also be put to instrumental use in the preparation of classroom tasks. For instance, the notion of *internal diversity* point to the need to develop activities that can be adapted by learners to their particular understandings and expectations. *Redundancy* highlights the importance of shared experiences and established standards of engagement. *Decentralized control* points to a need for *all* class participants to be attentive to emergent interpretations. *Organized randomness* foregrounds the importance of setting boundaries and rules—constraints that operate proscriptively rather than prescriptively. *Neighbor interactions* prompts attentions to the manners in which ideas might be represented and juxtaposed.

Again, that such conditions are met in a mathematics classroom is no assurance that complex possibilities will arise. However, a neglect of such conditions will provide a reasonable assurance the collective activity will never exceed any individual's insights.

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TECHNOLOGY AND MIDDLE SCHOOL STUDENT RESPONSE TO MODEL-ELICITING ACTIVITIES: A CASE STUDY

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Abstract: Ends-in-view problems often require students to design a generalizable tool for decision-making. To aid the development of such tools, in particular a consumer guide, spreadsheet software can be used to help students develop, test, revise, and refine their products. In the case described here, a seventh grade student uses spreadsheet software to design a consumer guide. The mathematical model used encompasses statistical concepts such as scaling and weighting of variables.

Mathematics education research has explored multiple methods of technology use in the classroom for many years (Kaput & Thompson, 1994). In particular, technology is often used as a computational or instructional aid for students (e.g., Shaffer, 1997; Wilensky, 1995). However, beyond the classroom, the use of technology is also associated with the design of sharable, reusable tools. For the purposes of this paper, a sharable mathematical tool is one designed with a particular purpose or end-in-view even though the exact nature of the tool is not known (English & Lesh, in press). There are particular criteria for designing a useful tool. In particular, it should be clear when, where, how, and why to use the tool in particular situations. For example, a consumer guide should help the consumer make a decision about what product to buy. The consumer guide should reorganize and aggregate data gathered from different products in order to give the consumer a recommendation about what to buy. A consumer guide should be clear about what data were collected and how the information was analyzed. The information should also be presented in such a way that the consumer easily understands the results. Such tools may be used to aid consumers with decision-making, report information, or aggregate information. Spreadsheet software can be used to aid the collection and analysis of data for developing the tool.

Model-eliciting activities are one example of a type of problem that can incorporate tool design and characteristics of ends-in-view problems. In the example given in this paper, a student works on a model-eliciting activity called the *Snack Chip Problem* to design a consumer guide for purchasing snack chips. The student uses Microsoft® Excel to design the consumer guide which allows individual consumers to rate different types of snack chips. First, I elaborate on salient characteristics of model-eliciting activities. Then, I describe the experience of a student working on the *Snack Chip Problem* and characteristics of the development of the consumer guide tool.

Ends-in-View Problems and Model-Eliciting Activities

Ends-in-view problems are tasks requiring students to develop complex products with specific criteria (English & Lesh, in press). The tasks are called "end-in-view" because the students know what the purpose of the product, but they do not know the exact nature of the product. The students are given requirements for the final product and must sort through available information to design the product. This type of task goes beyond standard word problems where the information given is clearly defined and the students are asked to find some lost tool or operation for transforming the givens to goals. In an ends-in-view problem, the task of the student is often to design the procedure for processing information and to determine what the givens and goals are. There are specific criteria for a successful product, but the exact design of the product is not specified. Often they are producing a sharable, reusable model. So, instead of an answer to a problem, the students are producing a generalizable procedure that can be used for decision-making in multiple problems. For example, the *Snack Chip Problem* is one example of an ends-in-view problem completed by a student described later. It states to students "In this investigation, you will be developing a consumer guide to help people determine which type of snack chip is the best to buy. It is your decision what to focus on in your consumer guide. Your consumer guide must help people in choosing any snack chip, not just the ones you use in this activity." (English, 2001) The students then complete a series of questions asking them to think about different types of variables connected to snack chips. The students often list aspects such as price, nutrition, value, taste, fat content, and calories among other variables. In particular, for the *Snack Chip Problem*, the students develop a tool consumers can use to decide what kinds of snack chips to buy. The students define the important variables and any representations used in the guide to help consumers. The students also collect and sort available data from actual snack chips. The students then use a spreadsheet package (in this case Microsoft® Excel) to develop a tool used to represent the data and help the consumer decide which type of snack chip to buy. When middle school students are given access to a computer with typical word processing and spreadsheet packages, they are capable of producing more complex products than with access only to basic calculators. The tool can be designed within the software packages available on the computer.

In the case of the *Snack Chip Problem*, the end-in-view is the consumer guide for purchasing snack chips, but the students need to interpret the givens and goals of the problem. The relevant variables (or givens) are not specified, and the students need to determine which variables are important and how to formulate the consumer guide (the goal). There are many different kinds of snack chips, and each package has a lot of information on it that could be incorporated into a consumer guide. This situation is similar to real-world problems ranking products from chips to dishwashers to universities. Different rankings incorporate different variables with different weights. The students need to collect data, sort data, and interpret data to meet the end-in-view and define the givens and goals. While processing the data, the students need to keep the end-in-view in mind. Namely, that they need to produce a consumer guide that helps consumers decide which snack chips to buy. However, the exact nature of the consumer guide is not defined.

One important difference between designing a tool and solving a problem is that tool design can incorporate cyclic development. In the case of the consumer guide for snack chips, a first iteration of the tool can be developed, tested, revised, refined, and tested again. Each of these phases represents a cycle of development. After each cycle, the tool becomes more refined and more sharable and reusable. The testing cycles also helps the student clarify underlying assumptions that go into the making of the tool. The end-in-view helps students determine whether the tool needs to be revised and how to revise the tool. Another important distinction is that students should be regularly asking themselves and others whether the tool works rather than when the answer is right.

One subset of ends-in-view problems are model-eliciting activities that require students to produce a complex tool, description, explanation or model describing a method for solving a problem for a particular client. Their final product is often contained in a letter describing their method for solving the client's problem. They need to describe the method with enough clarity that the client can read the letter and implement the method in a similar problem situation. The method should work not only for the particular task at hand, but also be generalizable to other situations the client may encounter.

An important characteristic of model-eliciting activities that is relevant to using technology and ends-in-view problems is self-assessment. (Lesh, Hoover, Hole, Kelly, & Post, 2000) Self-assessment in this case means that students should be able to examine their developing products and determine whether they are satisfying the criteria for a successful product. Students should also be able to determine what revisions are necessary. Spreadsheet software can simplify the testing of products, but students also need to be careful that they are entering data and formulas correctly by monitoring the results carefully. Spreadsheets also remove some of the computational work involved in data processing, but add the ability to incorporate more variables, add types of data and revise procedures quickly. For example, when solving the *Snack Chip Problem*, a student can enter data, and an equation for computing a score for each type of chip. If the student thinks that one variable needs to have more or less weight than the others, coefficients in the equation can be changed, and new scores can be computed quickly. This revision capability allows for a more refined product. If the student had computed scores by hand, he/she may not want to change the coefficients in the equation and recalculate all the scores. With a spreadsheet, the student can test and then revise procedures more easily to improve the design of the tool.

A second important characteristic of model-eliciting activities is that the problem should be contained in a context in which the students can make realistic assumptions. (Lesh et al., 2000) One part of the reality characteristic is that the problem should be real to the students. In this case, the students understand the need to make purchasing decisions. They also understand which variables might be important to examine (e.g. nutrition, taste, price) since they have either purchased or been with someone purchasing snack chips. So, the context of the problem is immediately accessible to the students. Another part of having realistic constraints and assumptions is to understand what tools would be available to someone in the "real world" solving the problem. In the case of the *Snack Chip Problem*, spreadsheet software would probably be available to collect and analyze data. Hence, for students solving the problem, a laptop with Microsoft® Excel gives the students the same tools that would be available to a marketing analyst disseminating information about snack chips. The computer software also allows for greater generalizability since it is easier to modify the tool than with paper and/or a calculator. The spreadsheet also allows for more complex representation of information in charts and graphs that can be used as part of the consumer guide.

The *Snack Chip Problem* is also somewhat unique in the types of variables that students can choose to aggregate. First, part of the task for the students is to determine which variables are important. Second, some of the variables are qualitative characteristics that may be quantified (e.g., taste preference) and some are quantitative (e.g., price, fat content). The students may divide the variables categorically (e.g., high and low price). For some variables, high values are good and for others low variables are desirable. The students may also have to scale the values up or down. Variables may also be grouped by type. For example, fat content, sodium content, and sugar content could be aggregated under a nutrition score. Students also need to consider variables with positive or negative relationship with consumer desires. For example, fat content may be subtracted because it is an undesirable characteristic of snack chips. For an ends-in-view problem, students must consider not only the numerical values of a variable, but the type and quality of the variable. Mathematically, these requirements fit within current National Council of Teachers of Mathematics (NCTM) standards for mathematics (NCTM, 2000).

Related to the variables used in the tool (in this case a consumer guide) is that there is a significant amount of data processing and statistical analysis that must occur to develop an effective tool. The students have to determine which variables are important to collect, how to collect the information, how to sort the information, how to organize the data in a spreadsheet, and then how to process the data to give results to the consumer. These phases are not necessarily completed sequentially. As discussed previously, there are often cycles of development in which the tool is revised, refined, and tested repeatedly. Keeping an end-in-view in mind helps the students decide on revisions and determine whether the tool they are developing is both mathematically and realistically useful and effective. The spreadsheet software simplifies revision and testing because the computational requirements are reduced. However, the students must be cautious in accepting results from the spreadsheet and recognize their role in entering data and designing the spreadsheet.

Student Solution Case Study

My goal for the case study described was to investigate the types of products the students produced given access to laptop computers. In addition, I was also interested exploring in the characteristics of the cycles students went through when developing their products, particularly when using technology. There are many studies and examples of paper-based activities (Doerr & Lesh, in press), but less work related to the incorporation of technology. The student described in this section provides one example of work on ends-in-view problems. The case study represents the beginning of work that can be done in this area, and more extensive research will be necessary to provide more detailed accounts of work in this area and illuminate characteristics of these activities.

The *Snack Chip Problem* was implemented along with a series of eight other model-eliciting tasks during a nine-week Saturday course for gifted fifth through eighth graders. The six students voluntarily enrolled in the course. They completed approximately one model-eliciting task per week during each two-hour class session. They often worked in pairs, but occasionally worked individually. Typically, they worked on the problem for about an hour to an hour and a half, and then made presentations of their solutions to the rest of the class. They began using the laptop computers during the first week of class. Although the classroom situation is fairly atypical, the site served as a forum for testing new types of activities and exploring the possibilities for technology use.

During the first class session, the students completed a survey asking about their motivations for taking the math class, their computer experiences, and their preference for mathematics. The survey served primarily as a tool to help me consider appropriate pairings of the students and understand their interests in taking the course. In a previous semester, I had not given a survey at the beginning of the course and had to rely on individual conversations with the students to determine their interests in mathematics and their computer experience. Knowing their prior experience with computers became especially important because students' comfort and experiences with computers can effect their development of products. Because the activities are designed to be accessible to a variety of students within the class, varied mathematical abilities did not concern me as much in terms of whether the students would be able to develop a product. The nature of the products may vary, but all the students could develop a product with the tools available.

My role in the classroom was as their teacher, but I was also collecting data for this study. To reach the goal of examining the types of products developed by the students, I collected videotapes or audiotapes of students working. I also made extensive field notes immediately after the class sessions ended. In the field notes, I described my observations about their work in pairs. I also noted as much about their cycles of development as possible. Due to my role as teacher in the classroom, it was difficult to make extensive notes during the class while working with the students and taking care of the technology. I also collected their final products to supplement my notes.

I attempted to minimize my intrusion into their work on the activities. I tried not to influence their products with my own opinions as much as possible. The only area where I played a significant role was in helping them work with the laptop computers. I provided any technology help they requested (e.g., formatting documents, finding mathematical functions in Microsoft® Excel, saving documents). For the first week, I chose a less mathematically challenging activity in order to help them adjust to the new requirements for these activities, the class organization, and to the laptop computers. As the weeks progressed, I tried to choose a variety of activities in terms of mathematical content (e.g. statistics, spatial reasoning, algebra) and contexts.

The *Snack Chip Problem* was solved during the last class session. Developing the consumer guides took the students approximately an hour to complete. To help them complete the problem, I provided eight different kinds of snack chips and their prices. The students could collect any data they needed from the packages and taste the chips if they wanted. At the end of class, they presented their guides to the rest of the class. Each pair had access to a laptop computer with Microsoft® Word and Excel. Typically, they used Microsoft® Word to type their final letters to the client. The students also used Microsoft® Excel for some problems if there was statistical analysis or other data processing.

The student in the transcript excerpts below, Alex, is a seventh-grade boy. He was originally assigned a partner for this session, but they started to work individually a few minutes after starting the activity because they had very different ideas about how to design the consumer guide. Alex asked me if he could work independently. During earlier weeks, I had allowed him and other students to work independently if they wanted to. Alex worked well with other students in general and was an active and enthusiastic participant in the class. For example, after a big snowstorm, he was concerned the class would be cancelled on Saturday. Alex had some computer experience and could use Microsoft® Excel and Word with very little assistance from me. In the survey from the first class, Alex said his favorite subject at school was math "because its not boring; not just memorizing things."

Alex used Excel to develop the consumer guide shown in Figure 1 and then typed a letter explaining the guide in Word. For example, Figure 1 shows a spreadsheet he developed to solve the *Snack Chip Problem.* The student designed the tool so that consumers could enter information about two kinds of chips and compare scores. The equation to find the total score found in the far right column was score = price – fat + (10*preference) – (calories/10). I provided the prices from the store where I purchased the chips for the students. Alex used the number of grams of fat per serving for the fat value. Alex tasted each chip and assigned a preference value between one and 10 with 10 being the best score. Alex used the calories per serving for the calorie value. The calorie values are divided by 10 because the values are usually above 100, and all the other variables had values less than 10. Fat and calories were subtracted because they negatively impacted the desirability of the chips (according to Alex and other students in the class). In developing the score equation, Alex assigned relative weights to variables, scaled variables, and developed a generalizable model that could be extended to different types of chips and other products.

Because Alex was working alone, I asked him more questions than usual as he worked on his guide. During the session, he talked to me, to other students, and to himself. At the beginning of his work, I made a few suggestions that were largely ignored because Alex had a clear idea about the product he wanted to produce. At the

Chip	0.88	11	8	160	53.88
	price	fat	prefrence 1-10	calories	
Chip	2.87	10	5	160	26.87
	price	fat	prefrence 1-10	calories	
1	NACHITOS		crunchies	45.78	
2	RITAS		gourmet herb	18	
3	PAPA'S		ritas	56	
4	CRUNCHIES		ridged	26.87	
5	TORTILLA CHIPS		tortilla chips	40	
6	LAWS		nachitos	71	
7	RIDGED		laws	27	
8	GOURMET HERB		papa's	53.88	

Figure 1. Alex's consumer guide for snack chips.

beginning, his guide appeared to allow only for entry of data about one type of chip at a time as only the first two rows in Figure 1 were on the spreadsheet. Without any prompting from me, Alex said, "I should put two in so you can compare two of 'em at the same time." An effective guide should show more than one type of chip, and he wanted to be able to see scores for two types of chips at once. I suggested he make a score column and reformat the spreadsheet so a consumer could enter more types of chip. He said, "But they're not like doing 'em all at one time usually because it's a guide since it said guide I think." Later, when he collected data from all the eight types of chips I had brought to class, he added the eight rows on the bottom listing the rank ordering of the chips and the scores.

The second type of interaction I had with Alex was as a tester of his consumer guide. Other students worked with partners they could use to test ideas. After developing his first draft of the consumer guide he said to me, "Ok give me two kinds of chips. Ok want to try it now? Want to try it to see if works?" I sat at the computer, tasted the Crunchies chips, and entered data into his spreadsheet. After we got the score (10.78), I asked him what the score indicated. We had the discussion shown in Figure 2.

Asking a second person test the guide with a new type of snack chip helped Alex find flaws in the first version of his guide. After the discussion, he noticed the coefficient he was using for the taste preferences was too low (7) and he needed to raise it to 10. Testing the tool caused a revision of the tool and a refinement of his algorithm for scoring the snack chips. At this point, his intent for the scores once they had been produced was not clear. However, my question and testing did aid a revision to the tool. After he changed the coefficients in the equation, I tested the tool again. This excerpt

A: 10.78
M: So what does that mean?
A: I need to change my times 7 back to a times 10.
M: Ok. So when you get that score up here, then what do you do?
A: What?
M: What do you do once you have the score?
A: Go to a different one. Because this is like to decide if you want different chips compared to other ones.

Figure 2. Transcript excerpt for chip scores.

is also an example of self-assessment since he felt that 10.78 was not a reasonable score and needed to be larger. In ends-in-view problems such as this one, it is notable that he placed more emphasis on whether the guide worked rather than whether it was "right.".This distinction can be made between designing a tool (the consumer guide) and finding an answer to a problem.

During later testing of the tool, he noticed that the Gourmet Herb chips that none of the students liked, including Alex, had a higher score than the Crunchies Chips. He had been computing scores for the chips by collecting data from the packages and tasting the chips to give a preference score. He was recording total scores for each type of chip at the bottom of the spreadsheet. In the excerpt in Figure 3, the Gourmet Herb chips received a score of 18.

He seemed perplexed that the Crunchies had received a lower score than the Gourmet Herb chips. He then noticed that he had used my taste preference score for the Crunchies and his taste preference score for all of the other types of chips. Before this point, he had not told me that the guide was designed for an individual. Alex seemed to assume that "consumer guide" meant individual consumer guide. Working with a partner may have helped him to develop a guide that would incorporate data from more than one person at a time. However, at this point, it also is not an unreasonable assumption that an individual may generate a guide and score chips. The excerpt also illustrates how during model-eliciting activities students need to grapple with assumptions related to the context and clarify those assumptions for users of their products. Real-world consumer guides typically include explanations of the variables and methods used for generating rankings of products in order to increase their validity and reliability to consumers.

A: They got an 18. They beat the Crunchies.
M: They beat the Crunchies.
A: But this is your preference. I liked the Crunchies.
M: I gave the Crunchies a 4.
A: What?
M: Oh. What happens then? Like if you have a different preference than I do.
A: Well it's supposed to be all from one person's preference.
M: Oh. So it's an individual consumer guide?

Figure 3. Transcript excerpt for preferences.

Other consumer guides could use iterative processes. For example, a guide may use pairwise comparison of chips in order to find the "best" snack chip. Two types are compared and the "winner" moves on to the next pair to be compared. In addition, stochastic processes may also be used to break ties between types of chips. Chips may also be ranked by each variable and then the rankings by variable may be aggregated. Rankings by individual variable illustrate how data may be re-represented and then combined. In addition, students may use charts and graphs in order to illustrate characteristics or scores for the chips. Again, alternate representations are typical of real-world consumer guides generated for products from dishwashers to universities.

The *Snack Chip Problem* also asks for a shareable, reusable consumer guide. An important criterion for a successful guide was that it should be useful for judging not only the snack chips available in the classroom, but also other types of snack chips. The consumer guide should also be usable by many different people buying snack chips. The use of Excel to develop the consumer guide increases the shareability and reusablity of the tool. It was also easy for the student (e.g., Alex) to modify or generalize the consumer guide. As an extension problem, the consumer guide that the students develop for the *Snack Chip Problem* could be used for either other types of consumer guides (e.g., cars, furniture) or the idea of a guide for comparing different objects could be extended to other statistical applications involving data analysis (e.g. giving grades, generating fair teams). Once students have developed a powerful tool for solving one problem, they should have a tool that will help solve other problems even in other contexts.

Areas for Further Research

This paper has explored one instance of tool design in a model-eliciting activity by a seventh-grade student using a spreadsheet to design a consumer guide. Further investigation needs to be done using this type of tool design problem in order to understand the nature of tool development more fully and the mathematics related to tool design. In addition, the student in this example was solving the problem individually. Model-eliciting activities are usually solved in small groups so further investigation could be done related to tool design by a small group. It may also be important to identify the types of skills (e.g., collaboration, new mathematical skills) that become important when solving ends-in-view problems and model-eliciting activities.

In terms of implications for teaching, there are still questions related to assessment and evaluation that would need to be answered. In particular, how can these types of problem be incorporated into the curriculum and how can the products generated by the students be used to document mathematical understanding? This issue is particularly complex given that this type of problem caters a range of mathematical achievement levels that go beyond traditional mathematical understanding. Namely, the development processes required for designing a mathematical tool go beyond finding the right procedure for solving a problem. The processes include the testing, revising, and refinement processes described above as well as the incorporation of multiple mathematical skills. In addition, once a teacher has identified certain mathematical skills, the question then arises about how to decide what to do next. In the case of the problem described here, a teacher could continue with the exploration of other statistical concepts or for other methods of data representation. Such instructional decisions may depend on the tools developed by the students as well new concepts the teacher may want to introduce. (For more about sequences of activities and other modeling related problems see Doerr and Lesh (in press).) Classroom discussion of the products may vary depending on the teacher's instructional needs as well.

Conclusion

Tool design is an area that has not been explored extensively in the middle-school mathematics curriculum. However, it is an area that allows students to use many different types of mathematical skills and to use a variety of different types of abilities. In addition, the students need to understand when and how different skills are needed and analyze work with multiple representations of data (e.g., from the chip bag to the spreadsheet to a chart). Tool design is also an important aspect of many mathematically intensive careers such as business, engineering, and science. Model-eliciting activities and tool design are also areas that incorporate the use of technology in ways that go beyond pure computation. The technology facilitates revision and testing of the tools and adds to the realistic constraints of the problems. Knowing when and how to revise a tool is a skill important in general problem solving and the design of complex products.

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CAUSAL REASONING IN THE PROCESS OF CONCEPT FORMATION: THE CASE OF LINEAR INDEPENDENCE AND LINEAR DEPENDENCE OF A SET OF VECTORS

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This study, still ongoing, is a part of a research project (Hristovitch, 2001) which investigates the role of metaphors, analogies and symbolization in students' conception of basic linear algebra notions in the case of linear dependence and linear independence of sets of vectors. The results of that project indicated that along with analogies and metaphors reflecting our everyday life experiences, students bring into mathematics the causal aspects of events and processes taking place in the physical world. In this paper I will focus on students' causal reasoning in the process of concept formation and will argue, based on the evidence from the data, that students' causal interpretations of structural (a-causal) relations in mathematics might become an impediment in learning at a higher level of mathematics such as linear algebra. Further, I will try to initiate a discussion on the role of causal reasoning in the process of concept formation with the aim of making initial steps toward developing a theoretical perspective of causal reasoning in mathematics.

Theoretical Perspective

Using Sfard's (1989, 1991, 1992) perspective on concept formation as a frame of reference, this study explores how students come to construct the notions of linear dependence and linear independence and what the obstacles to students' understanding of these concepts might be. Sfard claims that the process of concept formation begins with operational understanding, viewing a concept as a process, algorithm or action; the structural understanding gradually evolves from the operational by viewing a concept as a static construct or as if it is an abstract object. Further, she argues that the shift from an operational to a structural conception is an inherently difficult process for students in any area of mathematics. The problem is especially acute and worth consideration in the case of linear algebra due to the highly abstract nature of this field; even the learning of its very introductory notions requires a rather structural conceptualization and there is not much room for an operational conception.

It is widely accepted that metaphors, analogies and symbolization are central in the process of constructing mathematical knowledge. Metaphors assist us in understanding abstract formally represented mathematical ideas in terms of familiar, everyday experiences (Lakoff & Núñez, 1997; Presmeg, 1997; Sfard, 1994, 1997). Analogies allow us to understand the structure of one conceptual domain in terms of another (Gentner, 1989; Gentner & Holyoak, 1997; Holyoak & Thagard, 1989, 1997), thus help us in constructing models for theory development and problem solving. The inherently representational character of mathematics (Kaput, 1987a, 1987b) makes it impossible to consider the process of constructing mathematical ideas without looking deeply into the role of reasoning with symbols in that process. However, the project indicated that aspects of students' spontaneous analogies and semantic interpretations of metaphors and symbols might become an impediment in the shift from an operational to a structural conception.

Drawing upon these perspectives, this study explores the role of causal reasoning in the process of concept formation in mathematics. More specifically, the investigation is guided by the following questions: What is the role of causal reasoning in the shift from operational to structural conception? What is the interaction between causal reasoning and other aspects of the process of conceptualization such as metaphors, analogies, symbolization, and students' ability to justify (or explain) their solutions?

Methodology

This study employs a qualitative research design that integrates observation, interviewing and written samples. The data was collected in a large Midwestern state university where an Introductory Linear Algebra course for mathematics and science majors was offered every semester in two sections taught by two different instructors. Regular class observations were made starting at the beginning of the semester until the concept of interest was completely covered and connected to the material that followed; the lectures were audiotaped and field notes were taken during every lesson. In the course under observation, the students were presented with two definitions of linear independence and linear dependence: the first, rather informal (operational) definition was introduced at the beginning of the semester; the second more rigorous (structural) definition was introduced later in the semester, approximately one month after the first definition. The following are the definitions presented to the students:

First definitions: *A set of vectors is called a linearly dependent set of vectors if at least one of the vectors in the set that can be expressed as a linear combination of the other vectors in the set.*

A set of vectors is called linearly independent if none of the vectors can be expressed as a linear combination of the others in the set.

Second definitions (Test for Linear Independence): *A set of vectors* $\{v_1, v_2, \dots, v_n\}$ *is called a linearly dependent set if there are scalars* $c_1, c_2, \dots, c_n$, *not all zero, such that* $c_1v_1 + c_2v_2 + \dots + c_nv_n = O$.

A set of vectors $\{v_1, v_2, \dots, v_n\}$ *is called a linearly independent set if there are scalars* $c_1, c_2, \dots, c_n$ *such that whenever* $c_1v_1 + c_2v_2 + \dots + c_nv_n = O$ *follows that* $c_1 = c_2 = \dots = c_n = 0$.

Two quizzes were collected from all students – one, after the introduction of the operational definition, and the second, after the structural definition. In addition, two interviews were administered following each of the quizzes mentioned above with a subset of the participants in the study.

The participants of the study were 60 undergraduate students enrolled in this course as part of their major requirements. The number of participants (27 in section 1 and 33 in section 2) and their gender and ethnicity were determined entirely by the current enrollment in the classes. The students enrolled in these sections were asked to participate voluntarily in two interviews and only students who signed the consent form (giving permission for participation) were included in the study – 7 students from section 1 and 5 students from section 2, for a total of 12 students.

For the analysis of the quizzes and the interviews, this study employed grounded theory approach (Strauss, 1987) for coding, categorizing and possible theory saturation.

Findings

The analysis of the data shows that students exhibit elements of causal reasoning mainly in their responses on the first quizwhich was given to them after the introduction of the first rather informal definition of linear dependence and linear independence. For this reason, I will present more elaborately the results of this quiz and, later on, at the end of this section I will make some overall remarks on the results from the second quiz and the interviews. The following are the questions on Quiz 1:

Q1: What does the following statement mean to you? The set of vectors {A, B, C, D, E} is linearly independent.

Q2: Let {A, B, C, D, E} be a linearly independent set of vectors. If we remove two vectors from the set, say A and B, would the set of remaining vectors, {C, D, E}, be linearly dependent or linearly independent? (Justify your answer)

Q3: Let {A, B, C, D, E} be a linearly independent set of vectors. If we add a new vector F to the given set, would the new set of vectors, {A, B, C, D, E, F}, be linearly dependent or linearly independent? (Justify your answer)

Q4: What does the following statement mean to you? The set of vectors {M, N, K, R, V} is linearly dependent.

Q5: Let {M, N, K, R, V} be a linearly dependent set of vectors. If we remove two vectors from the set, say M and N, would the set of remaining vectors, {K, R, V}, be linearly dependent or linearly independent? (Justify your answer)

Q6: Let {M, N, K, R, V} be a linearly dependent set of vectors. If we add a new vector S to the given set, would the new set of vectors, {M, N, K, R, V, S}, be linearly dependent or linearly independent? (Justify your answer)

The responses to these questions indicate that students employ causal reasoning in various situations: when interpreting metaphors, when interpreting the symbolic representations of equations, and when gaining understanding of the novel situations in analogy to prior concepts and experiences.

1. In the definition of linear dependence and linear independence, the phrases "written in terms of" or "expressed as" are used in their literal meaning, signifying that a linear combination is just another representation of an already existing object (vector, matrix, or function). In contrast, the responses in the first quiz show that some of the students (7 students) made sense of the phrase "vector expressed as a linear combination" in terms of their everyday life experience and transformed it into rather metaphorical expressions as "formed by," "made up," "comprised of," "produced," or "combined to create." Thus, the syntax of the symbolic representation of vectors expressed as a linear combination was captured by metaphors which carry operational and causal character as well. For example:

(Student C13)

Response Q1: They cannot be comprised of each other.

Response Q5: Dependent. They are still dependent because they would not be affected by removing M and N from the set.

Response Q6: Dependent. A vector in the set can still be comprised of a combination of the other vectors in the set.

(Student B29)

Response Q1: One of these or multiple of it cannot be added to a multiple of another vector in the set to produce a third vector in the set. Or a multiple of one vector can equal another.

Response Q3: This depends on what vector F is, if F completes a linear combination with one of the existing vectors, the set is linearly dependent. Otherwise, the set remains linearly independent.

The students viewed "vector expressed as a linear combination" as a process in which a vector is "made up" out of the other vectors in the set as if they are its "components." In these examples we see how a static structure is viewed as a causal-like process of building up or constructing and, depending on the outcome of these constructions, a set of vectors might "become" dependent or independent. We should notice that the second student (B29) used the term "produced" rather metaphorically and this production-like view of "vector expressed as a linear combination" did not affect negatively his/her inferences. On the other hand, this same view ("comprised") might have led the first student (C13) to wrong conclusions in question 5.

Further, in the context of the definition, the words "independent" and "dependent" are used metaphorically, indicating only the relational character of the concepts of linear independence and linear dependence. When interpreting the definition of linear independence, some students (4 students) deepened the perceptual roots of the metaphors "dependent" and "independent" and transferred into the area of mathematics a sense of causality – "vector is affected by," "the vectors depend for their values" or "you can't do anything with one without knowing the others." The following examples show students' semantic interpretation of the "dependent" metaphor.

(Student C10)

Response Q4: The vectors rely on each other to exist.

(Student C5)

Response Q4: That means that all of these vectors depend on the others for their values. $c_1M + c_2N + c_3K + c_4R = V$ or any combination of terms. You cannot find one without having information about another.

(Student C19)

Response Q4: M, N, K, R, V depend on each other. You can't do anything with one vector without knowing the others.

(Student C18)

Response Q4: Means, each vector is dependent on the others. By changing one will affect the rest.

The students deepened the perceptual roots of the metaphors "dependent" and "independent," and transformed them into "rely on each other to exist," "depend on the others for their values" or "you can't do anything with one vector without knowing the others." The intention of these metaphors in the context of mathematics is to bring to the fore the implicit structural relations (the ground) in a set of vectors. The students though, influenced by their everyday life understanding of the words "dependent" and "independent" brought up an unintended meaning of the idea of relation carrying the flavor of causality. And again we observe, just as in the case of "written in terms of," the operational, process-like, causal character of students' understanding in the initial phase of conceptualization of the notion of linear dependence and linear independence.

2. The responses on this first quiz also show that, for some students (12 students), the understanding of linear independence and linear dependence develops in analogy to their prior experiences with sets and everyday life settings that resemble the novel situation they are faced with. In analogy to their experiences with material sets, some students reduced linear independence and linear dependence from a relational property of a set to an attributive property of individual vectors, viewing individual "dependent" and "independent" vectors as causal *carriers* or *agents* of the set property. Respectively, in some instances, a vector that can be expressed as a linear combination of other vectors was called "dependent" and a vector that cannot is called "independent." As a result, for 5 of these 12 students a set of vectors was expected to be dependent if each vector in the set is "dependent," and consequently wrong infer-

ences were made. In this way, a property of a set was reduced to a property belonging to each individual element of the set, and instead of useful analogy we were witnessing misanalogy with consequent wrong inferences. For instance:

(Student C10)

Response Q1: None of the vectors rely on the others to exist.

Response Q2: They would be linearly independent. They were independent before the others were removed, and the removed ones are independent.

Response Q3: We do not know because we were not told whether F was dependent of any of the other vectors.

Response Q4: The vectors rely on each other to exist.

Response Q5: Linearly dependent because they were to start with and the ones removed were also dependent.

Response Q6: Not sure, S might be independent the others still dependent.

In this example, linear dependence was interpreted as "vectors rely on each other to exist," which suggests that the student, although misinterpreting the definition, did not ignore completely the relational character of the concept. Nevertheless, from the responses on the other questions, it is evident that, in analogy to sets of concrete objects, the student considered *independence* or *dependence* of set as a property of individual vectors ("the removed ones are independent"; "S might be independent") and expected a set to be dependent or independent if each vector in the set is respectively "independent" or "dependent." From the response on Question 5, it is evident that the student viewed each vector in the initial set as "dependent" and respectively, without any considerations for possible linear combinations among K, R and V, the set of remaining vectors was qualified as "dependent" because "they were to start with." Further, we should notice that in Question 6 the student referred to vector S as "might be independent" without reference to the set to which it was supposed to be added. Respectively, because the others are "still dependent," the new vector was never actually joined to the initial set and the new set was not considered. Thus, we can infer that, in analogy to material sets the students considered individual vectors as carriers or agents of the property of a set and developed the following model: A set of vectors is independent if none of the vectors in the set is "dependent" on the others; A set of vectors is dependent if each vector in the set is "dependent" on the others. The fact that the first statement is true but the second false further complicated students' difficulties since for independent sets, the analogy to the material sets worked; for dependent sets, however, the analogy did not.

Further, 4 of the 12 students were realizing that for a set to be dependent not all vectors should be necessarily expressed as a linear combination, and that it is enough if "at least 2 vectors from the set {M, N, K, R, V} are linearly dependent." In other words, in accordance with the definition of linear dependence, the students modified their everyday life experience to suit the novel situation they were faced with. The new model allowed them to give a correct answer to all of the questions, and the analogy to sets of concrete objects was indicated only in the ways they phrased their responses, specifically in the use of the terms "independent" and "dependent" in reference to individual vectors. The arguments were abundant with phrases like "if all five were independent" or "at least one is dependent." Such phrases, and especially the often used expression "if M and/or N were the only vectors dependent in the set," are indications for a mapping between material sets and sets of vectors – the notion "vector expressed as a linear combination" is viewed as if it is a property of individual vectors. Because of the alteration, made in accordance with the definition of linear dependence (shifting from "each" to "at least"), the model developed by the students worked despite the focus on individual vectors. The student obtained correct answers on question 5 after removing the "only dependent" vectors, making possible that the "remaining 3 would be independent," and respectively, if one of the remaining is dependent on the others the "remaining three will be still dependent."

For one of the students, however, despite the modification in the definition of a dependent set (see Response Q4 in the example below), the analogy to sets of concrete objects led him/her to an erroneous response on question 5.

(Student C13)

Response Q1: Each vector A, B, C, D, and E belongs to its own vector space. They cannot be comprised of each other.

Response Q2: Independent. If {C, D, E} are independent with the set of {A, B, C, D, E} then by removing a, and b it will not affect their independence.

Response Q3: That depends on whether or not F is a vector dependent on any of the others in the set. If F is dependent then the set would be dependent, and vice versa.

Response Q4: A vector in the set {M, N, K, R, V} can be made up of a combination of the other vectors in the set.

Response Q5: Dependent. They are still dependent because they would not be affected by removing M, and N from the set.

Response Q6: Dependent. A vector in the set can still be comprised of a combination of the other vectors in the set.

Although locally (only in question 5), the student fell back to the understanding of the students in the first group (shifting back from "a vector" to "each"). Respectively, in his/her response there are no considerations of the possibility that the vectors K, R, and V might be unrelated in the new set – the vectors in the new set are "still dependent because they would not be affected by removing M, and N from the set."

Finally, some students (3 out of the 12) used slightly modified version of the analogy to material sets – they did not transform "vector expressed as a linear combination" into a property of an individual vector. Instead, they reduced the participation in a linear combination into a permanent (attributive) property of individual vectors even when a vector participates in a combination with a zero coefficient. Thus, those students considered the participation of a vector in a linear combination as the *carrier* or the *agent* of the dependence of a set. For example:

(Student B9)

Response Q1: Any given vector in the set of {A, B, C, D, E} cannot be written in terms of the other 4 vectors. There would be no way to manipulate them such that this would become true, even using 0's.

Response Q2: Linearly independent. If all 5 original vectors could be written in terms of each other, than any given group of the 5 could also. Since {A, B, C, D, E} is independent, so is any subset.

Response Q3: Linearly independent. If the set of {A, B, C, D, E, F} were linearly dependent, then that would force {A, B, C, D, E} to be linearly dependent.

Response Q4: Any given vector in the set can be written in terms of the other 4, using 0's if necessary, but two have to have non-zeros.

Response Q5: Dependent. If two vectors are removed, one may manipulate the equations so that the other three can be written in terms of each other.

Response Q6: Dependent. Even if S is unrelated, a 0 constant may be multiplied to it.

The above example illustrates this permanence of the participation in a linear combination. In the response on question 3 the student employed indirect reasoning erroneously to show that assuming the new set to be dependent will lead to the contradiction that the initial set is also dependent, completely ignoring the possibility of having a relation between the vectors in the new set. In other words, the ability to participate in a linear combination was considered to be a property of the individual vectors despite the changes in the set. Similarly, the response on question 3 suggests that in the student's view, even if two vectors were to be removed from a dependent set, "one can manipulate the equations so that the other three can be written in terms of each other." Thus, for the students, the participation in a linear combination is considered to be a permanent property of the individual elements of a set. Just as with the members of a team, even when some of them do not play at a given moment ("having a zero coefficient"), they are still members of the team.

3. When dealing with the problems in this quiz, students (38 students) focus their attention on the added and the removed vectors as a cause for the outcome without considering the new set as a whole – "if the removed vectors are dependent", "if the new vector is dependent itself" – as a result, wrong inferences were made.

(Student C18)

Response Q5: The remaining set will still be dependent. K, R, V were some how related to M, N. Therefore, removing M, N will affect the value of K, R, V.

(Student C22)

Response Q3: Not sure, you would have to check for the dependence or independence of F compared to the other vectors.

Response Q5: Could be either. By removing M & N you might be changing the dependency of the vectors to independent if M & N are the same or a multiple of {K, R, V}.

(Student B17)

Response Q3: It could be linearly independent or linearly dependent because it is possible for some linear combination of {A, B, C, D, E} to be equal to F. However, it is also possible for F not to be dependent of {A, B, C, D, E} as well, making it independent.

Response Q5: I could be either independent or dependent, depending on whether or not M & N are needed to form {K, R, V}.

As evident from the above examples, students attempt to achieve coherence of their explanations by trying to identify the reason (or the cause) for the status of the new set – in their justifications they focus on how the addition or the removal of vectors and respectively the properties of these vectors might lead to a "change" in the dependence or independence of the new set. This attention to the added or removed vectors as a cause for "maintaining" or "changing" the status of a set further indicates the operational character of students' understandings of linear dependence and linear independence at this stage.

Finally, I would like to make some remarks about the findings from the interviews and the second quiz. The responses on the questions in the first interview support the findings from the first quiz that some students (3 out of 12 interviewed) intuitively understand linear dependence in analogy to sets of concrete objects; and without paying much attention to the precise formulation of the definition, they transform "at least one can be expressed" into "vectors are dependent on each other" or "all of them are dependent on," thus, viewing the "dependent" vectors as causal carriers of the property of a set. The data from the second quiz and the following second interview does not provide much evidence for causal reasoning in the students' considerations. This should not lead us necessarily to the conclusion that the students have discarded causal inferences after the introduction of the structural definition, neither that they have shifted to structural understanding of linear dependence and independence, but rather that the structural character of the questions asked did not provide much room for employing everyday life intuitions. In the process of learning, the initial intuitions of some students gradually evolve towards a more structural understanding of linear independence. But I should point out that on the continuum between operational and structural understanding, only 10 students consistently exhibited structural understanding across all the problems given in the second quiz. Most of the students (40 students) remained with a rather operational conception, applying the structural definition occasionally and often not successfully. Some of them (12 out of 40) developed a procedural understanding of linear dependence and independence. They transformed the second definition into an algorithm of finding the raw-reduced-echelon form of a matrix, but were unable to apply the test in more abstract and complex situations. Finally, I should mention that 10 students did not grasp the notion of linear dependence and experienced difficulties in applying both definitions.

Discussion and Conclusions

As pointed out earlier in this paper, metaphors and analogies are vital for the process of concept formation – they assist us in grasping abstract mathematical ideas in terms of familiar everyday life experiences and allow us to gain understanding of the structure of one domain in terms of another. But it is evident from the results of this study that the use of metaphors and analogies in the area of mathematics is not unproblematic. Most mathematical definitions contain metaphors which have to be interpreted by students in order for the concept behind the definition to be understood. In the definition of linear dependence and linear independence, the metaphorical expression "dependent" when applied to a set, although having everyday life perceptual roots, intends to explicate the notion of "relation" (that there is a relation among the vectors in the set). When interpreting the definitions, specifically the terms "dependent" and "independent," students, influenced by their everyday life experience, deepen the perceptual roots of these metaphors and transfer into the area of mathematics a sense of causality: "vector is affected by," "the vectors depend for their values." Further, the focus on "vector expressed as a linear combination" in the operational definitions might have fostered students' everyday life intuitions to view such vectors as the "carriers" of the dependence of a set. As evident from the results on the analogy to sets of concrete objects, these intuitions led some students to view individual vectors as "dependent" or "independent," and others students, to transform vectors' participation in a linear combination into a permanent property of the vectors; thus encouraging the view that a set is dependent if each vector in the set is "dependent" or if each vector participates in a linear combination. As a result of these interpretations students made inappropriate inferences in several occasions. Thus, students' interpretation of a metaphor depends greatly upon students' understanding of the established 'norm' of language usage in the area of mathematics. Distinguishing between what is literal and what is metaphorical in the mathematical language amounts to understanding the nature of mathematics which leads us to the question: Is mathematics susceptible to causal explanations and causal agency? In the literature on philosophy of science and mathematics there are many rivaling theories about the existence and the cognizing of abstract objects. But despite that there is no leading epistemology that gives coherent and persuasive account of such objects, there is still some consensus on the view that mathematical objects are abstract and causally inert (Shapiro, 1997; Bunge, 1959) and that explanations in mathematics are outside the *causal nexus* (Kitcher, 1989). Thus, in the area of mathematics there are no "agents" or "carriers" of special properties, so whenever in mathematics we talk with language holding such meanings, it is only metaphorically.

But the ambiguities in interpreting metaphorical expressions cannot account for all the difficulties that students experience in the process of conceptualization. Students' analogy to sets of concrete objects has deep roots in their everyday understandings where the constitution and the properties of a set depend primarily on the common attributes of its individual elements. In a study of learning group theory, Dubinsky (1994) reports a similar phenomenon: students, ignoring the operation involved, "interpret a group primarily in terms of its elements, that is, as a set" (p. 273). In his view, it appears possible that students are assimilating the problem situation to an existing set schema.

In this study, in the case of linear dependence, students do not ignore the relations among the vectors in the set, but transform them as if they are an attributive property of each individual vector, thus the "dependent" vectors are viewed as the "carriers" or the "agents" of the dependency of the set. Students' difficulties in dealing with relations among the elements of a set might also be due to difficulties in shifting between distinct ontological categories (Chi, Slotta, & Leeuw, 1994). In their theory of conceptual change, Chi et al. (1994) claim that learning is difficult if students have to shift from initial understanding of a concept as belonging to the ontological category of "Matter" (sand, paint, human being) to an understanding of the concept as belonging to the category of "Processes." The shift could be especially difficult when the new understanding belongs to the subcategory, "Constraint-based-interactions," which are processes with no material causal agents (such as magnetic field and gravitational field). This perspective partly explains students' focus on individual vectors in the analogy to sets of concrete objects and their expectation that each vector in a dependent set is a "carrier" of the dependency, either by being "dependent" or permanently caring a participation in a linear combination. Students' early understanding of sets (or of collection of objects) originates in their everyday experiences with sets of concrete objects where the constitution of a set is usually determined by attributive properties carried by each material object in the set. Thus, for some students the transition to understanding a property of a set based on algebraic relations among the elements of the set, i.e., with no material carriers, could be very difficult. In fact, such difficulties can be experienced even by experts: for example, for years physicists were searching for a magneton, a material carrier of the magnetic field, or a graviton, a material carrier of the gravitation field.

Thus, in the process of learning new mathematical ideas, students attempt to achieve coherence and consistency in their understanding by grounding these new ideas in prior mathematical and everyday life experiences. But, as evident from the above findings, these prior concepts and ideas, although enhancing students' understanding at a certain level, they can hinder the shift from operational to structural conception and become an impediment in learning mathematics at a higher level.

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FRACTAL FILAMENTS: A SIMILE FOR OBSERVING COLLECTIVE MATHEMATICAL UNDERSTANDING

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This paper provides one response to the questions of the nature of and the portrayal of collective mathematical understanding, particularly as it might be observed in a classroom context. The view here considers collective understanding as an embodied, dynamical phenomenon portrayed using the simile of fractal filaments. Using the responses to a prompt by students in a third grade classroom (8 year olds), such a fractal and entailed fractal filaments are observed to be about patterns drawn from the actions and inter-actions of students as they engage in mathematical thinking. It is shown how self-similarity across scale and recursion, two key features of fractals, figure in the observed dynamical collective understanding. Some characteristics of this understanding include common constructive mechanisms and mathematical interpretation. As well as complementing recent work on dynamical personal understanding and on social interactions and mathematical knowing, the paper points to the usefulness of this view for teachers interested in observing and acting upon the dynamical collective understanding that emerges in their mathematics classes.

There has been considerable recent research on the relationships between various classroom interactions among the members of a mathematics class and the knowing acts of the individuals (Cobb, Yackel et al. over the past 10 years (e.g., Cobb, Boufi, McClain, & Whitenack, 1997); Goos and Galbraith et al over the last 5 years (e.g., Goos, Galbraith, Renshaw, & Geiger, 2000)). This paper discusses work and ideas that complement such approaches. It relates to a question raised at PME-NA by Martin (2001) of the possibilities for and the nature of collective mathematical understanding. Over the past 15 years Pirie and Kieren have been developing and using a theory in which mathematical understanding and its growth are understood as dynamical personal phenomena (e.g., Pirie & Kieren, 1994). Such Dynamical Personal Understanding (DPU) is observed to occur in action and inter-action. However, it is not simply about such action and inter-action per se but is about patterns in that action which are observed as varying modes of less formal and local or more formal and abstract understanding. If understanding is seen as a dynamical personal process then the question of collective understanding is problematic. What is it? How does it arise? How is it related to individual acts and social interaction? How might it be observed and or used in a classroom by a teacher or researcher? It will be the purpose of the proposed paper to explore the concept of collective mathematical understanding by observing it using the simile of a "fractal filament."

Theoretical and Observational Frames

The observational ideas in this paper related to Dynamical Collective Understanding (DCU) are inspired by a number of recent ideas about knowing, such as emergence, coemergence and dynamical systems. The particular observational simile, fractal filament and related consequences, will be posed within what Maturana would typify as objectivity-in-parentheses; that is, this observational frame and related ideas are part of a multiversal rather than a universal perspective (Maturana, 1988; Bohm, 2000).

We turn first to the idea of collective intelligence through emergence. In a recent book, S. Johnson (2001) reviews recent research showing how observed collective properties can be characterized as emergent from individual acts. For example, efficient ant colony food retrieval can be seen to emerge from repeated individual acts of ants based on 3 elements: ants' capability to make certain bio-chemical distinctions; trace pheromones of ant activity; and opportunities for ants to bump into other ants or their traces. Perhaps collective understanding can be observed as such an emergent, or we suggest, coemerging phenomenon.

If this is so, then collective mathematical understanding, taken as a dynamical and plastic phenomenon, will be observed to occur in the mathematical actions and particularly in the inter-actions of persons. This stance would be congruent with considering DCU as an emergent property. But DCU is not simply about actions with associated dynamical personal understandings or simply reflected in the nets of interactions among such persons in action. Like Kaufmann's (1995) idea that life is observed in patterns drawn out of the autocatalytic nets of molecules, DCU is observed in the patterns in the actions and inter-actions among knowers. Thus, DCU is observed in, and in fact is embodied in, inter-actions such as co-occasioning and conversation; DCU is about patterns in these and not properties of them per se. DCU is of a different implicate order (Bohm, 2000) from them, related to the whole collective and its total embodied setting rather than referenced to individual knowing per se.

However, as suggested, DCU can be observed to occur in settings where individuals are acting and inter-acting. Based on enactive views of cognition derived from the work of Maturana and Varela and some of our previous related research, we posit that one can *observe collective understanding* as a dynamical process emerging, in Johnson's (1987) sense above, from individual knowing acts based on three conditions: the individuals' capabilities for making certain distinctions which the observer sees as mathematical; observable re-presentations of such actions; inter-action among such individuals and with the environment, which includes the culture of the communities in which the persons in interaction are observed to exist (e.g., the culture of the particular classroom, the broader culture of historical and contemporary mathematics). Mathematical knowing and the related phenomena of personal and collective understanding are taken to be a 'bringing forth of a world of significance including mathematics' (Maturana and Varela, 1992). Such knowing, as well as DPU and DCU, coemerge in interaction with the environment including others such as other students, teachers and materials. Our understanding of this is illustrated by the model in Figure 1 (Simmt, 2000; Simmt & Kieren, 1999).

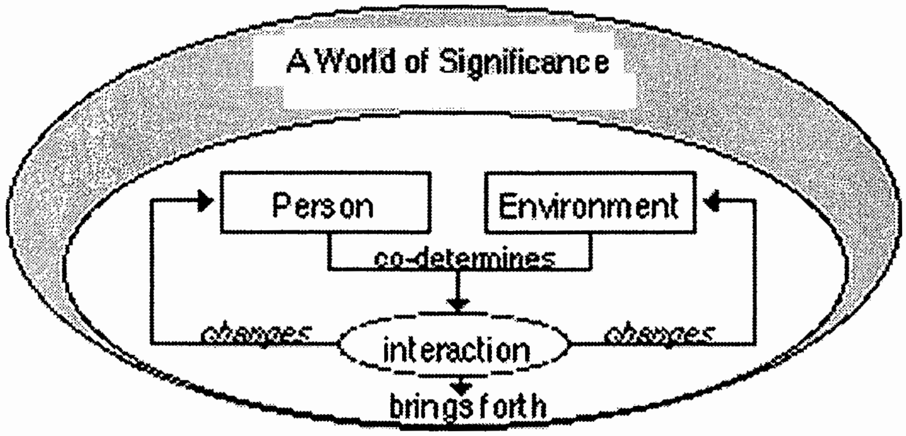

Figure 1. Knowing in interaction.

This diagram allows us to observe both the individual determinants and the social/environmental co-determinants of knowing all-at-once. Such knowing and the related understandings are observed then as coemerging phenomena. Following the various paths in this model suggests one can observe knowing from a constructivist or a social constructivist view depending how the influence of the environment on the individual is taken. Another feature of the model is important for the thinking in this paper. The knowing actions and representations of the individual can be observed as occasions for the understanding acts of others (for example, other students, the teacher or the researcher). These actions and representations also can occasion changes in the environment (for example, the ways in which the tasks or materials or mathematics is interpreted). It is in this way that the cognitive domain can be observed as enhanced through interactions (Maturana, 1991; Simmt & Kieren, 1999). As well, it is in this way that the DCU can be observed to emerge from and coemerge with individual understandings and interactions among the persons enacting them. It may well be that DCU can be observed as a feature of this plastic cognitive domain. As such it can be observed to occasion DPU as well as be seen as arising in a space of interacting individuals.

If we observe dynamical collective understanding in a setting what do we 'see'? Our suggestion is that if we take the model above and use it to interpret mathematics knowing in action then we note that collective understanding arising as above is like a 'fractal filament'. Assume that a fractal filament is a subset of a fractal object. The fractal may be thought of as a collective of connected nodes. Like the pieces of art which Peitgen (see Peitgen & Richter, 1986) derives from the Mandelbrot set, each node has its own character; yet each node is in some way similar to the others and, the entire Mandelbrot set demonstrates self-similarity across scale. While each filament of a fractal has its own character and look, yet each has critical features of the whole and can be seen as arising through a similar dynamical process. We are observing collective understanding in this way.

Why might we do this? We are viewing collective understanding as a dynamical, changing phenomenon. The model of knowing we are using observes such knowing both for the individual and the community in which it occurs as a recursive phenomenon. Fractals have been used in many places to characterize such dynamical recursive phenomena. Thus we are using it as a simile.

An Illustrative Sketch from Research

The following sketch is taken from the observation of children in an 'average' 3rd grade classroom of 8 year olds in a western Canadian city. It is from one site of an ongoing interpretive research programme being carried out using varied data sources from multiple sites. A feature of this research program is that data are collected and interpreted by several researchers and reinterpreted in an ongoing and recursive manner. In the class from which this illustration is taken, the children are coming to know fractional numbers. They have previously worked with such numbers that can be shown by fractions with denominators that are powers of 2. The children have observed and used the multiplicative proportional character of fractional amounts through the use of paper folding and related informal mathematical activities. In the work here they are looking at fractional numbers as 'muchnesses' emphasizing their additive character. One day earlier they had been introduced to a 'half fraction kit'—a set of inter-related rectangles such that they have 2 units worth of ones, halves, fourths, eighths and sixteenths (a subset of the 'half fractions'). Each of the children has a kit. The children work in groups of 3 to 5. It is the classroom practice of this teacher to have children illustrate their work for the whole class. Thus while each child creates her or his own 'solution(s)' they are also in direct contact with others in their group and there are many examples of work and other students' explainings available. As will be seen below, the children or at least nearly all of them are capable of making distinctions in fractional number settings (e.g. can work with fractional amounts as 'sums' of other fractional units; can transform units into other units); they are encouraged to make and do make observable re-presentations of their ideas in many and variously public forms; they do interact with one another and with elements of the environment in a community with an observable mathematical 'culture'.

The students are working from a prompt, *Missing Fraction Mysteries*: *"A fractional amount is missing or hidden. We know it is more than 1/4 and smaller than 3/4. What might it be?"* Quickly, nearly all (except two) of the students generate multiple 'solutions' to the 'mystery'. Most of the children exhibit individual Image Having dynamical personal understandings (Pirie & Kieren, 1994). That is, such children are seen to be able to construct a fractional quantity that matches the constraints of the mystery in some without being given explicit directions of how to do so and without having to observe the missing amount. In other words, they have an image of fractional numbers as additive amounts. Further, they are observed to produce numerous re-presentations and representations of their thinking. Finally, there is opportunity for significant student-student; student-teacher; and student-environment interaction as anticipated in the interactive knowing model above. Conditions are in place for observing the possibility of emergent collective understanding.

What is observable? Three sets of responses are illustrated here. There are many more pieces of the children's work that could be fitted into these sets, particularly the first one. Further, although each piece of work noted comes from a different child various of these children's works and those of many others would be classified in more than one of the three sets illustrated here. Thus the actions and the understanding in action of any child are not necessarily unique to one of these sets or any other set not here illustrated.

The illustrative set A-D represents only a small fraction of the individual understanding acts that could have been associated with it.

(A) Teanna solves the problem by laying a half-piece, an eighth-piece and a sixteenth-piece on top of three fourth-pieces and says "there".

(B) Sally (in Teanna's group) lays down a half-piece a 3 sixteenth-pieces, writes 1/2+3 1/16 [sic]. She notices that this is just 11 1/16 [sic], which the students, in chorus, shout is "just one sixteenth from three fourths".

(C) In another working group Donny is sent to the board with his solution 1/2 + 1/8 + 1/16 + 1/32 +[...] +1/1024 which he writes out and draws showing a very tiny rectangle as the difference from 3/4.

(D) Quickly both members of his group and more than 5 others make up solutions which resemble but are not like Donny's.

At one side of the room three children, two in one working group and one in another group, interact (E-H).

(E) Trying to be different Dane writes 5.9/8.

(F) One of his partners, Wally sees this and writes 5.99/8 and says "that's just a hundredth off" [of 3/4].

(G) Megan eyeing their work from the next group writes 5.999/8. When asked about it she doesn't know the name of the decimal fraction she has written but she says her solution must be even closer to 3/4.

(H) Soon the three of them are representing many fractional amounts such as 5.999999/8. Notice that unlike the children in A-D, these children make no direct or even reference to the half fraction kit pieces in these actions.

One of the members of Donny's group Sam starts a different kind of work (I-J).

(I) Sam takes a one-unit piece and places a fourth piece on one corner of it. Just adjacent to this piece he places a sixteenth piece. He draws a sketch of this

and writes 1 - 1/4 - 1/16. Like Donny he also does a like drawing and writing at the board.

(J) Many other students and some whole working groups of students are forming solutions like Sam's and many are writing subtractive sentences like his.

Observing Collective Understanding Using Fractal Filament as a Simile

It can be observed that the children above are engaging in mathematical actions and interacting about them. It has been shown that they are enacting various kinds of dynamical personal understandings (DPUs). But what might we make of Martin's question of collective understanding? How might observing A-J or each of the sets above in fractal terms help us observe and understand the dynamical collective understanding? If we try to interpret the collective understanding of at least A-J above as fractal-like then there should be some features of it that are self-similar across scale. That is, the collective understanding of the whole should have features that manifest themselves, perhaps differently in the three sub-filaments.

Some features of this dynamical collective understanding might be:

I) Common cognitive mechanisms: enacting fractional amounts as additively combinable units. This feature may emerge from the individual knowing structures and DPUs. Might we think of part of the collective understanding as this common approach to the fractional quantity?

II) Common form of language use: descriptive and analogical language use. Although the children use fractional symbols and language in ways that an observer would interpret as appropriate, this language use while certainly generative especially in filament E-H is descriptive and analogical in form; the symbols are put for other kinds of quantitative actions. Where additive symbols are used, the children, although many can, see no need to actually 'do' the addition. Further the children make inventive but grammatically and mathematically appropriate use of symbols. While language use and its related metaphors have a personal basis what we are noting here is that this form of language use is occasioned by the collective.

III) Common mathematical interpretations: in each of the individual acts characterized above and in each of the sets one can see that all of the 'solutions' to the mystery are amounts which are close to 3/4. Of course there is nothing in the prompt or the materials that demand such a solution. But it appears to be just the way "they see it and act on it." Could this be part of a *collective image having*?

IV) Occasional commonality: actions are occasioned by the kit and interactions. To an observer it appears that the whole class expects to learn mathematics from using materials or from observing the work of others. As the individual differences in the work above illustrate, individual work is not caused by the materials or teacher expected responses or by the work of others; the individuals appear to select elements from the rich sources on offer and transform them for their own use. Such occasioning appears to be an expectation of the participants in this community and we conjecture is a feature of the DCU.

These and other features might be used to form part of the characterization of the group's collective understanding. It is important to note that none of these features was explicitly discussed in class or by any of the groups themselves. Certainly none of these features were 'taught'. But they can be observed and in that observation can be taken to emerge through and coemerge from the knowing and understanding actions individuals, the re-presentations available and the inter-actions or 'bumpings' which occurred. Further if one looks at A-D; or E-H; or I-J each these features is present in these filaments as well, of course in very different forms. One can contrast the kinds of symbols used in each filament but also see the common informal nature of the language use as well as the commonness of mathematical interpretation and additive nature of the solution. Thus these emerging features of DCU can be seen as features of self-similarity across scale or across sub-filaments.

There are many ways in which collective mathematical understanding might arise or emerge. The fractal filament simile suggests that collective understanding might be observed as a recursive phenomenon or might arise recursively. In what ways, if any, is this the case here? It is clear that the re-presentations of earlier understanding actions are available to the individual, to members of each working group, and at least in some cases to the class as a whole as 'input' to later work. Further there is evidence that interaction as anticipated by the Simmt model occurs particularly in the filament E-H. That is actions of individuals and their re-presentations are not only observed as occasioned by others and the environment, but the actions of individuals can be seen to occasion actions in the group and change the environment (that is the task in that group is at least temporally a task about decimal fractions approaching six. The filament E-H offers a further illustration of self-similarity with the whole but with a different language emphasis. After creating a number of solutions similar to those in A-D, Dane deliberately tries to be different (E). Notice in E that the use of decimal in a ratio form is very different from A-D but yet shares the commonalties I-IV. Wally (F) is observed to be occasioned by Dane's representation but certainly does not copy it. Megan enacts an understanding (G) that carries further what she sees to be Wally's contribution to what is now a decimal interpretation of the original prompt and acts to initiate a continuing co-occasioning of mathematical actions in their group (H). The new actions recursively build on hence also change the interpretations of the previous understanding actions both mathematically and socially in the community. Dane's earlier "showing off" now becomes a part of the actions which underlie the collective actions E-H. Thus such features of emergent dynamical collective understanding is *contingently observed* be a recursive phenomenon maintaining its central character yet undergoing continual change. To an observer, the cognitive domain for the class is changing as well. That is the possibilities for personal cognitive, the mathematical, the linguistic and the social elements of knowing are changing for the collective and the features of DCU might be seen as elements in that dynamical cognitive domain.

While the collective mathematical understanding is observed here as a phenomenon of an implicate order, it can yet be seen as fully embodied in the collective, here the collective of the class. Because the collective is composed at another level of individuals the collective understanding is embodied in a personal structural sense in that as suggested by the Simmt model in Figure 1, individual knowing actions raise the possibility of occasioning changes in others and hence in the collective and hence collective understanding. But as seen particularly in feature IV DCU is also socially embodied just as changes in it emerge from inter-actions occurring in a mathematical/ social milieu. Finally, DCU is observed as mathematically and culturally embodied. It is particularly evident that the elements of practice in this classroom provided a useful research environment in which to observed both growth in DPU and DCU. Such collective understanding can be seen both as related to and recursively contributing to the living culture of the mathematics classroom. At the same it also is a vestige of the historical and contemporary communities of mathematical practice.

Consequences of and Possibilities From this View of Collective Understanding

In mathematics, fractals can be observed as markers of dynamical, recursive dynamical systems. One might then expect fractal features in observing collective mathematical understanding as a dynamical co-emerging phenomenon. We have tried to show in what ways the simile of fractal might be useful both in observing and in characterizing collective mathematical understanding. In particular we have tried to show just how features of this collective understanding might arise recursively and manifest themselves in a self-similar manner across sub-collectives or sub-filaments. Using this simile raises different research questions about collective understanding only a few of which are: In particular settings what are the characteristics of collective mathematical understanding? What are the similarities and differences in the collective understandings of different classes perhaps engaging with similar mathematical topics or curricula or similar prompts or tasks? In what ways are the qualities of the collective understanding of a particular class different depending on the curricular topic or the prompt offered? In what ways characteristics both of the individuals, the interaction and the classroom culture give rise to or support the collective mathematical understanding? What might be the nature of Collective Primitive Knowing as a basis for Dynamical Collective Mathematical Understanding? In what ways does the collective understanding or features of it occasion individual understanding in particular settings?

But using the concept of fractal filaments as an observational tool has consequences for teaching as well. First, it might possibly provide teachers with a tool to observe the mathematical understanding of their classes in a dynamical on-going manner. Using such a tool means that the teacher is made more aware at a class level of the recursive character of mathematical knowing and understanding. The teacher can also be aware of the features of collective understanding at a quite different level and character from individual understandings in action. Further she can become aware of the different manifestations of this understanding in the student actions underlying different filaments. For example, in this case the teacher might become more aware of the general features of the informal language use in the classroom and build this knowledge into her patterns of interacting with the class or various working groups in it. It might also prompt her to observe how the use of language changes over time and hence how this feature of collective understanding changes. Because the teacher is both a leader of and part of the collective in the classroom, she can use her observations to change her perception of mathematics as occurring in her classroom or of the nature of the mathematical task or prompt as lived out in the actions and inter-actions in her classroom, but also to change elements of classroom environment in ways that she thinks might affect the collective understanding. For example, the teacher in the setting discussed here might have reposed the prompt in such a way as to change the collective understanding that the solution must "approach" three fourths as a bound. More generally, if collective understanding is a coemerging feature of a self-organizing system, the teacher might be prompted to probe her role as both a learning member of that system and as a special 'catalyst' in it. This raises the broader question of the roles and effects of the teacher and the external and the lived curriculum on the collective understanding of a class.

In response to a question about the nature of and the characterization of collective mathematical understanding from Martin (2001) we have suggested using fractal filament as a basis for observing such understanding. The basis for the use of this tool was derived from recent ideas in emergence, self-organizing systems, and knowing as coemergent. Using the tool enabled us to observe features of collective understanding that arose under particular conditions in one classroom. The questions raised above suggest that further study of collective mathematical understanding may be interesting and useful persons interested in mathematics knowing and understanding in action, in the relationship of socio-cultural classroom features to such knowing as well as those interested in different approaches to thinking about the role of teachers with respect to the changing understanding in their classrooms.

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REMEMBERING AND FORGETTING IN THE MATHEMATICS CLASSROOM

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Students often have difficulty in transferring their mathematical knowledge into new situations. We offer one explanation for this phenomenon, by exploring the ways in which a group of students came to know about ideas of similarity in the context of a problem-based curriculum and contrasting how they acted when facing a new problem that might appeal to their knowledge of similarity. We hypothesize that lack of transfer into some novel tasks is a byproduct of the actions by which instruction organizes a public memory oriented toward transfer into a range of familiar tasks.

Among the suggestions for problem solving that Pölya (1948) provides in *How to Solve It*, a notable one is that of creating a "plan," considering one's previous experiences. The suggestion triggers the image of searching through a mental filing cabinet of past experiences, where all related problems have been labeled and organized, and using that data to inform attempts at the task at hand. But it is not so simple. Whereas searching for past experiences to plan and attempt novel tasks is a useful practice, remembering those experiences is not just a voluntary act—remembering and forgetting are in fact conditioned and constrained by the contexts of instruction. This paper examines an example as a way to move further a discussion on the role of public memory in mathematics instruction.

In the *Principles and Standards for School Mathematics* (2000), the National Council of Teachers of Mathematics (NCTM) emphasizes the importance of conceptual understanding in mathematics and characterizes it by one's "...ability to use knowledge flexibly, applying what is learned in one setting appropriately in another" (p. 20). But using that standard to gauge the quality of mathematical work in classrooms might lead to disappointing findings. For it is with astonishingly high frequency that researchers have documented students' inability to transfer previously acquired mathematical knowledge into new situations (Hiebert & Carpenter, 1992). Why does that happen? We ask that question in the context of examining one case where students failed to recognize the relevance of their existing knowledge of similarity while addressing a new problem. An explanation is proposed that ties students' recognition and remembering to the context within which mathematical concepts are developed.

Theoretical Framework

In order to frame an analysis of students' recognition of relevant previous knowledge, we use Vergnaud's notion of *conceptual field*. We use that notion to look at both the context of students' learning of the concepts of similarity, and students' actions when asked to attempt new problems that involved the concepts of similarity. According to Vergnaud (1996), a conceptual field is comprised of three components –the set of situations or problems (P) that give meaning to the concept; the operational invariants (I), or action schemes that put those concepts to work; and the symbols and symbol systems (S) used to represent the concept. The latter is meant to encompass linguistic and symbolic tokens as well as other standard inscriptions that might come in the form of figures, diagrams, graphs, or equations.

The recognition that a new problem is related to a previously experienced conceptual field may trigger the implementation of operational invariants from that conceptual field. That recognition may be triggered by the resemblance between new problems and those problems that characterize a conceptual field. Recognition may also be triggered by the presence of the symbols that are characteristic of that conceptual field. Moreover, misrecognition may be explained in terms of lack of perceived resemblance between problems or absence of recognizable symbols. Beyond establishing a means for exploring the understanding of students' perceptions of a particular concept, the use of Vergnaud's theory of conceptual fields allows one to analyze the progress of students during their work with novel tasks using those concepts. Having chosen a problem for students to work on and being aware of some of the experiences which have allowed students to build knowledge of the mathematical concepts involved in that problem, we can hypothesize feasible operational invariants that students might have access to. We can use those to understand what students actually do when they work out unfamiliar problems.

As concepts are encountered and developed through classroom instruction and interaction, one must carefully consider the role of the classroom, specifically the teacher, in the development of conceptual fields. One must remember that knowledge development and use in schools is as much an institutional as a psychological phenomenon. In particular, the way in which public records of past work are generated and accessed often responds to instructional goals and needs. Thus, whereas an observer might consider that a new problem calls for concepts that students have been exposed to, that problem might still fail to elicit students' recognition of those ideas. The implicit regulations of the instructional context in which students have experienced the target concept may help explain such misrecognition (Brousseau, 1997).

Methods and Data Sources

Because of our interest in examining the relationship between instructional context and students' recognition, two kinds of data were collected – classroom observations, and student and teacher interviews. Observations were conducted over a period of three months in a seventh grade mathematics classroom in a suburban, racially and economically diverse school district, taught by an experienced teacher, whom we call Ms. Briar. The classroom used Connected Mathematics, which specifically states as one of its goals to foster in students the ability to transfer mathematical knowledge into new situations (Lappan et al., 1996, 2002), and to use problems as contexts for learning (Lappan et al., 2002). Classroom observations followed the development of the "Stretching and shrinking: Similarity" unit (Lappan et al., 1998). In particular we focused on students' coming to know the theorem that states that the area of a similar figure "grows by the square of the scale factor" (Lappan et al., 1998, p. 27n), which we refer to as the Similar Figures Area Conjecture (SFAC). Fieldnotes from classroom observations, the textbook, and artifacts from classroom activities serve as sources for describing the context in which the concept of the SFAC was developed in this classroom.

After lessons covering the SFAC were completed, an interview with Ms. Briar was conducted to learn more about how she works to develop students' knowledge of mathematical concepts. A problem that, from an observer's perspective might be solved using the SFAC, was used to provide context for a discussion on Ms. Briar's expectations for her students' work in a different context. To examine students' recognition of the relevance of the SFAC in the problem chosen, four students were asked to work on the novel problem in a session led by the first author. Their conversations and reasoning will be examined here.

Knowledge of Similarity in the Seventh Grade Classroom

The conceptual field of the SFAC was developed through a series of investigations in the unit, Stretching and Shrinking (Lappan et al., 1998). The first investigation is designed to introduce the notion of similarity to the students and get them to compare similar figures to discover relationships between them. In Investigation 1, "Enlarging Figures", students construct a "two-band stretcher" by tying the ends of two identical rubber bands together to serve as a tool for creating similar figures. The stretcher is held down on a point *P*, the anchor point, and a pencil is inserted at the other end of the stretcher. Students are instructed to stretch the rubber bands so that the knot tying the two together outlines the picture. Doing this, the pencil traces out a new picture, whose image is twice as large as the original (Lappan et al., 1998, p. 6). The subsequent follow-up questions ask students to compare their "original figure" with their "enlarged image", in terms of the lengths of the line segments, the angle measures, the general shape, and the area of the two figures (p. 7).

The first author observed the discussion of the resulting comparisons students made between their original figures and the images they had created. Students agreed that the line segments were twice as long, that the angles remained the same, and that the general shapes were retained. What they did not agree upon was the relationship between the areas of the figures. One student asserted that the area of the second figure must be twice as big as the first, reasoning that all of the line segments were twice as long. A second student, whom we call Ryan, contradicted this statement, saying that the image was four times as big as the original figure. His reasoning was that he thought that you could fit four of the original figure into the image. The teacher adopted, "Ryan's Conjecture" and pursued another example. A simpler shape was chosen and the students were asked to guess how many of the original figure would fit into the enlarged image. Utilizing this new shape, a flag, students could readily see that the image indeed fit four of the original figure. Ms. Briar ended the period by telling her students that they would return to "Ryan's Conjecture" later in the unit.

The concept of area in similar figures was further developed in Investigation 3, "Patterns of Similar Figures" in the same workbook (Lappan et al., 1998). Students worked with "rep-tiles" during this unit. A *rep-tile* is "a shape whose copies can be put together to make a larger, similar shape" (p. 29). See Figure 1 for an example of a rep-tile, similar to that provided in the student workbook. Students were given several shapes and copies of the shapes with which to construct rep-tiles. The classroom activity deviated from the text here, as Ms. Briar asked her students to construct their own tables to record data about the scale factors of the shapes (two this time, going from original to rep-tile and from rep-tile to original). She also asked them to record two ratios--the ratio of the perimeter of the original to the perimeter of the rep-tile and the ratio of the area of the original to the area of the rep-tile. Here students had a chance to verify that "Ryan's Conjecture" held true and that when objects' perimeters are made twice as large, the area is increased by a factor of four. The teacher pointed out that fact was to the students and the class again referred to the conjecture as "Ryan's Conjecture".

Later in the unit, students constructed another rep-tile that was used explicitly to discover the relationship between scale factor and area. Figure 2 illustrates the problem from the *Stretching and Shrinking* text that students discussed (Lappan et al., 1998, p. 52). Students examined the relationship between scale factor in these three diagrams and created their own diagrams of what the fourth and fifth triangles of this sequence would look like. They discussed the relationship between areas and came to realize that the square of the scale factor told them the factor to multiply the original figure's area by in order to obtain the area of the enlarged image. They did not explicitly write down the formula as a mathematician might (e.g., saying that if the ratio between corresponding sides of similar figures a:b then the ratio between their areas is a^2:b^2). Yet, when determining the ratio of the areas of similar figures, students understood that they could square the scale factor and multiply it by the area. The formula "area * scale factor * scale factor" was written on the board and students worked through several examples, calculating areas of figures that were 8 times bigger, 10 times bigger, and so on. Ms. Briar reminded them that this was exactly what Ryan's Conjecture had stated for the case of a scale factor of two.

After developing the conjecture over the course of at least the five days where these three activities took place, students were asked to apply it in various problems. One such application was related to reading maps. Students were given a map of the state of Utah, furnished with a scale, and asked to determine an approximation of the perimeter and area of the state (Lappan et al., 1998, p. 45). In this instance, they could utilize their knowledge of the Similar Figures Area Conjecture and their calculated scale factors from perimeter to determine the area of the state. Another similar application of the conjecture relating measurements considered designing buildings (Lappan et al., 1997, p. 36); the problem's diagram is illustrated in Figure 3. In this problem students were asked to determine the area of the figure on the grid and, using the SFAC, determine the area of the figure if it were built as specified by the scale given (1cm =

40m). It is the context of this problem, successfully completed and explained to the observer by several students, which will closely mirror the problem presented to students during the problem solving session where we had intended to observe students' reasoning and actions as they approached an unfamiliar problem.

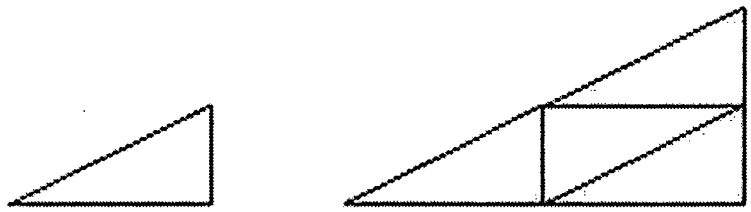

Figure 1. Rep-tiles. [Redrawn by the authors according to a figure in Lappan, G. et al. (1998, p. 29) ©1998 by Dale Seymour Publications]

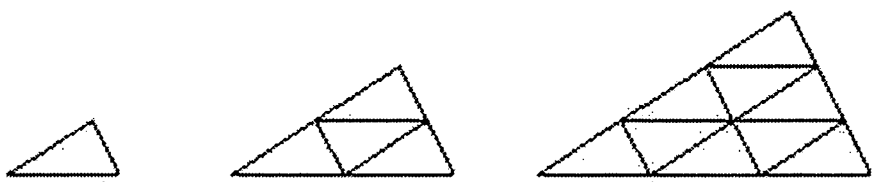

Figure 2. Building larger rep-tiles to find a pattern. [Redrawn by the authors according to a figure in Lappan, G. et al. (1998, p. 52) ©1998 by Dale Seymour Publications]

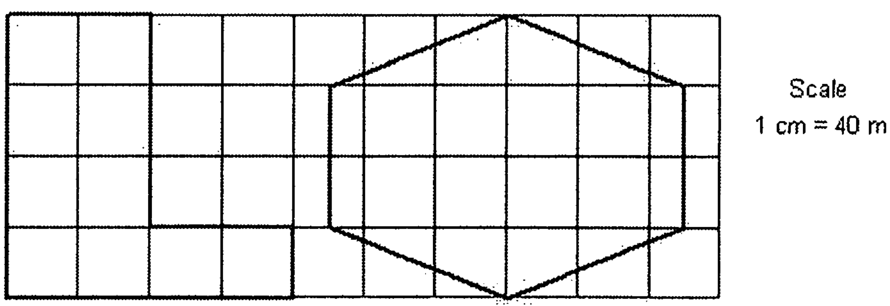

Figure 3. An Area Application. [Redrawn by the authors according to a figure in Lappan, G. et al. (1998, p. 29) ©1998 by Dale Seymour Publications]

We can use those activities to describe the conceptual field of similarity being developed in Ms. Briar's class, particularly as it relates to conceiving the SFAC. A variety of problem situations (*P*) gave meaning to the SFAC. Specifically, the problem set includes "rubber-band stretcher" problems, "rep-tile" problems, problems involving reading maps, and problems involving reading architectural plans for buildings. Generically, these problems involved comparisons between the areas of similar figures using the same unit and calculation of the area of a figure, using a given scale factor and the area of a similar figure. However, this problem set by no means exhausts the types of problems that could be created around the SFAC. For instance, students were exposed very minimally to the notion that the SFAC might be used in converting between area measurements of a given figure according to two similar units of measurement, even though the last examples given in class began to touch upon that issue. As we consider the language that came to be associated with similarity, we also note that the conceptual field developed in Ms. Briar's class excluded that kind of problem.

The proposition that we call the SFAC was never actually given a name in the classroom, only informally referred to as "Ryan's Conjecture". But Ryan's Conjecture, which referred specifically to the case where there is a ratio of two between sides and a ratio of four between areas of specific figures, does not fully describe the relationship that students learned during the lessons surrounding this concept. Instead, the relationship is more often signaled for use by the language tokens characteristic of the problems used during its development. Problems of comparison are indicated by the terms "scale factor", "original figure", and "enlarged image" and are seen repeatedly throughout the *Stretching and Shrinking* (Lappan et al., 1998) workbook; furthermore, such terms were heard repeatedly in reference to the need to use the SFAC by the first author during instruction. Indeed, the final version of the procedure associated with the concept of the SFAC was written on the chalkboard as $area_{image} = area_{original}$ * scale factor * scale factor. In the few problems that required students to transform measures using the SFAC, the problems required "comparison" or "calculation" and the term "scale" was always explicitly provided along with the numerical value to be used as the scale factor.

In addition to the language tokens used to point at the concept, there were representations associated to it. Students might recall the concept in terms of the images of the man they traced with the rubber-band stretcher, the rep-tiles which served as building blocks, or the maps and diagrams that gave them practice in using scale factors in real-life situations. More generally, the images that went along with the concept always included (actually or potentially) two separate figures, distinctly different in size, as source and target of the similarity.

The operational invariants associated with the concept included initially the counting of a given unit inside figures. We saw this first when Ms. Briar drew a flag which students handled by literally counting the number of original figures fitting inside. But even more clearly, we see the method of counting illustrated with the use of the rep-tiles which serve as building blocks for larger and larger figures, and which could easily be counted to determine the ratio between areas of different figures. In problems where they had to calculate an area students would use the formula $area_{image} = area_{original}$ * scale factor * scale factor. Students began to use the provided scale factor and an estimate of the area of the "original" figure in order to apply the formula and obtain the necessary measure of area that problems demanded.

The New Problem and the Teacher's Expectations

The problem chosen to elicit students' use of the SFAC (borrowed from Stein, 2001) presents an oddly shaped figure, drawn upon a grid where each unit square of the grid is designated as one squared centimeter. Directions indicated for students to determine the area of the figure in squared centimeters and in squared millimeters (see Figure 4). We foresaw this problem as calling for the recognition that the scale factor, in going from millimeters to centimeters, is 10. We foresaw the problem calling for recognition that the two area units involved are similar, that the SFAC helps determine the ratio between the area units, and that this ratio (100) is enough to determine the area of a given figure according to the two units.

Whereas the SFAC might be instrumental for students to solve that problem, there are several aspects of the problem that separate it from those that students encountered in class when they were learning the SFAC. First of all, the problem situation was somewhat new to the students. Although they had encountered grid diagrams asking them to change units in their work with the SFAC, they had not encountered a problem asking them to consider a conversion in squared units. Although the intended action was the same, one might also argue that there was little indication—in the language or the graphical representation—that this was an appropriate action to take. The rep-

Find the area of the shaded region in square centimeters and square millimeters.

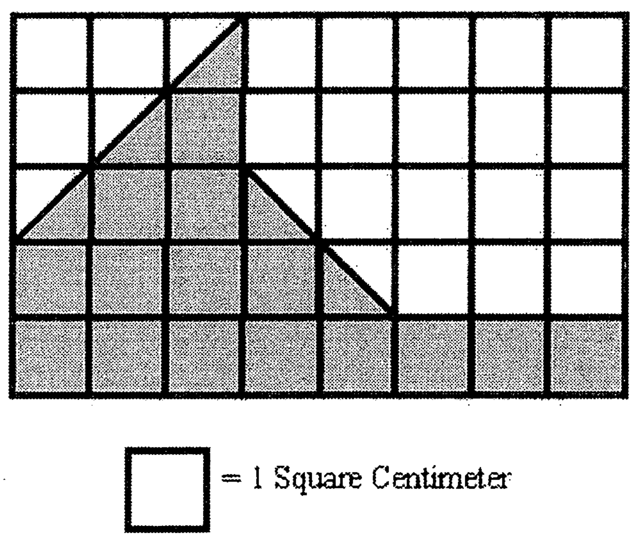

Figure 4. The problem solving task [Redrawn according to a diagram from Stein, 2001, p. 110; © National Council of Teachers of Mathematics].

resentation of this problem is different from their previous exposures in that not two figures are given or eluded to (an original and an enlarged or reduced one) but just one. Moreover, there is no scale relating cm^2 and mm^2 as they had seen in other problems such as the Utah problem. Finally, the language used in this problem is also different than in the problems used to develop the concept. Students are asked to "find the area" instead of "compare" areas; and there is no mention of "scale factor" in the problem.

The Teacher's Instructional Strategies and Expectations for the Problem Chosen

We were also interested in understanding how Ms. Briar had interpreted and implemented the activities that provided context for the development of the similarity concepts. So, an interview was conducted by the first author, to learn about the different strategies Ms. Briar had used to help students in the problems worked in the classroom. Ms. Briar emphasized the importance of establishing a rich and engaging environment with which to associate the concept so that elements of this environment could be called upon to elicit memories of a particular concept in later instances. Moreover, she emphasized the importance of establishing a diverse language with which to refer to concepts:

> One thing is if you've got a nice engaging activity, that's the kids own data, or innately of interest, or that type of thing...so that you can go back to that if you want to stress a point.... We develop an informal language that may make sense only to that class but you keep connecting it and take a more casual approach to the term.... I think the more that comes out of them, the more likely it is to stay.

It seems that such an environment was indeed created around the concept of the SFAC. Problems occurring throughout the development of the SFAC utilized students' own data and observations while addressing several different real life problem situations primarily through word problems, with settings as described previously. Moreover, as we have seen, Ms. Briar tried to introduce students to a variety of language referents for the concept. The conjecture, first hypothesized by a student, was labeled "Ryan's Conjecture". Throughout future development of area comparisons, this label continued until finally formally modified into the equation $area_{image}=area_{original}$ * scale factor * scale factor.

Ms. Briar also commented on the problem that the first author had chosen to give to students. Interestingly, when asked what she would expect her students to do with the task, she did not foresee that this problem would call for the use of the SFAC. Although she herself described her reasoning through the task using the SFAC, she felt that her students would attack the problem as an instance of conversion of units. This was despite the fact that similar problems, showing transformations across measurement units treated as similar figures, had been presented recently (within two weeks of the interview and problem solving session) in the text and discussed by students in small groups. Furthermore, she felt strongly that a great number of students would run into problems with converting centimeters squared to millimeters squared, relying instead on their schema of conversion from centimeters to millimeters according to which they would multiply centimeters by 10 to get millimeters. She said about those students who might see the problem as an instance of conversion of units, “what the most common error then would be if they thought...multiply by ten...cause there’s some kids who still don’t know the squared relationship”.

Students’ Work on the New Problem

Four students from Ms. Briar’s third hour mathematics class volunteered to participate in the study. The students, accompanied by the first author, took time away from a normal class period to work together in the school’s library on two problem solving tasks, including the task shown in Figure 4. The group included three Caucasian and one Asian–American student, two males and two females, and covering a broad range of mathematical abilities. Incidentally, one group member was Ryan, of “Ryan’s Conjecture”, the student who had made the initial observation of the SFAC and whose name was associated with the concept throughout the observation period.

The students were asked to work each problem individually for a period of about 5 minutes (when everyone had finished), at which time the group convened to discuss together the work which had been done. All four had correctly answered the first portion of the problem and had utilized the expected method of counting the shaded squares. In order to obtain the answer in squared millimeters, three of the four students utilized what they remembered about conversion of units to multiply their answer in squared centimeters by 10.

Kulp: So, what did everyone get for millimeters squared?

Ryan[1]: 175 millimeters. And I just moved the decimal.

Molly: I got 170.

William: I got 175...

Kulp: ...Okay, so how did you guys go from 17.5 to 175?

Ryan: Well, when anything’s multiplied by10 you just move the decimal over 1.

Joung was the only student who obtained a different answer to this question – 1700mm^2. Unfortunately, she could not explain her methods to the rest of the group, insisting only that her answer was incorrect.

It appeared as if the students had recognized the problem as calling for conversion of units and thus implemented operational invariants—moving the decimal point—that were relevant to those type of problems, learned long ago. In doing that, they had all run into a common error that occurs when converting areas. When asked if there was another way to approach the problem, a way to check their answers, the students remained silent. The students were prompted to consider scale when the interviewer drew in an additional block, this time much smaller and labeled as one squared millimeter. But it was only with the introduction of the particular language associated with the SFAC, in the form of the question "What would my scale factor be to get from here to there (pointing first at the one centimeter squared and then at the added one millimeter squared)?," that students were able to reconsider their answers. With the label "scale factor" introduced, Ryan immediately changed his original solution to the problem and the others were quick to follow from his explanation.

Some Conclusions

Why is it that the students failed to recognize this problem as an instance of similar figures? There seem to be several characteristics of our problem solving task that are different from those of the context in which students had developed and used the SFAC. First of all, there are key differences between the kind of problem we used and those used around the conceptual field of similarity in the classroom. Notably, the problem did not involve operating on a pair of figures, source and target of a similarity. Furthermore, the symbols involved included tokens that had not been customary in the classroom (squared millimeters and squared centimeters). The problem did not include symbolic cues that would point students to similarity, such as: "scale," "scale factor," or "compare" were not included in the problem.

However, one cannot completely alienate the problem from the range treated with similarity. There were certain characteristics of this problem that were (intentionally) similar to previously worked problems, including the problem described previously which used an architectural context to develop an understanding of conversion of units as a unique instance of the SFAC. It is also true that there were concrete differences in the setting of doing mathematics – the students were not in the classroom where they normally use their mathematics, and it was the first author rather than their teacher who was working with them. Yet, we feel that the observed misrecognition owes more to the characteristics of the context (including problem and symbols) than to the difference in settings. At any rate, because a goal of the problem-based teaching embodied in CMP is to foster students' ability to use mathematical knowledge in new situations that might take place outside of the classroom, we should be interested in examining carefully this misrecognition.

It is significant for this discussion to examine why it is that the actions of the students (treating the problem as a conversion problem and not as a similarity problem) would not have surprised their teacher--as she had forecasted that treatment, as well as the error that students ended up making. One possible explanation, suggested by the second author, is that these students' actions, which might be thought of as forgetting or as a display of their individual inability to recall and transfer knowledge they have learned, actually have instructional roots. Those instructional roots become visible when a problem is chosen that breaches the implicit contract (Brousseau, 1997; Herbst & Kilpatrick, 1999) between teacher and students by virtue of which recalling and transfer are made to work efficiently in the classroom.

Students' failure to identify the relevance of the SFAC is actually their success in reading the cues for a conversion problem. In this instance then, we see not an example of the students' forgetting their knowledge of the SFAC, but an example of a problem where the cues at hand are not similar enough to those which have become triggers for the SFAC concept during its development. Thus, the knowledge that students have of the concept cannot be accessed, whether to solve the problem or check the work that they have done on the problem. Only when a cue, associated with the concept, is introduced do students have access to their memory of the SFAC and its related actions. Teacher and students share a mutual understanding, built on the context of instruction that relates how students develop understanding of concepts and how they perceive the relevance of those concepts in problems encountered at a later time. In other words, because instruction must aim at transfer, instructional practices tend to include the identification of central contexts and conditions of transfer. As the potential contexts for transfer are many more than those identified in instruction and some of them might actually be contexts for the teaching of other ideas, it would always be possible to produce a misrecognition like the one observed. But whereas such misrecognition could be considered an interesting effect of instruction we want to stress that we are not suggesting that the particular instruction observed or the particular curriculum being used are to be blamed. Rather, we tend to think that it is a systemic effect that would be more or less observable in any kind of instruction that aims at ensuring transfer.

Certainly, this topic needs to be researched further. An understanding of the characteristics of tasks and the connection between the design of problems, lessons, and units that build multiple ways of organizing and accessing the public memory could have tremendous impact on the work of the teacher and how she negotiates with students the work to be done. And understanding such connections would hopefully allow one to have insights into how to promote the application of student's mathematical knowledge when they are faced with unfamiliar problem solving situations – in the classroom and outside of it.

Note

[1]All students' names are pseudonyms.

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THE 'AHA MOMENT': STUDENTS' INSIGHTS INTO THE LEARNING OF MATHEMATICS

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The moment of illumination – the AHA moment – has long been the basis for lore in mathematics. Unfortunately, these phenomena are reserved almost exclusively for the domain of practicing mathematicians. This study focuses on the impact of these AHA moments on students' affective domain. In particular, I examine the role of the positive emotion that accompanies such moments of illumination in changing the attitudes and beliefs of 'resistant' students. That is, preservice elementary school teachers who deem themselves to be incapable and/or phobic of mathematics and the learning of mathematics but are forced to take an undergraduate mathematics course as qualification to enter into a teacher education program.

At that moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it...

Poincaré (cited in Hadamard, 1954)

Barnes (2000) calls it the 'magical moment.' Polya (1965/1981) speaks of 'a sudden clarification.' Davis and Hersh (1980) dubbed it 'the flash of insight.' What they are all referring to is the moment in mathematics when an understanding is suddenly achieved. At this moment, what was previously confusing and elusive becomes clear. Learning has occurred, and it seems to have done so in an instant – a moment of learning.

For mathematicians, the 'AHA moment' – as I refer to it – is both an accepted and expected part of the problem solving process. We know that often the solution comes to us; there will be an illumination. And when it comes it is accompanied by feeling of joy and satisfaction. These experiences and the feelings that accompany them, however, are not unique to mathematicians – even within the field of mathematics?

Prologue (Motivation and Question)

Imagine a person who has a dislike for mathematics. Somewhere in their past is a negative experience with the subject – it may have been a single event, or it may have been a series of negative events. The nature of the experience is not clear, but what is clear is the impact that it has had on them as a learner of mathematics. They feel that they "were never good at mathematics', or that they 'can't do mathematics'. Perhaps they are even afraid of the subject, suffering anxiety at the thought of having to endure a mathematics course.

Such a person is not difficult to imagine. We encounter such individuals on a regular basis. They are our neighbor, or brother, our friend – and sometimes they are our student. Now imagine a whole class of such students – 100 plus individuals who, to a person, would describe themselves as either being incapable of doing mathematics, or having a phobia about the learning of mathematics, or both.

What impact would an experience of illumination have on the individuals in such a group? These are not mathematicians – yet they are working in the field of mathematics. They are not anticipating these moments of insight; they may not even be familiar with such moments. For many of them mathematics has long been a subject devoid of wonder, surprise, and discovery. As a researcher I became interested in what such a population would make of these moments of illumination in the context of mathematics. Where they experiencing them, and if so, what effect were they having on them?

Background

> In that moment when the connection is made, in that synaptic spasm of completion when the thought drives through the red fuse, is our keenest pleasure.
>
> Thomas Harris (2000, p. 132)

The learning of mathematics has classically been studied from the perspective of cognition. That is, the examination of what are the cognitive processes involved in learning, and how do they operate. However, the inability of such research to explain the failures of people in problem solving contexts who posses the cognitive resources necessary to succeed has prompted the re-evaluation of the role of the affective domain in the learning of mathematics (Di Martino & Zan, 2001).

The affective domain is most simply described as feelings – the feelings that students have about mathematics. In general it is understood that the affective domain is comprised of beliefs, attitudes, and emotions (McLeod, 1992). The beliefs are just that – what students believe to be true about mathematics (c.f., beliefs that mathematics is 'difficult', 'useless', or 'all about memorizing formulas'). Ajzen (1988) defines attitudes as "a disposition to respond favourably or unfavourably to an object, person, institution, or event" (p. 4). Attitudes can be thought of as the responses that students have to their belief structures. That is, attitudes are the manifestations of beliefs (c.f., attitudes such as 'math sucks', 'I can't do math', or 'I was never very good at math').

Research has shown that both beliefs and attitudes are relatively stable within an individual learner. That is, they change slowly. Thus, changing negative attitudes or misconceived beliefs is a difficult and slow process. On the other hand, stable also implies that such negative feelings are slow to form. Emotions on the other hand, are relatively unstable (Eynde et al., 2001). They are rooted more in the immediacy of a situation or a task and as a result are often fleeting. Students with generally negative outlooks can experience moments of positive emotions about the task at hand and, conversely, students with generally positive outlooks can experience negative emotions.

Because of their stable nature, research has focused primarily on the role of beliefs and attitudes on the learning of mathematics (Eynde, De Corte, & Verschaffel, 2001). The results indicate that attitudes – and as a result, beliefs – are strongly linked to school achievement (Leder, 1992; Ponte, Matos, Guimarães, Cunha Leal, & Canavarro, 1992) Attitudes and beliefs can be viewed as the gatekeepers to learning. That is, before a student can even begin to engage in the learning of mathematical content they have to first decide that they are both willing to learn and capable of learning the presented material. Once that has been decided any residual beliefs and attitudes regarding the learning of the content will continue to affect how they actually learn it. For example a student with the belief that mathematics is about 'memorizing rules' will approach new content matter from the perspective of identifying a rule, mastering the use of that rule, and then memorizing that rule – regardless of the intended cognitive outcome of the lesson.

Conversely, the relatively unstable nature of emotions has resulted in little research being done on the impact of this component of the affective domain on the learning of mathematics (for exception see DeBellis & Goldin, 1999, 1993). This is the goal of this study. I am interested in how the positive emotions elicited by an AHA moment affects the beliefs and attitudes of a student and as a result affects their learning of mathematics.

Methodology

Participants in this study are pre-service elementary school teachers enrolled in a 'Foundations of Mathematics for Teachers' course that is requisite for entry into a teacher education program. The course runs for the length of one semester (13 weeks) and has four contact hours each week. The data for this article comes from the 112 students enrolled in the course at the time of the study. These students could, for the most part, be described as resistant. That is, they are resistant to the fact that they have to take a mathematics course – as many describe themselves as math-phobic, or at least math-incapable. They do, however, appreciate the fact that they need to learn the course content, as they are highly cognizant of the fact that they need to fulfill this requirement in order to advance into teaching.

The course, itself, is generally regarded as an'unpacking' (or 'repacking') course, as it focuses on examining elementary school mathematics from a more general and encompassing perspective. The hope is that such an approach would allow the students to build connections between individual strands of mathematics and facilitate a deeper understanding of the topics as they make sense of all that they see.

At the time of the study I was the teaching assistant for the course. Although the course is taught in a lecture format the students are provided with support in the form of a tutorial lab. The lab is open to students from 20 to 30 hours per week and is manned at all times with between one and four teaching assistants. There are weekly homework assignments and students are expected to work in groups of three to five students to complete these. The group work is not only expected, but also facilitated in the group centred atmosphere of the tutorial lab where the students meet regularly as groups. It was in this lab setting that I had my contact with the students.

The data for this study came from two sources. The first of these is through informal discussions with students as they worked in the lab setting. Students were engaged in a discussion early on in the course regarding experiences of illumination in mathematics. At the time there was no explicit purpose to these interviews other than to use them as a way to enter into a dialogue with the students in an attempt to get to know them. The discussions were very informal and can best be described as conversations.

The result of these initial interviews can be summarized as producing two very prominent trends. The first trend was that students, by enlarge, knew what was meant by an AHA moment – even without definition. Many had alternate names for it referring to it as 'when it clicks' or 'that spark'. Some students assumed I was referring to 'when you suddenly remember the name of someone or something that was on the tip of your tongue, but you couldn't remember'. Only a very few students needed me to clarify what I meant by an AHA moment. However, I did provide clarification for everyone, if only to situate the phenomenon, and the discussion, in mathematics.

The second theme that emerged from the informal interviews was how few of the students could recall an AHA moment in the context of mathematics – many of them claiming to never have experienced one. It was this second result that turned out to be the more interesting one from the informal interviews. However, it wasn't until I began to witness, and hear about, the AHA moments that students were experiencing in working through the course content that I realized that some of these students may be experiencing this phenomena for the first time.

It was this realization that prompted the creation of the second instrument for the expressed purpose of capturing some of these experiences. As part of the course requirements the students were obliged to produce an end of term project. It was decided that one option for this project was to write about an AHA moment they had experienced in the context of mathematics during their participation in the course. This option, as communicated to the students, was prefaced with an anecdote that was meant to describe the type of phenomenon that they were invited to write about.

> 'I had been working on the problem for a long time without any progress. Then suddenly I knew the solution, I understood, everything made sense. It seemed like it just CLICKED!'

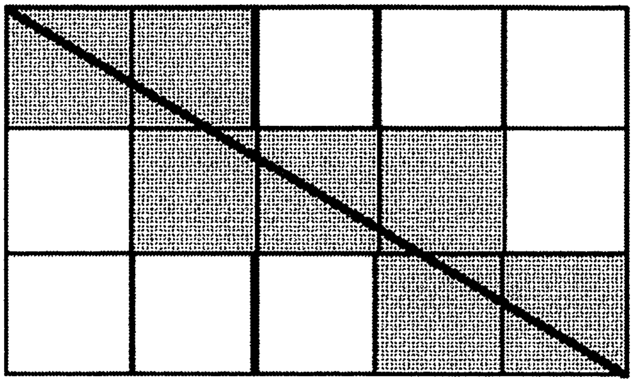
This was followed by some very specific criteria pertaining to mathematical context, content, and understanding, as well as a conclusion regarding the value of the AHA moment in the teaching and learning of mathematics.

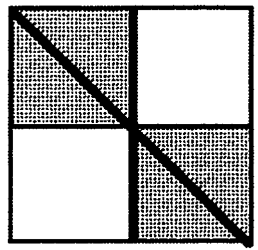
As it was not certain that everyone could claim to have experienced an AHA moment the students were offered an alternative to this assignment. They could, if they wished, engage in a problem-solving task. In order to be fair this option was open to all students regardless of their experience with an AHA moment. The alternate assignment was:

Of the 112 students enrolled in the course, 76 students chose to write about their

On a squared paper draw a rectangle and draw in a diagonal.

How many grid squares are crossed by the diagonal?





In the case of a 3x5 rectangle or a 2x2 rectangle above, we can simply count. However, can we make a decision about a 100 x 167 or a 3600 x 288 rectangle? In general, given N x K rectangle, how many grid squares are crossed by its diagonal?

AHA moment. Although the assignment covered a wide range of attributes surrounding their experience (see above) this study focuses on an aspect of the assignment that was not asked for. All but one of the students who chose to write about their AHA moments also felt compelled to relate how the experience made them *feel* – and in many cases, the effect that these feelings had on them. I use the term 'feelings' to represent all three of the elements of the affective domain – beliefs, attitudes, and emotions. What follows is a presentation of some of these responses.

Student's Responses

As already mentioned there were 76 students who chose to write on their AHA moment. Of these, five students misunderstood the instructions and wrote on illuminations that they had experienced in earlier mathematics courses – usually in grades five to 12. However, for the purposes of this study this did not disqualify their responses. As also mentioned, all but one of these 76 students mentioned how they felt when they experienced the AHA moment – even if to only say 'I felt great'.

However, not all of the students portrayed an accurate understanding of what was meant by an AHA moment. At least one student took the assignment to mean that she was to construct an AHA moment. That is, she came into the lab and sat down at a table and tried to create an understanding of Venn diagrams – a topic that until then had troubled her. In fact, through my tutelage she did manage to construct an understanding of Venn diagrams and then, unbeknownst to me, proceeded to write about this process for her assignment. This would have been fine if there had, indeed, been a moment of illumination within the process – but there wasn't. Instead, there was an observable slow awakening to the concept. Evidence of thinking of the AHA moment as a slow dawning of understanding was also present in the writing of two other students. Andrea even went so far as to rename the moment the 'AAAHA moment'.

Andrea: My AHA moment has come slowly – not all at once, but little by little I am grasping the concept.

The remaining students all presented experiences that were consistent with what was expected in the spirit of the assignment. Although, as already mentioned, everyone expressed their feelings in some capacity what follows is a presentation of some of the more prominent presentations organized according to the themes that emerged from analysing the data. For the purposes of brevity the mathematical context of the AHA moment has been left out.

Anxiety

Although the topic of anxiety was not brought up in the context of the AHA moment 34 students felt it necessary to mention how they felt about mathematics, or about taking a mathematics course as a way to provide a baseline for their discussion on change feelings.

Both Jennifer and Stephanie reflect on how they feel about having to take this course in order to be able to enter into the teaching program. While Jennifer states a dislike for mathematics Stephanie expresses a fear of the subject matter.

Jennifer: I have never been a person that likes or even enjoys math at all, so the idea of having to take this class if I wanted to teach wasn't very appealing to me. So I came into the course with the preconception that it would be just like any other math class that I had taken.

Stephanie: When I entered Math 190, I felt that fear in my stomach return. I needed this course to enter teaching so the pressure was on.

Marcie reflects on her experience in mathematics in general – going all the way back to her negative elementary school experiences.

Marcie: I was feeling emotions that should not have even existed in grade school.

Change in Beliefs

There were 39 students who displayed a change in their beliefs through the experience of the AHA moment. Only a handful of these actually used the term 'belief' in one form or another to express this feeling. Nonetheless, the change in their beliefs comes through in the articulations of their feelings.

Susan expresses how the experience has changed her beliefs on both her ability to solve problems and by which process she uses to produce a solution.

Susan: The AHA moment is inspiring. It makes students believe that they solved that question through reasoning and deep thought, and inspires him or her to seek more of these moments to obtain a sort of confidence and further knowledge.

James reflects on how the absence of these experiences may have contributed to his belief that he was not good at mathematics.

James: For myself, I wish that I'd had more of these moments in my earlier years of high school then I would maybe not have so readily decided that I was not good at math.

The belief of what 'it takes' to be good at math is altered for Lena as she expresses that she now sees that it is not an issue of intelligence.

Lena: Knowing that I could stare at a problem and in time I would understand, gave me more confidence that I could be successful in math. It really is not an intelligence issue.

What is interesting is the variety of beliefs that were affected by experiencing illumination in the context of mathematics. Although most of them centre on students' conceptions of their abilities to do mathematics some students expressed that they no longer see mathematics as being a collection of closed skill tasks that can be solved quickly, as expressed by Kyla.

Kyla: I used to think that if you couldn't get it right away you didn't know how to do it. This is the longest I've ever worked on a problem. I had just about given up when it just came to me.

Although Kyla's response was similar to that of one other student, her response is unique in that she arrived at this conclusion in the context of doing the other option for the final project. Kyla had not intended to write on an AHA moment and so chose to pursue the problem solving option of the assignment. It was during her work on this problem that she had, what she claims to be, her first 'mathematical AHA moment'.

Change in Attitude

Because attitudes are the manifestations of beliefs it was sometimes difficult to discern the two. That is, almost every expression of a change in attitude had a discernable change in beliefs associated with it – and has been counted in the 39 responses discussed above. However, this does not preclude the richness of the attitudes that were changed. Charlotte and Stephen express a change in optimism and expectations, respectively.

Charlotte: I have a better attitude now; I'm more optimistic. This is helpful in learning as complete thought processes can be impeded by a dejected attitude.

Stephen: Also, I enjoy math now. I feel like this success stimulated more success. Now I have raised my expectations in math.

However, there were five student responses that clearly show a change in attitude without a discernable change in beliefs. This is best demonstrated in Kristie's comments.

Kristie: I must admit that math is challenging for me ... after the AHA moment you feel like learning more, because the joy of obtaining the answer is so exhilarating. It almost refreshes one's mind and makes them want to persist and discover more answers. It gave me the inspiration and the determination to do the best that I can do in the subject.

Kristie has most definitely changed her attitude about the pursuit of mathematics in that she is feeling inspired and determined to succeed in the course. What is not clear is whether or not this is as a result of a new belief that she can succeed.

It is clear from the data that the positive emotion that accompanies the experience of the AHA moment has the power to change both beliefs and attitudes. This however does not explain the mechanism by which this is achieved.

Analysis

Almost all of the participants alluded to a sense of accomplishment that accompanies the AHA moment – most actually using the word 'accomplishment' to describe the feeling. However, it should be noted that although successful completion of a problem is often a by-product of a moment of insight it is not exclusive to this phenomenon. Problem solving is often achieved through the direct application of problem solving tools and strategies available in a student's repertoire of mathematical experiences. This does not mean that they have experienced a moment of illumination. The feeling of accomplishment that accompanies the solution of a problem is most likely associated with one of two things – the production of a solution and/or the achievement of greater understanding. In either case, the phenomenon of an AHA moment can be detached from the feeling of a sense of accomplishment. This is nicely demonstrated in Deanna's statement.

Deanna: After I understood the question and I had completed it, I felt as though I had accomplished something. I felt as though I was somewhat complete in my understanding of the problem.

Having made the distinction between feelings of accomplishments and moments of illumination there is a wealth of research that indicates that repeated success and feelings of accomplishment contribute to a change in attitudes and beliefs. Furthermore, if an AHA moment is one of the phenomena that contribute to a sense of accomplishment then it could be said that accomplishment is one of the intermediaries by which attitudes and beliefs are changed. Unfortunately, this does not provide a greater description of the mechanism by which the change occurs.

However, there is a clue in what Deanna doesn't state and Elizabeth does that may provide some insight into the mechanism of emotional change.

> Elizabeth: AHA moments are those great moments of deeper understanding and clarification of problems where incorrect or incomplete understanding is overcome. These moments inspire us and encourage us to keep going despite the frustration and anxiety that often tends to overwhelm us in times of difficulty when attempting to solve a problem.

Elizabeth articulates very nicely that the AHA moments 'inspire us and encourage us to keep going'. And it is the AHA moments themselves that provide the impetus for this inspiration. This expression of the role of AHA moments was shared by 28 of her classmates.

Are attitudes and beliefs, in fact, stable entities? If they are then it requires successive successes to change negative attitudes and beliefs. The AHA moments provide the inspiration and encouragement to embark on this path of successive positive experiences and in so doing provides an image of feasibility to the student. This would account for changes in both students' attitudes about their abilities to do mathematics and their beliefs about the accessibility of mathematics.

There is another explanation of the mechanism by which changes to attitudes and beliefs are facilitated. Ironically, it is a belief in the potential for change. It could be said that what makes mathematics enjoyable to pursue is the potential of positive emotion – the anticipation of that 'keenest pleasure'. In many ways this is a belief in the potential for change – emotional change. For David, however, the belief in the potential for change is not rooted in the emotions but in the beliefs themselves. His experience with AHA moments provides him with the hope that one day mathematics will make sense.

> David: The moment of comprehension is what keeps 'wannabe' mathematicians in the game. The hope that one day, in one instant, the world will mysteriously come into alignment and math will make sense.

Conclusions

The moment of illumination, the AHA moment, that instance when the connection is made is part of the culture of mathematics. They are the *fishing tales* that mathematicians tell. But they are not the exclusive property of practicing mathematicians. Their power to transform attitudes and beliefs towards the learning of mathematics makes these instances of insight an indispensable resource in the fostering of mathematics students. That they should be taken advantage of is indisputable. The order of business should now be – how to use them? That is, how are we going to orchestrate our students' learning environments to best facilitate the potential for illumination?

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USING PHYSICS AS A TRANSITIONAL TOOL IN THE LEARNING OF CALCULUS

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Physical representations of calculus concepts link students' experiences in the physical world with concepts in the calculus classroom. This study describes how students use physics concepts as transitional tools to aid in their understanding of calculus concepts. Transitional tools are objects in the environment or experiences that both separate the learner from another physical object and strengthen his/her understanding of the object using mathematical contexts such as symbols and graphs. The results of this study indicate that when students use physics as transitional tools they use physics in one of four (not necessarily disjoint) ways: Contextualizers, Example-Users, Mis-Users, and Language-Mixers. Each of these categories will be discussed and examples of students in each category will be contrasted.

Introduction

Physical representations of calculus concepts link students' experiences in the physical world with concepts in the calculus classroom. Many research reports allude to the importance of physical representations in the learning of calculus concepts (Nemirovsky & Noble, 1997; Thompson, 1994), but few specifically address the role of the physical representation in students' understanding of calculus. Many students experience the mathematical concepts of average rate of change, derivative, and integral in physics classes as they study concepts such as motion, force, and electricity.

This paper describes how students use physics concepts as transitional tools to aid in their understanding of average rate of change, derivative, and antiderivative. The focus of this paper is on describing the classification schemes that emerged during my study of students' use of physics as transitional tools in the learning of calculus. The primary research question I investigated was: How do students draw upon physics concepts to inform their understanding of rate of change, derivative, and integral? The setting for the study was an interdisciplinary Calculus/Physics[1] program. I will begin by describing the Calculus/Physics program and my involvement with the program.

The Calculus/Physics Program

The present study grew out of research done as part of an evaluation of an integrated Calculus/Physics program. The Calculus/Physics program was developed around two overarching mathematical themes: change and superposition. Additionally, the Calculus/Physics curriculum development was informed by recent research in the areas of calculus and physics learning (c.f. Ferrini-Mundy & Graham, 1994; McDermott, 1984) and research in the area of problem solving (Arcavi, Kessel, Miera & Smith, 1998; Schoenfeld, 1985). Each week the students met two days a week with the physics instructor in a laboratory and two days with the mathematics instructor in a computing laboratory. The final meeting session of the week featured both instructors and emphasized connections between calculus and physics that were salient during the week. A typical class session involved group activities, computer work, experiments, and a short lecture or class discussion. The calculus topics were reordered and coordinated with the physics content. The four basic threads of calculus (function, continuity, derivative, and integral) were discussed first with polynomial functions and then again for other classes of functions (logarithmic/exponential, trigonometric) as they arose in the physics curriculum. This reordering of the calculus topics allowed for a more unified calculus and physics curriculum.

As part of the evaluation of the program, I conducted clinical interviews with students enrolled in the Calculus/Physics course. A preliminary analysis of the clinical interview data revealed that Calculus/Physics students tended to use physics terminology and evoke physics concepts as they solved various calculus tasks. The research conducted during the evaluation of the Calculus/Physics program coupled with a review of the literature led me to further examine students' use of physics as a tool in understanding calculus concepts.

Theoretical Framework

In order to make sense of how students come to conceptualize calculus concepts, it is important to consider both how the students experience the concepts and how the students mentally organize information about the concepts. The idea that symbols represent information is a central theme in mathematics. However close scrutiny of what it means for symbols to 'represent' has led to the examination of the question: *Representation of what, for what purpose?* (Vergnaud, 1987) and more fundamentally, *What does it mean to represent?* The complexity of the task of attempting to answer the above questions can be attributed, in part, to the myriad of meanings of the word 'representation'. In particular, a struggle over the seeming duality created by the colloquial use of the word representation is evident in the literature (c.f. Janvier, 1987).

Nemirovsky and Noble (1997) put forth the notion of transitional tools as part of their emerging psychological perspective that allows for the analysis of an individual's constructive activity by challenging the convention that any given object or picture must reside either inside or outside a person's mind. By rejecting the notion that a visualization must be either internal or external, Nemirovsky and Noble (1997) overcome the common difficulty that arises from the need to describe whether a visual image is internal or external to the student. Transitional tools are objects in the environment or experiences that both separate the learner from another physical object and strengthen his/her understanding of the object using mathematical contexts such as symbols and graphs. For example, a student who talks about the motion of a cart on a track to help him/her conceptualize properties of the derivative is using the cart and track as transitional tools. Note that the cart and track are tools that reside both internally (in the student's memory of the cart's motion on the track) and externally (the physical existence of the cart and track).

Metholodogy

The present study was conducted at a large, public university in the Northeastern United States. Eight students enrolled in an integrated Calculus/Physics program participated in the study during the 2000-2001 academic year. The eight students in this study consisted of seven male and one female student. The gender balance is reflective of the gender balance in the Calculus/Physics class during the 2000-2001 academic year. This paper will focus on three students, Rob, Terry, and Todd.

Data Collection

The research plan consisted of three main data gathering parts: (1) conducting semi-structured task-based interviews (Goldin, 2000), (2) participant-observation in the Calculus/Physics course, and (3) obtaining copies of students' in-class notes, in-class activities, homework assignments, and examinations. The data was gathered in order to collect information about how the eight students were using physics as they worked through calculus problems. The interview tasks were designed to elicit information about how the students used physics to help them solve calculus problems presented in various contexts. Some of the interview tasks were presented to the students without a physical or other context. For example, students were given a graph of a generic function, f(x) and asked to sketch the graph of the derivative f'(x). The intention behind such problems was to determine if students might talk about a 'generic' problem in a physical or other context. Other interview tasks asked students to find such quantities as average velocity, change in momentum, and gravitational force.

The classroom observations focused on the language used by both the instructor and the Calculus/Physics students. In particular, I was interested in the extent to which the students invoked physics examples and language as they discussed abstract mathematical topics. Finally, the students' work was collected to gather more evidence about how the students were using physics to help them solve calculus problems.

Data Analysis

In the tradition of Stake (1995) who claims that, "There is no particular moment when data analysis begins" (pp. 71), data analysis was an ongoing process throughout this study. The major source of data in this study was the student interviews. The audiotapes of the students' interviews were completely transcribed and checked for accuracy throughout the study. Initial analyses of the interviews were used to inform future interview questions and help focus the classroom observations. Observation notes and notes taken during the clinical interviews were also transcribed.

Three main types of qualitative data analysis were employed in this study: micro-analysis (Strauss & Corbin, 1998), within- and cross-case analyses, and triangulation of data. During the micro-analytic phase of analysis, it was my goal to generate provisional hypotheses concerning students' uses of physics in their conceptualizations of calculus concepts. The result of the microanalysis was the development of a scheme that classified each student according to his/her use of physics. The classifications are listed in Table 1 with a short description of each. The Language-Mixer category had not emerged at this stage in the data analysis; rather it was during the within-case analysis that the Language-Mixer category surfaced.

The stability of the classifications was tested by means of a detailed within-case analysis of each student. Episodes in the interview transcripts were first coded for evidence of physics use. Then a second layer of analysis was conducted, re-coding the physics episodes according to the Physics Use Classification scheme. During the analyses of the individual students, a new category of physics use emerged: Language-Mixers. This code was developed to account for data that did not fit into any of the other categories. Students were classified as a Contextualizer, Example-User, Language-Mixer, or Non-User based on the Physics Use Classification code that appeared most often. For example, if the majority of episodes for a student were coded as "Contextualizer" then the student was classified as a Contextualizer.

Three independent researchers re-coded selected portions of the data. The independent coders agreed with the original codes 89% of the time for the first layer of coding. The independent coders agreed with the original codes 91% of the time for the second layer of coding.

Results and Discussion

As a result of the microanalysis, four categories emerged and were refined during the other stages of analysis. The emergent classification categories are Contextualizers, Example-Users, Language-Mixers, and Non-Users (see Table 1).

These classifications refer to the manner in which the students use physics concepts as tools to aid in their conceptualization of calculus concepts. The Physics Use Classifications emerged because I observed marked differences in the way students were using physics as a tool to help them solve calculus problems. The classifications are not hierarchical or necessarily disjoint. Rather the classifications are used as a way to describe student behavior.

The examples that follow serve to highlight differences between the Contextualizer, Language-Mixer, and Example-User categories. The Non-User is not addressed in this discussion since the Non-User does not employ physics as a tool to aid in conceptualizing calculus concepts.

Average Rate of Change Tasks

Students were asked to compute the average rate of change between different pairs of x-values on the graph given in Figure 1. Most students solved the problem by mapping out y-values that corresponded to the given x-values and then finding the difference in y-values and dividing by the difference in x-values. The students' explanations of this process and their thinking differed substantially.

Rob was classified as a Contextualizer because of his frequent tendency to discuss calculus problems in a physical context. For example, Rob offered the following explanation for his answer to finding the average rate of change between x = -1 and x = 2.

Table 1. Physics Use Classification Scheme

Physics Use	Description
Contextualizer	Student works and talks through calculus problems as if it were a physics problem. Majority of technical vocabulary used to solve problem is physics terminology. There is evidence that student is thinking about the problem in terms of physics.
Example-User	Student uses physics examples to justify solutions to problems or to help make sense of part of the problem. Actual problem at hand is solved using mathematical concepts. Student does not submerge the problem in a physics context. Majority of technical vocabulary is mathematical terminology.
Mis-User	Student's use of physics misconceptions interferes with student's solution to the problem. Student uses physics misconception to incorrectly solve the problem at hand.
Language-Mixer	Student intersperses physics and calculus terminology as he/she solves problem. Student does not immerse problem in physics context or use physics examples to justify solutions or help make sense of problem. Rather, student intermingles physics and mathematical language as he/she solves the problem.
Non-User	Student does not use physics concepts to language to solve calculus problems.

> One of these axes is distance, the y axis...They use the distance so it [the y-values] would be 4 and 1. And what you basically do is, I think, just like a slope...it would be the change in distance, so 4 – 1 would give you 3. And put that over the change in...x. Which would be 2 and 1, so 2 – 1. That would give you 1. So it's 3 over 1.

Although he was presented with unlabeled axes, Rob decided early on that the vertical axis represented distance. Rob's comment, "They use the distance..." indicates that he was reading the problem as if it were a physics problem. Although Rob mentioned slope, it was immediately followed up by an expression about distance. Rob's interpretation of the problem as asking about distance traveled indicates that he was immersing the mathematics in a physical context.

Terry was classified as a Language-Mixer since he typically infused mathematical language and physics terminology as he discussed his solutions to calculus problems.

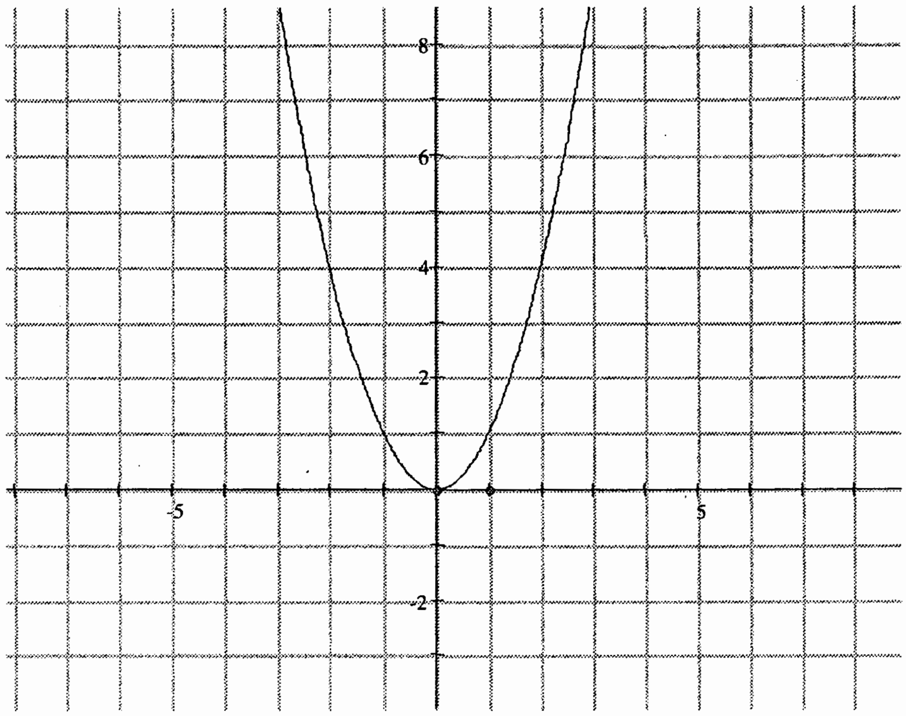

Figure 1. Students were asked to compute average rates of change between various points on this graph.

Terry's solution to the same average rate of change problem offers a different perspective:

> Between x_1 and x_2, their position, so that's position (1, 1) and at x_2 it's (2, 4). If I remember right, it's rise over run. And so it should be 4 – 1 over 2 – 1, which would be 3 over 1. So it would be 3.

Terry's used the word 'position' here to pinpoint the location of the coordinates (1, 1) and (2, 4). Terry's inclusion of ordered pairs in his discussion indicates that he was using the word 'position' to refer to the placement of the ordered pairs in the Cartesian plane, rather than as a distance. Terry used the mathematical 'rise over run' to indicate that the average rate of change is calculated by subtracting the difference of the y-values (rise) and dividing by the difference of the x-values (run).

Todd was classified as an Example-User because he often talked about physics examples to justify his answers. Todd infrequently used physics concepts or examples to help him as he solved a calculus problem. Todd was asked to compute the average rate of change between x = 1 and x = 2 on the graph shown in Figure 1. Todd solved the problem by calculating the difference in y values and dividing by the difference in x values and then added the following commentary when he was asked to discuss how he could justify his answer of 3:

> Well, for me that 3 means that it's increased its speed or position – whatever this graph is – by 3. So maybe it's moved 3 spaces forward on a checker-

board or maybe it's a car going 3 miles per hour faster. It's just something that's increased by 3.

Todd used ideas from physics and from his everyday life experiences not to get a handle on his numeric answer. Todd did not use physics concepts or vocabulary to solve the problem; rather he employed physics concepts as a way to make sense of the result of his mathematical procedures.

Rob, Terry, and Todd used physics as tools in different ways as they solved an average rate of change problem. Rob used physics as a way to trigger or motivate the mathematical tasks that he used to solve the problem. Terry used physics as part of his vocabulary and Todd used physics as a means to make sense of his mathematical solution.

Sketching Derivatives and Antiderivatives

Students were given graphs of functions, derivative functions, and antiderivative functions and asked to sketch related graphs. As observed in the Average Rate of Change tasks, the students' explanations of their solution processes differed significantly.

As I met with Rob throughout the year, I noticed that he, more than any of the other students, tended to talk about most of the problems in terms of physics concepts. For example, Rob was presented with the graph of f(x) in Figure 2 and asked to sketch a graph of f'(x). Rob explained his thoughts on the problem:

> ...and you'd assume that this is, in terms of physics, this would be position. And you'd be trying to solve what's happening with the velocity. So, position – it seems to be increasing and it's falling, well slowly. Like in terms of t, it's increasing, it seems to be slowing down and then it turns around and goes backwards.... So I want to show that in my graph.

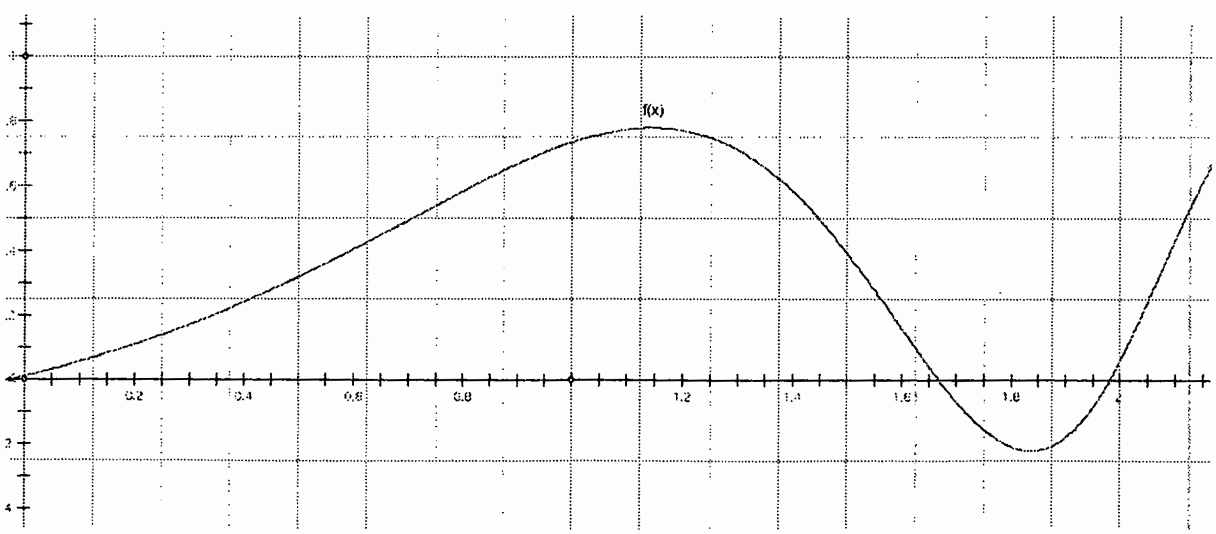

Figure 2. Students were given this graph of f(x) and asked to graph the derivative, f'(x).

Rob went on to talk about a specific physical situation as he tried to make sense of the graph around its maximum value.

> *But to me it seems that - if it was a ball that you pushed across a table and it was a distinct v[elocity]...distance is decreasing and then it stops, turns around, goes backwards. Time is still going.*

Rob also labeled the axes of the graph as distance (y-axis) and time (x-axis) and he also labeled the axes of his solution graph as velocity (y-axis) and time (x-axis). Rob used the axes labels to help him make sense of the physical situation. Rob remembered his experiences describing and analyzing the motion of a ball rolling on a table and used those memories as tools to help him solve this specific calculus task. Rob talked about the problem strictly in terms of velocity and position – a physical representation.

Terry, on the other hand, did not show evidence of using physics concepts as tools to solve a similar problem. Recall that Terry was classified as a Language-Mixer; he often interspersed mathematics and physics vocabulary as he explained his solutions to problems. Terry was asked to sketch the antiderivative of the function in Figure 3. As he 'read' the graph from left to right, Terry said, "*It's got positive slope and it's zero, and it goes to another positive slope. And it goes faster and slower...And over the interval [before the graph crosses the horizontal axis] is increasing and then it is decreasing.*"

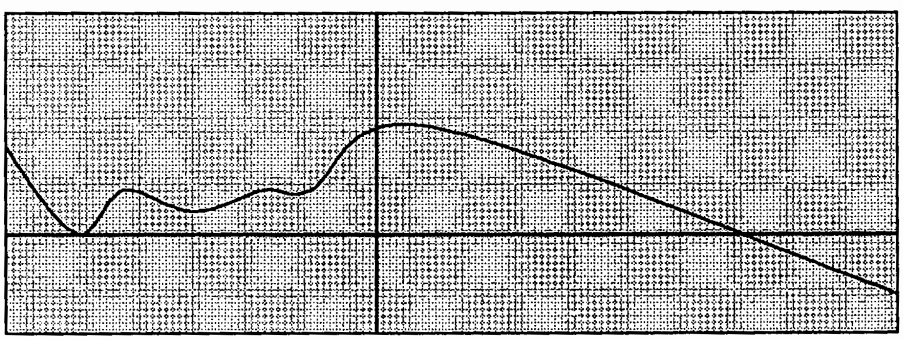

Figure 3. Students were asked to sketch the antiderivative of the function shown above.

Terry combined mathematical vocabulary (positive slope, increasing, decreasing) and physics descriptors (faster and slower) as he solved the problem. Terry first talked about the slope of the graph, then using the physical descriptions 'faster and slower' to describe motion, and finished his discussion talking about intervals of the function increasing and decreasing. Terry did not show evidence of using his knowledge of physics concepts to help him solve this problem. Rather, Terry's use of physics was limited to the use of some physics terminology to help him describe the graph. Terry appeared to use mathematical properties of the graph, such as intervals that the function was increasing and decreasing, to help him solve the problem. Such mathematical vocabulary was absent from Rob's discussion of the problem, indicating that Rob did not depend on the mathematical properties of the graph, but rather the physical properties of the graph, to help him solve the problem.

Todd did not use physics in any way as he solved similar derivative and antiderivative tasks. When presented with the task in Figure 2, Todd first talked about his general method for dealing with such tasks.

> So the derivative is basically the slope of a tangent line at a point. So I'm going to take a few points, plot where the tangent line – the numerical value – is, and then do a basic graph from there. So first off I'm going to find the points where the slope equals zero.

Todd did not employ any tools from physics to help him solve the problem or discuss his solution. Rather, Todd used his definition of derivative as a tool to solve this problem. Todd defined the derivative as the slope of the tangent line and he quite literally solved the problem using this definition. Todd's strategy for graphing the derivative was to choose points on the graph of the function, estimate the slope of the tangent lines to those points, and then plot those estimates on a graph of the derivative. Markings on Todd's examinations and homework assignments indicated that he employed this strategy regularly throughout the year.

Rob, Terry, and Todd used physics as a tool in different ways to solve derivative and antiderivative problems. Rob immersed the calculus problem in a physical context and solved the problem based on his memories of experiencing physical phenomena and his understanding of physics concepts. Terry used physics vocabulary as a tool to help him communicate his method of solving the calculus problem. Todd did not appear to use physics in any way to help him make sense of, solve, or communicate his solution to the calculus problem.

Conclusion

The Physics Use Classification Scheme emerged from the analysis of the data and was developed to describe the major ways in which students use physics as tools to help them understand calculus concepts. Rob, who was classified as a Contextualizer, immersed the calculus problems in a physical context in order to solve them. He used his memories of experiencing physical phenomena as a starting point for which to solve calculus problems. Terry used physics vocabulary as a tool to help him communicate his ideas about calculus concepts. Todd sometimes used physics as a way to make sense of his answers to calculus problems.

Rob, Terry, and Todd relied on their understanding of physics in different ways as they solved calculus problems. Rob, Terry, and Todd each exhibited a different level of commitment to using physics to help him solve calculus problems. The Physics Use Classification Scheme offers an organized way to talk about a student's level of commitment to using physics as a tool for understanding calculus. A natural question arises as to the benefits of being labeled in one category versus another.

Future research should begin to address this sort of question by further examining and comparing the level of calculus understanding of students in each of the Physics Use Classifications.

Note

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VISUALIZING AND UNDERSTANDING VARIATION

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This work presents an exploration on the mediation role played by the TI-92 calculator. Our interest is located at the concept of variation studied by means of algebraic and, mainly, dynamic representations. In this respect, the idea of visualization plays an instrumental role in the paper. We distinguish two levels of visualization: perceptual and structural, which function as tools to understand the variation phenomena set in a geometric context. We present fragments of a longitudinal study on variation and visualization. The cognitive behavior of the students, when working with the tasks, put forward the transformation of computational tools into conceptual instruments.

Introduction

We will introduce some results from a longitudinal study on visualization and variation mediated by the TI-92 algebraic calculator. Our experimental results come from questionnaires and interviews with 15-17 olds, in which it was observed that students have deep difficulties when working with the algebraic and the visual representation of variation.

To measure the impact of these computational technologies *as tools for learning*, it is important to identify difficulties students have, when they come to school, and the ones they have when they leave. In our schools, students have their first contact with the idea of variation at about 12-13 years of age. This happens in contexts involving proportional variation using tabular and graphical representations.

Conceptual Framework

Computing environments provide settings to study the evolving conceptions of students and teachers as they use the tools of the environment. For instance, our students refer to mathematics as a set of symbolic expressions. Accordingly, knowledge of mathematics means being able to use procedures to transform a symbolic expression into another symbolic expression. Graphing tools may produce then, a shift of attention from symbolic expressions to graphic representations.

At first, *mediation* of the calculator has an amplification effect upon the student's cognition. It becomes fundamental to understand the nature of knowledge of students that emerges from their interactions with those mediating tools. This interaction provides additional *expressive power* to her/his ideas and, eventually, an enhanced computing capacity. At occasions, this amplified capacity makes evident for the student her/his conceptual misunderstandings by offering a richer vision of a certain mathematical phenomenon (Berger, 1998). Working with computational tools in school media leads us to face the work from two different angles: as *amplifying* tools and as *cognitive reconceptualizing* tools. These amplification and reconceptualization processes can be illustrated in the following way: The amplification process is similar to the function of a magnifying glass. Through this lens, we can enlarge objects visible at first sight. Magnification does not change the structure of the objects that are being observed, however, on the other hand, the reorganization process can be compared to the act of seeing through a microscope. The microscope allows us to observe what is not visible at first sight and, therefore, to enter a new plane of reality. In this way, the possibility of studying something new and of accessing new knowledge arises.

We distinguish between *to see* as "recognizing the content in a graphical representation" (this concerns *amplification*) and *to see* understood as "recognizing structural relationships in a graphical representation" (this concerns *re-conceptualizing*). The latter meaning is what we call *to visualize*.

Perceptual variation is linked with physical operations and *representational variation* is linked with symbolic operations.

In our experimental work, the calculator played the role of a probe to explore students' cognition. The calculator allowed us to (partially) penetrate into the students' mind looking for information on their mathematical thinking.

Experimental Results

Experience One

At the beginning, we designed five questionnaires trying to identify the notion of variation by means of algebraic and graphical expressions. The questionnaires were applied to a population of about 400 students (15-17 olds). Just to give a flavor of this work, let us introduce a questionnaire designed to explore variable magnitudes using algebraic expressions.

Two students, Maria (M) and Juan (J) go to school. Maria walks in such a way that her distance from school can be represented by $M(t)= t^2 - 4t + 4$ (she begins walking form home); Juan's distance from school can be represented by $J(t)= 3t + 4$ (walking from home too). The variable t obviously denotes time (in minutes).

The following questions were asked:

1) What is Juan's (or Maria's) distance from school when he begins walking?
2) Is there an instant when Juan's distance from home is 6 meters? In that case what how long is the time spent?

From the first level (15 olds), 99 % of the students were unable to answer correctly any of the questions. 77 % of the students from the third level (17 olds) gave partial correct answers or could answer correctly one of the questions. A general conclusion that can be extracted from this work, is that students have serious difficulties dealing with algebraic models. They find it hard to give meaning to this kind of representations.

Experience Two

In this section, we introduce a (videotaped) task aimed at examining the mediating role (Chassapis, 1999) of the calculator in the expression of knowledge that students constructed during geometric activities. We were probing the cognitive activity of students, to try to discover ways in which object manipulation help students understand the invariant properties of a geometric object. This is a key task when researching variation.

We suggest that the students' expressions of coherent and meaningful mathematical propositions prove that they have constructed the relevant relationships or, in other words, the structural links that constitute a figure (Moreno & Block, 2002).

Looking at Figure 1, the teacher asks the students to do the following:

Teacher: Drag point B to the right and then to the left along the arc, but look carefully and try to answer these questions:

1) Does angle B change when you move point B along the arc?
   Take point A and drag it to the left and to the right. Does angle B change?
2) Take point C and drag it to the left and to the right. Does angle B change?

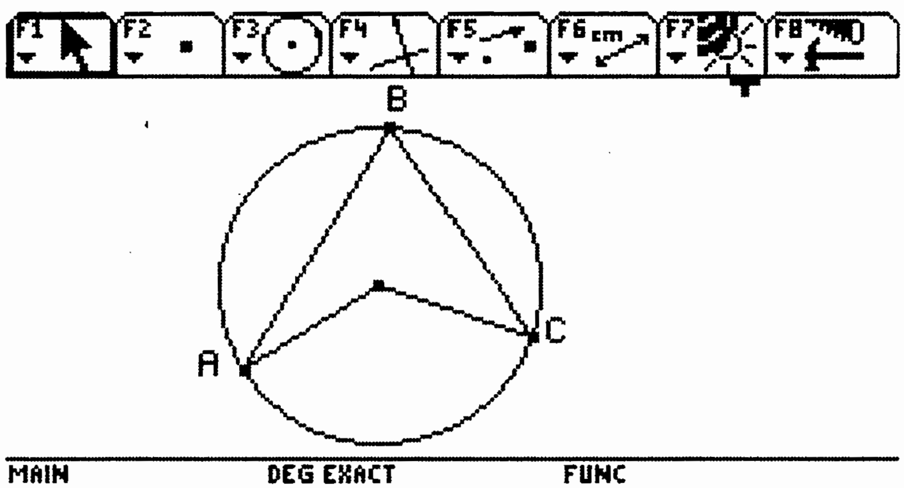

Figure 1

Before the students started to complete the task, the instructor questioned them about their previous knowledge on the subject. Only one student knew that the angle B, in Figure 1, remains constant as long as one does not move points A or C. Interestingly, the students did not know how to explain this behavior. None of the participants had met the central angle theorem or that a triangle inscribed in a semicircle is always right. The following are some of the participants' answers taken from the task sessions:

Felipe: It looks as if the angle doesn't change even though point B is moving!

Manuel: Let me see, I can't see... maybe...

Felipe and Manuel are talking about Figure 1, while dragging point B to the left and to the right in their calculator screen.

Teacher: Do you think that the angle will change?

They both answered yes, and this is exactly what the rest of the group was expecting.

Teacher: (addressing the entire group) Observe what is changing and what is not changing, and try to keep on doing the task. If you find something interesting for you don't hesitate to tell me.

The participants worked for 20 minutes on this task. Then the teacher proposed the following construction and asked the corresponding questions:

> Draw the segments from A and C to the center O of the circle (in Figure 1). Drag point A to the left and to the right; observe angle B as well as the angle formed by the segments that connect points A and C to the center O of the circle (central angle). Repeat the operation with point C.
>
> Move point A or point C until they are collinear with O, the center of the circle. How is angle B changed?

Felipe and Manuel, moved point A until it was collinear with point C and with point O. Finally, they moved point B, showing the teacher what they were doing at every moment. The rest of the participants observed what Felipe and Manuel had found.

Felipe: It looks like when the angle in the middle is 180°, angle B is 90°!

Teacher: Why are you saying so?

Manuel: We have already tried it, and it seems that way. Look!

After 20 more minutes, nobody could further expand the argument about the rightness of the angle B. Then the teacher proposed the students to measure and label the central angle and angle B. After 15 minutes, the students called the teacher and showed him a table in a notebook, one column showing the values for angle B and the other showing the values for the central angle.

Felipe: One column is almost twice as large as the other!

Teacher: How can you express what you found?

Felipe: The center angle is two times greater than the other.

Teacher: Just like that?

Felipe: Ah ... Within a circle the center angle is two times greater than the other.

The aim of these teaching sessions is to study how students express their arguments from the exploration of the different elements in the figures provided. Of course, exploration and expression are possible, in enhanced ways, because of the *executable* representations provided by the software (Cabri-Geometry, in this case). The central angle theorem is a conceptual tool to link circles, rays, radii and tangents, to create a *local organization* (Moreno, 1996) from a fragment of geometric knowledge.

ExperienceThree

Let us discuss another (videotaped) task posed to students from third level (17 olds).

Given a circle with center *O* and radii *OA* and *OB*.

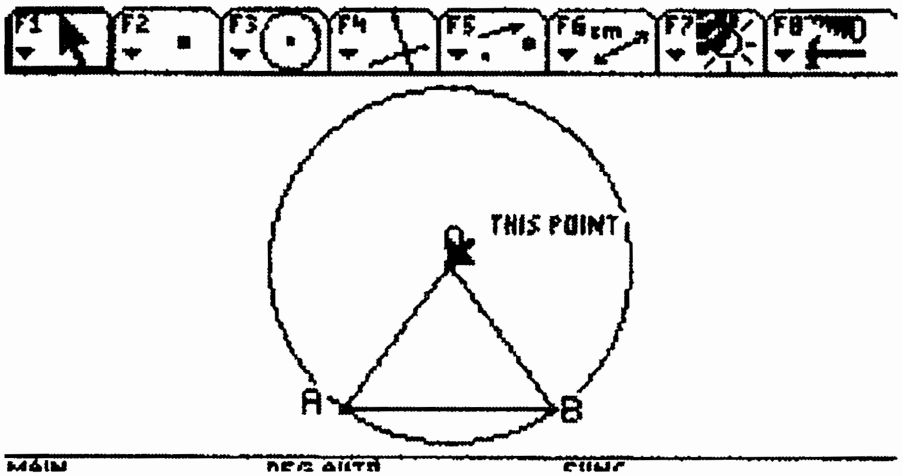

Figure 2

Consider the triangle *OAB*. Keeping *O* and *A* fixed, move *B* on the circle. The instructor asks if moving *B* changes the area of triangle) *AB*. Jacobo's and Luis' answer (before seeing the dynamic process made feasible by the calculator) is: "The area is changing." Afterwards, they arrive at the conclusion that the largest area is obtained when "the triangle seems to be right." The instructor asked *what* were they looking at: "Inside the triangle" was the answer. Let us notice that this activity is far from trivial.

When the students *observed* that point *B* moves slowly on the circle Luis modified his first answer. Then he said: "the area remains the same." Jacobo did not agree with this new answer, but could not give an argument against it.

The instructor, trying to mobilize their ideas, asked if the triangle's perimeter was changing. Luis answered: " If the perimeter changes, then the area changes." Luis and Jacobo agreed: "the area is changing and the largest triangle must be right" sustaining this statement in that the change of area depends on the central angle. Luis said: The greater the angle *AOB*, the lesser the area.

The next moment, the instructor (I) asked:

Instructor: What happens to the area when the central angle *AOB* diminishes?

Luis: It gets larger

Instructor: But eventually the area gets smaller…

Luis: No.

These fragments of the dialogue shows that the students' perception (*to see,* first definition) is fairly correct, but when they have to produce an argument (trying to reach the level of *visualization*) their reasoning becomes confusing. Dragging the figures (or an element of a figure) on the screen generates a sequence of figures in real time. Manipulating this sequence discloses for the student a "mathematical behavior" of the figure that is tangible from the perception level. The student has the opportunity to integrate this behavior into a general image, a kind of general or generic picture. This lets the student to access to the visualization level, that is, recognizing structural relationships in a graphical representation. From the dialogue above we can appreciate that the students answers vary as they explore the figure with the help of the dragging capability of the dynamic geometry software. Even if their answers do not show a full understanding of the mathematics involved, they were able to overcome the plain perceptual level (reading the figure at a superficial level). They found a key structural

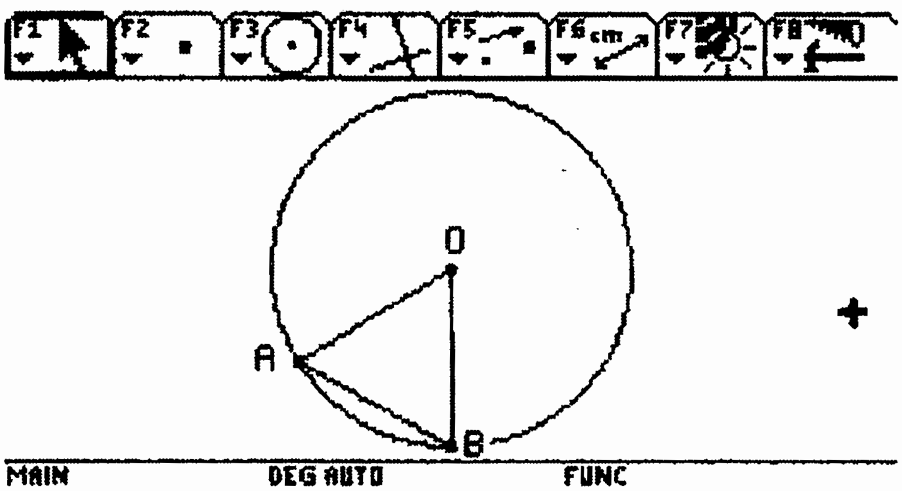

Figure 3

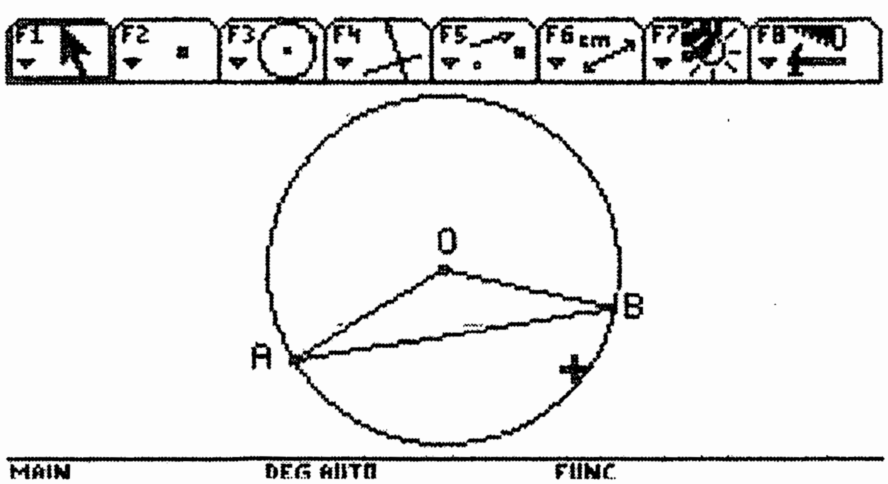

Figure 4

relationship hidden (to them) in the figure: The role of the central angle, as illustrated by the following two figures.

When the point B moves to the right, then the central angle at o increases. Increasing the angle changes the area of triangle AOB. What is specific of a drawing is incorporated into a generic image. Thus, visualizing is a generalization process.

Articulating the sequence of figures into a whole, makes feasible a new reading of the figures involved. The student can see changes in the sequence (the area, for instance) that belongs to the sequence, that is, to the generic image. This technology provides a new representational infrastructure with high potential for geometry teaching and learning.

Using technology should provide students richer representational infrastructures and their corresponding processing. This should open a window of opportunities to explore mathematical situations posed to them.

Final Remarks

The use of calculators, in particular, algebraic calculators, is important to promote a mathematical way of thinking that is consistent with the practice of the discipline.

> Electronic technologies –calculators and computers- are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, facilitate organizing and analyzing data, and compute efficiently and accurately. They can support investigation by students in every area of mathematics.... When technological tools are available, students can focus on decision-making, reflection, reasoning, and problem solving. (NCTM, 2000, p.24)

When students make use of technology in their learning experiences, it becomes important to document methods and strategies that appear as fundamental in their mathematical thinking. We should be able, eventually, to answer questions that include:

What kind of mathematical thinking can be enhanced with the use of technology while learning the discipline?

Students' growing familiarization with computational tools allows these tools to be transformed into mathematical *instruments* in the sense that computational resources are gradually incorporated into the student's activity. We suggest then, that exploring with computational tools eventually allows students to realize how the mediational role of these tools helps them to re-organize their problem-solving strategies (Guin, D. & Trouche, L. 1999). Working with the virtual versions of mathematical objects provided by the algebraic calculator, promotes the constructive activity of students. Indeed, these virtual versions produce the sensation of material existence, given the possibility of changing them where they exist, that is, on the screen. We suggest, then, that exploring with computational tools eventually allows students to realize how the mediation role of these tools helps them reorganize their problem-solving strategies. In the absence of these, it is almost not feasible for students to establish a conjecture, nor are they capable of producing a formulation associated with their explorations and express it in the language of the computational medium in which they are working. The computing environment is an *abstraction domain* (Noss & Hoyles, 1996), which can be understood as a scenario in which students can make it possible for their informal ideas to begin coordinating with their more formalized ideas on a subject.

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SEEING U.S. REFORM TEACHING IN A JAPANESE CLASSROOM: MUTUAL ADAPTATIONS IN FIRST GRADERS' LEARNING TEENS ADDITIONS

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In order for a mathematics classroom to be truly effective, allowing students to invent their own methods is not enough. Students can also be supported to learn more effective methods while their understanding is developed further. Using a Japanese Grade 1 classroom as an example, we illustrate how students' ideas are valued, how reflection and analysis of different methods deepen student understanding when connections are made among different methods, and how students gradually develop ownership of the culturally-valued method in a sense-making environment. The study found that through the three phases in the unit of addition with totals in the teens, students shifted from using various methods based on their prior knowledge of embedded numbers to making tens to add numbers.

Perspectives

The recent U.S. reform movements propose a new vision of mathematics classrooms where individual students' ideas, interests, and experiences are valued while classrooms as a community develop their understanding of mathematics together (National Council of Teachers of Mathematics, 1999). This sometimes gives a false image to teachers that while students are inventing their own strategies and constructing understandings, teachers should not teach at all (Fuson, in-press). This perceived dichotomy confuses not only teachers about the potential effectiveness of instruction but also students about the expectations they should have for their learning. We propose that in order to establish a truly effective classroom environment, we need both --- student-generated ideas and good teaching. When students' ideas are valued and connected to culturally-valued ideas through classroom activities in a sense-making environment, we are teaching for understanding.

The Vygotskian view of learning focuses on the ways in which the child's spontaneous informal knowledge is bridged to formal knowledge that is valued in the community through interactions with a more-experienced partner (Vygotsky, 1978). The ideas exchanged between the child and the partner are gradually internalized by the child while he/she moves along his/her learning path, and the mutual engagement of the child and the partner creates the necessary support for the learning. This is not a passive process in which the child "receives" new knowledge, but rather is an interactive process in which both partners continuously assess and change the patterns of their interaction to make progress toward the goal. Rogoff (1990) calls this "interdependent goal-directed activity" where both partners work together to reach a certain goal. Through guided participation in the inter-dependent and goal-directed activity, a shift occurs where the ownership of the activity is gradually transferred from the partner to the child, and the child increasingly becomes a more active agent in his/her own learning.

In a classroom setting, multiple learners are making their negotiations simultaneously as they engage themselves in learning activities. Cobb and Yackel (1996) illustrate the "global shift in classroom mathematical practices" using the example of a group of second graders learning the base-ten system. While students shared individual solutions to the problems, the classroom as a community developed a taken-as-shared method, and individual students' engagement in mathematical activities with their teacher and peers generated the support for their learning. Thus, students actively contributed to the evolution of classroom mathematical practices as they reorganized their individual mathematical thinking, and at the same time, their participation in the activities provided opportunities and limitations in such cognitive reorganizations.

This study attempts to extend our existing knowledge of effective classroom teaching by examining the ways in which students' ideas and a classroom-taught method are connected, negotiated, and understood through classroom activities in a Japanese mathematics classroom. Japanese classrooms present many characteristics of U.S. reforms, and examining how learning takes place in an effective Japanese classroom may help us understand how learning takes place in mutually adaptive ways.

Methods

Twenty-five (25) Grade 1 students and their classroom teacher Mr. Otani participated in the study (the real name of the teacher is used because of his request; all student names are pseudonyms). The study was conducted at a full-day Japanese School in a suburb of a midwestern metropolitan city. The school is operated by the Japanese Ministry of Education and closely follows the Japanese National Course of Study. Administrators and teachers are sent directly from Japan through the ministry, and the instructional language is Japanese. The school houses approximately 200 students in 1^{st} through 9^{th} grades. These students typically come to the U. S. due to their fathers' work in the metropolitan area. The families who relocate under these circumstances typically stay in the United States for 2 to 5 years and go back to Japan afterwards. They are different from other ethnic minority groups in the United States who immigrate to stay. This Japanese community puts much effort into preserving the culture in their children's lives because they wish their children to have successful experiences when they go back to Japan. For that reason, maintaining Japanese ways of teaching and learning is considered to be important for the community.

Observation fieldnotes, videotapes, notes from interviews with Mr. Otani, and classroom artifacts (copies of worksheets, tests, quizzes, etc.) are the data sources for this study. The unit focusing on addition with totals in the teens was chosen to be the focus of the study because the focus of the unit—learning of the break-apart-to-make-ten (BAMT) method—is considered culturally important because it provides a foundation for students' learning of multi-digit addition and place value understanding in the future (e.g., for 9 + 4, take 1 from 4 and add to 9, add 10 and the 3 remaining in the 4 to make 13). The unit consisted of 11 lessons over a three-week period. The classroom was observed daily during these lessons.

While the classroom was observed, careful fieldnotes were taken to record (1) the general lesson procedures, (2) teaching-learning activities and their structures, (3) kinds of student participation in the activities, (4) students' engagement and reaction to the activities, (5) verbatim whole-class and small-group discussions (as many as possible of the latter), (6) questions and responses of Mr. Otani and of students, and (7) conversations among researcher, Mr. Otani, and students before, during, and after the class. The lessons were also videotaped to support and supplement the observation fieldnotes for the analysis of the foci stated above.

Data from the observation fieldnotes were analyzed to illustrate (1) how different student ideas emerged in the course of the instruction, (2) how the taken-as-shared classroom mathematical method was negotiated and developed in the classroom, and (3) how individual methods and classroom practices enabled and constrained each other as they guided students' learning.

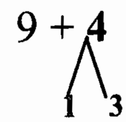
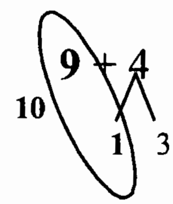
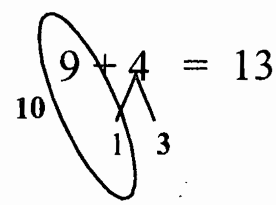
In order to identify the changes in the methods used in the classroom, fieldnotes were coded for the problem-solving steps students took in the whole-class context and individually in independent work as they learned to use the BAMT method for addition. Representational drawings, teacher's questions, and finger use that supported the learning were also identified. Special attention was paid to the methods students used, difficulties students had, and the adjustments Mr. Otani made to support students with these difficulties. Table 1 presents the steps (1), (2), (3), and (4) students took to solve addition problems using the BAMT method and the visual representational support in the textbook and used on the board by Mr. Otani. Fieldnotes were coded using the steps (1), (2), (3), (4) to illustrate general changes in students' use of the method in the classroom.

Results and Discussion

As with U.S. lessons, mathematics lessons taught in Japanese schools vary in terms of their structure and/or purpose. Some lessons are open-ended in nature, and students are encouraged to explore and investigate a significant mathematical problem on their own. Other lessons are more focused on skill building. Although the Japanese lessons that attracted U.S. attentions in the Third International Mathematics and Science Study (TIMSS) were largely open-ended and investigative (TIMSS, 1997), Stevenson and Stigler (1992) also reported that some Japanese lessons they had observed were mainly of repetitive practice and seatwork.

The present study found similar variation among lessons within the unit and that these different types of lessons worked together to support different phases of student

Table 1. The Break-Apart-To-Make-Ten (BAMT) Method (Example: 9+ 4)

Steps	Description of steps	Visual representational drawings
1	Realize that 9 needs 1 more to make 10	$9 + 4$
2	Separate 4 into 1 and 3	
3	Add 9 and 1 to make 10	
4	Add 10 and 3 to make 13	

learning. There were roughly three phases within the 11-lesson unit as students learned to add numbers with totals in the teens. The unit started with an open-ended question and student sharing of ideas, shifted to developing understandings of and skill with the BAMT method, and then focused on gaining fluency with the method. The shift was gradual and was supported by Mr. Otani through questions and continued sharing of different ideas. Students maintained the ownership of their learning through the unit. The balance between exploration of various methods and practicing a certain method appeared to be important in the classroom as student learning was guided to bridge from their spontaneous ideas to a more culturally-valued method.

For the first phase (lessons 1 and 2), classroom time was spent largely for sharing of different ideas of students. Students as well as Mr. Otani worked to understand what they had brought to the classroom learning setting. For the second phase (lessons 2 and 4), students worked to apply various student-generated methods to different addition situations to deepen their understanding of those methods guided by questions by Mr. Otani. Students reflected and analyzed the methods across different numbers. As they increasingly become aware of the benefit of using the BAMT method, they voted on that as the classroom method to be used. For the final third phase (lessons 4 to 11), students spent time gaining fluency with the BAMT method. This not only provided opportunities to develop skills but also strengthened understanding by students of each of the steps in the method. Students gradually gained ownership of the method as their own. The discussion of each phase follows.

Phase One: Sharing of Student Methods

From the very beginning of the first lesson, students' ideas and contributions drove the lessons to a considerable extent. All the contributions were recognized and appreciated by Mr. Otani and peers, and students actively shared and discussed different methods and ideas. Although Mr. Otani was the one who guided and assisted student discussions for a productive interaction, it was also very clear that students were important mathematical contributors and thinkers in the classroom.

For the initial introduction of the unit, Mr. Otani showed a group of 9 blue magnetic counters and a group of 4 red magnetic counters side by side on the blackboard. Some students immediately shouted out the answer "13!" Mr. Otani then initiated discussion by saying, "Some of you are quick in telling the answer, but who can share with the class your thinking?"

Several students raised hands to share their ideas. Sakiko went up to the board when called and moved one red counter to add to the blue group.

Sakiko: From 4, I add 1 to 9 and make 10.

Mr. O: So, the 9 became 10 and 4 became 3?

Sakiko: Then we know 10 and 3 make 13. We learned that before.

Mr. Otani summarized Sakiko's method on the board and continued to ask for other students' contributions to drive the discussion.

Mr. O: Anyone did this differently? 9 + 4? Nobuhiko?

Nobuhiko: 9 + 4 is … at first, 3 and 4 is 7.

Students: What? What are you saying? We don't understand! (Overlapping comments)

Mr. O: Will you say it again, Nobuhiko?

Nobuhiko: Took 3 from 9 … (moves 3 counters from 9).

Mr. O: Took 3 from 9?

Nobuhiko: Add that 3 and 4 … (put 3 and 4 counters together)

Mr. O: So, 9 is …

Nobuhiko: 9 became 6. Separate 9 into 3 and 6.

Mr. O: OK.

Students: Oh, that's what he meant. I know it now. (Overlapping comments).

Nobuhiko: Then, 7 … I mean …

Mr. O: 7?

Nobuhiko: Became 7. 4 and 3 became 7, so 7 and 6 is 13 (points to the counters)

Mr. O: 13. 7 and 6 make 13. You like 7 + 6, don't you?

Students: Nobuhiko has remembered those partners for a long time!

Mr. O: Is that so? (Friendly laughter)

Students: Yeah, he always does this way. This is Nobuhiko's secret method!

Students worked to understand and evaluate Nobuhiko's solution method as he made a contribution to the discussion. His seemingly incomprehensible idea at the beginning ("9 + 4 is … at first 3 and 4 is 7") was gradually clarified by guiding questions from Mr. Otani. As other students accepted his method as a valid mathematical approach, their ideas become connected. While Nobuhiko worked to articulate his idea to communicate with his classmates, he took time to explain one step at a time, and careful reflection on his own thinking helped develop mutual understanding in the community. Other students also actively evaluated the quality of Nobuhiko's explanation by demanding that he describe the process of his thinking with clarity, and by doing this, individual students in the classroom and Mr. Otani negotiated and tried to establish the shared criteria for what a good mathematical solution and explanations of such a solution should be.

Koichi then shared his method as he changed the addition situation into 5 + 5 + 3. Tadashi volunteered to show the class how he counted unitarily, from 1 to 13 to get the answer. Following his contribution, two students shared different counting methods: Tetsuhiro counted by 2s, and Tomoko counted by 3s. Although counting was not the official topic of the lesson, Mr. Otani allowed students time to explain, and he then summarized to the whole class how counting by 1s, 2s, and 3s were different from one another.

Mr. Otani then summarized the different approaches shared by students at this point. The break-apart-to-make-ten (BAMT) method was called the "Sakiko method," changing into 7 + 6 was called "Nobuhiko method," 5 + 5 + 3 was called "Koichi method," and the different counting methods and other recomposition methods that used different break-apart pairs of addends were also named after the students who shared those particular methods. The Sakiko method (BAMT) was the primary method to be taught in this unit according to the curriculum, but Mr. Otani spent approximately equal amounts of time explaining each method at this point by asking students questions.

Thus, at this beginning of the unit, Mr. Otani welcomed students' different ideas and allowed room for diverse methods. The sharing process was important at the beginning of the unit because it provided opportunities for students to review previously learned concepts, demonstrate their competence, and set the stage for future exploration. Student discussion was carefully directed by Mr. Otani to focus around the process of solving the problem, and that provided opportunities both for the students who already knew the answers and for those who were experiencing such large problems for the first time. The students' sharing of methods was the driving force of the discussion, and they spent time developing clear explanations to show their thinking with the visual representational supports on the board.

Phase Two: Applying Student-Generated Methods for Different Numbers and Developing Understanding of the BAMT Method

Next, Mr. Otani extended students' understanding by using the new addition situation of "8 + 5" (this problem requires a partner of 2 to make 10 with 8). He asked students to apply the methods they had come up with as a class to solve the new problem.

Mr. O: Who can explain again how to solve this (8 + 5) using the Sakiko method?

Sanshiro: We can separate 5 into 2 and 3, and make 10 with 8 and 2, then add 3 to make 13.

Mr. Otani summarized Sanshiro's explanation on the board using the counters and the visual representative drawings (see Table 1) while verbally reviewing the steps by asking students. Mr. Otani then continued to focus on giving careful descriptions for different methods using different representations (numbers, drawings, counters, etc.) as students shared their methods. Students were given opportunities to deeply think about, analyze, recognize, and discuss these methods.

Mr. O: Will anyone explain other methods?

Yutaro: Nobuhiko method was different. He moved 2 blue (counters) and added to 5 (red counters) and made 7 and 6.

Mr. O: (Moves counters as Yutaro suggested,) so we had 7 and 6. Any other way?

Koichi: Move 3 blue ones aside to make groups of 5, 5, and 3.

After Mr. Otani demonstrated the Koichi method with counters, he drew representations for each of the methods side by side on the board for comparison.

Koichi: Any way we did, we got 13!

Students: (Overlapping comments, they agree all the answers were 13.)

Mr. O: I see how 7 + 6 is 13, and 5 + 5 + 3 is 13, too. But I want you to look at Sakiko's method here. Why do you think she made 10?

Akemi: ... (Quietly,) maybe because we learned that before?

Mr. O: Let's listen to Akemi now. Will you say that louder?

Akemi: Because we learned that 10 and 3 is 13 in the first trimester.

Mr. O: So, Sakiko method may be the easiest because we have learned this before. Did we learn 7 + 6 before, too?

Students: (Overlapping comments) We did! We didn't!

Mr. O: I don't think we learned 7 + 6 in the classroom. So, I think making 10 may seem like the easiest method to us now because of what we know already.

By applying the BAMT method in different situations (9 + 4 and 8 + 5) while also making reference to other students' methods, the connections between the BAMT method and the ideas students brought with them were strengthened and their understanding of the BAMT method as well as the addition situation grew. Subtle but important negotiations occurred in their math discussions as they worked to find the "easiest" method to be used in the addition situations. When students voted for the method they thought the easiest to use in this addition situation, the Sakiko method of making ten received the most votes, and all other methods also got several votes.

Students also took time to think about and continued to discuss the reason why the BAMT would be the easiest method. Guided by Mr. Otani's questions, they explored the base-ten number structure and the importance of "tens" in our number system. Mr. Otani wrote the numbers 0 to 19 on the board in two rows (0 to 9, 10 to 19, with the teen numbers under the 0 to 9) and asked students if they noticed any pattern.

Mr. O: This may be something you had known even before you started first grade. Let's read this together (he points to the numerals as students read them).

Students: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.

Mr. O: Does anyone see a pattern?

Hiroshi: ... 10 is 1 and 0, 11 is 1 and 1.

Mr. O: What do you mean, Hiroshi?

Hiroshi: 10 is under 0 and is (written as) 1 and 0. 11 is under 1 and is (written as) 1 and 1.

Mr. O: Is everyone OK with this? It seems like Hiroshi noticed the way we write numbers?

Makiko: … We only use 9 numbers to write numbers, like 1, 2, 3, 4, 5, 6, 7, 8, 9, like that.

Students" (Overlapping comments) What is she saying? We don't understand!

Mr. O: … I think what Makiko is saying is that we use those same numbers repeatedly to write different numbers. Let's see how we do this. 1, 2, 3 … (point to numbers on the top row, 0 to 9, as students read along).

Students: 1, 2, 3, 4, 5, 6, 7, 8, 9 …10? (Seeing the numeral 9) 9? 10? (Discussion among themselves, some students seem confused)

Mr. O: Look up here. OK? We may take this for granted, but we have 10 numerals, 10 numbers, and all the numbers are written with these 10 numbers. So, … after 9, the next number is …

Students: 10!

Mr. O: 10, because after 9, we don't have a new number. And, we think 10 as a chunk, and put that chunk in the tens place … so making 10 as we add numbers make senses here.

The students actively made sense of the addition situations throughout the phases one and two. Mr. Otani valued diverse thinking of students by asking them to share different methods, maintained ownership of ideas by referring to the methods using the students' names, and encouraged critical thinking by having different students explain and apply the method in a new addition situation. Although voting in this initial lesson divided students into different camps, this voting for "the easiest method" continued on to subsequent lessons until on Day 4 all students agreed that BAMT method is the most useful method in the addition situation. This helped individual students to express their choices, and with Mr. Otani's gentle guidance concerning the generality of this method, students gradually made their own decisions for their learning as they came to experience the generality and effectiveness of the BAMT method and recognized the value of this culturally-important method.

Phase Three: Gaining Fluency With the Method

After the third lesson of the unit, the classroom learning focused increasingly upon the practice to gain fluency with the BAMT method. The general steps that are involved in solving an addition problem using the BAMT method are summarized in Table 1. The steps (1), (2), (3), and (4) specify the process students took to solve the problem using the method. Mr. Otani's questions for the whole class or for individuals elicited descriptions of steps of the methods and not just answers. Representational drawings accompanied the teacher's questions to help students visualize the steps they took to solve the problem.

The whole-class practice was the essential learning experience for the students. During it, Mr. Otani varied his levels of support to meet the learning needs of the whole-class and of individual students effectively. This whole-class practice was sometimes done as all the students shouted out answers together to Mr. Otani's question, or at other times individually as students responded to Mr. Otani's question while the whole-class supported the individual students by shouting out the whole-class feedback of "It is OK," or "It is not OK."

This shifting between whole-class and individuals-in-whole-class practice was done in a way to effectively support individual students' learning and engagement. As the classroom community worked to solve addition situations with 9s first (9 + n) and then moved on to 8s (8 + n), and 7s and 6s, Mr. Otani first used the whole-class practice where all students were encouraged to energetically call out answers together. In these activities, Mr. Otani typically had several problems written on the board, and as he pointed to one problem at a time, students energetically said answers together to the question about the steps of the method. Mr. Otani then moved to ask individual students the same questions as an extension of the previous whole-class exercise, while students who were not answering supported the responding student by giving the oral "OK" feedback. At the final stage, students worked independently or as pairs in their notebooks to solve problems. Mr. Otani varied this pattern sometimes. When he felt that students were not engaged enough in individual practice, he then switched to whole-class practice so that all students could participate together.

While the whole-class practice helped the students move forward as a group, the teacher individuated support when students gave answers individually. While the class engaged in individual practice in the whole-class context, Mr. Otani varied his support to meet individual student's needs.

The purpose of this last phase of the unit when lessons focused on practice of the BAMT method was not merely drilling for skill. It provided important learning opportunities and helped strengthen understanding of individual students at different levels. An analysis of the growth during the unit of six target students (Murata, 2002) found that individual students' movement differed from one another, and continuous classroom practice provided supports and opportunities for them to revisit the same concept many times when necessary. While students took time to practice and gained fluency with the BAMT method, the shift occurred that made the method become truly owned and understood by students, each through their own learning path.

Conclusion

The example of Japanese Grade 1 classroom illustrated a learning environment where students' ideas are valued, opportunities are provided to investigate different ideas and to make connections among them, and much time was given for students of different levels to develop and maintain the ownership of the culturally-valued method taught in the classroom. Although 8 out of 11 lessons were spent for the Phase

Three of the unit, lessons in the last phase often started with the student review of the BAMT method, sometimes with comparison with other methods, guided by the questions of Mr. Otani. Even after most students became familiar with the method, Mr. Otani always helped them review the steps of the method using multiple representations when new addition numbers were introduced in lessons. The unit began with students' sharing of ideas at the beginning, followed by reflection and analysis of different methods, and concluded by students' using and practicing the culturally-valued BAMT method as their own.

Acknowledgments

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PROBLEM POSING: WHAT CAN IT TELL US ABOUT STUDENTS' MATHEMATICAL UNDERSTANDING?

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Setting the Scene

In 2001, at Snowbird, Utah, members of the μ-group[i] from the University of British Columbia, under the guidance of Dr. Susan Pirie, engaged in an innovative, continuous session of linked papers which consider the notion and nature of mathematical understanding and reviewed a number of different theories and approaches to this phenomenon. We presented, discussed and contrasted a variety of key ideas in this area and considered their usefulness and applicability in analysing the mathematical actions of pupils working in school classrooms. We also presented in detail how we undertake our video analysis and for this, the reader is referred to the paper by Pirie in last year's proceedings entitled "Analysis, Lies and Video-tape." This year we intend to use a slightly differently structured session to enable even greater audience participation, especially by those who are unable to attend the entire session. We will still be show-casing the collaborative work of the teachers and mathematics educators who are members of the μ-group, but we will offer an interactive symposium space in which to discuss three papers and three posters, all presented together in one long session. Participants will be invited to examine the presented research, and to enter into dialogue with the authors both as a group and individually.

As part of a SSHRC[ii] funded project into the effects of Problem Posing on ways of understanding mathematics, μ-group members have taken a common mathematical prompt—"If the answer is 72, what is the question?—to a wide variety of students and explored the mathematical understanding revealed as the students grappled with posing problems from their own, individual, mathematical perspectives that fit the provided situation. Each researcher presented the problem as they saw fit to their students, given the students' level of mathematical knowledge and sophistication

Our session as a whole will encompass papers and posters presented in interactive ways by:

Susan Pirie, with a review of the group's theoretical position and methodological stance;

Katharine Borgen, whose work is with Grade 9 students tackling mathematics in English which is their second language;

Lionel LaCroix, who has taken the common prompt out of the classroom to an industrial-trades worker of 25 years experience of using mathematics in the workplace;

Donna Jenner, who has taught and studied Grade 2/3 students working individually and in small groups.

Shelley Saltzman, who has worked with a small group of Grade 4 students with learning difficulties in mathematics;

Shevy Levy, who approached pairs of students of a wide variety of ages;

Amy Bamford, who challenged her Grade 12, enriched curriculum class.

Lyndon Martin will act as a discussant for the session.

Theoretical Framework

Since 1987 Thomas Kieren and Susan Pirie have been involved in a collaborative research project to develop, test and refine a theory for the growth of mathematical understanding. Rather than think of understanding in terms of singular (or even multiple) acquisitions, or test for it merely by noticing differences in pre and post instructional behaviours, our theory allows one to observe growth of understanding as a dynamical, non-linear process. It offers a way to examine the cognitive activities of students as they 'come to know' mathematical concepts and therefore also offers the possibility for a fine-grained analysis of the problem-posing process, not so far undertaken.

The theory (Pirie & Kieren 1991, 1994a, 1994b, 1996) posits eight layers of understanding together with the cognitive activity of 'folding back' as crucial to growth of understanding. As illustrated in the figure, these layers each contain all previous understanding and are each contained by all outer more sophisticated understanding. A student's growth of understanding is traced out by a 'back and forth' pathway across the layers. The core of all understanding is one's Primitive Knowing, defined as everything one knows *except* knowledge already created on the particular topic under consideration, which is, of course, understanding at one of the other levels *for that topic*. Three of the inner layers are defined as Image Having, Property Noticing and Formalising. At the Image Having level, for any particular topic, students are observed to hold one or more mental image (often very specific, limited and context dependent), which they

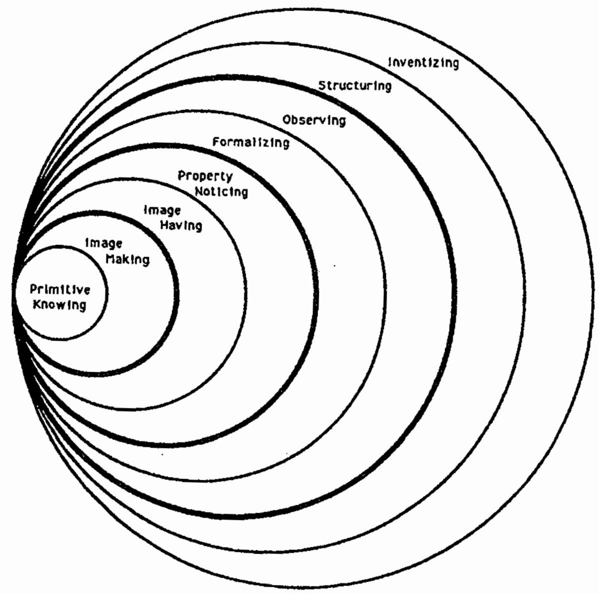

Figure 1. The Pirie-Kieren model for the dynamical growth of mathematical understanding.

are able to employ when working on mathematical tasks. At the Formalising level, students are seen to be working with generalisations of concepts, and they no longer need to relate back to the specific mathematical contexts that gave rise to their understanding. Between these two layers lies that of Property Noticing. Here students are engaged in the reflective activity of examining their images: making distinctions and connections between them. It is this level of cognitive activity that enables students to formalise their mathematical knowledge with comprehension, and it is this level that is so often ignored by advocates of discovery-based learning.

Problem Posing

In mathematics, "problem posing" refers to both the creation of questions in a mathematical context and to the reformulation, for solution, of ill-structured existing problems. Over the past two decades, problem posing has received only sporadic interest within the field of mathematics education, and this focused mainly on students writing story problems in a given context or with given data. Cudmore (2000) tends to show that the value of this work is limited in terms of growth of mathematical understanding. Research has also been largely focused at the elementary grades, and therefore involved comparatively unsophisticated mathematics and questioning. (See a review of the literature by Silver (1994) and more recently work published by Brody and Rosenfield (1996) and English (1997)). To pose mathematical questions at any level, however, involves more than being able to do the mathematics. It requires some understanding of the mathematical concepts involved - at the very least a feel for when a concept can be appropriately invoked. As a part of the SSHRC funded project, Pirie has developed an additional, somewhat more specific meaning for problem posing. We have asked students to pose, or create, mathematical questions in a specific topic area that they have studied, "like the ones you did in the text book" when all they are given is "the answer from the back of the book." We have hypothesised that learning to problem pose might also enhance students' learning for understanding. That is to say that, in coming to ask questions on mathematical concepts, students might come to understand those concepts in a more generalised, less context-dependent way.

Methodological Stance and Methods of Analysis

Since the aim of our research is to tease out the "specific structure of occurrences" (Erickson 1986) of problem posing, the methodological approach is an interpretative one, based in case studies and ethnographies, using participant observations, video-stimulated recall techniques, clinical interviews and analysis of students' written work. Since so little is known about the nature of problem posing as a cognitive activity, such a methodological approach allows us to view the process of data collection as "progressive problem solving, in which issues of sampling, hypothesis generation and hypothesis testing go hand in hand" (Erickson 1986). This approach enables decisions of which cases to focus on to be taken as the research progresses and we are building a "portfolio" of examples of growth of mathematical understanding that illustrate learners working at the Property Noticing level. Pirie's specific form of problem posing is proving very productive in providing evidence of work at this level.

Each of the researchers videoed their students "problem posing" and used multimedia techniques, mainly based around VPrism software, to assist in the analysis of the data from the perspective of the mathematical understanding revealed by the students through use of the "answer is 72" starting situation. As can be imagined, some very different and interesting features have emerged from the research and these will all be open for discussion at the session.

The Portfolio

There is clear evidence from our data (and that of many others) of a disconnected mental leap that many students are required to make as they move from having constructed some mental image(s) of a concept, to its formal representation, without having grasped the relationships and features of their images that enable this abstract generalisation of their concrete experiences. This transition between levels of understanding is not well understood by the mathematics education community. Action and discovery learning, while enabling students to form richer mental images for mathematical concepts, have failed to provide the link, that many students lack, between action-tied images and generalised formalism. Asking students to pose mathematical problems within a given topic, rather than to answer well constructed teacher/text questions, appears to provide a much needed, connective bridge to formal generalisation with understanding.

The specific form of problem posing that we are using currently was developed out of an incident where students were working from a standard text book, with questions in the topic of quadratic equations such as: Solve $5x^2 + 3x + 0.25 = 0$, which simply needs students to remember the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A question posed by one of the students in the class, however, - "Why does 3.5 'solve' that equation?" - led to a class discussion around the meaning of "solve" in this context and what the formula was actually 'doing'. From this Pirie constructed the question:

"If the page giving the exercise had been torn from your text book, but you still had the answers at the back of the book could you reconstruct the quadratic equations? For example, if for question (1) the solution was 3.5, what was the quadratic equation?"

This notion of giving students an answer in a specific mathematical context, and expecting them to create a "standard" question to fit it was then expanded to other topics. This is proving to be a very effective method of provoking learners to think at the Property Noticing level in order to connect the images for the topic that they have built up, with the formalisations that they have been using to answer the questions in the text book exercises.

The rest of the contributors to this μ-group presentation will illustrate, using a few of the examples in the portfolio, the widely differing ways in which a single prompt can provoke or reveal growth of understanding at a whole range of ability and topic levels.

PROBLEM POSING IN A SECOND LANGUAGE

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Mathematics has often been considered "language free." At the same time, problem solving and problem posing are considered appropriate means of determining students' conceptual understanding of mathematics. This article considers the working of four relatively capable mathematics students in Sweden, whose *primitive knowing* was developed in Swedish, who were asked to pose a problem in English. Results indicate that for these students, an obstacle to their ability to express their mathematical understanding was their need to concentrate on the correct grammatical presentation of their work. This distracted from their concentration on the mathematical presentation, possibly making them look less capable.

An individual's first language is the language through which he/she initially perceives the environment, through which first concepts are developed, and through which identity as a person is achieved (Pattanayak, 1986). While many consider that mathematics is an absolute with universal symbols, making language less important in the understanding of it than in other subjects, mathematics has a very precise language with subject specific meanings that are often combined with symbolic notation (Scholnick, 1988; Mestre, 1988; Lass, 1988; Kimball, 1990). The ability to express oneself mathematically requires logical and abstract thinking (van Heile, 1986).

Due to the considerable amount of immigration around the world, the language of communication in the classroom is often not the first language of the student, or even the language spoken in the home. While students who come from a home where English is the language of communication will be aware of, although probably unconsciously, the linguistic structures that affect meaning, this is not necessarily true for second language learners. Thus, while many second language students are placed in a regular mathematics class long before they are placed in a regular class where language is thought to be more significant, they may lack knowledge of the linguistic structure that gives meaning to the expressions encountered in the mathematics classroom (Cuevas, 1991), possibly placing them at a cognitive disadvantage.

Standards based mathematics promote sense making by being comprehensive and by developing ideas in depth in a coherent and motivating manner (Taftan, Ryes & Wasman, 2001). The 2000 National Council of Teachers of Mathematics (NCTM) Standards have stressed the importance of conceptual understanding in mathematics as a component of proficiency. This implies an understanding that is beyond the simple manipulation of symbols, allowing, and even requiring, students to express their understanding verbally. However, teachers often find that, while students may be able to perform the mathematical algorithms appropriately, they often struggle in their attempt to describe the mathematics they have done or to explain why they used a particular approach. This inability to verbalize their procedures may not necessarily indicate a lack of conceptual understanding, but rather, may be due to the abstractness and theoretical nature of many mathematical concepts (Miller, 1993).

When students work at mathematics in a language that is not their first language and which is not the language spoken at home, the ability to express their understanding is even more of a problem than for first language students because they often "think" the mathematics in their first language (Borgen, 1998). Thus, when second language students seems to perform less well than first language students, it may be that they simply lack the linguistic ability to express their cognitive understanding (MacGregor, 1993; Adler, 1995). In order to put the mathematics into context, they must, to borrow an expression from the Pirie and Kieren (1994), *fold back* to terminology and images formed and described in their first language, and may attempt to transpose and translate these to the new situation. Attempts at translations may not help as direct translation seldom, if ever, provides the same meaning as the original expression. Also, the students might not have encountered the relevant mathematical terminology in their first language. Translation may thus create disjoint connections between cognition and verbal expression, a discontinuity of ideas from one language to the other. So, when second language students are asked to express their mathematical understanding, even though they may have learned the requisite terminology in the second language, they may simply be unable to verbalize their thoughts because the foundational concepts are understood in their first language. Discussion of the new material cannot therefore take place in a coherent manner.

The emphasis on understanding being placed on mathematics by NCTM Standards has lead to an increased emphasis on problem solving. This, of necessity, requires understanding of the language and of the context of the question. The area of problem solving, therefore, is one in which second language learners often have difficulty. They thus often rely on patterning solutions based on problems previously done in class (Borgen, 1998).

Problem posing is often seen as a means by which students can display their level of understanding of a mathematical concept. In forming a question for which the answer is known, it is often assumed that students can display their understanding more clearly than if they are simply asked to solve a question.

This paper will consider a problem posing situation involving students for whom English is not their first language. I will consider the question they created and their solution to it, although the two do not agree, and discuss the effect that using a second language might have had in the problem-posing situation.

Answer is 72: What is the Question?

The Swedish Setting

The setting for this study was Umea, a medium sized city in the northern part of Sweden. Four grade nine students, Klara, Stine, Nathaniel and Maria, agreed to take part in a videotaped session. It is important to note that in Sweden, the use of English as a language for communication is promoted in many ways. In most schools, in the first grade, students learn songs, colours and some useful words. By grade four they start a more formal approach by learning to read and write in English (S.-M. Kuoppa, personal communication, 2002). Most television programs imported from English speaking countries are not dubbed, as is the norm in many European countries, but rather are presented in English with Swedish subtitles. And, although Sweden has a strong rock and pop culture of its own, not only are North American pop songs widely circulated and listened to, as far back as the early 1970's the well known Swedish rock group, ABBA, recorded in English as have, more recently, Ace of Base and Roxette, to name a few. Swedish Metal bands that record in English include Inflames, Children of Bottom, Opeth and many more. Thus, most Swedish people have been exposed to the English language on many levels, and, by the time students are in grade nine, they appear fluent in oral English.

The Problem Posing Session

When I met the four students involved in this session, I spoke to them briefly to ascertain their level of English and to explain what was planned during the session. As expected, their understanding of English, and their ability to express themselves in it created no major obstacle. I explained to them that I was from the University of British Columbia and that I was part of the MU Group, a research team of teachers, university students and professors who were interested in mathematical understanding and that we were presently working on a particular aspect of that understanding. Specifically, we were interested in something called "problem posing." Therefore, in this session, I would not give them a question, but rather, I would give them a solution and they would be asked to create a question that suited it. The catch was, however, that my particular interest was in mathematical understanding in a second language. Therefore, I was asking them to create and discuss the question in English as much as possible. However, if at any point they could not find the expression they needed in English, they should naturally speak in Swedish. I had made the assumption that since they had learned mathematics in Swedish their primitive knowing (Pirie & Kieren, 1994) would be in Swedish, that they might not know all the appropriate English terms, and that they might need to use each other as "sounding boards", knowing what they wanted to say (in Swedish) but not knowing how to express it (in English).

I had had a chance to peruse the textbook that the students had been using to determine what work they had covered during the year. Although I could not read Swedish, I was able to ascertain the gist of what they had done by the mathematical symbolism. I had also talked with the teacher who indicated that the most recently covered unit was a review of fractions. I therefore suggested to the students that they could use this as a start, or they could choose any area of mathematics where they thought they could come up with a suitable question. Thus, the students began their solution, or question making, by discussing a setting in which Ana and Alex buy some apples. They briefly considered a scenario in which one person buys more apples than another. Klara then suggested that they should come up with "the mathematic (sic)" first, and then make the "problem." The students briefly discussed the format the "equation" should take. (In transcriptions, **bolding** is used to indicate that words were stressed orally, --- for a pause, and … to indicate missing sections.)

Klara: *Well, if we should come up with like the mathematic, the mathematic problem first then it, then put it in the words.*

Klara: *Maybe we can, so if the answer is seventy-two. We can have like --- seventy-two something times something divided by something and the answer is something and then you have to count out what, what x is and x will be seventy-two.*

Maria: *Uh hu*

Klara: *Do you get it?*

Nathaniel: *Yeah*

Klara: *So, uh, x is seventy-two. OK, seventy-two, then we can have seventy-two times something that works. --- Seventy-two times --- Four maybe? Or four --- OK seventy-two then its x times four and that is two hundred and eighty-eight. Two hundred eighty-eight. That's divided by ---*

Stine: *Two hundred eighty eight*

Maria: *It could be like Ana buys*

Nathaniel: *Its divided in four you get seven --- You get --- no*

Klara: *No, but if you divide in four you get seventy-two. And that's the answer, but*

Nathaniel: *Yeah, I know, I know*

Klara: *Uhm if you divide it by --- maybe --- six you get forty-eight, so if it is x times four divided by six equals forty-eight, what is x?*

Maria: *Uh hu.*

Nathaniel: *Seventy-two*

The students thus formed their "equation" and went on to form a problem using this form as their solution. The students' decision regarding the format of this "equation", it seems, dictated their thinking about the problem, and they considered it as the solution. The actual problem they created did not agree with the solution they had predetermined. Specifically, the problem the students posed and their solution are presented in Figure 2.

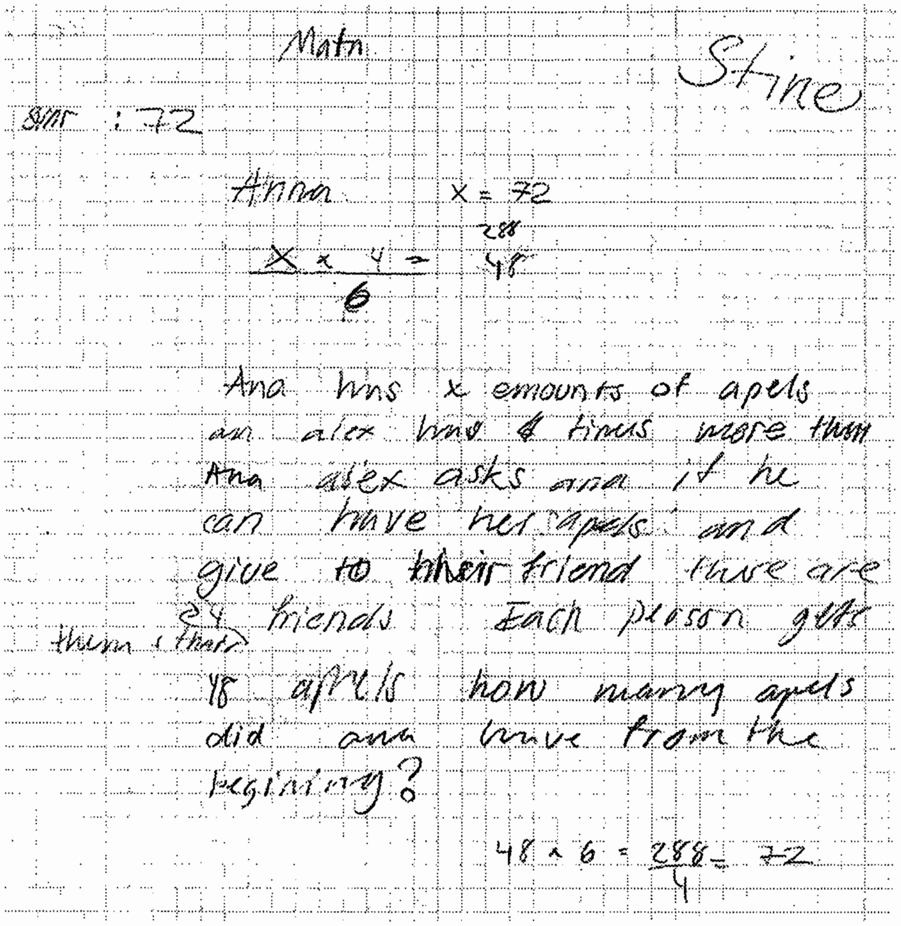

Figure 2. Problem posed by the students.

Rather than transcribe the entire discussion, in this next section I will state some of the inferences I made with respect to the students' performance of mathematics using English as their language of communication and will give the transcription of only those statements that prompted me to arrive at these inferences.

Inferences

The problem posed was a mere "copy" of problems from previous mathematics work. Little creativity went into formatting the question. The students had decided that they would form a "word" problem and discussed how to form it:

Nathaniel: *We gotta write a problem and solve.*

Klara: *Mmmm. Well should we like --- apples and that's x and that times something is seventy-two or shall we do like another ---"*

...

Stine: *We ca --- They always --- Ana and Steve always bla bla bla and umm umm umm are going to give us some apples. (At this, all the students laughed.)*

Later, they specifically indicated that they had simply formulated a question similar to those that they were accustomed to doing.

Nathaniel: We make like example from the book. They are all ---

Klara: They are always taking Ana. They have to divide apples or potatoes or whatever.

It is unclear whether they would have been more creative if they had been asked to form a question in Swedish. However, what is apparent that they simply used an existing model to format their ideas – that they felt that this was a suitable type of question to form in a mathematics class.

The students, it seems, needed a mathematical model and only then could they make a problem (albeit incorrect) around this "equation." As noted earlier, Klara had indicated that they come up with the mathematics first, and then created a situation in which the equation $4x/6=48$ applied. Following was a set of disjoint statements in which the students seemed to be trying to formulate their thoughts as to how the question could be worded.

Klara: ... But then you have to have like Ana has x amount of apples --- and uh

Nathaniel: Times five by four times more

Klara: No, like Alex has four times more than Ana

Nathaniel: How much?

Klara: Ans --- so maybe

Nathaniel: And then they

Klara: And then they ---

Stine: So they put them together and [She swore in English](laughter)

Nathaniel: Well, if we do like and they put them together

Klara: Ana. Put them together and divided by divided by six

Nathaniel by six persons

Klara: Yes, by six, by six persons

Nathaniel: By? And how, yeah, and how how many apples

Klara: Does Ana have, yeah

Few, if any, theoretical mathematics concepts were expressed during the problem posing session. The students simply tried to state a problem that suited their "equation."

Correct grammar seemed more important than correct mathematics. Once the format of both the equation and the problem were outlined, the students continued to discuss the actual situation, but concentrated more on the grammatical structure of the problem they were creating than on the mathematics involved, being sure that numbers and tenses agreed.

Klara: *So that's Alex asks Ana, asks Ana if he can have hers --- hers and divide them --- to her, and give --- give to his friends.*

Nathaniel: And give them to his friends

Klara: *And give to his friends*

Nathaniel: ***Six*** *friends*

Klara: *Or,* ***to*** *give to his friends. For hers to*

Nathaniel: *Hers apples*

Three girls: ***Her*** *apples*

Nathaniel: *Her, I mean*

Klara: *Give it to her his um ... And together Alex and his friends are --- Alex's* ***five*** *friends*

Maria: *Yeah. Are divided them*

Klara: *along if if five friends, then he gets some, too. Or maybe there are four friends and they divided by*

Maria: *by friends.*

Klara: *And by* ***their*** *friends, not her friends, not his friends, his friends, umm her friends,* ***their*** *friends.*

Nathaniel: *Their friends are four*

Klara: ***Their four*** *friends. Or maybe just like them and their friends --- them and their friends --- and their* ***four*** *friends*

She again corrected Nathaniel.

Nathaniel: *Alex asked Ana if he can have her apples and give to his four friends."*

Klara: And give to ***their****.*

In these latter two incidents, Klara seemed to be making sure that the grammar was appropriate to the numbers. Later she also corrected herself from *"did Ana had"* to *"did Ana have,"* showing a concentration on tense, thus the importance of grammar.

Once the "equation" was formed, the students considered it the solution to the problem they had posed without considering how they would actually do the problem had it been presented to them. No thought was given to the relationship between the problem (words) and the mathematics (equation). When asked to find the solution to the question they had created, Klara read the question (not verbatim).

Klara: *Ana has x amounts of apples and Alex has four times more than*

Ana. *And Alex asks Ana if he can have her apples and so they can divide them to* **their** *four friends also. And so then each person, when they have divided them, each um gets forty-eight apples. How many apples did Ana have from the beginning?--- So then you take forty-eight that everyone gets times six 'cause there are six persons. Then you get* [uses calculator] *two hundred and eighty-eight. Yeah. And two hundred and eighty-eight divided by four, 'cause you got Ana's ... four that's around*

Nathaniel: *Divided by four is. Divided by four is seventy-two. So we have the answer that is seventy-two*

Klara: *Ana has seventy-two apples from the beginning. Ta da*

During this conversation, all four wrote down the solution as they had discussed it in their posing situation. They did not realize that their working and solution did not match the problem that they had posed. Notice that Klara had to clarify again in her mind that there were six people, and stated the four *"'cause you got Ana's."* However, she did not identify the fact that Ana had x apples, and it was Alex who had four times more, or that, when they were put together, there would be $5x$ apples. Thus, the students did not really rework the problem, but simply used the numbers that they knew.

Again, it is unclear whether this was due to the use of a second language, forcing them to concentrate on the linguistic structure of the problem created, or because they already "knew" the process or equation through which they got the solution. This area of the research needs more study in other situations and should be compared to similar posing situations for first language students.

Discussion

The students involved in this discussion were described by their teacher as being mathematically able and seemed to have a good command of the second language, English, in which they were asked to pose a question. However, their *primitive knowing* (Pirie & Kieren, 1994) was constructed through learning in Swedish. Although they had been advised that they could use Swedish in their discussion if it helped them clarify their meaning, they did not feel the necessity to do so. It seems that there were no situations in which they felt that they could not express themselves adequately in English. They concentrated more on the language of presentation than on the mathematical presentation, seeming to want to be sure the problem was grammatically correct. The result was a solution that did not agree with the actual problem posed. It agreed with their original "equation" and did not consider the details they inserted into the problem, nor was the problem at all creative. It was simply a copy of a typical textbook problem. It may be that, since the students had been taught mathematics in Swedish, they had not encountered more complex mathematical terms in English, so that their ability to express themselves was limited to the terminology and concepts that they understood in English. It could also be that their concentration on the correct grammatical presentation deterred them from concentrating on the mathematics.

While one must be careful not to take the situation out of context or to generalize from one case, one must still consider the similarities between this situation and the situation of students new to a country where the language of the classroom is not the student's first language. Many of the students placed in an English speaking mathematics classroom in North America have a much poorer command of the English language than do these students. Learning to converse in a second language requires approximately two years of extensive exposure, while learning to communicate technically requires closer to seven years (Cummins, 1979). Thus, for second language learners, concentration on the correct use of language (conversation) may detract from their ability to concentrate on the mathematics involved (technical). Understanding mathematics involves an interplay of conceptualizations and verbalizations that must be meshed together to create meaning. If the languages of the two (conceptualization and verbalization) are different, the verbalization, the need to understand and be understood, and the need to express oneself as appropriately as possible, may take precedence over the conceptualization. In concentrating on the language, students may not be able to attend to the mathematics.

In a problem-posing situation, the linguistic structure of the problem being posed, as they are trying to make meaning of the language used as well as the mathematics involved, may be of greater significance to second language students than the mathematical structure of the problem and its solution. Therefore, problem posing may not be a suitable method for determining their level of mathematical understanding.

A TRADESMAN'S MATHEMATICS UNDERSTANDING IN A MACHINE SHOP: INSIGHTS PROVIDED BY THE PIRIE-KIEREN THEORY

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This case study examines the understanding of ratio used on the job by a successful tradesman as he designed and built a milling machine gear mechanism. The tradesman had limited formal mathematics and technical training and in problem solving he made extensive use of hands-on materials, trial and error and the freedom to try out an answer and modify it as needed. Using the Pirie-Kieren theory of mathematical understanding, it was determined that the tradesman worked at the levels of image making, image having and property noticing. The power and utility of these problem solving methods evident in this study and, from the perspective of the Pirie-Kieren theory, the necessity to work through these levels of understanding in the process of developing the higher levels of understanding associated with the school mathematics challenges the authenticity and validity of much of what is done in schools in the name of developing students' understanding of mathematics and teaching students to become problem solvers.

This case study examines some of the mathematics understandings used by a tradesman in solving problems encountered in his work. This individual is of particular interest in that, while highly regarded for his expertise within his industry, his formal mathematics and formal industrial training have been minimal--he developed the vast majority of his technical knowledge on the job in a variety of industries over his career, and he has been self employed and working independently in his own machine shop for most of the past 25 years. (The participant, now 81 years of age, began working full time at age 14 in a garage after receiving three years of elementary school education.) Thus, many of his mathematical understandings and problem solving methods have been constructed in response to particular problems encountered on the job, making use of the tools and equipment at hand, and reflect a limited degree of prescribed school taught methods. It should be noted that a minimum of research has been published which focuses on the mathematics and problem solving performed on the job in an industrial setting. (e.g., Scribner, 1984a, 1984b, 1984c) It is also the purpose of this study to examine the usefulness of a type of novel and open-ended question to initiate dialogue with an individual for the purpose of revealing mathematics understanding. The particular question used in this study was, "The answer is 72, what is the question?"

Mathematical understanding was examined in this interpretive case study through the lens of the Pirie-Kieren dynamical theory of mathematical understanding (Pirie & Kieren, 1994). In comparison with situated cognition, for example, which focuses on identifying and comparing mathematics practices in different settings, the Pirie-Kieren theory facilitates the analysis of even brief mathematics problem solving episodes in terms of a developmental perspective of individual mathematical understandings.

The volunteer subject was interviewed over a number of days in his workplace and asked to talk about his use of mathematics and problems that he solved in his work. These exploratory discussions were initiated, in part, using the question indicated above. On the basis of what was said, further questions were created *in situ* and posed by the researcher to encourage the participant to elaborate on the types of problems that he solved and the use of mathematics in his work, his ideas and his methods for performing these and his criteria for assessing the appropriateness of his methods and results.

Audio recordings of the interviews, all written work produced by the participant, photocopies of printed reference materials cited and photographs of artefacts referred to during the discussions were retained for analysis. These data sources were reviewed by the researcher, significant episodes were identified and coded in the interview dialogue, and verbatim transcripts were constructed of many of these episodes. All of these information sources were then used in concert to construct the case description. Finally, the conclusions constructed by the researcher were shared with and validated in further discussion with the tradesman himself.

A number of features of this man's use of mathematics and mathematical problem solving figured prominently in the discussions. These included the use of charts for useful numerical data and unit of measure conversions, the use of various specialized measuring tools, the use of tools and machinery in the problem solving process, the use of standardized and prefabricated components when building things, and the use of conventions and "rules of thumb" specific to machining. In all cases the appropriateness of his work was tested empirically—did it do the job?

For the first interview, the tradesman brought and presented a number of charts that he used on a regular basis. He explained that he didn't really do much math on the job, instead he relied on these charts and others like them in his machinist's handbook. A number of these charts were posted prominently on the walls in both the machine shop and office work area. The charts that he brought included: industry standard drill sizes in inches to three decimal places and the drill sizes needed for tapping holes of different sizes (i.e., making a thread within a hole for a bolt), measurements for standard male threads of different sizes, the dimensions for various shapes and sizes of steel bars available for the manufacturing process, the dimensions for locating between five and 12 holes equidistant around a unit circle, recommended torque values for fastening bolts of various sizes and various metal strengths, the required lathe settings to create standard sized threads, and a standard trig table which included the formulas for determining all possible unknown values within a right triangle given one side length and one other piece of information. It was the tradesman's view that if you didn't have the required information at hand, you could always look it up in a machinist's handbook or equipment manual. In a subsequent discussion, the tradesman agreed with the researcher's conclusion that using these charts minimized the need to perform multi-step calculations, use formulas and perform any form of algebraic manipulation.

Throughout the interviews the tradesman also explained how he used specialized tools and/or equipment for: measuring the thickness or diameter of an object to the nearest ten thousandth of in inch, measuring the depth of machine treads (used in conjunction with a special chart), setting material to be machined at a predetermined and precise angle of inclination in a vice, precisely rotating and securing a work piece to be machined using specialized chuck, as well as how he used the digital readouts on a number of his machines to locate the center of a piece of work, to the nearest thousandth of an inch, by taking an average of the co-ordinates of the outer edges, and how he used these same digital readouts for converting between imperial and metric measures of length. Like the charts described in the paragraph above, the tradesman used specialized tools and his shop equipment to perform routine mathematics tasks in ways which simplified his work and minimized his effort.

Another prominent part of the discussion related to conventions or "rules of thumb" and the use of available standard or prefabricated tools and components in his work. On technical drawings for example, dimensions given in decimal form were understood to be accurate to a thousandth of an inch with tolerances of plus/minus " a thou"—the standard unit of measure for amounts less than an inch. Decimal values of tenths, hundredths and ten thousandths of an inch were all expressed in terms of thousandths of an inch. Dimensions given on technical drawings in fraction form were understood to have tolerances of plus/minus 1/64 of an inch or plus/minus 15 thou. (Author's note, 1/64 = 0.015625) When cutting steel on a band saw a rule of thumb for selecting the right type of saw blade was that ratio of teeth on the blade to thickness of the material be no less than one and a half to one. This was to prevent the work from vibrating as it was being cut. And, when cutting a keyway slot in a hole (for securing a gear or wheel onto an axle), the width of the slot depended upon the standard sized cutters that were available to do the job and it was a rule of thumb that the depth of the keyway slot was half of it's width. Threads were always cut using standard metric or imperial thread sizes because of the tools and equipment used to perform these tasks, pre-existing and standard material sizes and prefabricated components were routinely used or modified for use in the design of objects to simplify the manufacturing process. And lastly, all of the cutting, milling, drilling and grinding operations could only be performed as permitted by the machinery available. The effect of these practices was to set a highly codified context in which problem solutions were conceptualized. For example, when approached about building a specialized latch for a garden gate, the tradesman's first response was to reframe the proposed design in terms of the material and components that he currently had in stock, and in terms of standard sizes and standard machine operations.

Unlike school mathematics or school problem solving where the answer is at the back of the book or the teacher is the arbiter of solution correctness, the tradesman was able to test the appropriateness of his efforts by trying them out and checking to see if they functioned as they were supposed to, and in the end they always did! Whenever a design did not meet the intended criteria, this indicated simply that the process was not yet finished. In fact, it was not uncommon for a client to return after a job had been completed requesting that further modifications be made to the device that had been made because new ideas arose from seeing it and using it.

If the answer is 72, what's the question?

When asked what question could be answered in his work with the answer 72 the tradesman, after an extremely long pause, explained that three of the gears, in a complex gear mechanism that he designed and built some years earlier for manufacturing hand reamers on his milling machine, each had 72 teeth. A reamer is a tool used for shaving off a minute amount of metal inside of a hole to make it a precise size after it has been drilled, or to resize a hole precisely after it has become worn. To make reamers, pieces of round bar are first cut to length and small center holes are drilled in each end to a depth of approximately a quarter of an inch. These pieces are then held by their center holes and passed lengthways through a rotating cutter on a milling machine to make between six and twelve evenly spaced indentations or "flutes" in the reamer between four and a half and five inches in length at one end. The reamers are then hardened and the metal which remains between the flutes is then ground to an exact size to serve as cutting blades. A large portion of the interview time was then spent discussing the design criteria and construction of the gear mechanism used on the milling machine in this manufacturing process.

The gear mechanism was designed in order to perform three operations for milling reamers of different sizes: 1) to allow for the reamers to be rotated or "indexed" on their center axes after each pass through the cutter so that a predetermined number of flutes could be cut, 2) to rotate the reamers slightly as the they passed through the cutters to produce a helix or spiral cut pattern in the flutes and to adjust the degree of rotation for different sizes of reamers, and 3) to hold, index and rotate up to four reamers at a time for simultaneous machining. The power to the gear mechanism was fed from a shaft on the table of the milling machine.

The tradesman explained that when he designed the gear mechanism he drew upon his experience using various combinations of gears--which he changed manually on an old lathe in order to cut helical threads. He had often used a gear with 72 teeth with various smaller gears to reduce the speeds on the lathe. He chose to use three 72 tooth gears with three smaller gears in the gear mechanism for his milling machine to obtain the slow rotation needed for turning the reamers as they passed through the cutter. The choice of 72 tooth gears in particular allowed for simple gear ratios with many different sizes of small gears, these particular gears met the size and strength requirements of the job and by choosing three gears of the same size he would need only to design and build one kind of bracket to hold each of them in place. The 72 tooth gears were paired with of a 16 tooth, a 36 tooth and an 18 tooth gear which the tradesman explained " all divided 72 evenly." He explained further,

> My math wasn't that great, that I could figure things out in advance, to come to the exact ratio.... I wasn't interested in accuracy, where one or two teeth would make a difference, but I had to be in a certain range. I assembled a couple of them and then checked the results and then assembled one more and then I made a note of it, it was a matter of trial and error. This was pretty much the process.

Two other gears were needed to complete the mechanism, one at the beginning of the gear sequence and one near the end. Initially two specific gears were selected to provide the appropriate rate of rotation of the reamer to produce the helical pattern

Size	Flutes	Power Gear	1st Stud	2nd Stud	3rd Stud	Spindle Gear	Index turns
1/4" up	6	27	72/18	72	72/16	48	1/2
7/16"	6	24	72/18	72	72/16	48	1/2
1/2"	8	24	72/18	72	72/16	64	1/2
5/8"	8	36	72/18	72/36	72/16	64	1/2
11/16"	8	27	72/18	72/36	72/16	64	1/2
1 1/8"	8	24	72/18	72/36	72/16	64	1/2
1 5/16"	10	27	72/18	72/36	72/16	80	1/2
1 3/4"	12	18	72/18	72/36	72/16	64	3/4

Figure 3. The gear configurations for machining various sizes of reamers.

needed for a one inch reamer and to allow for the indexing required to make eight flutes. Then over a period of many months, other gears were purchased, one or two at a time, for use in the beginning and end positions of the gear sequence to obtain the spiral patterns required for reamers ranging in diameter from one quarter of an inch to one and three quarter inches for reamers with six to twelve flutes. The specific combinations of gears that worked best were then recorded on a chart for future reference. A recreation of this chart is included here as Figure 3. When asked if, at the start of the design process, he was sure that the speed reduction and spiral cut features of the gear mechanism would work as needed using the initial three pairs of gears, the tradesman answered "No. Probably 80 percent sure."

When asked about how well his gear mechanism worked for making reamers of different sizes, the tradesman explained that, "I ran into some problems with the one and one eighth inch reamer. I still haven't figured out how to give it enough spiral on the flute. It's not quite enough. There's enough to get by. It's not that critical."

The tradesman was asked if he could relate the numbers on the gear combination chart for the gear mechanism. To get at this the researcher posed the question, "If part of this chart was missing, could you use the existing numbers on the chart to figure out the missing ones?" He responded by explaining that he would set the machine up with a finished reamer in place and observe how it moved using different combinations of the gears until he found the combination which gave the required indexing and spiral.

Assessment of Mathematical Understandings

The focus of this analysis is the tradesman's understanding of ratio as evidenced in his description of designing and building the gear mechanism for his milling machine and, as mentioned earlier, mathematical understandings evident during the interviews were assessed using the Pirie-Kieren theory. In order to make the case that the tradesman was able to work at least to the level of property noticing, evidence for the levels leading up to and including this level will be presented in turn.

The level of image making was evident when the tradesman tried different combinations of gears to adjust the speeds of both his lathe and milling machine, and when he worked with different combinations of gears by hand, counting the resultant number of turns of the small gears for each turn of a 72 tooth gear. The level of image having was evident when the tradesman commented that linking 16, 18 and 36 tooth gears with 72 tooth gears all resulted in simple gear ratios because these numbers "all divided 72 evenly." Finally, the level of property noticing (specifically the property of equivalent ratio) was evident when the tradesman responded that a 90-30 gear combination would act the same way as a 72-24 combination when he was asked why he chose to work with 72 tooth gears instead of, for example a 90 tooth gear attached to a 30 tooth gear. When asked about the possibility of using other gear combinations, he explained that a 36-18 gear combination would work in the same way as a 72-36 combination. Elsewhere the tradesman explained that linking a large gear with a small gear would reduce the speed in a machine and that the larger the difference in size, the greater the change in speed.

There was no direct evidence of the level of formalizing. At this level, a research subject is able to abstract a method or common quality from his previous image dependent know how and would be ready for, capable of enunciating and capable of appreciating a formal mathematical definition or algorithm. It is unknown whether or not the tradesman was ready for this as it did not arise as a part of the conversation. It is this researcher's conjecture that the tradesman would be able to follow and appreciate a formal mathematical explanation of his gear mechanism, but this remains only strong hunch.

It is asserted that the tradesman did not work at level six, that of observing at which a research subject is able to reflect upon and co-ordinate formal activity and express patterns as theorems. When asked if he had a way to relate the numbers on the gear chart, he could not without deferring or "folding back" to actually using the gear mechanism with a reamer which is the level of image making. The process of folding back, which is vital to the growth of understanding, is a crucial feature of the Pirie-Kieren theory.

Using the Question, "If the answer is 72, what is the question?

It appears that the researcher had luck on his side when he posed this question to the tradesman. The number 72 had come up in the interview a few minutes earlier when the tradesman showed the gear configuration chart for the gear mechanism to the researcher. When asked if he could think of another question from his work with 72 as the answer, the tradesman could not. A review of all of the printed material from the interviews did not reveal any other meaningful instances of the number 72 and the tradesman reported that measurements of 72 thousandths of an inch and 72 inches were very unusual in the kind of work that he did. And, when asked what other number might make this question more meaningful for him, he picked things that were constant terms in his calculations, specifically pi and diameter--a value typically given or known in this man's work, not something to be determined. His response to this latter type of open ended question for revealing mathematics understanding appears to relate to the context specific way that this tradesman used mathematics and solved problems.

Conclusion

The mathematical understandings revealed in this real-world and authentic problem solving context are significant for a number of reasons. First, little detail is known about the kind of mathematics and mathematical problem solving that is done in industrial settings such as a machine shop. These findings provide evidence that that a tradesman can work effectively, at least some of the time, with little formal mathematics. Second, the context in which this tradesman solved problems, specifically with outcomes that he valued highly (i.e., the desire to get the job done) and a need to be self reliant stands in marked contrast to the context in which mathematics and mathematical problem solving are done in a mathematics classroom. It is reasonable to conclude that these non-mathematical dimensions of problem solving contributed greatly to the tradesman's persistence as a reflective problem solver, which ultimately has led to his effectiveness without the need for more sophisticated understandings of mathematics. And lastly, the viability of the methods that were developed and used successfully in this setting, namely the use of hands-on materials, trial and error and the freedom to try out an answer and modify it as needed reveal the power and utility of these methods for problem solving. These methods are seldom allowed in school classroom, and yet an individual can use them to solve complex problems effectively. Furthermore, the Pirie-Kieren theory indicates that these types of activities, which were associated with the image making, image having and property noticing levels of mathematical understanding, are essential in the development of higher levels of mathematical understanding customarily associated with school mathematics. This finding challenges the authenticity and validity of much of what is done in schools in the name of developing students' understanding of mathematics and teaching students to become problem solvers.

It is clear that further research on the use of mathematics in industry would further our understanding of the development of mathematics understanding beyond the classroom; the connections, or lack thereof, between classroom mathematics and the workplace; and the design and implementation of mathematics curriculum, particularly for students who may enter a technical field when they finish school.

STORIES OF "72"

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This study illuminates the rich, flexible ways in which primary children search for and access current knowledge of number or further develop their mathematical understanding of number while working on a novel problem task. A group of four grade Two and Three was videotaped as they developed a question or task where the answer is 72. The seventy minute videotape was then analyzed using the Pirie-Kieren theory and model for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994a). Two aspects of the theory, folding back and primitive knowing were useful in describing and analyzing the children's actions. The notion of 'interventions' provided a powerful vehicle for considering how the actions and interactions of the teacher occasion the emergence of mathematical understanding. The notion of story as articulated by Borasi, Sheedy & Seigel (1990) was also useful in describing and analyzing how the idea of story impacted how the children approached and develop their problem solutions of stories.

By focusing the lens on a group of four children, this study illuminates the ways in which some primary children search for and access current knowledge of number or further develop their mathematical understanding of number while working on a novel problem task. This study is part of a larger research project headed by Dr. Susan Pirie that is focused on the individual's growth of mathematical understanding in diverse settings. In developing an analysis of the students' activity, the paper draws on elements of the Pirie-Kieren theory for the growth of mathematical understanding and on the power of story in developing children's mathematical understanding as elaborated by Borasi et al. (1990).

The class and the small group were videotaped as the children worked on a task that asked them to create a question for a given answer of 72. The children were taped and observed during the seventy-minute session and the author took some field notes. The notes are brief however, as the author is also the classroom teacher who is seen on the tape interacting with the featured group of four students. This is a mixed ability, combined Grade Two and Three class consisting of 11 Grade Three and 10 Grade Two students. All the children were asked to record their thinking on paper about the question related to an answer of 72. The videotape is focused on four students, three Grade Three and one Grade Two children as they discuss their understanding of the number 72 and how to create a question where the answer is 72. Like the rest of the class, this group recorded their personal approach on paper. The class has engaged in a variety of tasks in the previous months through calendar math, specific activities and explorations designed to develop their knowledge and understanding of number in flexible ways. In other classrooms in previous years, the children became used to seeing and solving story problems where all the information is included in the story and solving means to merely use the given information. These experiences with the typical story problem form an obstacle in this atypical problem situation. Now they are asked to invent the information for the story. 72 is not part of the story but is its resolution. On the previous day, the children and myself informally discussed how many different questions we could ask someone where the answer is 25. During this discussion it became apparent that the concept of question was very difficult for them to use in a practical way. Many of their responses were equations that "equaled" 25. I began using the phrases 'story question' and 'task question' as a way to focus their thinking on how to create a question that did not include the answer of 25. This same language was used to pose "the answer is 72, what is the question?" task on the following day.

In order to analyze the students' mathematical understanding of number, use was made of the Pirie-Kieren theory and model for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994a). This theory has been presented and discussed at a number of previous PME and PME-NA meetings (see for example Pirie & Kieren, 1992; Martin, Towers, & Pirie, 2000). The Pirie-Kieren theory provides a way to analyze, describe and account for the growing understandings of individual learners of mathematics and, through the notion of 'interventions', provides a powerful vehicle for considering how the actions and interactions of the teacher occasions the emergence of mathematical understanding. Two aspects of the theory, folding back and primitive knowing were useful in describing and analyzing the children's actions. Primitive knowing refers to the mental images a child "holds in her head" when approaching a new mathematical task. Folding back describes how the learner, when faced with a novel question or task that is not immediately resolvable, will return to an inner level of understanding (like primitive knowing) for concepts such number. This mental action involves reviewing and reshaping earlier understanding in the context of the present problem situation (Pirie & Kieren, 1992). As well, the notion of story as a powerful vehicle in helping children examine and represent their mathematical understanding, as articulated by Borasi et al. (1990) was useful in describing and analyzing how the idea of story impacted how the children approached and developed their problem solutions or stories.

What follows is a detailed analysis of the videotaped session featuring the activity of the group of four students. The analysis includes a description of the children's activity to provide the reader with a flavour of what was observed.

The notion of using story as a context for this task is strong for all the children and is evident in the way many of the students frame the task for themselves. In our mathematics' classroom, books are frequently shared and discussed and the concept of story is used extensively when learning concepts. When reading books together, the children are invited to talk about the mathematical ideas they see embedded or emerging from the story. At other times, the children are asked to "tell the story behind numbers" and to write about what they are learning and have understood. As with other researchers such as Borassi et al. (1990) and Whitin and Whitin (2001), stories in books, personal stories and a variety of writing is valued in our classroom as a powerful way to learn mathematics and to represent one's mathematical thinking and understanding. As well, our pre-discussion shifted from using the word "question" to what "story task or story question" you could ask someone that would result in an answer of 25. The word "question" is confusing for most of the students. As indicated earlier, their experiences with problems has developed an expectation that problems are stories where all the information one needs to find a solution is present in the given story. Now they are asked to write such a story themselves where the only information is the solution. In this initial discussion, it is also clear that many of the children do not understand what is meant by a question. This is not surprising when I recalled all the instances when various children would launch into personal stories in response to a presenter's invitation to ask her questions.

To set the stage, the children are asked to recall our discussion the previous day about the various ways that we could ask someone "to do something", a story question, a story task or activity, that would result in an answer of 25. The students then are asked to consider 72 as an answer and are invited to select from a variety of manipulatives to help them create their question, task, story or activity either with their group members or on their own. They are also given paper to record their thinking so that all the children's work could be examined later.

After carefully examining all the children's recordings and sorting them into categories, it was clear that the small group's activity could be used as a way to discuss how various children understand the number 72. What follows is a discussion of how each child (Noel, Elizabeth, Cole and James) understands 72 and use what he or she knows to create a story or a story game situation that represents 72 or has a goal of 72. The discussion will also suggest instances when individual children grow in their understanding through interactions with each other or with myself.

James and Cole both create similar stories about a store that sells items like cupcakes, muffins or cookies. They both use their selected manipulatives to initiate a story idea. At first, it looks like they are merely "playing." The arrows that are constructed are unrelated to the stories they eventually develop on paper. As described by James, the arrows are merely "thinking arrows, they help me think." But as they progress through the task, the materials seem to represent story elements rather than specific mathematical concepts and processes. It appears that James and Cole are engaging in a meaning making process where their understanding of number is being developed and represented as specific story elements (Borasi et al., 1990). The unifix cubes, tiles and dual coloured chips help them shape the story and the question that is posed at the end of their stories. As the two boys describe their stories, it is evident that their understanding of number is very different. In James' story problem, he begins by describing a store that has 72 cupcakes to sell. James appears unable to conceive of 72 as being an answer. Perhaps due to previous experiences with typical story problems, he sees the need to include 72 in his story as part of the story question. In contrast, Cole's story about a store includes concepts of money, multiplication, addition and subtraction as he describes how the shopkeeper sells various amounts of muffins and cookies (ten cents each). Cole's story also describes how the shopkeeper suffers the theft of differing amounts of money at two different points in the story before he poses the question of "how much does the person have?" Cole has flexible, multifaceted images for the number 72 that he accesses mentally and represents as elements of his story. Through mapping (Pirie & Kieren, 1992) it is possible to draw a "map" of Cole's activity and to identify, track and analyze the growth of Cole's understanding with the aid of the Pirie-Kieren model (see Figure 4 for the pathway mapping Cole's growing understanding).

The videotaped recording gave me a way to analyze Cole's actions with the manipulatives, his conversation with group members and our discussion together, providing me with a way to trace his movement between the layers of the model. Cole is at a level in his mathematical understanding that the Pirie-Kieren model describes as image having. He no longer needs to perform certain actions with materials to demonstrate concepts such as addition, subtraction, multiplication or division. Rather, Cole can manipulate his mental images as if they were objects to compose his story. The previously described events are all image having activities. Cole has also crossed what the Pirie-Kieren model describes as the "don't need boundary" between the image making level and the image having level. Cole's understanding of "the number 72" is not tied to a specific image. Rather he is able to flexibly and mentally create story details that require him to mentally calculate the total earnings from the sale of various amounts of muffins and cookies and to track the total earnings after losing various amounts of money through theft. Cole recognizes that 72 can be represented by groups of muffins and cookies, as differing amounts of money and as mathematical processes where 72 is the result. The Pirie-Kieren theory describes this level of understanding as property-noticing, a level where learners recognize properties of a concept such as "that 72 can be represented by multiplication or by the processes of adding and subtracting amounts of money." Unlike Cole, property noticing is absent in James' understanding. James prides himself on being able to quickly calculate equations. Numbers are for calculating. He does not seem to make the connection between calculating and the reason for calculating or in other words, the story behind the numbers.

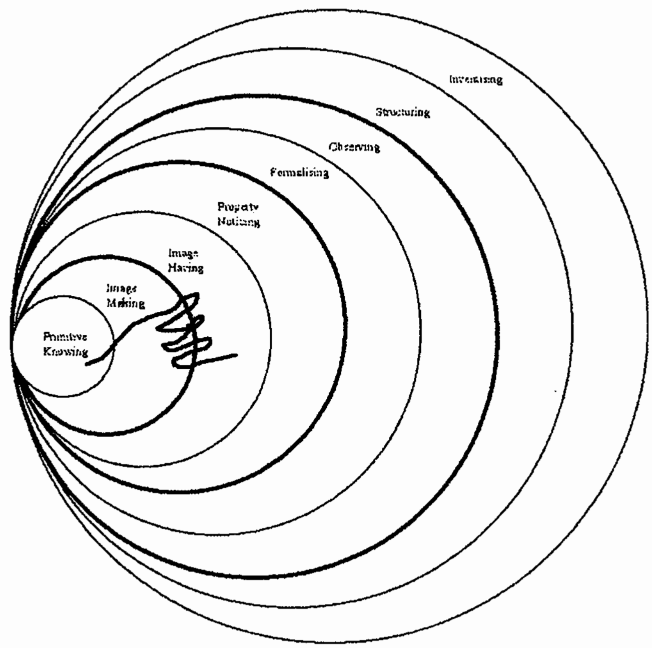

Figure 4. A mapping of Cole's growing mathematical understanding.

James and Cole worked independently, there was virtually no discussion of their stories either between the boys or with myself. In contrast, manipulatives and prolonged discussions were central in how Elizabeth and Noel created their games. The game idea was a radical departure from the approach used by other children to weave a story around the number 72. This is a highly atypical context for exploring number for these children. There is no guiding information or story leading them to a correct answer. They have to imagine and write the story themselves. Creating games was the way Elizabeth and Noel respond to this challenge. These children used their manipulatives extensively to make and explain their individual games. Ongoing discussions with each other and with myself prompts both Elizabeth and Noel to engage in specific actions that results in the emergence of richer, personal understandings for number. Both children recognize that 72 can be broken down in a variety of ways but their progress is slowed by the need to "fold back" to examine, strengthen or create images for specific concepts such as multiplication and division. The number of players in Noel's game changes progressively as his images for multiplication and division become more developed. The number 72 is initially represented in his game as the number of stops a player's bus can make and the total number of passengers that get off the bus at the various stops. Each bus holds four people (the bus is eight multilinks snapped together with four bingo chips on top) and players race each other to see how many people they can "put off at the most stops." These story elements represent Noel's primitive knowing about the structure of games which he uses to build images for the number 72. After beginning to make rows of pennies (a penny represents one stop), Noel asks "what is half of 72?" He appears to be trying to create a game where two players try to race each other to see which one can get to their half of the stops first. Noel needs to fold back from image making activities to his primitive knowing that an even number can be broken into two groups. He then uses this for the number 72. My intervention is intended to help him make an image of how 72 can be divided into two groups, an intervention described by Pirie & Kieren as invocative. After placing dual coloured chips into 'partners' while counting by twos to 72, two rows are created with 36 chips in each row which I believe "shows" Noel that half of 72 is 36. However, it is not apparent to Noel as he continues to count all the chips then one row of chips and eventually the number of pennies in his rows to establish and re-establish the number he seeks. He folds back to primitive knowing about how to count objects to 72, one by one. Noel seems to be working at the image making level, trying to create an image that will help him to build the game but he returns to an earlier understanding of number to help him create the images he needs to create the game. As he continues to develop the game for four players then six players, our discussions center on the total number of stops in the game and the number of stops in each of the four then six "alleys." I am trying to grasp his current level of understanding. While this is the intention of my validating intervention, I am not sure that Noel responds in the manner the intervention intends. At times, he would begin another image making action such as counting or placing more pennies on rows or would simply wander off! By the time Noel scribes his directions to me for how the game is played, it is apparent that he has worked out how 72 can be developed into a game for two, four or six players.

The goal of Elizabeth's game is also 72 people being dropped off at stores but three people disembark at each store. Each player has a "train" of 10 connected multilink cubes with 10 coffee beans on top of each train. Coffee beans represent the people and dual coloured chips represent the stores, one row of red chips or "stores" and one row of yellow chips or "stores. Elizabeth attempts different ways to set up the game with the various materials using her primitive knowing about numbers. When we begin to talk about her game, she tells me she has "too much" because 40 and 40 is 80. She is referring to the two rows of 13 chips with 3 beans on each one and a single bean at the end of the row or 13 x 3 + 1 = 40 plus 13 x 3 + 1 = 80 for the two rows. Our conversation centers on having her explain what she is doing and thinking. Elizabeth has folded back to her image for the quantity 80 but this is not a useful image. Elizabeth very quickly says she needs to figure out "how many pairs of chips with three coffee beans on top would be 72." She needs a new richer image that will help her to continue designing the game. Without further conversation, Elizabeth continues to add chips to the rows until she has counted by three's to 72. She needs an opportunity or space to consider what she has done and is thinking about and how to proceed. In folding back, Elizabeth may have accessed images she has for number and developed richer images (image making) to guide her in completing the game.

This study suggests that primary children approach problem tasks with rich, diverse and flexible levels of mathematical understanding. In this study, the children were asked to use what they understand about number in a situation they had never encountered before. In previous experiences, problems had been solved by using the information in the story problem. Now they are asked to create the information or story where the answer would be 72. For some children like James the novel problem situation remained an obstacle preventing him from seeing the answer of 72 as separate from the story information. For other children like Cole, stories provide the means to represent their thinking and understanding. Composing a story provides children with a way to "generate and elaborate mathematical ideas" and to engage in problem posing in a manner that honours current thinking while encouraging a deeper understanding of mathematics (Borasi et al., 1990). The context of story helped children like Elizabeth and Noel to grow in their mathematical understanding of number. While weaving details of the game, these two children used current number knowledge and evolving knowledge to develop the games. At the same time, the story game context created a need for new images to guide the creating process. Teacher interventions play a key role in supporting the growth of the children's understanding by providing thinking space. The intention of teacher interventions is to provide children with opportunities for them to independently construct or modify their personal images. Often teacher interventions are meant to verbally prompt children to return to their conceptual images, to create new richer images that will guide their outer actions. However such interventions can also create a space where the child is given a quiet place for reflection unhindered by time or conversation. An unusual task occasioned the creation of oral and written stories as the primary children grappled with how to create a problem task or question where the answer is 72. While writing their stories, some of these children also grew in their understanding. The stories of 72 authored by these young children reveal an understanding of number that is both diverse and complex.

...AND THE ANSWER IS 72: MATHEMATICAL UNDERSTANDING THROUGH OPEN ENDED TASKS

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This poster focuses on children's mathematical understanding of basic operations in mathematics, such as addition, subtraction, multiplication as well as estimation and place value, when they are given an open-ended task.

The research involved the videotaping of an interview with a group of grade 4 students. This particular group is comprised of children, about whom their teacher has some concerns regarding their understanding and performance of mathematical skills. It is for this reason that these particular children became members of the focus group. The data was analysed through the use of the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding. (Pirie & Kieren, 1994).

The task posed was to make up a problem for which the answer is 72. The students were divided into two groups of three and were initially given about 15 minutes to come up with as many problems as they could. They were each given paper and a pencil, and had access to their textbooks.

It is worth noting that in the three months that I worked with, and observed this class, the students had no exposure to any sort of open-ended activities or problems. In addition, I never observed any group activity during math class. The students' sole direction in learning and understanding has been by way of a single dimensional approach to the learning of a particular algorithm. In the time that I was in the classroom with these students they did a short unit on estimation. The textbook presented the idea of estimation, for the most part, by a series of addition and subtraction exercises, whereby the students were asked to give, not an exact number, but rather an estimated one. The exercises were generally not situated within an appropriate everyday context.

It is clear that the mathematics used by the students in the new problem situation reflected their prior learning experiences. In general, the students did not come up with situational problems, but rather exercises in addition and subtraction and, in some cases, multiplication. When asked to come up with their own problem, not one of the students entertained the idea of coming up with a problem involving the concept of estimation. When it was suggested that the students think about using their estimation skills in the creation of their problems, one girl said, "I can't do estimating." One of the boys said, "Estimation is just guessing." In this school year the students spent several weeks practising algorithms for addition and subtraction, and it is these algorithms and rules which form the pupils images for number operations. The poster will elaborate on the nature of images and explore how they were able, and unable, to use them in creating questions which led to an answer of 72.

IS IT AN ALGEBRA QUESTION? – ANALYSIS OF STUDENTS' STRATEGIES WHEN SOLVING A NUMBER THEORY PROBLEM

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The theoretical underpinnings of this study came from the Theory for the Dynamical Growth of Mathematical Understanding (Pirie, 1988; Pirie & Kieren, 1989, 1994). The Pirie-Kieren theory provides a way to visually trace the thinking actions of individuals working mathematically, and the growth of their mathematical understanding in a non-linear process related to *particular mathematical topics*. The theory posits *Primitive Knowing* as the inner layer of the model, which contains all a student's prior understandings that may influence the growth of the *new topic* being considered. In addition, it has been shown that when faced with a problem, students need to *fold back* to an inner layer of understanding, often their primitive knowing, to enable them to extend their current inadequate and incomplete understanding.

Several questions have arisen which form a focus for the paper: 1) How do students solve a non-routine mathematical problem? 2) What connections, if any, do students make to their primitive knowing when introduced to a new problem category? and 3) to what extent do students' existing understandings assist or hamper their growth of understanding of a new problem category. It is the purpose of the poster to discuss possible answers within the context of observing how students solve a particular number theory problem, a topic not included in school curricula. The problem was:

Alfie: Can you guess the ages of my three children? The product of their ages is 72.

Beatrice: That's not enough information. Give me another hint.

Alfie: My youngest child is more cheerful than my twins.

Beatrice: Now I can figure it out!

Six pairs of middle school to university level students were engaged for one-hour sessions in informal settings. The participants were asked to think aloud as they worked and responded to a word problem. All sessions were videotaped and transcribed. The students were encouraged to use pencils and papers to record their work. The analytical model for examining the videotape data emerged from the research activity of Davis, Maher & Martino, (1992), Mercer (1995), Pirie (1996, 2001) and Powell (2001). Analysis of the transcripts focused on characterizing students' individual behaviour as well as identifying trends or similarities across their actions. Analysis of students' actions, inter-student discussions, and their written work were used to produce a description of the students' growth of understanding as they worked through the mathematical task. With the investigation of each individual, clearer patterns of students' mathematical growth of understandings emerged. Most students considered the problem "algebraic", and tried to solve the problem through a two-equation system. It was evident that existing understandings of algebra did not assist their growth of understanding of a new problem category. In my poster I will illustrate some of the students' errors and discuss possible answers to the research questions.

THE ANSWER IS THE QUESTION: THINKING BACKWARDS WITH EXPONENTS AND LOGS

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This poster investigates students' growth of understanding during an open-ended problem solving session. Through focusing on this topic review session involving senior level high school students, I investigated the different images held among students for exponents and logarithms. I also investigated the different ways in which students accessed these existing understandings as they were required by the problem solving situation. More specifically I considered the ways in which students recalled and used their existing understandings when faced with a novel problem solving opportunity.

The theoretical framework I employ is the Pirie-Kieren Theory for the Growth of Mathematical Understanding. (Pirie & Kieren, 1994) This theory provides a means of analysing the growth of understanding as students revisit their existing images of exponents and logarithms and try to apply these understandings in new situations.

The students involved in this research are grade 12 level mathematics students in an enriched class at a private Catholic high school in North Vancouver, BC. The students are capable and motivated mathematics students. The students are working on an open-ended problem solving opportunity as a review of a recently completed unit on exponents and logarithms. The unit included topics such as exponential functions, the laws of logarithms, mathematical modelling, geometric sequences, and logarithmic functions. Students are placed in small groups and are posed with the problem "The answer is 72. What is the question?" The students are further challenged to create "challenging" questions to puzzle their classmates and problems that review each of the concepts in the unit. The mode of inquiry is video recording of classroom small group work in problem solving. Two digital video cameras were placed around the classroom, each one focusing on the discussion of one small group for the duration of the session. The video data was analysed to produce a meaningful picture of the growth of understanding of each small group of students as they worked on the problem.

The poster offers an analysis of this video footage by mapping the growth of understanding of each of the three groups through the levels of the Pirie-Kieren model. It also highlights the common features and the differences in the growth of student understanding for the different small groups. Particular attention is paid to the place of 'folding back' to prior understanding and to the working of students at the Property Noticing level of the theory.

Notes

[i]This is a group of teachers, faculty members and graduate students from British Columbia, Canada who work regularly together and are undertaking and interested in research into the growth of mathematical understanding at all stages and levels of learning, from early childhood to the work place.

[ii]This research was supported by the Social Sciences and Humanities Research Council, Grant Number 410-1999-0159.

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LOOKING FOR A CIRCLE AND FINDING AN APPLE: THE ROLE OF TOOLS AND BODILY ACTIVITY IN MATHEMATICAL LEARNING[1]

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As part of a larger study investigating students learning dynamical systems, this report examines how one university student developed powerful ways of knowing acceleration with a tool called the water wheel. Our goal in this analysis is to better understand the use of tools and bodily activity in mathematical learning and how they suggest alternative characteristics to knowing. In particular, we develop the idea of "knowing-with" as it relates to the use of tools and bodily activity. In so doing, we contribute to a line of research that seeks to develop constructs and perspectives for inferring the quality of students' mathematical experiences.

Introduction

In this paper we elaborate on a particular type of knowing that Broudy (1977) called "knowing with." He proposed that this form of knowing is different from two other types of knowing that are usually proposed: "knowing that" and "knowing how." Knowing-that is declarative knowledge, the type that is typically expressed in verbal assertions and theory-like elaborations. Knowing-how is performative and gets expressed in actual performances. A typical example used to make the distinction is, say, that of the tennis player who masters certain types of serves by being able to do them (knowing-how) and the tennis analyst who describes what bodily abilities enables certain players to be good at those servings (knowing-that). The phrase "knowing with" means that there is something else that is a central part of knowing: the thing one uses or becomes familiar with[2]. An analogy might be the light that illuminates the page we are reading. As we read we are not consciously analyzing the light that makes the text visible for us. Nor is the ambient light a component of our knowing how to read. Similarly, someone who is or becomes fluent with a tool or a technique may illuminate with this fluency a way to look at a phenomenon, neither explicitly "applying" certain facts or theories about the tool or technique (which would be knowing-that), nor executing procedures that are part of its use (which would be knowing-how).

In reality all three forms of knowing play out together with more or less relative prevalence. For example, take the case of knowing a foreign language. Becoming a fluent speaker in a foreign language entails, in addition to knowing how (e.g., utter correct expressions appropriate to the circumstances) and knowing that (e.g., stating grammatical rules), developing certain views and sensitivities regarding the things talked about. These views and sensitivities enable us to grasp humor, poetry, word games, and many other phenomena that are difficult or impossible to translate—they constitute what we would call knowing-with the foreign language.

We propose that knowing-with is an essential and not always recognized aspect of developing fluencies with tools and techniques in mathematics education. As an elaboration and refinement of what Nemirovsky, Tierney, and Wright (1998) refer to as "tool perspective," knowing-with suggests an important alternative characterization to knowing, which may be useful for teachers and instructional designers interested in understanding more deeply the role of tools and bodily activity in mathematical learning.

Theoretical Orientation

As a general orienting perspective, we draw on the theoretical orientation of symbolic interactionism as described by Blumer (1969). From this perspective meaning is neither located in a thing, like a water wheel or a graph, nor does meaning arise through a person's psychological organization. Rather, meaning develops through and is located in interactions—interactions with others and self-interactions. Compatible with this perspective is Nemirovsky's (1994) notion of symbol-use. Symbol-use, in contrast to symbol-system, refers to "the actual and concrete use of mathematical symbols by someone, for a purpose, and as part of a chain of meaningful events" (p. 390). We define symbols broadly to include conventional and unconventional inscriptions, physical objects, gestures, physical activity with a device, and utterances. Because gestures often co-occur with speech and because a speaker typically does not differentiate between the gestures she uses and the words she says (McNeil, 1992; Varela, Thompson & Rosch, 1991), we found this broad definition of symbol necessary in our attempts to make sense of the complicated process of construing meaning.

The Study

We conducted a total of eight, 90- to 120-minute open-ended individual interviews with three students, Jake, Monica, and Kyle. Each student had completed three semesters of calculus and had taken or was taking differential equations. In the interviews students worked with a physical device called the water wheel (described below). All interviews were videotaped and transcribed. Summaries of each interview were developed and compared across all interviews. In this report we focus on an approximately 16-minute episode with Jake because it was most helpful in our understanding the role of tools and bodily activity and for framing the mathematical learning of the other two students. Just as in the case of the foreign language, Jake's use of the water wheel entailed all forms of knowing; however, the episode we have selected allows us to trace with particular detail his knowing-with the water wheel.

As shown in Figure 1, the water wheel consists of a clear, circular acrylic disc holding 32 plastic tubes around its perimeter. The disc is mounted on an axle and is free to rotate a full 360° and tilt between 0 and approximately 45°. In the middle of the disc are two concentric clear plastic cylinders that contain a variable amount of oil that acts as damping for the system. The amount of oil can be adjusted by raising or lowering the oil container shown on the right side of Figure 1.

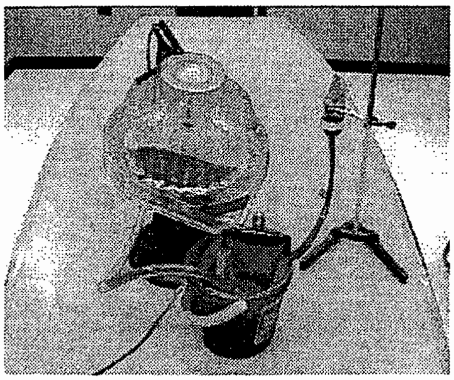

Figure 1. Water wheel.

Blue colored water from a bucket with a submersible pump (with adjustable flow rate) showers into several contiguous tubes. When tilted, the tubes that receive water are on the "higher side" of the wheel. Each tube has a small hole at the bottom that allows water to drain out, which then collects in a drip pan and is directed back into the bucket containing the submersible pump for a continual flow of water into and out of the tubes. When the wheel is tilted and there is even a slight variation in the way water is distributed in the tubes, gravity causes the wheel to rotate. The rotation can be constant, periodic, or something more complicated depending on different parameter settings (tilt, pump flow rate, and oil level) and on the amount and location of water in the tubes. During periodic motion, water in the tubes tends to collect on one side of the wheel in a bell-shaped form. Students in our study referred to this as the "heavy spot." Figure 1 shows this heavy spot near the bottom left of the wheel. In the next section of the paper Jake simulates the heavy spot with stacks of marbles that can be inserted into the tubes. When connected to a computer, optical sensors collect data with real-time displays of angular velocity versus time, angular acceleration versus time, and/or angular acceleration versus angular velocity. For a further description of this tool see Nemirovsky and Tinker (1993).

Discussion and Analysis

Revisiting Earlier Conclusions

At the start of the second interview Jake revisited his conclusions from the first interview about the rotational motion of the water wheel depicted by three different velocity-time graphs (a constant positive velocity, a varying sinusoidal-like velocity graph in the first quadrant, and a varying sinusoidal-like velocity graph along the t-axis, all shown in Figure 2). For each velocity-time graph, Jake knew-how to simultaneously trace the graph with his left hand while enacting an imagined diagram of the wheel with his right hand. These rotational motions were constant, fast-slow, and back-and-forth, respectively). The snapshot shown in Figure 2 depicts Jake engaged in this process of being the wheel. Jake's knowing-how to coordinate the water wheel motion and the velocity-time graphs was mediated by his knowing-that when the velocity-time graph crossed the t-axis the water wheel would reverse direction, that local maximums and minimums correspond to when the heavy spot is at the bottom of the swing, and that the sign of the velocity indicated clockwise or counterclockwise motion.

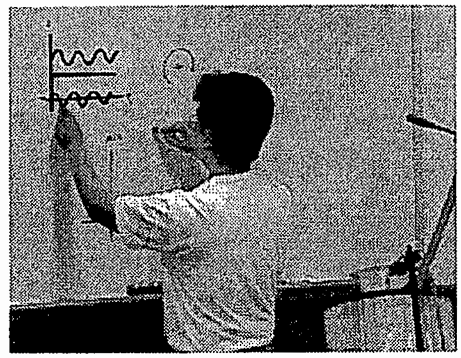

Figure 2. Velocity-time graphs.

An additional example of knowing-that is evident when Jake next revisited his development in the first interview of the acceleration versus velocity graphs corresponding to each of the three different velocity versus time graphs (see Figure 3). Jake recast the three velocity-time graphs as a point along the positive velocity axis, a circle centered along the positive velocity axis, and a circle centered at the origin. When creating his circle shaped graphs in the acceleration versus velocity plane, Jake capitalized on his knowing-that the acceleration is zero when the velocity is maximum or minimum and his knowing-that acceleration is positive when the velocity is increasing. Rather than being the wheel, it was through these calculus-based relationships that Jake recast the velocity-time graphs as acceleration-velocity graphs. Jake knew acceleration as a consequence of his knowing-that it is the derivative of velocity.

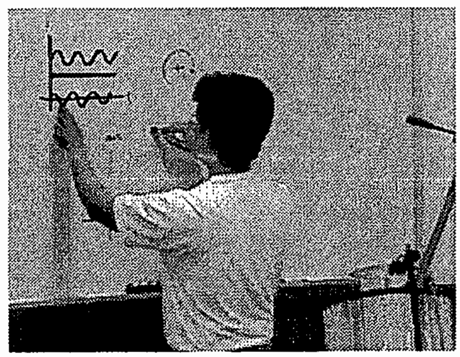

Figure 3. Acceleration-velocity graphs

Computer-Generated Graphs

As an entirely new activity, Jake next explored computer-generated graphs of acceleration versus velocity by physically moving the wheel with his hand. For the three types of motion (constant, fast-slow, and back-and-forth) he generated acceleration versus velocity graphs that were consistent with his expectations. For example, Jake said, "According to this [points to the board and traces out the circle centered on the positive velocity axis], if I speed up and slow it down, speed up and slow it down, it should be somewhat, kind of like a circle." After moving the wheel with his hand in a fast-slow, fast-slow manner (see Figure 4), he obtains a computer-generated graph that "seems to follow the general pattern." Jake goes through a similar process for the other two types of rotation and is satisfied with the overall consistency between his expectations and the computer-generated graphs.

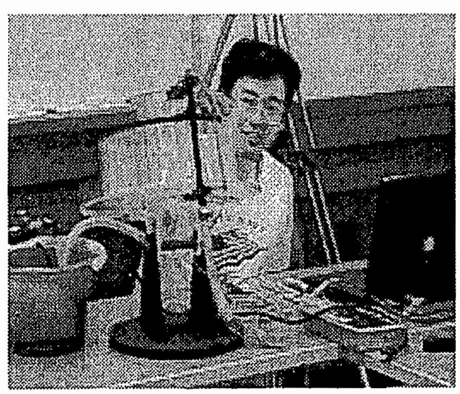

Figure 4. Computer-generated graphs.

Chris then offered Jake stacks of marbles taped together to simulate a "heavy spot." After inserting the marble stacks into three contiguous tubes, Jake continued his examination of computer-generated acceleration-velocity graphs for the case when the wheel moves back and forth without ever completing a full revolution. As Jake investigated the way in which the acceleration versus velocity curve evolved over time with the marble stacks, he knew-that the wheel would eventually come to rest and expected to see the acceleration-velocity graph "wind down to zero." Because the software only collects data for a set amount of time, this did not happen. Jake decided to adjust the starting position of the wheel so that the simulated heavy spot was relatively close to its resting position, and thus he could more or less observe the acceleration-velocity graph winding down to zero. This is part of developing a tool perspective (Nemirovsky, Tierney, & Wright, 1998) where a person figures out what is and is not a consequence of the tool.

Finding an Apple

Jake's tool perspective develops further in the next excerpt, where he ascertains whether an unexpected acceleration versus velocity curve is or is not an accidental byproduct of the marble stacks. Recall that Jake related the back-and-forth rotation to the circle centered at the origin in the acceleration-velocity plane.

(1) Chris So, what if we, um, take our heavy spot and started further, like up here [close to the top of the wheel] [Jake: Oh, OK.] Just when it's about to, ready?

(2) Jake Yeah.

(3) Chris Huh. Why do you sit back and fold your arms, Jake? [Laughs.]

(4) Jake Well, I was thinking why the acceleration went down momentarily [points to two lowest places on graph in the third and fourth quadrants – see Figure 5]

(5) Jake Ohhhh.

(6) Chris So, it did something like [Chris draws an apple shape on the board similar to that in Figure 5]

(7) Jake Yes, yes. Like, an apple. [Chris: Like an apple] Yeah.

(8) Jake You know, I was wondering why I have, I, I mean, right now it seems like going pretty circle. But, especially like, like that, the initial, that initial, uh, run. Acceleration seem to, uh, or, or the, uh, acceleration,

(9) Jake Let me think, this is positive acceleration [traces out counter clockwise motion over the wheel] and this is a negative acceleration [traces out clockwise motion over the wheel]. OK, um.

(10) Jake So, initially, it went this way [with his left index finger he traces out the start of the curve in the first quadrant]. So, it was positive [his right hand traces out counterclockwise motion over the wheel]. It [acceleration] became negative somewhere around here [at the point where the wheel changes direction]. And, so this is the point when the, uh, velocity is zero and the acceleration is at the max. So, it must be at top

when, when the acceleration wasn't so big too much. And . . . huh [folds his arms and sits back].

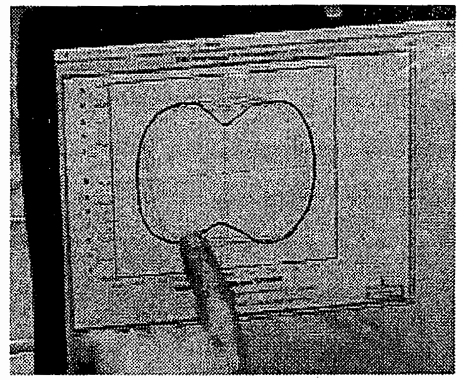

Figure 5. The apple shape.

This excerpt begins with Chris suggesting they start with the heavy spot closer to the top of the wheel (1). The resulting computer-generated acceleration versus velocity graph is a surprise to Jake because instead of a circle shape graph, which is what he expected, the initial portion looks like an "apple" (7). Jake is curious to figure out why the "acceleration went down momentarily" (4), where down refers to the two lowest places on the acceleration versus velocity curve. In Figure 5, Jake points to one of the "down" spots.

Jake's use of the term "down" is a feature of the shape of the acceleration-velocity graph. Down does not appear to be related to the motion or rotation of the water wheel, either in terms of velocity or acceleration. The initial momentary down portions of the graph and the later graph that is "going pretty circle" (8) is a way for Jake to distinguish between what is unexpected and what is expected. Jake's attention to the differences in shape (circle versus apple) is significant because it gave him reason to pay attention to acceleration as it pertains to the actual motion of the wheel rather than derived from velocity.

As Jake enacts the motion of the wheel with his hand (9), he states that movement in one direction corresponds to positive acceleration and movement in the other direction corresponds to negative acceleration. Although Jake says acceleration, his conclusions are actually compatible with velocity (where counterclockwise is positive). The incorrectness of his statement about acceleration aside, Jake continues in (10) to coordinate his hand enactment of the motion of the wheel with both acceleration and velocity. In relation to different types of knowing, we see this as another example of Jake's knowing-how to produce motion according to certain graphs.

Theory-Like Elaboration for the Apple

Jake then clears the screen and repeats the data collection, looking back and forth from the computer-generated graph to the water wheel and then staring intently at the wheel.

(1) Jake Um. See, at the top, it was maximum acceleration. At the, at the, the, whenever, whenever it [the wheel] reaches the top, so, [Chris: OK.] at, whenever it swings down and reaches the top, it's, uh, maximum. It, at the maximum [gestures in a grabbing and pulling motion] acceleration, you know, this way [points and gestures in a counter clockwise direction]. Towards this direction [repeats gesture].

(2) Chris OK, I'm listening.

(3) Jake OK. Now. I think the reason why this went up [points to the dimpled part of the apple shape in the lower half plane on the computer-generated graph – see Figure 5] was because of the, uh, little momentum here. You know, I mean, when it came up, it went kind of, like this [jiggles a marble stack in one of the tubes], you know, because, you know, this thing was pulling down [gestures in a pulling motion]. But, the, uh, the thing was still trying to go that way. I mean, I, I can show you.

We characterize Jake's theory about the jiggling marble stacks accounting for the surprising apple shape as another example of knowing-that. Similar to when a tennis analyst carefully watches a player serve and theorizes why he continues to double fault (e.g., he keeps dropping his eyes), Jake is theorizing about something that he is noticing about the marble stacks. We think that this episode also illustrates Jake's knowing-with (illustrating again how different types of knowing often play out together).

We see two types of gestures in (1) that are particularly important – a grabbing and pulling gesture and a pointing and swirling gesture. In between the two words "maximum acceleration" Jake gestures in a grabbing and pulling motion followed by a pointing and swirling motion with his hand while he says, "You know, this way, towards this direction" and then repeats the pointing and swirling gesture. We see these two different gestures as important to Jake's knowing acceleration with the wheel. As Jake's grabbing and pulling motion followed by his pointing and swirling motion suggests, he is knowing acceleration with the water wheel because he is noticing the forces acting on the heavy spot, imagining what it is like for the marble stacks standing up in the tubes wanting to go one way while the wheel wants to go another way (3). These two gestures suggest a knowing-with the tool that is neither declarative nor performative.

Knowing-With the Water Wheel

In an effort to support his marble-jiggling theory, Jake then leaves the room to get tissues to stuff in the tubes to prevent the marble stacks from shifting. Upon returning with paper towels, Jake wraps the marble stacks in the towels so that they fit snuggly in the tubes. He generates a new acceleration-velocity curve and, much to his surprise, finds that the apple shape is still present, concluding that the apple shape is not the result of the marble stacks jiggling inside the tubes. At Chris's suggestion to try "some other graphs," Jake decides to look at an acceleration-time graph and is as surprised at what he sees as he was when he saw the apple shape. It is likely that Jake expected to find a sinusoidal type graph, but instead found a graph with a "wicked move," which he sketched on the board behind him (see Figure 6).

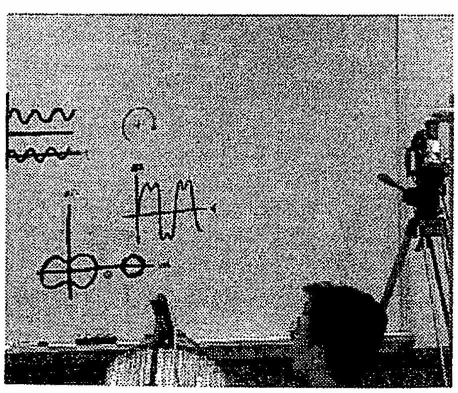

Figure 6. Acceleration-time graph.

As the discussion surrounding the wicked move continued, Chris asks Jake, "Does it [the wicked move] happen when it's like this?" [setting the wheel in motion such that the back and forth oscillation is less than 180 degrees]. Chris's question at once refers to an unaccounted for aspect of acceleration and the spatial-temporal position of the water wheel. The phrase "when it's like this" merges both aspects of time ("when") with aspects of position ("like this") for the heavy spot. We see this as an example of fusion (Nemirovsky, Tierney, & Wright, 1998), where qualities of a graph are blended with actions that give rise to these qualities. Chris's question was not ambiguous or confusing to Jake as evidenced by the fact that Jake began moving the wheel with his hand to illustrate where he sees the maximum acceleration (at the top of the wheel's swing as before) and where he sees zero acceleration (at the bottom). Jake goes on to explain:

(1) Jake OK. When it's up here [when the heavy spot is at the top of its swing], that's when the acceleration is the, uh, maximum. And, over here [bottom of the wheel], the acceleration is pretty much to zero.

(2) Chris How do you know where the acceleration is maximum?

(3) Jake Um. Well. One thing that I just noticed is that, uh. [Jake moves the wheel around so that the heavy spot is in different location] It seems to, uh, it, it seems to be proportional to the, well, see, as this thing goes up, it seems to . . . what? One, one thing I'm noticing [Jake carefully moves the heavy spot with his hand, feeling the force at different angles] is that, well, see, as I move this, I feel more weight right now, you know. But, the, uh, weight is almost zero up here [at the top]. Um. So, in other words, you know, up here [top], I don't have to apply, uh, as much force. As a matter of fact, it pretty much stays there. But, if it's in, you know, the middle. But, over here [heavy spot at the side], is, the force is like, the, uh, like a maximum. Um, um. And, uh, over, over here [bottom], of course, you know, the force is pretty much zero. So, uh, I don't know, force due to gravity, I think, it's, uh, like, like, almost zero. And, over here [side] it, it seems to be most, the force seems to be most strongest. ... And, so, I think it has to do with the, um, the, uh, how do I say it? The, not, not the amplitude, well, maybe it's the amplitude. But, the, uh, angle of the oscillation. So, in other words, if I start, like, up around here [at the side], it should go, it shouldn't become like an apple shape. That's my theory. I mean, of course, I'll try it.

In (2), Chris asked Jake how he knew where acceleration is at its maximum. This question and Jake's response highlight how experience with a tool like the water wheel affords a certain way of knowing a mathematical idea like acceleration that is different from knowing-that acceleration is the slope of the tangent line or knowing-how to move the wheel in coordination with a velocity-time graph. When sensing and feeling the water wheel, Jake's way of knowing acceleration grew in him as he physically

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moved the wheel with his hand. He says, "I feel more weight now" (3). By carefully attending to variations in the force he had to apply to keep the heavy spot in different positions, Jake distinguished when and why the wheel would experience a maximum acceleration. This excerpt exemplifies Jake's knowing-with the tool. Much like how knowing-with a foreign language entails developing particular cultural subtleties and nuances enabling one to grasp humor or poetry, Jake's knowing-with the water wheel enables him to develop certain views and sensitivities to force and angular position of the heavy spot that enable him to grasp acceleration. In this example, Jake neither explicitly applied Newton's law of motion (which would be knowing-how) nor did he state rules governing the relationship between velocity and acceleration (which would be knowing-that). Rather, by being the wheel, where he imagines what it is like for the water wheel, Jake is knowing-with the water wheel.

Concluding Remarks

As alluded to earlier in the paper, we see knowing-with as an elaboration and refinement of tool perspective. As described by Nemirovsky, Tierney, and Wright (1998), "Adopting a tool perspective involves emulating the tool's sensitivity to certain aspects of motion and not to others, ascertaining conditions under which the tool is useful, and recognizing patterns of significance in the tool's products" (p. 125). The tool perspective idea is a way to characterize how becoming fluent with a tool sensitizes one to aspects that are salient. Our analysis in this report helps to see the connection between this idea and forms of knowing that the idea of tool perspective does not finely distinguish.

We also highlighted in our analysis the important role of actually being the tool. In our different examples, being the tool was an important part of Jake's developing views and sensitivities about motion, velocity, force, graphs, and acceleration. For example, when Jake was coordinating the motion of the wheel with the different velocity-time graphs, his gesturing with his right hand was a form of being the wheel as a means to tell the story of the actual motion of the wheel. In the theory-like elaboration for the apple example, Jake's grabbing, pulling, and swirling pointing gestures was an integral part of his developing sensitivities to force, acceleration, and motion. In our last example, Jake's touching and sensing the wheel—his being the wheel—were inseparable from his knowing acceleration with the tool. In these examples, being the tool is similar to how Ochs, Jacoby, and Gonzales (1994) describe how physicists create interpretive journeys through graphical representations, where they talk and gesture as if they are physically traveling through the inscription.

In this case study Jake anticipated that for back-and-forth rotational movement of the water wheel, the maximum acceleration would occur at the instant the wheel changed direction and that the corresponding shape of the acceleration versus velocity graph would be circular; but surprisingly learned otherwise. In the approximately 16-minute episode in which this story unfolds, our analysis highlights three different types of knowing: knowing-that, knowing-how, and knowing-with. As we illustrated in our analysis, these different types of knowing are not disjoint and often play out together. Our analysis also underscores knowing-with as an essential aspect of developing fluencies with tools in mathematics education. Knowing-with a tool illuminates a way to look at a phenomenon, which is different than either carrying out procedures or applying particular facts or theories. Rather than a direct application of previously learned ideas and methods, knowing-with a tool like the water wheel is a process akin to organic growth that involves the whole body. Through his different gestures and physical interactions with the water wheel, Jake perceived and sensed forces and motion—imagining what acceleration is like for the water wheel. These developing fluencies are what enable him to grasp acceleration in ways that enrich and complement his knowing-that acceleration is the derivative of velocity and his knowing-how to coordinate different graphs with the motion of the wheel. Our work contributes to developing the knowing-with construct by adding a needed degree of specificity in relation to the use of tools and bodily activity in mathematical learning.

Notes

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[2]With a different language, Polanyi (1962) made similar distinctions.

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COMPARING INSTRUCTIONAL STRATEGIES FOR INTEGRATING CONCEPTUAL AND PROCEDURAL KNOWLEDGE

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We compared alternative instructional strategies for integrating knowledge of decimal place value and regrouping concepts with procedures for adding and subtracting decimals. The first condition was based on recent research suggesting that conceptual and procedural knowledge develop in an iterative, hand over hand fashion. In this iterative condition, conceptual and procedural lessons were interleaved. The second condition followed the common ordering of conceptual lessons before procedural lesson (concepts-first condition). All lessons were presented via a computer-based intelligent tutoring system and seventy-two sixth-grade students participated. Students in the iterative condition made greater improvements in procedural knowledge and comparable improvements in conceptual knowledge, compared to the concepts-first condition. Students in both groups did better when problems were presented in money contexts rather than symbolically. Both the iterative ordering of lessons and presenting problems in money contexts reduced students' digit alignment errors when adding and subtracting decimals.

Introduction

Competence in mathematics rests on children developing and linking their knowledge of concepts and procedures (Bisanz & LeFevre, 1992; Hiebert, 1986; Hiebert & Wearne, 1996; Silver, 1986; Sowder, 1992). However, competing theories have been proposed regarding the developmental relations and instructional importance of conceptual and procedural knowledge. Whether concepts or procedures develop first has been hotly debated within psychology (e.g., Gelman & Williams, 1998; Siegler, 1991; Siegler & Crowley, 1994; Sophian, 1997). Similarly, the "math wars" for how we should improve mathematics instruction have pitted conceptual understanding against procedural skill as the most important goal of mathematics instruction (Mathematically Correct, 2000; National Council of Teachers of Mathematics [NCTM], 2000). The mathematics education research community has begun to move beyond this dichotomy and recognize the importance of each type of knowledge. However, *how* conceptual and procedural knowledge are related is still not well understood. We propose that conceptual and procedural knowledge influence one another in mutually supportive ways and build in an iterative process. The purpose of this study is to compare instruction on decimals that is based on these mutually supportive, iterative relations to instruction based on the common sequence of concepts before procedures.

Developing Conceptual and Procedural Knowledge

Recent research indicates that conceptual and procedural knowledge often develop in an iterative process, with improvements in one type of knowledge leading to improvements in the other type of knowledge, which leads to further improvements in the first type of knowledge (Rittle-Johnson, Siegler & Alibali, 2001). For example, limited understanding of some domain concepts guides students' attention to important problem features and facilitates generation and use of a correct procedure. In turn, developing this procedural knowledge leads to improvements in conceptual understanding, perhaps by freeing cognitive resources for noticing patterns and relations, highlighting the importance of certain problem features, or revealing certain misconceptions.

Applying this iterative process to instruction could aid learning for several reasons. A "big idea" in reforming mathematics instruction is developing students' conceptual understanding of mathematics (NCTM, 2000). Although the general wisdom of developing students' conceptual understanding in a domain early in instruction is intuitive and an improvement over old practices, there are several reasons why iterating between conceptual and procedural instruction could lead to even greater learning.

First, presenting all the conceptual material first could overwhelm students' working memory and lead to confusion. Rather, some level of procedural fluency reduces working memory load (e.g., Shrager & Siegler, 1998) and could facilitate further improvements in conceptual knowledge. Second, iterating between conceptual and procedural lessons could help highlight the relevance of each lesson type for the other, avoiding the common problem of students not integrating conceptual and procedural instruction (e.g., Resnick & Omanson, 1987). Finally, iterating between varied, but related, tasks could support appropriate generalization of concepts and procedures, and thus reduce overgeneralizations (applying a concept or procedure in an inappropriate way) and undergeneralizations (failing to transfer to appropriate tasks) (Anderson, 1993).

Although this iterative instructional approach appears promising, past research does raise several cautions. First, conceptual understanding can be used to generate and choose good procedures (e.g., Geary, 1994; Gelman & Williams, 1998; Hiebert & Wearne, 1996), so it may be important to begin procedural instruction after students have firm conceptual understanding. Furthermore, some studies have found that prior procedural instruction can interfere with learning concepts (e.g., Warrington & Kamii, 1998). Thus, it is important to compare the effects of an iterative ordering of conceptual and procedural instruction to sequential ordering of conceptual instruction before procedural instruction (see Table 1).

Using Context to Elicit and Build Upon Informal Knowledge

An additional big idea in mathematics instruction is eliciting students' prior, informal knowledge through real-world contexts (NCTM, 2000). Real-life story problems can be easier for students to solve than matched symbolic problems, partially because the context helps students avoid common errors (Carraher, Carraher, & Schliemann, 1985; Koedinger & Nathan, 2000; Rittle-Johnson & Koedinger, 2001). Nevertheless,

Table 1. Lesson Orders for the Two Conditions

Concepts-first condition	*Iterative (context-first) condition*
1. Contextualized concept lesson	1. Contextualized concept lesson
2. First abstract concept lesson	2. Contextualized procedure lesson
3. Second abstract concept lesson	3. First abstract concept lessons
4. Contextualized procedure lesson	4. First abstract procedure lesson
5. First abstract procedure lesson	5. Second abstract concept lesson

national assessments indicate that U.S. students do poorly on story problems and often do worse on story problems than on symbolic computation problems (Baranes, Perry, & Stigler, 1989; Lindquist, 1989). It is clear that not all story contexts elicit students' informal knowledge. Furthermore, there is little guidance for how to integrate context into conceptual and procedural instruction. Context could facilitate learning of particular types of knowledge. For example, context could facilitate links between conceptual and procedural knowledge because the grounded, informal context makes it easier to see the relations between the two types of knowledge. In this case, introducing procedural instruction in context immediately after conceptual instruction using the same context could facilitate links between the two types of knowledge. Context could also help students to bridge from grounded, informal concepts to abstract, mathematical concepts. This would suggest presenting instruction on formal mathematical concepts immediately after contextualized conceptual instruction. The two instructional conditions in this study contrast these two potential benefits of presenting problems in context (see Table 1).

In summary, we compared alternative instructional strategies for integrating knowledge of decimal place value and regrouping concepts and procedures for adding and subtracting decimals. We evaluated whether iterating between instruction on decimal concepts and procedures, initially within money contexts, led to greater learning than instruction that covered decimal concepts in context and in the abstract before lessons on decimal procedures. We also evaluated whether money contexts elicited informal knowledge and supported better performance on a decimal assessment. We predicted that iterating between lessons on decimal concepts and procedures would lead to greater learning and that providing money contexts would improve performance.

Method

Participants

The intervention was a component of a sixth-grade mathematics curriculum that we are developing. Four classes of sixth-grade students at our two pilot schools participated. Each class period was assigned to one of the two conditions. Eighty-three students began this study and 72 completed the intervention and assessments, with an equal number of students in the two conditions.

Intervention

The current study focuses on our computer-based intelligent tutoring systems for decimal concepts and procedures. The Cognitive Tutors provide on-demand, step-specific help at any point in the problem-solving process and immediate feedback on errors (Koedinger, Anderson, Hadley, & Mark, 1997). We designed three conceptual lessons on decimal place value and regrouping and three procedural lessons on adding and subtracting decimals.

In the conceptual lessons, students were asked to enter a number in a place value chart and then to show the value of the number in novel ways using regrouping (e.g., 6.0 as 6 ones, as 5 ones and 10 tenths, etc.; see Figure 1). In the first conceptual lesson, the problems were presented in a money context and using money terminology for the place values (e.g., dimes, pennies). On later conceptual lessons, the problem format was the same, but no context was given and symbolic place value names were used (e.g., tenths, hundredths).

In the procedural lesson, students were given word problems that required adding or subtracting two decimal numbers (see Figure 2). Students entered the numbers in a chart and completed the computations. In the first procedural lesson, problems were

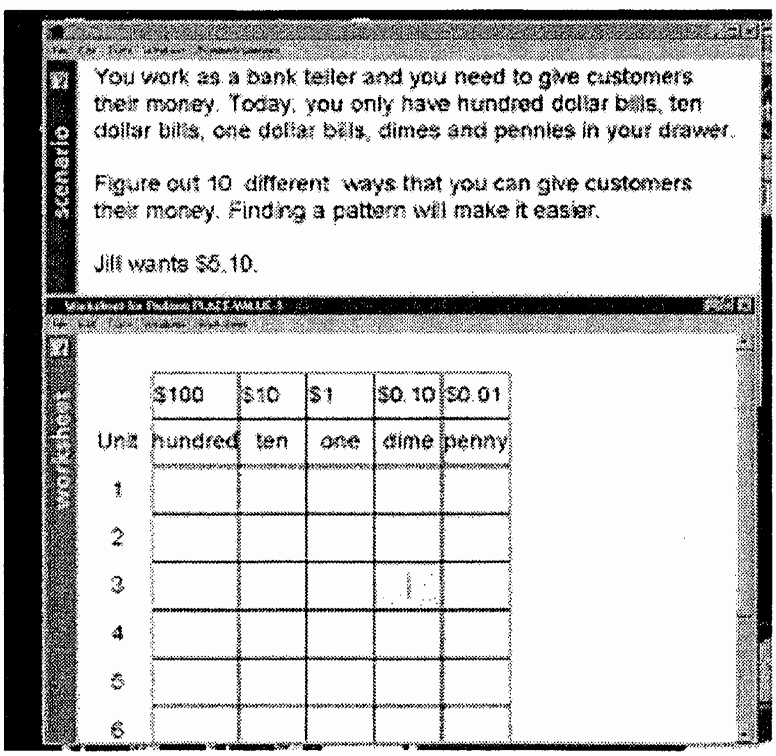

Figure 1. Screen shot of a problem in the contextualized conceptual lesson.

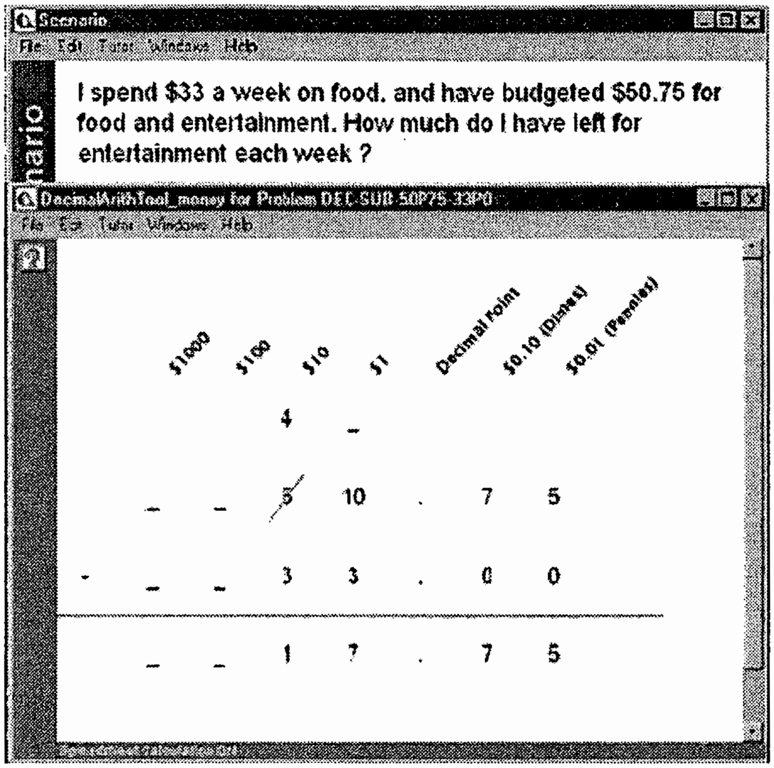

Figure 2. Screen shot of a problem in the contextualized procedural lesson.

presented in a money context, and monetary place value column labels were included on the chart. In the second lesson, problems were in non-money (often unfamiliar) contexts and the standard place value labels were included on the chart. In the third lesson, the chart did not include place value labels.

The order of these lessons varied by condition. Both orders began with the conceptual lesson on place value and regrouping in a money context. In the *concepts-first* condition, the other two conceptual lessons were presented next. Then, the three procedural lessons were presented, beginning with the lesson in context. In the *iterative* condition, the second lesson was the procedural lesson in a money context. Then, students completed the second conceptual lesson, followed by the second procedural lesson, and so forth (see Table 1).

Students worked on the Cognitive Tutor twice a week for four to eight weeks, with an average time of 4 hours and 10 minutes spent on the decimal lessons.

Assessment

The pretest and posttest followed a 2 (Knowledge type: conceptual or procedural) x 2(Context: money or none) x 2 (Learning type: learning or transfer) design, with 1 learning and 2 transfer problems in each of the other cells, for a total of 12 questions. See Table 2 for an example of each question type. We generated two versions of the

Table 2. Example Assessment Items

	Conceptual		Procedural	
	Learning	Transfer	Learning	Transfer
Money Context	Show 5 different ways that you can give Ben $4.07.	2 dimes are worth how many pennies?	You had $8.72. Your grandmother gave you $25 for your birthday. How much money do you have now?	You buy a super-size candy bar for $1.12, a bag of chips for $3.39 and a pack of soda for $4. What is your total cost?
No Context	List 5 different ways to show the amount 4.07	2 tenths are worth how many hundredths?	Add: 8.72 +25	Add: 1.12 + 3.39 + 4

assessment so that the same problem was presented in a money context and with no context (kind of "Difficulty Factors Assessment" see Koedinger & Nathan, submitted).

Results

An alpha value of 0.05 was set as the criteria for all statistical analyses.

Pretest

We conducted a repeated-measures ANOVA on percent correct at pretest, with knowledge type, context and learning type as within-subject factors. Students solved more problems correctly when they were presented in a money context, rather than without context (63% vs. 31% correct), $F(1,82) = 114.18$. As shown in Figure 3, the benefits of context were greater on conceptual knowledge items than on procedural knowledge items, $F(1,82) = 5.25$. Comparisons of performance on individual items suggested that the money contexts reduced alignment errors when adding and subtracting (3% vs. 18% of relevant problems) and elicited knowledge of the relations between different place values, such as the number of pennies in a dime, on the conceptual items. Finally, students did better on the procedural knowledge items than on the conceptual knowledge items (59% vs. 44% correct), $F(1, 82) = 9.96$, and on the transfer items than the learning items (56% vs. 44% correct), $F(1,82) = 13.44$, though in both these comparisons other factors were not controlled and may account for the differences.

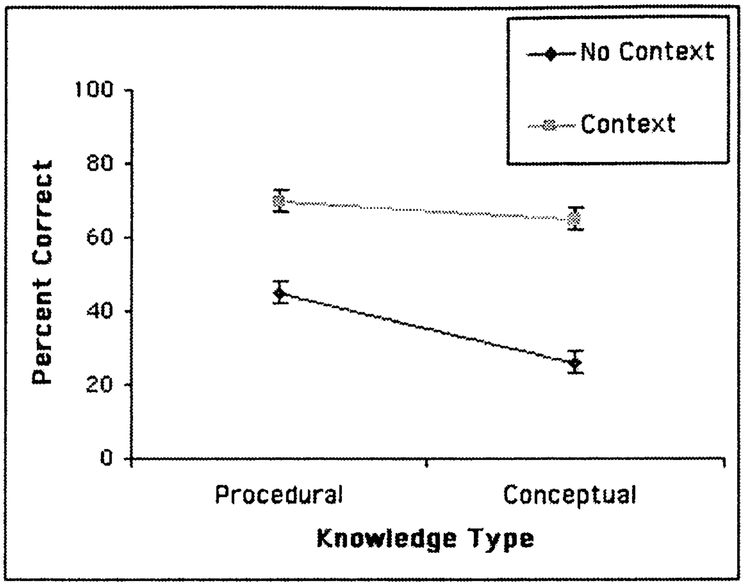

Figure 3. Effect of context on conceptual and procedural items at pretest.

Effects of Condition on Learning

For each student, we calculated his or her gain score as the percent correct at posttest minus percent correct at pretest. To evaluate the effects of condition on learning, we conducted a repeated-measures ANOVA on the gain scores, with condition as a between-subject factor and knowledge type and learning type as within-subject factors. There was a main effect for condition, $F(1, 72) = 3.92$. As shown in Figure 2, students in the iterative condition made greater gains than students in the concepts-first condition (23 vs. 13 percentage point gain). There was also an interaction between condition and knowledge type, $F(1, 72) = 4.91$, such that the effect of condition was mostly on the procedural knowledge items. The effect of condition did not differ between the learning and transfer items. Comparisons of performance on individual items suggested that the iterative ordering was most beneficial on items where the "align numbers on the right" pre-conception seemed most difficult to override - when a two-digit whole number is added to a two-digit decimal number (e.g. 8.72 + 25). Students in the iterative condition made a 25% gain on this item, compared to no gain on this item in the concepts-first condition. Furthermore, students in the iterative condition only made alignment errors on 6% of relevant arithmetic problems at posttest, compared to 14% in the concepts-first condition.

Why did the iterative condition lead to better learning of decimal arithmetic procedures? Informal observations and quantitative data from the intervention suggested that the iterative ordering highlighted the links between the conceptual and procedural lessons. The conceptual task of representing a number in many different ways was novel to the students and very challenging. To solve a single conceptual problem during the intervention, students spent 6 to 7 minutes, made an average of 6 errors and 1 or 2 help requests. The procedural task of adding and subtracting decimals was a familiar task for sixth graders and was much easier for them (2 to 3 minutes per problem, and less than 1 error and 1 help request per problem). Informal observations suggested that recognizing that ideas of borrowing from decimal subtraction could be applied to the conceptual task of representing a number in multiple ways facilitated understanding and performance on the conceptual task. If this was the case, students in the iterative condition should have done better on the later conceptual lessons, which were presented after a procedural lesson for these students (see Table 2). Although not a statistically reliable difference, students in the iterative condition outperformed students in the concepts-first condition on the second and third conceptual lessons. Students in the iterative condition took less time per problem (7 min vs. 8.2 min), made fewer errors (6.2 vs. 8.3 errors per problem) and asked for less help (2.1 vs. 2.7 help requests per problem) than students in the concepts-first condition (recall that during the intervention, students worked on a problem until it was correct). Students in the two conditions had similar performance on the second and third procedural lessons. Linking to students' knowledge of borrowing during the conceptual lesson may have strengthened this knowledge and helped to build connections between conceptual and procedural knowledge.

Discussion

Both presenting decimal problems in money contexts and iterating between conceptual and procedural lessons on decimals led to greater success. First, these results contradict the common belief that word problems are harder than symbolic problems (Koedinger & Nathan, 2000). Students solved more problems correctly when they

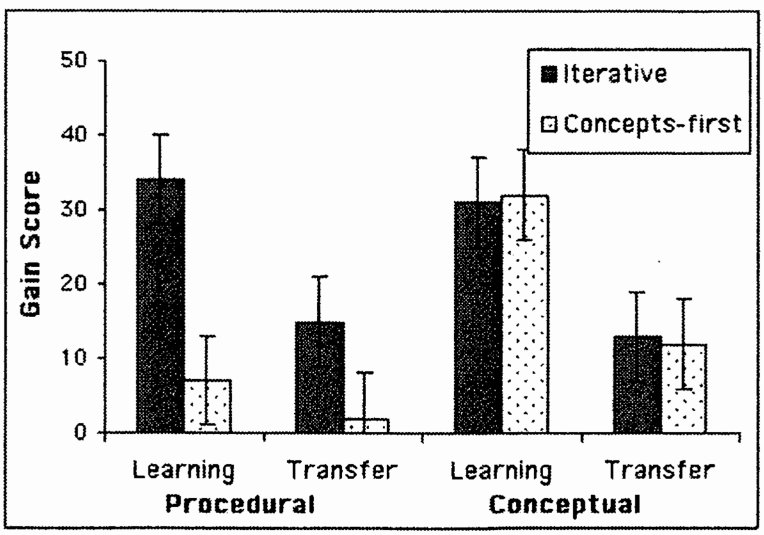

Figure 4. Effect of condition on gain scores on the conceptual and procedural assessments.

were presented in money contexts, possibly because it tapped students' informal knowledge of needing to align digits of the same place value before adding and subtracting and of the relations between different place values for re-grouping.

Our findings also suggest that reform efforts to develop students' conceptual knowledge before their procedural knowledge need to be refined. Iterating between conceptual and procedural knowledge lessons, first in money contexts and later without money contexts, led to greater learning than presenting all the conceptual lessons first. This iterative sequencing was particularly beneficial for procedural knowledge. It seemed to help students avoid an over-generalization of the "align digits on the right" procedure that works with whole numbers. Introducing the procedural task early, and interleaving it with conceptual instruction, seemed to help link and strengthen knowledge shared by the conceptual task and the procedural task. These findings support an iterative model for the development of conceptual and procedural knowledge and help move beyond the debate on which type of knowledge develops first or is more important.

These findings also illustrate the interplay between research and practice. Theoretical constructs and laboratory research on the development of conceptual and procedural knowledge and on the effects of context guided the design of our classroom-based instructional intervention. This intervention led to improved student learning, and the classroom data provided empirical evidence to constrain theories on learning and teaching conceptual and procedural knowledge.

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EXPLANATIONS OFFERED IN COMMUNITY: SOME IMPLICATIONS

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As the responsibility for providing explanations in the mathematics classroom shifts from the teacher to the students, a need arises for mathematics educators to better understand the implications of this transformed learning environment. Using a vignette and student work from a 7th grade mathematics class, we discuss some implications of creating learning opportunities based on student explanations. We note that explanations offered in this context can be understood as explanations in-action or explanations as re-presentations; that is, we make a distinction between explanations offered in the moments of meaning making and the explanations offered to others as a summary (or finished product) of ones own understanding. Further, we discuss how offering explanations is a relational activity and one that has ethical implications within a community.

The last two decades of research in mathematics education have demonstrated the value of nurturing students' mathematical thinking within mathematics classes that function as communities of learners (e.g., Cobb & Bauersfeld, 1995). However, such environments create challenges for the teachers as the responsibility for providing explanations shifts from the teacher to the students—or rather, to the teacher/students community as the teacher continues to play a distinct role in orchestrating the activities and the discourse in the classroom. Ball and Wilson (1996) offer a striking example of how encouraging and validating student solutions can leave the teacher in a difficult position when the students' responses are incorrect. "In considering curriculum, and in contemplating what it means to connect with students, teachers face a complex set of issues over what counts as worth knowing and what is entailed in respecting students as learners" (p. 186). In our work, we are interested in extending previous research to address and understand the conditions for and implications of creating a community of learners who engage in mathematical activity together.

In this paper, we turn our attention to student explanations offered in a classroom community. Through a vignette illustrating the events that arose in a 7th grade mathematics lesson we consider the manner in which explanations were offered and tease out some implications of creating learning opportunities based on student explanations. Specifically, we will address the features of the explanations offered in this context, the means by which they were accepted or rejected, and the possibilities for mathematics knowing that arose in offering and responding to explanations.

Theoretical Frame

Our research is based on the premise that mathematics is a human activity of personal, social and cultural consequence; hence, mathematics and mathematics knowing have ethical significance and implications. Such a premise is important for mathematics educators and learners—indeed for all who engage in mathematics—because it situates mathematics within our actions and interactions with other humans. Broadly speaking, the intent of our research program is to explore the ethics of mathematics knowing as it arises in teachers' and students' actions and interactions in the context of the mathematics classroom.

In our research, we heed Bateson's (1972) advice that we view ethics not as a question about whether the means justify the ends but as a question about the ecology of the act. "We have to find the value of a planned act implicit in and simultaneous with the act itself, not separate from it in the sense that the act would derive its value from reference to a future end or goal" (p. 160). Although a teacher may plan to deliberately do one thing and not another because of the potential consequences of those actions, ultimately, most of the teacher's actions and interactions are enacted in the moment (Mason, 1999; Varela, 1999). Therefore, the ethical implications of mathematics knowing are primarily situated in the moment. Our study fits within and draws on a body of research that attends to the mathematics knowing of children and demonstrates how understanding develops in a classroom community (Bauersfeld, 1988; Cobb, Yackel & Wood, 1992; Confrey, 1999; Yackel & Cobb, 1996). Like Ball and Wilson (1996), who are concerned with the relationship between teaching as a knowledge endeavor and teaching as a moral enterprise, and Lampert (1992) who is interested in the synthesis of effective and responsible teaching, we too are interested in studying the ethical features of teaching and learning mathematics. However, by focusing on the ethics of the moment by moment actions and interactions in the mathematics classroom our goals are to broaden our understanding of the ethical implications of teacher interventions, mathematics curricula, and mathematical explanations when offered in the classroom community. This paper is necessarily an incomplete contribution to this significant conversation of the mathematics education community. It is incomplete in that it is a response from a particular perspective and it is incomplete in that it generates new understandings and questions from which we can continue to explore.

Mathematical Explanations

Our focus in this paper, in particular, arises out of our understanding of explanation as a relational phenomenon (Maturana, 1991). Mathematical explanations are a response 'to' someone *and* something. That is, the act of explanation is a response to an explicit or implicit question raised within the interactions among self-other-subject. The question arises, not from one of these sites, but in the 'dashes' or interactions in between. Explanations are also offerings 'for' someone, including ourselves, in an attempt to either provide an answer to the specific question posed or to broaden understanding of the topic of concern in that moment. Explanations are also responded to 'by' someone. The listener listens for the adequacy of the explanation according to some explicit and imposed or implicit and negotiated criteria. Underlying how mathematical explanations are offered and responded to are a set of assumptions, expectations and emotional responses. The explanations offered and the criteria by which they are assessed, establishes and continually reconstitutes what is viewed as acceptable mathematical practice and discourse in the classroom community.

Gordon Calvert (2001) distinguishes between two forms of explanations that occur in the context of mathematics classrooms: *Explanations in-action* and *explanations as re-presentation*. Explanations occurring in the course of mathematical activity are predominantly stated *in-action*. Explanations expressed in-action are offerings or invitations to oneself and others in an effort to broaden understanding in and for that moment. The explanations do not entail a search for an optimal expression and they are not evaluated against criteria of completeness or perceived correctness, but they are judged according to whether they are plausible, viable and consistent with previous experiences. Explanations stated in-action provide a layer of understanding in response to an immediate question or concern so the participants can continue on in their investigation. The explanation may raise new questions or may provide a means to further their investigation into a larger question. Explanations viewed as acceptable are incorporated into subsequent actions and are reexamined and revised when they no longer prove to be plausible. By interacting with the other through explanations in-action we make space for the other to exist beside us. We create the possibility for community.

Explanations as re-presentation are generally evoked when students are asked to summarize or re-present their explanations to other members of the learning community after a response has been formulated to a terminal question. The formulation involves a search for an optimal expression that encapsulates their activity and purposefully removes known errors and tangential activities. Upon presentation, the listeners' criteria for acceptance shifts from its usefulness for further activity to its perceived correctness and completeness to the question initially posed. Such explanations provide a mechanism for sharing the disparate activities in a classroom and potentially give rise to further questions. Explanations as re-presentations generally are offered within established communities—communities in which criteria for acceptance of the explanation are shared within that community, however loosely or well defined. In the case of mathematics classrooms the explanations offered by students and teachers have particular features that are observed to be mathematical. It is also the case that differing explanations and disagreements may provide opportunities for the community to broaden its understanding as a whole and to 'correct' inconsistencies between offerings of its members.

Returning to the implications and ethics of the situation, we note that whether the explanation is evaluated against criteria of viability and usefulness or correctness and completeness, it is immaterial without the response of the listener. If the listener accepts the explanation of the other then he or she accepts the other as a member of the community in which the explanation is offered. This leads us to ask: What does it mean to accept, reject or disregard the explanation of the other? What are the implications of such responses?

Methodology

Our research program is dependent on the intersection of three research studies, each led by a different researcher, which focus on the interactions among teachers, learners and curricula in mathematics classes. Each of the researchers brings a particular interest to the projects at the same time as being interested in the questions of the other two researchers. We collaborate because we have found it fruitful to cross our research questions and interpretations with the others. Out of this blend comes new interpretations; ones we may not have come to if each of us were focused by the same research question or restricted to a common research site. In our research we interpret student actions and interactions in mathematics classes where students have been invited to offer explanations and listen to the explanations of others; we interpret teacher responses and responsivity in an effort to study teacher responsibility; and we interpret the curriculum that is brought forth in the lived experiences of the mathematics classroom.

In the site from which the data referred to in this paper are taken the researcher/teacher has conducted a year-long teaching experiment in which notions from complexity theory have been transformed and are being used to inform instruction (see Davis and Simmt, this volume). In this study the researcher/teacher takes responsibility for preparing for instruction, teaching the lessons and assessing student understanding. Data collected from these activities are transformed through transcribing audio and videotapes, scanning selected pieces of students' working papers and isolating specific video clips. A key feature of the data selection, transformation and analysis is that at least part of it is done in collaborative sessions where the research teams from each site work together on questions associated with their individual interests, the interests of the others and the questions and meanings that arise in the interactions among the researchers.

Explanations for Teaching/Explanations for Meaning Making

In this section we offer a vignette and some students' work to illustrate how explanations were used for teaching and for meaning making. The vignette has been created from the data compiled during a 50 minute 7th grade lesson which was part of a unit on integer concepts and operations. The students' comments and displays illustrate the range of responses students offered to the class in the large group discussion. During that large group discussion a number of conceptions of integer multiplication were offered and developed. Student work shown in figures 1 and 7 illustrate multiplication as x groups of y integer disks; figures 2 and 8 suggest an understanding of multiplication as repeated addition/subtraction; figure 3 represents integer multiplication as hops on a number line; figure 4 illustrates a variety of connections within the topic; figure 5 demonstrates a procedural approach to multiplication; and figure 6 is a statement of rules for multiplication.

Vignette

The students working in pairs were given the task to find the result of ${}^{+}3 \times {}^{-}4$ and were directed to "show how you know." Immediately after the students paired off they began working. The researcher/teacher walked around the room and passed out double-sided disks to those who requested them. Many of the students took out their personal number lines. Earlier in the year the students had considered multiplication of whole numbers as repeated addition and they had just completed a study of integer addition. It was assumed that the students likely had considered multiplication as x groups of y objects in their elementary school mathematics. Prior to this lesson the students had not done any integer multiplication in this class. However, it is obvious that a couple of the students had been introduced to it before. The students worked for about 20 minutes developing their explanation (in-action). Then each pair of students prepared a display on a large sheet of paper which was subsequently put up on the bulletin board. As is frequently the case in the class, there were a variety of explanations offered by the students.

"Tara, tell me what you did here," the teacher asked as she pointed to Tara's display (figure 1).

"Well, $3 \times {}^{-}4 = {}^{-}12$, because the three groups of negative four chips equals ${}^{-}12$ chips."

"Is there anyone else who has an explanation like Tara's?"

Cathy responded, "Ours is. We have three groups of ${}^{-}4$ that makes ${}^{-}12$" (not shown).

Another student added, "We have ${}^{-}4 + {}^{-}4 + {}^{-}4$. That equals ${}^{-}12$" (figure 2).

"That's a little differently said," the teacher suggested. "Isn't it Karen?"

"Well, it's really the same," Karen replied. "We just didn't draw the chips."

"Actually, ours is the same. We drew the four groups, but we explained it more," commented Tim (not shown)..

"Who did this one?" the teacher asked as she pointed to another display which appeared to use repeated addition (figure 3).

"That's ours," Michael said.

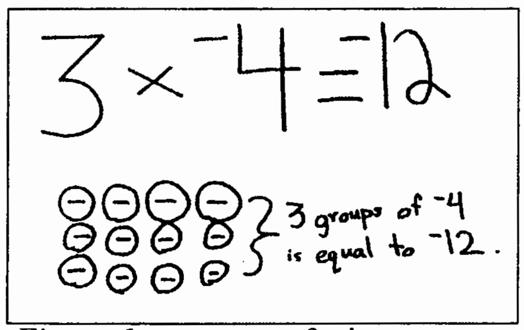

Figure 1. x groups of y integer disks.

$$+3 \times -4 = -12$$

$$-4 + -4 + -4 = -12$$

Figure 2. Repeated addition/subtraction.

"Why is there a zero?"

"He said we needed it." Michael said as he pointed to Aaron.

"Well, you need to start somewhere," Aaron explained. Aaron and his partner used a number line in their explanation. "That's ours over there" (figure 4).

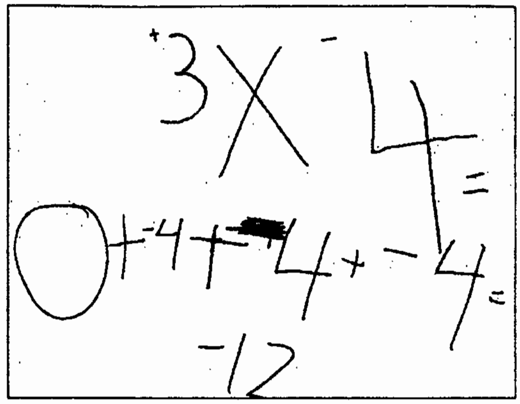

Figure 3. Repeated addition connected to number line.

"Oh, I see what you mean. On the number line you start at zero," the teacher said. At the far side of the room, one student's hand slowly went up. "Yes, Tim?"

"Actually, it doesn't matter where you start," Tim said slowly as though he was thinking through his idea.

"What do you mean?"

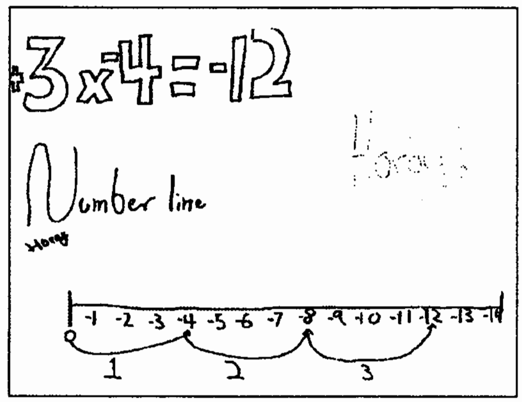

Figure 4. Hops on a number line.

"No matter where you start, negative four three times is always negative twelve," he asserted. "Even if you start somewhere else on the number line" suggesting a vector interpretation. Tim's idea did not get picked up by any of the students in the class but it certainly did add to the possibilities that had been offered.

"What else have we got here?" the teacher asked looking around the room at the posters. "This one is interesting" (figure 5). Katie and Kelsey's poster offered a very "teacherly" display of their thinking with all parts of the poster neatly labeled and a few connections made to language and ideas they had studied before.

"That's ours," said Katie.

"We did it two ways," Les explained (figure 6). "First we did it with our calculator and then we did it like them."

As the class time was coming to an end, the teacher scanned the posters hang-

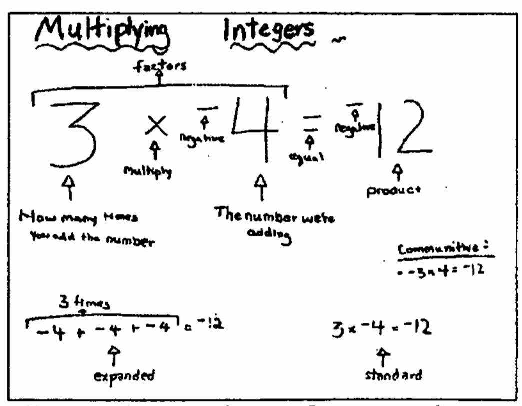

Figure 5. A variety of connections.

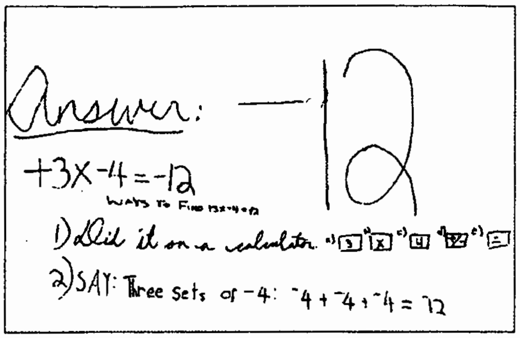

Figure 6. A procedural approach.

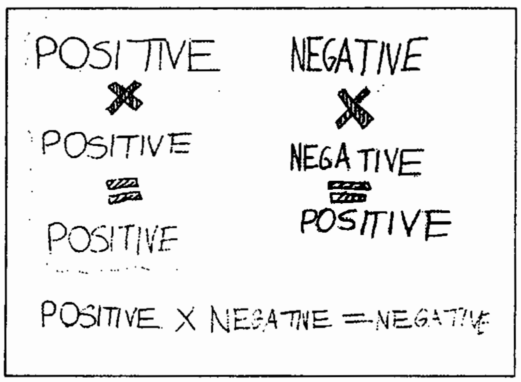

Figure 7. A statement of rules.

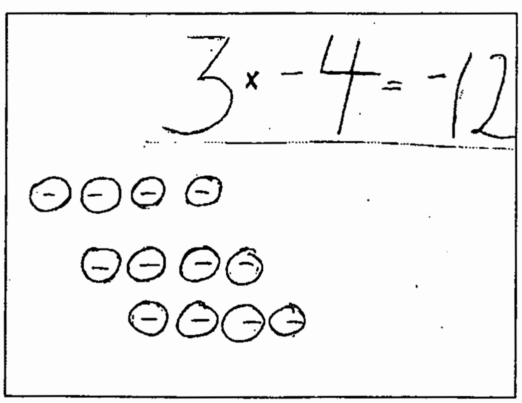

Figure 8. Offered as different from the others.

ing that had not yet been discussed. "Here is another one that is different" (figure 7).

"That's ours," Jessica replied. "I learnt that last year. We wrote the equations on the back," Jessica added as though this helped with the explanation.

Wrapping things up the teacher said, "Well, it looks like the ones remaining are like the ones we have already discussed."

"No, ours is different," Samantha interjected (figure 8).

"Which one is yours?"

Samantha pointed to a poster that showed three groups of four negative disks.

"How is yours different?"

"Well, three times negative four is negative twelve because three times four is twelve and then it is negative because the 4 is bigger," she replied quite confidently.

The teacher hesitated, "Oh, it is? Tell me. What would four times negative three equal?"

After a brief moment Samantha responded, "Positive twelve."

A rumble started. Many of the students started speaking at once.

"No, it's negative twelve."

"It's commutative."

"It's still negative."

What a way to end the class!!

"Jared. You have been asking what's for homework. Here is your homework. I want to know what 3 x –4 equals and what 4 x –3 equals and you must be able to explain your response. We'll continue on from here tomorrow."

Interpretations

While there are many interesting perspectives to take on the explanations offered, we address the following aspects in our interpretations of the students' displays and comments: features of the explanations; possibilities that arose in the offering; and responses to the explanations.

Features of Explanations

The students' displays could be described as explanations as re-presentations. That is, in the sharing activity described above, the students' activities and discussions in their small groups were re-created on chart paper as a response to the question posed. The explanations were created as "finished products" for discussion. While there was a possibility that the explanations provided might not be communicated clearly or thoroughly or that they might spark new ideas, there was no expectation, as they were being created, that the students would need to return to the explanations to work on them further.

From the few examples offered in figures 1 - 8, it can be seen how carefully the students created the displays of their explanations. These displays were neat and orderly; they provided illustrations and, in many cases, text to elaborate their points. These explanations served the immediate purpose of sharing ones thinking about a particular concept to people who were not privy to the original discussions in which the explanations were formulated. It is evident that the students knew that for an outsider to understand their explanations they would need to create an explanation that was both coherent and illustrative. Hence, simply the teacher's demand for a public explanation had consequences for the mathematical knowing that emerged.

The students' displays were also created in anticipation of listeners who held both explicit and implicit criteria for acceptance. In this particular classroom culture the students are aware that their displays are to intentionally communicate what they know. Most groups appeared to respond in similar ways to the explicit "show how you know" criteria. That is, they needed to not only answer the question "what is $^{+}3 \times {}^{-}4$?" but they had to respond to the more implicit concern, "the answer is (in this case) $^{-}12$ because ..." by drawing on their previous experiences in this and in other mathematical communities. We see that students responded by using their prior understanding of number and operations, their recollection of procedures and rules, and by using available tools (e.g., calculator and number lines). Further, it is interesting to note that in figure 5 the students who created that display also appeared to respond to a more general concern, "Show *what* you know about multiplying integers" by labeling symbols and providing definitions.

Possibilities that Arose in the Offering

The linking of explanations by way of same, similar or different conceptual principles provided important dimensions to the possibilities for learning and possibilities for the culture of the learning community. Although these were explanations as representations, they also served the purpose of meaning making at the collective level. The mathematics community itself was brought forth in those acts of explaining and listening. It was our observation that one of the implications of offering mathematical explanations in class was that mathematical concepts (e.g., integer multiplication) and a mathematics community co-emerged in the interactions. Like Maturana (1988), we note that community is formed as participants come to share criteria for accepting explanations. In other words, we observed workings of the criteria of acceptance in the classroom community and the ways in which the implicit criteria for accepting explanations were negotiated and articulated by the community in the process of sharing.

Responses to an Explanation

As illustrated in the vignette, the students were able to comment on their explanations and the explanations of others, as one by one their posters were the focus of attention. Students were eager to explain their displays to the others and most students in the class respectfully listened to the explainers. As students heard the explanations of others, a number of them commented that their explanation was similar: "Mine is like that." "You've already done one like mine". In other cases, students noted that their explanations (although they looked the same to the teacher and other students) were different to the others that had already being described. For example, two posters had three groups of four red disks (representing negative) with the mathematical equation $3 \times {}^{-}4 = {}^{-}12$ written across the top of the sheet (figures 1 and 8). But after hearing the explanation of how the picture showed the result was ${}^{-}12$, one student, Samantha, suggested that her poster (figure 8) was different than the one that looked very similar (figure 1). When asked how it differed Samantha responded, "Ours is ${}^{-}12$ because the 4 is negative and it is bigger." Surprised by her explanation, the teacher asked, "What would 4×-3 equal?" The student quickly responded, "Positive 12." This immediately caused a reaction among the members of the class, as more than one student insisted that this would be -12 as well. A few even tried to explain with the model they had offered. Unfortunately, this response came very late in the class and the student objections ended with the dismissal bell. The students were packing up as the researcher/teacher asked them to think about the products $4 \times {}^{-}3$ and $3 \times {}^{-}4$ and to be prepared to explain "how they know".

In another case, a student took up the explanation of an other and extended it far beyond anything that had been discussed up to that point (and beyond what the teacher had thought about in her planning—see Davis and Simmt, in this volume). In that case, Tim, after hearing Michael and Aaron's discussion about why a zero was included in Michael's equation (see figure 3) argued that it did not matter where to start, $3 \times {}^{-}4$ is always ${}^{-}12$.

Ignoring or paying little attention to a particular explanation is also a response. Notice how the teacher did not pay much attention to the "rules" for multiplying integers. The teacher deliberately passed over this poster a number of times and it was not until late in the class when the students interjected with it that it was introduced to the class discussion. In the case of figure 6, where the students carefully noted the keystrokes they used on their calculator and then offered repeated addition as a second way of showing how they know, the students boasted that they did it two ways. An intervention the teacher made when the students were working in pairs may have triggered them to emphasize that they did it two ways. When the teacher was circulating among the students as they worked, she noted that the students had written the keystroke sequence for the calculator. At that point she indicated to the boys that although this method provided a way of finding the product it did not illustrate an explanation of why $3 \times {}^{-}4 = {}^{-}12$. The students then added the repeated addition illustration. What implicit criteria of acceptance are being created by such actions? The criteria of acceptance are constantly evolving within the classroom community as teachers and students alike offer possible explanations, remain silent about some and challenge yet others.

Conclusions

The act of explaining and offering an explanation is central in the mathematics classroom. As instructional practices move from ones in which teachers take responsibility for explaining, to instructional practices where the full classroom community is responsible for explanations, mathematics educators will need ways to think about these new environments. In this paper we have offered illustrations of possibilities as well as language to talk about the features of explanations; illustrated possibilities that arose from students preparing and sharing explanations, and responding to explanations within the classroom.

Our research suggests that the explanation is one feature of a mathematics class that is fostered by and fosters mathematical communities because it puts people in relationship with one another through their ideas. Explaining and listening in a mathematics class brings forth the community through the development of relationships that arise through making space for the other, and it brings forth a world of mathematics within which the particular group exists, as criteria for acceptance of a mathematical explanation emerge in action. In these ways, we claim the acts of explaining and listening and the resultant artifact—the explanation—are critical to mathematics education. However, once we begin to think about explanations in this way we find ourselves immersed in questions of ethics as we explore how students through their mathematical actions make space in their worlds for others (Varela, 1999). In other words, we suggest that offering an explanation in a grade 7 mathematics class is an ethical act because as Maturana and Varela (1992) note “every act in language brings forth a world created with others in the act of coexistence which gives rise to what is human. Thus every human act has an ethical meaning because it is an act of constitution of the human world (p. 247).”

Note

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FOCUSING ON KEY DEVELOPMENTAL UNDERSTANDINGS IN MATHEMATICS

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Although mathematics educators uniformly are in favor of teaching for understanding, what is meant by "understanding" varies greatly. In this paper, I elaborate the construct of "key developmental understandings," understandings that distinguish among levels of development of a concept. The construct does not represent an area of inquiry that is new to mathematics education. Rather it offers an abstraction of productive activity that has gone on in particular research projects on particular mathematical topics. With this abstraction, we can examine current research, curricula, and teaching. Using a classroom example involving fractions, I illustrate how focusing on key developmental understandings leads to a particular, potentially useful type of thinking and direction for inquiry.

- ***Teaching***. Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.
- ***Learning***. Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. (National Council of Teachers of Mathematics, 2000, p. 11)

Recent discourse in mathematics education has coalesced around the importance of focusing on and fostering students' mathematical understanding. This agreement among mathematics educators has led to a commitment to generate new learning goals for students that are less skewed in favor of skill and facts learning, less focused on what students do and more on how they think. Thompson and Saldanha (in press) asserted that, "conditions for improved instruction entail an enduring discussion of what the community intends students learn" (draft p. 33). As part of this discourse, I introduce the notion of *key developmental understandings* as a way to think about understandings that can be useful goals of mathematics instruction.

The vast majority of mathematics educators would likely report that they endeavor to promote mathematical understanding. However, what they mean by "understanding" varies greatly (Sierpinska, 1994). The word is used so broadly that in mathematics education that it is of little value without further explanation. One of the most common uses of *understanding* is "knowing why something is true or appropriate" (e.g., why the denominator "remains the same" when we add two fractions that have like denominators). Certainly, we want mathematics students to have such knowledge. However, " knowing why" although characteristic of someone who understands, does not specify the important issues in developing understanding, and thus, is inadequate for specifying the goals of instruction. A second common perspective on understanding is characterized by Hiebert and Lefevre's (1986) definition of *conceptual knowledge*: "knowledge that is rich in relationships. . . . a network in which the linking relationships are as prominent as the discrete pieces of information" (pp. 3-4). Although this is descriptive of understanding, it offers little guidance for determining useful goals for student learning.

In the next section, I use an example involving fractions to distinguish and elaborate the construct of *key developmental understandings*. My choice to coin a new term is based on my conclusion that "understanding" is likely to remain a term that lacks specificity and is therefore of little value in a scientific discourse. It is important to note that characterizing learners in terms of their understandings, it is not a claim that these understandings *exist* in the learner. Rather, it is a way that an observer (researcher, teacher) can impose a coherent and potentially useful organization on her experience of the learner's actions (including verbalizations).

An Understanding of Fraction

I have often engaged teachers in thinking about what it means to have an understanding of fractions. Typically they respond that it means knowing that a fraction, *m/n* (my language), means that a whole is divided into n equal parts and that we have m of those parts. Consider this description in light of the following classroom episode.

In a fourth-grade class, I asked the students to use a blue rubber band on their geoboards to make a square of a designated size, and then to put a red rubber band around one half of the square. Most of the students divided the square into two congruent rectangles. However, Mary, cut the square on the diagonal, making two congruent right triangles. The students were unanimous in asserting that both fit with my request that they show halves of the square. Further, they were able to justify that assertion.

I then asked the question, "Is Joe's half larger; is Mary's half larger, or are they the same size?" Approximately a third of the class chose each option. In the subsequent discussion, students defended their answers. However, few students changed their answers as a result of the arguments offered.

I will offer an interpretation of these observations and then use this fraction example to ground the discussion of key developmental understandings. The students that argued that either the rectangular or the triangular half was larger conceive of halves as an *arrangement* in which a whole is partitioned into two congruent parts. They do not understand that partitioning a whole into two equal parts creates a new unit whose size, relative to the original unit (whole), is determined. That is, they do not understand that "one half" indicates a *quantity* (amount), not just an arrangement. Thompson & Saldanha, (in press) describe this limited view of fractions as an additive rather than a multiplicative relationship.

Educators, who understand a fraction as a quantity, find it difficult to think about this limited conception of one half (as an arrangement). I offer the following way of thinking about this limited conception. One can partition a square into two rectangles with any cut parallel to one of the sides. Any such partition will create two parts that can be compared to each other and which sum to the whole. However, in the case where the partition results in equal parts, an important part-whole relationship is determined (from the perspective of those who understand it) – a new, specified unit of quantity is constituted. This special relationship between the part and the whole, created by equal partitioning, is neither obvious nor automatic to the young learner who is just beginning to explore fractions. It is an example of a *key developmental understanding*.

What is a Key Developmental Understanding?

I use the above example of fraction understandings to point to two characteristics of key developmental understandings.

A first characteristic is that key developmental understandings involve a conceptual advance on the part of the learner. By conceptual advance, I mean a change in the learner's ability to think about and/or perceive particular mathematical relationships. In the fraction example, understanding that equal partitioning creates specific units of quantity represents a significant advance allowing the learner to conceive of and act with fractions in more powerful ways. Those who had this key understanding found the question of which half is bigger to be trivial. For those who did not have this key understanding, not only could they not see that the two differently shaped halves were necessarily equal, they could not follow the reasoning of those who could.

This last observation leads to a second characteristic: An indication that a key developmental understanding is involved is that those students without the knowledge do not tend to acquire it as the result of an explanation and/or demonstration. A key understanding is an important developmental step, one that is essential to recognize as a goal of instruction.

How Does One Identify a Key Developmental Understanding?

Mathematics educators often cannot identify key mathematical understandings by examining their own mathematical understandings. What were key developmental issues early on in the development of their understandings are not apparent as they examine their now sophisticated understandings. Indeed, many key understandings develop without the learner's awareness that a conceptual advance has taken place. In the fraction example, I indicated that frequently educators consider one half to be one of two equal parts of a whole. Whereas this is a reasonable way to describe one's idea of one half, it is virtually useless in guiding instruction towards a concept of a fraction. The young learner can easily learn that the parts must be equal (usually assimilated as "identical" and based on their notion of fair sharing) and because they have whole number concepts, they can learn the correspondence between the whole numbers and the parts involved. Thus, it is relatively unproblematic for learners to identify a typical shaded rectangle with particular vocabulary ("one half") and symbols ("1/2"). None of this represents a transition from thinking about whole numbers to thinking about fractions.

The most reliable way to identify key developmental understandings is to observe learners engaged in mathematical tasks in order to specify understandings that can *account for differences in the actions used by different learners to perform the same task.* In the fraction example, contrasting the learners who claimed that the triangular half was larger with the learners who claimed that the triangular half was equal to the rectangular half creates a context for identifying a key aspect of understanding fractions. This was in fact the strategy of Piaget (c.f., 1952) and his colleagues.

To summarize: a key mathematical understanding is a conceptual advance that is important to the development of a concept. It identifies a qualitative shift in the learner's ability to think about and/or perceive particular mathematical relationships, that is a change in the assimilatory structures that the learner has available. The emphasis here is on "ability to think about and/or perceive." I am not referring to a missing piece of information that affects one's performance. Rather, I emphasize that without engaging in a developmental process, the learner lacks a particular mathematical ability. The set of key developmental understandings specified for a concept is different than the taken-as-shared knowledge that the mathematical community has about the concept. Although there may be some overlap, these sets are often quite distinct. For example the formal definition of a rational number is quite different from the understanding of a fraction as a quantity discussed earlier.

One additional point is important to emphasize. Key developmental understandings can be specified at different levels of detail. If we return to the example of fraction-as-an-arrangement and fraction-as-a-quantity, we can compare that specification to the more detailed work of Steffe (2002) and Tzur (1999). The level of detail specified for a key understanding is adequate if it serves to guide the effort for which it is needed (e.g., teaching, curriculum, further research).

Key Developmental Understandings and Prior Work in Mathematics Education

Significant work has been done in in particular areas of mathematics education that could be characterized as research on key developmental understandings. Piaget (c.f. 1952) and Steffe (cf. Steffe, Cobb, & von Glasersfeld, 1988) have both directed major research programs of this type. Other examples include Thompson (1994), Tzur (1999), Heinz (2000), Simon and Blume (1994), Confrey (1994), and Behr, Harel, Post, and Lesh (1994). Thus, my postulation of the construct, key developmental understandings, is aimed at articulating a distinction that can be observed among existing examples. Although these examples exist and are valuable parts of the mathematics education literature, such research is still not widespread. Further, attention to key developmental understandings is generally not the basis on which curricula are developed. My elaborating this construct has three related purposes:

1. As mentioned in the introduction, the mathematics education community has used the term *understanding* in a nebulous fashion, and therefore the term has lacked specificity and power.

2. By abstracting from existing examples and naming the construct of *key developmental understandings*, I wish to call attention to a powerful basis for mathematics instruction.
3. By elaborating this construct, I aim to promote explicit discussion of and increased inquiry into key developmental understandings and their role in mathematics instruction.

Although some of the research done in the last 25 years has contributed to our knowledge of key developmental understandings, much more work of this type is still needed.

The Role of Key Developmental Understandings in Mathematics Education

Understanding mathematical development involves having a sense of developmental steps and understanding how learners can progress from one step to another. Identifying key mathematical understandings is a way of specifying developmental steps. As key understandings are specified, inquiry into how these understandings develop and how they can be fostered becomes possible. Research is needed that can identify key understandings and describe developmental processes for many mathematical concepts.

A focus on key developmental understandings represents a significant change from a more conventional focus on students' knowledge. The former engenders a different set of questions for inquiry and a different direction for assessment and instruction. I use the fraction example again to illustrate this point.

If a mathematics educator identifies understanding of one-fourth as knowing that it is one of four equal parts that make up a whole, what kind of assessment might she generate? One possibility is that she creates a series of diagrams for which the students must decide if each represents one-fourth. Some of the diagrams would have one of four congruent parts shaded. Some would have more than one of the four congruent parts shaded. Some would have a different number of congruent parts and some would have unequal parts. (The reader is probably familiar with such assessments.) This assessment is appropriate for the specification of one-fourth as "one of four equal parts of a whole." That is, the assessment provides evidence of whether the student knows that four parts are involved, that they must be equal, and that we are attending to one of those parts. However, if one begins with an articulation of fraction-as-a-quantity as a key developmental understanding, then Figure 1 becomes an important assessment item, potentially discriminating between a student who understands a fraction as an arrangement and one who understands a fraction as a quan-

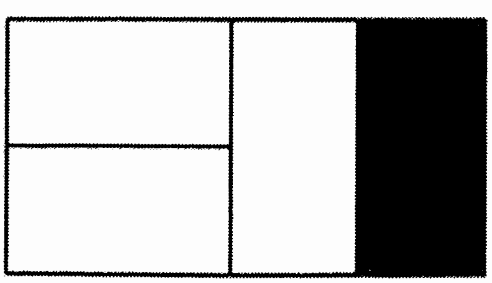

Figure 1

tity. (This claim is made acknowledging the weakness of drawing conclusions from a single assessment item.)

Continuing the comparison of these two views of understanding, a conventional approach to a student's difficulty with a problem is to ask, "What is it that the student does not know?" The approach based on key developmental understandings is to ask, "How is the student who can solve the problem different from the student who cannot?" This is a subtle, but important distinction having to do with whether the focus is on t*he mathematics as seen by the one who understands it* or on *distinctions in the learners' understandings of the mathematics.* Again, let's use our fraction scenario. A response to the classroom situation might be to conclude that, "the students do not know that the shape of the one-half does not matter." This seems to describe the situation. Based on this conclusion, the intervention might encourage the students to cut up the triangular half and superimpose it on the rectangular half. This exploration might be expanded to other shapes and other fractions. The expectation would be that the students would eventually conclude that the shape of the fraction is not important, that one-half of any shape is the same size. This intervention has many of the features associated with current mathematics education reform efforts, hands-on problem solving, student exploration, attention to patterns, and students drawing their own conclusions. However, what is missing is the key understanding that underlay differences in the students' response to the original task.

The students who originally claimed that the halves were the same did not make that claim because they had cut up a rectangular half previously, nor because they were visualizing such a cut. They knew instantly and with certainty that the halves had to be the same size. The pedagogical intervention that involves cutting up the triangle is aimed at getting students to *know that* the two halves are in fact the same size. The intervention does not engender the anticipation that they *must be* the same size. That is, the intervention does not deal with conceptualizing that the result of equal partitioning is a new unit that has a fixed relationship to the original unit (whole). Tzur (1999) described a teaching experiment with students aimed at promoting this key understanding. Because it is focused on an underlying understanding that is key to the students' development with respect to fractions, the interventions are quite different from the fraction-cutting activity described in the preceding paragraph.

The type of question asked by one who is interested in key developmental understandings, "How is the student who can solve the problem different (conceptually) from the student who cannot?" leads to a particular type of inquiry. For example, one of the more intractable problems in mathematics education is how to foster growth in students who use an additive comparison for situations involving multiplicative comparisons (ratio). Defining ratio and creating lessons to make sure students have all facets of the definition are not likely to foster the understanding needed. An inquiry into the (developmental) differences between the student who can identify multiplicative comparisons and the one who cannot (or cannot for particular types of situations) is more likely to produce a breakthrough in this area.

The construct that I have proposed in this paper does not represent an area of inquiry that is without precedent in mathematics education. Rather it offers an abstraction of productive activity that has gone on in particular research projects on particular mathematical topics. With this abstraction, we can examine current research, curricula, and teaching. With respect to research, much more research into key developmental understandings is needed in most mathematics content areas. When we examine available curricula, including recent reform-based curricula, we generally do not see an orientation toward key developmental understandings. Finally, thinking in terms of key understandings is not commonplace among teachers of mathematics at any level. This is a potentially important focus for mathematics teacher education.

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RE-CONCEPTUALIZING PROCEDURAL KNOWLEDGE: THE EMERGENCE OF "INTELLIGENT" PERFORMANCES AMONG EQUATION SOLVERS

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This paper explores the development of students' knowledge of mathematical procedures. Students' tendency to develop rote knowledge of procedures has been widely commented on. I postulate an alternative, more "intelligent" endpoint for the development of procedural knowledge, where students choose to deviate from established solving patterns on particular problems for greater efficiency. The purpose of this study was to explore the instructional conditions that facilitate the emergence of this outcome. Students with no prior knowledge of formal linear equation-solving techniques were taught the basic transformations of this domain. After instruction, students engaged in problem-solving sessions in two conditions. In the treatment group, students completed "alternative ordering tasks," where they were asked to re-solve previously completed problems but using a different ordering of steps. Completing alternative ordering tasks was found to lead to more intelligent solving.

Introduction

For much of this century, mathematics educators have sought to address students' tendency to view school mathematics as a series of procedures to be memorized. Researchers in mathematics education concur that (a) procedures learned by rote are easily forgotten and error-prone; and (b) the learning of procedures must be connected with conceptual knowledge in order to foster the development of understanding (e.g., Hiebert & Carpenter, 1992). The National Council of Teachers of Mathematics (NCTM) has articulated this emphasis on conceptual learning by calling for decreased attention to "memorizing rules and algorithms; practicing tedious paper-and-pencil computations; memorizing procedures ... without understanding" (NCTM, 1989, p. 71); and "rote memorization of facts and procedures" (NCTM, 1989, p. 129).

There is little doubt that the rote execution of memorized procedures does *not* constitute "mathematical understanding." However, there are other ways in which a procedure can be executed other than by rote, some of which could be characterized as "intelligent" or even as indicative of "procedural understanding" (Greeno, 1978). But few prior studies have considered procedural outcomes other than rote knowledge, much less explored its development. This paper attempts to map out this terrain: I examine the development of students' knowledge of mathematical procedures, with particular emphasis on examining learning outcomes other than rote execution.

Perspective

Fundamentally, executing a procedural skill requires that one have knowledge of its component steps and the order in which these steps should be applied. But not all performances of a skill are the same. In particular, skillful execution in mathematics can mean two very different things. On the one hand, skillful execution involves being able to use procedures rapidly, efficiently, with minimal error, and with minimal conscious attention; in other words, to execute a procedure automatically or by rote (Anderson & Lebiere, 1998). On the other hand, being "skilled" means being able to select appropriate procedures for particular problems, modify procedures when conditions warrant, and explain or justify one's steps to others; that is, to execute a procedure thoughtfully or deliberately (Ericsson & Charness, 1994; Karmiloff-Smith, 1992), "relationally" (Skemp, 1976), "mindfully" (Brown & Langer, 1990; Langer, 1993), or "intelligently" (Ryle, 1949).

Although acknowledging that both notions of "skilled" are important and necessary (National Research Council, 2001), mathematics educators have had difficulty integrating these two competing visions of mathematical proficiency. The tension between these two visions is a foundational issue in mathematics education: it not only pertains to our educational goals for students, but also speaks directly to what it means to know and to do mathematics.

While the first outcome for successful skill execution (automaticity) has been frequently examined by cognitive scientists (Anderson, 1982; Anderson & Fincham, 1994; Anderson & Lebiere, 1998), "intelligent" execution of procedures has been less widely studied and is thus the focus of this paper. I begin by articulating what I mean by intelligent execution of procedures. I then describe a study that explored the development of this capacity.

Framework

What does it mean to intelligently execute procedures?). I have proposed elsewhere that two central features of intelligent execution of procedures are (a) flexibility, and (b) innovation (Star, 2001a, 2001b). Flexibility refers to the ability to use a wide range of mathematical procedures in order to generate the best solution for particular problems (Beishuizen, van Putten, & van Mulken, 1997; Feltovich, Spiro, & Coulson, 1997). Flexible solvers have knowledge of standard solution procedures, but they also choose to use alternative or non-standard procedures on certain problems, when doing so results in a better or more efficient solution. Metaphorically, flexible solvers have more tools in their procedural toolbox.

Another feature of intelligent execution is innovation (Gick, 1986; Ryle, 1949; Simon & Reed, 1976). Innovation refers to the ability to use steps within a procedure in atypical ways in order to produce a more efficient solution. An innovative solver is able to use the individual steps of a procedure in ways other than that suggested by a standard solution. Metaphorically, innovation refers to the ability to use the tools in one's toolbox in non-standard ways that do a better job of performing certain kinds of tasks.

Both innovation and flexibility can be seen in the example solutions shown in Table 1. Note that the three problems in Table 1 are almost identical, but they have been solved using three different solution strategies. Problem A is solved using a standard solution method, one that is commonly and explicitly taught as *the* way to solve linear equations in US schools. Problems B and C could have been solved using this exact same method, but the solver has recognized and capitalized on the opportunity to use different strategies for these two problems -- strategies that are at least as good as the standard method but that could not be used in problem A. A solver who chooses to use the three solution strategies shown in Table 1 on this set of three problems is demonstrating flexibility; she has knowledge of multiple solution procedures and can select the most appropriate one for a particular problem.

Table 1. Example Solutions to Three Linear Equations

Problem A:

$$\begin{aligned} 4(x+1)+2(x+2) &= 3(x+4) \\ 4x+4+2x+4 &= 3x+12 \\ 6x+8 &= 3x+12 \\ 3x &= 4 \\ x &= 4/3 \end{aligned}$$

Problem B:

$$4(x+1)+2(x+1)=3(x+4)$$
$$\begin{aligned} 6(x+1) &= 3(x+4) \\ 6x+6 &= 3x+12 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Problem C:

$$4(x+1)+2(x+1)=3(x+1)$$
$$\begin{aligned} 6(x+1) &= 3(x+1) \\ 3(x+1) &= 0 \\ x+1 &= 0 \\ x &= -1 \end{aligned}$$

Innovation can be seen by looking closely at the solution steps used in problem B. In her first step, the solver has combined the terms $4(x + 1)$ and $2(x + 1)$ to yield $6(x + 1)$. The way that this solving step, "combining like terms," is more typically used is to combine variable terms (such as $4x$ and $2x$) or constant terms (such as 4 and 2), as was done in the standard solution method seen in problem A. In problem B (and also in problem C), the solver has used the same "combine like term" step, but in an atypical way that results in a solution that is arguably better. This atypical or non-standard use of an equation-solving step is what is meant by innovation.

Framing intelligent equation-solving in this manner raises the question of how innovation and flexibility develop. This question is largely unexplored. Basic skill practice has been linked to the development of rote knowledge (Anderson, 1982; Fitts, 1964), but the development of more flexible knowledge appears to require a different kind of practice, which has been referred to as "deliberate" (Ericsson, Krampe, & Tesch-Romer, 1993). One hypothesis for what such deliberate practice looks like comes from studies where participants were asked to solve a problem repeatedly in order to observe changes in their solutions that emerged with practice. There is ample evidence that solving a problem multiple times can lead to more automatic execution (e.g., Simon & Reed, 1976; Anzai & Simon, 1979; Blessing & Anderson, 1996; Koedinger & Anderson, 1990). However, there is also reason to hypothesize that, under certain conditions, re-solving previously completed problems can lead to more intelligent solving (Blöte, Klein, & Beishuizen, 2000; Krutetskii, 1976).

In the present study, I test this hypothesis by utilizing a task I refer to as the "alternative-ordering task." Participants are asked to re-solve previously completed problems but using a different ordering of steps. In this task, students are not merely practicing the same solution over and over again, but instead are generating, comparing, and evaluating the effectiveness and efficiency of different solution procedures. There is reason to speculate that such a task may lead to more intelligent solving, in the form of greater innovation and flexibility.

Goals

In my prior work (Star, 2001a, 2001b) I demonstrated that intelligent execution of procedures, as described above, exists in school-aged learners, and I explored the development of this capacity among solvers working individually on a paper-and-pencil task. In the present work, I build upon these initial findings by examining the development of intelligent execution among groups of students in a simulated classroom setting.

Method and Data Sources

Thirty-six 6th grade students volunteered to participate in this study. Students attended one-hour experimental sessions in groups of six for five consecutive days. The mathematical domain that I chose to use in this study was linear equation solving. A pre-test on linear equation solving was administered on Day 1. Students were then given a brief 30-minute scripted lesson on the steps used to solve linear equations (e.g., adding a constant to both sides of an equation, adding like terms, etc.). Following instruction, students were given a post-instruction test on these steps.

At the conclusion of instruction, all six students were randomly assigned to a treatment or a control group. Both groups devoted the sessions on Days 2, 3, and 4 to equation-solving practice. During these three problem-solving sessions, students solved a great variety of equations, some of which were very straightforward (e.g., $2x + 4 = 10$), while others were much more complex (e.g., $4(x + 2) + 6x + 10 = 2(x + 2) + 8(x + 2) + 6x + 4x + 8$).

In the problem-solving sessions, students engaged in alternating cycles of individual work followed by group discussion. The treatment and control groups differed only in the content of the group discussion. In the treatment groups, the discussion centered on students comparing their solution methods, discussing the differences between solution methods, and generating alternative solution methods. In the control groups, students participated in a discussion of identical length and concerning the same problems. However, the discussion focused on the correctness of numeric solutions and methods of checking numeric solutions. On Day 5, students in both conditions were given a common post-test.

Results

The most interesting result was that students in the treatment and control groups differed in the ways that they chose to approach equations by the end of the study. Significant differences emerged in both students' flexibility and innovation.

With respect to flexibility, treatment students were significantly more likely to become flexible solvers than control group students: In particular, treatment students were more likely to use several different solution methods on the post-test problems, while control students were more likely to rely upon a single solution for all problems. The repeated equation-solving practice that control group students received resulted in the discovery, for each individual, of a dependable, favorite solution method that was used on many subsequent problems. Sometimes a student's favorite solution method was an efficient one; however, in other cases, students reliably and consistently used solution methods that were quite inefficient. For example, Table 2 shows the solution strategy that Billy (a student in the control group) used on several problems toward the end of the study. Billy repeatedly moved variable and constant terms back and forth, from one side of the equation to the other. Despite the inefficiency of this approach, it is one that Billy used consistently. In contrast, treatment students, ostensibly because they were repeatedly asked to consider alternative ways that equations could be solved, did not settle into a favorite, consistently used solution strategy. Rather, treatment students varied the ways that equations were solved in the post-test depending on the specifics of individual problems, demonstrating knowledge of multiple solution strategies. Treatment students were not content merely to solve an equation using an algorithm that was known to always work for them; they tried multiple approaches in order to arrive at the best solution for a particular problem.

Table 2. Billy's Solutions to Problem 4.2

Billy:	$3(x+1)+6(x+1)+6x+9$	=	$6x+9$
	$3x+3+6x+6+6x+9$	=	$6x+9$
	$3+6x+6+6x+9$	=	$3x+9$
	$3+6x+6+6x$	=	$3x$
	$3+3x+6+6x$	=	0
	$3+6+6x$	=	$3x$[a]
	$3+6+3x$	=	0
	$6+3x$	=	-3
	$3+3x$	=	0[*]
	$3x$	=	-3
	x	=	-1

[a]Indicates an error in how a transformation has been applied.

This increased flexibility appears to be related to innovation: there is evidence that treatment students in this study were more likely to innovate than control group students. Recall that innovation is the use of an equation-solving step in an atypical way that results in a better solution for a particular problem. 81% of treatment students showed signs of emerging innovation on at least one problem attempted during the 3 problem-solving sessions, while only 15% of control group students did so. Innovation is illustrated in Table 3, which shows Anna's (a student in the treatment group) solutions to equations she encountered in the second and third problem-solving sessions. Note the difference between how Anna solved problems 2.2 and 3.2, two problems that are structurally identical. Anna's solution on problem 3.2 is more efficient than the one used on problem 2.2, in that she chose to divide to both sides as a *first* step (on 3.2) rather than a last step (on 2.2). (Dividing to both sides as a last step is the typical way that students solved equations such as this one.) Anna's use of the "divide to both sides" step" atypically is an example of an innovation. Anna came to this knowledge as a result of the treatment: the generation, comparison, and reflection on multiple solution strategies. Control group students were significantly less likely to produce innovative solution strategies such as this one illustrated in Table 3.

Discussion

Table 3. *Anna's Solutions to Problems 2.2 and 3.2*

Problem 2.2:	$3(x+1) = 15$
	$3x+3 = 15$
	$3x = 12$
	$x = 4$
Problem 3.2:	$3(x+2) = 21$
	$x+2 = 7$
	$x = 5$

This study provided evidence that engaging in alternative ordering tasks, which involved re-solving previously solved equations using a different ordering of steps, led students to believe that equations could be solved in more than one way and that some strategies were better than others. Treatment students' cognizance of multiple ways that equations can be solved led to an increase in their ability to innovate, where innovation refers to the use of a step in an atypical way that results in a better solution. The ability to innovate was also related to increased flexibility in treatment students' solutions, where flexibility refers to a reluctance to rigidly adhere to the exact same solution sequence when solving similar problems. Students who did not experience this treatment were more likely to develop one solution method that was rigidly adhered to on all problems.

This study adds to the literature on equation solving by shifting the focus from students' errors to the capacities that successful performers exhibit. A review of the literature on the use of mathematical procedures (with its emphasis on cataloging the multitude of errors that students make) suggests that the most important feature of success in this domain is the ability to rapidly execute error-free procedures. The present study suggests that another important feature of a successful solver is the ability to intelligently use procedures; that is, to selectively choose to deviate from standard and practiced methods in order to produce even more efficient solutions. Students who have capacities for innovation and flexibility have more sophisticated knowledge of equation solving transformations that only emerges in their application. This outcome of learning procedures has not previously been considered in the mathematics education literature.

Implications. The fact that the alternative ordering task was effective in this study suggests that these results could be used to inform classroom practice in several ways. First, during instruction on equation solving (and other symbolic mathematical procedures), teachers should frequently and regularly ask students to re-solve previously completed problems using a different ordering of steps. The multiple solutions that are generated in such a task can then be compared and contrasted. The study described in this paper suggests that the incorporation of such tasks will result in substantial gains in students' ability to innovate and be flexible.

Implicit within this recommendation is a caution *against* direct and explicit instruction on the "standard solution method." Especially for novice learners, teachers should avoid labeling any one solution method as being the best way, the right way, or the only way. Benefits that arise from engaging in the alternative ordering tasks come when students think carefully about how to generate additional solution strategies and how to compare multiple solution strategies. Students come to their own conclusions about the features that identify one solution as different from another (e.g., efficiency), and direct instruction on a standard, efficient procedure would appear to subvert this process.

A challenge that necessarily accompanies these recommendations concerns student motivation. Many teachers find students to be uninterested in learning how to solve equations, and so it might appear that asking students to re-solve previously completed equations would further reduce already low levels of motivation. This is certainly a valid concern. However, there is a great deal of evidence from the elementary grades that such concerns can be addressed. Many examples exist of classrooms where a climate has been created that incorporates the features that are integral to the recommendations detailed above: student collaboration, the sharing of multiple solution strategies, and the group comparison and evaluation of mathematical procedures and reasoning (Ball, 1993; Chazan & Ball, 1999; Lampert, 1990). There are fewer examples of this kind of classroom environment at the high school level, particularly related to the instruction of mathematical procedures. The study described here suggests that there is much to be gained from efforts to make such changes at the secondary level.

Conclusion

Procedures are an integral component of mathematics. While fluency is certainly one educational outcome, this paper has identified another in the ability to vary the ways that one uses procedures on particular problems in order to arrive at maximally efficient solutions. Krutetskii captured this distinction as follows:

> Incapable students are marked by inertness, sluggishness, and constraint in their thinking in the realm of mathematical relations and operations. ... Mathematically able students are distinguished by a greater flexibility, by mobility of their mental process in solving mathematical problems. It is expressed in a free and easy switching from one mental operation to another qualitatively different one, in a diversity of aspects in the approach to the problem to problem-solving, in a freedom from the binding influence of stereotyped, conventional methods of solution, and in the ease in reconstructing established thought patterns and systems of operations. ... Very typical of capable students is a striving for the most rational solution to a problem, a search

for the clearest, simplest, shortest, and thus most 'elegant' path to the goal. (Krutetskii, 1976, p. 282-3)

While incapable students -- those with rote knowledge of procedures -- are relatively easy to find, intelligent solvers present a much more significant challenge. The study described in this paper represents a first attempt to re-conceptualize procedural knowledge so as to include such a relational outcome. If flexibility and innovation in the use of procedures are integral to our educational goals for students, further investigation of the development of this kind of procedural knowledge is vital.

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STUDENT-TO-STUDENT QUESTIONING IN THE DEVELOPMENT OF MATHEMATICAL UNDERSTANDING: SIX HIGH SCHOOL STUDENTS MATHEMATIZING A SHELL

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This paper describes the nature of student-to-student discourse questioning as learners construct a mathematical model for the growth of a spiral shell. Students' natural language becomes mathematically formalized as students' questioning patterns evolve. Questions posed by students are viewed as pivotal mechanisms in students' discursive drive toward building deep understanding.

This paper describes continuing research on the nature of student-to-student discourse questioning as learners negotiate and assimilate strategies to construct a mathematical model for the growth of a spiral shell.[1] Research on the use of language by students doing mathematics has generally been framed by classroom contexts and uniquely developed symbolic systems. Student discourse as negotiation of mathematical understanding has been described as being grounded in the situated social context of a classroom. However, little attention has been paid to specific linguistic aspects of students' natural[2] language use in mathematical reasoning that give rise to mathematical language and understanding within a classroom domain (Morgan, 2000). Language structure patterns in mathematics classrooms evolve, as do structural patterns learned and used by speakers in the processing activity of language acquisition (Langacker, 1999). Language structure patterns within institutional contexts such as classrooms "appear to be characterized by asymmetric distribution of questions and answers among the participants" with questioning dominated by the teacher and answering identified with the role of student (Drew & Sorjonen, 1997).

Questioning in mathematics classrooms traditionally occurs when the teacher poses questions to students and occasionally, students may pose questions to the teacher. Significant research has been conducted on teacher questioning of students (Maher & Davis, 1990; Maher, Davis & Alston, 1992; Maher & Martino, 1992; Martino & Maher, 1999; Martino & Maher, 1994). Teachers whose practices include listening carefully to the language of students, following and fostering the growth of student ideas, may recognize distinctions between the kinds of questions students do ask. For example, teachers may interpret certain student questions as solicitations to the teacher to repeat a previous statement, or questions may be perceived as requests for the teacher's expert evaluation of student performance, queries for procedural instruction, or questions that suggest conditional possibilities.

Recently, research has shifted from teacher centered questioning, toward examining the significance of student-to-student questioning in the development of mathematical understanding. This shift appropriately coincides with an increased emphasis in student centered learning as students propose mathematical ideas and actively engage in small group learning experiences (National Council of Teachers of Mathematics [NCTM], 2000). Current research, which investigates student interaction without teacher intervention, suggests that students ask distinctly different types of questions as they work together to build understanding and develop mathematical language and models (Walter, 2001). Central to this research is inquiry into how mathematical understanding is shaped and promoted by the structure of questioning patterns in student-to-student discourse.

Perspective

One perspective on the learning of mathematics views human experience as situated and constituted within social contexts. This perspective minimizes the consideration of individual learning in isolation from the social contexts in which learning is situated. Therefore, from this point of view, individual knowledge is concurrently constructed with internalized social interaction.

Instructional environments affect learning and shape the ways and extents to which learners participate in classroom discourse. Discourse analysis contributes to our understanding of language as social filter (Brown, 2001). In other words, participation in a discourse is filtered by the extent to which a participant appropriates language and symbols peculiar to that discourse (Doerfler, 2000; Philips, 1988). Traditional classroom instructional environments often suppress mathematical discourse between students and filter out the experiencing of mathematics as active model building. Mathematics as model building is a central process in the work of professional mathematicians and a major thrust identified in the NCTM Principles and Standards (2000).

The development of mathematical language within student discourse may best be examined within the context of observable social factors. Therefore, the use of videotaped student interactions as one data source "can make visible subtle nuances in speech as well as non-verbal behaviors" (Powell, Francisco, & Maher, 2001). Thus, identification of subtle differences in student-to-student questions is made possible through analysis of student interactions captured on video. For the purposes of this paper, the issue of particular interest is how student-to-student question patterns evolve and influence the emergence of meaningful mathematical language.

Method

Six high school students, working together in pairs or as a group, were videotaped with one camera. The videotapes were digitized, compressed, and saved as MPEG1 files. The video files were time-code linked with databases of video content description, critical events, verbatim transcript of student utterances and coded events within the qualitative research software vPrism (Walter & Maher, 2002; Powell et al., 2001). Video content descriptions were carefully created to broadly convey as objectively as possible what occurred on video. These descriptions are important contextual referents and are included in the episodes presented here. Critical events were identified as specific video clips that captured and conveyed peculiarly significant utterances or behaviors germane to inter-student questioning and mathematizing the spiral. From the vantage created by selected critical events, analysis proceeded by re-examining what factors contributed to a particular event and what impact the event might or might not have had in subsequent student interactions.

Iterative re-examination of events is a principle feature for the development of detailed storylines in this qualitative model of analysis. During this process, codes were developed, modified, and refined to trace themes and patterns in student-to-student questioning. Differences in the nature of student questions became evident as question units were identified and contextually analyzed and re-analyzed. Question codes were developed to include: (QA) question that checks attunement between participants' understandings and seeks demonstrated mutual agreement; (QI) interrogative question for information that is not procedural; (QP) procedural question; (QC) confirmation request by participant regarding participant's own conceptual understanding, differs from attunement by not demonstrating concern for "the other's" understanding; (QS) speculative question that posits potential; and (QR) rhetorical question. Students' mathematical inscriptions and journal entries provided triangulation of data.

Data and Analysis

This qualitative study presents analysis of several video vignettes of six high school students who were invited to participate with other students in a summer pre-calculus institute as part of a fourteen-year longitudinal study of children's mathematical thinking conducted by Carolyn A. Maher, Rutgers University. Students gathered for four hours each day for two weeks to do mathematics in a simulated classroom setting.

Students were presented with a sequence of open-ended mathematical modeling tasks during this time. One task involved polar coordinates, a topic with which most of the students had little experience[3]. The task featured a photocopy of a fossilized spiral shell that students held and inspected. The fossil had been sliced in half to reveal the intricate, segmented internal structure that anciently housed and protected the growing organism. Students were asked to find the center of the shell, draw a ray, organize a table of measurements and say what they could about "r as a function of q."[4] The modus operandi in the longitudinal study was for students to freely work with one another while researchers, taking field notes, respectfully observed. Students were customarily not given instruction on procedure or content, although a three-minute intervention did take place approximately twenty-one minutes after students began working on the task.

The six students spotlighted in this study developed three mathematical models. This paper presents an analysis of the evolution of questioning patterns in the first forty-five minutes of student discourse during the initial development of one model, a tabular representation of concentric circles (Figure 1).

After the task was introduced, students spent the first seven minutes negotiating agreement on where the center of the spiral should be located, whether "r" represented radians or radius, and what "r as a function of theta" meant. Students asked twenty-five questions during this interval which were primarily attunement (10) and interrogative (9) questions. Attunement questions focused on agreement for the location of the center while interrogative questions attempted to refine understanding of what theta, radian, and radius meant. Students continued to refer to the written task as the authoritative source in justifying their ideas. Student language is very generalized, such as when Michelle refers to the center as "this little hole" and Angela refers to the spiral shell as "the thing".

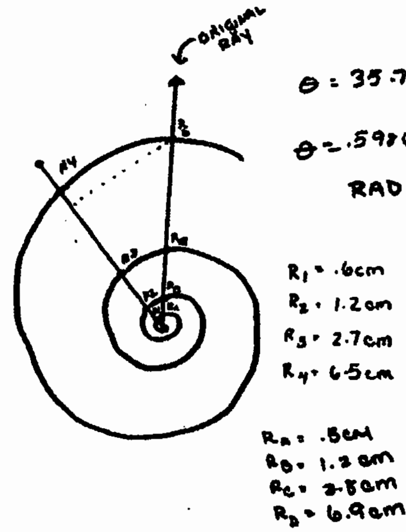

Figure 1. Students' initial representation for concentric circle model of spiral shell.

Students agreed to draw the ray "straight up" after locating the center of the spiral. The following excerpt demonstrates that the notion of radius was more difficult to resolve because it was not immediately apparent to the students whether "r" meant radius or "r" meant radians. Students discuss the meaning of r and theta and radians, and radius. They discuss which radius should be measured.

12:08:28	Michelle	What can you say about r as a function of theta? Like if you have like say two radians, what does r stand for? (QI)
12:25:18	Robert	Radians
12:27:17	Michelle	Well, no, isn't that theta, isn't theta the radians? (QC)
12:31:27	Angela	r is radius, is it not? (QC)
12:34:01	Sherly	That's what I thought.
12:35:10	Michelle	Okay, but what's theta? (QI) It can't be measured in degrees.
12:39:02	Sherly	They want theta.
12:40:19	Robert	You gotta measure in radians
12:42:08	Michelle	Right, so
12:44:00	Angela	Why can't you measure in degrees? (QS)

12:46:04	Sherly	'Cause it says lets measure distance in centimeters and angles in radians.
12:50:24	Robert	And something about like it depends on, um,
12:53:23	Angela	Alright, well, what's the radius of this thing? (QI)
12:57:18	Robert	That's what we've got a ruler for.
12:59:22	Sherly	'Cause we could do this one or you could do like the whole thing.
13:03:14	Michelle	Yeah, but its still not the whole thing.
13:04:10	Sherly	That's true.
13:10:03	Michelle	Maybe it means from the center to the outside. [Michelle suggests they measure from the center to the outside. Ashley asks a question to which Michelle responds.]
13:13:19	Sherly	Wait, you could also go from the center to that. [Sherly points to the copy of the shell with her pen.]
13:15:25	Angela	It's a spiral, it keeps going. It doesn't have a radius.

Angela's language changed, as she referred to the spiral instead of "the thing" while Michelle and Sherly continued to refer to the spiral as "thing". Angela pointed out that a spiral does not have a radius; it just "keeps going". These students had limited experience with polar coordinates, but had stronger intuitions about radii and circles. Perhaps Angela's understanding of radius and circle prompted the mathematical refinement of her language by requiring distinctions between "circle" and "spiral".

As discussion continued, Michelle suggested that they measure along the ray from the center of the spiral to various points where the ray intersected the spiral. Angela agreed and each student then traced the spiral and began measuring various lengths along the ray in centimeters. Question types alternated between interrogative and attunement when it became apparent that different measurements were being obtained. Michelle exclaimed that she had a brainstorm and turned toward Robert to share her idea. Students' language patterns include overlapping conversations, reflected in the excerpt presented here.

18:23:01	Michelle	Oh, I have an idea! I have a brainstorm.
18:27:21	Sherly	Okay, what does table r as a function of theta mean? (QI) Bob! [Sherly directs her question to Robert.]
18:32:12	Michelle	Okay, let me explain it to you 'cause you'll probably understand me. [Michelle attracts Robert's attention by reaching out toward him with her right hand as she speaks. She proceeds to explain her idea. Both students lean toward the table and each other as they speak.]

18:35:12	Michelle	Okay, say this is point five, right? (QA) [Michelle and Robert are leaning forward and Michelle is gesturing with her pen in a circular motion on top of the photocopy of the spiral. She looks directly at Robert and smiles as she asks her question.]
18:38:22	Michelle	And say that was the circle or whatever, would we measure...like.. radians? (QS)
18:45:18	Victor	It's what...circle...what [Victor is talking with Sherly in another conversation.]
18:47:02	Michelle	It would be point five centimeters right? (QC) Around the circle.
18:50:12	Victor	Ooohh
18:51:26	Robert	But there's like, isn't there like six pi radians and
18:56:29	Sherly	What
18:57:17	Robert	Should it just be six pi times point five? (QI) I don't know. [Robert asks for information about what the answer should be.]
19:00:15	Michelle	No, it's two pi radians
19:02:10	Ashley	Yeah
19:02:23	Victor	It's what? (QI) [Victor now participates in the conversation between Michelle and Robert.]
19:04:03	Sherly	Yeah, two pi radians [Sherly supports Michelle's assertion.]
19:05:17	Robert	Yeah, yeah, two pi... [Robert, Michelle and Sherly determine that there are two pi radians in a circle.]
19:06:07	Sherly	Good job.
19:07:07	Robert	..'cause its six point somethin'. [Robert represents pi as a decimal.]
19:11:12	Sherly	Wait. Can we just stop for a second? ... see where we're heading? (QA)
19:15:00	Robert	So do we just take that and multiple it by the point five? (QP) [Robert asks again if they multiply by point five, this time as a procedure.]
19:18:27	Victor	Yeah, that's right

19:19:24	Michelle	I don't know
19:21:09	Angela	No, cause its not like it's like a circle. Its not. It's a spiral. Its not a... [Discussion about using a circle vs. spiral. Angela disagrees.]
19:26:03	Robert	It doesn't matter.
19:27:18	Michelle	I don't think...I know it's not perfect, but I think that everyone should just, you know, assume, say it is a circle. So why don't we have one circle that's point five centimeters and then one circle with one point two centimeter radius? (QS) [Here is the stated beginning of a model that develops into the final model presented by the group, even though Michelle is absent the next two days.]
19:40:11	Angela	Well, see then it will be like, you know. [Angela explains why she doesn't agree with Michelle's idea. She draws four concentric circles.]
19:44:19	Michelle	Yeah, but doesn't matter.

Michelle articulated a model for the spiral—circles with different radii—as a speculative question. Although Angela initially disagreed, it was this concentric circle model that the group developed for their final presentation given three working days later. Members of the group also constructed two other models over the next three days as students continued to work and ideas were shared between groups.

The following excerpt demonstrates that careful analysis of questioning in discourse between students depends on context. Questions taken out of context may appear to be identical in nature when they are different utterances since they function differently.

50:28:14	Sherly	See, look, its sixty degrees. So, sixty degrees is how many radians? (QI) [Sherly is looking at her calculator and speaks to Victor. Victor is looking at his calculator, leans over and looks at Sherly's and says, "Yeah, um, that means."]
50:31:04	Angela	I'm sorry. What did you say? (QI) [Angela responds to Michelle. Conversations between individuals are overlapping.]
50:32:27	Michelle	I'm saying, go out with another line
50:35:02	Victor	Sixty degrees is how many radians? (QR) Well [Victor repeats Sherly's question and then picks up a calculator.]

50:41:19 Victor Sixty degrees times, uh, pi over one-eighty degrees. [Victor explains and Sherly is first looking down and then looks up toward the ceiling and rolls her eyes wide from side to side as she listens.]

Sherly's question, "Sixty degrees is how many radians?" was an interrogative for information. Victor's question was rhetorical since he was repeating Sherly's question verbatim rather than asking the question as an original interrogative.

Results

During the first forty-five minutes working on the Placenticeras task, students asked one hundred forty-eight questions. Forty-four percent were interrogatives, twenty-three percent were attunement, sixteen percent were procedural, seven percent were confirmative, seven percent were speculative, and three percent were rhetorical. As discourse developed over time, the distribution of question types changed. Prior to Michelle's speculative question about using concentric circles to model the spiral, forty-seven questions were posed. Thirty-eight percent of those forty-seven questions were attunement, thirty-six percent were interrogative, eleven percent were procedural, eight percent were speculative, and six percent were confirmative. After the notion of concentric circles as a model was introduced, until the session concluded, students asked one hundred one questions, forty-eight percent were interrogative, nineteen percent were procedural, sixteen percent were attunement, seven percent were confirmative, six percent were speculative and four percent were rhetorical.

Students' natural language became mathematically formalized as students distinguished between meanings for radius, radians, and spiral. Questions posed by students are shown here to be active, pivotal mechanisms in students' discursive drive toward building deep understanding. Under certain circumstances, student-to-student questioning patterns probe beyond procedure to identify mathematical relationships and develop meaningful representations.

As research continues to investigate student-to-student interactions, new insights may be gained into the structure of student discourse, the kinds of questions that students ask in constructing mathematical knowledge, and how teachers might tailor practice to capitalize on the kinds of questions students ask and facilitate development of mathematical understanding in classroom situations. This research has important implications, linked to classroom practice, that procedural instruction in mathematical modeling does not adequately address the nature of student questions, particularly with respect to the ways that these precalculus students build mathematical understanding.

Notes

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[2]Use of the word "natural" refers to the circumstances in which researchers are not directly instigating student discourse. Interviews of students by researchers would not be considered natural student language (Wood & Kroger, 2000).

[3]After reading through the task, one of the students spontaneously remarked that five of the six students had studied polar coordinates for one day during precalculus math class the prior year.

[4]Robert Speiser, Brigham Young University, developed this task, Placenticeras, for an honors calculus course.

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DEVELOPING STUDENTS' UNDERSTANDING OF EXPONENTS AND LOGARITHMS

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In this paper, we describe instruction designed to teach students about exponents and logarithms and report a pilot study to test the effectiveness of this instruction. Based on the theoretical work of Dubinsky and Sfard, we postulate a set of mental constructions that a student could make to understand the concepts of exponents and logarithms. We then describe computer and paper-and-pencil exercises designed to induce students to make these constructions. We report a pilot study assessing the efficacy of these exercises. Students receiving our instruction outperformed students receiving traditional instruction across a variety of measures, including performing basic computations, recalling formulae, explaining why rules of exponents and logarithms are true, and answering conceptual questions.

Introduction

Exponents and logarithms are important mathematical concepts that are useful for modeling and understanding population growth, radioactive decay, and compound interest. Further, exponential and logarithmic functions are central concepts for many college mathematics courses, including calculus, differential equations, and complex analysis. Unfortunately, research indicates that students' understanding of these concepts is quite limited (e.g., Confrey & Smith, 1995). In particular, students often forget many properties of exponents and logarithms shortly after they learn them and can seldom explain why these properties are true (Weber, in press).

While mathematics educators have proposed instructional techniques to supplement or replace traditional pedagogy of exponents and logarithms (e.g., Confrey & Smith, 1995; Rahn & Berndes, 1994), to our knowledge, the efficacy of these techniques has not been assessed. The purpose of this paper is to describe instruction designed to teach students the concepts of exponents and logarithms and to report a pilot study in which we tested the effectiveness of this instruction.

Theoretical Framework

Students are often told that the exponential operation represents "repeated multiplication" (e.g., $2^3 = 2\times2\times2$). However, as many researchers have pointed out (e.g., Confrey & Smith, 1995; Lakoff & Nunez, 2000), this conception is inadequate to perform much of the reasoning that we associate with exponents and logarithms. For instance, to a student who can only view exponents as repeated multiplication, expressions such as 2^{-1} and $2^{1/2}$ will be nonsensical, as you cannot multiply a number by itself negative one or one half times. In this paper, we attempt to teach students about exponents and logarithms by first having students understand exponentiation as a process, then having them view exponential and logarithmic expressions as results of applying this process (i.e., b^x represents the number that is the product of x factors of b), and finally generalizing this understanding for cases where the power in an exponent is not a natural number. We discuss the theoretical underpinnings behind these constructions below.

According to Dubinsky, a mathematical operation can be understood as an action or as a process (Dubinsky, 1991; Asiala et al., 1996). An *action* is a repeatable physical or mental transformation of mathematical objects to obtain other objects. Students limited to an action understanding of an operation can apply this operation only in response to an external cue explicitly detailing what steps to make. In the case of exponents, students with only an action understanding can do little besides calculating an exponent when they are given a specific base and power, and only if the power is a positive integer. After repeating an action and reflecting upon it, the student may interiorize the action as a *process*. Individuals with a process understanding of an operation can imagine the result of the transformation without actually performing the corresponding operation and can construct a new process by reversing the steps of the original transformation. A student with a process understanding of exponents can imagine b^x as a number that is the result of applying the operation of exponentiation, an ability that we believe is necessary to understand the rules of exponents, and can imagine reversing the process of exponentiation to obtain the process of taking a logarithm.

Many researchers have noted that expressions such as b^x have multiple meanings: this expression can be viewed as an operation- multiply b by itself x times- or it can be viewed as a mathematical structure- the number that is the result of applying the process of exponentiation. It can also be viewed as a function, a family of functions, or a string of symbols, depending on the context in which it is used (e.g., Sfard & Linchevski 1994). In particular, Sfard (1991, Sfard & Linchevski, 1994) distinguishes between an operational understanding of a concept- which focuses on its algorithmic nature- and the structural understanding of a concept- which treats the result of a process as an object in its own right. While students are generally capable of acquiring an operational understanding of a concept, acquiring a structural understanding appears to be quite difficult. In our view, an operational understanding of exponents would be tantamount to understanding exponential expressions as calls for repeated multiplication, while a structural understanding would be interpreting b^x as *the number that is the product of x factors of b* and $\log_b m$ as *the number of factors of b that are in the number m.*

At this point, a critical reader may question why a student needs to understand b^x as a mathematical object. According to Sfard, possessing a structural understanding of mathematical objects is necessary to reason about mathematical concepts because it makes one's knowledge of the concept more compact and intuitive (Sfard, 1991, p. 23). We believe that this is the case with exponents and logarithms. Consider the following rule about exponents: $b^x b^y = b^{x+y}$. A student with a structural understanding of exponential expressions could interpret this equation as "The product of x factors of b and y factors of b is (x+y) factors of b". Perhaps a student could also explain this equation as a statement about processes, at least in theory. (e.g., You obtain the same result if you multiply b multiplied by itself x times with b multiplied by itself y times or if you multiply b by itself (x+y) times). However, such an explanation would be longer and less intuitive. Hence, we believe that possessing a structural understanding of exponential and logarithmic expressions greatly aids students in understanding their properties.

After a student acquires an operational understanding of b^x as "the number that is the product of x factors of b", that student must generalize that understanding to account for cases where x is not a natural number. How students may do this is an interesting and difficult question, but it is beyond the scope of this paper. Discussion of this topic can be found in Confrey and Smith (1995), Lakoff and Nunez (2000), and Weber (in press).

Instruction Used in Our Study

In this section, we describe instructional activities designed to lead students to make the mental constructions we describe above. After receiving this instruction, we would like students to be able to complete routine tasks that are traditionally associated with exponents and logarithms. That is, students should be able to perform basic computations with exponents and logarithms and be able to recite and apply the "rules" of exponents and logarithms. We would also like students to acquire a deeper conceptual understanding of these topics. Students should understand why the rules of exponents and logarithms are true and they should be able to use their conceptual understanding to answer traditional and non-traditional questions.

The first goal of our instruction is to have students understand the act of "taking an exponent" as a process (as described in section 2 of this paper or in Asiala et. al. (1996)). Research indicates that a particularly effective way for leading students to construct this understanding is to have students write a computer program to apply this operation. (Tall & Dubinsky, 1991). Researchers argue that in writing a computer program, students are forced to reason about and explicitly describe the steps of an operation. In doing so, students are likely to reflect on the steps of an operation and interiorize it into a process.

In our instruction, students were first taught the structure of a basic for loop and were given a for loop program in MAPLE that performed multiplication of integers as repeated addition (or repeated subtraction, in the case where a factor is negative). Students were asked to write a similar program that performed exponentiation as repeated multiplication (or repeated division). The students worked in pairs to write these programs. The instructor would answer questions and would help the students with MAPLE syntax, but would not help the students in other aspects of the program writing. All students completed this task in a one hour class period.

Our next goal was to have students understand b^x as the number that is the product of x factors of b and $\log_b m$ as the number of factors of b that are in the number m, a construction that Sfard coins *reification*. Sfard (1991, 2000) emphasizes the role that names and symbols play in acquiring a structural understanding of a mathematical concept. We designed paper-and-pencil worksheets in which the student was asked to describe exponential and logarithmic expressions as mathematical objects. They were also given exercises in which they were required to use a structural understanding of the exponential expressions to complete. With the judicious use of worked out examples, the students had several prompts to help guide their work. Some examples of these exercises are given in the Appendix. Students worked in groups of two or three to complete these activities. After the activities were completed, they were discussed, handed in, corrected, and returned to the students. This continued until all the activities were complete.

A skeptical reader may question whether something as trivial as paper-and-pencil exercises and classroom discussion can trigger a mental construction as sophisticated as reification. The only response we can offer at this time is the results of our study indicate that it can. Students who received this instruction appeared to be more capable of treating b^x as a mathematical object than students who received traditional instruction. We will describe these results in the next section.

Evaluation of Our Instruction

To evaluate the effectiveness of our pedagogy, we conducted a pilot study in which we implemented this instruction in a college algebra and trigonometry course and then compared the performance of our students with students who received traditional instruction on a set of interview questions.

Methods

Participants

Two groups of students from a regional university in the southern United States participated in this study. The experimental group of students was enrolled in the first author's college algebra and trigonometry course. The control group of students was enrolled in a separate section of the college algebra and trigonometry course taught by a different instructor. The instructors of the control and experimental groups spent roughly equal time reviewing exponents and logarithms. 15 students in each course volunteered to participate in this study.

Procedure and Materials

Three weeks after receiving instruction, the students were interviewed individually. During the interview, the students were asked the following questions:

Basic computations

B1. What is 2^3?

B2. What is $\log_2 64$?

B3. What is $\log_x x$?

B4. $\log_9 729=3$. Use this information to find $\log_3 729$.

Rules

R1. $b^x b^y$ can be simplified to what? Why?

R2. $\log_a x^r$ can be simplified to what? Why?

R3. How can you express the square root of x as a power? Why?

Conceptual questions

C1. Is $(1/2)^x$ an increasing function or a decreasing function? Why?

C2. Is $(-3)^{10}$ a positive or negative number? Why?

C3. Is 5^{14} an even number or an odd number?

C4. How would you find $\log_5 78125$?

Results of the Pilot Study

The number of correct responses for each of the Basic Computation and Rules questions is presented in Figure 1.

Basic computation questions- Every student was able to compute 2^3, indicating that all participants had some basic notion of exponent. As can be seen from Figure 1, the participants in the experimental group performed much better than their counterparts in the control group on the remainder of the Basic computation questions. No student in the control group was able to answer questions B3 or B4.

Rules questions- The data in Figure 1 indicate that students in the experimental group were able to recall more rules of exponents and logarithms than students in the control group. The difference between the groups becomes more pronounced when one examines the students' responses when they were asked why the rules of exponents and logarithms were true. Not a single student in the control group was able to explain why any of the rules of exponents were true. However, students in the experimental group were often able to give an explanation of why the rules were true. For instance, eight students were able to explain why $b^x b^y = b^{x+y}$. One typical response from a student was "Because we're having b x amount of time and y x amount of time, so when you set it up...it's basically like your adding up all the repititions of b". Six students in the experimental group were able to explain why $\log_b x^r = r \log_b x$ and why $x = x^{1/2}$. The latter result was particularly impressive, as why $x = x^{1/2}$ was never explicitly discussed in our activities.

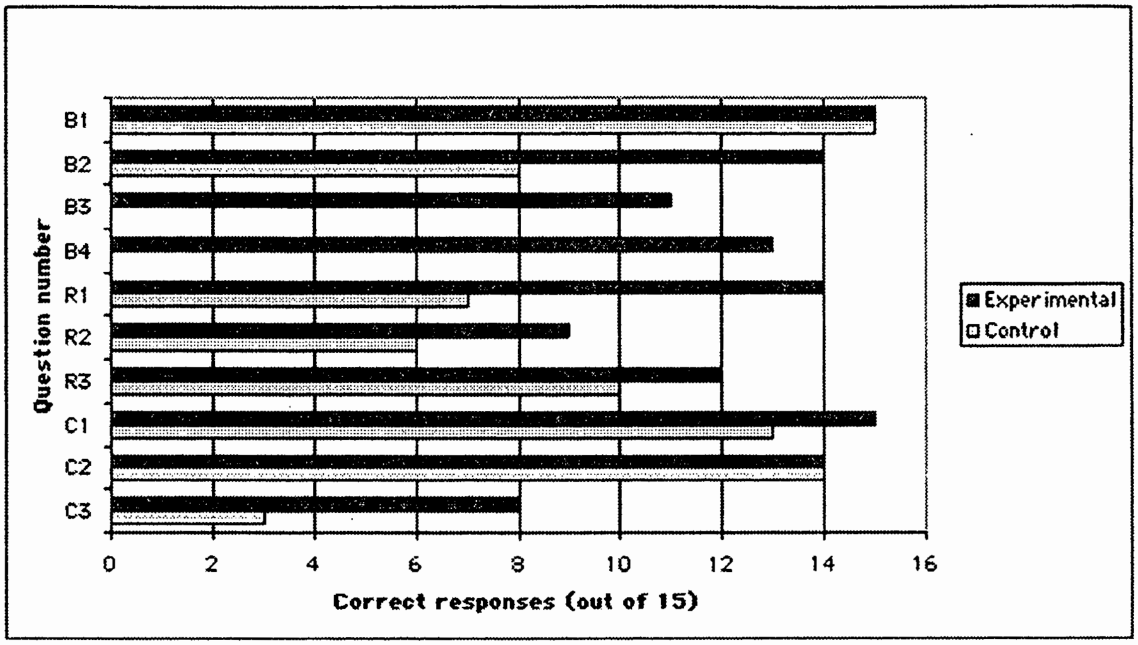

Figure 1. Performance on interview questions.

Conceptual questions- Both groups generally performed well on questions C1 and C2; they both were aware that $(1/2)^x$ was a decreasing function and $(-3)^{10}$ was a positive number. However, students in the experimental group were better at stating why these statements were true. When asked why the first property was true, no student in the control group could give an adequate response. Most students relied on looking at specific examples, usually only looking at the cases where x is one or two. However, five students in the experimental group were able to explain why this rule was true by using their understanding of exponentiation as a process. For instance, one student said, "If you keep multiplying [by one half], the number is going to keep getting smaller and smaller". Likewise, many students in the experimental group noted that the number $(-3)^{10}$ could be decomposed into the product of positive numbers, while such responses from the control group were uncommon.

12 students in the control group believed that 5^{14} would be an even number, generally because they believed an odd number raised to an even power would be even. The three students who correctly stated that 5^{14} would be odd conjectured this by looking at simple cases such as 5^1, 5^2, and 5^3. 8 students in the experimental group also answered this question correctly. While four also did so by looking at simple cases, four other students offered deeper explanations such as, "An odd to any power is always going to be odd... 'cause you keep on multiplying by an odd number, so it can never turn even" and "It'll be odd. If you multiply two numbers ending in five, it's going to end in five. So 5 to the anything will end in five".

When asked how to compute $\log_5 78125$, eight students in the control group knew this was tantamount to solving the equation $5^x = 78125$, but none could offer anything more than this. In contrast, several students in the experimental group offered responses that demonstrated an understanding of logarithms as reversing the process of exponentiation. For example, four students suggested repeatedly multiplying by five until the result reached or exceeded 78,125 and another suggested dividing 78,125 repeatedly until he reached one. The number of repetitions required would be the answer.

Reconstruction of forgotten knowledge- Perhaps the most promising result from this pilot study was that students in the experimental group often could not recall properties of exponents and logarithms, but were able to use their conceptual knowledge of these topics to reconstruct these rules. For instance, three students in the experimental group initially believed that $b^x b^y$ was equal to b^{xy}. (This was also a common mistake in the control group). When these students attempted to explain why this rule was true, they wrote out b^x as x factors of b and b^y as y factors of b. At this point, the students realized that there were (x+y) factors of b in $b^x b^y$, so the correct answer must be b^{x+y}. In contrast, this phenomenon did not occur with any of the students in the control group. In fact, not one student could explain why a single rule of exponents was true. There were several other instances of students in the experimental group using their conceptual knowledge to correct an initially erroneous response. For instance, one student initially believed that $(1/2)^x$ would be an increasing function, but then realized that as x grows, "we are going to be taking half of it more often, so it will be getting smaller." This illustrates an important point. As time passes, one's knowledge of symbolic rules will generally decay. If one has a deep understanding of the concepts involved, these rules can be reconstructed. If not, the rules cannot be recovered.

Discussion

Summary of Our Data

Students who received our instruction performed better than students who received traditional instruction at performing basic computations, recalling rules, and explaining why the rules of exponents and logarithms are true. They were also better able to answer questions that required them to use their conceptual knowledge of these topics.

Limitations of this Study

We consider the results of this pilot study to be encouraging. However, we should note that any conclusions drawn should be tentative, due to shortcomings in the design of this study. First, the author of this paper also served as the instructor of the experimental students and the investigator in the experiment. This raises a host of methodological concerns. Perhaps the instruction to the experimental students was superior to that of the control students, but only because the instructor of the former group of students was more motivated or more able. Perhaps the students in the experimental group performed so well in an effort to please or impress the investigator, who also happened to be the individual who assigned their grades at the end of the semester. Students in the control group would feel no such obligation. Also, although there was no *a priori* reason to suspect that students in the experimental group had more knowledge of exponents and logarithms prior to instruction, this possibility cannot be dismissed since no pre-test was given to these students. Clearly these concerns need to be addressed before definite conclusions can be drawn.

How do we know that the experimental students' performance was due to our instruction? First, many of these students' responses were analogous to the way that they completed our worksheet. For instance, to answer question B4, one could simply combine several rules of exponents. (e.g., $\log_9 729 = 3$. So $9^3 = 729$. $(3^2)^3 = 729$. $3^6 = 729$. $\log_3 729 = 6$). However, not one student solved the problem this way. All 13 students who answered this question correctly wrote 729 as 9×9×9, and then noticed that each 9 "split" into a pair of 3's so 729 can be written as 3×3×3×3×3×3. Thus there were six 3's in 729. The language that the experimental students used was also indicative of their thinking. For instance, many students spoke of there "being six threes in 729", indicating that students were thinking in a way that was consistent with our worksheets. Students' explanations of why the rules of exponents and logarithms were true were also consistent with their work on our worksheets.

Second, students' responses to our conceptual questions were consistent with our theoretical analysis. We predicted the conceptual questions could best be answered if the student had a process understanding of exponents. Many of the students' successful responses explicitly drew on this understanding.

Conclusion

In the past decades, our understanding of how students acquire mathematical concepts has increased immeasurably. This study marks our first attempt to apply the influential learning theories of Dubinsky and Sfard into the classroom. Using their theories, we postulated a set of mental constructions that a student could make to understand exponents and logarithms. We then designed instructional activities to lead students to make these constructions. We described the results of a pilot study in which we assessed the effectiveness of this instruction. The results are encouraging; students who completed our instructional activities outperformed students who received traditional instruction in a variety of measures, including recall of formulas, simple computation, and specification of why the formulas are true. Perhaps most significant is that students can use their deep understanding of these topics to reconstruct forgotten symbolic knowledge. However, due to limitations in the design of our study, any conclusions drawn from this study should be tentative. We are attempting to replicate the results of our pilot study in a more controlled setting with more students. If our attempts are successful, this study will be the focus of a future report.

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Appendix

Sample exercises from our worksheet (Desired student responses given in **bold**).

Describe each of the exponential expressions in terms of a product and in terms of words.

4^3 = **$4\times4\times4$ = the number that is the product of 3 factors of 4**
b^x = **$b\times b\times b\times\ldots$ (x times) = the number that is the product of x factors of b**

Simplify each of the expressions below by writing each exponential term as a product. Summarize each simplification in words.

b^2b^4 = **$(b\times b)\times(b\times b\times b\times b) = b^6$**
The product of 2 factors of b and 4 factors of b is 6 factors of b.
bb^x = **$b\times(b\times b\times b\times\ldots$ (x times)) = $b\times b\times b\times b\times\ldots$ (x+1 times) = b^{x+1}**
The product of b and x factors of b is (x+1) factors of b.

THE COMPARISON OF TEACHERS' AND STUDENTS' ROLES IN TWO DIFFERENT TEACHING ENVIRONMENTS IN MATHEMATICS

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The roles of the teacher and the students in grade five mathematics classrooms in two different teaching environments, one involving collaborative small-group work and the other involving conventional teaching, were analyzed in terms of their nature and circulation. The results of the analysis indicated that the student's role shifted from listener to active participant in different forms in the classroom. The role of the teacher, however, shifted from information giver to facilitator.

Introduction

Roles in teaching and learning differ when mathematics is taught by different methods. In conventional teaching environment, there is a transmission of body of knowledge, independent of the learner, from the teacher to the students (Hoyles, 1985). The teacher asks the questions and directs the students, as the main decision-maker of the teaching. The teacher is responsible for the classroom and the materials, and maintaining the order (Cobb, Wood & Yackel, 1995).

The social interaction perspective considers mathematical understanding as ability to form, reflect and use a view of mathematical idea and combine several ideas into one's own framework, besides challenging and rejecting the several ideas in a logical base (Hoyles, 1985). Hence, mathematical meaning becomes a product of social interaction (Voigt, 1994) and discussion (Neyland, 1994).

In collaborative small-group learning, students are involved in a mutually established network of ideas and explanations for the given tasks (Damon & Phelps, 1989). Expert members are forced to listen to and reflect on the novice members' ideas and find a way of expressing the tasks to the novices in a communicatable way. Teachers are only facilitators in this process and, allow and encourage students to extend their abilities and take risks for new experiences and materials (Moll & Whitmore, 1998).

The purpose of this study is to investigate how the role structures in teaching mathematics change during the teaching of conceptual tasks by collaborative small-group learning. The groups formed on the conceptual tasks were mixed ability groups. The role structures in terms of the nature of roles and their distribution will be sought in order to understand how students can communicate the mathematical understanding effectively to others compared to the teacher's communication.

Method

This study involves analysis of classroom interactions, particularly the roles of the teachers and the students in teaching in two classroom settings: One involving the collaborative small-group approach, and the other the conventional teaching approach. The experiment was carried out on fractions in both classes at the same time in 2000-2001 academic year. In conventional teaching class, the fractions were taught by conventional teaching approach, whereas in the collaborative small-group class, the fractions were taught by assigning the students to five-member collaborative groups.

The fractions in the grade five mathematics included the conceptual topics such as kinds of fractions, converting the fractions to each other, equivalence of fractions and ordering of fractions; and operational topics such as addition, subtraction and multiplication of fractions. The topics and their sequence, the exercise book and the homework assigned to the students were the same for both classes. The treatment in two classes was different only in terms of the method of instruction. In the collaborative small-group class the group worksheets prepared by the researchers were used. The students in two classes took the regular exams and the quizzes prepared by the school mathematics department.

The worksheets used in the collaborative small-group class were developed according to the findings of the literature review, the objectives of the National Curriculum, and the previous pilot study conducted in 1999-2000 academic year. The worksheets aimed to teach the fractions concepts and the operations with fractions by using the collaborative small-group approach. Students were required to work as a group on the tasks, such as writing rules or solving exercises, by making decisions by consensus, ensuring that all group members contributed their ideas and suggestions, and seeking assistance primarily from each other. Most of the worksheets were supported by manipulatives such as colorful unfolded cardboards representing the wholes, colorful pieces of cardboards representing fractions in continuous forms, and play dough representing fractions in discrete forms, to aid students' discussions with concrete examples.

Data Sources

The study was conducted on two fifth grade classes, each with 30 students, in a private elementary school. Mathematics were taught by the same teacher. The lessons of both classes were tape-recorded. Here, one class session from each class was transcribed and examined as they are the representative of all other sessions. In the collaborative small-group class, the recorder was placed on a group's desk in order to gather data of group discussions when they work on the worksheets. Each lesson lasted 40 minutes. The classroom interactions in both classes were transcribed and used as data source. One class session of the two classes when the same concept was taught were focused to be interpreted. The possible roles that might likely to appear in the interactions within the students, or between the students and the teacher in both classes were predicted in terms of teaching and learning components, such as explaining, questioning and confirming. The data were coded, recoded and reviewed by the researchers and final version of the codes was determined.

Results and Discussion

In the conventional teaching class, the teacher has the dominant roles such as questioning, explaining and directing the discussion. She informed the students about the concept and related subconcepts by writing on the board before introducing the concept. Then, she asked questions or posed exercises to the students related with the definition or the concepts. The main role of the students in this class was limited to responding to the teacher's questions or writing the answer of the exercises on the board. The teacher either confirmed the students' responses, or warned them to correct their responses usually by directing the students by leading questions. Finally, the teacher generalized or summarized what she had talked about and asked students write the generalizations.

In the collaborative small-group class, however, additional roles such as gaining attention, initiating, explaining, correcting, confusing and conflicting appeared and all of the roles were shared among the group members during the group discussion. The students explained the tasks, posed questions to their friends and generalized the rules by discussion. The ones who did not talk in one task talked in the other, so the listener role as well as the explainer role seemed to change among the members. Even the students with lower ability level had the explainer role from time to time. All group members contributed to the construction of knowledge and the generalization process. The conflicts arose were removed effectively through the discussion. Alternative suggestions to the tasks were quite frequent, which brought multiple perspectives to the process.

Collaboration brings different roles to the teaching environment than those appear in the conventional teaching environment. This research indicated that conventional teaching was seemed to be not very effective in inserting the multiple perspectives that were initiated by the additional roles that students have in the collaborative small-group environment.

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PRESCHOOL CHILDREN'S NUMERICAL THINKING IN PART-WHOLE SETTINGS

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A model of preschoolers' part-whole reasoning in the context of the first five numbers is proposed, based on data from previous research, and analyses of individual interviews with 14 preschool children (median age 3 years 10 months). A major cognitive tool at work in many of these children seemed to be an ability to visualize, not just static configurations, but sequences of actions, when outcomes of such actions were hidden from view. Problem situations where one or more items are added or removed from a small set appear to be potent experiences for the development of numerical part-whole reasoning, just as part-whole reasoning and integration operations serve as foundations for solving informal addition and subtraction problems.

Children younger than five years are able to deal with problem situations involving small quantities (Mix, Huttenlocher, & Levine, 2002). They are able to discriminate between sets of different numerosities in this range, have been found able to do so from the first months of life, are able to match small numerical sets across modalities, and are generally able to apprehend the numerosity of sets of items up to five without needing to count—a process known as subitizing. Preschoolers have intuitive ideas about addition and subtraction.

Conceptual Analysis of Preschoolers' Part-Whole Reasoning in the Context of Small Quantities

Qualitative Comparison Scheme

A fundamental and innate ability to make gross comparative judgments of quantities is present in children from the earliest months. Perceptual strategies including length and density can be used in making judgments, although conservation of discontinuous quantities has not yet been achieved. Children know that observing items being added to a collection will have the effect of increasing the size of the collection. A cognitive structure for making these kinds of judgments possible has been called the protoquantitative increase/decrease schema (Resnick, 1989). Children in division situations involving the sharing of discrete items will partition a set with little apparent attention to equality of subsets.

Quantitative Comparison Scheme – Up to 4

Provided the number of items is perceptually accessible, and the number is in the subitizable range, young children are able to focus attention on quantitative relationships arising from physical transformations of a defined collection, and will use counting to enumerate small sets if needed. When counting behavior is observed, children typically count each item deliberately; usually in response to the adult's question "how many?" However the last word response may not have cardinal meaning. Children in this stage are able to perform integration operations (Steffe, von Glasersfeld, Richards, & Cobb, 1983) on perceptually available material only, in the range 1-3.

Quantitative Comparison Scheme – Representational Stage

Small sets can be represented internally using visual imagery when such sets are not perceptually accessible. Children in this stage are able to perform integration operations on representations of sets of 1-3 (visualized or figural), so that they are able to perform simple calculations mentally. For quantities beyond 3, perceptual material needs to be available. They also begin to communicate their ideas of quantity through use of finger patterns. Thus when asked how many items are hidden from view under a plastic cup, a child may respond by showing the number of fingers corresponding to the number thought to be hidden. Ability to anticipate the outcomes of transformations yet to be observed is limited.

Numerical Comparisons

Children at this level are fluent in solving problems involving quantitative relationships within small collections. Prior experiences can be drawn upon to anticipate outcomes of transformations within small collections. These children can be said to have coordinated the logic of class inclusion with cardinal knowledge for the "intuitive" numbers (Piaget, 1941/1965). Thus they are aware that one is contained in two, but two is not contained in one, on up, to four or five. Development toward reversible part-whole operations is well advanced for problems involving small numbers. Emergence of symbolic activity does not depend on acquisition of conventional symbols, but such acquisition is well established and facilitative. In sharing situations, children use systematic many-to-one allocations, and check their resulting shares using counting methods.

A major cognitive tool at work in many children seemed to be an ability to visualize, not just static configurations, but sequences of actions, when outcomes of such actions were hidden from view. Success enumerating hidden items depended not only on an ability to visualize quantitative outcomes, but also keep track of two levels of outcome, unless perceptual feedback was needed. In children of this age, such visualizations are assumed to be predominantly spatial in nature. Can preschoolers make sets of 1-5 in their mind in the absence of physical material? Evidence from this study would suggest that this is so. How spatial representations of quantitative transformations evolve into numerical entities (if in fact this is universally the case) is yet to be explained. Children in this study were less successful enumerating more than 3 hidden bugs, supporting Starkey's (1992) observation of a discontinuity between 3 and 4 in children's numerical development. However, this discontinuity may just be relative to the ages of the children studied.

We learn that problem situations where one or two items are added to or removed from a small set appear to be potent experiences for the development of numerical part-whole reasoning, just as part-whole reasoning and integration operations serve as foundations for solving informal addition and subtraction problems. Part-whole reasoning and addition and subtraction operations in whole number contexts are reflexive cognitive skills. That is, facility with part-whole reasoning enables children to conceptualize, and be successful solvers, of addition and subtraction problems. Experience solving a variety of addition and subtraction problems enables development of part-whole reasoning, and a critical engine: progressive integration operations.

A continuing mystery requiring further study, is how numerical information is processed. A capacity to switch focus from individual items to a grouping or uniting of items, depending on the person's goals, seems fundamental. An integration operation has been proposed to explain how cardinal understandings are made possible. We are aware of the dynamics of part-whole reasoning where a subset is cut out from the whole while the whole set is kept in focus. We suppose that the logical operations of class inclusion are important here. The reverse situation, where a whole is rendered a part, by the conjoining of other items to make a new enlarged whole, prefigures the symbolic statements we know as addition of whole numbers.

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PROMOTING COLLABORATIVE LEARNING IN MATHEMATICS THROUGH STUDENT PRESENTATIONS: A STUDY BASED ON ACTION RESEARCH

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Study Question

Does having teachers and students share part of the teaching responsibilities encourage and facilitate collaborative learning in the mathematics classroom?

Purpose and Framework of Study

The background for this study is illustrated in the following teaching vignette:

I have just returned a quiz my Calculus students have written. Instead of providing the correct solutions myself, it is my normal practice to have students who have scored full marks on specific questions take up the solutions with the class. A student heads to the front of the room and proceeds to explain correctly how she arrived at her solution. Her explanation is clear and so is her written solution on the board. After she takes her seat, another student raises his hand and asks me to explain the solution again. Other students vocalize their support for this request. I am careful to grant this request by giving the solution in exactly the same manner as the student who had originally given the solution, trying to use the same techniques and words that she has just used. When I am finished, the students that did not seem to understand the solution the first time all smile and nod their heads in understanding.

This study is based on Action Research. A widely accepted working definition of Action Research is: "a form of self-reflective enquiry undertaken by participants (teachers, students or principals, for example) in social (including educational) situations in order to improve the rationality and justice of (a) their own social or educational practices, (b) their understanding of these practices, and (c) the situations (and institutions) in which these practices are carried out" (Carr & Kemmis, 1986). The research findings can be applied to a mathematics curriculum to encourage student presentations and promote collaborative learning.

Methods of Inquiry and Data Sources

Research Setting

The study took place in two grade 13 Calculus courses that I teach this year at a secondary school in Toronto, Ontario, Canada. The school that I teach in is a termed school, so I have the same group of Calculus students for the entire 2001-2002 school year.

Research Design

From November 2001 to December 2001, during the first term of the school year, the following three types of data were collected:

Student evaluations of presentations made by their peers. There were six student presentations made in each class and the entire class filled out evaluation forms after each presentation. On the evaluation forms, students had to identify themselves in order to ensure that the forms were taken seriously and properly filled out.

Self-evaluations of presentations by students. Each student filled out a presentation self-evaluation form after his or her presentation. The presentation self-evaluation form was identical to the one filled out by each student's peers, except that the former was titled "Self-Evaluation".

Student questionnaires. The entire class filled out questionnaires after the presentations. The questionnaire was designed to determine both a student's attitude and disposition towards math generally and towards presenting and learning from his or her peers specifically.

The presentation evaluations and the questionnaires both had a quantitative component and a qualitative component. In the presentation evaluations, made by the class, the presenter and myself, a presentation mark based on six components was scored out of a possible 30 points. The evaluations also asked for comments in terms of the strengths and weaknesses of the presentation.

In the questionnaire, a rating scale was used to determine a student's attitudes and disposition towards learning, especially in terms of their math education. As well, the questionnaire asked students for written responses in terms of how they felt their Calculus class was being taught.

The three types of data were collected and analyzed to determine connections that would help answer the study question. The data was examined quantitatively and qualitatively for correlation by comparing: (1) a presenter's current course grade compared with the quantitative scores from the peer evaluation, the teacher evaluation and the self-evaluation of the presentation; (2) the qualitative responses to the presentation evaluations in terms of the strengths and weaknesses of the presenter; (3) the summary of the quantitative scores on the questionnaire; and (4) the qualitative responses to the questionnaire.

Results of Study

The findings from this Action Research study demonstrate that collaborative learning can be promoted through student presentations. This type of learning takes the emphasis off the teacher as the sole knowledge giver in the classroom. Through watching peer presentations, students realize that there are sources in the classroom other than the teacher to whom they can turn to for help. Moreover, students discover that by making presentations they can better consolidate their knowledge of course content material and become better equipped to share that knowledge with their peers.

The findings on comparing a presenter's current course grade with his or her quantitative score from the peer evaluation illustrate that making a presentation will almost always improve a presenter's overall course grade. The presenter's peers benefit from the delivery of course content material and the presenter benefits by achieving a higher grade.

The qualitative responses to the presentation evaluations in terms of the strengths and weaknesses of the presenter demonstrate that when course content is delivered through peer presentations, students gain an appreciation for understanding different perspectives on the question. The degree to which a student benefits from his or her own presentation is determined by how strong a student he or she is in the subject matter. The above average students benefit by taking advantage of good presentations skills to increase course grade, while at the same time consolidating knowledge. Weak students benefit by being able to attain a presentation mark higher than their course grades achieved through quiz and test performance. The very strong students benefit by consolidating their deep understanding of the mathematical content, together with being able to practice their presentation skills. In addition, perhaps by undertaking qualitative tasks such as making presentations, all students will consolidate their knowledge and achieve higher marks on their quantitative evaluations. If this is the case, then making presentations may be even more beneficial for all students that participate.

The summary of the quantitative scores on the questionnaire illustrates that the consensus of the students is that presentations are beneficial to their understanding of the course material. The presenters benefit because they have to understand the subtle nuances of the topic. The presenter's peers benefit because they are exposed to another perspective on how to approach the topic. Although students generally agree that presentations are beneficial, there are still a number of students that are not reaping these benefits because they are not participating in the presentations. Further studies need to investigate how all students can be motivated to participate in the presentations so that they can benefit from them. The qualitative responses to the questionnaire also provide additional support for the suggestion that presentations help students better understand the content material. These responses shed light on why some students choose not to participate in the presentations. Students who feel that they are particularly weak in the content area and students who feel uncomfortable presenting in English are not participating.

Overall, the results of this Action Research study demonstrate that presentations are beneficial both to the presenters and to the presenters' peers. Almost all math students will see their overall course grade improve if they make presentations. In addition, making a presentation requires a student to fully understand the material that he or she presents. The presenter's peers have an opportunity to see how someone else tackles a question and to learn that their fellow students can be a source of knowledge for them. When students are encouraged to take responsibility for their own learning, collaborative learning through student presentations can be a powerful tool that helps everyone in the classroom.

Relationship of Study to the Goals of PME-NA

This study based on Action Research addresses the major goals of PME-NA by attempting to link the research in collaborative learning to the practice of teaching mathematics in the classroom. This study is specifically based on linking research and practice, which is the theme of this year's PME-NA conference. Moreover, the study examines links between quantitative and qualitative evaluations and explores the motivations and rationales for students to participate in non-traditional learning activities within the mathematics classroom.

As a full-time secondary mathematics teacher and a part-time doctoral candidate, I appreciate the opportunity to make changes in my teaching practice based on the research and literature to which I an exposed in my own studies. I believe that by attempting to better understand the process by which students become accountable for their own learning, we can help ease the transition to post-secondary school. To be successful in post-secondary institutions, students require skills such as the ability to engage in collaborative learning and to take the initiative. If these skills are honed in secondary school, students will be better equipped to handle the challenges of post-secondary school. By supplementing research findings with actual classroom practices, we will be able to gain a better understanding of the psychological aspects of teaching and learning mathematics, which will have useful implications for the mathematics classroom.

EMPHASIZING UNDERSTANDING IN MATHEMATICS CLASSROOMS

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What is the value of mathematics? While this question has been debated in the past, I would argue that current conditions in the world make one answer particularly reasonable. The value of mathematics is in the understanding that it provides about our physical and social environments. The science of mathematics may be beautiful and mental training may still be a valued outcome of mathematics instruction. However, the ability to use mathematics to develop understandings of things like global economic systems, technology, environmental science, music, art, and other things is the primary reason that most people need to know mathematics.

Many new curriculum materials emphasize the use of contexts in teaching and learning mathematics. In general, the primary role of the contexts in these materials is limited to serving as a catalyst for developing mathematical statements. Once a context has been translated into mathematical statements it is moved to the background so that emphasis can be placed on the mathematics to be learned. This makes some sense given that the full development of a context may divert attention from the mathematics to be learned. However, limiting the role of contexts in this way also limits the role of mathematics. Mathematics is used to compute an answer but is not used to form a frame for reasoning about and understanding the contexts.

Understanding and Mathematics

While there may be many different reasons to teach and learn mathematics, the primary use of mathematics is in developing understanding of our physical and social environments. I would argue that it is the use of mathematics to develop understanding that should influence the reasons and the ways that mathematics is taught in schools. This is consistent with Dewey's (1973) interpretation of the relationship between the learner and the curriculum. The attitudes, motives, and interests that have led to the development of mathematics can be found in the learner's attempts to understand his or her physical and social environments.

This focus on understanding in teaching and learning mathematics is also supported by the important role that mathematics plays in structuring our physical and social environment. As Skovsmose (1994) has argued, mathematics is absorbed into parts of our basic conceptual system and so assumes a power to format parts of our social environment. Geometric series and compound interest is an example of this formatting power. Lakoff and Núñez (2000) argued that our unconscious understanding of many mathematical concepts is supported by our ability to use metaphor to map the structure and meaning of everyday contexts onto these mathematics concepts. This subconscious connection between mathematics and our environment makes it all the more reasonable to expect that mathematics would play a central role in shaping and understandings our environments.

It is the conscious use of mathematical metaphors as a resource for reasoning that motivates learning and developing mathematics. The use of metaphor allows us to extend our understanding of prior experience to new experiences that share structural similarities. Dewey (1973) argued that it is concepts that allow us to extend our understanding from one thing to another. Thus, by mathematical metaphor I mean the formal mathematical concepts that are the subject of school mathematics instruction. However, as Dewey also argued the meanings of these concepts are most fully realized when they are used to develop understandings of things that are relevant to the learner's current situation. These arguments elevate the role of contexts in mathematics teaching and learning from the background to the foreground.

A Note on Understanding

There are many ways to understand a context. One may develop a spiritual understanding, a historic understanding, an economic understanding, or any combination of these and other kinds of understanding. The kind of understanding that is developed depends on the kinds of metaphors that are used and of course on the reasons one seeks the understanding in the first place. Thus, I am not arguing that mathematics is the only ways to understand a context but only that mathematics provides a particular kind of understanding that is often valuable.

There are four basic requirements that motivate a need for mathematical understanding. First, the learner must identify individual parts or components of an object or context. Second the learner must question the relationship between the parts that have been identified. A third requirement is that the learner questions the function of the object or context in a larger context. The final requirement is that the learner questions how the parts, the relationships, and the functions, change over time. It is in the effort to address these four aspects of an object or a context that the value of mathematics as a source for reasoning and understanding can be realized.

When questions concerning parts, relationships, function and temporal change are raised then mathematics variables, equations, operations, and systems become metaphors that we can use to understand many different physical and social contexts that share structural characteristics with these mathematics constructs. Thus, developing deep conceptual understanding of the structure of mathematics is key to being able to apply mathematics toward developing greater understanding of our physical and social environments. Also developing the ability to map meaning between mathematics and environmental structures should provide increased understanding of the mathematics concepts being used. It is within this framework that we can begin to investigate the fuller use of contexts in the teaching and learning of mathematics.

Current Use of Contexts

An initial analysis of secondary mathematics and post-secondary mathematics texts has been conducted. I make no claim here that there is not some general goal of developing mathematics as a resource for understanding that is addressed in the texts that have been analyzed. My initial analysis was only concerned with the explicit use and treatment of contexts and understanding in the texts. The results reported here are preliminary but I believe characteristic of the results that the continued analysis will reveal. However, a further analysis that considers measures of more implicit uses of contexts in mathematics texts is ongoing.

Two evaluations of the mathematics texts are summarized here. How many different contexts are introduced in a typical unit or section and to what extent if any is a context discussed after the initial problem is solved? The first measure provides insight into how well the contexts are developed in the texts. The second measure determines the extent to which using the structure of mathematics concepts to understand the structural characteristics of the contexts is a goal.

General results indicate that current mathematics textbooks do not emphasize development of understanding of the contexts they introduce to support instruction. This is uniformly the case for traditional algebra and trigonometry textbooks. Each example and each exercise introduces a new context. Knowns and unknowns in the contexts are identified and represented with variables, an equation is developed and solved, and the context is not discussed any further. Mathematics is treated as the science of procedures for finding answers, contexts are treated as props for developing equations, and developing mathematics as a resource for reasoning about social and physical environments is not an explicit goal.

Curriculum materials such as Contemporary Mathematics in Context introduce fewer contexts but this is largely because there are fewer examples and exercises in the texts. The reduction in exercises is replaced by a stronger emphasis on developing contexts that are more meaningful to the students. More attention is given to understanding the process whereby social and physical contexts are converted into mathematical expressions and equations. This is accomplished largely by asking a series of related questions about the contexts, however, as with the more traditional texts, there is no explicit effort to relate the structure of the mathematics being used to solve the problem to the general structure of some aspect of the contexts. Mathematics is still largely limited to the computation of answers.

As stated these comments reflect my initial analysis of mathematics texts. Curriculum materials such as Contemporary Mathematics in Context and Data Driven Mathematics place a much stronger emphasis on the use of context and provide a learning environment that could support the development and use of mathematics as a reasoning tool to develop understanding of social and physical environments. A more thorough analysis of these texts that considers the implicit use of contexts may indicate that this is in fact a reasonable expectation.

While the results reported here are only preliminary they are sufficient to emphasize a need for increased attention on the ways that mathematics is used in curriculum materials. Treating mathematics as a one-dimensional computational device limits the students' understanding of mathematics and may adversely impact many students' motivation to learn mathematics. When our goal is to understand physical and social phenomena then mathematics is a tool for understanding. If we view history, chemistry, biology, engineering, or everyday life through a mathematical lens then what we develop is a better understanding. Conversely if we are not trying to understand anything then mathematics may be of little practical use beyond simple measurement and computation.

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DEVELOPING PEDAGOGY FOR WIRELESS CALCULATOR NETWORKS[1]

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Introduction

More than two decades and of research and experience supports the idea that computer and calculator technologies can have an important role to play in supporting and effecting student learning (Heid, 1988; Kaput, 1992; Kutzler, 1996; Papert, 1980; Waits and Demana, 1999). The development of Classroom Communication Systems (CCSs) is providing new possibilities for technologies to play a fundamental role in creating and supporting effective learning environments. The TI-Navigator, from Texas Instruments, is a wireless CCS and its advent brings the power and potential of CCSs into K-12 classrooms in a novel, flexible and mobile way. The pedagogical potential of CCS technology is still in its development stage but preliminary research suggests considerable benefits to active student participation in class and collaborative inquiry in the classroom (Abrahamson, Davidian & Lippai, 2000; Bransford, Brophy, & Williams, 2000; Davis, 2002; Dufresne, Gerace, Leonard, Mestre, & Wenk, 1996; Mestre, Gerace, Dufresne, & Leonard, 1997; Wenk, Dufresne, Gerace, Leonard, & Mestre, 1997). The present study is designed to illustrate the potential of CCSs in facilitating effective teaching and in creating effective learning environments.

Theoretical Framework

Bransford, Brown, and Cocking (1999) developed a framework for designing effective learning environments. The design of such environments is based on principles which form the basis for a model of learning with specific implications for teaching called How People Learn (HPL). Consequent to these principles Bransford et al. propose that effective learning environments should be learner-centered, knowledge-centered, assessment-centered, and community-centered.

Aspects of a teacher taking a learner-centered approach include the extent to which the teacher uses questions, tasks, and activities to show existing conceptions that students bring to the classroom, and the extent to which teachers exert an appropriate amount of pressure on students to think through issues, establish positions, and commit to positions. A knowledge-centered approach manifests itself in a focus on conceptual understanding, and the diagnosis and remedy of misconceptions. Assessment-centered instruction concentrates on formative assessment to provide feedback to students and to teachers on student conceptions. Finally, a community-centered approach is reflected in, for example, class discussion, peer interaction, and non-confrontational competition. (Bransford et al., 1999).

Description of the Technology

From the mid-eighties to the present day Classroom Communication Systems (CCSs) have been a slowly evolving technology consisting of a network of palm-type devices or graphing calculators controlled by a central computer. CCSs consist of four main parts: (a) a computer which is operated by a teacher at the front of a classroom and which runs a software package; (b) an LCD panel, television, or other type of projection system which displays information; (c) student devices which may be calculators, computers, palm pilots or organizers; and (d) a network which connects student devices to the teacher's computer, interprets communication protocols, and sends tasks to and from students and the teacher. Using a CCS students can send in answers to multiple choice questions, send in alpha numeric answers to questions, send in lists of numbers based on measurements, participate in simulations controlling an on-screen icon or, so-called, turtle.

Methodology

The participants in this study are ten high school mathematics and science teachers and their students. The teachers are experienced in teaching with graphics display calculator technology and were trained in the technical aspects of operating the TI-Navigator system as well as pedagogical techniques and possibilities available in a networked classroom. Before the training teachers were surveyed in open question format about their pedagogical practices and will be surveyed at the end of the school year about possible changes in their practice. A series of visits was undertaken to their classrooms where the TI-Navigator system is being used.

Shortly before the visits to classrooms teachers and students completed Likert-style surveys specifically designed to elicit their views of the extent to which TI-Navigator classrooms reflect the HPL model (i.e., the extent to which TI-Navigator classrooms are learner-centered, knowledge-centered, assessment-centered and community-centered). During the visits, formal observations of classes were made using the VaNTH Observation System (VOS) (Harris & Brophy, in progress). This system is designed to capture classroom interactions reflecting the HPL model. Interactions in the classroom are recorded and categorised as examples of one of the centerednesses. Responses to the Likert-style surveys were used as the basis of protocols for focus group interviews with students and individual teacher interviews. The interview protocols were designed, therefore, to gain further insight into the TI-Navigator classroom as exemplification of HPL effective learning environments.

Conclusions

The role that CCS technology has to play in implementing HPL principles in the design of an effective learning environment centres on the free flow of information in the classroom: CCS technology gives teachers more information on what students are thinking, gives students more information on what other students are thinking, gives students more information on their progress, and supports sharing of information to facilitate collaborative learning. The accumulation of effects of the use of the TI-Navigator on student feeling, motivation, behaviour and classroom dynamics suggest that the TI-Navigator can have a positive influence on student learning. These findings illustrate the potential of CCSs in facilitating effective teaching and in creating effective learning environments however, more research is needed to develop and refine these initial understandings.

Note

[1]Supported by a grant from the National Science Foundation.

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WHAT DO STUDENTS ATTEND TO? OBSERVING STUDENTS' MATHEMATICAL THINKING-IN-ACTION

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In this report, I discuss some preliminary findings from an ongoing research project. From an enactivist inquiry, the research study seeks for an understanding of the nature of students' mathematical thinking: what students attend to in mathematical tasks and what is the dynamics of this attention. The study involved 7 students who were engaged in extra curricular mathematical tasks. Data included observations and video recordings. Manipulative materials, symbolic records, embodied problems posed and the other students' understanding are seen to influence what students attend to. Also, particular forms of utterances and written work are seen to indicate differences and shifts in what students bring forth from the tasks.

In recent studies on mathematical thinking researchers acknowledge that exposure to well presented mathematical concepts does not guarantee that students will come to think mathematically. In addition to psychological and mathematical influences, researchers recognize experiential, social-cultural and institutional influences on students' learning (Kieren, 2000; Lave & Wenger, 1991; von Glasersfeld, 1995). In order to understand why students fail to learn seemingly well-taught concepts some researchers have studied discrepancies between students' idiosyncratic conceptions and images, and the "standard" mathematical concepts and definitions. However observer constructs such as these discrepancies, cognitive obstacles or conflicts and children's misconceptions that are based on the acquisition model of learning inadequately explain the complexity of learning (Davis & Sumara, 2000; Sfard, 2001). This research that is based on enactivist inquiry aims at understanding students' mathematical thinking by studying what students attend to and what problems students pose as they engage in mathematical tasks. The study relates to studies that view mathematical knowing as coming to see what is mathematically significant, such as Mason (1989) and Sfard (2001). The study, nonetheless, emphasizes that perception is prejudiced, and that it holistically includes listening, imagining, touching, expressing etc. in addition to seeing. That is, perception is inseparable from conception.

The ongoing research has so far involved seven junior high students in extra curricular sessions. In each session paired students engaged in non-routine, variable entry mathematical tasks. I observed as well as participated in an interventionist teacher mode, by offering ongoing prompts (Towers, 2001). At the end of each session, in a conversational interview, I encouraged students to talk about their actions and interactions. The analysis and interpretation in this study involves gaining some insights into the embodied and co-emergent nature of students' mathematical thinking and, consequently, how it is occasioned.

From the enactivist perspective, cognition is embodied—it is all-at-once mathematical, evolutional, psychological, biological, socio-cultural and institutional (Kieren, 2000). Cognition is viewed as perceptually guided actions; it is an act of specifying, of in-forming the relevant features of the environment (Simmt, 1998). In embodied cognition, our actions and what we attend to are taken to be inseparable (Varela, Thompson & Rosch, 1991). Thus the nature of students' mathematical thinking and actions (conceptions) and what they attend to (perceptions) as they engage in mathematical activities are co-implicative. What a learner may attend to is not only a feature of the "object" of attention. It is also determined by the structure of the learner—the biological, social and historical *readiness* (to use Bruner's term) to attend—and the environment, which includes other learners and the teacher. Unfortunately, in analyzing students' difficulties in understanding concepts, say of fractions or of limit, researchers have hardly focused beyond the structural properties of mathematical tasks. Bruner (1996) remarks that what we attend to is one of the many possible worlds. There is need to hermeneutically study how students' situational factors and structures come to bear as students bring forth mathematical worlds during mathematical activity. Transcripts of the sessions together with students' written work were analyzed. In the analyses, I paid close attention to students' actions and interactions: What they said and how they said it—voice inflection, body language, metaphors and analogies; and what they wrote and how they wrote it—the representations, visualizations, and form and content of the symbolic records. In an attempt to develop ways of observing mathematical thinking-in-action the interpretations sought to explore the selective nature of students' perception that filters the relevant mathematical meaning from what is taught.

Three themes emerged from the data from the project and from a related classroom research. First, what students attend to seemed to be intertwined with: the manipulative materials that the students worked with, the symbolic records that the students kept and the other students' understanding. On a moment-to-moment basis the concrete materials and symbolic records played an evocative role as sites for mathematical interpretations besides being re-presentations of already formulated thoughts and actions. As presentations the materials and written records opened up spaces for learners to articulate the at once unformulated. Often students who worked collaboratively appeared to be sharing objects of attention to the extent that most shifts in attention were collective and seamless. Second, subtle variations in the use of concrete materials at times gradually called forth different thoughts and actions. For instance students assembled fraction kits[1] by seeing the strips as discrete objects rather than as proportions of a whole piece of paper. This way, even when they got the same answer, the students over played the ratio aspect—number of pieces—at the expense of the quantity aspect—size or length of pieces—of fractions. In another task the students manipulated the dominoes and chips to find out numbers that could be arranged as a string of consecutive positive integers. As they continued to make sense of the numbers, they posed different problems and their focus shifted from the geometrical shapes of the numbers to the pattern of the first digit in the arrangements. Third, loud, animated and proactive utterances, and the episodic nature of students' writing at most times indicated shifts in attention to what was mathematically significant. This relation might suggest that emotions are braided with thinking-in-action.

As the research study proceeds, I hope to pursue the themes further. From ecological perspectives, mathematics teaching might have to do with prompting students to notice certain aspects of their worlds and to interpret those elements in particular ways (Davis, Sumara, & Luce-Kapler, 2000; Mason, 1989). In which case the teacher's responsibility is to attend to the learners (so as to attend with the learners) as they make sense of their mathematical worlds. The exploration of how students interpret mathematical situations might not only offer to researchers and to teachers developed ways of observing what students attend to, but also open up possibilities "to attend with" the students so as to enhance mathematical thinking.

Note

[1]For a description of the fraction kit manipulative see Kieren, Davis & Mason (1996).

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FIVE AFRICAN EIGHTH-GRADERS IN WEST HARLEM TACKLE A DIVISION PROBLEM

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This paper is one of two that focus on the use of mathematical inscriptions. Both papers give case studies of how groups of learners grapple with important concepts in surprising ways. In each case, despite large differences in age and setting, understanding what the learners *write and draw* supports key steps toward understanding how they learn. In this paper, we consider data from an eighth-grade classroom in a multicultural urban school. We follow a series of three inscriptions, tracing carefully how they are built and used.

The work presented here is part of a larger investigation about the building and communication of mathematical ideas, which continues a sequence of investigations by the first two authors (Speiser & Walter, 1994, 1996, 1997, 2000). We chose here to write about a multiracial, multicultural urban classroom to investigate a particular hypothesis: the need to focus on communication might *facilitate* building collective discourse about models, proofs, strategies, explanations, and justifications. In this note we concentrate on how five students in this classroom built and used three *inscriptions*, first to justify a particular solution, and then to help present their thinking to the class.

For us, a representation—for example an inscription—is a *presentation*, perhaps to oneself as part of an ongoing thought, or perhaps to others as part of an emerging discourse. We analyze inscriptions from two closely related viewpoints: (1) to more clearly understand how learners build both personal and socially negotiated meanings, and (2) to document, based on these meanings, the kinds of reasoning, especially the kinds of argument and justification, that the learners use, share, and debate. We report data from a single two-hour classroom session, led by the fourth author (Alejandro Rivera) in Intermediate School 195. Located in West Harlem, IS 195 is an urban, overpopulated, multicultural school, whose students include many recent immigrants. Thirty-six students participated in the session we report, on March 30, 2001. They came from the Dominican Republic, Mexico, Ecuador, Yemen, Ivory Coast, Senegal, Guinea, Guinea Bissau, and Mali. In this session, students spoke Spanish, French, Arabic, and at least three indigenous African languages in addition to the English they were learning. To communicate, students and teacher often had to translate through a chain of several languages. We videotaped the session with one digital camera, and photocopied student written work. The task reported here (Figure 1) was adapted from *Mathematics in Context* (1997): "Four sub sandwiches will be shared among 6 students. Draw lines to show how the students can divide the subs so that each student gets the same amount of sandwich. What fraction of a sandwich does each student get?"

Figure 1. The submarines as shown in the students' task.

In their initial exploration, graphic and numerical inscriptions are seen to emerge together in the course of animated conversation, first to refute a proposed but incorrect solution, then to justify and later help communicate a more grounded and also correct response. The first proposed solution of 1.5 emerged by dividing six students by four sandwiches. Abibatou, reading the problem statement, insisted, "The problem says, uh, show the graph… Draw, draw lines, draw lines." On her own, she drew lines (Figure 2, left) to connect one student with 1.5 sandwiches and did not finish the inscription, very likely because six iterations of the segment representing 1.5 would obviously go beyond 4.

Five minutes later, the rest of the group still did not seem ready to dismiss the proposed solution of 1.5, which Abibatou rejected earlier. As shown below, Abibatou linked a numerical inscription (a calculation) to her graphic representation by adding 1.5 six times (see Figure 2, middle). She then used her numerical inscription to refute the proposed solution 1.5. Writing emphatically, she explained, "Ya 1, … 1 point 5, 1.5, 1.5, 1.5, 1.5, 1.5. See, see that, that's not equal to four sandwiches." Indeed, the six ones in the left column of her written calculation visibly (note, too, her repeated use of "see") gives 6. In our view, the inscription, especially its iterated 1s, allows Abibatou to reduce her refutation to the simple counting of six 1s. We can view the inscribed calculation, therefore, in two ways: in the background as the addition of six decimals, and in the foreground as the counting of six 1s.

The transcript indicates that Abibatou was aware that the correct solution was 4 divided by 6. She used a revised form (Figure 2, right) of her first graphic inscription to show her group (and later the class) that this solution was correct. In other words, use of a new inscription helped to ground, this time, a proof. Her proof was based on multiplying rather than on counting. Aniuska now invites Abibatou to explain her argument.

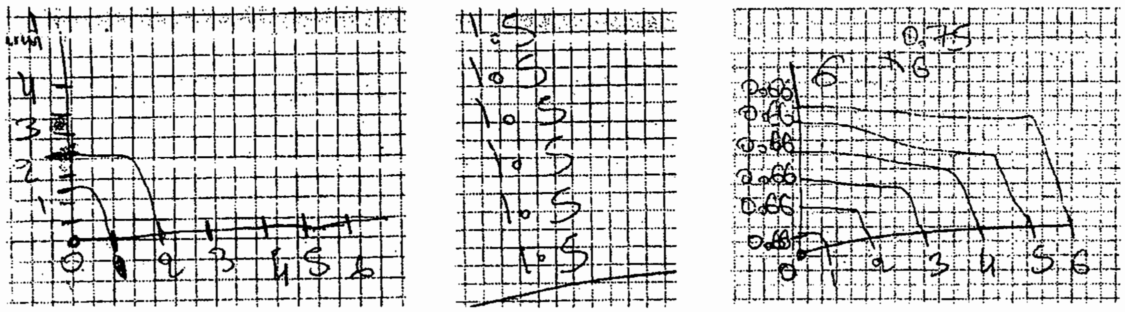

Figure 2. The three inscriptions.

Abibatou: Uh, and four is the, the number of the sandwiches. And six is the number of students and, and uh, and the graph. And when you, ... press in the machine, four divided by six, uh, equals zero point six six six. And, when you, ... do the answer times six equals four...

Rivera: Alright. Is that clear? Are there questions? So from that graph we conclude that one person, one student gets zero point six six of a sandwich. Right? According to the graph. I don't know. One person gets zero point six six.

Abibatou and Aniuska agree. Note that the third inscription (Figure 2, right) can also be read in two ways: as a scaled representation of six .66s iterated to give 4, and as the representation of a 1-1 correspondence between six students and the parts of sandwiches (.66 sandwiches apiece) they will receive. In our analysis, the first reading may present part of Abibatou's thought to herself, while the second reading, the presented correspondence, became part of the group's discussion with Rivera, above, and eventually the group's presentation to the class.

This group's presentation almost surely follows a different trajectory from that expected by the writers of the given task, and indeed different from paths followed by other students in the room. The work we have described abstracts away from the realistic aspects of the problem situation, and reformulates the task at a more formal level. Their discussion quickly boils down to four key steps: establishing a criterion for validity, refuting incorrect solutions, justifying a correct one, and presenting the correct solution both with proof and with a graphic inscription that they use in part to reconnect their numbers to the concrete problem setting. This solution grew from their particular experience, perhaps much of it gained earlier (in French-speaking schools) in Africa. Its presentation in this classroom, in our view, helped to enrich the discourse that Rivera sought to foster. In this discourse, the English which these students strove to master afforded public opportunities to discuss and sometimes reinterpret key inscriptions. Those inscriptions (Doerfler & Maher, 2001), as components of the students' discourse, both support and clarify the students' hard-won words.

The need to build instruction based on learners' prior experience and current thinking, in evidence above, hardly seems to need discussion. How to *meet* that need, however, has been the subject of considerable debate. Impacted urban classrooms surely stand to benefit from close attention by researchers, given the challenges, inequities and needs clearly in evidence. In the longer study, this particular analysis contributes evidence that students in this classroom used a variety of quite sophisticated strategies, to discover and construct solutions and to anchor proofs and explanations, first within their groups and then in presentations to the class. Based on these analyses, we can assess our main hypothesis about the function of communication. If strong emphasis on communicating *reasoning* can help students to build important mathematical understanding here, in this impacted urban setting, then, given appropriate conditions and sufficient time, they could help to foster students' learning anywhere.

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JAPANESE CHILDREN'S UNDERSTANDING OF TEN AS A UNIT

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An exploratory study was conducted to investigate Japanese children's (ages 5.5 to 7.3) understanding of ten as a unit, using the framework developed by Steffe and his colleagues. We will discuss insights gained on Japanese children's understanding of ten and possible influence of the language structure.

Introduction

International studies, both large and small scale, have shown that students from Far Eastern countries such as Japan, Korea and China consistently outperformed their counterparts from the West. Some researchers have noted that the number words in those countries reflected the base-10 numeration system very closely, and such language structure may be a contributing factor for their superior performance. For example, Fuson & Kwon (1992) discussed how Korean language structure may have helped Korean children to develop efficient methods for simple addition and subtraction problems. Miura & Okamoto (1989) suggested that the Japanese children's cognitive representation of numbers positively affected their understanding of place values. Some of these researchers speculated that the number word structures in Far Eastern languages may have contributed to their children's performance.

On the other hand, research by Steffe and his colleagues suggest children's concepts of units play a central role in their understanding of numbers (Cobb & Wheatley, 1988; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). However, there has been no study that investigated Asian children's understanding of ten as a unit. Thus, a small scale, exploratory study was conducted to analyze Japanese young children's concepts of ten.

Research Questions

The current study has two major purposes. First, the study explored whether or not the theoretical framework proposed by Steffe and his colleagues in their work with children in the United States might be useful in analyzing Japanese children's understanding of numbers. The second purpose is to gain additional insights into Japanese children's understanding of numbers.

Theoretical Framework

In order to truly understand our number system, children must be able to conceive ten as not only a collection of ten ones but also an entity in itself. Moreover, they must be able to count and increment by tens. Thus, children must understand ten as an iterable unit to make sense of base-10 numeration system. Steffe and his colleagues' work with children in the United States revealed that children's number concepts development was heavily influenced by units with which children can work. In our base-10 numeration system, the ability to work with ten as a unit plays an important conceptual step in young children's number concept development. In particular, Steffe & Cobb (1988) hypothesized five unit types of ten children construct: (1) numerical composite, (2) abstract composite, (3) ten more, (4) repeatable, and (5) iterable.

Of these three, numerical composite, abstract composite and iterable unit of ten appear to form significant demarcation points in children's understanding of ten as a unit. Ten as a numerical composite unit means that children consider ten as simply a collection of ten units but not *one* unit of ten. In that regard, ten is not different from any other number. Ten as an abstract composite unit, on the other hand, means that children can consider ten as a collection of ten units and one ten *simultaneously*. However, children with abstract composite unit of ten rely on the availability of suitable materials to re-present ten as a unit. Thus, they are still unable to anticipate the results of counting activity nor counting by ten does not signify incrementing by ten more ones. When children can finally free themselves from the need for materials to re-present unit of ten, they are said to have constructed ten as an iterable unit. In this study, we focused our analysis on these three types of ten as a unit.

Methodology

A convenient sample of 12 Kindergarteners, ages ranging from 5 years and 5 months to 6 years and 3 months, and 12 first graders, ages from 6 years and 8 months to 7 years and 3 months, participated in this study. These students attended the Kindergarten and the Elementary School affiliated with the College of Education of a national university in Japan. Therefore, these students are by no means a representative sample of Japanese children.

Each child was individually interviewed. There were four types of interview questions for children in both age groups. These tasks were adopted from Steffe and his colleagues' work and involved individual counting squares and ten-strips (rectangular strips on which 10 squares were marked off). These interviews were semi-structured in that there were a set of common questions for each age group. However, based on children's response, the interviewer asked additional questions, and or or altered the size of numbers. Each interview took approximately 20 minutes.

Counting (CT) tasks introduced children to the materials. It also investigated how children count and how far they could count. In Screened Tasks Sum (STS), children were shown several counters/ten strips, and they were asked how many counters they see. The interviewer then informed children how many counters were hidden below the screen and asked if they could find out how many counters there were in all. In Screen Tasks Missing Addend (STMA), the set up was similar. The difference was that, instead of informing children the number of hidden counters, the interviewer told children how many counters there were altogether. Children were then asked to figure out how many counters were hidden. Finally Tens Task (TT), children were asked to find the total number of counters as counters were laid in front of them. See Table 1 for the numbers used in the standard tasks for each grade.

Table 1. Numbers Used in the Standard Questions in Each Grade.

Task	Kindergarten	Grade 1
CT	3; 7; 10; "10"; 14 ("10" + 4); 20 ("10" + "10")	7; 10; 14 ("10" + 4); 20 ("10" + "10")
STS	[20] + 3; [5] + 10; [30] + 20	[7] + 11; [31] + 8; [23] + 15; [28] + 7; [8] + 43; [36] + 29
STMA	[ ] + 5 = 25; [ ] + 10 = 7; [ ] + 30 = 50	[ ] + 5 = 25; [ ] + 10 = 17; [ ] + 30 = 50; [ ] + 7 = 35; [ ] + 43 = 51; [ ] + 18 = 43
TT	6; "10"; 3; "10"; "10"; 4; "10"; "10"; "10"; 2; 5; "10"; "10"; "10"; 2	6; "10"; 3; "10"; "10"; 4; "10"; "10"; "10"; 2; 5; "10"; "10"; "10"; 2

Note. "10" in CT and TT indicates a ten strip. Numbers inside brackets in STS and STMA indicate the number of counters hidden underneath the screen. The symbols, "+" and "=" are used for our convenience. Formal addition language was not used by the interviewers.

Data Analysis

All interviews were videotaped then transcribed. We used both videotapes and transcripts in their analysis of the data. In CT and STS, children's counting activities were noted. In particular, we asked, "Can children produce the decade number sequence (10, 20, 30...) correctly?" "Can they count on, and if so, can they count on from any number, or from a decade number?" "Can they coordinate the decade number sequence and the standard number word sequence (1, 2, 3...)?" In STMA and TT, we focused on children's use of ten as a unit. In particular, we were interested in how children counted ten strips.

Presentation

During our presentation, we will present the summary of our analyses. We will then share with the audience our assessment of the usefulness of Steffe et al.'s framework in analyzing Japanese children's understanding of ten as a unit. We will also share insights gained from this study on Japanese children's understanding of ten as a unit as well as any possible influence of language structures.

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IN THEIR OWN WORDS: TEACHER AND STUDENT REFLECTIONS ON A WIRELESS NETWORK OF HANDHELDS

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This paper illustrates areas of impact on education (anonymity, confidence, teaching options, use of data and engagement) by uses of a classroom network of handheld devices. A series of interviews was done with a group of students, and their teacher. This group used the network on a daily basis for the entire 2001-2002 school year. Excerpts from those interviews help to tell the story of this technology.

Reflections

In the networked classroom, students can submit answers to be considered by the class without their identity being associated with that information. Anonymity facilitates the ability to explore answers in a non-threatening way, freed from *who* sent in an answer; students are able to explore *what* the answer is.

Teacher: It just promotes a lot of discussion and everybody's free to discuss it because kids can be criticizing an equation that they themselves wrote and nobody would know.

Students identify with their response, icon, data, etc. that shows up in the group display. With time, this representation of self in the group space can give the students a sense of how they are doing relative to the class as a whole.

Interviewer: Does that, in the other classes where you don't know how other people are doing (Student 12: Right), you don't know if you're the only one (Student 12: Right), does that raise your anxiety level any...?

Student 12: Yeah it's scary, because I think I'm the only one...I'm looking at my test, I think I'm the only one who got a 60 or whatever. And the couple of kids around me I'll know what they got but then I have no idea how anyone else is doing, because it's all privately done. Not that I need to know their test grades, but I'd like to know, how I stand. Am I the only one who needs help? And then you feel embarrassed to be the one raising your hand all the time, be the one staying after class because you think you're the only one. So, here, it's a lot more comfortable. You're not embarrassed in front of the other kids.

The ability to gather responses on all questions from all students gives important knowledge to the teacher. That knowledge then gives the teacher options for how to proceed in class.

Teacher: It's great to know, where the kids are, actually it's not always great because sometimes it's pretty depressing to see where the kids are. There was something I did this year in one of my classes and I asked if there were any - I thought I had done a fine job - I asked if there were any questions, nobody had any questions and I just had an inkling, And I said okay well log on and lets check. And I believe two kids got it right so obviously they didn't have a clue what they were doing and I went back and re-taught.

An interface, which scaffolds the teacher and students' navigation of the device, and the ease with which the network moves and shares students' data, makes data intensive; standards based activities more accessible to teachers.

Interviewer: In the class what do you think you can do now that you really couldn't do before?

Student 7: I'd say with the labs, the labs that we do. I don't think we would be able to do them as easy and as quickly as we can now. Like the Music Lab and everything that we just did. It just goes so much quicker. Instead of having "Okay, copy this from here..." you know it's just "Okay it's in the calculator, send it," everybody gets it, it's a lot quicker and easier.

Not raising your hand or avoiding eye contact no longer lets a student off the hook for participation. The network enables all students to be more engaged in the classroom.

Student 14: Well, with collecting since she collects the data from everyone, she checks, it's an easier way for her to check homework without having to single anyone out and it's also, the whole class is doing the work now and learning it and no one can really get away with just faking any of the work.

TRINITY PARADIGM OF INTELLIGENCE

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The intention of this research report is to unfold a unified concept or the essence of human intelligence. What is intelligence? This study is grounded in the transpersonal phenomenological methods of Braud and Anderson (1998). We believe these methods are viable approaches for exploring individual's experience of intelligence and intellectual potential. Our interviews consisted of three parts. The first part was a personal story. The second part dealt with an example of an intelligent person and elaboration on that idea. And, the third part was elaborations on the role of human intelligence in achieving a life purpose as perceived by each of the participants.

After analysis of the transcripts and data reduction, three main categories emerged: (1) Intra-personal, (2) Interpersonal, and (3) Transpersonal. Intra-personal attributes are within the realm of a person. Such a personal boundary encases and connects those elements that interact at the cellular, tissue, organ, and finally whole body function. The second category is Interpersonal attributes. These represent interdependence and interconnectivity between individual and others/environment. The third category that emerged was transpersonal. Interconnection of one to oneself and then to friends, family, community, society, environment, and ecosystem through a realization of a sense of wholeness which enables one the freedom to get closer and ultimately unite with her/his spiritual essence.

Although each one of these attributes may look different at first glance, they all share a common essence of interconnection or interdependence. For example, "analytic ability" and "mathematical ability" denote one's ability to compare and contrast qualities and quantities of events, situations, numbers, equations, occurrences, and data. In the case of "information processing," it denotes one's ability to gather and perceive information, data, or input, process, and compare them with what is already stored in memory, and finally transform them to an output. Again, the essence of such processing is the ability to interconnect input, throughput, and output. Human intelligence is where all our thinking, acting, and feeling originate. Therefore, the holistic theory of TPI has significant implication in mathematics education and mathematical literacy. The goal of many research and implementation efforts in mathematics education has been to promote learning with understanding. Helping our students to realize who they are, emphasizing the process of "human-becoming," with interdependence being our main intention, empowers our students to place mathematics into meaningful context in their daily living. This way they are endowed with more creative ways of problem locating and problem solving. In addition, students become more confident in their abilities to recognize patterns and relationship among patterns. Also, this approach helps them to become more intuitive thinkers, considering alternative strategies, and discovering/inventing more intelligent solutions.

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PICTURES OF CALCULUS KNOWLEDGE NETWORKS: DATA AND SOFTWARE

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This talk describes our findings from a study on the conceptual development of calculus students and emphasizes our attempts to construct visual and schematic diagrams of student understanding and the connectedness of their ideas. In particular, we hope that by introducing graphical and computer-aided methods, we can begin to study student spontaneous reasoning as an emergent phenomenon in its own context and not merely in relation to more official or formal expectations. We will display graphical representations of mathematical problem solving and introduce the software tools used to create them. We will need board space for paper displays and, if possible, a laptop projector.

The importance of the connectivity of student mathematical understanding has been a strong theme in recent work of such varied authors as Li-Ping Ma, Steve Monk and David Tall. However, when trying to qualitatively analyze the richness and connectivity of student knowledge, as either a teacher or a researcher, it is highly tempting to do so without interpreting student knowledge as either competent or deficient with respect to a selection of possible official mental organizations (e.g., the conventional metaphors of Cartesian graphical representations or function analysis). Indeed, one essential difficulty is that avoiding such preconceived structures requires gaining access to student's forming and dynamic knowledge networks, in which ideas may be connected tenuously and inconsistently.

In our study, we analyze student understanding of limit and covariational reasoning in terms of their facility with different metaphorical contexts and we attempt to characterize and diagram the connectivity of student ideas using a combination of qualitative analysis and computer-aided text and network analysis. We introduce these techniques in an attempt to acknowledge and capture the role of informal reasoning in the development of rich connected understanding, a topic on which there is much lively disagreement (for instance between neo-Piagetian APOS theorists who consider informal reasoning to be a temporary scaffold, to Vygotskians who assign a central dialectical role for intuitive reasoning, to Lakoff and Nunez who posit that mathematical reasoning is a series of metaphorical abstractions carrying structure from more basic notions which are all eventually rooted in the most basic embodied experiences).

We believe that the analyses presented are promising first steps towards an analysis of student use of language in reasoning that give evocative approximations of these fleeting but essential mental connections. Our analysis amplifies a careful semantic analysis of local relations between mathematical ideas of students by using computers to analyze the larger patterns that emerge from such fine-scaled analysis, and indeed to constructing visual representations of these larger patterns.

The study was performed during the academic year 2000-2001, and had both a large-scale and fine-scale component. We gathered data from a class of 120 students enrolled in a first year calculus course at a major southwestern public university. We conducted task-based interviews with 14 students. Each student was interviewed and audio taped three times during the year for about an hour per session.

THE NATURE AND ROLE OF TEACHER INTERVENTION WITH RESPECT TO DISCOURSE IN THE MIDDLE SCHOOL MATHEMATICS CLASS

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Verbal discourse, defined as, "purposeful talk on a mathematics subject in which there are genuine contributions and interaction" (Pirie & Schwarzenberger, 1988, p. 460), is recognized by the National Council of Teachers of Mathematics [NCTM] and by social-constructivist theorists as providing potentially important scaffolding to the learning-teaching process (Cobb, Yackel, & Wood, 1992; NCTM, 2000; Vygotsky, 2002). Professional and theoretical literature points to the importance of the teacher's role in helping to make classroom discourse meaningful, but also notes that the teacher's voice has been underrepresented (Chazan & Ball, 1995; Lampert & Blunk, 1998; NCTM, 2000). The teacher's role is a critical one in the orchestration of discourse in the mathematics classroom (Chazan & Ball, 1995).

The most common form of classroom discourse is 'triadic dialogue' that includes IRE (initiation, response, evaluation) or IRF (initiation, response, follow-up) (Cazden, 1988; Knuth & Peressini, 2001). Lotman (1988) categorized discourse as univocal (one-way transmission of knowledge) and dialogic (dialogue with its aim being the construction of meaning) (cited in Knuth & Peressini, 2001). Teachers' interventions within the IRE/IRF structure have been shown to play a significant role in whether the discourse tended more toward dialogic or more toward univocal (Wells, 1999). Knuth and Peressini (2001) presented observations from mathematics classes that supported the importance of the teacher's role in the development of dialogic discourse, but did not present a detailed linguistic analysis of the IRE/IRF structure. Wells (1999) presented an analysis of classroom discourse supported by linguistic theory, but did not examine discourse within mathematics classes.

This study applied sociolinguistic tools to examine the teacher's role in discourse within mathematics classes, focusing on the teacher's interventions within the IRE/IRF structure. Two middle school mathematics teachers noted for their use of instructional programs aligned with the NCTM Standards were interviewed and observed. Classroom observations were audiotaped, videotaped, and transcribed. Pre- and post-observation interviews provided background for understanding the teachers' perspectives about classroom discourse. The transcribed classroom observations were coded, focusing on "moves" (Wells, 1999) that were found to be critical in distinguishing dialogic and univocal discourse.

The results indicated that the teachers in the study valued classroom discourse as noted in the NCTM Standards. Although discourse was evident throughout the classroom observations, it was predominately univocal in nature, i.e., the teachers conveyed meaning to the students, but the dialogue indicated that the students generally did not generate meaning for themselves. There were times when the discourse tended toward dialogic, but the classroom discourse was rarely categorized as truly dialogic. Future studies will search for themes that explore the tendency toward univocal discourse in mathematics classes and the conditions that lend themselves to the appropriate use of dialogic and univocal discourse in effective mathematics instruction.

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STUDENTS' UNDERSTANDING OF SOUND WAVES AND TRIGONOMETRIC REASONING IN A PROJECT-ENHANCED ENVIRONMENT

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This paper reports a study that examines gains made by students participating in a project-enhanced, Industrial Electronics (IE) high school course unit that was designed to promote the developmental understanding of sound waves and trigonometric reasoning. The unit was designed to foster common content learning (via benchmark activities) by all students in the class, and to help students gain a deeper conceptual understanding of a specific topic through group project work.

Benchmark activities included student participation in the analysis of motion and musical waveforms using probeware and student exploration of wave phenomena using wave simulation software. Other activities were also used to bridge the ideas of triangle trigonometric ratios to the graphs of sinusoidal curves, which lead to the understanding of frequency, period, amplitude, and wavelength concepts.

The group project work was initiated by asking students to investigate the question, "What salient features are required in order to reproduce sound?" After preliminary student research on this question, students were asked to generate and then investigate their own driving question, which was derived from the original research question. The three groups emerged with projects of radio waves, hearing implants, and the sound quality of compact discs.

Both quantitative and qualitative data were collected in order to access student gains in understanding of sound waves and trigonometry. Quantitative data was comprised of pre-and post-tests while qualitative data involved videotaped clinical student interviews, classroom observations, and videotaped classroom activities of the High School Industrial Electronics classroom. Special focus was placed on student use of benchmark activities to assist their project work, and how project work facilitated benchmark lesson understanding.

Two students clearly showed that group project work, enacted in conjunction with benchmark activities, enabled understanding of course content due to ability to connect and relate in a bi-directional manner their individual group project work with the benchmark activities. One student, Evan, from the Radio Waves group obtained the deepest understanding of superposition when he connected superposition ideas obtained from his research on radio waves with superposition ideas visited during benchmark activities. Evan made the realization that a radio wave is formed through superposition of a modulating wave (he called sound wave) and a carrier wave. This student went on to score perfectly on all superposition problems on the post-test, and obtained the largest gain score on these superposition test items.

The second student was a member of the hearing implants group. While examining normal modes on a simulated string, he verbally described how the normal mode was like a vibrating eardrum. Later during a post-implementation interview he explained how he could find the period of a given graph by relating it to the first harmonic (fundamental mode) on a string, which he had also previously related to a vibrating eardrum. During the post-interview, this student scored perfectly on questions involved with finding the period of a given waveform, and he made the largest gain score going from the pre- to post-trigonometry test.

This paper provides insight into future ways to design a project-enhanced classroom that ensures opportunity for students to learn specific content goals of a course, while simultaneously becoming "experts" of their own research topics. Final results showed that students who had a driving research focus, thought about and connected their group project work with benchmark activities, made the greatest gains in conceptual understanding. This project implementation design demonstrated the need for both the benchmark and group project implementation features.

Proceedings of the
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North American Chapter of the International Group for the

Psychology of Mathematics Education

Volume 3: Research Reports, Short Oral Reports, Poster Presentations

PME-NA XXIV

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Athens, Georgia

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History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implementation thereof.

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1983	PME-NA 5	Montreal	ED 289 688
1984	PME-NA 6	Wisconsin	ED 253 432
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1986	PME-NA 8	Michigan	ED 301 443
1987	PME-NA 9/PME 11	Montreal	SE 055 633
1988	PME-NA 10	Illinois	SE 056 278
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1991	PME-NA 13	Virginia	SE 352 274
1992	PME-NA 14/PME 16	New Hampshire	SE 055 811
1993	PME-NA 15	California	ED 372 917
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International

Year	Meeting	Location	ERIC Accession No.
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1981	PME 3	England	ED 226 956
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1981	PME 5	France	ED 225 809
1982	PME 6	Belgium	ED 226 943
1983	PME 7	Israel	ED 241 295
1984	PME 8	Australia	ED 306 127
1985	PME 9	Netherlands	ED 411 130-131
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1988	PME 12	Hungary	ED 411 128-129
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1991	PME 15	Italy	ED 413 162-164
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1993	PME 17	Japan	ED 383 536
1994	PME 18	Portugal	ED 383 537
1995	PME 19	Brazil	ED 411 134-136
1996	PME 20	Spain	SE 064 730-734
1997	PME 21	Finland	ED 416 082-084
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1999	PME 23	Israel	ED 436 403
2000	PME 24	Japan	ED 452 031-034
2001	PME 25	Netherlands	SE 065 634-637
2002	PME 26	England	Not yet available

Preface

It is with great pleasure that we at the University of Georgia host the 24th annual meeting of PME-NA as we also hosted the 4th annual PME-NA meeting exactly 20 years ago in 1982. A quick glance at the differences between then and now shows how the field has grown and changed: The proceedings that year were about 250 pages long, and there were 34 papers presented in 7 topic areas. This year the proceedings will be nearly 2000 pages long with over 200 presentations in 15 topic areas. Despite the growth of the field and the organization, the hallmarks of collegiality and open intellectual exchange remain.

The theme of this year's conference is *Linking Research and Practice*. The theme is intended to highlight the interplay between the ways that research is used in practice and the ways that research grows out of practice. The invited plenary speakers were asked to address the theme in their areas of expertise by challenging the audience to think critically about the research we do–the questions we ask, the methods we use, the contexts in which we do research, the people with whom we do research, how we communicate the results of our research, etc. We hope that those in attendance as well as those who will read these papers in years to come will be stimulated to think deeply about our roles as researchers and consumers of research.

We received over 250 proposals for sessions at PME-NA and are grateful for the work of the many reviewers who helped shape the program. We undertook only structural editing (format and references) on the final papers so as to leave intact the integrity of the authors' work.

We wish to express our appreciation to the many people at the University of Georgia who have made these volumes and this conference a possibility, including, but not limited to, Patricia S. Wilson, Margaret Caufield, Elizabeth Platt, Salli Park, Bernice Peters, Teresa Banker, Nancy Williams, Joseph Allen, Brian Wynne, and all of the faculty and graduate students in the Department of Mathematics Education.

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Preservice Teacher Education

PREPARING PROSPECTIVE ELEMENTARY TEACHERS TO FOSTER CONCEPTUALLY BASED MATHEMATICAL UNDERSTANDINGS: A STUDY INVESTIGATING CHANGE IN PROSPECTIVE TEACHERS' CONCEPTIONS RELATED TO MATHEMATICS TEACHING AND LEARNING

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This study reports first year findings from a multi-year research project that examines changes in prospective elementary teachers' conceptions related to conceptually based mathematics teaching and learning. This study is the result of a *design experiment* (Brown, 1992) advanced between traditionally distinct departments of mathematics and education in a large, state-supported university. Participants include 300 elementary education students during mathematics content and general pedagogy courses. Results suggest that most prospective teachers changed their conceptions of mathematics and its teaching and learning during this program. Specifically, changes were made in two areas: appreciation of role of content knowledge in designing and implementing instruction, and perception of mathematics as more than rules to be memorized.

This research paper reports first year findings from a multi-year (2001-2004) study that examines changes in prospective elementary teachers' conceptions related to conceptually based mathematics teaching and learning. This study is the result of a *design experiment* (Brown, 1992) advanced between traditionally distinct departments of mathematics and education. Our efforts respond to a current dilemma in teacher education. Students often leave teacher education programs with the same preconceived notions about content, teaching and learning as when they enter (Ball, 1988; Murphy, 1990). One contributing factor is that many teacher education programs, particularly at the elementary level and in mathematics, do not connect across content and method throughout the entire program (Ishler, Edens, & Berry, 1996). Although students traditionally take courses in mathematics, general pedagogy and mathematics education, these courses often do not stress conceptual understanding of content. Many programs do not make conceptual understandings explicit enough to challenge previous conceptions of mathematics, nor do they place these new understandings within the larger context of teaching (e.g., Borko et al., 1993).

Our designed program seeks to assist prospective teachers in developing an orientation that promotes conceptually based mathematics instruction. We are defining "orientation" to include a teacher's knowledge, beliefs, and understandings about mathematics and mathematics teaching which frame his/her thinking and decision-making as it relates to practice (Magnusson, Palincsar, Marano, Ford & Brown, 1999). To this aim this study investigates the following questions: (1) WHAT changes occur in prospective elementary teachers' orientations toward mathematics and its teaching and learning?, and (2) HOW can an integrated program that promotes conceptually based understanding facilitate change in prospective teachers' orientations?

This paper examines findings related to the first research question.

Theoretical Framework

This study draws on areas of research in mathematics teacher education related to teacher orientation and teacher change. Prospective teachers struggle with how to balance their mathematical anxiety and narrow mathematics orientation with their professed desire to teach in ways that reflect reform expectations (e.g., Benken & Wilson, 1996, 1998; Borko, Davinroy, Bliem, & Cumbo, 2000; Thompson, 1992; Wilson & Goldenberg, 1998). National (e.g., NCTM, 2000) recommendations call for an approach to mathematics teaching that allows students to communicate, problem solve, and engage in cooperative, conceptual mathematical activities. This shift toward more " inquiry based" instruction assumes teachers view mathematics as a tool for thought, rather than a set of rules and procedures to be memorized. However, prospective teachers' orientations are deeply embedded and difficult to change (Ball, 1990; Richardson & Anders, 1994); they are unlikely to make adjustments in their thinking without intervention and deliberate support.

Research on teacher change indicates that programs can facilitate changes in orientation within the context of educational reform (e.g., Richardson & Placier, 2001). To help prospective teachers develop their orientations the connection between content knowledge, beliefs, and pedagogical knowledge must be considered (Stipek, Givvin, Salman, & MacGyvers, 2001). Knowledge of subject matter and general pedagogy help form pedagogical content knowledge (Marks, 1990), and therefore both types of knowledge must be examined.

To develop orientations that promote teaching mathematics for conceptually based understanding, we developed a model that incorporates the relationship among three primary tenets critical to mathematics teacher education: mathematical content knowledge, pedagogical content knowledge (Magnusson, Borko, & Krajcik, 1999; Shulman, 1987) and mathematics teaching practice (Fennema & Franke, 1992) (Fig. 1). This model illustrates the important role mathematical content knowledge plays in forming one's orientation, which then serves as a lens through which pedagogical content knowledge, and later mathematics teaching practices, are developed. Our experiment addresses the connections among these elements early and jointly within the elementary education program experience. Although research has looked at one or two of these components in isolation (e.g., Brown & Borko, 1992; Hashweh, 1987; Wood, Cobb, & Yackel, 1991), few have investigated the interactions among all three (Fennema & Franke, 1992).

Orientation's Role in the Teaching of Mathematics

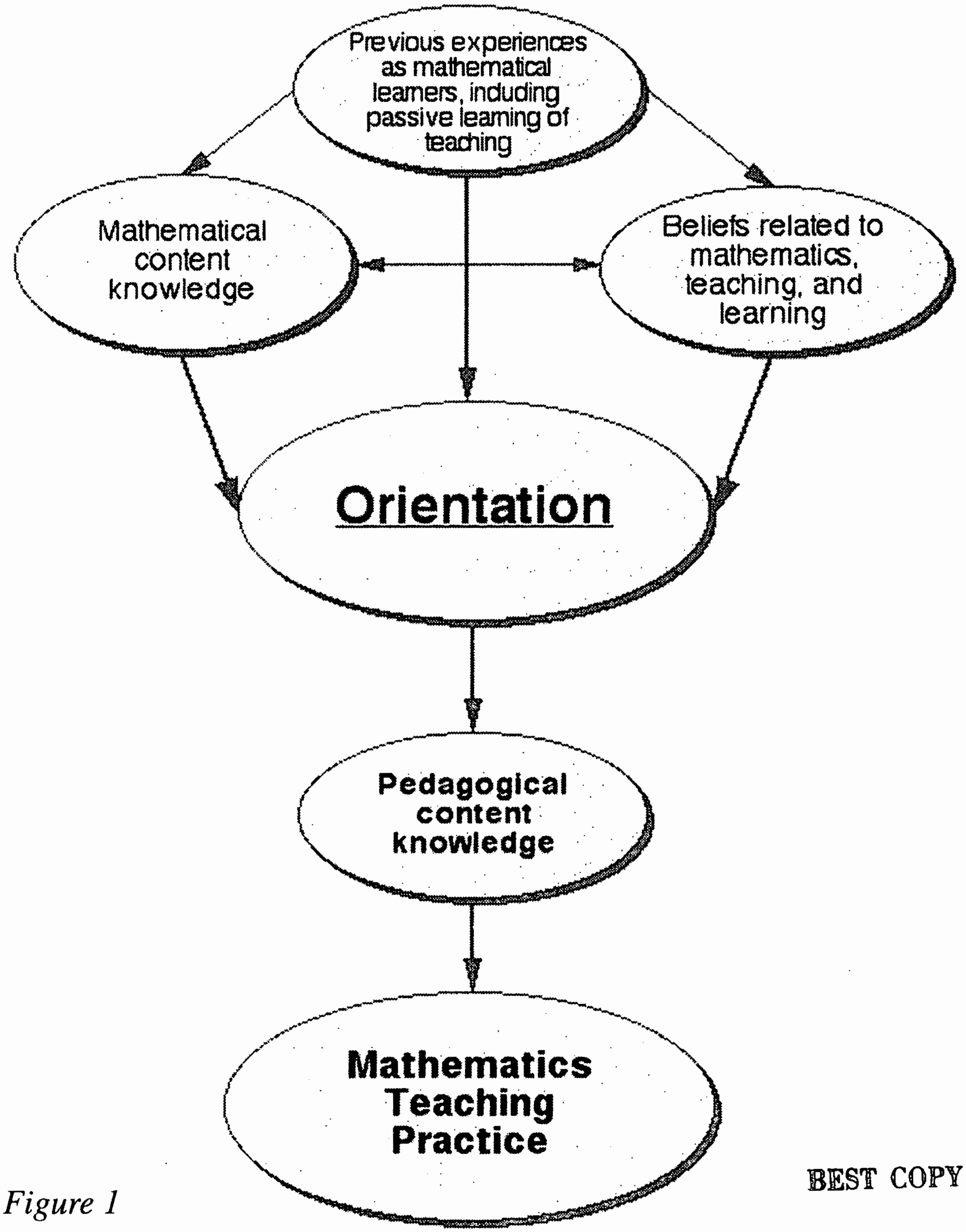

Figure 1

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Methodology

This study takes place at a comprehensive institution serving over 14,000 students. With approximately 300 elementary education undergraduates graduating per year, the elementary education program is considered a substantial program both within the university and across the country.

Within the required elementary teacher education courses, there are three primary courses within which we are focusing our efforts related to mathematics education reform: mathematics content course, professional development course in instructional design, and mathematics methods course.

Design Experiment

The goal of our design experiment is to create an elementary mathematics teacher education program that addresses and develops students' orientations as a systemic objective. To allow time to foster new orientations we aligned previously disconnected courses in mathematical content and pedagogy throughout the required mathematics strand. Course goals reflect our common orientation toward teaching and teacher education, including active knowledge construction, opportunities for on-going reflection, focus on enduring mathematical understandings, and modeling teaching practices that support these tenants. These goals are critical in achieving teacher change (Richardson & Placier, 2001).

Data Sources

This paper examines data collected from elementary education students (n=300) during the first year (2001-2002) of a three-year research project. Project data sources include: (1) orientation surveys, (2) course artifacts, (3) content knowledge exams, and (4) semi-structured interviews. This paper explores results from surveys and course artifacts.

Orientation surveys were distributed at the beginning and end of two courses: mathematics content and general pedagogy. The content survey garners information about participants' overarching conceptions related to mathematics and its teaching. The pedagogy survey asks participants to elaborate their understanding of teaching and learning. Most survey questions (e.g., "Mathematics involves mostly facts and procedures to be learned," "It is important for teachers to have a thorough understanding of the subject he/she is teaching") use a Likert-type scale (1-5). End of course surveys also ask participants to reflect on their experiences related to specific course activities, and how these courses may have influenced mathematics orientation. Administering surveys at the beginning and end of these courses allows for interpretation of changes in individual participants' orientations and comparisons across individuals throughout the program.

Artifacts (course assignments) were collected from *experimental* sections (taught by principal researchers). Assignments were designed to allow students to make explicit connections between being a learner of mathematics and a prospective teacher of those mathematical understandings. Assignments deliberately create cognitive conflicts (Pajares, 1993), which challenge students' conceptions, facilitate reflection, and provide opportunity for expanded thinking.

Analysis: Three Phases

(1) Data was analyzed using *direct interpretation* (Stake,1995) to garner emergent themes and patterns within individuals to understand the substantive changes in prospective teachers' thinking. Specifically, coding illustrated what changes occurred in prospective teachers' orientations toward mathematics, as well as mathematics teaching and learning.

(2) Data was *aggregated* (Stake, 1995) across individuals to understand growth in orientations across all individuals within each course.

(3) What was learned in the second phase was used to make comparisons between experimental and non-experimental sections.

In essence, case studies will be done at three levels: individual students, course section, and course design.

Validity issues are addressed by the following: (1) data is triangulated across multiple sources, (2) coding is done independently by two researchers, allowing for cross-validation of results, and (3) external validity of coding is explicit in the multi-stage nature of the research project.

Results

Preliminary results suggest that most (approximately 85%) of the prospective elementary teachers in experimental sections changed their orientations towards mathematics, as well as its teaching and learning. Specifically, changes were made in two areas: (1) appreciation of role of content knowledge in designing and implementing instruction, and (2) perception of mathematics as problem solving. In the first area, participants recognized that their own understanding of content is critical to the design and implementation of instruction. As one prospective teacher stated, "We all have different ideas about different subjects, and everything we teach will be influenced by our own understandings and beliefs about that subject." By the end of pedagogy course activities, participants also saw content knowledge as being an important attribute of a "good" teacher. This reflects a dramatic change from themes during the beginning of the course; most participants only referred to affective attributes, e.g., "caring," "kind," and "loves children." In addition to recognizing the important role of their own understanding of content in instruction, these prospective teachers also referred to their students' understanding of content as being a primary goal of instruction.

In the second area, participants changed their views of mathematics as a discipline. For example, many participants who understood mathematics as static and comprised of rules to be memorized, later described it as complex, changing, and open-ended. By the end of the mathematics content course, almost all participants referred to mathematics as involving problem solving, using multiple approaches, and as necessary to solve problems in real-life situations. As one participant stated, "I now [after taking this course] realize there are different ways to approach problems." Changes in orientation were less evident in non-experimental sections.

Comments made by prospective teachers in end of course surveys and on course assignments indicate that the collaboratively designed activities helped to scaffold this change. For example, many participants explained that writing about their understandings forced them to reflect upon and connect those understandings. Even during the mathematics content course (often taken prior to the instruction and design course) participants commented on the need to understand content in flexible ways to best help their students understand these concepts. As one participant stated, "The more methods and strategies I learn in problem solving, the easier it will be to get concepts across to students." These findings suggest that when programs are purposely designed to understand and address students' thinking about content and pedagogy early and connectedly in a program, changes can be made in mathematical orientations that will ideally lead to conceptually based teaching.

What we find compelling related to these results is that while these prospective teachers of mathematics grew in their understanding of and orientation toward mathematics and recognize the important role these understandings play in implementation of instruction, they often *design* mathematics instruction to reflect a more directive style. Reasons include difficulty of mathematics as a subject and wavering confidence in their developing mathematical orientations. Our continued research project will now follow these prospective teachers into their elementary mathematics methods course and student teaching experience. We anticipate that students will strengthen these tenuous connections as they experience teaching mathematics in more reform-oriented ways.

This study contributes to what is understood about prospective teachers' thinking concerning mathematics and its teaching and learning. Within this design experiment our shared vision of conceptually based mathematics education was able to be realized by collaboratively incorporating efforts early and connectedly in the teacher education process, thus stimulating interdisciplinary research in mathematics and elementary teacher education.

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GROWTH IN MATHEMATICAL UNDERSTANDING WHILE LEARNING HOW TO TEACH: A THEORETICAL PERSPECTIVE

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This theoretical paper outlines a conceptual framework for examining growth in prospective teachers' mathematical understanding as they engage in thinking about and planning for the mathematical learning of others. The framework is based on the Pirie-Kieren (1994) Dynamical Theory for the Growth of Mathematical Understanding and extends into the pedagogical realm by assuming that growth in mathematically specific understanding for teaching is a dynamical, leveled but not linear, transcendentally recursive process of reorganizing ones knowledge. Data from a preliminary case study is shared to illustrate how the framework can be used to provide a lens for examining growth in mathematical understanding within the context of learning to teach high school mathematics.

Determining ways to challenge and extend prospective teachers' ideas about *school mathematics* (the mathematics they will teach) is recognized as one of the most important matters to be considered by mathematics educators today (Conference Board of Mathematical Sciences (CBMS), 2001; Sowder et al., 1998). To break the perpetual cycle of inadequate knowledge of school mathematics, teacher preparation leaders were advised to modify their programs of study to provide better preparation for future teachers in both *university mathematics* (typical college mathematics courses) and school mathematics. This situation leaves teacher preparation program leaders faced with deciding exactly how mathematics preparation can be accomplished both efficiently and meaningfully without neglecting the much-needed time for pedagogical development.

Research results (Berenson & Cavey, 2000; Bowers & Doerr, 2001; Cavey, Berenson, Clark, & Staley, 2001; Clark, 2001; Ma, 1999) suggest that engaging prospective teachers in *teaching tasks* (thinking about and planning for the mathematical learning of others) may be an effective way of addressing both content and pedagogical developmental needs. Bowers and Doerr (2001) engaged prospective and practicing teachers in computer-based activities as learners of the mathematics of change (rate of change) and then as teachers of rate of change and observed both mathematical and pedagogical insights made by the prospective and practicing teachers as they were engaged in both types of activities (as learners & as teachers). Berenson and Cavey (2000), Cavey, Berenson, Clark, and Staley (2001), and Clark (2001) conducted preliminary studies to investigate the plausibility of using an experimental curriculum, "lesson plan study" (LPS), to promote growth in prospective teachers' understanding of school mathematics and teaching strategies. As LPS participants, prospective secondary mathematics teachers engaged in multiple conversations with others (researchers and peers) about teaching a specific secondary mathematics topic. Such conversations occurred over five weeks on the same lesson topic and seemed to promote growth in prospective teachers' understanding of school mathematics (Berenson & Cavey, 2000) and teaching strategies (Cavey et al., 2001; Clark, 2001).

The use of these teaching task methodologies leads to theoretical questions about prospective teachers' growth in mathematical understanding. As such, a conceptual framework was developed to provide a lens for examining prospective teachers' growth in mathematical understanding as they participated in LPS. Here, I briefly describe each component of the conceptual framework, the first activity of LPS, and provide an example to illustrate how the framework can be used.

The Pirie-Kieren Model: Growth in Mathematical Understanding

The Pirie-Kieren (1994) Dynamical Theory for the Growth of Mathematical Understanding depicts understanding as a dynamical, leveled but not linear, transcendentally recursive process of reorganizing ones knowledge. The model is a theory *for* the growth of understanding of a specific mathematical topic by a specific 'person'[i] over time and is comprised of layers of sophistication in thinking that describe the mental activities necessary for growth in mathematical understanding of a particular topic. It is assumed that a learner comes to a particular learning situation with primitive knowledge [all knowledge *not* related to the particular topic] as well as some knowledge of the particular topic, identified by some outer layer of thinking. The seven outer layers of thinking are: Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring, and Inventising.

Pirie and Kieren (1994) asserted that each of the seven outer layers "is composed of a complementarity of *acting* and *expressing*" where "acting encompasses all previous understanding, providing continuity with inner levels, and expressing gives distinct substance to that particular level" (p. 175). Both acting and expressing involve mental as well as physical actions. Through acting, the learner may reflect on how their previous understanding applies to a new learning situation. When expressing, however, the learner makes it clear to oneself or others what knowledge was gained. The terms used for acting/expressing complementarities within the image making, image having and property noticing layers are doing/reviewing, seeing/saying, and predicting/recording.

Critical to the theory is the idea of recursion—that learners revisit layers of thinking in the process of extending their mathematical understanding. Layers are revisited with more sophisticated thinking along one thread of a particular topic in an attempt to broaden and deepen knowledge of that topic. Folding back is the term used to describe the mental activity of accessing one's more primitive knowledge to construct mathematical understanding at an outer layer of thinking. This notion exemplifies the idea that an individual's mental activities do not move in one direction. Rather, an individual functioning at an outer level of understanding will repeatedly return to an inner level to extend their mathematical understanding (Martin, 1999). For example, in formalising his or her understanding of fractals, a mathematician may fold back to his or her knowledge of complex numbers to understand why fractals portray self-similarity. In comparison, an elementary student may fold back to his or her knowledge of whole numbers to develop a rule for finding a common denominator while formalising his or her understanding of the fraction concept.

A deeper look at folding back (Martin, 1999) reveals the complex nature of this process. In particular, not all acts of folding back are necessarily effective in extending mathematical understanding. Research results indicated that the effectiveness of folding back depends on both the structure of the environment and the individual learner and that folding back tends to be more effective when the learner is prompted to fold back to collect specific information (Martin, 1999). This type of folding back, collecting, "occurs when students know what is needed to solve a problem, and yet their understanding is not sufficient for the automatic recall of useable knowledge" (Pirie & Martin, 2000, p. 127).

Extending the Pirie-Kieren Model into the Pedagogical Realm

The framework is extended into the pedagogical realm by assuming that *teaching mathematics understanding* (understanding enacted when making decisions about the mathematical learning of others) draws upon three mathematically specific understandings: mathematics, mathematics teaching strategies, and mathematics learning. While it is obvious that teaching mathematics understanding draws upon other primitive knowledge types, such as understanding human behavior, the research for which this framework was developed is primarily focused on examining the development of the aforementioned mathematically specific knowledge domains.

It is assumed that growth in such understandings can be modeled as Pirie and Kieren (1994) modeled growth in mathematical understanding and that prospective teachers begin their teacher preparation programs, specifically their first methods course, with some level of understanding of mathematics, mathematics teaching strategies, and mathematics learning. Indeed, beginning prospective teachers typically have an outer layer understanding of mathematics and mathematical learning developed through coursework and other life experiences and images of teaching strategies developed through observation. Such outer layer understandings can constrain additional growth in prospective teachers' understanding of school mathematics (Berenson & Cavey, 2000) when prospective teachers do not see the need to revisit certain mathematical ideas. See Figure 1 for an illustration of how the Pirie-Kieren model is extended into the pedagogical realm.

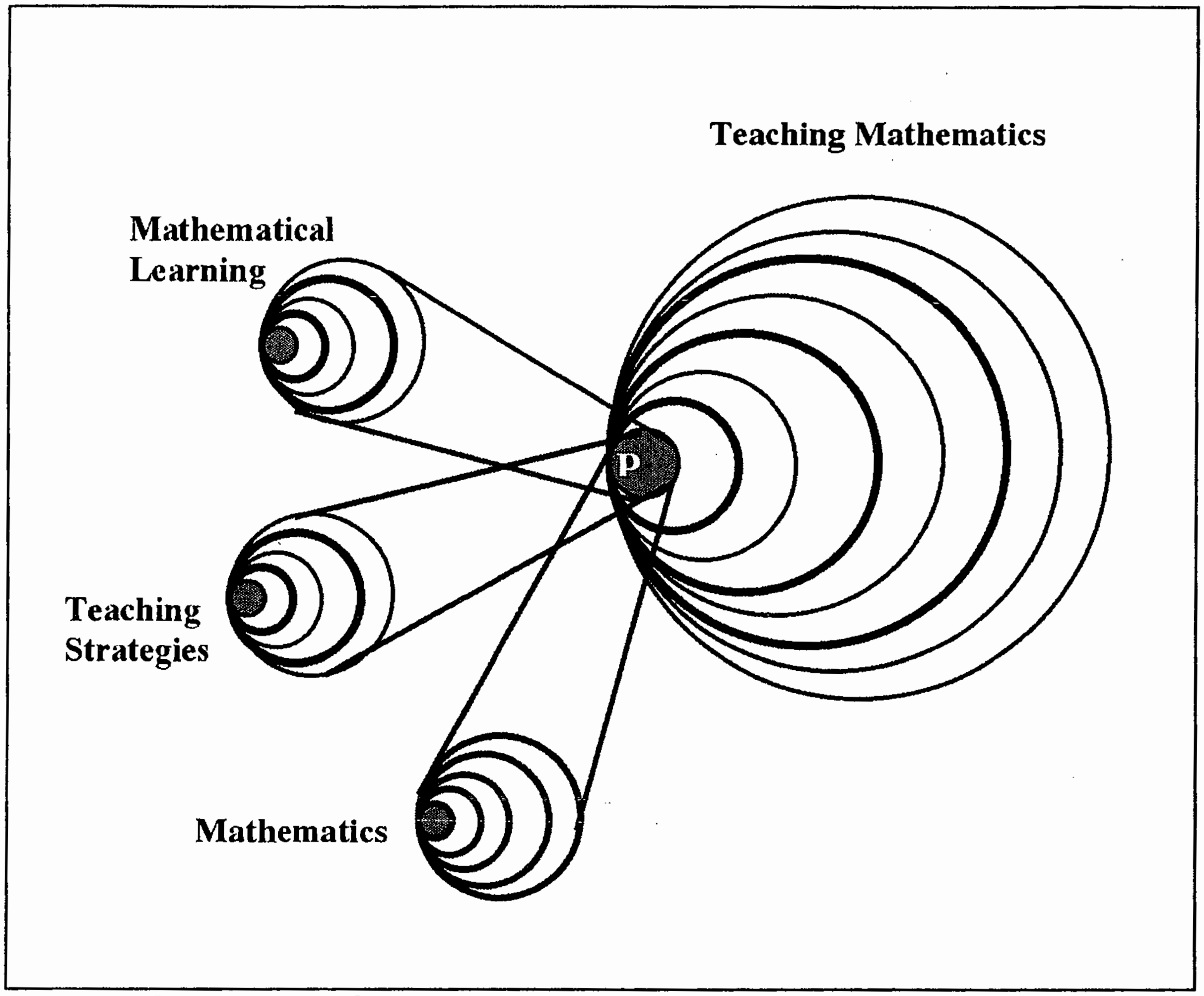

Figure 1. Content-Specific Primitive Knowledge Domains Accessed While Learning To Teach Mathematics.

Right Triangle Trigonometry LPS: The Framework In Action

Data from a preliminary case study is used to illustrate how the framework provides a lens for examining growth in mathematical understanding while one prospective teacher participated in a LPS. This LPS was conducted at the beginning of an introductory methods course for prospective secondary mathematics teachers at a large public university and focused the participants on teaching right triangle trigonometry. Data for one participant, Molly (pseudonym), are shared from the initial planning activity [preliminary interview, lesson planning, and post-planning interview], in the form of videotaped interviews and written artifacts. Table 1 contains a summary of the tasks included in the first activity of LPS focused on teaching right triangle trigonometry.

Table 1. Initial Planning Activity of the Right Triangle Trigonometry LPS

Task	Description	Duration
Preliminary Interview	Videotaped conversation between a researcher and a prospective teacher about what the prospective teacher remembered learning about right triangle trigonometry and preparation for lesson planning.	20 min
Creating Plan 1	Immediately after the preliminary interview, the prospective teacher was left alone with resources (texts, manipulatives, etc.) to plan a lesson to introduce right triangle trigonometry to a high school geometry class.	45 min

Initial Growth in Understanding Right Triangle Trigonometry

Image Saying

Even though Molly had difficulty remembering what and how she learned right triangle trigonometry, she shared two images of her understanding during her first meeting with the interviewer (the preliminary interview). One image was based on the memorization of an acronym whereas the other image was centered on using right triangle trigonometry to solve for an unknown angle measure. When Molly was asked about what she remembered learning about right triangle trigonometry, the following dialogue occurred.

Molly: I remember cosine, sine, and tangent.

I: Okay. Go ahead and write that down while you're explaining this. I want to know everything that you remember. [Molly wrote 'cos', 'sin', and 'tan' on her paper, with 'tan' directly below 'sin' and 'sin' directly below 'cos'.]

Molly: I don't remember all that much.

I: That's fine.

Molly: One thing that stands out is they told me something like socatoa. I'm trying to remember which one goes with which one. [Molly first wrote 'SOCA', then erased 'CA' and wrote 'HCAHTOA' so that 'SOHCAHTOA' was written across the top of her paper.] If I have this right, then this would mean that if you use sine, then you have opposite over hypotenuse, and if you use cosine, you have adjacent over hypotenuse, and if you had tangent, you had opposite over adjacent.

The subsequent conversation indicated that Molly recalled using right triangle trigonometry to determine an unknown angle measure. When asked about how right triangle trigonometry is used, Molly stated, "If you have a right triangle, the sides of it to try to figure out the angles, is all I remember." She seemed to be recalling a way of using right triangle trigonometry to determine an angle measurement by using the measurements of two sides of a right triangle.

Molly was clearly not confident in her ability to recall these or other ideas related to right triangle trigonometry. When asked about what she remembered about what her teacher showed her, Molly remarked, "All I remember is that sohcahtoa thing that stands out to me. I just remember cosine and sine." Molly recognized her limited memory and planned to 'fix' the problem through review. After Molly received instructions for the lesson-planning component, she asked, "Can I use these books to refresh?" Such an inquiry indicates that Molly planned to fold back and collect information about right triangle trigonometry for the purpose of planning her lesson.

Folding Back to Collect

As Molly created her first plan, she folded back to collect mathematically precise information concerning many ideas related to right triangle trigonometry. As indicated by her lesson plan notes, Molly folded back to collect 1) the definition of a right triangle, 2) the Pythagorean theorem, 3) definitions for sine and cosine, 4) the relationships between side lengths for 30-60-90 & 45-45-90 triangles, and 5) a definition for similar triangles. All items collected, except for the definition of similar triangles, were incorporated into her first plan.

Molly's collecting indicates that she valued mathematically precise definitions in her own learning and provides insight into her growth in understanding of right triangle trigonometry. Just the fact that four items were collected, in addition to the definitions of sine and cosine, indicates that Molly's right triangle trigonometry understanding thickened to include connections to the Pythagorean theorem, right triangles, relationships for special right triangles, and similarity. Collecting more precise definitions for sine and cosine indicates additional thickening in her understanding. By considering how these items were incorporated into her first plan, I gained additional insight on how her understanding of right triangle trigonometry changed.

Molly's first plan seemed to be geared towards solving for missing parts of right triangles. By starting the lesson with the definition of a right triangle, she set the stage for the context of the lesson. The Pythagorean theorem was viewed by Molly as a way "to calculate the unknown measure of a side of a right triangle given the measures of the other two sides." For Molly, this led directly to the definitions of sine and cosine since she thought of right triangle trigonometry as a way to find an angle measure based on the measures of two sides. In other words, she thought of both the Pythagorean theorem and right triangle trigonometry as means for 'solving for missing parts of right triangles'. Molly thought of the Pythagorean theorem as a means for using two side lengths to determine the third side length, whereas right triangle trigonometry was thought of as a means for using two side lengths to determine an angle measure. The relationships among the sides of triangles with angle measures 30-60-90 and 45-45-90 triangles seem to have been included to simplify the process when working with these types of triangles. Hence, Molly's understanding of right triangle trigonometry was thickened by connecting to mathematically precise information in relation to 'solving for missing parts of right triangles'.

Folding back to collect mathematically precise definitions for sine and cosine thickened Molly's understanding of right triangle trigonometry further. In her written notes, sine and cosine were referred to as ratios of the measures of sides. She wrote, "Sine is the ratio of the measure of the leg opposite the acute angle to the measure of the hypotenuse." She also referred to sine and cosine as ratios during the post-planning interview, indicating that she was image saying and had indeed remade her image of sine and cosine.

In essence, by the time she finished planning her first lesson, Molly collected mathematically precise definitions and relationships for topics she understood to be immediately connected to 'solving right triangles'. Collecting this information and incorporating the ideas into her lesson thickened Molly's knowledge of right triangle trigonometry.

Growth in Understanding Similarity

Image Saying

In her notes for plan 1, Molly wrote, "Similar triangles have 3 angles of 1 triangle congruent to 3 angles of another triangle and the measures of their corresponding sides are proportional." However, she had not incorporated the idea into her first plan and she struggled to explain the concept when prompted to do so during her second meeting with the interviewer. When Molly was asked how she might help a student understand similarity, she suggested a plan that started with giving the student the definition and then showing some examples. When asked to describe similarity she stated, "It's when two triangles have angles that are similar to one another." Molly was not confident in this response and quickly added, "I would have to look that up. I'm not sure to be honest with you," which indicates that Molly was interested in collecting more information on similarity.

Folding Back to Collect

The interviewer immediately prompted Molly to consider two triangles drawn on a separate piece of paper and asked how she would know if they were similar. Molly drew two right triangles and the following dialogue occurred.

I: Do they have to be the same size to be similar?

Molly: [Pause] No.

I: If they don't have to be the same size, what makes them similar?

Molly: If they are both right triangles, with maybe their angle measures the same, 45-45 or 30-60. As long as all three angles are the same, then they're similar, but they can have different lengths.

This dialogue seemed to help Molly rethink the relationship between corresponding angles of similar triangles, thereby remaking her image of similar triangles. It appears that she folded back to mentally collect part of the definition for similar triangles she had written in her lesson plan notes.

Summary of Molly's Growth in Mathematical Understanding

During the first meeting with the interviewer, Molly exhibited a limited understanding of right triangle trigonometry. In fact, Molly started the LPS with two images of right triangle trigonometry, the SOHCAHTOA acronym and the image of using right triangle trigonometry to determine angle measures. However, opportunities to share her mathematical images seemed to help Molly become aware of gaps in her own mathematical understanding and subsequently plan to collect mathematical information. In addition, as Molly made images for teaching right triangle trigonometry, she folded back to her mathematics understanding to collect, revise, and construct images of mathematics related to right triangle trigonometry. In addition, during subsequent LPS components, Molly repeatedly folded back to revise her understandings of right triangle trigonometry and related primitive knowledge domains. In particular, Molly revisited the ideas of ratio, similarity and the Pythagorean theorem in relation to right triangle trigonometry. An illustration of the primitive knowledge domains that Molly accessed while extending her mathematical understanding during LPS is provided in Figure 2.

Discussion and Implications

By placing the learning of school mathematics within the context of teaching tasks, prospective teachers rely on their teaching mathematics understanding and are thereby forced to fold back to their understandings of mathematics, teaching strategies, and mathematical learning to make decisions about the mathematical learning of others. However, there is much to be learned about the effectiveness of using teaching tasks in adding depth, breadth, and thoroughness (Ma, 1999) to prospective teachers' mathematically specific knowledge for teaching. Particularly, more research is needed to ascertain the effectiveness of using LPS to extend prospective teachers' mathematics teaching understanding.

The framework presented here provides a lens for examining growth in mathematically specific components of teacher knowledge (mathematics, mathematics teaching strategies, and mathematics learning). Initially developed to examine growth in prospective teachers' mathematical understanding as they participated in LPS, the framework can also be used to examine growth in prospective teachers' understanding

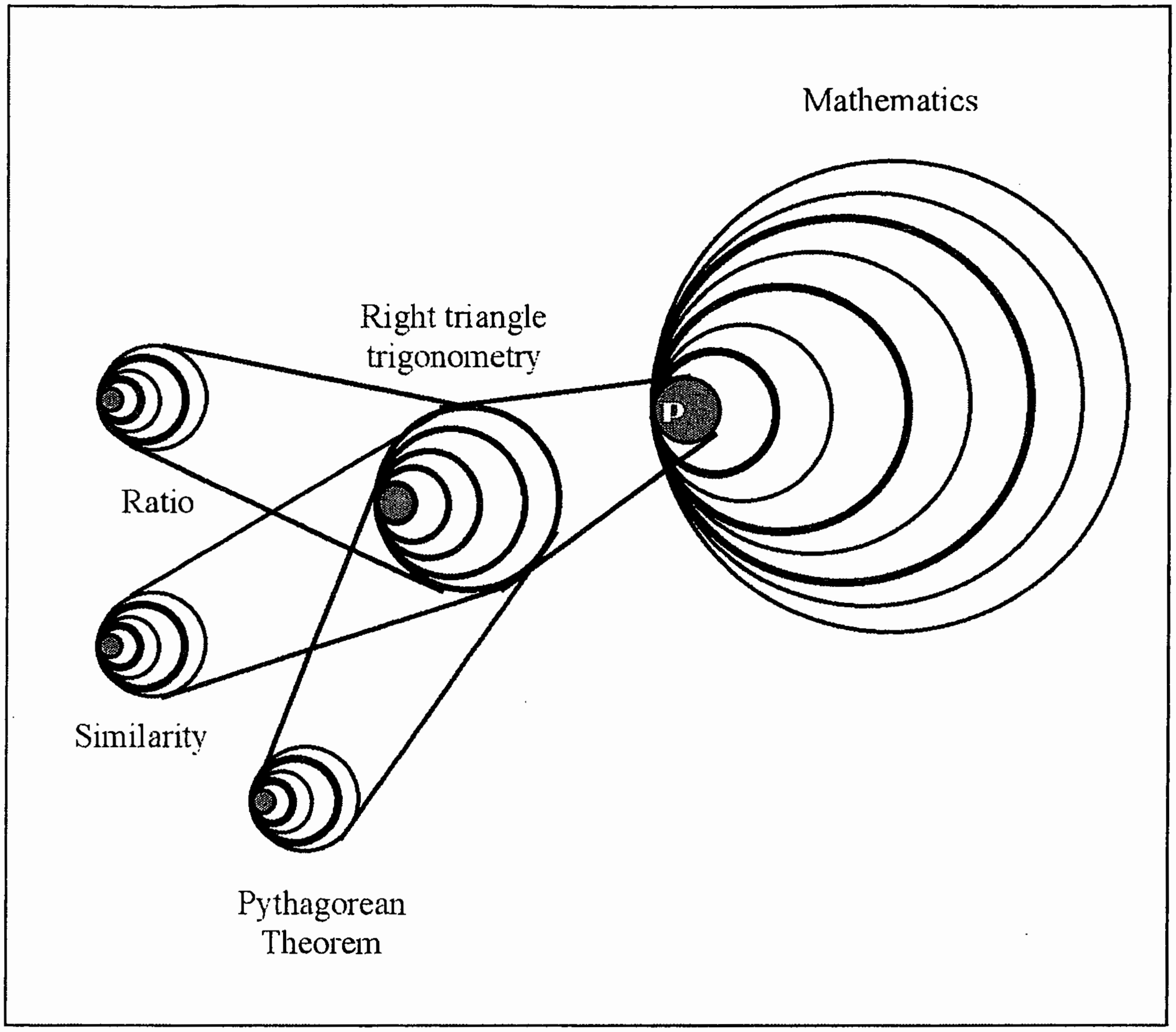

Figure 2. Primitive knowledge domains in mathematics understanding that Molly accessed during LPS.

of teaching strategies and/or mathematics learning. In addition, the framework can be used to examine growth in teaching mathematics understanding for any learner engaged in making decisions about the mathematical learning of others. Applying the framework to other teacher-learning situations is expected to contribute to understanding how teachers learn to teach mathematics. At a very minimum, future applications of this framework must consider the thoughts and actions of prospective teachers when interacting with k-12 students.

Note

[i]'Person' can refer to an individual learner or any size group of learners. Essentially, it can be thought of as a 'learner unit', comprised of any number of individuals that are jointly learning.

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UNDERSTANDING CHANGE: EPISTEMOLOGIES OF THREE FUTURE TEACHERS OF ELEMENTARY SCHOOL MATHEMATICS

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The purpose of this study was to understand 3 prospective teachers' epistemologies of knowledge in general and then specifically in the knowledge domains of mathematical knowledge (MK), mathematics learning knowledge (MLK), and mathematics teaching knowledge (MTK) as they participated in an early mathematical field experience at an elementary school. The analysis revealed that the participants' epistemological views for knowledge in general were helpful in understanding changes (or no changes) in epistemological views in MK, MLK, and MTK. Although the 3 participants shared many common experiences, how they perceived the experiences was very different.

Prospective teachers frequently encounter ideas about teaching mathematics that differ from their own mathematics learning experiences. Models that describe epistemologies (Baxter Magolda, 1992; Belenky, Clinchy, Goldberger, & Tarule, 1986; Kitchener, 1983; Perry, 1970/1999) are helpful in understanding how individuals perceive these ideas. Such models highlight that individuals who adhere to different epistemological stances interpret and respond to the same learning environment in different ways. Related work in mathematics education (Cooney, Shealy, & Arvold, 1998, Copes, 1982) supports this view. Understanding prospective teachers' epistemologies can contribute to theory about learning to teach mathematics, which will aid in the design and implementation of teacher education programs.

The purpose of this study was to understand three prospective teachers' epistemologies, or views of the nature and justification of knowledge, in general and then specifically in the knowledge domains of mathematical knowledge (MK), mathematics learning knowledge (MLK), and mathematics teaching knowledge (MTK) throughout their teacher education program. In this paper I report the results from the first semester of the study.

Theoretical Framework

Models that describe epistemologies indicate that individuals who adhere to different epistemologies have different orientations regarding who counts as a source or what counts as evidence for knowledge and to what extent that knowledge is certain. Essentially, the extent of certainty of knowledge ranges from knowledge perceived as certain and free of context and human values to knowledge as relative and based in context and personal interpretations. Sources of knowledge are characterized as predominantly external (e.g., perceived authorities), predominantly internal (e.g., personal experiences, intuition), or a blend of both. What counts as evidence for knowledge is directly related to source, in that an individual assumes that information is correct on the basis of what an authority says, personal experiences (it worked for me), intuition (it feels right), reasoned judgment (this is how we agree to resolve differences), or some combination thereof.

Another characteristic for describing one's epistemology is an individual's willingness to entertain multiple perspectives. First, if sources of and evidence for knowledge tend to be based on perceived authorities, for example, then an individual tends to assume that a question has only one right answer. A prospective teacher who adheres to such a view may be inclined to look toward instructors of mathematics education courses for *the* way to do mathematics or *the* way to teach mathematics (e.g., Mewborn's (1999) participants in Stage 1). He or she may prefer learning environments in which discussions of multiple perspectives on a topic are not promoted.

Second, a learner may tend to focus on internal sources of knowledge. The term *internal* is intended to imply a focus on self and intuition or a focus on others as intuitive individuals. Certain answers exist *for oneself* on the basis of personal experiences and intuition. Certain answers exist *for others* on the basis of their personal experiences and intuition. Multiple perspectives are considered legitimate, and knowledge is considered uncertain because of one's inability to fully understand the experiences of others. A prospective teacher from this perspective may appreciate whole and small-group class discussions in mathematics education courses but might be unwilling to evaluate others' views because he or she believes that everyone has the right to his or her own opinion or that everyone has his or her own interpretation of an experience (e.g., Brenda in Chauvot, 2001, or see Wilson & Goldenberg's (1998) discussion of *extreme relativism*).

Third, a learner may expect environments that promote multiple, legitimate perspectives in some contexts but not in others. A prospective teacher may expect a multiplistic environment in a methods course but not in a mathematics course. Evidence for knowledge varies according to the context and the individual's tendencies toward accepting external or internal knowledge sources.

Methodology

The study began when the prospective teachers participated in an early mathematical field experience designed to challenge traditional beliefs about mathematics and its teaching and learning. Purposeful sampling strategies (Patton, 1990) were used to select Jenna, Debbie, and Gail. Their responses on an open-ended belief survey showed that they held different epistemological stances. For example, the variation in their rationales to "When you are a teacher, will you ever ask your students to solve a new kind of problem without first showing them how to solve it?" indicated that sources of mathematical knowledge varied across participants. Gail said that she will not pose such problems, and she relayed a desire for students to imitate the teacher, whereas Debbie will, and she argued that the children could be successful without help from the teacher. Jenna will, as well; her rationale indicated that she intended to "fix" children's thinking to move toward a predetermined mathematical goal.

A university instructor directed the field experience for 2 1/4 hours per week at an elementary school. Three main components of the field experience were (a) video clips illustrating different ways children approach mathematical tasks, (b) carefully designed interviews with children, and (c) small and whole-group discussions and writing assignments about the videos and the interviews (see Philipp, Thanheiser, & Clement, 2002.) All three participants were concurrently taking the first of four mathematics courses for elementary teachers.

Data sources for this study were Baxter Magolda's (1992) Measure of Epistemological Reflection (MER), a semistructured interview, and a class-log writing assignment. All interviews were audiotaped and transcribed. Data from the field experience and the mathematics course included the pre- and post-belief-survey, a mathematics autobiography, and writing assignments about the videos and the interviews.

Data analysis of the MER followed Baxter Magolda's (2000) recommendations for interpreting participants' responses. The additional data were used to confirm or disconfirm this analysis. The data were then reviewed again to ascertain more specifically who or what the participants viewed as sources of and evidence for MK, MLK, and MTK, and to what extent they viewed that knowledge as certain.

Results

Sources, Evidence, Certainty of Knowledge

The analysis pertaining to knowledge in general revealed that both Jenna and Gail had multiplistic tendencies. Both preferred classes that focused on ideas or concepts rather than on factual information, because the former required more "thinking" (as opposed to memorization) and were often structured in such a way that exposed them to their peers' thinking and ideas. In terms of evidence for knowledge, both Jenna and Gail showed reliance on the instructor for sorting through ideas. Gail, however, also seemed to expect some form of rationale from her peers. She liked that small-group activity "forces everyone to have an opinion about a particular topic and be able to defend it." Gail frequently spoke of a "personal interpretation" that seemed to influence what counted as evidence for knowledge. She preferred courses about ideas because, she said, "I am able to make an interpretation of the discussions based on my own personal viewpoint. There [are] many possibilities for a correct answer, based in part on your own understanding of the subject matter."

Gail preferred to be what she called an *active listener* in her courses:

> I like to listen to everyone's point of view. I do have a point of view, but I don't like sharing it. I compare my point of view to their point of view and just make a conclusion out of it.

Gail's position held across all her courses whereas Jenna described her role in the classroom as dependent upon the content. For example, Jenna argued that her role in her history and geology courses was "pretty much sitting and listening"; there was no room for discussion.

Debbie preferred courses that focused on factual information because "there's no gray area." She believed that her interactions with peers or instructors should remain at a minimum. She saw no need to attend a class if the instructor intended to merely follow the book. She preferred "sitting back and listening" and rarely asked questions in her courses because frequently her peers asked questions for her. Debbie did not value her peers as knowledge sources; they served as tools for obtaining knowledge from her instructors. She trusted what her instructors had to say about the content and her own ability to teach herself (from the textbook), if needed. Like Gail, she felt that her role in the classroom was the same across all her courses.

Sources, Evidence, Certainty of MK

The initial data indicated that for all three participants, sources of and evidence for MK for themselves and for children resided with the instructor, the textbook, or both. For example, Gail described her past mathematics learning experiences as "the more the teacher was able to explain a topic to me with as little confusion as possible, then I would be fine." All three participants acknowledged that mathematics was more than a set of rules and procedures. They reported that learning mathematics should be more than memorization and that they wanted children to understand "the whys" of algorithms. The participants gave no indication that they viewed MK as a knowledge domain that involved context or personal interpretation.

The mid- and end-of-semester data provided opportunities to ascertain changes (or no changes) in the participants' epistemological views in MK. All three claimed that the field experience allowed them to see that one can approach mathematical problems in multiple ways and that this notion was different from their own mathematics learning experiences. For example, Gail commented favorably about a video clip in which the teacher asked students to share their strategies with the entire class. She noted,

> When I was their age, it was the teacher who was the sole instructor of the class, leaving little room for insight amongst my classmates.... [I liked how] the other students along with the teacher can compare and contrast different methods and see which ones are wrong or right.

Her comments indicated that Gail had begun to value children as their own sources of MK and that the children and the teacher should decide what counted as MK. Gail admitted that mathematics was not as clear-cut as she had thought and that mathematics involved abstract thinking as well: "It's abstract in the way you arrived at the answer and how you think of the problem and how other people [think of the problem]." In fact, there were several indications that both Jenna and Gail wanted children to be their own sources of MK; personal interpretation was involved, and "what made sense" served as evidence for that knowledge.

As another example, Jenna's post-response regarding whether or not she would ask students to solve a new kind of problem without first showing them how to solve it was "Yes. So they can come up with their own algorithms that make sense and work for them." Jenna made similar comments in papers she wrote about her interviews and in her descriptions of her own learning of MK throughout the semester: "It's a lot of how you see things, and how you interpret things, instead of just, 'Do it this way. Memorize it.'" Jenna and Gail seemed to now have a more personal approach in their thinking about MK.

Compared to Jenna and Gail, Debbie gave fewer indications of epistemological change in the domain of MK. One theme that was consistent with Debbie was a focus on the importance of conceptual understanding. However, Debbie seemed to continue to see herself as the source of MK for her future students. She felt that familiarizing herself with different ways children approached mathematical problems would help *her* "explain how to do things." She did not value having the children share the multiple strategies. "I think that it is important to only show the kids a single way to do it until they have an advanced understanding of the type of problem they are learning."

Sources, Evidence, Certainty of MLK and MTK

Essentially, MLK and MTK *became* knowledge domains for Jenna and Gail. They had not previously thought about either how children think about mathematics or how to teach mathematics. Jenna commented, "I have never thought of teaching math as very important, or never a big deal in the classroom. [I now] realize the importance of how you teach math."

For both Jenna and Gail, viewing the video clips, working with children, working with their peers, and receiving guidance from the university instructor all served as sources of MLK and MTK. What counted as evidence for this knowledge were mostly their (now) deeper understanding of mathematics ("I finally understand this algorithm.") and their interviews with the children ("She did exactly what we talked about in class!").

Debbie described the field experience as interesting and informing. The video clips of whole-class lessons and of teachers reflecting on their practice were the most valuable to her. She appreciated the "hands on" environment and opportunities to see how kids think about mathematics. But she felt that she had not engaged in teaching: "I'm not teaching anything. I'm just picking their brain." She described some of the coursework as "busy work" and "extremely repetitive." She went on to say,

> We just interview them. There's no social aspect of it. It's pretty much get the job done and, "How did you get that answer?" and then they'll tell you. You take notes, and then you do write-ups ... [about] what we learned and

what the kid learned. Kind of b.s. [*sic*] your way through it, and in 2 pages, you're done.

Debbie did not value the field experience in the same way as Jenna and Gail did. She fulfilled the course requirements, but not in a meaningful way.

Final Comments

Jenna and Gail's multiplistic tendencies coupled with the field experience provided opportunity for acknowledging children's mathematical thinking and considering implications for mathematics and its teaching. Continued data collection will indicate whether and why these changes are sustained. Perhaps a better forum for Debbie would be a context that is more aligned with her (single) notion of what teaching is about and how one learns to teach mathematics. She seemed to discount the field experience because of its structure. She may be one who needs mathematics teaching successes and more importantly failures within whole-class instruction that might prompt her to re-examine her view that the teacher is the sole source of MK.

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PLACING CHILDREN BETWEEN PROSPECTIVE ELEMENTARY SCHOOL TEACHERS AND MATHEMATICS: THE MATHEMATICAL EARLY FIELD EXPERIENCE

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To address the problem that prospective elementary school teachers (PSTs) go through even well-designed mathematics courses in a perfunctory manner, we propose a new model, the Children's Mathematical Thinking Experience (CMTE). The CMTE was designed to enhance PSTs' beliefs about mathematics and children's mathematical thinking, by having PSTs work in principled ways with individual children early in their teacher-training experiences. The PSTs come to attend to children's mathematical thinking and then to the mathematics, because they care about the children with whom they work. Our analysis indicates that the CMTE enhanced PSTs' beliefs, and did so to a greater degree than more typical early field experiences.[1]

One cannot teach what one does not know, and so a prospective teacher must understand mathematics to be in a position to teach it. However, we find problematic the idea's translation into practice if prospective elementary school teachers first learn mathematics content and only later consider issues of teaching and learning. First, we know that developing deep understanding of the mathematics of elementary school is far more difficult than was once thought (Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998). Second, our experience has been that even when prospective elementary school teachers (PSTs) attend a thoughtfully planned course designed to engage them in rich mathematical thinking, too many of them go through the course in a perfunctory manner. Many PSTs' conceptions about what mathematics is-- a fixed set of rules and procedures--along with their perceptions of how children and adults learn mathematics--by being shown how to solve problems in a prescribed step-by-step fashion--interfere with the more conceptual, meaning-making goals of mathematics courses designed for PSTs. We thus believe that there is a better place to begin working with prospective elementary school teachers. As a major part of a large-scale study involving five treatments and 164 PSTs, we have studied a new model for integrating mathematical content and pedagogy, the Children's Mathematical Thinking Experience—Live (CMTE-L[2]). PSTs, while taking their first mathematics course for elementary school teachers, engage with children's mathematical thinking by interviewing and tutoring elementary school-age children.

The purposes of this paper are thus to provide (a) a rationale for providing PSTs opportunities to work with children early in their programs, (b) a brief description of the CMTE-L, and (c) evidence that the CMTE-L changed PSTs' conceptions about mathematics and how children learn it to a greater degree than more typical early field experiences.

Why Focus on Children's Mathematical Thinking?

The widely accepted notion that children learn by building upon their existing knowledge might be taken as an axiom of learning. David Ausubel stated, "If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows" (quoted in Hiebert & Carpenter, 1992, p. 80). We adapt Ausubel's comment to include not only what the learner knows but also that about which the learner cares. We believe that we can facilitate the process by which PSTs come to learn mathematics by beginning with what PSTs care most about—children. We place children (rather than children's thinking, for example) at the center of caring because we believe that for most PSTs, the initial caring is a phenomenological act of concern for the whole child versus for a particular characteristic of the child. We hypothesize that when PSTs engage children in mathematical problem solving, the PSTs come to care about children's mathematical thinking. PSTs begin to see how children think about mathematics and come to recognize that children solve problems in varied and sometimes mathematically powerful ways. It is then that PSTs' caring will extend to mathematics, because they realize that to be prepared to understand the depth and variety in children's mathematical thinking, they must themselves grapple with the mathematics.

How Do We Focus on Children's Mathematical Thinking?

Our approach to integrating mathematics content and pedagogy is through the CMTE-L we created to provide PSTs opportunities to interview and tutor children in mathematics and to reflect upon the process. The CMTE-L is a course held weekly for 2 1/4 hours in a classroom on-site at a local elementary school. The course addresses children's mathematical thinking and mathematics, and although it is neither a mathematics course nor a mathematics methodology course, it combines aspects of both. The CMTE-L is different from a mathematics course because the mathematics studied is not an end in itself but instead arises from the PSTs' work with children. Furthermore, the MEFE is different from a mathematics methodology course because we do not attempt to help students learn to teach a group of students. For example, we do not discuss lesson planning or classroom management. We do not consider how children might learn from one another, because PSTs work with only one child at a time, a model selected for two reasons. First, by working with only one child, PSTs can focus solely on one child's mathematical thinking. Second, when experiencing difficulties understanding the child's thinking or considering ways to support the child, the PSTs must grapple with the child's reasoning, because neither the child nor the PST can turn to other children for help (cf. Thompson & Thompson, 1994). In the CMTE-L, PSTs work directly with children in about half of the 13 sessions; other times they analyze previous sessions with children, plan subsequent sessions, discuss video clips of children solving mathematics problems, or consider more general issues related to children's thinking or mathematics. The PSTs in the CMTE-L are concurrently enrolled in the first of four mathematics content courses required of PSTs.

In the first part of the course, the PSTs examine the mathematical thinking exhibited in young children's solution strategies for various types of mathematics problems (Carpenter, Fennema, Franke, Levi, & Empson, 1999). The PSTs learn to conduct an assessment interview and then, working in pairs, conduct an interview with a primary-grade student. Then they focus on children's understanding of place value and, in the last part of the class, on children's understanding of rational numbers. After each interview, the instructor leads a class discussion focusing upon what the PSTs learned, struggled with, or found surprising during their interviews. The interview sessions are interwoven with content discussions, in which some points are illustrated by using video clips.

The individual interviews and tutoring sessions, accompanied by reflections on the experiences, motivate the PSTs and help them to see that mathematics is a complex subject to teach and to learn. When they see, firsthand, a child struggle to make sense of some of the ideas and then see children on video clips struggle with the same ideas, they begin to recognize the difficulty of the topic and realize that it merits their attention. We hope that this compelling evidence shapes their beliefs and ultimately motivates them to attend to the mathematical content.

What Evidence Do We Have to Support a Focus on Children's Mathematical Thinking?

Research Design

The data described in this section comprise a subset of the data collected as part of a large-scale study involving five treatments and 164 PSTs. The study was conducted in Fall 2001. The description of *all* treatments and data collected is beyond the scope of this paper, and because of the project timeline, the analysis provided here is preliminary. We do, however, provide evidence that the CMTE-L did shape the PSTs' conceptions about mathematics and mathematics teaching and learning, and did so to a greater degree than other treatments.

Data Sources

To investigate the CMTE-L's effects on PSTs' beliefs about mathematics and children's thinking, we analyzed data from several sources. One source was an open-ended-response computer survey designed to assess PSTs' conceptions about seven beliefs related to mathematics and children's mathematical thinking. Each belief was measured by at least two items; the scores for the items were merged, and a single score was assigned for each belief. The blinded data were coded by trained coders external to the project. The inter-rater reliability for the coded data averaged 84%. Coders resolved disagreements through discussion of the data as related to the scoring rubric. We discuss change data, representative of the magnitude of change scores on all beliefs from pretest to posttest, for one belief: *The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics (Belief 7).* Both during and after the semester, we interviewed small numbers of PSTs individually and in groups. We have also used end-of-course written-survey data to triangulate the results.

Sample and Treatments

All project participants were paid and were randomly assigned to one of five treatments. We limit this analysis to two treatments, the CMTE-L (n = 50) and an early field experience, the Mathematical Observation and Reflection Experience in a Reform Classroom (MORE-R[3], n = 22). PSTs who engaged in the MORE-R observed mathematics lessons of teachers known to engage in reform-teaching practices. We compared the data for PSTs in these two groups for two reasons: (a) The MORE-R provided PSTs access to children in classrooms of reform-oriented teachers, and so one might expect that the conceptions about children's mathematical thinking and mathematics would be shaped more positively for the MORE-R PSTs than for the CMTE-L PSTs, and (b) sending PSTs to schools to observe is a common early teaching-related experience for PSTs; thus, we compare a more typical model to the proposed one. We consider the MORE-R an enhancement from the typical field experience in that teachers were carefully selected and known by project team members to engage in aspects of reform-oriented mathematics teaching.

Data Analysis

To give a sense for the magnitude of the change in CMTE-L students' conceptions, we provide data about one belief, stated above, on children's thinking. This belief was measured by three items, one of which follows.

a) Which fraction is larger, 1/5 or 1/8, or are they the same size?

b) Your friend Jake attends a birthday party at which there are five guests who equally share a very large chocolate bar for dessert. You attend a different birthday party at which there are eight guests who equally share a chocolate bar exactly the same size as the chocolate bar shared at the party Jake attended. Did Jake get more candy bar, did you get more candy bar, or did you and Jake each get the same amount of candy bar?

Which of these two problems is easier for children?

__*a* is easier than *b*

__*b* is easier than *a*

__Both items are equally difficult.

Please explain your answer.

This item is intended to measure PSTs' beliefs about whether real-world contexts support or hinder children's thinking. Many adults believe that children are best supported in their mathematical thinking by working primarily with symbols, but evidence shows that real-world contexts better support children's thinking (Carpenter et al, 1999). To receive the highest score on this item, a respondent would select the choice *b is easier than a* and explain that the real-world context supports children's thinking whereas the symbols may actually interfere. On the pretest, 74% of CMTE-L and MORE-R PSTs received the lowest score possible on this item, indicating a belief that solving word problems is more challenging for children than interpreting symbols. One PST wrote, "Story problems tend to get children confused and allow mix ups to happen." That PSTs initially hold this belief is hardly surprising; during CMTE-L class discussions, many acknowledged their own difficulties in solving word problems and did not think that experiences with contexts could help young children to make sense of mathematics. Only after interviewing young children and seeing them solve a variety of problems in context did some of the CMTE-L PSTs begin to think differently about how children might come to understand mathematics.

Results

Across all seven beliefs measured, more CMTE-L PSTs than MORE-R PSTs increased their belief scores: More than 2/3 of the CMTE-L PSTs' belief scores but fewer than 1/2 of the MORE-R PSTs' belief scores increased. Furthermore, the CMTE-L PSTs' increases were greater than those of the MORE-R PSTs. For Belief 7, 67% of the CMTE-L PSTs' belief scores increased at least 1 point (on a 5-point scale), compared with 52% of the MORE-R PSTs' scores. The percentage of CMTE-L PSTs' scores that increased at least 2 points (36.7%) was more than 1 1/2 times the percentage of MORE-R PSTs' scores (23.8%) that did so, and the ratio was even greater for those whose scores increased 3 points: 12.2% of CMTE-L PSTs' scores increased 3 points, 2 1/2 times the percentage of such MORE-R PSTs' scores (4.8%).

One CMTE-L PST who on the pretest received the lowest score on the item above, received a high score on the posttest by responding, "[Choice *a*] might get confusing for kids because…they might think that since 8 is bigger than 5 then [1/8] is the bigger of the two numbers. …[Choice *b*] is easier because it is put into real life context. This way the child will be able to think about himself being at the party and be able to know which one is larger."

We hypothesize that the CMTE-L PSTs' greater improvement compared with MORE-R PSTs' can be accounted for because the CMTE-L PSTs worked, in principled ways, directly with one child at a time. This experience was motivating, both because they worked with a child about whom they cared and because they had to be prepared to elicit, understand, and support the child's mathematical thinking. Furthermore, the CMTE-L PSTs had opportunities to debrief their interviews and hear from classmates about their experiences working with children. The instructors attempted to focus these discussions on typical, interesting, or puzzling thinking that children exhibited. Through the thinking of the children, the instructors could engage the PSTs in discussions about mathematics. This combination of experiences influenced the beliefs of more than 2/3 of the CMTE-L PSTs.

Although the MORE-R PSTs' belief scores improved, both the number of PSTs who changed and the extent of change were less dramatic than for the CMTE-L PSTs. For some readers this result may seem surprising inasmuch as the PSTs visited classrooms, observed reform-oriented teachers, and reflected on their experiences in written form after each observation. However, this model has limitations. For instance, although the selected teachers were known to engage in reform-oriented teaching practices, such teaching is multifaceted, and some teachers may have exhibited fewer of its features than others. Also, classrooms have many aspects, so that for PSTs to focus primarily on children's mathematical thinking is challenging. Other than considering the mathematics of the learner, MORE-R PSTs could focus on, for example, classroom management, manipulative selection, personality of the teacher, bulletin boards, arrangement of desks, textbook use, and so on. Thus for PSTs to focus on our objective, children's mathematical thinking, was difficult. In contrast, in the CMTE-L the focus on children's thinking was explicit, and tasks and video clips were constrained with that focus.

Although not all students who participated in the CMTE-L greatly improved their belief scores, 95.7% of the CMTE-L PSTs described the experience as either *very valuable* or *valuable.* The CMTE-L PSTs whose belief scores did not improve much may have enjoyed and valued the experience of working with children but may not have shifted this appreciation to mathematics and children's thinking. What experiences could have supported these PSTs to change? is an open question.

Conclusion

When our PSTs worked with children and attempted to analyze their mathematical thinking, most came to see that children can approach mathematics in a variety of creative and nontraditional ways and that children must build upon an understanding of fundamental, underlying concepts. The PSTs thus began to view mathematics as a web of interrelated concepts and procedures, and they recognized that children could solve problems in ways that were different from adults' ways. With this understanding, many PSTs came to care about mathematics, not as a discipline of intrinsic interest but as a means to enable them to complete their caring relationship with a child engaged in mathematical thinking. Near the end of the CMTE-L, a student commented on this change in attitude:

> I can't even explain in half of a page how much my views have been affected. When I worked at elementary schools [while] in high school, I never understood why the kids did not get what I was saying. I can't wait to go back [to my mom's classroom] and help the kids with math. I am so much more confident in my understanding of math. I have a greater appreciation for math … now. I get worried just thinking about how I am going to make my kids *really* understand place value and math. Instead of dreading teaching math, I am now really excited (and nervous) about teaching math. (Jackie, 12/05/01)

In this paper we provided a framework to explain how we sought to address the problem that the study of mathematics content is separated from and precedes the study of pedagogy for undergraduate PSTs. We integrate mathematics content and pedagogy earlier for PSTs by building upon the interests that drew PSTs to teaching: interest in children. Our initial data are promising. These deliberately chosen and highly structured experiences in the CMTE-L support PSTs' engagement in and motivation to learn mathematics to prepare them to support their future students' mathematical thinking.

Notes

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[2] The Children's Mathematical Thinking Experience—Live (CMTE-L) stands in contrast to a separate treatment, the Children's Mathematical Thinking Experience—Vicarious (CMTE-V). In the CMTE-V, students watch video clips of children solving mathematics problems, but do not work directly with children.

[3] The Mathematical Observation and Reflection Experience in a Reform Classroom (MORE-R) stands in contrast to a separate treatment, the Mathematical Observation and Reflection Experience in a Traditional Classroom (MORE-T).

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AUTHENTIC APPROACHES TO LEARNING ASSESSMENT STRATEGIES: BEGINNING TEACHERS' PRACTICE IN CLASSROOMS

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In the project described in this paper, the synthesis of research into situated learning was used to design and develop a multimedia learning environment to teach assessment strategies in mathematics to preservice teachers which was intended to transfer to teaching practice. The paper describes a follow-up studywhich investigated the influence of using the authentic approaches presented in the multimedia learning environment on the teaching and assessment practices of the same people as beginning teachers. Practicing teachers, who as preservice teachers participated in the earlier study, were interviewed with regard to the influence that the authentic approach had on their first and second years of teaching and assessing mathematics. Analysis of data indicates that a complex mix of influences and environmental factors impact upon the teachers' use of assessment strategies in their beginning years of teaching, and that preservice pedagogical beliefs are not always translated to beginning classroom practice.

Introduction

The persistent failure of beginning teachers to access pedagogical skills and strategies acquired during teacher training is an ongoing problem for teacher educators throughout the world. In mathematics, despite the variety of strategies recommended, the *Third International Mathematics and Science Study* indicated that teachers generally continue to limit their approaches to traditional methods (Lokan, Ford & Greenwood, 1996). Traditional classroom approaches reflect a view of mathematics as a set of discrete, hierarchically arranged facts and skills; a view of learning mathematics as replication and repetition; a view of teaching mathematics as exposition and practice; and a view of assessing mathematics as paper and pencil testing for the sole purpose of grading and ranking (Niss, 1993; Stephens, 1992). This lack of progress may result from ineffective teacher-training courses (Borko, et al., 1992). Even newly-trained teachers with non-traditional beliefs about mathematics teaching and learning return to traditional teaching approaches (Brown & Borko, 1992; Raymond, 1997).

One reason for the long-term ineffective practical outcomes of such university courses, is according to Resnick (1987), a result of too little engagement with genuine situations, and too much emphasis on theoretical perspectives. Genuine situations such as school practice, may enable some student teachers to experience classrooms which reflect curriculum reform. For many, however, their school practicum experience is often a reaffirmation of the traditional approaches that are already ingrained from their own schooling (Comiti & Ball, 1996). For those who do observe and engage with teachers using current approaches, the extent and range of approaches seen is limited to a few—the ones with which their supervising teachers are familiar and comfortable implementing. One way to align university teaching and learning more substantially with the way learning is achieved in real-life settings is to base instructional methods on more recent theories of learning which reflect this shift, such as *situated learning* (e.g., Brown, Collins, & Duguid, 1989; Collins, Brown, & Newman, 1989; Lave & Wenger, 1991; McLellan, 1996). Collins (1988) defines situated learning as: 'the notion of learning knowledge and skills in contexts that reflect the way the knowledge will be useful in real life' (p. 2).

Nine critical characteristics of situated learning environments were defined from the extensive literature on the subject and used to inform the instructional design of learning environments (described in detail in Herrington & Oliver, 2000). The learning environment, in reflecting a situated learning approach, needed to:

- provide an authentic context that reflects the way the knowledge will be used in real-life;
- provide authentic activities;
- provide access to expert performances and the modelling of processes;
- provide multiple roles and perspectives;
- support collaborative construction of knowledge;
- provide coaching and scaffolding at critical times;
- promote reflection to enable abstractions to be formed;
- promote articulation to enable tacit knowledge to be made explicit; and
- provide for integrated assessment of learning.

A multimedia program was developed to enable teachers and pre-service teachers of mathematics to explore issues of assessment, entitled *Investigating Assessment Strategies in Mathematics Classrooms* (Herrington, Sparrow, Herrington, & Oliver, 1997). Twenty three assessment strategies suitable for use in K-12 classrooms were identified and grouped in the following categories: *Questioning, Interviewing, Testing, Problem solving, Reporting* and *Self-assessment,* and presented in a multimedia format. Five complex, authentic activities were designed consisting of memos and other documents, to enable students to explore the resource within the parameters and constraints of a realistic task. The interface (see Figure 1) simulates the front part of a classroom with the resources represented by appropriate metaphors (such as video cassettes, filing cabinet drawers, folders on the desk) giving direct access to the following resources:

1) Video clips of teachers using various assessment techniques
2) Video clips of teachers' comments on the strategies

3) Video clips of children's comments on the strategies to present their own thoughts
4) Interviews with experts in the field to provide theoretical perspectives
5) Reflections by third year preservice teachers to provide practical advice
6) Text descriptions of each assessment category
7) Teachers' resources and children's work samples
8) An electronic notebook to enable students to copy text and to write their own ideas
9) Problems and investigations to enable the students to complete authentic tasks.

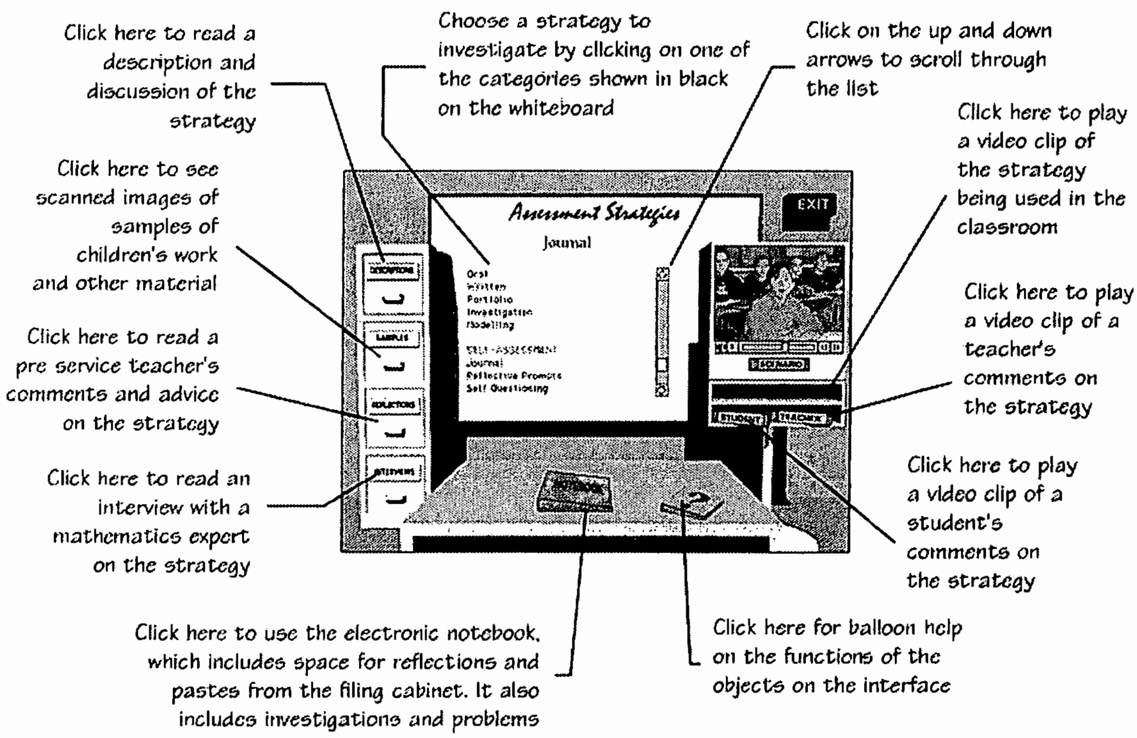

Figure 1. The interface of *Investigating Assessment Strategies in Mathematics Classrooms* showing constitutive elements.

In the first study, preservice teachers explored a variety of assessment strategies appropriate to K-12 mathematics classrooms using the multimedia assessment program. The students spent three weeks of the semester examining strategies within the context of a complex and sustained authentic activity, where in small collaborative groups, they prepared and presented a report on a new assessment plan for mathematics in a school. In order to do this, they were asked to consult with experts, look at what was happening in classrooms, talk to teachers, talk to students and so on—all of which could be done 'virtually' from the CD-ROM.

Three pairs of students were interviewed and observed as they used the multimedia program and the analysis of talk revealed that students used a substantial amount of higher-order thinking as they worked with the assessment program (reported in Herrington & Oliver, 1999). A study of the transfer of a variety of assessment strategies to classroom practice was conducted with the students as they completed two weeks professional practice in schools approximately five weeks after the completion of their work on the assessment program (reported in Herrington, Herrington, & Sparrow, 2000). All the students were required to teach mathematics classes in this practice, and it was expected that they would have the opportunity to implement different assessment strategies. Both students and supervising teachers in the schools were interviewed and the comments were transcribed and analyzed. Transfer was thought to have occurred if firstly, students using the interactive multimedia program on assessment had a good understanding of the types of assessment appropriate in the mathematics classroom and were able to articulate this understanding; and secondly, they employed a variety of the assessment techniques shown in the program, as opposed to the predominant use of pencil-and-paper tests (Cognition and Technology Group at Vanderbilt, 1993).

Analysis of the data shows that all the students could speak knowledgably and confidently about assessment, and all the students used a variety of techniques to assess children's understanding. As was expected, all were influenced strongly by the supervising teacher in the schools, many of whom had planned assessment in advance of the students' arrival. However, even when assessments had been planned in advance, all students used techniques that they were able to use without the contribution or agreement of the supervising teacher—such as checklists, anecdotal records and open interviews, which students could use individually with children as they circulated around classrooms. Five of the six students attributed their use of assessment techniques directly to the interactive multimedia program.

These very positive findings were qualified by two mitigating factors: the brevity of a two week professional practice and the substantial influence of the supervising teacher. In the words of one student: "I'm not the qualified teacher. I'm in their situation, in their room, conforming to their rules. So I can't just suddenly say 'Hey, let's do some oral assessment'".

Many of the students in the first study were inhibited in the choice of assessment strategies by the influence and authority of their supervising teachers, in a way which may not have been an issue if the students were practising teachers with their own classes. In order to investigate this issue further, a follow-up study of those students who had gained employment was carried out when the students were in the second year of teaching.

In the follow up study reported here, interviews were conducted with students from the first study who had gained employment as teachers. These four teachers were in their second year of teaching, three in secondary schools Evie, Rowan and Zoe (pseudonyms used) and one in a primary school, Debra. Three teachers (two female, one male) were teaching mathematics in private schools in the metropolitan area of Perth, and one female, Evie, was teaching in a remote outback Government school in the north of Western Australia.

Participants were interviewed at their schools, for approximately 90-120 minutes each using an interview protocol described by Denzin (1989) as *a non-scheduled standardized interview*. Questions and probes were determined in advance, but there was flexibility in adjusting the interview protocol. Students were questioned about: their beliefs about assessment; their knowledge and use of assessment strategies in mathematics; the influence of school policy, colleagues, and national guidelines on their assessment practices; and the influence of their teacher training on their current practices. The interviews were recorded and transcribed for analysis. Transcripts of interviews were analyzed using techniques of qualitative analysis recommended by Miles and Huberman (1994) and Eisner (1991). Transcripts were summarized for each student and placed in a matrix that allowed comparisons to be made for each student's response to each interview question. A further summary of these responses is given below.

Results

Beliefs About Assessment

The secondary teachers felt that generally assessment practices had changed since they were at school with more emphasis on problem solving, investigations, and project work. Evie believed that many students had difficulties in adapting to different types of assessment, particularly those students that lacked motivation and showed poor attendance. She noted that open book exams had become a common practice at her school.

Zoe felt that the use of graphing calculators resulted in more conceptually based questions and was concerned about the lack of reliability of tests and grading. Rowan commented that there was too much testing in his school and would like to see more of an emphasis on investigations and problem solving with younger students.

Debra, a year one primary school teacher, enjoyed the use of varied approaches to understanding students and felt that there was less use of worksheets than when she had gone to school. She believed her assessment to be diagnostic identifying areas where individual children needed assistance.

Self Reported Assessment Practices

Two secondary teachers, Rowan and Zoe, reported that testing was used as their main form of assessment, ranging from multiple choice tests to conceptual tests using graphics calculators. They also indicated that they used group projects, posters and oral reports. The two other teachers, Evie and Debra, reported on using tests, observations, checklists, portfolios, projects and individual interviews. Debra and Rowan had used self assessment techniques where children reflected on their learning by drawing faces to express their understanding and by making review sheets for exams. Zoe indicated that she had tried using journals but found them unsuccessful because students found it difficult to give thoughtful reflections.

Perceived Influences on Assessment Practices

Debra relied heavily on the advice of a more experienced teacher and support from the Principal. Evie also felt that advice from colleagues was more helpful than professional development organized by the district office.

Zoe attributed her assessment approaches to mathematics education courses completed at university. However, the school had a policy of encouraging similar assessment strategies across grade levels which she indicated limited her use of alternative approaches.

All the teachers recalled having used the interactive multimedia in their mathematics education course and felt that it was a helpful and important resource – one with which they were able to learn about different forms of assessment in a collaborative and enjoyable way. Debra, in particular, felt that she would be able to find good use for the program now that she is teaching.

Discussion

In both studies the teachers appeared to have a sound understanding of alternative assessment strategies and ways in which they could be implemented. While the first study presented constraints in employing these strategies on professional practice, (such as limited time and the influence of the supervising teacher), as teachers they were also faced with cultural and practical constraints. Constraints include the requirements of the mathematics department in the school, State Curriculum guidelines, issues of practical classroom management, personal issues such as the amount of time that can be devoted to planning, and convictions about the suitability of certain types of assessment for certain students. As such, the teachers' pedagogical beliefs do not always translate to classroom practice.

Analysis of data indicates that a complex mix of influences and environmental factors impact upon the teachers' use of assessment strategies in their beginning teaching. For example, Zoe commenting on her practicum supervising teacher's use of assessment strategies in 1996, appeared to have a range of alternative strategies in mind which she failed to use in her teaching practice in 2000:

> *Interview with Zoe (1996):* It's opened my eyes a lot more ... and also watching my [supervising] teacher and really disagreeing with a lot of the assessment strategies he'd use. He only used pencil and paper assessment strategies. Of course I didn't say anything, but I'd sit there thinking 'Oh remember what we learnt'.

Interview with Zoe (2000): Testing. That's the main approach. With years 8, 9 and 10 mainly topic testing, and then at the end of their year we do an exam to get them ready for Year 11 and 12 … If you put it on a piece of paper, you know if they can interpret it. I don't think that sort of testing is fantastic for every student … but I can't think of another really appropriate assessment task.

The enthusiasm Evie had in 1996 to try a range of assessment strategies in her own classes, by 2000 had been overwhelmed by significant social and cultural problems:

Interview with Evie (1996): There were only limited types of assessment that I could use [on teaching practicum], but hopefully in the future I'll be able to use a wider range of the ones that were on the multimedia. Hopefully I'll be able to ... start journals and things like that.

Interview with Evie (2000): I'd like to vary a lot of things, such as I'd like to do a lot more collaborative work with the kids, group work. And probably even presentation type stuff, where kids can actually demonstrate or explain their findings, whether it be an investigation or even project work … I just find that all these fantastic ideas that I come up with usually seem to backfire when I use them in the classroom. I don't see myself as being a really boring teacher … [and] I'm not blaming it on the kids—but they just lack any self-motivation. And it's not just me … all the learning areas are having the same problems with the same kids. I think the problem is they don't see the importance of education … there is a high rate of kids dropping out, and girls falling pregnant at a young age which is really sad.

While the teachers' pedagogical beliefs do not always translate to classroom practice, this is not to conclude that teachers' beliefs about assessment revert to traditional beliefs. All the teachers in the study, when prompted on various assessment strategies indicated they had tried alternative approaches and that they would be willing to try these techniques in the future. One teacher indicated that she would try journals as an assessment strategy when she could better explain to the students how to keep them. The use of a situated learning environment (such as that provided on the CD-ROM) during teacher training may provide contextualised knowledge about assessment strategies but this knowledge will not necessarily be applied in teaching practice either as preservice or novice teachers because of intervening constraints. The scenarios aimed at developing pedagogical knowledge but did not directly focus on potential constraints that needed to be taken into account. This could be overcome by including these aspects as part of the problem that needed consideration, and could equally well be evident as teacher guidance when the student teachers discussed their solutions to the scenarios.

The apparently powerful social, cultural and environmental pressures with which new teachers must deal—pressures which cause many to revert to more traditional modes of teaching to deal with them— could also arguably be ameliorated by providing ongoing professional development. The professional development of beginning teachers is recognized as a necessary process in the long-term development of 'full-fledged professionals' (Schoenfeld, 2002, p. 22). Without support, these teachers can find it difficult to adapt to change, and can readily abandon approaches to teaching and learning emphasized in their university courses and current curriculum initiatives (Schoenfeld, 2002).

However, such support needs to be focussed. It is apparent the teachers in this study retained their pedagogical beliefs and practices that they had learnt during their university mathematics education course using the multimedia but were often constrained in applying this knowledge in the initial teaching environment. Professional development that simply revisits approaches encountered at university may well be ineffective because it does not address the range of constraints that are faced by novice teachers. On the other hand, novice teachers may benefit from support in *constraint management* where the problems of novice teachers could be aired and discussed in a similar way that could happen with preservice teachers. Rather than face to face discussion of potential problems, the Internet has the potential for providing a forum where constraints could be openly discussed and resolutions reached in a collegial environment.

Conclusion

The studies described in this paper have revealed an ongoing discrepancy between the preparation of teachers and the reality and demands of classroom teaching. The use of situated learning environments has gone a long way towards ameliorating the cognitive gap between theory and practice, but it has not been entirely successful (on its own) in promoting sustained transfer to the real teaching situation. Practical constraints and real-life restrictions on teachers' practice need to be accounted for more substantially and incorporated more fully into the situated learning model. Further research on these factors, and inservice professional development as described here, may help to ensure that pedagogically sound practice is learned, applied and sustained throughout teachers' professional life.

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MATHEMATICIANS' PERSPECTIVES ON THE MATHEMATICAL KNOWLEDGE NEEDED BY ELEMENTARY PRESERVICE TEACHERS

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This study explored what two mathematicians believe elementary preservice teachers need to learn in mathematics courses to enable them to be effective teachers of mathematics. The participants used the word "knowledge" not to mean particular facts or identifiable pieces of information, but a more global sense of the subject and how one thinks about and learns mathematics. Three major themes emerged from the data. These mathematicians strived for their preservice teachers to be life-long learners of mathematics; to think about mathematics as a creative process; and to start with definitions and axioms to obtain the first two goals. Implications for practice and future research are discussed.

Purpose of Study

It sometimes seems that mathematicians and mathematics educators are at war about what preservice teachers need in terms of mathematical content. As Judith Sowder (2000) highlighted in her address at the NCTM research pre-session, to circumvent a war requires that both sides "come to understand and respect the value system of the 'other' side" (pp. 2-3). Such was the purpose of this exploratory study, to investigate what mathematicians see as crucial for preservice teachers to learn in content courses in an effort to understand what is valued by mathematicians. Given this understanding, mathematics educators have a better chance of seeing how our perspectives can fit with, and not fight against, the "other side."

Theoretical Perspective and Related Literature

Research suggests that a teacher's view of mathematics has a direct impact on how the subject is taught (Thompson, 1992). There have been numerous studies about the beliefs held by elementary, middle school, and secondary classroom and preservice teachers about mathematics and the teaching and learning of mathematics. This study investigated mathematicians' viewpoints about what preservice teachers need to learn in mathematics courses. Consequently, this paper focuses on a group of mathematics teachers that largely has been ignored in the past: college mathematics professors who contribute to preservice teachers' understanding of mathematics and indirectly to their understanding of mathematics teaching and learning.

The discussion regarding the mathematical knowledge and preparation of teachers has gone on for quite some time. Back in the 1960's, the movement was to enhance teachers' and students' knowledge of set theory. Deborah Ball (1990a; 1990b) identified shortcomings in preservice elementary and secondary teachers' mathematical knowledge. Borko, et al. (1993) found that an elementary preservice teacher who had a stronger mathematics background than most elementary preservice teachers was still lacking in the knowledge needed to effectively teach elementary school mathematics. Liping Ma's (1999) work has awakened many to the idea that the type of mathematical knowledge needed for K-12 teachers might be a blend of content knowledge and pedagogical knowledge. In fact, the American Mathematical Society [AMS] in cooperation with the Mathematical Association of America [MAA] has published a report that calls for a "rethinking of the mathematical education of prospective teachers within mathematical sciences departments at U.S. two- and four-year colleges and universities" (CBMS, 2001, p. 4). In particular, the report highlights that many mathematicians see aspects of mathematical knowledge for teaching as content for methods courses while education faculty see these aspects as fundamentally tied to mathematical content. As a result these issues are often overlooked. Consequently, it is argued in this document that it is "vital that faculty in the mathematics department and the school of education agree on substantial expectations for student learning and achievement in these mathematics courses" (p. 7).

The *Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics [NCTM], 1991) suggests preservice teacher education should reflect the standards recommended for K-12 students. The standards proposed for K-12 students (i.e., *Principles and Standards,* NCTM, 2000) emphasize a balance between learning procedural and conceptual knowledge so that students have a more relational understanding (Skemp, 1978) of the mathematics being studied. But what do mathematicians believe preservice teachers need to learn in mathematics courses to enable them to be effective teachers of mathematics? What do they see as the "big ideas" in content courses for preservice teachers?

Methods and Data Sources

The participants in the study were two mathematicians, M1 and M2, who teach content courses for elementary and middle school preservice teachers at a regional university in the south. The mathematics department at this university has a history of communicating and working with the mathematics educators in the education department. While we do not agree on every issue, there is the atmosphere of respect for other people's ideas. The program within which these mathematicians work is democratic in the sense that the people who will teach the same course decide as a group on the textbook that will be adopted for the course. They also decide on the specific topics that will be covered in the course. Nonetheless, in reality, the same course is usually very different depending on the professor and his/her emphases and teaching strategies.

The data were collected from two semi-structured interviews, numerous informal discussions, and classroom observations. One of the mathematicians, M2, also observed one of my methods courses during the same semester I observed his content course. M2 also shared with me a dissertation written by Eyles (1998) at the University of Texas about a teaching approach M2 uses in his mathematics courses. This dissertation included a transcript of an interview with M2 about this approach to teaching mathematics.

The study was designed to investigate people's perspectives and not to test specific hypotheses, thus, a qualitative design was appropriate. Data collection and analyses were guided by analytical induction techniques (LeCompte & Preissle, 1993; Patton, 1990). I began the study by observing the mathematicians' classrooms and engaging in discussions with them afterwards about various episodes. I created the first interview protocol based on these observations and discussions. The interviews were transcribed and those along with the field notes from classroom observations were coded. As I began to analyze the data, themes and relationships began to immerge. In informal discussions in person or via e-mail, I was able to ask questions to help modify and clarify the hypotheses that were developing. The second interview centered on portions of *The Mathematical Education of Teachers* (CBMS, 2001), in particular, the chapters regarding recommendations for content courses for elementary preservice teachers.

Results

The participants in this study were very genuine about their desire to improve mathematics education in schools by making an impact on their preservice teachers' knowledge of mathematics. They were convinced that what they were doing in the content courses would result in better teachers of mathematics at the K-8 levels. However, both were also quick to say that his way was not the *only* way to approach teaching mathematics to elementary preservice teachers. Both mathematicians gave overall favorable reviews of the CBMS document and were satisfied that what they were doing in these courses was in line with the document's recommendations. M2 explained,

> When I read [the CBMS document], and then thought about what it is I do, it looks like I read this *before* I made my courses, in a sense, that many of the points in here, I claim are purposely addressed without knowing that those were actual initiatives, because of what I want to do to or for perspective students. (M2, interview)

Data indicated that these mathematicians strived for their preservice teachers to be life-long learners of mathematics; to think about mathematics as a creative process; and to start with definitions and axioms to obtain the first two goals.

These mathematicians wanted their students to see mathematics as a creative process, not a body of procedures to memorize:

> I want them to see mathematics as a creative activity. I want them to see it as something that is different perhaps than what they may have come to believe it is. That it is not a collection of formulas and rules, but it is a way of reasoning and a way of thinking in attacking problems. (Interview, M1).

One day in class, as M1 reminded the students that he was expecting them to understand and explain why something was true, a student blurted out, "Are you an outcast among your colleagues?" (classroom observation). This student went on to explain that he could not fathom that this might be an acceptable thing to do in a mathematics class. The student was also frustrated that this was his last mathematics course and he was just now being asked to learn why.

> When discussing students' evaluation of the courses, M1 described that one of the complaints I get from the students is they'll say, this was more critical thinking than what I need to do in the classroom or what I am going to teach or...they will say, this was too much reasoning. Fine, I figure I have been successful... until you do that, it is going to be procedure. And that is the message that has to get through to people who are going to teach.

How the mathematicians attempted to encourage their preservice teachers to think about mathematics as a creative process was very different. M1 posed problems that students had never seen before in an effort to encourage his students' progress. If they were successful in taking what they knew and working on something novel, he was confident that they would begin to see that mathematics was about reasoning. M2's teaching wholeheartedly supported this idea. His approach to teaching was based on the Moore Method, a method of teaching named after R. L. Moore of The University of Texas that resembles the Socratic method. In this approach, the teacher begins with certain axioms and the students build the mathematics from those axioms. Problems are posed and students are randomly called on to respond, with the expectation that other students in the class will listen carefully to their classmates' responses and speak up if there are any mistakes made. In talking about his teaching approach, M2 maintained,

> The issue is we identify some things as being central. And what mathematics is about is to find out what the consequences of those ideas are. And then succession is the whole game, what else has to be true. And these things are not something that are imposed on you, these are the things that you discover. That's what the thinking is all about. (M2, interview)

The mathematicians believed that one of the most important aspects of their courses was to instill life-long learning of mathematics in their students. M2 emphasized that his students needed to develop "habits of mind" that would enable them to be life-long learners of mathematics. The premise behind this idea of life-long learning is that since preservice teachers cannot learn all the mathematics they will ever teach, they need to learn how to learn mathematics so that when they are confronted with content that they do not understand, they have the "mindset," "habits of mind," or strategies with which to figure it out. It is this very idea that these mathematicians argue is the creative process in mathematics – taking something that you do not know and figuring it out with ideas that you do know.

> I don't think the content is that important – if we presume that they know high school math.... I think there is a foundational level that you can get even through something like a [content course for preservice teachers] – by foundational, I mean this idea of definitions, axioms, applying definitions, developing things from *some* axiomatic process. The one I use is set theory. I take set theory as sort of the basis upon which we do everything in [the course]. And so, it doesn't matter what those bases are, as long as somehow there is a basis and from there you build. I think they need *that* experience. (Interview, M1)

This mathematician was not concerned about content knowledge, per se, but was more interested in the students becoming competent using definitions, axioms, and theorems to create something. In many of our discussions, he argued that if a student is capable, he/she should be taking calculus, not the mathematics courses for elementary preservice teachers. His perspective was on the *process* of doing mathematics.

In explaining why this emphasis would make these students good teachers of mathematics, he explained that:

> When they teach mathematics, they will understand maybe the significance of what the definition does for a mathematician as opposed to someone else. For someone else, the definition might be a way to use a word, whereas for us, it's a way to build theory. The elemental definitions provide the links. So if you make them go all the way back to definition, then when they're teaching their kids, they will see what those connections are, see what the links are. (M1, interview)

He was convinced that if someone came through their calculus course, that they would be able to figure out any elementary mathematics because of the creative experiences they would have had in the calculus course.

Both mathematicians were adamant that these courses for elementary preservice teachers must be taught at a college-level, which meant using an axiomatic approach. M1 explained his developmental approach in which he allowed his students to use only the definitions that had been discussed in class to tackle problems. In explaining how he introduced the meaning of fraction, he said

> The way we get to it is that it is a number that allows us to solve the equation Ax=1 where A is an integer. We know that we have problems that come up where we have to solve that so the question is what is necessary in order to do that. (interview)

M2 claimed that mathematics "begins with facts and definitions" (interview). His teaching centered around formal logic and language, the "tools of the mathematician" (course handout). While I sat in his class and listened to students hesitating to respond to his inquiry about "what is an existential quantification which is logically equivalent to: 'It is not the case that (if x is a number, then x*x > 1)'" (test item) and as they tried to use the formal language he had imposed on them, I wondered if the students really understood what it was that he was attempting to get them to do.

In explaining why he emphasizes formal logic in his courses, M2 argued that formal logic allows people to clearly communicate about the mathematics. He maintained that formal logic is of utmost important and thus must be placed at the forefront of the course so that he has ample opportunity to "teach it again and teach it again and teach it again" (interview).

Discussion and Implications

It seems that these mathematicians may not be on the same page as many mathematics educators, but they may at least be in the same book. Two of the big ideas that these mathematicians identified seem to be consistent with what the "reform" is calling for – life-long learners and a realization that mathematics is based on reasoning. If mathematicians and mathematics educators can agree about the "big ideas," at least that is a starting point.

It seems apparent that the mathematicians value reasoning and justification, but to focus so heavily on axioms seems to hinder many students' development. If the axioms and definitions are not meaningful to students, what kind of foundation are they starting from and where will that allow them to go? What is imperative for teachers to develop is this habit of mind that working mathematicians possess – a propensity to figure things out given what they already know. However, the starting points need to be more meaningful to students. I contend that the definitions and axioms used by these mathematicians were perceived by many of the preservice teachers as arbitrary, artificial, and abstract. A class using an axiomatic approach seems nothing less than the "typical" mathematics class in which they do not understand what the teacher wants them to do so they memorize the material with very little understanding.

If an axiomatic approach in content courses for preservice teachers is non-negotiable, it could lead to a positive aspect if this approach is explicitly used to help these students become aware of their assumptions when they make a decision or cling to a belief in any realm. If preservice teachers understand this subtle reality of axioms, it might help them to see that mathematics is not a set body of rules and procedures that never changes and must be memorized as absolute truths.

It is clear that maintaining the status quo by continuing to teach traditional mathematics content courses will only continue the vicious cycle in which students develop only a shallow understanding of mathematics because their teachers hold a shallow understanding of the content. A significant idea that I walk away from these mathematicians with is that students need to *learn* to learn mathematics. With so many topics crammed into a traditional content course for preservice teachers, there is little time for students to really *do* much mathematics. So instead of attempting to cover several topics in a quick and cursory fashion, why not design content courses around a few topics that can be and are explicitly related to school mathematics. In such an environment, there is time for students to engage in *doing* mathematics – something they need to do to understand that mathematics is something you do rather than something you memorize. The type of work that mathematicians do is foreign to most people because of most people's experiences in mathematics classes. Consequently, with fewer topics needing to be covered, it is more likely that students can develop an understanding of how to use definitions in mathematics and to begin to develop the habits of mind similar to working mathematicians.

These mathematicians argued that if elementary preservice teachers are capable of taking calculus or "higher level" mathematics courses, we should allow those courses to substitute for the content courses for elementary preservice teachers. If those courses developed the spirit of doing mathematics in these preservice teachers, I might agree with their contention. However, given the current environment in which teachers work, when will these teachers have the time to develop a deeper understanding of the content they are teaching? Why not address the issue of teaching preservice teachers to learn mathematics using content that is explicitly tied to the content they will be teaching?

As a result of the work between mathematics and mathematics education at this university, we have discussed a couple of options for these early mathematics courses. We have discussed the possibility of offering an educational seminar that runs concurrently with these earlier mathematics courses. These seminars would be designed to make explicit the connections between the mathematics content the preservice teachers are learning and the content they will be teaching. Because of the existing working relationship between mathematics and mathematics education, this could prove to be an effective compromise. The department has also planned to pilot textbook materials for the earlier mathematics courses for PreK-8 preservice teachers in which set theory is not used as the guiding framework and in which the course is a mixture of content knowledge and pedagogical content knowledge. Definitions are a mainstay in this curriculum, however, they are definitions that teachers use in teaching elementary topics and concepts. Thus, the definitions are likely to be meaningful as well as relevant to the preservice teachers.

The mathematicians in this study were pure mathematicians. A future study could explore the perspectives of applied mathematicians. Given that applied mathematicians begin with the physical experience and model physical phenomena, they may believe different starting points other than axioms are more appropriate for preservice teachers.

One thing that has become clear to me as a result of engaging in this exploratory study is that it is essential that mathematicians and mathematics educators become allies. We must learn to work together in order to break the vicious cycle that limits the mathematical knowledge of our teachers and students.

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ENHANCING TEACHER CANDIDATES' GEOMETRY UNDERSTANDINGS: COMBINING THE USE OF TECHNOLOGY, A CONCEPT DEVELOPMENT ACTIVITY, AND DISCOURSE

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Teacher candidates participated in a computer-based project designed to enhance their knowledge of geometrical terminology as they used a concept development activity. The aggregate findings as indicated in three case studies reveal that knowledge of the links between two-dimensional and three dimensional figures increased (a) when participants' recognized they were learners, (b) as participants actively engaged in a conceptual development activity, and (c) through peer discourse.

Background

Historically, many K-8 teacher candidates have experienced limited exposure to geometry in their own K-12 schooling experiences (Usiskin, 1987). Fueled by these weakened mathematical experiences, they often lack conceptual understanding of the link between two- and three-dimensional geometrical figures. Consequently, these teacher candidates rely mostly on the real-world use of terms rather than on more precise mathematical definitions. Several complementary activities were designed to help alleviate these misconceptions and build conceptual understandings to strengthen their geometrical knowledge. The presenter will initiate session discussion regarding (a) the need for teacher candidates to correct misconceptions and enrich their understandings of geometry, (b) the use of computer programs as a tool of learning, (c) the significance of using a concept development activity to promote learning, and (d) the benefit of peer discourse in learning.

Purpose of the Research

The purpose of the research conducted was to investigate the instructional strategies of using computer software combined with a conceptual development activity and discourse to help strengthen a teacher candidate's understanding of the links between two- and three-dimensional geometry and related geometrical terminology.

Theoretical Framework

This investigation was guided by several compatible theories: (a) the constructivist learning design, (b) an adaptation of the learning cycle model, (c) the van Hiele model of geometric thought (NCTM, 1987), (d) Vygotsky's research involving guidance of good analyzing questions from a teacher (1994), and (e) the appropriate use of the computer as a tool of learning (Baker, 1997). The instructional strategies were influenced by the constructivist learning design (Knuth & Peressini, 2001). Within this learning theory, learners make personal meanings as they create shared interpretation with others. The teacher-facilitator orchestrates classroom events that promoted exploration and examination of mathematical topics. The researcher who is a teacher educator modeled interactive teaching tactics that involved the teacher candidates actively constructing their knowledge in multiple ways (Cagnon & Collay, 2001). Since, the researcher was aware that much learning or construction of knowledge takes place through social interactions, the teacher candidates problem solved in pairs, verbalized their thinking, and capitalized on these social interactions.

Two headphones were connected to each of the five classroom computers so the teacher candidates could use the software, interacting through normal dialogue, without the audio portion of the CD-ROM being heard in the classroom setting. The other students were involved with other autonomous geometry activities. This format allowed the teacher educator/research to interact throughout the activity time frame with the pairs of teacher candidates at the computers as well as other groups doing compatible geometrical activities. She employed a variety of analyzing questions to prompt the teacher candidates to dialogue about their understandings. The questions and the learners' comments as well as paired dialogues were captured on audio tape recorders located at each computer.

The learning cycle model (Atkin & Karplus, 1962) was originally comprised of a three-step cycle (a) exploration, (b) conceptual introduction, and (c) concept application. The adapted model, shown in figure 1, was designed to enable the teachers candidates to formulate and test new geometrical knowledge by incorporating computer software (Maxwell, 1999). By implementing the four components of *explore*, *specu-*

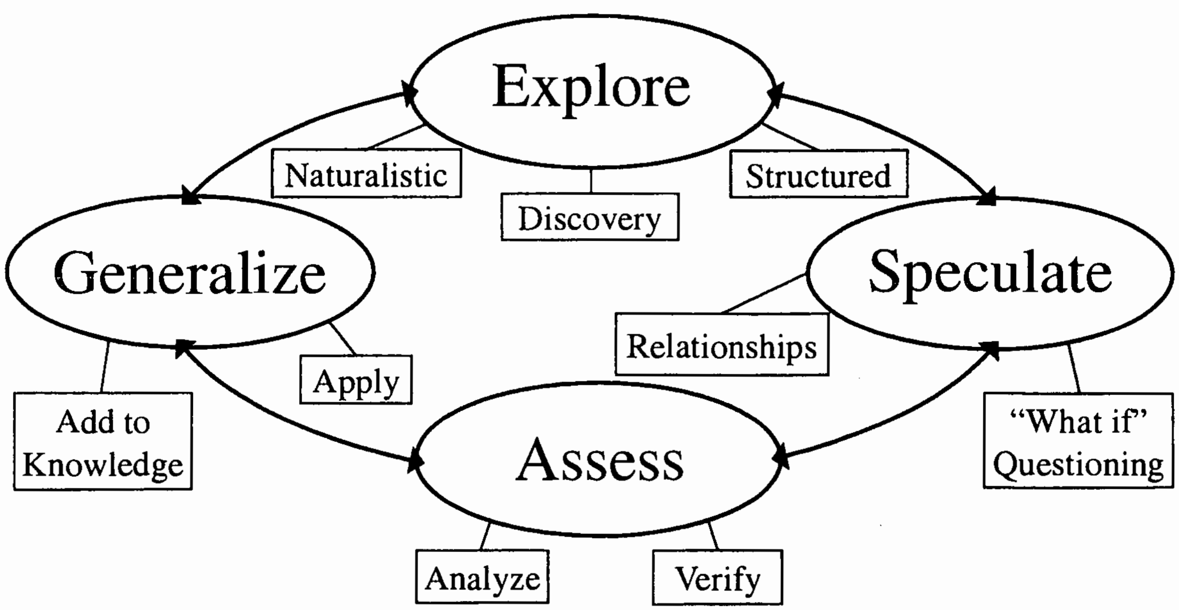

Figure 1. Conceptual Learning Cycle (Maxwell, 1999).

late, *assess*, and *generalize*, the math educator hoped that the teacher candidates could increase their conceptual understandings about geometry.

Elementary school teacher candidates should involve their future students in informal geometry during the elementary and middle school years. However, since teacher candidates have been instructed through rote memorization, they need to improve their spatial understandings of geometry prior to teaching concepts. According to Clements and Battista, "Spatial reasoning . . . consists of the set of cognitive processes by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated" (Clements & Battista, 1992, p. 420). The geometry component was guided by the van Hiele model of geometry thought research that confirms that distinct levels describe students' geometrical conceptual development from elementary school to college (NCTM, 1988). The van Hieles' believed that progressions through the five levels could be achieved through carefully guided experiences (NCTM, 1987). Elementary /middle school students were to progress through the first three levels under the guidance of their knowledgeable teachers.

Table 1. The Van Hiele Model of Geometric Thought (Hatfield, 2000).

LEVELS	CHARACTERISTIC	3-DIMENSIONAL EXAMPLES
Level 0 Visualization	Student reasons about visual whole, recognizing the figure.	Student can identify a *cube, prism, pyramid,* and other three-dimensional figures.
Level 1 Analysis	Student reasons about an informal analysis of the parts or attributes.	Student identifies *face* and *non-face,* the shape of the *face,* can count the *edges* of a three-dimensional figure.
Level 2 Abstraction	Student can form abstract definitions. Shapes can be related to one another.	Student can tell the differences between a *pyramid* and a *prism* through analysis of parts. Student can create a definition of each.
Level 3 Deduction	Student reasons formally within the context of a mathematical system.	Student proves theorems about two- and three-dimension figures
Level 4 Rigor	Student compares systems based on different axioms.	Student studies in absence of concrete models, sees abstractness

In this experience teacher candidates viewed two-dimensional *nets* and manipulated three-dimensional figures to help them learn about the links between two- and three-dimensional models. Later they define a *prism* through accomplishing a concept development activity. By using the computer as a tool of learning, the teacher candidates could develop their own cognitive understandings (Baker, 1997).

Research Design

This study was conducted using qualitative methods. The researcher determined that the naturalistic classroom setting of teacher candidates experiencing cognitive dissonance and beneficial discourse with completing a concept development activity might help to increase geometrical knowledge. The following research questions were addressed:

1. What happens to teacher candidates' understandings of the relationship between two-dimensional and three-dimensional geometric figures as they interact with peers, use the computer as a tool of learning, and complete a concept development activity?
2. By using a computer employing an experiential approach, can teacher candidates strengthen their conceptual understandings of two- and three-dimensional geometry?
3. As teacher candidates interact with peers and note misconceptions, what part does attitude play toward learning more about geometry?
4. Frequently, teacher candidates are confused by multiple meaning words as they grapple with understandings of concepts previously misunderstood. How do multiple meaning words influence their geometrical understanding of figures?

The study site was a purposefully selected mathematics methods class with five computers located in a classroom of a university in the Mid-Southern United States. In this study, forty teacher candidates from two K-8 mathematics methods classrooms participated in a computer-base project.

During two consecutive, three-hour, mathematics methods classes, pairs of teacher candidates were challenged to complete a self-sustaining activity guide containing prompting questions with an imbedded concept development activity. This concept development activity lasting one-hour was developed by a mathematics educator/researcher to increase the students' comprehension of the link between two- and three-dimensional figures. This activity incorporates a progression of a four cyclic strategy: exploration, speculation, assessment, and generalization through an interactive approach. Ultimately, the pairs of teacher candidates were to define a prism using the attributes discovered. The teacher candidates were briefly introduced to three-dimensional items in order to become aware of the various multiple-meaning words such as *face* and *net*. The teacher candidate pairs rotated to the computers as vacancies occurred, while other groups of teacher candidates participated in compatible geometry activities (Hatfield, Edwards, Bitter & Morrow, 2000). Audio recorders were placed at each computer; two earphone headsets were connected to each computer. Teacher candidates used the CD-ROM software product entitled Might Math™, by EdMark© (1995), as a tool of learning. The superhero theme in the Mighty Math™ software designed for elementary and middle school children teaches concepts connecting two- and three-dimensional geometry. Learners can learn terminology by experiencing what that terminology means. By manipulating the geometric figures through the use of the program, the learner can see geometry from a holistic point of view, as opposed to the frequently memorized definitions. The concept development activity Prism Priming guides learners of any age to recognize patterns that can lead to understandings.

Data Sources

Data were collected from four sources: teacher candidates' written answers to various prompts as recorded on the activity sheets and quizzes, audio tapes of their active computer experiences and discussions, teacher candidates' extemporaneous writings located on journals/papers, and the teacher educator's field notes and documented responses to questions. Data were analyzed through constant comparative methods using grounded theory. The researcher used pseudonyms to provide anonymity to the teacher candidates as the case studies were developed. The teacher candidates' reactions to learning experiences conformed to one of three categories: floundering, gaining competence, and refining concepts. Although the three cases represent a slight amount of aggregate responses, the primary basis of each case was one teacher candidate of each category.

Three Case Studies

The elementary/middle grade teacher candidates were apprehensive as we talked about their understandings of elementary geometry. Their prior experiences had been filled with anxiety, particularly as they tried to make sense out of the plethora of definitions, postulates, and theorems. Several learners admitted that geometry seemed like a blur, while others indicated it was a nightmare experience that they really wanted to dismiss. Timidly, a small minority acknowledged they had never had a formal course in geometry. Only a meager number of teacher candidates recognized geometry's importance to understanding mathematics more completely. However, the teacher candidates wanted to avoid this important mathematics subject. As they sat at the tables with their peers, they chatted nervously, dreading the impending activity.

Adventure Preparation

Θ_1: Diedre Nowell cocked her head in a distinctive manner, chatting to her friend and confidant, Sherry Mix. This course was just one more in her quest to become a teacher. She assumed that if she exercised her normal charming mannerism, she would succeed. After all, this was just one course in a series that marked her progression in the Master of Arts of Teaching (MAT) degree designed for second career individuals. She believed she would accomplish some amount of understanding and be closer to her goal. Although her lack of knowledge of mathematics content troubled her somewhat, she internally hoped she had an acceptable knowledge of geometry, especially to teach elementary students. She would grin and bear this experience, hopefully getting help from her best friend, Sherry.

Θ_2: Bobbie Canard glanced at the other five women at her table. She believed that they all knew so much more than she did. The others laughed and commented to one another while she sat quietly, contemplating what little she knew. GEOMETRY—the word frightened her! In fact, math in general frightened her! Bobbie had contemplated that taking this math methods class with her friends would be helpful. Now, she was having doubts. She judged that they might find out how inept she was at math. She hadn't told anyone that she still had to complete a math deficiency in her undergraduate program—It was her secret! If she could just *hang on* to complete this class.

Θ_3: Vicki Stevens quietly listened to the discussion. She reflected about her understanding of geometry—she couldn't even recall any positive aspects about the subject, just lots of memory work. Learning had always come fairly easily to Vicki. However, now she questioned if she had really learned much of anything except various procedures. Nevertheless, she was willing to try various aspects of this course, just hoping that she could deepen her understanding of the math topics. The use of the computer as a tool of learning sounded fascinating to her. She admitted to herself that within most classrooms she had only seen the use of the computer as a reward or to do mindless practice.

Computer Capers—Learning the Program

Θ_1: Diedre giggled excitedly as she and Sherry started the program. To her, this was like playing at a video arcade. She had only seen children use skill/drill math computer activities that were only a step above the worksheets she had done in grade school. Dr. Gee, the activity caricature, was explaining the connections between two-dimensional and three-dimensional figures. She was talking so fast that it was hard to follow. Diedre triggered the appropriate yes or no button, rather flippantly answering the questions that Dr. Gee asked. The two women huddled together trying to make sense of the program as well as trying to answer the questions on the activity sheet. It concerned Diedre that Sherry didn't seem any more competent than she was in geometry. She realized that she might have to shoulder more of her load on this assignment. The tasks required the two of them to concentrate, read, comprehend, and do what was asked. The two women capriciously explored the software, which later caused them to misunderstand aspects. For example, the women skipped over exploring some indicated items, as they did not want to risk answering an aspect incorrectly. The professor approached on occasion, using prompting questions to help them focus on the task at hand. Clearly, this activity was challenging Diedre's ability to follow directions and to learn at the same time. She wished the professor would stand nearby and individually guide them on this journey.

Particularly puzzling was the new vocabulary needed for understanding. Dr. Gee had given the instructions to create the designated components of the task so fast. The women learned the meaning of the word net by listening to the conversations of others at nearby computers. Each woman questioned the meaning of the word face, and sought help from the facilitator. By the time they manipulated the cube's net to place the various required items on the faces, Diedre had learned to rotate the accompanying three-dimensional figure to see the other faces. Yet, Task 2--Prism Priming, seemed more puzzling. The two women had to synthesize their learning differently. As indicated in several different sources, Diedre was troubled by this new learning method. She was good at memorizing components of written sections of books and to dutifully regurgitate these ideas on a future test. Yet, Diedre suspected that her understanding of the connection among aspects of geometry was weak.

Θ_2: Since Bobbie was uncertain as to how to use the CD-ROM, she allowed Melinda Moore, her selected partner, to take the lead. With Melinda as the principle typist, she believed that her contribution would be minimal. It was not that she didn't want to learn, but instead she felt more comfortable taking a secondary role, especially when she felt less than confident in both geometry and in using the computer. "That's cute," Bobbie commented as the Dr. Gee caricature appeared on the screen querying the two women about the links between the net and the three-dimensional solid. Initially, neither woman knew some of the figures, particularly the distinction between the pyramid and the prism. They simply guessed, chuckled nervously when they were wrong, and tried once more. After learning about the rotation buttons from the facilitator, they began investigating in more depth: "A pyramid has a point, but this one has two long lines. . . . a cube is equal on all sides. . . . Is a triangular prism the same as a pyramid? . . . I don't think there is something called a square pyramid. . . . But, look at its bottom (Bobbie repeated three times). . . . Could we have a round pyramid?"

Bobbie and her partner Melanie felt secure in their understandings of the software as they started the concept development activity called Prism Priming. The two women chatted excitedly as they noted that the definition of the prism consisted of "rectangles that are connected by a similar shape on the top and bottom." Interestingly, she now identifies a face as "the whole area of one side of a shape." To her, a shape is a three-dimensional figure. This indicates that a generic meaning still supersedes a technical meaning. Bobbie is just beginning to notice the differing attributes of two-dimensional and three-dimensional figures.

Θ_3: Vicki Stevens and her partner, Betty Paluchi, excitedly began the Mighty Math™ software by clicking on the caricature, Dr. Gee. They quickly became engrossed in the examination of folding the net into a solid. Dr. Gee asked, "Does this net match this solid?" By clicking either the yes or no button, they could respond appropriately and learn immediately if they were correct. Look, it's a "three-dimensional figure, all laid out . . . if you fold it up, it makes a solid," exclaimed Vicki. Some figures were easy to determine—just count the parts or faces. Other three-dimensional figures were more difficult. They had learned to "turn it over to find out about this one," or click on the hint button. Vicki blurted out, "See, we were right! Oh, I get it now!" The women explored the *level* button and acknowledged that they were indeed at the beginner's level. Before they moved to the Exploring Lab where they designed their own decorated cube, they paused to help others around them. Clearly, Vicki and Betty felt confident about this innovative computer program. Additionally, Vicki recognized the power of learning in a different style. "We discovered things by ourselves instead of being told everything. And, as a result, more learning took place," she speculated. This concept development activity gave her an understanding of the meaning of constructivism since she had experienced the power of this type of learning.

Understandings of Geometry

Θ_1: Diedre admitted that she "had a negative experience with geometry in her school days, because it was always *so boring*." She indicated that "I have never been spatially adept. I get turned-around easily in new cities and am not able to read maps very well." Furthermore, she had difficulty identifying the various components of the three-dimensional figure, such as a face or edge, except when prompted to do so. Although she decided that she "probably would have liked it more if my teacher had made it more interesting," she appreciated "the way this program gave us a three-dimensional visual of our geometric shapes." One of the Tasks—Prism Priming—was above her level of understanding, as it required her to recognize the shapes of the various faces, draw these and record their number on a chart. Sherry was helpful at this point so the task could be completed. "I really liked the concept of the program . . . but I would imagine that young children would become very confused the first time they used the program." Was she really talking about herself?

Θ_2: Bobbie and Melanie worked together well, complementing each other's strengths. Both solicited the other's opinion and choices of answers throughout the adventure. Neither pointed fingers at partner's errors but explored the software program learning from each other and Dr. Gee. They were evenly matched partners in both temperament and math knowledge sharing useful dialogue that assisted each with geometric development. Bobbie recognized that a pyramid has a *point* and could have different bases. She kept insisting that her partner "look at its bottom" when naming the type of pyramid. She used an analogy to characterize her understanding of a net after looking at several examples. "A net is like a [sewing] pattern." However, she used technical language to describe a net on the activity sheet. Initially, they had difficulty creating the decorated cube. But when they accidentally erased their creation, they started over rapidly completing the project. Bobbie carefully drew the triangular pyramid required in "Creative Ideas" activity. She helped her partner, Melanie, revise her misconception that all pyramids had to have a triangular base. Now, Melanie was aware of square and hexagonal pyramids.

Θ_3: Vicki and Betty enjoyed their geometry adventure. Vicki admitted she did not agonize at all with this activity "because I was enjoying the new adventure rather than dreading geometry. This computer program disguised the very fact that it was all about geometry. I probably learned more about shapes in this allotted time than I ever did in my school years." Later, Vicki wrote in her journal, "When we worked on the computer assignment, I noticed . . . we were actively involved, we engaged in problem solving, cooperative learning took place, and we learned to figure it out for ourselves. I was familiar with some of the terminology used—such as net, and two-dimensional and three-dimensional figures—because I had read about these geometric concepts in the text. The computer, however, gave me a much better understanding of what I had read. For example, after seeing a net on the computer and manipulating various three-dimensional objects, I had a much clearer image in my mind of what the text[book] was conveying." When asked to make her own definition of a prism on the activity sheet, "I realized what constructivism is about. It helped the light bulb go off in my head. It was such a fantastic way for me to learn a definition. I designed it myself. . . . By constructing the definition myself, I am not likely to forget it. Besides it was much more fun than looking up a definition in a textbook and then trying to memorize it for a test."

Linking Language to Misconceptions

Θ_1: The language Diedre used when discussing three-dimensional geometry points to her van Hiele low level of understanding just as the van-Hieles predicted. She referred to a sphere as a three-dimensional circular object, like a ball. She continued to improperly use the terms net and face. At times she was unable to distinguished between two-dimensional and three-dimensional, using these terms interchangeably. Several weeks after the activity was completed, she was unable to differentiate between a prism and a pyramid, stating that a prism was a pyramid as well as a three-dimensional triangle. Diedre was unaware that words could have more than one meaning.

Θ_2: Initially, Bobbie lacked the sophistication of language to be considered higher than van Hiele Level 0 with regards to three-dimensional figures. However, she was beginning to look at the various attributes of the solids as indicated by her awareness that a pyramid could have different shaped bases. The Multiple Meaning Words document shows that Bobbie had a clear understanding that words could have multiple meanings. However, she still possesses some misconceptions. Although Dr. Gee used the word solid throughout the program, with many models that were transparent, Bobbie still believes that "a solid is a three-dimensional shape that is filled."

Θ_3: As shown by the Multiple Meaning Words document, Vicki recognizes that words can be generic or technically connected to mathematics. Several weeks after the she explored the Dr. Gee program, Vicki defined a net as "a pattern that can be made by covering a three-dimensional object with paper and having no overlapping pieces." Her two definitions—a solid as "a three-dimensional figure having length, width, and height" and articulating the same definition of a prism as stated on her activity sheet—provide the indication that she has stabilized her van Hiele level as level 1—analysis. She could be considered at the beginning stages of van Hiele, level 2, since she is beginning to develop her own definitions of the various three-dimensional figures with ease. She is starting to visualize the attributes of the solids, recognizes faces and the positioning of parallel planes, but needs to concentrate on understanding the significance of edges.

Findings

The data collected reveal some changes in each of the teacher candidate's conceptual understandings of geometrical terms with increased recognition of two- and three-dimensional forms in the real-world as well as acknowledgment of differing definitions in the mathematical and real-world contexts.

Θ_1: Diedre's geometrical development hinged on her willingness to explore and learn in depth, more than just acknowledging her weaknesses. However, Diedre's geometric development was hampered because she simply went through the motions of answering the activity sheet questions, rather than enthusiastically choosing to learn more geometry. Although the two partners dutifully answered many of the questions, they explored a minimal amount of geometry with the software program. Diedre failed to recognize that through her own exploration she could increase her understanding of geometry. She operated in a teacher-mode" only rather than being a learner herself. Her attitude and view that the mathematics methods course is just one more hurdle to overcome on her way to becoming a teacher prevented her from maximizing her learning. Her non-awareness of multiple meaning words indicates that she compartmentalizes content subjects.

Θ_2: Initially, Bobbie was anxious about her understanding of geometry, but attained knowledge during the experience with the combination of perseverance and shared communication about understandings with Melanie. Additionally, she was a willing participant in seeking knowledge from this program. As the activity continued, she relaxed, enjoyed the adventure, and had no pretense of excellence. She was interested in learning more herself. After she felt comfortable with the program itself and turned the solid figures to inspect all faces, she participated in the concept development activity. Bobbie experienced some progress in understanding the differences between two- and three-dimensional figures. Now, she recognizes that nets are like patterns that create the three-dimensional solids. Additionally, she is distinguishing the shapes of the faces and how they make the sides of the figures. Although she possesses some misconceptions herself, she noticed other misconceptions. She gently guided her friend to look at the bottom of a pyramid to distinguish its type. Evidence from the Multiple Meaning Words document shows that she was not hampered by the generic meaning of words used, but transcended this problem to learn more about geometrical figures. Still, she needs more experience in visually recognizing all three-dimensional shapes before she can be classified at another van Hiele level.

Θ_3: Although Vicki's understanding of geometry was initially more advanced than some of her peers, she faced this geometry session with apprehension. However, Betty's cheerful attitude helped as they began the program. Vicki's geometrical development advanced as she recognized that she didn't have to memorize the various definitions, but simply answer the questions asked, fill out the table, and connect the ideas together. She seemed to absorb the attitude of her peer, relaxed, and began to learn differently. She strengthened her foundation of geometry by recognizing the various attributes of the solids and how these are connected to the two-dimensional nets. This experience could have resulted in restricted improvement. Three situations prevented this: Vicki's positive attitude and interest toward wanting to learn more geometry, her willingness to tackle the previous procedural understandings by using the constructivist paradigm, and her absorption of the bubbly nature of her partner. She stated "I have learned that, when teaching students, technology can be used for much more than just word-processing and games."

Conclusions

The three individuals had varying degrees of success. More importantly, this success seemed to be contingent on their attitudes toward learning more than their acknowledgment of their limited understandings. The individual who did not view herself as a learner was hampered throughout the experience. The two other participants, who were committed to learn in this constructivist manner, explored, conjectured—sometimes inaccurately, tried varying ideas, assessed their understanding by questioning and seeking peer advice, and generalized their suppositions. From these interactions these women increased their geometrical understandings of the links between two- and three-dimensional figures.

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GROWTH AND DEVELOPMENT OF PRE-SERVICE ELEMENTARY TEACHERS' MATHEMATICAL KNOWLEDGE

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We detail the mathematical growth of a pre-service elementary teacher of the period of a semester. This student had an unusual combination of very poor basic computational skills and a capacity for higher level problem solving involving algebraic notation. We discuss the extent to which one or two semesters of mathematics content, however well-structured, is sufficient to develop deep understanding, much less pedagogical content knowledge, particularly when growth of mathematical reasoning and understanding are coupled with incorrect or inappropriate algorithms which have become automated over several years of use. We provide evidence that this student acquired new insights and understandings about mathematics and experienced changes in her attitudes and beliefs. However, we do not know whether the changes in attitudes and beliefs are short- or long-term nor whether what she learned is truly "owned" by her.

Introduction

We are concerned, in this study, with the development of deep understanding of mathematics by pre-service elementary teachers. Those who teach mathematics content courses for pre-service elementary teachers are aware that, as they begin the mathematics content course, pre-service teachers' attitudes to mathematics are generally instrumental, focused on formulas and correct answers. We discuss and analyze what constitutes evidence of the development of deep understanding of mathematics by focusing on a young woman enrolled in a mathematics content course for pre-service teachers at a community college. This case study was carried out in a mathematics classroom and has important implications for linking research and practice. The issues raised in this study focus on what is theoretically desirable in a content course for pre-service teachers versus what might be practically attainable in one or two semesters. Effecting change in classroom practice based on research is not always straight-forward (McGowen & Vrooman, 1998). Yet research that provides evidence for what student growth in understanding and competence is attainable, and what the limits and applicability of those attainments are, is vital if mathematics content courses are to impact pre-service teachers deeply.

Many authors have addressed the question of growth of knowledge and growth of understanding in mathematics (for example, Ball & Bass, 2000; Clarke, Helme & Kessel, 1996; Davis & McGowen, 2001; Hiebert & Carpenter, 1992; Ma, 1999; McGowen, & Davis, 2001; Mewborn, 1999; Pirie & Kieren, 1994; Simon & Tzur, 1999; and Tall, 1995). The issues discussed in the literature acknowledge the importance and desirability of the growth of conceptual understanding and technical competence. What is largely not addressed, however, is the extent and degree to which desired growth can be supported and encouraged in one or two semesters, and to what extent such a limited exposure to mathematics content for pre-service elementary teachers can impinge on decades of relatively negative experiences.

From a more quantitative angle, Hake (1998) and McGowen & Davis (2001) have utilized a measure of relative increase, pre-test to post-test, for whole classes and for individual students. This statistic—*gain factor*—is calculated as the ratio *gain = (post-test% – pre-test%)/(100% – pre-test%).* Indications are that this statistic is related to student growth in a more cognitive sense (Hake, 1998; McGowen & Davis, 2001).

The main purpose of this study is not simply to re-iterate that pre-service teachers generally consider mathematics to consist of formulas and correct answers. It is also about how a particular person—Holly—develops considerably deeper understanding of mathematics over a semester, and how Holly's knowledge of mathematics may yet be insufficient for her to teach elementary mathematics effectively.

Methods

The principal question of this study—"how does a pre-service teacher develop a deeper understanding of mathematics over a semester?"—is not easy to answer, and we need to indicate how our modes of inquiry will do that, and how our theoretical framework supports our question. At Harper College, prospective elementary teachers are required to pass a pencil-and-paper competency exam of basic arithmetic computations with a grade of 80% as a condition of successful completion of the first mathematics content course of a two course sequence. Students take this exam during the first week of the semester. They have two additional opportunities to take and pass the exam during the semester. Written work in the form of in-class tests, out of class assignments, and student reflections was also collected from students. Holly's written work was examined and analyzed in detail. Holly's work was analyzed in detail because of extraordinary incongruities in poor basic computational skills and outstanding higher order thinking skills. The individual gain factors were calculated for all students and Holly's gain factor located in the distribution for the class as a whole. Holly was also interviewed in relation to both her in-class and out of class written work.

Results

On her initial attempt the first week of class, Holly correctly answered six of thirty questions (20%). Her answer to the first of the following two problems is shown in Figure 1. She did not attempt the second.

Only nine of the forty-six students who took the competency exam the first week of the Fall 2001 semester gave a correct response to problem (a). Thirty-two students—including Holly—gave incorrect responses to this question. Five students made no attempt to answer the question. Fourteen different incorrect responses were given for this problem, including Holly's answer of 266. Fifteen different incorrect responses were given for the second problem.

(a) 31/2 x 2 8/15 (b) 1 3/4 ÷ 1/2

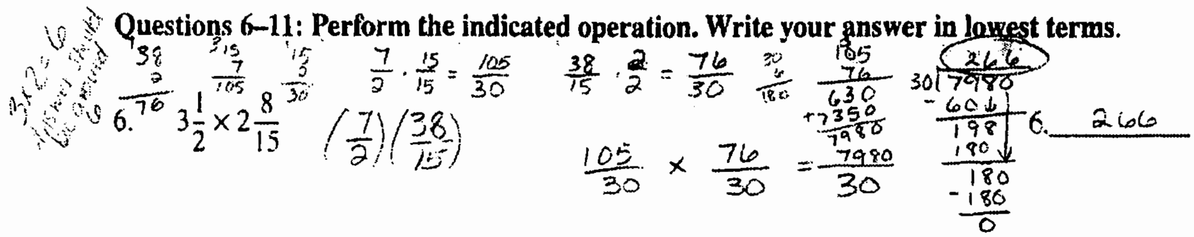

Figure 1. Holly's response to question on initial competency exam (Week 1).

On her second attempt eight weeks later Holly, like many others in the course, more than doubled her initial score to 50%. For Holly and the other 28 of 46 students who scored less than 30% initially, doubling the initial score on the second attempt midway through the semester indicates significant improvement but does not come close to satisfying the department's required passing score of 80%. The work on her 2nd attempt of the competency exam Question 6 (multiplication of two mixed numbers) is difficult to explain (Figure 2). Her work for Question 17 does not justify the answer and we are left wondering how it was actually determined. Though her work on Questions 27 and 28 is procedural and not as efficient as one might hope, she has

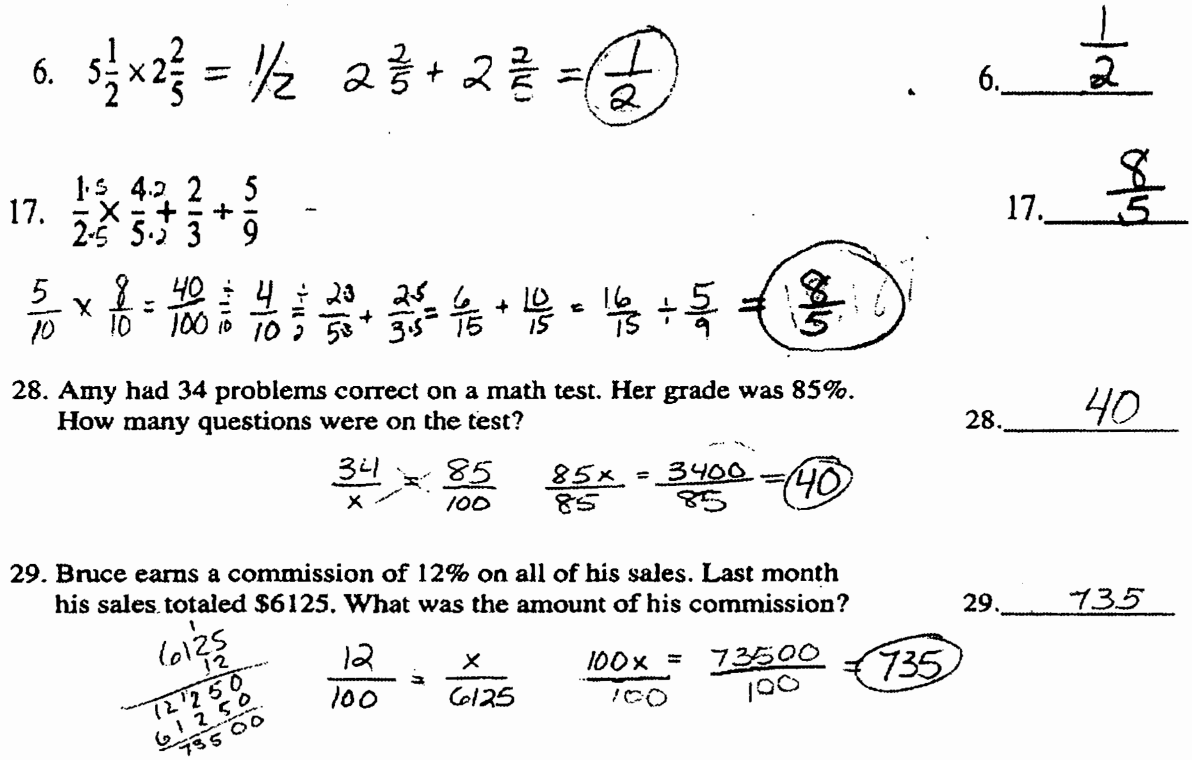

Figure 2. Responses from Holly's second attempt on the departmental competency exam (Week 9).

represented these two problems appropriately and has written correct answers to these questions which she had not even attempted eight weeks earlier.

In an interview after taking the competency exam for the second time, Holly mentioned that she had studied the textbook's explanation of the theorem of addition of fractions with unlike denominators and the definition of multiplication of fractions to figure out how to do problems like Question 17, above. Holly had been unable to attend any of several review sessions due to scheduling conflicts so she attempted to learn a procedure she knew she did not know on her own using her textbook as reference.

In her mathematical autobiography, written at the beginning of the 16 week course, Holly describes her previous experiences with mathematics as follows:

> Out of all the subjects that I have had to endure over the course of my education, Math has always been the most difficult for me to understand....The basis of all my math problems stems from the fact that I never learned the basics in mathematics....If I am asked a simple question such as: What does six times seven equal? I literally have to stop and think about it for a minute before I can come up with an answer.

Holly's responses to questions on an exam taken during the 12th week of the semester provide a startling contrast to her work on the initial basic skills exam and her midterm second attempt on this exam.

Question 5:

Consider the set of numbers A = {5, 6, 7, 8} with a made-up operation called *. The rules for * are shown in the table below.

a) Is * commutative for set A? Justify your response.

Student response:

Yes. * is commutative.

5 * 5 = 8 6 * 5 = 7 7 * 5 = 6 8 * 5 = 5
5 * 6 = 7 6 * 6 = 6 7 * 6 = 5 8 * 6 = 8
5 * 7 = 6 6 * 7 = 5 7 * 7 = 8 8 * 7 = 7
5 * 8 = 5 6 * 8 = 8 7 * 8 = 7 8 * 8 = 6
5 * 6 and 6 * 5 both have the same result of 7.
7 * 5 and 5 * 7 both have the same result of 6.
5 * 8 and 8 * 5 both have the same result of 5.
6 * 7 and 7 * 6 both have the same result of 5.
6 * 8 and 8 * 6 both have the same result of 8.
8 * 7 and 7 * 8 both have the same result of 7.

Therefore, the set A is commutative since this is true for all.

*	5	6	7	8
5	8	7	6	5
6	7	6	5	8
7	6	5	8	7
8	5	8	7	6

Figure 3. Operation table for question 5.

Recall that the gain factor for an individual student is the ratio (post-test% – pre-test%)/(100% – pre-test%). Classes with average high gain factor (great than 0.5) correspond generally to reform type classes rather than more traditional chalk and talk type or drill type classes (Hake, 1998). Further, individual student growth in understanding of mathematics, as well as basic skills improvement, sees to be connected with higher individual gain factors (McGowen & Davis, 2001). Holly's individual gain factor was the 5th highest in a class of 33 students, and was a striking 0.85 compared with a class average of 0.58. From this perspective alone we should be alerted to possible significant cognitive growth in Holly's understanding of mathematics.

Holly's final grade (87%) was the sixth highest of the thirty-three students who took the final exam. Her final self-evaluation submitted with her portfolio in the sixteenth week provides confirmation of the growth of her mathematical understanding, and documents her changed attitudes and beliefs about mathematics and what it means to learn mathematics:

Reflecting on ... mathematics in August versus today now almost sixteen weeks later there is a significant difference on the way that I personally approach a math problem. In August I would: do the computation, focus of my attention being on the numbers within the problem, and usually not considering the content of the problem at hand. Once I received what I thought was the correct answer I would move on to the next problem and start the whole process over again. Now in December I focus my attention on the entire problem not just the numbers within it, I now look for alternative strategies and select the most appropriate and consider the context of the problem. Once I get a result I go through a series of reality checks: is this answer reasonable? Does this answer make sense in the context of the problem? Then I interpret my result by reflecting upon it and consider are there any other answers, strategies, possible explanations for the problem? When this process is complete only then move and I then on to the next problem to repeat the steps I just mentioned.

Holly shows evidence of higher order thinking and algebraic skills, which is particularly puzzling in light of her inability to carry out basic arithmetic computations. Evidence of her ability to think a problem through and reflect on her answer conceptually comes from her work on prime number values of a polynomial. Holly was interviewed after the Second Unit Exam had been submitted and asked to explain how the answer to the question below was determined. She said that both she and her partner had experienced an "AHHHH" moment. When asked to explain exactly what they had noticed that produced that AHHHH moment, Holly pulled out a copy of the spreadsheet she had created for testing various values of n in the equation and provided the following explanation:

- Proof and Justification was another method that was learned and relearned over the semester. In a mathematical problem there lays a claim setting restrictions on the problem's possible solutions and which must abide by these restrictions to be labeled as a correct solution to the problem. Example six on the group exam gives such an example of claims and their relation to the process of Proof and Justification: *Does the expression n ^ 2 – n + 41 give only prime numbers? Explain your reasoning and justify your answer.*

- No, this formula does not always generate a prime number. If n = 41, the square root of the sum is 41. 41 ^ 2 – 41 + 41 = 1681 and 1681 = 41^2.
- Our first approach to this problem was actually computing the expression and plugging in numbers up to 108 for n to determine if the result was a prime number. Next we looked at the actual number 41 more closely and tried other prime numbers in its place in the expression above to see if it would also generate prime numbers.
- This did not occur when we replaced 41 with any other prime number, therefore we knew that there was something special about the number 41 in this particular expression. Having this information on the number 41 created an AHHHHH moment in our minds and we decided to plug in 41 for n to see if the expression worked. This proved false and we found an example that proved that this expression was not true for all.
- In this problem the claim that existed was the number 41 giving only prime numbers, in other words is this prime number true for all numbers that you plug in the expression. To disprove a claim that there exists only prime numbers give one counter example, and now the claim true for all is proved false and is justified as well."

Holly's ability to reason mathematically by week 12 of the semester—to appropriately and effectively argue using a proof by exhaustion, to make and test conjectures, to recognize and represent relationships symbolically, and to clearly articulate what was done, both verbally and in writing—to make sense of the mathematics—provides a startling contrast to her earlier work on the departmental competency exam. Further evidence for this is provided by her responses to Question 11 on the same exam (Figure 4):

11. Find the smallest counting number that has the following properties: When it is even divided by 2, the result is a <u>perfect square</u> number, and when it is divided by 3, the result is a <u>perfect cube</u>. Document all efforts and justify your conclusion.

Must be Divisible by 6, therefore "a" and "b" must be multiples of 6.

and $n = b^2 \times 2$ Since $a^3 \times 3 = n$ will have the smaller multiple of 6, we started

$n = a^3 \times 3$ by cubing 6.

$6^3 \times 3 = 216 \times 3 = 648$

Then divided 648 by 2 - 648 ÷ 2 = 324, took the square root = 18.

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Figure 4. Question 11: Unit II Exam (week 12).

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Student response:

… even [written by the phrase *divided by 2*] and $n = b^2 \times 2$; $n = a^3 \times 3$.

[*The number*] must be divisible by 6. Therefore "a" and "b" must be multiples of 6.

Since $a^3 \times 3 = n$ will have the smaller multiples of 6, we started by using 6.

$$6^3 \times 3 = 216 \times 3 = 648$$

Then divided 648 by 2 648/2 = 324, took the square root = 18.

Holly's work on the open response final exam provides evidence of her growing ability to think flexibly and of her improved competence with fractions acquired during the sixteen weeks (See Figure 5).

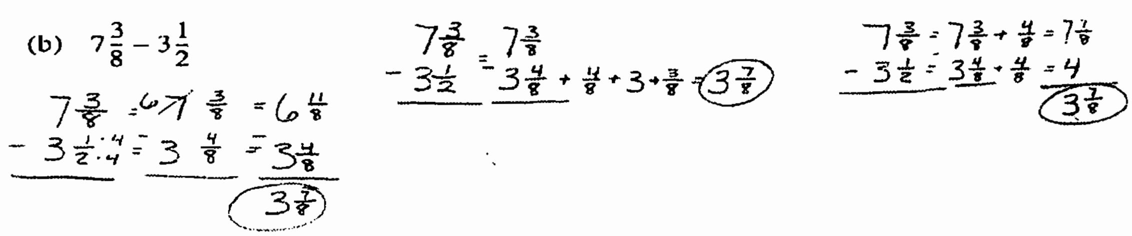

Figure 5. Holly's work on fraction problems in the final examination.

The week before the final exam, Holly still had not passed the competency exam with a grade of 80%—she had a 60%. Yet, a week later, on the final exam, she was able to not only do computations correctly with similar fraction problems, but demonstrated her ability to do the calculations in a variety of ways. She was given an *Incomplete* as her course grade and allowed to enroll in the subsequent course subject to passing the departmental final within six weeks of the next semester start-up. She did so and passed the competency exam with a score of 83%.

Analysis

How does our data illuminate the issue of the growth of deeper understanding of mathematics by pre-service elementary teachers? Being able to use various algorithms correctly and selecting an efficient or elegant method of solution are often considered indicators of growth. For us, indicators of growth in deeper understanding of mathematics means that students are thinking more flexibly and systematically, able to articulate what was done, justifying why a method was used, generalizing their results, recognizing and building on connections between problems in different contexts and between various mathematics topics, and recognizing the role of definitions and proofs. Holly's work on the competency exam contrasted with her classwork and exams during the semester challenges us to re-examine our traditional notions of evaluation and assessment. What constitutes evidence of learning? What do our traditional methods of assessment tell us?

How much of what she has learned will Holly retain and use over time? On the one hand, Holly, like many of her colleagues, has struggled to re-learn computational algorithms which had become deeply-fixed habits of automatic, incorrect responses learned previously, and she experiences debilitating mathematics and test anxiety under stressful conditions like the high-stakes competency exam. Despite these obstacles, Holly, and many others, demonstrate the ability to develop more flexible ways of thinking as they investigate problems and reflect on their work during the semester. They make sense of the mathematics they are learning and build connections—having realized that in order to make connections, they "have to have something to connect." Their language grows more precise, what they focus attention on initially changes, and mathematical marks become truly symbolic. The change in attitudes and beliefs, coupled with their growth in understanding and improved competency is evidence of the changes that have taken place.

The inability of preservice teachers like Holly to demonstrate competency of basic arithmetic computations at the beginning of the semester has not proven to be an accurate predictor of their ability to think mathematically—to represent problem situations, make connections, justify their work, use mathematical arguments, or to think flexibly and systematically. Since Fall 1996, only 4 of 206 pre-service elementary teachers who attempted the course passed the departmental competency exam on their initial attempt with a grade of 80% or higher of those who enrolled in the first content course. One hundred fifty-five of the 206 students completed the course and took the final exam—a 75% completion rate. Of those who completed the course, 96% (149 of 155) were successful, i.e., passed the course with a grade of C or better and passed the departmental competency exam with a score of 80% or better on either their second or third attempt.

Discussion

What are reasonable expectations of students who enroll in a mathematics content course with deeply held beliefs about mathematics and what it means to learn mathematics that reflect a strictly utilitarian perspective of mathematics that often limits their mathematical vision? The complexity of the task in determining answers to our questions is illustrated by an examination of the work and comments of one elementary pre-service student. If, as seems clear to us, deeper understanding develops over time, should we focus our efforts on providing students with pedagogically sound experiences of what it means to learn mathematics and then direct subsequent efforts to building on and transforming that foundation knowledge of mathematics into pedagogically useful content knowledge? Can students like Holly think about how to use their mathematical knowledge in the classroom when their mathematical foundation lacks both conceptual understanding and skill competency in the mathematics that is the basis for what they will be teaching? Holly's initiative in opening and following the textbook to carry out fraction calculations is to be applauded—she wants to be a teacher and is highly motivated. However, her interpretation of the text is an example of "a little learning is a dangerous thing." What is symbolic and meaningful for those with the mathematical knowledge to interpret the symbols and an understanding of the implicit assumptions about the product in the denominator is, in the given context, simply a rule to follow in order to get the answer for students like Holly who attempt to interpret the marks with limited mathematical understanding.

What constitutes evidence of deep understanding? Is it being able to recognize which algorithm a student is using? Is it being able to calculate an answer using various algorithms and being able to choose between known alternative methods and selecting the most efficient/elegant method of solution? Is it being able to articulate what was done, why the particular method used is valid? Is it demonstrating the ability to recognize connections given isomorphic problems in different contexts? Is profound understanding attainable in a sixteen week semester? Over two semesters? By the time the pre-service teacher graduates? If a deep understanding of the mathematics that pre-service teachers need to know takes years to develop, then we need to consider what is possible over time and what is reasonable to expect of pre-service teachers.

The pre-service elementary students who enroll in mathematics content courses at many institutions — particularly at two-year colleges and other institutions with open enrolment policies—are in need of a mathematics content foundation to think with before they are able to think about how to use their mathematical knowledge in the classroom. If our goal is the development of deep understanding of mathematics by those who will be our future teachers, then the various constituencies involved in teacher preparation need to align their instructional goals and practices across institutions so that the development of profound understanding can develop over time. There is, in our view, no doubt that this is a developmental issue, and one that is compounded because it is about development in mathematics by adults for a professional purpose. It seems to us that there are profoundly important educational issues at stake here, and that much focused and penetrating research is required to answer a number of fundamental questions. We believe the results presented here are of significance to those interested in the growth of mathematical knowledge of pre-service elementary teachers. Above all else, we feel, this is a question of considerable interest for the mathematics education community.

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TEACHING MATHEMATICS WITH TECHNOLOGY: THE ROLE OF HIGHER EDUCATION IN LINKING THEORY WITH PRACTICE

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Today's technology standards (ISTE, 2000) challenge teacher education programs across the nation to address the need to produce computer literate teachers for the 21st century who are confident in their ability to choose and incorporate instructional technology into their classroom teaching. The purpose of this paper is to present a description and an analysis of the affects of training sessions on pre-service teachers' ability to appropriately select mathematics instructional technology for use in the everyday classroom. A group of 51 pre-service teachers participated in training sessions where they used a critical eye to evaluate the features of various PreK-12 mathematics software and websites for instructional purposes. Quantitative analysis methods were used to assess the pre-service teachers' prior mathematics learning environments and specific experiences using technology; while qualitative analysis methods were used to search for patterns of affect on the pre-service teachers' beliefs and confidence in using mathematics software and websites to create a technology-based learning environment.

Today's technology standards (ISTE, 2000) challenge teacher education programs across the nation to address the need to produce computer literate teachers for the 21st century who are not just knowledgeable of the internet, word processing programs, spreadsheets, and presentation software, but are also confident in their ability to incorporate instructional software and websites into classroom teaching. Levine and Donitsa (1998) state that computer literacy involves not only the knowledge, understanding, and value of technology, but also a positive attitude in one's ability to apply the theory related concepts to classroom instruction. Unfortunately, Lederman and Neiss (2000) report that technology courses that are part of teacher preparation programs often place emphasis on pre-service teachers learning about technology rather than on learning how to teach their subjects with technology.

Significant changes in teaching and learning have emerged due to the integration of technology and curriculum. Weglinsky (1998) reports data from the 1996 National Assessment of Educational Progress in Mathematics that find both teachers' professional development in technology and the use of computers to teach mathematical concepts and higher order thinking skills to be positively related to academic achievement in mathematics. Despite these results and the growing access to technology, Kent (2001) reports the U.S. Department of Education as calculating only 20% of 2.5 million public school teachers feeling comfortable using technology in the classroom. Of the full-time faculty, 99% have access to computers and the Internet somewhere in their schools. Of that group, across all grade levels, only 39% reported using computers or the internet "a lot" to create instructional materials; 34% used them for record-keeping; and less than 10% used them to access lesson plans, do research, or investigate best practices.

Neiss (2001) reports the National Council of Teachers of Mathematics as pinpointing technology as an essential component of the Pre K-12 mathematics learning environment due to its ability to influence the mathematics that is taught as well as enhance students' learning. Kerrigan (2002) has found the capabilities of mathematics software and websites to include promoting higher-order thinking skills, developing and maintaining computational skills, introducing children to collecting and analyzing data, facilitating algebraic and geometric thinking, and showing the role of mathematics in an interdisciplinary setting. As one reflects on the powerful influence that mathematics instructional technology can have on teaching and learning, the role of higher education becomes evident.

Drier (2001) states that teacher educators are challenged with the task of preparing pre-service and in-service teachers to incorporate technology in ways that will develop a deep understanding of mathematics for themselves as well as for their students. Haughland (2000) states that how computers are used is more important than if computers are used. Selection of appropriate software and websites is key to the ultimate success of integrating instructional technologies into classroom teaching. Therefore, the need for teacher preparation programs to serve as catalysts for the implementation of technology into the mathematics classroom is vital.

The purpose of this paper is to present quantitative and qualitative research efforts that investigate the affects of mathematics software and website evaluation training sessions on pre-service teachers' awareness of, confidence with, and appreciation for the use of technology in the teaching of mathematics. A group of 51 pre-service teachers enrolled in a required sophomore year 15-week long technology course participated in training sessions where they learned how to use a critical eye to choose appropriate mathematics software and websites for use in everyday classroom teaching. Patterns of affect on the pre-service teachers' beliefs and confidence in using software and websites to create a technology-based learning environment in the mathematics classroom are reported in this paper. Findings are shared in order to provide teacher educators with information that may serve them when planning programs that will prepare educators to effectively integrate technology into classroom instruction.

Abilities, knowledge, and skills in teaching with technologies are emphasized in the preparation of teachers so they may make informed decisions about which technologies to use for teaching specific topics of the mathematics curriculum. The primary goal of the training sessions described in this paper was to engage pre-service teachers in evaluating mathematics software and websites in order to prepare them to select and implement appropriate instructional technology into the mathematics learning environment. Projected outcomes included enhanced awareness of and appreciation for the capabilities of technology in the learning process as well as heightened levels of confidence in selecting and implementing software and websites into the teaching of mathematics. This conceptual framework stems from the constructivist approach that proposes the development of insights and skills to support future teachers in making informed and rational decisions when confronted with instructional choices concerning technology and the mathematics classroom.

Initial Assessment

Halpin (1999) states that teachers often teach as they were taught. Therefore, prior to the training sessions, an investigation into the pre-service teachers' past mathematics learning environments as well as their knowledge and specific experiences concerning technology was conducted in order to gather information about prior learning. A survey was administered that sought information concerning the mathematics pedagogy and tools that the pre-service teachers' experienced as students at the Pre K-5, 6-8, and 9-12 grade levels. The first part of the survey required the pre-service teachers to indicate the presence or absence of specific features in their prior learning of mathematics. These features were explained and included the following: teacher-centered instruction, student-centered instruction, convergent questions, divergent questions, conceptual development, computational skill building, cooperative learning, memorization and drill exercises, journal writing, applications, interdisciplinary connections, problem solving strategies, manipulatives, games, texts, and worksheets. The second part of the survey focused on the pre-service teachers' prior uses of instructional technology as well as their self-confidence to select appropriate software and websites for use in the mathematics classroom. They were asked to supply the specific software titles and websites they had previously used at school, home, and/or work, along with an explanation concerning their reason for using the specified software and websites. They were also asked to indicate their self-confidence in evaluating the quality of instructional software and websites, and to comment on how they would select appropriate software and websites for use in the mathematics classroom. After the survey was administered, interviews with the pre-service teachers were conducted where they could speak candidly about their survey responses. These interviews provided the opportunity to gain a deeper understanding of the pre-service teachers' written responses.

Data collected from the surveys and interviews indicated that the majority of pre-service teachers' mathematics learning environments focused on computational skills rather than on the development of conceptual understandings, problem solving strategies, and higher order thinking skills. Memorization and drill of basic facts, definitions, and formulas was common at all levels while activities such as journal writing and mathematical games were practically non-existent. Their responses revealed that the most common experienced teaching technique at the Pre K-5, 6-8, and 9-12 levels was teacher-centered instruction where lecture from the teacher was basically the sole means of communication. Discourse amongst students was rare. Their conception of cooperative learning was where a group of students "sat together" to work on their examples and ask each other for help when needed, rather than a group of students working as a team, each dependent on the other, to accomplish a common goal. Most of the pre-service teachers were used to convergent questions where one correct answer and method of solution were present. Opportunities for discussion concerning multiple methods of solution and multiple correct solutions/representations made possible by the use of appropriate divergent questioning and problem solving situations was not a common memory for many of the pre-service teachers. The use of manipulatives declined rapidly as the grade levels increased. Common learning tools were textbooks and worksheets. Applying mathematical concepts to the real world and/or other discipline areas was a rare event for these pre-service teachers. Many commented with statements such as "I couldn't see how I would ever use some of the math I learned in school," and "Many times I found myself frustrated because the math I was learning had nothing to do with anything else I was learning."

The most familiar computer application reported by the pre-service teachers was word processing. Most had used Word Perfect for typing research papers. A small number had created spreadsheets with Excel and designed presentations with PowerPoint. A vast majority of the pre-service teachers had used the internet, but only for the purpose of e-mail. Few had conducted research on the internet. Using technology for instructional purposes was not widespread. Of the few pre-service teachers who had used software for instructional purposes in school, only one did so on a regular basis in science class. The others reported sporadic use of software that didn't relate to classroom topics. The software they used at home was geared towards test preparation for standardized tests. Those who used software at work did so in an after-school program where children, ages 4-6 used reading and math readiness programs. All of the pre-service teachers felt insecure in their ability to evaluate mathematics software and websites. Their comments revealed that the majority would not know where to begin if asked to evaluate instructional software and websites for the mathematics classroom. Many thought that the mathematics software and websites were just games the students could play once the content was mastered.

Description of Training Sessions

The training sessions were designed to prepare the pre-service teachers for entrance into their own classrooms not only aware of the capabilities of mathematics instructional technology, but also with the knowledge and confidence to appropriately select and integrate software and websites into their teaching of mathematics. The sessions began in a whole group instructional setting where two 1-hour class periods were spent discussing how to evaluate mathematics software and websites. The pre-service teachers were guided to use criteria and consider the following when viewing the software and websites: the mathematics standards covered, use of higher order thinking skills, ability to sustain interest, grade level, ease of use/free from technical flaws, ability levels, reading level, use of demonstrations/models/visual aids, appropriate color/sound/graphics, conceptual development, skill building, problem solving strategies, applications, interdisciplinary connections, as well as accurate information and interactive features/feedback. The pre-service teachers were also instructed to look for the degree of access to background information, evidence of effectiveness, the presence of record-keeping, and complete documentation/site supporters. After the introductory discussion about the criteria one should use to evaluate software and websites, the pre-service teachers as a whole group were guided by the instructor to evaluate two pieces of mathematics software and two mathematics websites. During the guided discussion, each of the criteria was addressed for each of the viewed software and websites. The similarities and differences existing amongst the software and websites were discussed as well as their appropriate uses in the mathematics classroom. The dominating strengths of the viewed pieces of instructional technology were noted and used to begin a discussion concerning the value of distinguishing between those programs and websites best suited for whole group instruction and those suited for small group instruction. In addition, the software and websites most effective for purposes such as conceptual understanding, skill building, applications, and interdisciplinary connections were addressed. For example, the pre-service teachers reasoned how the software and websites that contained visual displays and multiple representations would be appropriate for developing mathematical concepts while those that engaged the students in real-life situations such as owning a business and investigating architectural designs were best suited for problem solving, applications, and interdisciplinary connections. The pre-service teachers considered the interactive feedback features where students were guided to arrive at correct answers as well as multiple methods of solution when possible. They also looked for record-keeping aspects of the software and websites as a form of student assessment. Throughout the discussion the feedback from the pre-service teachers was recorded and used as a means to summarize their findings.

The pre-service teachers were then paired as a team and asked to evaluate in the same manner as before two additional pieces of software and two additional websites during the next two 1-hour class sessions. Each pair was given different pieces of software and websites and was instructed to take notes on their findings. Their notes were later used to prepare for class presentations concerning the software and websites that they evaluated. Throughout this assignment, the students were observed to take note of their level of engagement as well as their reactions to the evaluation procedure. Visits were made to each of the pairs to discuss the instructional purpose of the software and websites they were evaluating. When evaluations by all teams were completed, the next two 1-hour class sessions were spent discussing their evaluations. Each of the teams presented their findings to their peers based on the given evaluation criteria. They justified their evaluations by demonstrating specific features of the programs and websites that caused them to reach particular conclusions. Feedback from their classmates was elicited. The discussions that followed each of the presentations gave the pre-service teachers an opportunity to voice opinions, share insights, and become actively involved in the decision making process involved in selecting instructional technology for the mathematics classroom.

The same student pairs were given the assignment of developing a technology-based lesson plan that incorporated either a piece of software or a website as their main technological teaching tool. They were also asked to incorporate an appropriate follow-up activity to the lesson that would incorporate a piece of instructional technology that was not used in the main portion of the lesson. For example, if a team developed a lesson about fractions using a piece of software, their follow-up activity used a website concerning fractions. The students were given time throughout the remainder of the semester to plan while guidance and feedback was given by both the instructor and fellow classmates. The teams made decisions concerning issues such as whole group versus small group instruction, alignment of software and websites with lesson objectives, and methods of assessment. The lessons were presented using PowerPoint, and selected software and websites were viewed. The rationale for choosing particular software and websites was discussed by the presenting pair. The presentations demonstrated their educated use of the given criteria as well as their collaborative efforts in the decision making process. As before, feedback from their classmates was shared. The discussions at this point in the training sessions portrayed the pre-service teachers not only as active participants in the evaluation process, but also as practitioners of technology-based mathematics lessons.

Conclusions and Recommendations

Upon completion of the training sessions, the pre-service teachers were asked to reflect on their experiences evaluating mathematics software and websites and incorporating them into lessons. During interviews, each of the pre-service teachers was asked to comment on both the positive and negative aspects of the training sessions, their viewpoints concerning the value of integrating technology into the mathematics curriculum, and their level of self-confidence in selecting and incorporating instructional technology into classroom mathematics teaching. Collected data indicated that the vast majority of pre-service teachers viewed the sessions as a completely positive experience. Of those responding that the training sessions were not completely positive, the only negative aspect reported were technical difficulties that infrequently occurred when a program wouldn't run or a website wasn't responding. Their responses revealed that they appreciated having a structured set of criteria that they used consistently throughout the evaluation process. The pre-service teachers enjoyed working with a partner and sharing their findings with others. They indicated that the "support system" that the design of the sessions allowed for gave them the opportunity to brainstorm with another person and refine their ideas. The majority viewed the development of technology-based lessons as a hands-on experience that put theory into practice. Many of the pre-service teachers made comments such as "It was helpful to have us all present our evaluations. This way we saw more than just what we were assigned," and "By actually incorporating a piece of software and a website into my own lesson, I feel I did what I learned to do."

Patterns in the pre-service teachers' responses were noted and organized into three categories: heightened awareness of the capabilities of technology, stronger levels of confidence in selecting software and websites, and deeper appreciation for technological resources. The majority of the pre-service teachers responded that they were more aware of the uses of instructional technology in the mathematics classroom as a result of the training sessions. They specifically noted software and website capabilities such as: development of mathematics concepts, skill building, use of problem solving strategies, demonstration of mathematical applications, and provisions for interdisciplinary connections that they were not aware of prior to the training sessions. The pre-service teachers as a whole commented that they were more confident in the process of selecting software. They indicated feeling better about their ability to choose instructional technology that was appropriate for age, ability level, and mathematical topic. Quotes included "We didn't just accept the first thing we saw," "We selected tools that were relevant, motivational, and would really benefit kids," and "We were able to pick it apart and see both the positives and the negatives." The majority of the pre-service teachers viewed instructional technology as a tool that can enhance the mathematics learning environment and provide a much richer experience than they had as students. They reported feeling at ease knowing that they electronic technologies would support their efforts to create a mathematics classroom comprised of hands-on exploration, reflection, and problem solving.

Conclusions rendered from the findings of this study warrant not only recommending the inclusion of instructional technology evaluation skills into teacher preparation technology courses, but also that those learned skills be used in higher level methods courses to build awareness of resources and enhance teaching techniques in all disciplines. Teachers hold the key to the effective use of technology in the classroom. Promoting awareness of, appreciation for, and confidence in one's ability to select and use instructional technology can help put theory into practice and increase the numbers of those who teach with technology and thus enhance classroom teaching and student achievement.

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PRESERVICE TEACHERS UNDERTAKE DIVISION IN BASE FIVE: HOW INSCRIPTIONS SUPPORT THINKING AND COMMUNICATION

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In this study, preservice elementary teachers, working in two groups, solve division problems using different number systems in base five: a place-value system and an additive system. In the first system, the students' algorithm displays symbolically the numerosities on which the formal steps are based. To divide using the second system, the students' quotient emerges as a missing factor, found by filling blanks in a multiplication algorithm that the students had developed first. Their construction takes advantage of special features of their number system, features that place value numbers do not share. To analyze the work considered here, we focus on the students' inscriptions, both as parts of thinking and as parts of an evolving discourse about methods and their grounding.

This paper is one of two that focus on the use of mathematical inscriptions.[1] Both papers give case studies of how groups of learners grapple with important concepts in surprising ways. In each case, despite large differences in age and setting, understanding what the learners write and draw supports key steps toward understanding how they learn.

In this report, we follow two student presentations about whole-number division, using contrasting number systems in base five. We have drawn these student presentations from extensive data gathered in an extended teaching experiment in the mathematics content sequence for elementary education undergraduates at our university. In this teaching experiment, now in its fifth year, we have worked with classes of about twenty-four students for two semesters at a time. Thinking ahead, not just to the later methods course but also toward our students' subsequent professional development, our experimental teaching focuses primarily on how our students work collectively to solve challenging, extended problem tasks and then present and justify proposed solutions. We have avoided lecturing. Instead we have emphasized selecting and proposing tasks we hoped would trigger the development of important mathematical concepts by the learners. We have been especially interested in how the need to build convincing explanations, arguments, and justifications might help to deepen student inquiry and discourse.

Background and Guiding Frameworks

Research by the first two authors, begun in 1995, has centered on how mathematics learners construct, represent, explain, debate, reconsider, reconstruct, and justify their mathematical ideas. In our work so far, the rich variety of representations that learners build (including graphs, diagrams, written symbols, gestures, or specific language use) in order to develop and convey ideas, results, and lines of reasoning has been especially striking. For us, a representation is a *presentation*,[2] perhaps to oneself, as part of an ongoing thought, or perhaps to others, as part of an emerging discourse (Speiser & Walter, 1997). Here we focus on *inscriptions*: things that people write and draw.

The approach we take to classroom research builds from the guiding principle that learners must adapt their current understanding when they encounter problems where their prior understanding proves inadequate. Hence we view the growth of understanding as a process of assimilation and accommodation. In particular, we design instruction, primarily by selecting, developing, and posing problem tasks that challenge students to revisit, reconsider, and perhaps to reconstruct prior ideas, techniques, knowledge, and understanding, in detail and at length. We see the building and discussion of inscriptions as essential to the building and communication of important mathematical concepts.

Rationale

The need to build instruction based on learners' prior experience and current thinking hardly seems to need discussion. How to meet that need, however, has been the subject of considerable debate. For preservice elementary teachers, the need to gain strong understanding of the mathematics they will help children to learn presents, we think, two major challenges: our students' prior experience, often through instruction which has failed to meet their need to build important understandings through their own successful efforts, and the need to help our students make productive contact with successful work and thinking, not just by themselves or by their peers, but also by young learners.[3] In our teaching experiment, we and our students both take challenges head-on. Here we concentrate on how our students meet the first one: to build a new, more helpful, mathematical experience.

Data Sources and Task Design

The student work reported here took place in October 2001, in two sections of our experimental mathematics course. Both sessions were videotaped by one camera that followed focus groups, filmed student presentations, and also panned around the room. The data we discuss are drawn from student presentations. We collected students' written work after each session. Further, at the end of the semester, students submitted journals and reflective essays, which we have also found useful. Key events from the videotapes were transcribed and checked by three people, in preparation for analysis. The task reported here was the number system task developed in 1995 and described in Speiser & Walter (2000). Before the data given here, each student group built a number system using the symbols O, A, B, C, and D, with two requirements: (1) the given number system should be compatible with standard base 5 blocks, and (2) the given system should allow them both to count and do arithmetic. The student work reported here discusses long division in two different number systems: a place value system in base 5, and an additive system based on an extended set of blocks.

Methods or Modes of Inquiry

The interpretation of inscriptions and their use will help us to track important changes in the work and thinking of the learners. Individuals might build them, for example, to present aspects of their own thinking to themselves. But, once an inscription has been made, groups of learners can then work with it, either socially or individually, in different ways. We will treat inscriptions, suitably interpreted, as parts of thoughts that can be readily inspected, both by learners and by us. As explicit objects of construction, argument, reflection, reconstruction, and perhaps debate, their analysis can help to anchor narratives about the learners who have worked with them. These narratives, in turn, may help to clarify, and possibly support, hypotheses about the learners' work and thinking. Hence, we will surround given inscriptions with transcript data, background information, and interpretations, through which ideas contributed by specific individuals can be seen as informing (or perhaps failing to inform) the discourse of a group.

Data and Analysis

In our classroom, we had already used a set of base 10 blocks. In the statement of the new number-system task, however, we asked our students to imagine a set of base 5 blocks, indicated by line drawings on the task sheet. The base 5 blocks were designated as follows. The 1cm cube was called a *piggy*, by then a longstanding local usage, explained by Speiser and Walter (2000). The next three blocks, in ascending sizes ($5 \times 1 \times 1 cm^3$, $5 \times 5 \times 1 cm^3$, $5 \times 5 \times 5 cm^3$) were respectively called *longs*, *flats* and *solids*. We now present and analyze two data segments, in which two different student groups explain division in their (very different) number systems.

The first inscription to be analyzed appears in Figure 1 below. Five students have built a place-value system in base 5, where the symbols O, A, B, C, and D represent the cardinal numbers zero through four, respectively. Thus, for example, CDD represents three flats, four longs, and four piggies, giving 99 if we convert to base 10. They have posed the problem CDD divided by D to illustrate their method of division. This is shown in Figure 1, at the right, as one student, Heather, writes as she explains:

> So, we have D's into C's, and D's won't go into C's. So, we need to break the C's into the long position. So, that's a 1, that's [smiles, laughs] A [i.e., one] C. [Writes 5 tally marks below the first D in the dividend.] B long [Writes 5 more tally marks below the first 5.] And C, long. [Writes 5 more tally marks below the first 5, then draws three arrows from the C to the three groups of tally marks.]

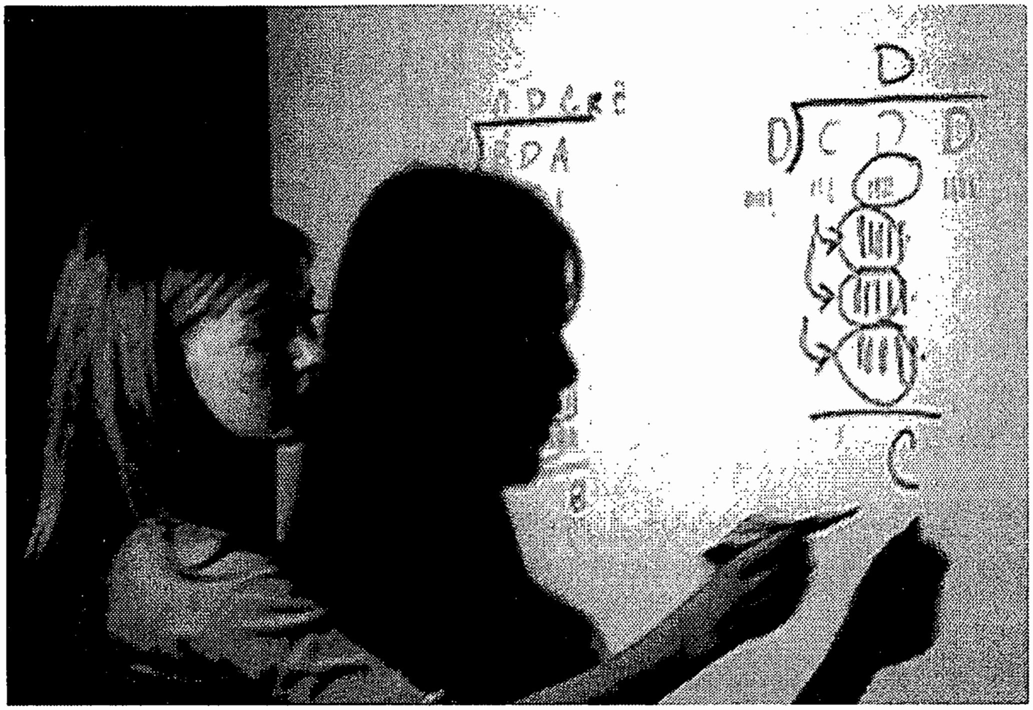

Figure 1. Heather's inscription.

> So this is how, this is converted, over…
>
> So and we need D groups. So, here's a D group [Circles the four tally marks on the row just underneath the D.] Here's a D group, with that one, that's not in there [Circles the four leftmost tally marks on the 2nd row underneath the D.] Here's a D group [Circles the four leftmost tally marks on the 3rd row underneath the D.] Here's a D group [Circles the four leftmost tally marks on the 4th row underneath the D.] And we have C left over. [Marks each uncircled tally mark in that column with a dot, draws a line beneath the column, and writes C below the line.]
>
> So we have D groups. [Writes D above the division bar.]
>
> With C left over [Points to the C at the bottom.] and then the D drops down. [Writes the D next to the C and then writes the tally marks below D and then below the C.]

The C in CDD has been decomposed into three groups of five tally-marks, placed below the D in the next column. Groups of four (i.e., D) tallies have been circled, leaving a vertical column of "I"s, each marked by a dot. Once these D's have been accounted for, the three "I"s marked with dots give C below the horizontal line, and then D is brought down. This is quotative division: this method (Speiser, Walter, & Lewis, 2001) asks how many Ds can be taken out of CDD. The use of tallies seems to symbolize[4] more abstract objects, that the numbers denoted O through D can be interpreted as counting. In the transcript of the students' conversation, these abstract counters are identified, at different times in different ways, with particular base 5 blocks. The tally marks make clear that Heather, as spokesperson, as well as other members of her group, are counting with composite units (Olive & Steffe, in press; Steffe, 2001), much a s children have been seen to do when working with fractions.

In a second data segment, a different group of students use an additive base 5 system, somewhat analogous to Roman numerals, to develop a completely different algorithm: to find the quotient as a missing factor. In their inscription, they successively fill blanks in a previously-developed algorithm for multiplication, where one factor and the product have been given, in order to determine the second factor. Their method depends on the additive nature of their number system, in which products emerge naturally as rectangular arrays (see Figure 2, right for an example).

This group's number system works as follows. The symbols A, B, C, D, and O are used to represent different sizes of blocks. The symbols A, B, C, and D represent, respectively, the piggy, the long, the flat, and the solid. This group invented a further *imaginary* block, equivalent to five solids placed end to end, which they denoted by the remaining symbol O. They called this final block a *big long.* Numbers are then represented additively (somewhat as in the Roman system), by juxtaposing symbols. Thus DDCBBAAA represents two solids, one flat, two longs, and three piggies. Adding and subtracting lead to quite straightforward algorithms. Multiplication, however, becomes very interesting.

To multiply, the group first built a table (Figure 2, left) to tabulate products of the basic blocks, which they called the *chart.* Then they found more complicated products by constructing rectangular arrays. For example, the product of CBBBAA by CBBAA led to the inscribed algorithm (Figure 2, right), a mosaic of rectangular arrays. The product, after combining the blocks indicated, is OODDCCAAAA.

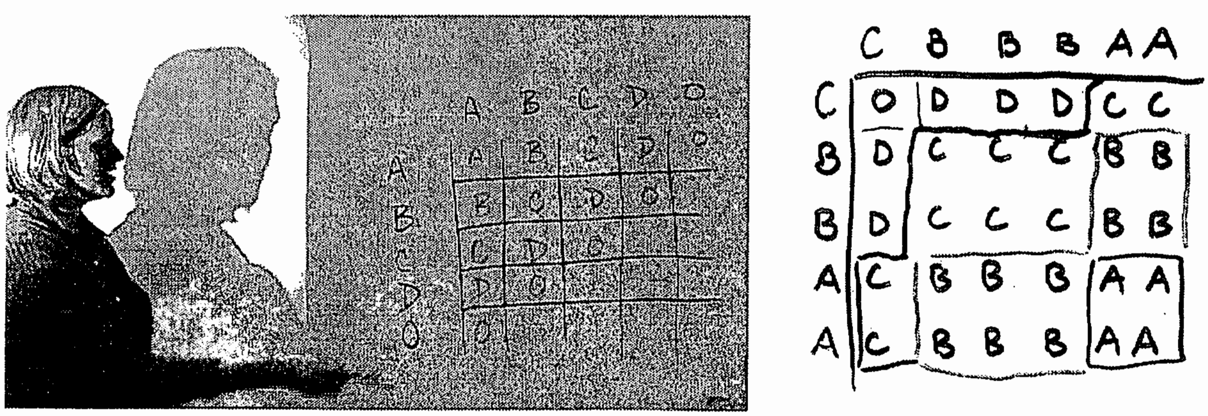

Figure 2. Left: the products of the basic blocks. Right: the chart for CBBBAA times CBBAA.

To divide, two group members, Jana and Heather (a different Heather from the first example), reversed the multiplication process they had just demonstrated, dividing the product, OODDCCAAAA, by the factor CBBAA. To do so, they created a blank multiplication diagram (which we will call the *workspace*), wrote the divisor CBBAA down the left-hand side, and proceeded to determine the second factor step by step, by filling in the workspace one column at a time depending on successive entries for the second factor at the top margin of the workspace. To make this clearer, Figure 3 (left) shows one step of the process: beginning the second column of the workspace.

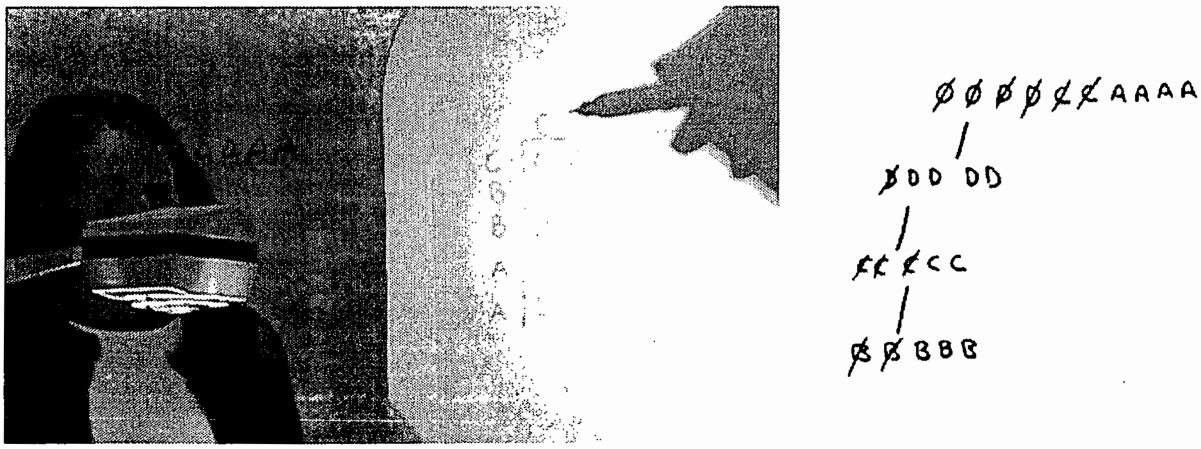

Figure 3. Left: the workspace, with Jana ready to fill in the second column. Right: the corresponding tree.

To the right, we see a further inscription (Figure 3, right), which we will call the *tree*: a diagram the students used to keep track of the given product (that is, the dividend, OODDCCAAAA), with entries crossed off successively as they were either moved into the workspace or replaced in the tree (indicated by vertical lines) by equivalent collections of smaller blocks. To summarize the process, the first O in the product was moved into the upper left corner of the workspace. By the chart, we have O = CxC. Therefore, the first C of the divisor (CBBAA) forces C into the first place in the quotient of the workspace. Then the first column of the workspace can be filled in by the products given in the chart, and finally the first O in OODDCCAAAA is crossed off in the tree, and then the further entries (just found) below O in the first column in the workspace.

In the following data segment, Heather and Jana discuss the first C in the workspace quotient, and then the entries in the first column of the product which appear below it.

Jana: Umm, to figure out what would go on top, you can see that over here an O and a C would be another C. So you can work that way with our chart. So the C would go on top there.

Then, we see that a C times a B on the chart is a D ... D, C, C. So now we can cross off D, D, C, C and we have left, what we have left is O, A, A, A, A.

Umm, we have an O left [referring to the tree], but if we put the O there [in the workspace, where the D appears in Figure 3, left] all we would have left after that would be 4 A's and we couldn't fill in the rest of the column, so we are going to have to go one lower than that to a D. [Writes a D to the right of the O inside the product in the workspace.]

And so in order to do that, we break this O [in the tree] down into 5 D's. [Heather crosses off the first O, 2 D's, and 2 C's in the tree. Then, she writes 5 D's below the 2nd O in the tree. Meanwhile, Jana, at the workspace, turns toward Heather.]

Heather: This many D's.

Jana: This many D's. [Displays 5 fingers.]

We read this data segment as a presentation, for the other students in the class, of how the steps described here were discovered. In particular the last two lines of dialogue look (at least to us) like a spot check for the number of D's that give an O, which, by definition, will represent the number of flats that constitute a solid. At this point *three* inscriptions are in play: the workspace, the chart and the tree. The latter two, in a sense, are used as tools for building up the workspace, where the sought-for quotient is to be assembled. Each inscription, in a different way, presents important parts of their authors' thinking to themselves, as they made decisions about how, step-by-step, they should proceed. Here, in our view, they serve a somewhat different function, to help the speakers narrate their sequence of decisions to the class, perhaps anticipating questions from the floor.

Very soon, two students, from other groups, in different places in the room, indeed pose questions. These questions will trigger *shifts* in how the three inscriptions (perhaps especially the workspace) work within the student discourse.

Erin: [From across the room.] So, the answer is just going to be what's on top of the line, right?

Jana: Right.

Erin: Okay. [Jana looks back at Heather, who is writing on the board.] ...

Erin: And what's the stuff below again? It's like your ...

Jana: This in here? [In the workspace, Jana points to the area inside the lines.]

Erin: Yeah.

Jana: That's our ... what we're dividing, [Points to the dividend problem statement written at the top of the overhead.] we're breaking it down until, until it fills ... the square perfectly.

Jana: Because, yeah, does that make sense?

Andrea: [From one side.] How did you get the first O in the first place? I don't remember that.

Jana: We just took our, largest num…, our largest letter which was an O and stuck it in there. [Points to the first O in the dividend in the problem statement.]

Andrea: Is that what you always do? Just take the largest one?

Jana: Right, and if, and if, if we break everything down and realize that ... um, we don't have enough to fill the one column, then we would go to the one below, ... the, the next largest num…, letter after that. Okay?

The inscriptions, which at first served their authors as presentations of parts of the authors' thinking, are now reframed to anchor a collective conversation about the choices that the authors made to carry out key steps. On what basis were these choices made? We see a shift in tone, from more formal presentation to less formal interchange. Where first the inscriptions seemed to organize and tabulate important bits of information, now they help to locate, and make more explicit, lines of reasoning. These lines of reasoning connect important calculation steps to the group's chosen meaning for division: finding a missing factor by reinterpreting the dividend as a suitable product. It is important to note that finding missing factors by filling blanks in multiplication templates looks extremely awkward in place-value arithmetic. Our students took advantage of important special features of their *additive* number system in order to invent their method.

Conclusions

Our data suggest ways that the construction of inscriptions can help to trigger mathematical discoveries, and also facilitate communication, through the ways in which inscriptions are designed and then interpreted. Our analysis suggests in some detail how inscriptions can begin as presentations, to oneself (or to a few collaborators) of parts of one's own thoughts, but then may be reinterpreted (in effect restructured) to serve the later purpose of communication to a larger group. In the student discourse we report, communication emphasizes *reasoning* to justify conclusions given, and to explain the *choices* made in finding them. The students we have followed here first build, and then communicate original and quite sophisticated mathematics. Our analysis contributes evidence, as did the first two authors' monograph (Speiser & Walter,

2000) that the mathematical potential of preservice *elementary* teachers may be far greater than is generally assumed. Clearly, given the data here, we need to learn still more about the complex uses of inscriptions, both for solving problems, and, through discourse, for building shared and lasting understanding.

Notes

[1] The other is Speiser, Walter, Melnick, Rivera, and Christensen (2002).

[2] See also Speiser, Walter, and Maher (2001) as well as Doerfler and Maher (2001).

[3] See also the research monograph (Speiser & Walter, 2000) for related student explorations, as well as extended discussion, by both students and researchers, about how learners revisit and restructure prior mathematical and pedagogical understanding.

[4] See Cobb, Yackel, and McClain (2000)

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PRE-SERVICE MIDDLE-SCHOOL TEACHERS' CONSTRUCTION OF LINEAR EQUATION CONCEPTS THROUGH QUANTITATIVE REASONING

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We report on a semester-long teaching experiment with 16 pre-service middle-school teachers. During the teaching experiment, we developed and refined a sequence of teaching activities that focused on solving linear equations embedded in quantitative situations containing unknowns. We designed the activities to support middle-school students' development of algebraic reasoning, but found that the activities were also challenging for pre-service teachers. In particular, our pre-service teachers had significant difficulties understanding and coordinating the core concepts of division of fractions, linear and area measurement, unknowns, and linear equations. By the end of the teaching experiment, only 7 students demonstrated a deep understanding of linear equations and their solutions. Of these 7, 3 were able to explain solutions to linear equations by operating on length quantities in rigorous ways, and 4 others were able to explain solutions to linear equations by operating on length and area quantities in rigorous ways.

Student learning in algebra is a perennial concern in U.S. mathematics education. In recent international studies, U.S. middle-school students' achievement in algebra lagged behind that of their peers in high-achieving countries (National Center for Education Statistics, 2001b; Peak, 1996). Despite increases in average scores on the National Assessment of Educational Progress (NAEP) exam in 2000, 34% of the nation's eighth graders scored below a "basic" level of performance in mathematics (National Center for Education Statistics, 2001a). Moreover, in response to claims that algebra predicts success in college, there have been recent calls for "algebra for all." For reasons such as these, the RAND Mathematics Study Panel (2002) proposes a research program focused on the teaching and learning of algebra.

Past research on the teaching and learning of school algebra has often emphasized algebra as generalized arithmetic, a problem solving tool, the study of functions, or a language for modeling (Bednarz, Kieran, & Lee, 1996). Past research has also documented student difficulties with each of these perspectives on algebra (Kieran, 1992; Leinhardt, Zaslavsky, & Stein, 1990; Monk & Nemirovsky, 1994; Moschkovich, 1998; Nemirovsky, 1994; Schoenfeld, Smith, & Arcavi, 1993). In summarizing existing research on the teaching and learning of school algebra as of 1992, Kieran (1992) argued that algebra requires new understandings difficult to acquire in part because they are abstract and in part because they can conflict with existing interpretations of symbols that students have developed through years of solving arithmetic problems. Because students must develop algebraic reasoning using their prior understandings, including their past experiences with arithmetic, Kieran's argument raises practical and theoretical questions about approaches to teaching that can support students' construction of algebraic reasoning.

Further research on the teaching and learning of algebraic concepts and algebraic reasoning in the elementary and middle grades is pressing not only because of U.S. student performance in international studies and the RAND report, but also because the *Principles and Standards for School Mathematics* (The National Council of Teachers of Mathematics, 2000) state that an algebra strand should start in the elementary grades and have increased emphasis particularly in the middle grades. Our research focuses on preparing middle-school teachers to teach early algebraic concepts and operations. We report on an initial teaching experiment in which pre-service teachers were the students. During the teaching experiment, we developed and refined a sequence of teaching activities that focused on solving problems about quantitative situations containing unknowns, and we used the activities as a means of exploring the teaching constraints and affordances that the 16 students presented to us as we engaged them in the activities.

Theoretical Framework

We regard quantitative reasoning as the basis for algebraic reasoning. Thompson (1994) characterized quantitative reasoning in terms of *quantities* and *quantitative operations*. A quantity is a scheme consisting of an object, a property of that object, and an appropriate unit or dimension with which to measure that property. Quantitative operations are mental operations by which one forms new quantities in relation to one or more already-conceived quantities. Thompson (1994) pointed out that quantitative operations are not the same as numerical operations used to evaluate quantities. For the present study, we take *algebraic reasoning* to be quantitative reasoning about constant and varying unknowns. For a student to reason with *quantitative unknowns*, they must have both a scheme consisting of an object, a property of that object, and an appropriate unit of measurement, and the ability to imagine the result of that measurement, even if it cannot actually be completed.

At its core, our teaching experiment focused on solving linear equations of the form:

$$\frac{a}{b}x = \frac{c}{d} \tag{1}$$

We restricted our examples to ones in which a, b, c, and d were integers. At the beginning of our teaching experiment, the pre-service middle-school teachers could solve such equations using an "invert and multiply" rule, but they could not explain why the rule worked. We consider such solutions to be arithmetic at best, because one multiplies two known quantities, $\frac{b}{a}$ and $\frac{c}{d}$. Ma (1999) reported similar results in her study of elementary-school teachers. In our teaching experiment, we tried to support our students' reasoning about length and area quantities to solve such equations. The examples we discuss in this paper show that such solutions require operating with quantitative unknowns and thus require algebraic reasoning. Thus, our course emphasized the coordination of core, and difficult to learn, concepts including division of fractions, linear and area measurement, unknowns, and linear equations.

Methods and Data

Our research was based on a whole-class teaching experiment (Cobb, 2000; Steffe & Thompson, 2000) with a class of 14 college seniors and 2 masters degree students studying to be middle-school mathematics teachers. The students were enrolled in a semester-long course we taught on the teaching of algebra to middle-school students. During each class we posed problems to students, gave students time to work on the problems either individually or with a partner, and then had whole-class discussions during which students presented and discussed their solution methods, including any drawn representations that they may have used. We circulated and talked to students when they were working alone or with a partner to get a sense of how students were thinking about the problems and the strategies that they were using. At the beginning of the course, students solved problems using pencil and paper. During the second half of the course, students solved problems using Geometer's Sketchpad. We introduced Geometer's Sketchpad so that students could construct dynamic representations of varying length and area quantities. After each class we (the two teacher-researchers) discussed what had happened, hypothesized about aspects of problems and solutions that students did and did not understand, and planned activities for the next class to test those hypotheses. Our results emerged from our interactions with students during class and our analyses of students' written homework, Geometer's Sketchpad files, and exams.

Analysis and Results

The pre-service teachers in our teaching experiment faced significant challenges when trying to solve Equation (1) with understanding. In particular, our students had significant difficulties understanding and coordinating the core concepts of division of fractions, linear and area measurement, unknowns, and linear equations. By the end of the teaching experiment, only 7 students demonstrated a deep understanding of linear equations and their solutions. Of these 7, 3 were able to explain solutions to linear equations by operating on length quantities in rigorous ways, and 4 others were able to explain solutions to linear equations by operating on length and area quantities in rigorous ways.

One of our paradigmatic activities was to find the length of a segment if $2/3^{rds}$ of the segment was 5/7 cm. In an early phase of the course, students made sketches of the situation and, through whole-class discussions, adopted the sketch in Figure 1 as conventional. The graphical representation for $(2/3)x$ was produced by positing a *hypothetical* segment of length x cm., taking 2/3rds as an operation on that segment, and equating the resulting length with a second segment known to be 5/7 cm. Positing an unknown length, operating on that length by taking 2/3rds of it, and equating the result to a known length proved difficult for our students, initially. Nevertheless, through whole-class discussions, students developed an understanding of this first step of an algebraic solution to Equation (1).

To complete the solution, students had to partition the segment of length 5/7 cm. into two parts and then iterate one of those two parts three times to produce the segment of length 15/14 cm. At first, finding one half of the 5/7 cm. length was difficult for our students. They could imagine cutting the 5/7 cm. length into two equal parts, but were unsure what the resulting length would be. Through whole class discussions, the students came to understand that they could find the length by dividing each of the five 1/7 cm. lengths into two 1/14 cm. lengths and that five of the 1/14 cm. lengths was the same length as half of the 5/7 cm. length. Once the students understood how to determine that half of the 5/7 cm. length was 5/14 cm., they could then iterate that length three times to arrive at the final length of 15/14 cm., which was also the length of the original segment, x cm. This constituted a novel way of dividing fractions for the students, and we judged from in-class explanations and homework that at least some of the students learned how to solve linear equations in this way.

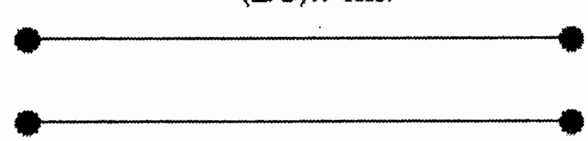

Figure 1. A linear representation for (2/3)x = 5/7.

We hypothesized, however, that for some of the students who did learn to solve linear equations as explained above, the underlying quantitative reasoning was restricted to the linear situation shown in Figure 1. To test our hypothesis, we had students solve the same $(2/3)x$ = 5/7 equation using *lengths of sides* and *areas of rectangles*. We had students draw rectangles in Geometer's Sketchpad to facilitate their construction of varying quantities. Figure 2 contains a typical sketch of a unit square, the lower left vertex was at the origin of the coordinate plane. By moving the point x on the x-axis, students could illustrate the variable length quantity, x cm., and the variable area quantity,

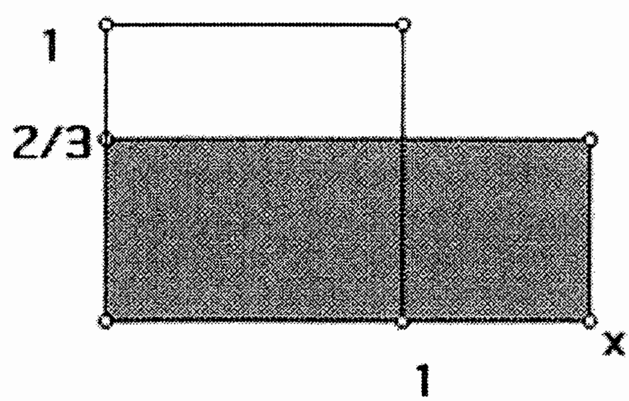

Figure 2. A Geometer's Sketchpad sketch of a unit square and a variable rectangle of area $(2/3)x$ cm^2.

(2/3)x cm^2, for themselves. To solve, students had to locate the length x cm. so that area of the shaded rectangle in Figure 2 was 5/7 cm^2.

A Final Exam Question

On the final exam, we gave the item in Figure 2 along with the representation shown in Figure 3 because the students had used this kind of representation in class when working on similar activities. The text of the question asked:

> Use the sketch to find the value of x when $(2/3)\cdot x = 5/7$. Be sure to write your answer in fractional form and explain how you produced it. In your explanation, be sure to explain the logic in how to solve the equation.

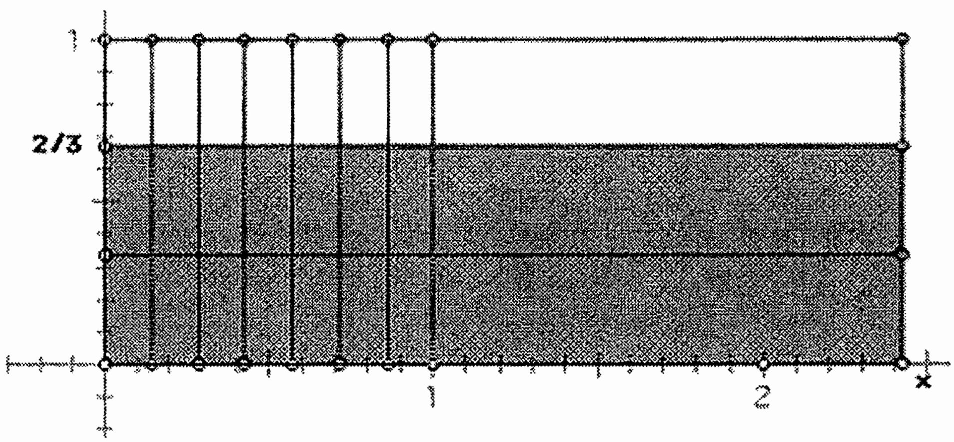

Figure 3. Placing x so that (2/3)x cm^2 was equal to 5/7 cm^2.

Students' responses varied from those exhibiting no quantitative reasoning, to those involving arithmetic quantitative reasoning (reasoning in which all quantities except the final answer are known), to those involving algebraic quantitative reasoning (reasoning that involves operating on unknown quantities). The paragraphs below summarize students' responses.

Non-quantitative or Arithmetic Quantitative Reasoning

Three students simply stated that x was 15/14 based on their prior knowledge to invert and multiply when solving such equations. These students did not give any further explanation of their solution. We characterize these responses as non-quantitative reasoning because none of the numbers was explicitly linked to a length or area quantity.

Three other students used arithmetic quantitative reasoning based on quotative division. One of these students constructed the representation shown in Figure 4 and gave the following ambiguous explanation:

One could rephrase this question and ask how many times will 2/3's go into 5/7's. One can see by looking at the drawing that in 5/7's there are 21 possible areas (we think the student meant that there are 21 sub-rectangles in the unit square). Take 2/3's of that and you have 14 possible areas. When trying to put 2/3's into 5/7's you could only 15 shaded 14ths [sic] will fit in the drawing shown below (we think the student meant that 15 of the 14 sub-rectangles in 2/3^{rds} fit into 5/7). This would make your answer: 15/14.

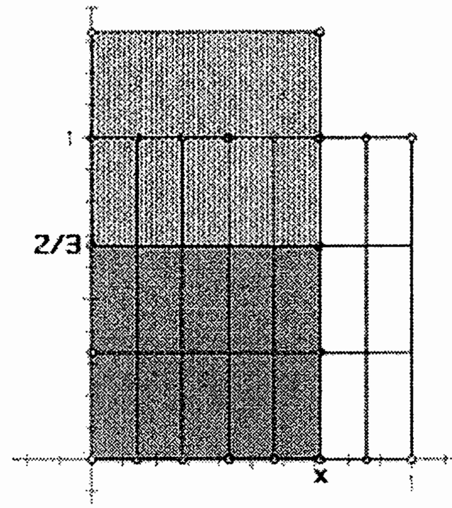

Figure 4. One student's arithmetic quantitative soluction.

Thus, this student focused on the 21 subdivisions of the unit square and then compared the 14 subdivisions of 2/3cm^2 to the 15 subdivisions of 5/7 cm^2. Just how the student understood connections between the number of subdivisions in the two rectangles and her final answer of 15/14ths remained unclear, but she may have provided an a posteriori explanation using her concept of quotitive division for the answer she knew must be right. The second student in this category also used her concept of quotitive division appropriately when finding how many times 2/3^{rds} would go into 5/7^{ths}, but the third student did not reorganize his whole number concept of quotitive division in situations involving fractions. In fact, there was strong indication in other parts of the course that he had not constructed the concept of an unknown.

Algebraic Reasoning

Of the students who used algebraic reasoning, three gave solid, correct explanations in which all quantities were linear, as in Figure 1, but not in the case where area was involved, as in Figure 3. Thus they appeared not to have reconstructed their algebraic thinking in contexts that used both length and area quantities. Four other students compared a common subdivision of 2/3^{rds} and 5/7^{ths} explicitly linked to area. The difference between these four students and the student whose work is shown in Figure 4 was that they began by constructing the rectangle of area (2/3)x cm^2, understood that (2/3)x referred to the area of a representative of all rectangles that could be made by moving x along the x-axis, and asked what length x would give the same area as 5/7 cm^2. Thus, these students made clear distinctions between length and area units. These students solved the task of Figure 3 by connecting the four pairs of numbered sub-rectangles as shown in Figure 5. Thus, they matched each of four sub-rectangles in the top row of 5/7 cm^2 to four sub-rectangles in 2/3^{rds} cm^2. Because there was a fifth sub-rectangle in the top row of 5/7 cm^2, the students knew at this point that the value of x was greater than 1 cm. The students used the fifth shaded subdivision in the top row to determine that that x was 1/14th cm more than 1 cm, or 15/14ths cm. Thus (2/3) cm. • (15/14) cm. = 5/7 cm^2.

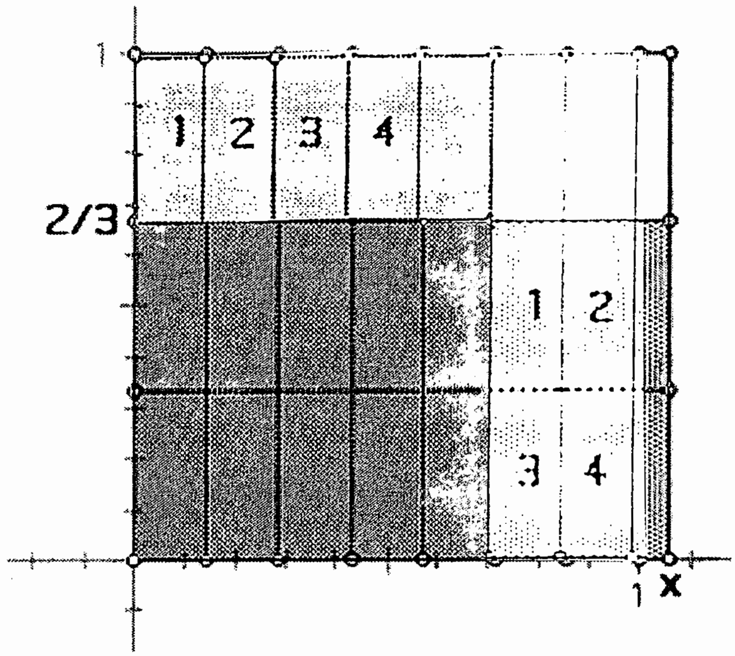

Figure 5. Four students' algebraic quantitative solution [numbers on subdivisions added].

Discussion

Although students' difficulty understanding division of fractions is legendary, only rarely has this difficulty been connected to constraints on subsequent paths by which students can construct solutions to linear equations. If emphasis is placed on students constructing solutions to linear equations, careful attention needs to be given to the kind of mathematical thinking that is involved in such constructive activity. In particular, solving Equation (1) through quantitative reasoning requires operating on unknown quantities. Thompson's (1994) view of quantity involves a unit to measure a property of some object. In the case of lengths, constructing a quantitative unknown entails generating an image of a segment and a posited unit length with which to measure the segment. Positing such measurement assumes that the student has constructed a sequence of such unit lengths that can be used to partition the segment, and that the unit length can be iterated enough times to make a new segment of equal length to the original segment (Steffe, 2002). Such reasoning is fundamental to the solution shown in Figure 1 and Figure 5.

The concept of quantitative unknown helps us explain why the distinction that four students made between linear and area units in representing the product of two fractions was so significant. We hypothesize that these students were able to construct the quotient of two fractions using area situations because they could consider x as a quantitative unknown; that is, they could consider x to be the result of measuring a segment in terms of a unit length. These students could then posit the product of 2/3 and x as the product of two lengths and ask when this product was 5/7 cm^2. Thus these students could operate on the result of measuring x in terms of a unit length without actually carrying out that measurement.

The equations and sample solutions discussed above illustrate that about one third (five) of the pre-service middle-school teachers in our teaching experiment had developed relatively deep understanding of Equation (1) in the context of length and area quantities by the end of our semester-long course. At the same time, about one quarter (four) of our students did not exhibit such abilities on the final exam. The remaining students (seven) fell in the middle—they could construct solutions to Equation (1) using situations that contained just length quantities, but they could not yet do so using situations that contained both length and area quantities. This observation indicates that approximately $5/12^{\text{ths}}$ of students had learned to operate as if they had constructed the quantitative operations necessary for producing he quantity symbolized by x in the equation $(2/3)x = 5/7$. But they didn't seem to understand why operating in the particular they did operate was a matter of logical necessity. That is, they didn't seem to be able to reflect on the way they operated and take it as input for analysis.

The finding that some teachers' understanding of fractions, and division of fractions in particular, is superficial is consistent with other findings like those reported by Ma (1999) for elementary-school teachers. We argue that teachers' difficulties with fractions constrain not only their teaching of arithmetic topics, but also the paths by which they can support their students' learning of core algebraic topics, including solutions to linear equations.

The teaching experiment reported in this study contributes centrally to the goals of PME-NA by shedding light on the opportunities and challenges of using quantitative problem situations to support pre-service middle-school teachers as they develop deep understandings of algebraic concepts that they will be asked to teach as standards-based instructional materials make their way in to middle-school classrooms. By deep understanding of algebraic concepts, we mean the ability to operate on known and unknown quantities to solve equations without resorting to the application of a numeric rule, such as "invert and multiply." The teaching activities that we used in our teaching experiment were designed not only for use by us with our pre-service teachers but also, potentially, for our pre-service teachers to use with their own middle-school students in the future.

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A PRELIMINARY LOOK AT THE DEVELOPMENT OF BEGINNING MATHEMATICS TEACHERS ACROSS MULTIPLE COMMUNITIES

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An instance where the internship of a secondary school mathematics teacher education student was roughly aligned with her methods coursework and her own beliefs about mathematics teaching provided the context for this analysis of the developing identity of a beginning mathematics teacher. The analysis draws on both situativity theories and established ideas about what is important for teachers to learn by combining Wenger's conception of identity development as learning within communities of practice with Shulman's heuristic of important teacher knowledge in a unit of analysis that we call mathematics teacher identity. The case study points to positive effects from involving prospective teachers in multiple communities where individuals share a common goal of reforming instruction. Specifically, such interaction can nurture aspects of the prospective teachers' mathematics teacher identities that are oriented toward reform, and thus assist them in developing into capable, confident, and committed reformers.

Introduction

Learning to teach involves the coalescing of an individual's history of interacting in a variety of communities playing other roles—student in the classrooms of one's own schooling, student in academic teacher education communities, and intern in another teacher's classroom—into a new role as professional teacher in its various attendant contexts (Bohl & Van Zoest, 2001). How teacher educators can best promote the development of reform-oriented teachers given these multiple influential factors remains an unsolved puzzle.

Much is known about the negative influences of interning in environments that compel interns to conform to a non-reform status quo (Cohen, 1991; Frykholm, 1995; Graham, 1997; Zeichner & Liston, 1987). Until quite recently, this negative impact of status quo contexts was exacerbated by the dearth of reform-oriented teachers working in reform-supportive schools where interns could be placed. The advent of reform curricula and professional development to implement those curricula has benefited the situation. Recent evidence suggests that professional development that supports curricular and policy changes results in classroom modifications leading to increased student achievement (Cohen & Hill, 2000). This study investigates the developing identity of a beginning mathematics teacher who participated in several reform-supportive communities.

Theoretical Framework

Our theoretical framework draws on situativity theories and on established ideas about knowledge that is important for teachers to learn. In particular, we combine Wenger's (1998) conception of identity development as learning within communities of practice with Shulman's (1987) heuristic of important teacher knowledge in a unit of analysis that we call *mathematics teacher identity* (Bohl & Van Zoest, in press). This model of identity combines (a) the knowledge in a person's head, (b) the communities in which individuals participate and the perceptions of people in those communities, and (c) the beliefs, commitments, and intentions that are stretched between the individual and those communities. *Self-in-mind* is a combination of one's knowledge and one's beliefs/commitments/intentions. *Self-in-community* is a combination of one's beliefs/commitments/intentions and one's perceptions of him- or herself in participation (see Figure 1). We use the term knowledge to refer to ideas that are more or less universally socially accepted and thus not open for public debate. We refer to ideas that are not universally held and thus are subject to public debate as beliefs. Commitments and intentions encompass one's desires to either act or not in response to particular situations, as well as impetus and justifications for doing so. We take learning to be the development of self-in-mind and self-in-community, and we focus on a condensed version of Shulman's (1987) list of important teacher knowledge: content/curriculum (what is to be taught), pedagogical (who is to be taught and how they should be taught), and professional participation (how to interact productively in the communities outside of the classroom that are concerned with teaching). Our distinction between, and subsequent combination of, the in-the-brain and social factors of learning allows the study of a broader swath of the cognitive-social continuum of personal development than is normally considered in research on teachers.

Methods

This is a case study of "Alice," a student in a reform-oriented secondary school mathematics education program who interned at a school using the Core-Plus Mathematics Project (CPMP) reform curricula. The site was chosen because it provided the best available opportunity to study an instance where the internship of a secondary school mathematics teacher education student was roughly aligned with her methods coursework and her own beliefs about mathematics teaching. In particular, the faculty at the school had developed a sense of professional community where the atmosphere was generally one of communal support for professional growth and curricular reform.

Data for this study were collected prior to, throughout, and following the internship studied. They include: interviews during the internship with Alice (II1-II6), her mentor (MI1, MI2) and Alice and her mentor together (JI1-JI4); planning sessions between Alice and her mentor (P1-P8); classroom observations followed by interviews (CI1-CI4); Alice's journal entries (J1-J25); and her audiotaped reflections (R)

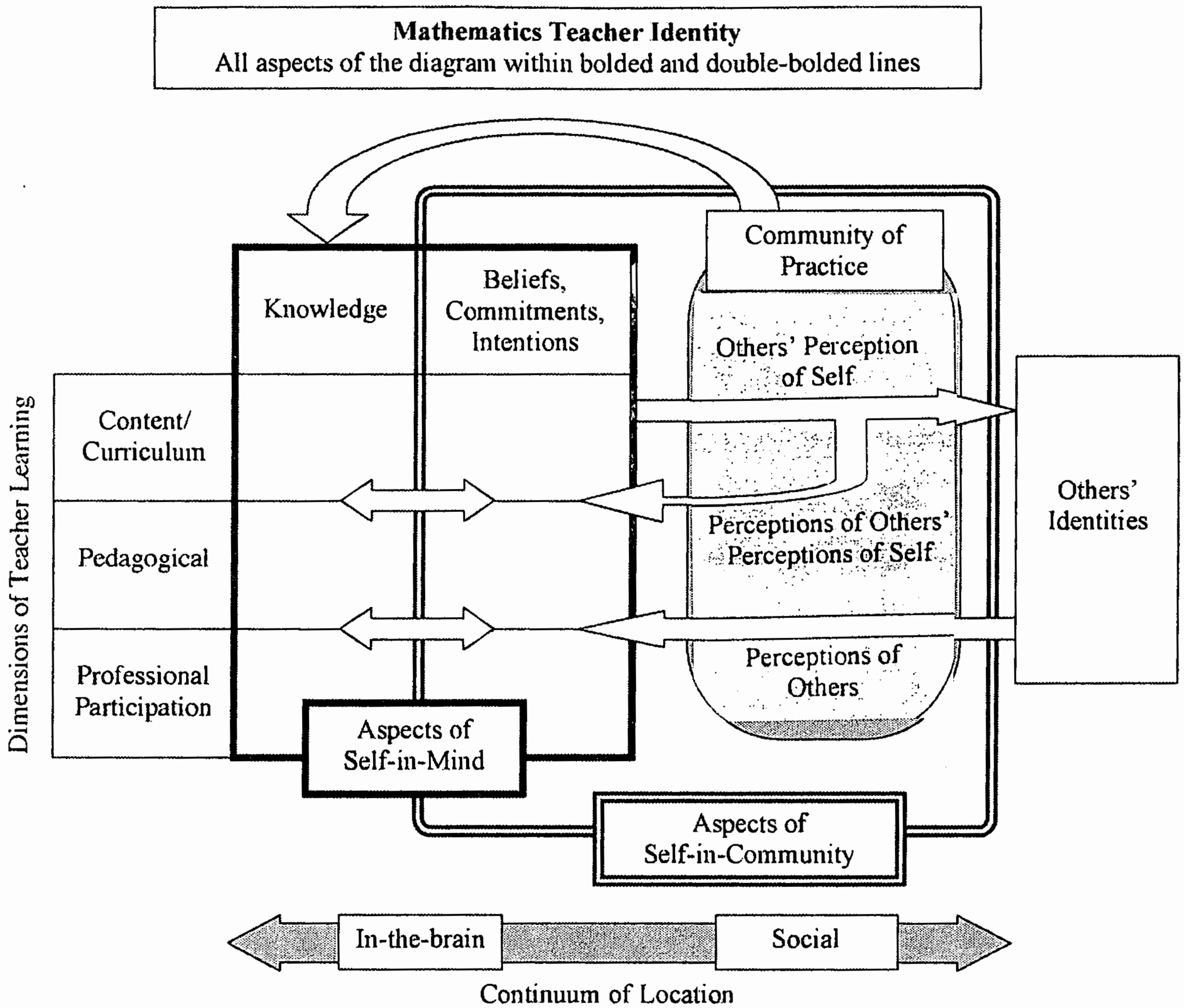

Figure 1. Identity as combination of aspects of self-in-mind and aspects of self-in-community (Bohl & Van Zoest, in press).

and an interview (PI) during her first teaching position. All data except the videotaped observations were transcribed and entered into a qualitative data analysis computer program (*NUD•IST*). The data was coded using indicators (Brown & Dowling, 1998) that mapped to the key components of the Mathematics Teacher Identity framework. The indicators were developed as the data was analyzed and were refined to reflect emerging issues and relationships.

Evidence

Alice began her internship with a sense of excitement and idealism that was quickly tempered by the realities of the classroom:

> I was able to help a few students today when they were doing group work. It felt really good but I can tell that I am not really secure with myself yet. I

> feel very unsure of my abilities and of the course material. Yet I know that I have a good foundation of knowledge for both. Somehow I always find myself wanting to check the answer book first before helping a student. I don't think I am ready to handle letting things 'flow where they want.' I want an open classroom with new alternatives but I still need to know where the world is headed. It's like hanging onto a thread of control as fiercely as I can. (J1, 62-78)

This quote encompasses many of the arenas in which Alice developed as a mathematics teacher and captures some of the tension she experienced among her knowledge; beliefs, commitments, and intentions; and perceptions. From the beginning she had a vision of herself as a reform mathematics teacher. However, throughout the internship she struggled with what that meant and how to make it a reality. In the following, we provide a glimpse into the development of Alice's identity along the three dimensions of teacher learning through her participation in several pro-reform mathematics education communities.

Content/Curricular Dimension

Alice characterized herself as an "ideal student" in secondary school who took advanced classes. She had been particularly successful in mathematics. However, she had a revelation during her undergraduate work when she realized that although she had always earned good grades in mathematics courses, she actually had developed very little deep understanding of the content of those courses. She had, rather, learned processes for arriving at correct answers without understanding the mathematics behind the processes. At that point she began to commit to reform in mathematics education, and made it a personal goal to teach her own students so as to enhance their understanding as well as their performance. This aspect of her self-in-mind dovetailed with her self-in-community as this commitment was supported by her three mathematics methods courses in at least two ways: 1) they gave her the opportunity to experience for herself learning mathematics for understanding, and 2) they provided information about curricular materials designed to develop student understanding.

During Alice's internship, the use of the CPMP materials and her mentor's and school's support of teaching for understanding reinforced her experiences in her coursework. Despite this overall resonance, however, some incoherence appeared in the details of implementing reform. We concentrate here on the way in which Alice's learning of mathematics during her internship created some dissonance between her self-in-mind (knowledge and beliefs/commitments/intentions) concerning the learning of mathematics and her self-in-community (those same beliefs/commitments/intentions and her and others' perceptions) regarding the same concerns.

At the beginning of the study, Alice was asked to describe the "perfect internship." She said that it should be:

> Not just about learning how to be a good teacher, but [the interns] learning to make the connections in the curriculum that they've never made before. That they're still learning, so that they get a sense that they're always learning about the material. Because I know that I don't go in pretending that I know everything about math. (II1, 304-310)

There were many examples of how her internship, and particularly the use of the CPMP curriculum, created the context for Alice to continue to learn more mathematics (Van Zoest & Bohl, in press). The newness of some of the topics meant that Alice and her mentor were together learning and clarifying the mathematical content for themselves. More often, though, it was Alice who was insecure in her content knowledge. In these cases, her mentor served as her teacher. The following excerpts, the first from an investigation dealing with percentiles and the second with developing mathematical models for scheduling, give a flavor of these interchanges:

> I: Now when you say like 40 percent or 40th percentile, cause I get a little bit confused and I just have to go through it. Is it correct to say that you are taller [pause]
>
> M: than 40 percent of the students. And 60 percent of the students are taller than you. Now that can be a little bit misleading because it probably should be 40 percent are as tall or shorter than you, [pause] and [it could be] 60 percent are as tall or taller than you. (P1, 211-220)
>
> ***
>
> M: See the earliest I can start task B is 3 days, the latest I can start it is 6 days, so that I have it done. Now this one, the earliest and latest are one and the same. But this task here I've got a choice.
>
> I: Right.
>
> M: Because I really have 3 days to play with that I can hold off starting this, because it's only going to take 6 days.
>
> I: So that [pause] okay, so you're earliest start time would be 3. Your slack time is 3. And the latest start time is 6. Okay.
>
> M: In a critical path you don't have any choices. (P2, 1906–1921)

In most instances in which Alice's mentor was helping her to clarify her mathematical understanding his approach was more directive than facilitative. His lack of use of questions in these interactions was in direct contrast to the way in which he and Alice approached their planning and their classroom teaching (Van Zoest & Bohl, in press). As a result, the way in which Alice learned mathematics during her internship was more reminiscent of her own experiences as a student in mathematics classrooms than the investigative experiences that she intended for her students to have or that she had experienced in her mathematics methods courses.

Confounding her mentor's directive approach to teaching her was Alice's resistance to engaging with the mathematics in favor of focusing on issues of teaching the mathematics. Although she desired to help students understand mathematics better by engaging them with it, she apparently did not see a need to engage with it herself. In contrast, she perceived that her mentor was interested in the mathematics. Alice reflected on the differences between them and the confusion it was causing her:

> I think that we may end up teaching a lot alike but for different reasons. He looks more into the math part of things while I seem to be more concerned with the "how to learn this math" and "how to manage" kind of things. I just don't know anymore; I feel so confused. (J13, 208-213)

This difference particularly frustrated her when she perceived it leading to overemphasis on the mathematics during their planning sessions:

> . . . he was more concerned with, "well does this power model fit it [better] with $1/x^2$, or is the $1/x^3$ [better]?" To me, I wasn't concerned about which one was actually better, I just wanted [the students] to be doing it. So it's hard because I get impatient because I really want to know how to teach it. (II4, 496-501)

The researcher's perception was that Alice and her mentor had similar goals, but that the mentor was not successful in making his goals clear to Alice and Alice was not experienced enough to understand that focusing on the mathematics can lead to understanding how to teach it. There is no evidence that Alice and her mentor discussed their difference in approaches to the mathematics or the problems it caused her. This tension between Alice's self-in-community (the mismatch between her and her mentor's perceptions of the role of teachers' engagement with mathematics) and self-in-mind (the dissonance between her beliefs/commitments/intentions regarding her students' learning of mathematics and that of her own) consumed a large amount of Alice's energy and, in particular, many of her journal entries.

Pedagogical Dimension

Two examples of Alice's development along the pedagogical dimension stand out from the analysis. The first involves her struggle to operationalize her perception of teacher as facilitator:

> I just don't know how to become that wonderful facilitator, yet give them enough direction so that every kid can see what steps are what so that they can't say, 'Well you didn't write that down. You didn't go over that in class.' That type of a thing. I don't know. I just don't know what it is that I need to fix. (CI2, 127-133)

One approach that Alice and her mentor tried was to break classroom investigations into short series of problems. They structured their classes so that the students worked in small groups on one or two problems for a fairly short period of time (generally less than ten minutes), and then participated in a whole-class discussion about their solutions. Observations documented that doing this minimized the investigative nature of the curriculum by focusing students on completing the problems instead of making connections among the mathematical ideas within them. Alice was uncomfortable with results of this "experiment" and listed it as one of the things she wanted to modify: "Like with the stopping and starting. I need to figure out some way that they will work in groups and get the work done" (CI3,196-197).

In an example of negotiating their joint practice, Alice and her mentor re-evaluated their approach and shifted to allowing the students more time to investigate. Alice considered this a watershed moment for her and described what happened:

> When we let them work for 20 minutes or so and then brought them back and go through things, things went so much better. They were so much more attentive and they seemed to have better questions to ask. Instead of when we broke it up into 'here's one problem, here's another problem.' That just doesn't work well. (PI, 998-1004)

The second example of Alice's development along the pedagogical dimension involves the more specific issue of using questioning as a pedagogical technique. Alice had learned about the importance of questioning to reform teaching in her methods courses. This resonated with her own experiences, thus she incorporated it into her self-in-mind. However, despite her commitment to teaching using questions, she found it difficult:

> [T]he thing that I am unsure of is learning how to draw what you want them to understand, out of them, through questions. Because it's so easy to be directive with them, but I want them to come up with the answers. (P1, 585-589)

In her journals Alice made many references indicating a conscious effort to improve her questioning. Her efforts were supported by her mentor who, despite his 31 years prior experience, refers to himself as being in the same place as she (i.e., as a learner) relative to reform teaching:

> I think the main thing that we both struggle with is we still haven't gotten to the point where we are letting the students do enough. We're still doing too much directed teaching and not enough of getting them totally engaged in the discovery process and coming up with; but I think, again, that's just going to come with time. And practice. And again, a lot of it is just learning with time and experience those questions to ask that are going to solicit that response that we need to get from them. It's so much easier to just tell them. And the problem with that obviously is generally speaking they don't learn it from being told. (P3, 456-467)

The recorded conversations of Alice and her mentor are filled with references to their shared commitment to questioning as a teaching technique and their efforts to improve their questioning. This congruence between Alice's self-in-mind and self-in-community both encouraged her to continue to improve and comforted her when she wasn't as successful as she had hoped. It fortified her personal commitment and allowed her to "know" she was capable of improving. Toward the end of the experience Alice noted that she was feeling more at ease "being able to think while I'm up there," specifically as related to formulating productive questions in response to students' questions and their answers to questions (P7, 426-426). Her mentor's evaluations and our observations of her classroom corroborate her perceptions.

Professional Participation Dimension

A key goal of Alice's final of three methods courses was to introduce the prospective teachers to participation in a community of mathematics teachers. The class itself was designed to function as such a community with the students working together as colleagues on projects related to the teaching and learning of mathematics for understanding. Beyond that, they were required to attend a conference on mathematics education and opportunities were provided for them to interact with experienced teachers who were in the process of reforming their mathematics teaching. Alice excelled in this course and incorporated many of the ideas about reform mathematics teaching into her self-in-mind.

By design, Alice's intern teaching placement reinforced the experiences that she'd had in her methods courses. She was able to see things that she had learned about in other communities—such as the CPMP curriculum and a focus on understanding—in action. She described one of the unintended consequences of this as she reflected back on her internship:

> I kind of look back on [intern teaching] and say, wow, those teachers were really together and they were very progressive and they wanted a lot of changes. I mean, they really, really liked Core-Plus mathematics. I don't think I realized how, umm, how great of a situation I was, I just never knew it could be anything different. I mean, that was the way it was for me and for some naive reason I thought that it was that way everywhere—that all teachers wanted, you know, all these great reforms and new math and things like that. (J26, 16-26)

Alice's first teaching position, as a semester-long substitute in a middle school, made it clear that that was not the case. At her new school, a senior colleague was a vocal challenger to reform. He opposed changing what he perceived to be successful traditional approaches to mathematics instruction. In particular, he admonished Alice to stick with the skill worksheets that her predecessor had used and not to try anything new. After she mentioned to him that she had intern taught in a CPMP classroom, she quoted him as approaching her the next day to say, "Well, I just want to let you know that I don't think that you should do any of that fancy stuff because parents might question your ability and your background. And these kids really need structure" (PI, 131-134). When this teacher disparaged reform mathematics curricula in the teacher's lounge in Alice's presence, the other teachers seemed to support his views. This made her uncomfortable because, although she didn't agree, she didn't want to antagonize her new colleagues. It also confused her because she observed that the other departments were very much involved with improving their teaching in ways that paralleled the mathematics reform movement. It seemed to her that the mathematics department didn't fit with the rest of the school—that somehow people believed that math was different from other subjects and kids weren't able to engage with it in the same ways.

During this time Alice maintained her connection with her internship school as well as with some of her colleagues from the university. Her district was also part of a large National Science Foundation-funded Local Systemic Change Project. This allowed her the opportunity to interact with teachers from other schools in her district who were using reform curricula and who wanted to see changes made in her middle school. This encouragement, fortified by the support of one administrator in her school, served to counter the discouragement she received from her middle school colleagues. With the backing of this broader community of practice, she worked hard to create the kinds of experiences for her students that a reform curriculum would have provided. Alice attributed her willingness to undertake this type of reform to the knowledge and confidence she had gained during her internship. Specifically, she felt that because of her experiences her "opinions [about reform mathematics instruction] are worth something" (II6, 1525-1526)—an example of the strength of her self-in-community regarding reform teaching. She held her experiences with the CPMP curriculum as exemplars of good reform teaching and learning, and utilized them as a baseline for comparison with the curriculum she was developing.

Alice's mentor predicted that she would have this kind of influence and attributed it to the environment in which she had interned:

> Over all it was just an awesome experience. Because Alice is ready to deal with change. She did learn change. She doesn't know that but she did learn what change is all about this year. I think she saw a lot of change. She will go on and do wonderful things. (MI2, 783-787)

In the years since, Alice has continued to play the role of mathematics education reformer at her middle school. Partly as a result of her considerable effort to align the middle school's curriculum with its high school's CPMP sequence, her school is now piloting the Connected Mathematics Project (CMP) materials. These materials are also aligned with the NCTM Standards, and Alice played a key role in convincing her administration that CMP would serve its students well.

Discussion

Here we will use the mathematics teacher identity framework (Bohl & Van Zoest, in press) to interpret the evidence collected about Alice's development across the dimensions of teacher knowledge as she participated in multiple communities in route to becoming a teacher.

Education students and intern teachers are peripheral (Lave & Wenger, 1991) members of the education community. That is, they are not party to the entire practice of teaching. Moving developing teachers toward the center of the teaching practice is the primary justification for the internship. Central to any practice is the very act of defining, along with other central participants, the practice itself. This "joint negotiation of practice" (Wenger, 1998, p. 84) is the process by which those involved determine how to best do the job at hand. In this case, Alice was given ample opportunity to learn about and participate in this process across multiple communities. Her university course work made her aware of the new and changing theories about how mathematics should be taught. In her contact with the local reform community, she was exposed to teachers who actively identified themselves with the role of negotiating the job of teaching mathematics. Further, her mentor identified openly not only as a learner of his practice, but also as an active participant in figuring out how to do his long-held job better. In the episode where Alice and her mentor experimented with the length of group work during classroom investigations, she had the opportunity to figure out for herself how to best implement in an actual classroom community the theories she had taken on as part of her self-in-mind. This exposure to the negotiating of what it means to teach mathematics from early on in her education seemed to help her understand both change and learning as natural parts of effective teaching. This combination of practice and community support also seems to have allowed Alice to build her knowledge base regarding what works and what doesn't, and to substantiate her beliefs, commitments, and intentions regarding her identity as a reformer.

Another central aspect of a community of practice is its "regime of mutual accountability" (Wenger, 1998, p. 81). This is the set of goals and standards (both written and unwritten) by which people within a practice judge their own and others' performances with a community. Alice realized that she had not really understood much of the mathematics at which she had succeeded, and decided that she wanted to teach for understanding. In this realization she found perhaps the first and most powerful common ground between her own beliefs, commitments, and intentions and those of the reform mathematics community of which she was just then becoming aware. From that point on, she was exposed throughout several communities to other standards, beliefs, and goals related to reform mathematics, and numerous individuals who were striving to teach within those parameters. Included among these was her mentor, who allowed her to participate with him (although not always successfully) in a joint effort to live within the reform regime of accountability. Their commitment to that regime is indicated by their responses to their failures, which was to critique their own performances rather than to question the overriding goals of reform.

Moving from a focus on the effects of the broader communities to a more specific focus on Alice's participation with her mentor highlights some tension in Alice's development worth examining. One of the primary aspects of mathematics teacher identity is one's perceptions of oneself and of others' perceptions within the community. Generally, in cases where Alice perceived that her and her mentor's goals were aligned, she made fairly smooth and steady progress towards increasing her knowledge base, and her beliefs, commitments, and intentions were maintained, and perhaps bolstered. By contrast, in instances where Alice perceived some dissonance, she was forced to rethink her beliefs, commitments, and intentions. As an example, in the case of substantiating the role of teacher as facilitator, Alice understood that both she and her mentor agreed on its importance. Thus, Alice was able to both resolve (in the immediate sense) the dissonance caused by the failure of class discussion to go as planned, and to contribute to her knowledge base by experimenting with different approaches and monitoring their success. On the other hand, regarding the importance of learning mathematics, Alice perceived that she and her mentor were working at cross-purposes. In this case, the development of both components of her self-in-mind seemed to be limited, as she never connected the importance of her students engaging with the mathematics with doing the mathematics herself. The main difference between these two examples was Alice's perceptions of her mentor's perception of the situation. Although from the researcher's vantage the mentor and intern's goals and commitments in both examples had the potential to be compatible, the intern perceived resonance in the first instance and dissonance in the second. The impact of this difference in perception on her ability to develop highlights the influence a community of practice can have on one's self-in-mind. It also points to the importance of attending to perceptions during intern teaching and reconciling them when they might interfere with the intern's ability to learn.

In the end, although Alice would not realize it until later, the dovetailing of her mathematics teacher identity self-in-mind with the regime of accountability of reformers, and her ongoing exposure to others committed to it, had a major impact in terms of solidifying her commitment to reform. Alice's ability to act on her reform-oriented beliefs and intentions despite a local community focused on maintaining the status quo attests to this. The strength of Alice's developing self-in-mind combined with her increasing identification with, and active participation in, broader reform-oriented communities of practice allowed her to become an agent of change in a situation where it would have been much easier for a new teacher to simply toe the line despite her own beliefs. This contrasts with the common and well-documented responses of interns either modifying their own selves-in-mind or simply accepting discontinuities between their selves-in-mind and selves-in community.

Conclusions

Through the combination of opportunities to become involved in the negotiation of the practice and early and on-going participation with others who subscribed to a similar set of beliefs, commitments and goals, Alice enhanced her mathematics teacher identity as a reformer. This included strengthening her content and pedagogical knowledge, fortifying her beliefs and commitments concerning reformed teaching, and developing a perception of herself as a potentially successful reformer within the community of mathematics teachers. Reform-oriented mathematics teacher identities are fundamental to sustained improvement of mathematics education. Although research has demonstrated that a reform-based teacher preparation program is not sufficient to generate this type of identity in beginning teachers (e.g., Frykholm, 1995; Parmalee, 1992), it does appear that aligning the internship with reform-based university coursework, in combination with engaging the intern in a broader reform community, has the potential to do so. This study highlights the importance of involving prospective teachers in communities that share a common goal of reforming and of investing resources in developing such communities. When this happens, aspects of prospective teachers' mathematics teacher identities that are pre-disposed to reform are nurtured and can influence their future actions as teachers and supporters of reform. In particular, teaching competence, commitment to teaching for understanding, and confidence as a reformer can all be enhanced.

Note

[1]The transcript codes are (Data Source, Line Numbers).

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USING RESEARCH ON CHILDREN'S COGNITION TO ENHANCE PRESERVICE TEACHERS' KNOWLEDGE OF DIVISION

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In this study, I examined the knowledge of prospective elementary teachers who participated in a mathematics course where considerations of content were driven by examples of the development of children's thinking. Using resources derived from the Cognitively Guided Instruction project, students in the course examined the mathematical ideas inherent to children's thinking. My analysis of interviews conducted at the end of the course revealed that my students not only tended to emphasize conceptual ideas rather than just procedures, they also placed a high premium on developing this kind of understanding in their future pupils. After a brief discussion of the context of the course, I will present some of the data that illuminate the strengths and limitations of my students' knowledge.

Background

In the past decade, there has been widespread concern about the subject matter preparation of teachers of mathematics. Much of the research on teachers' knowledge has focused on their level of knowledge in various mathematical domains and the influence their knowledge has on their teaching practice. It has become evident that while teachers may be proficient at performing mathematical tasks, many do not possess an understanding of the conceptual aspects of the content that they are teaching, even at the primary grades (Ball, 1991). There is evidence to show that teachers with less developed subject matter knowledge tend to focus primarily on mathematical procedures rather than the concepts underlying them. Without a strong understanding of the mathematics, such teachers were not able to adequately respond to children's questions or novel ideas (Leinhardt & Smith, 1985; Stein, Baxtor & Leinhardt, 1990). Little research has been conducted to examine how to help teachers strengthen their mathematical knowledge.

At the university level, teacher educators are faced with the challenge to provide programs for future teachers to help them develop the mathematical knowledge that they will need. Recently, there have been recommendations that teacher education programs actively engage preservice teachers in the mathematics of the elementary curriculum (Conference Board of the Mathematical Sciences, 2000; National Research Council, 2001). Traditionally, it has been thought that the content of the elementary curriculum is easy and requires little attention at the university level. Programs for future teachers emphasized advanced mathematics coursework in order to deepen their subject matter understanding. It has been found, however, that many students who had advanced coursework at the university level still had difficulty explaining the conceptual aspects of the fundamental ideas of elementary mathematics (Ball, 1991). As a consequence, many teachers entered their practice not having examined the material they are to teach since they themselves first learned it. Programs that do not explicitly examine the mathematics of the primary grades risk sending out teachers who are ill prepared to meet the demands of current reform agendas.

Subject Matter Knowledge for Teaching

It has become recognized that the knowledge base required for teaching must be conceptualized in a much different way than it is in other mathematical professions (Shulman, 1987). Rather than compacting ideas together to facilitate their use in other endeavors, a teacher must be able to unpack ideas in such a way as to facilitate learning (Ball & Bass, 2000). Teachers must be able to examine the fundamental components of a mathematical idea and make connections to other mathematical ideas and the trajectory of the curriculum (Ma, 1999). Typically, math courses do not engage students in this kind of activity. Courses for prospective elementary teachers need to help teachers to develop a flexible conceptual understanding of the mathematics they are to teach.

While it seems that a conceptual understanding of the mathematics is a key element of a teacher's knowledge base, it alone is not sufficient. In order for their knowledge to be useful for teaching, a teachers' mathematical knowledge must be developed in tandem with their knowledge of learners and pedagogy. A teacher must use their mathematical knowledge to anticipate and evaluate students' ideas or misconceptions and be sensitive to the ways that children learn about mathematical ideas. To be relevant, the mathematics that future teachers engage in should be situated within the context in which it will be used Brown, Collins, & Duguid, 1989).

Description of the Project

This paper will highlight a portion of a study that sought to examine the impact on teachers' knowledge of using research about children's thinking as a major component in a mathematics content course for prospective elementary teachers. The subject area that I will focus on in this study is division. Division serves as an important bridge to the development of rational number understanding. It has been observed, however, that prospective teachers come to their teacher education programs with a fragmented understanding of division (Ball, 1990).

This course was the first of three content courses required for elementary education majors at a large Midwestern university. There were 36 students in the two sections of this course taught by the author. During the units on number operations, I engaged the students in video and written material developed from the Cognitively Guided Instruction (CGI) project (Carpenter, et al., 1999). As this was a content course and not a methods course, I directed the students attention primarily to the mathematical ideas inherent to children's thinking. Much of the students' time with division was spent examining the differences between partitive and measurement division situations and their influences on children's strategies. The students also spent time examining the mathematics behind children's invented strategies for solving division problems.

Data Collection

Throughout the semester, I kept copies of all the students' written work as well as my own field notes and journal entries. I consistently reviewed these materials as the semester progressed. Near the end of the semester, individual clinical interviews were conducted in order to obtain a picture of their mathematical knowledge. Due to scheduling problems and technical difficulties, only 27 of the students in the course were interviewed. The questions for the interview were adapted from those developed by Deborah Ball (1990). For this paper, I will only discuss the results from the questions involving the division concept. The students were first asked to tell how they would respond to a child's question about what division by zero is. Second, they were asked to solve and provide a representation for a division problem involving fractions. Each student's responses were probed to obtain a picture of their conceptual understanding of the subject. As the problem situations were in a teaching context, I also probed to ascertain their thoughts about teaching these ideas.

Results

After several rounds of analysis, a three level scale was developed into which each student was placed. This was done twice, once with regard to the students' conceptual understanding of the material, and once with regard to their perceptions of how these ideas might be taught.

Knowledge of Division by Zero

The students' responses to the hypothetical child's question about the answer to the problem seven divided by zero fell into one of three categories. The students were sorted into the levels accordingly if they were unable to provide an answer (level 1), created an accurate representation but were unable to correctly interpret it (level 2), or for accurately explaining the problem (level 3).

Level One

Students whose responses were at level one were either unable to provide an answer to the problem, or recalled the answer but were unable to provide an explanation for their answer. Of the 27 students interviewed, 10 (37%) were placed at this level. Only two of these students correctly identified division by zero as being undefined. When asked to explain, they could not offer a response. The other students relied on incorrectly remembered rules such as "anything divided by zero is zero".

Level Two

Students at level two were able to create a representation that correctly modeled division by zero, however when probed, they were unable to correctly interpret what the representation meant. Four students (15%) gave responses at this level. Two of the students provided a representation consistent with a partitive division model. In this model a set of size seven is to be placed equally into zero groups. Since there are no groups, it is impossible to say how much each group gets. When asked about their representation, both students became confused and reinterpreted the situation as being multiplication with zero groups of seven or seven groups of zero. The other two students provided measurement representations where one asks for the number of sets of size zero inside a set of size seven. Both students created representations where zero is being taken out of seven but mistakenly interpreted the answer to be zero because nothing is being taken away.

Level Three

The remaining 13 (48%) of the interviewed students were placed at level three for accurately explaining why division by zero is undefined. All but two of these students explained the problem through a word problem. Six of the students described a partitive division situation. While the situation that the offered was correct, few of these students were able to provide a precise interpretation of the situation and corresponding answer. Many of these students recognized that this representation might be confusing with the answer interpreted as being zero. Two students talked in detail about the difficulty. Both recognized that they needed to get across the idea that since there are no groups to partition into, it is impossible for the partitioning to take place in the first place. Five students offered measurement division models. The students offering this representation had less difficulty interpreting the answer. Two of these students thought about the problem in terms of collecting a number of zeros to make seven. The other three thought in terms of breaking up seven into groups of zero.

The last two students at level three did not rely on word problems to explain their solution. One student made use instead of the relationship division has with multiplication. She explained that you could not multiply any number by zero to come up with an answer of seven. The other student relied on thinking about division by zero as a limiting process, explaining that as you divide by smaller amounts, the quotient becomes larger.

Teaching Division by Zero

The students were also probed to discuss how they would handle the situation should one of their students ask such a question. As with their knowledge, the responses were placed into one of three levels. The students either emphasized telling rules (level 1), giving an explanation of the concepts (level 2), or engaging the child in the mathematical ideas (level 3).

Level One

At level one, the students felt that it would be the most productive to tell the student the (sometimes erroneously stated) rule. At issue for many of these students was their perception that young children would not be able to understand an explanation. While some students indicated that this idea might be able to be explained at a higher grades, they still felt that it would be best to not risk confusion. One student, Jenna, expressed:

> Then, when you get older you say it is undefined or something. If they were young enough, I would just say it was zero. I don't know, try not to confuse them. I don't even know what it is all about.

Similarly, Kirstin believed that young children would not be able to understand:

> I think at an age, like sixth grade age or elementary age and you're learning the easier addition, subtraction, multiplication and division, you don't even think to ask why. It is kind of just there for you to look at and believe. I don't think at an elementary age you are taught to ask why.

At issue for many of these students was their own inability to understand the content. As a result, these students felt it would be best to avoid the idea completely.

Level Two

Of the 27 students, 16 (59%) talked about wanting to try to explain the concept to the child in one way or another. These students were classified at level 2. In each case, the emphasized their desire to help the child make sense of the problem. None of these students indicated that they would try to engage to child to work on the problem on his or her own or with other children. There was some variability in the ways that the students indicated they would explain. Two of the students appealed to completely erroneous explanations, like misinterpreting the relationship between division and multiplication. While their explanations were incorrect, they appealed to more than just a remembered rule. Some of the students remarked that they would make use of counters or pictures to help the child visualize the division process. Others indicated that they would use a story problem to put their explanation in context. None of these students indicated that they would use the word problem as a device to encourage exploration on the part of the children.

Level Three

The remaining four students were placed at level three. Like the students at level two, these students talked about their desire to help the child make sense of the problem. Rather than just providing a model for the child, these students also indicated that they would engage the children into thinking about the problem on their own. A primary characteristic of their responses was their intention to probe the student for his or her own understanding. For example, Helen responded:

> I guess I would first ask them what they thought and then be like, present a different division problem and see if they could point out the similarities. And if they couldn't, see why they can't.

While these students indicated that they would try to help the child through questioning, their inexperience with children inhibited their ability to be specific about what they would ask or what kinds of responses they might anticipate.

Knowledge of Division of Fractions

The second interview question under consideration involved creating a representation for the division problem 1 3/4 divided by 1/2. While creating representations can be thought of as a pedagogical activity, it is also inherently mathematical. If one is to create a context to model a mathematical idea, one needs to have an understanding of how that mathematical idea fits together. In this instance, one must have an understanding of both division and rational number concepts. As with division by zero, the students' responses were placed into one of three levels based on whether they were unable to provide a correct representation (level 1), provided an inappropriate representation (level 2), or provided an accurate representation (level 3)..

Level One

Seven students (26%) were categorized at level one. These students were either unable to provide a representation or they provided a representation that did not model the division problem. Two students were unable to any word problem for the division problem. Of these, one was unable to provide an answer to the problem either. Many of the students mistakenly created a word problem for division by two rather than division by a half. None of the students who solved the problem correctly were concerned that the word problem would result in a different answer. One student, Barbara, was fixated on the invert an multiply rule and instead created a story for multiplication by 2. While this gives the same answer as the division problem, the action taking place is quite different. Barbara recognized this, however, and remarked:

> I don't see it as dividing by a half, I see it as multiplying by two. So I don't think that I can come up with a problem where dividing by a half makes sense, because I don't see this as dividing by a half.

One student, Tiffany, correctly conceived of division by a half as finding the number of halves in 1 3/4, but had difficulty with the rational numbers. Her problem was:

> If I need one and three fourths batches of something, and all I have in my cupboard is a half a cup, how many times would I need to fill the half cup to get to one and three fourths batches?

Tiffany confounded the units of the two fractions creating a word problem that cannot be solved.

Level Two

Responses classified as level two were accurate representations of the division situation, but the context used would suggest an answer other than 3 1/2. Three students

(11%) were placed at level two. The problems these students ran into stemmed from the use of units that are not typically thought of as being divisible. Consider Jason's problem: "Well, if we have 1 ¾ pizza and each person would like to each [have] ½ of a pizza. How many people can we serve?" In this case, the appropriate answer would just be 3. The two other students used money in the form of coins.

Level Three

The remaining 17 students (63%) were able to provide an accurate word problem for the division problem. Of these, five students provided a word problem that would fit into a partitive division model. For instance, Emily's problem was, "Betsy had one and three fourths of pizza. This was half of the total amount there was. How many pizzas were there?" When asked why she thought that this word problem was 1 3/4 divided by 1/2, Emily responded:

> If you look at it again with the items and groups thing, then you are going to label on and three fourths as your items and half as your groups. So your items are one and three fourths and that's making uo on group which is half of the total. You have half a group here. So what is the one total group?

One of the key benefits of using the partitive model of division is the ease with which one can connect the word problem to the invert and multiply procedure.

Fourteen of the students at level three gave a measurement division problem. For instance, Jackie's problem read. "Say you need one and three fourths cups of sugar for a recipe, and you only have half a cup, so then how many half cups will you need?" Most of the students used measuring devices in their word problems. This context was particular helpful because as a physical entity, a measuring device can represent both a whole and a fraction at the same time. For instance, a half-cup measuring device could be thought of as half of a one cup unit, or as a whole half cup unit. This ability to flexibly unitize has been found to be an important aspect of learning how to work with rational numbers (Behr, Harel, Post, Lesh, 1992).

Teaching Division by Fractions

During the interview, the students were asked about how they might teach the division by fractions concept and whether they thought that creating a context for the problem would be important. Three levels were identified very similar to the levels identified for teaching division by zero. The students either emphasized procedures and rules (level 1), emphasized providing clear explanations of the concepts (level 2), or emphasized facilitating exploration (level 3).

Level One

Six students (22%) gave responses at level one. All of these students indicated that they would prefer to focus on having the children become proficient with the dicision by fractions algorithm. While some of these students thought that word problems might be beneficial, most thought that they would get in the way. They were afraid that a child would not be able to extract the meaning from the wording and would end up making the problem too complex. For them, the worded setting provided extraneous information and distracted from what they saw as the core problem of calculating the answer. Moreover, it was felt that word problems were too time consuming, where classroom time could be better used focusing on repetition and worksheets.

The two students at level one who did see some benefit in using word problems did not do so necessarily to promote conceptual understanding. Instead, these students saw word problems as a way to capture children's attention and motivate the learning of computational procedures. William echoed this point when he stated:

> I remember that if I didn't see when I was going to use this stuff, I just tuned out. So maybe they would see the word problem and see how easy it is to do the problem just flipping the fraction.

Level Two

Fifteen of the students (56%) interviewed were placed at level two. While many of these students felt that children should learn the invert and multiply procedure, these students also expressed that it would be important to explain what it division by a half means. For these students, the word problems served as useful tools to support the teachers' role in providing explanations to the children. Of these fifteen students, four indicated that word problems alone might be problematic. As with students at level one, they feared that chjldren might get confused with the word problems. Unlike the students at level one, however, these students wanted to do more than just tell the students the invert and multiply rule. Many of these students were concerned that they themselves did not feel that they were given the chance to understand what the procedure meant and felt that they would need to still provide some kind of explanation. Instead of just word problems these students suggested using pictures, activities and demonstrations to illustrate the concepts. Those students who did think that using word problems would be a beneficial way to help children learn the concepts also echoed the sentiment that visualization is important. In each case, the student talked about themselves creating the pictures or models from the word problems.

Level Three

The remaining six students (22%) were placed at level three. These students echoed many of the sentiments espoused by students at level two. Additionally, these students indicated that a primary benefit of using word problems would be their ability to help children develop these conceptions on their own rather than directly from the teacher. Consider Diane's statement:

> I think it helps kids use their minds when they think of stuff on their own when they aren't just given a certain way of doing things. I don't think that

> teaching is just telling kids how to do that. I think it is helping kids understand why it is they do certain things. Getting them to figure out things on their own.

Similarly, Emily emphasized the importance of allowing children to construct their own meanings:

> I think there's a lot to be said, first of all for the creativity, and just why stifle a child's understanding and creativity if they've got a way to do it, if they've got a better way to understand it. It's not really fair that we are forcing something on them.

Both of these students felt that by giving word problems, they could allow children to explore the ideas in their own ways.

The other students emphasized the benefit that word problems would have in classroom discussion. They felt that letting children discuss their solutions to the problems would allow the class to become exposed to different ways of thinking about the problem. Many of the sentiments expressed by these students mimic the structure that I established in the classroom. One student made explicit reference to way that I organized the class around working in groups and sharing strategies with the entire class.

Discussion

In Deborah Ball's (1990) study of prospective teachers' knowledge of division, only one of ten prospective elementary teachers were able to provide a correct explanation for division by zero. None of these teachers were able to provide a correct representation for division by fractions. In Ball's study, the students predominantly focused on procedures and rules and felt that citing these rules sufficed for an adequate explanation. Liping Ma (1991) reported similar results with inservice teachers. This is a marked contrast to the results presented above. The students in this course appeared to have a much stronger grasp of the conceptual idea of division.

It should be noted that during the course, division by fractions was a topic of study that I had the students engage in. They had been asked to create representations for these problems before. Division by zero, however, was not discussed explicitly. Almost 50% of the students in the study were able to provide a correct explanation for why division by zero is undefined. For most of these students, they were applying their understanding of division models, something that was a topic of emphasis in the course.

While these students focused a great deal on conceptual aspects of the problems, it is clear that there are still gaps in their understandings. These teachers did not evidence the kind of flexible knowledge that Liping Ma (1999) observed in the Chinese teachers of her study. The students in my study generally only offered one strategy for examining the problem. Moreover, my students rarely made connections to other related mathematical ideas. For instance, though many of the students were able to use their knowledge of division models to guide their explanations, few of the students tried to connect the situation under consideration to multiplication or to division with whole numbers.

One of the central themes of the students stated approaches to teaching was a desire to help children understand what they are doing rather than doing mathematics by rote. While talking about this many of the teachers referred back to their own experiences regretting that they had never had a chance to understand the mathematics. While many of these students emphasized the importance of understanding the ideas behind the mathematical rules, few indicated that this kind of understanding is something that children could construct for themselves. For many of these teachers, their approaches toward teaching was still very teacher centered and directed. It is unclear, then, how long such a focus on conceptual mathematics would last when they are actually teaching. Moreover, there is evidence of a potential source of conflict in the teachers' stated beliefs. With regard to division of fractions, none of the students indicated how their word problem could be used to illuminate the invert and multiply procedure. While they developed an understanding of what division by fractions might mean, none of the students connected it to the procedure. As many of the students also stated that it would be important for children to know the procedure, it is unclear what they might emphasize when teaching this material.

Finally, this study was conducted primarily to try to ascertain the mathematical thinking of students who had experienced a non-traditional class where content and pedagogy were intertwined. Much more research needs to be conducted to ascertain the specific effects of a treatment such as this. Research also needs to be conducted illustrated the long tern effects of instructional programs such as this one.

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THE INFLUENCE OF CULTURE ON LOGICAL REASONING

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This study investigated the impact of culture on logical reasoning. It found evidence indicating that the cultural values and beliefs held by a group of prospective elementary school teachers influenced the way they analyzed and solved two logic problems.

Introduction

The work of scholars such as Bishop (1988, 1994), Cooney (1985), D'Ambrosio (1985, 1988) and Thompson (1984, 1992) made the mathematics community reconsider some long-held beliefs concerning the relationship between culture and mathematics. What emerged from these studies was a greater understanding of how cultural factors such as language and beliefs affect the practice and outcome of mathematics education. Evidence from these studies show, for example, a strong connection between teachers' beliefs and their pedagogical behavior. Moreover, mathematics educators were made more aware of subtle culture conflicts which some students experience in the classroom, and the effect of these conflicts on the students' participation in mathematics. The present study investigated links between cultural beliefs and logical reasoning among pre-service elementary school teachers.

Method

The subjects for the study were 39 elementary education majors including one ethnic minority student. The students were enrolled in the third and last mathematics course for their program. The other two mathematics courses treat the content of contemporary K – 8 mathematics at an advanced level; they emphasize reasoning, problem solving, and introduce students to logic. For this study, the students were given the following two logic problems to solve three weeks apart.

1. One of four girls has emptied a cookie jar. These are their statements:

   Alice: "Betsy did it

   Betsy: "Doris did it."

   Candy: "I didn't do it."

   Doris: "Betsy lied when she said I did it."

   If only one statement is true, which of the girls took the cookies?

2. At a party of truth-tellers and liars, you meet a new friend. He tells you that he has overheard a conversation in which a girl revealed her identity, saying she was a liar. Is your new friend a liar or a truth-teller?

The problems, adapted from Fixx (1978, p. 40) were selected because they are fairly simple, and require only deductive reasoning to solve. The students were allowed to use any resource they had in order to complete each assignment. They were asked to describe their thinking about the problem, write down details of their attempts to solve it, whether successful or not, and justify their conclusions. They had a week to work on each problem outside class. After the students had submitted their solutions to each problem, I led the whole class in analyzing some of the submissions, including acceptable solutions, solutions with reasoning errors, as well as some that employed unacceptable or illogical strategies.

All 39 students submitted solutions to the first problem; 38 did the second. The handwritten solutions to each problem were carefully analyzed and classified on the basis of the conclusions and the validity of the supporting arguments. Cultural underpinnings of incorrect solutions and invalid arguments were noted.

Results

Twelve of the solutions submitted for the first problem were correct. Some of the solutions were quite elegant, and all were based on sound arguments. Eight solutions to the second problem were of the same quality. Thus, less than 33% of the solutions submitted for each problem were based on valid arguments. An analysis of the students' written responses indicated that cultural values and beliefs seemed to hold sway over much of the students' thinking about these problems. It seemed that many of the students approached these problems with a baggage of culturally based premises that made successful solution of the problems virtually impossible. Here are some examples.

In the first problem, some concluded that Betsy emptied the cookie jar, presumably because she had three "strikes" against her, more than any of the other girls. Strike one: she was a suspect in a crime. Strike two: she blamed the crime on somebody else; the justification for this strike was the popular belief that people who accuse others of crimes for which they too are suspects, do so because they "probably have something to hide." The third strike was her failure to deny the claim by Doris that she, Betsy, was a liar. Clearly, this pattern of reasoning has more to do with baseball and popular culture than logic.

Several students used another kind of invalid reasoning to arrive at the correct conclusion that Candy emptied the cookie jar. The students used a process they named "reverse psychology." This procedure called for reversing the statement made by each suspect. Since the reverse of Candy's statement is "I did it", these reverse psychologists needed and sought no further proof that Candy was, indeed, the cookie monster.

In still another set of solutions, some students also came to the right conclusion by employing a type of reasoning often used on multiple-choice exams when the examinee has no clue about a particular question. A student on her 4th attempt at the problem, put it this way. "I looked at the problem like a multiple choice question. Betsy's, Alice's & Doris' statements all have something in common. They are all attacking another person. So I chose the one choice that didn't have anything in common with the other statements. Therefore, Candy is the one who took the cookies"!

Similar patterns of reasoning, obviously influenced by cultural beliefs and practices, were observed in the solutions to the second problem as well. In one case, students stated that there wasn't enough information to "judge" the new friend. They considered it unfair to judge a guy "so early in a relationship" on the basis of a single, insignificant incident.

In another case, the respondents said the new friend was a truth-teller even though they believed that he was telling "a story" about the girl who supposedly revealed her identity. After all, "nobody tells the truth (or lies) all the time." Consequently, how one judges the new friend "depends on whether you see the good or the bad in people."

Yet, another group of students concluded correctly that the new friend was a liar, again, not on the basis of any logical argument, but rather because he was an "eaves-dropper" and therefore, not trustworthy!

Conclusion

The results of this study suggest that cultural practices and beliefs do influence some people's ability to reason mathematically. Is the degree of this influence related to the level of the knowledge of mathematics? The data from this study did not provide a clear answer. Although the mathematically strong students in the study sample produced most of the correct solutions, they also supplied some of the unacceptable answers. As I reflected on why many participants in this study were unable to come up with acceptable solutions to the two problems, I remembered what one of them wrote:

> "I think that when trying to solve a problem like this, you bring your own experiences and preconceptions to the problem. You base your decision on your own life experiences."

I believe that this quote accurately describes how many other students approached the problems. Faced with two unfamiliar problems that required no computation, many of the students were neither able to put aside their preconceptions, nor base their decisions solely on deductions made from the given data using the rules of logic.

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EFFECT OF EMBEDDED COTEACHING MODULE ON ATTITUDES OF PRESERVICE ELEMENTARY TEACHERS

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The objective of this study was to determine if attitudes of education majors at a four-year university toward inclusionary coteaching practices could be changed by embedding an intensive coteaching module within a mathematics methods course for preservice elementary teachers. The rationale for the study was based on preliminary survey findings obtained in 1999. At that time, students were asked to evaluate their overall attitudes toward inclusion by rating their perceived level of preparation, knowledge, and enthusiasm to coteach in an inclusive general education classroom. General education majors were consistently more neutral for coteaching.

Theoretical Framework

A major focus of national educational reform has been the ongoing effort to include students with disabilities in the least restrictive academic environment. The inclusive education movement, largely fueled by the legislative mandates of the Individuals with Disabilities Education Act (IDEA), requires states to establish procedures to assure that students with disabilities are educated, to the fullest extent possible, in general education classrooms (Lesar, Benner, Habel, & Coleman, 1997). In an inclusion model, special education teachers work alongside of general education teachers to deliver a single merged system of instruction, thereby unleashing the "best of both worlds" (Skrtic, 1996).

Gately and Gately (2001) explain coteaching as an ongoing developmental process whereby skills in interpersonal communication, curriculum, instruction, and assessment are gradually acquired. Fostering the desire for collaboration dialogue takes time (Hargreaves, 1992), and therefore it has been suggested that competencies in coteaching, collaboration, and inclusive instruction may need to be shaped early on during the teacher's preservice educational experience (Webb & Romberg, 1994). Although some of these preservice models described in the literature do indeed show promise (Farlow, Caseau, & Gurlin-Jones, 1996; Lesar et al., 1997; Munby & Hutchinson, 1998), it appears that many students in teacher preparation programs claim that they receive minimal opportunities to observe collaborative teaching in an instructional context.

Method

A three-week intensive coteaching module was embedded into one section (n = 13) of a mathematics methods course for preservice elementary teachers at a four-year liberal arts university. Another section (n = 14) of this same course, taught by the same instructor, did not receive the coteaching module and was used as a control group.

During the coteaching module, a university faculty member with a background in special education joined the course instructor. Together they taught the preservice teachers the basic models of coteaching and demonstrated coteaching in practice as they taught course content related to teaching basic arithmetic facts, place-value, and whole number computation for students in elementary school. The control class that did not receive the module on coteaching was taught the same mathematics topics by the course instructor alone.

At the end of the semester, approximately six weeks after the end of the coteaching module, both classes were given a survey to assess knowledge of, preparation for, and enthusiasm toward inclusionary coteaching practices. Using a modified version of the *Teacher Attitudes on Inclusion* developed by Monahan, Marino, and Miller (1996), a 34-item survey allowed students to rank their perceptions of inclusion on a Likert-type scale ranging from (1) strongly disagree to (5) strongly agree. Items taken from the original questionnaire were modified in their wording to be more appropriate for undergraduate students rather than current classroom teachers. Additional items were also added to assess the respondent's overall knowledge of inclusion as a service delivery model. *T*-tests were used to compare the mean scores for the coteaching group and the control group on knowledge of, preparation for, and enthusiasm for inclusionary coteaching practices.

Results and Conclusions

A comparison of mean scores on survey items concerning attitudes about inclusionary coteaching practices for both the group receiving coteaching instruction and the control group that did not receive coteaching instruction are given in Table 1. There was a statistically significant difference in mean scores in each of the three areas surveyed. The group receiving coteaching instruction had higher mean scores on items relating to knowledge, preparation, and enthusiasm about coteaching. Although it may not be surprising that the sense of knowledge and preparation for coteaching instruction are improved by the mere exposure to inclusionary coteaching practices, perhaps most exciting is the notion that preservice faculty can also directly impact enthusiasm for coteaching.

The objective of this study was to determine if embedding an intensive coteaching module within a mathematics methods course for preservice elementary teachers could change attitudes of education majors at a four-year university toward inclusionary coteaching practices. The results of this study indicate that attitude differences can be made, most markedly in the knowledge and preparation preservice teachers perceive they have for coteaching. To a lesser extent, preservice teachers can also have their enthusiasm and receptivity to coteaching influenced by instruction about coteaching. Since coteaching may be the most effective method of instruction for special education students in the inclusion setting, it appears that including coteaching modules in the preparation of preservice teachers may represent best practice.

Table 1. Mean Score Comparisons for Knowledge, Preparation, and Enthusiasm for Inclusionary Coteaching Practices for Coteaching Group and Control Group

	Mean	Standard Deviation	*t*-statistic	p-value
Knowledge				
Coteaching	4.11	0.2447	8.30	< 0.01
Control	2.37	0.6506		
Preparation				
Coteaching	3.56	0.7736	4.17	<0.01
Control	2.31	0.5474		
Enthusiasm				
Coteaching	4.07	0.4305	4.23	<0.01
Control	3.38	0.3999		

1 = strongly disagree 3 = neutral 5 = strongly agree

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A PRE-SERVICE TEACHER'S GROWTH IN UNDERSTANDING RATIO, PROPORTION AND RATE OF CHANGE USING LESSON PLAN STUDY

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Focus of Study

This study is part of an extensive 3-year longitudinal study was to examine the ways in which undergraduates who are preparing to teach high school mathematics come to change their understanding of the high school content through lesson plan study. We were trying to find out in what ways does lesson plan study change pre-service teacher's understanding of ratio, proportion, and rate of change. Berenson, Cavey, Clark, and Staley (2001) stated that teacher preparation needed to include instructional teaching strategies that are associated with the content of high school mathematics. Here we define the content of high school mathematics as topics up to and including Algebra-Trigonometry/ Pre-Calculus class.

Conceptual Framework

The conceptual framework was Pirie and Kieren's model of mathematical understanding. "[T]he growth of mathematical understanding is a whole, dynamic, leveled but non-linear, transcendently recursive process. This theory attempts to elaborate in detail the constructive definition process of organizing one's knowledge structures"(Pirie & Kieren, 1994, p.166). Descriptions that are relevant from this model include primitive knowledge, image making, image having, formalized knowledge and folding back. More specifically, the Pirie and Kieren theory was used in coding the transcript of the interview. We looked for instances in which the pre-service teacher went through the various stages of the theory and began to change her understanding of ratio, proportion, and rate of change.

Methodology

Karen was a student in her first mathematics education methods course. She participated in two iterations of lesson plan study (LPS) occurring over a 16-week semester. Each iteration of the LPS contained three distinct components. The first component was an individual interaction study that included a preliminary knowledge base interview, a lesson planning activity and a post interview. The second component was a group interaction study, which included a lesson planning activity, a group interview, and a group presentation to their fellow classmates. The final component had the student produce a reconstructed view of their individual lesson plan (Berenson, in press).

Here we report on the results of the first component of Karen's rate of change LPS cycle. She was asked to plan a lesson to introduce the concept of rate of change to an Algebra I class and relate that concept to ratio and proportion. Karen's lesson plan was viewed as acting in context while content competencies were needed to complete the task. The sources of data were transcriptions of two videotaped interviews, student notes of the planned lesson, and descriptive artifacts.

Results

In the pre-lesson interview, Karen discussed her Primitive Knowledge about ratio, proportion and rate of change in a procedural manner.

Ratio

K: "When I think of *ratio*, I think of 2 to 3. Like if you had two of one thing and three of another..."

K: "You have 2 to 3 and 2:3, and sometimes you could write a fraction 2/3."

Proportion

K: "...if you had different *proportions* of things, like if you had 8 red umbrellas and 10 black umbrellas and you wanted to know the proportion of red umbrellas to black umbrellas you would say 4:5"

K: "I think a lot of things deal with *proportion*. You know things being equal to each other."

Rate of Change

K: "The only thing I can remember doing with *rate of change* before then was maybe with slope or something. I can't really think of anything, it just reminds me of Calculus."

K: "I think basically if you were doing it to basic kids who had never heard it before, take two *rates*. If you had one rate and another rate and you are going from one to the next and you want to how they changed, you would subtract them from each other or something."

During the post plan interview, Karen described her lesson plan to the interviewer, who then asked clarifying and probing questions concerning her ideas about ratio, proportion, and rate of change. Karen seemed to be in the stage of Image Making when she began to replace her simplistic explanations with reasons for making connections for her students.

K: "Okay. Well before I would go into rate of change or rates, I would talk about ratios and proportions first, because that leads to it."

K: "A ratio is an ordered pair of numbers that are used to express a comparison between the numbers or measures. So it's a comparison of numbers or measures."

At this point Karen seems to Fold Back to her Primitive Knowledge and states, "...I might start with fractions because all fractions are ratios." Later in the interview Karen starts Image Having about how to teach and clarifies her new concept of ratio. Karen began to explain the part-whole and whole-whole relationships when dealing with fractions a concept that she used to help her clarify her new ideas about ratios.

K: "Right. Then I would go into putting up like if you had 2/3. Then you have two parts to three parts of a whole....If you have 3 apples and 2 are red and 1 is green. What is the ratio of red apples to green apples? It's 2 to 1 because you have two red apples to one green apple."

K: "If you have 20 kids in a class, which that's your whole, and 8 are girls and 12 are boys. The 8 that are girls are part of that whole and the 12 that are boys are part of that whole. But now if you were asking the ratio of boys to girls, it is 8:12 [Corrected to 12:8]. That's a ratio that's not a fraction."

After discussing her new ideas about ratios, Karen began to explore the topic of proportions. Karen moved from a stage of Primitive Knowledge in her first explanations of proportions to one where she began Image Making. In this process she began to replace her primitive knowledge with definitions and examples that she felt comfortable discussing.

K: "I would probably say a proportion is a statement of equality between two ratios."

K: "Like if you have a fraction that's 6/10 then it is equal to 3/5. That's a proportion, 6/10 = 3/5."

K: "...Like if you have four pieces of candy are $1.00, how much are twelve pieces of candy? And however you set it up, you would say like if you put candy to $1.00, put candy on top of $1.00 and that would be equal candy over here over dollars, which would be 12 pieces to how many dollars."

After stating new images of proportion she then Folded Back to her Primitive Knowledge and used other tools to assist her explanations, "Then I would go into cross multiplication you know cross multiply."

The lesson-planning activity appeared to have contributed to her growth of mathematical understanding of ratio and proportion encouraging her to Fold Back a number of times to revise her images. However, Karen's idea of rate of change seemed to remain unchanged.

K: "...but when I think of rate of change, I think of slope, having two rates and

> going from one to the next. Or if you have two rates, one's one rates in the same thing. Like if you're in miles per hour and you have one mile per hour, then you go, you speed up to go to the next miles per hour. What was the change in those two rates?"

These results from the lesson-planning task support Towers' (2001) conjecture that students' understanding is partly determined by teacher interventions. Interventions such as lesson-planning activities may allow students to reflect on their own knowledge and build new images. Through these reflections and discussions with colleagues and teachers, students could realize some of their own limitations and misconceptions about how to teach and what it means to teach mathematics.

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BRIDGING THEORY AND PRACTICE IN PRESERVICE MATHEMATICS TEACHER EDUCATION

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Ball (1988) notes that the mathematics teaching theory presented in mathematics education courses rarely influences "what university educators do with *their* students. [...] University classes are typically "delivered", even when they are *about* the problem with delivering knowledge to children in schools. As one teacher remarked wryly, she had been in courses where she got *lectures* on constructivism" (p. 28). It seems reasonable to expect that teacher education courses model the learning theory they espouse.

Constructivists generally agree that knowledge is actively constructed by the learner rather than passively absorbed from the environment. Implying the belief that students actively construct their understanding is the idea that "present strategies for teaching may need to be reversed; knowledge often should emerge from experience with problems" (NCTM 1989). Certainly students construct understanding even in passive lecture settings. However, if we want education students to learn how to recognize, analyze and engage in solving mathematical and pedagogical problems "then their instructional environment should be one where that is what they do" (Greeno, 1988, p. 518). Also, research indicates that for significant change to occur in teachers' beliefs and practice, teachers need experiences where they engage in practical inquiry and reflection about mathematics and mathematics teaching (McGowen & Davis, 2001; Stipek, Givvin, Salmon, & MacGyvers, 2001).

Linking a Mathematics Methods Course and Practicum Experiences

The mathematics teacher education course for secondary school preservice teachers described here was based on the assumptions that: 1) teacher agency and personal reform beliefs are an integral part of education reform; 2) teachers learn to use their personal agency in reforming education through experiences where they practice using their personal reform beliefs in their teaching; and, 3) opportunities for reflection promote a dialectic relationship between beliefs and practice. The course involved preservice teachers in outlining their personal mathematics reform beliefs and selecting and becoming more knowledgeable about an area of mathematics education that supported their beliefs (such as computer assisted learning, the use of manipulatives, problem-solving approaches, cooperative learning, etc.). They researched their area of interest, refined personal beliefs, and developed four detailed lesson plans. They implemented at least two of the lesson plans during their practicum (and videotaped one of these) and designed a resource package for other teachers to use.

Preservice teachers also reflected on their beliefs and practice. During the practicum, they kept a weekly journal of their efforts to apply their beliefs to classroom practice. After the practicum, they wrote a narrative of their experience. They also discussed the relationship between their reform beliefs and practice in their analysis of the lesson they videotaped. For many preservice teachers, the practicum is an experience where they put aside their personal beliefs and learn to adapt to the status quo. What they think and believe becomes secondary to existing classroom and school cultures. This course offered preservice teachers opportunities for practicum experiences where they also tried to improve on the status quo.

Confrey (1990, p. 107) lists "the promotion of student autonomy" as one of the components of a constructivist learning environment. However, is teacher agency, and an accompanying focus on teachers' reform beliefs, appropriate for the preservice stage of teachers' professional development? Fuller (1969) identified three stages of concern teachers go through as they learn to teach: survival concerns, teaching style concerns, and pupil concerns. Recognizing that, generally, certain concerns may tend to be more predominant in one stage than another is very helpful in supporting preservice teachers during their practicum experience. However, developmental descriptions of teacher growth are sometimes interpreted in terms of what preservice teachers can or cannot do at each of the stages identified. Such an interpretation questions the view that personal reform beliefs, and an accompanying focus on teacher agency, is an appropriate concern for the majority of teachers in the preservice teaching "stage". Scardamalia and Bereiter (1989, p. 43) concede that a case can be made for omitting such higher-level concerns "in preservice education and introducing them through inservice education, after teachers have passed through the 'survival' and 'mastery' stages and are ready to deal with impact." However, they also note that "it has been shown that once teachers are entrenched in problem-minimizing approaches it is very difficult to dislodge them". In our experience, preservice teachers can and do learn in reform-based settings.

Encouraging Investigation and Collaboration in Lecture Settings in Mathematics Courses

Preservice mathematics courses often take place in large classroom environments which have the potential to contradict the aims of reform-based learning. Encouraging and facilitating the active participation of students in constructing their own learning is necessary to provide students with the opportunity to experiment, self-reflect and interact. Organizing students into learning groups at the beginning of the year can offer them opportunities to work together on an exploratory activity at some point during each class. The groups are also helpful in terms of distributing materials (or returning assignments sorted by group number) quickly. While it may be difficult to have sufficient manipulatives for large classes, the groups can take an active role in constructing their own sets of items such as base ten blocks, fraction bars and algebra tiles. Sugar cubes make great substitutes for centi-cubes or tiles, and many other activities can be done with home made materials. After investigating, the whole class environment can then be used to share and generalize results. The small groups may also be used as a forum for presentations of individual projects on various topics. Students are asked to submit a proposal for a presentation on a topic in reform-based mathematical learning, which they will present to their group. The students are provided with resources describing reform-based learning and are encouraged in risk-taking and trying new ideas and topics in novel ways. The presentations themselves are graded within the peer group, using checklists developed by each group. Peer grades are submitted anonymously, and averaged, and a participation mark is also given for handing in the peer evaluation forms. All of this detailed evaluation has another important feature – students always come to the 'lectures'. While there is some recording to be done, this phase has no "marking" to be done by the instructor. The remaining project grade is determined based on a written report with background references on the activity graded by the instructor.

Students claim to enjoy group activities, and it seems to spark a new interest in mathematics and mathematics teaching. In fact, for many students with a history of feeling unsuccessful in mathematics in school, the reform-based experience is both exciting and rewarding. As one student put it recently, "it feels like the first time I've taken math." Students are encouraged to choose a topic they think they will really enjoy working on, however unusual, and topics have ranged from the mathematics of rainbows to the symmetry of the human face (with active group participation of course!). Students also gain first-hand experience in alternative evaluation practices. Often students try to outdo each other with active, fun and engaging topics and creative presentations, and many provide their group members with packages of ready-to-use resources. Similar "special interest groups" can also be used to investigate a particular topic, or create a particular product. An example would be a course resource book prepared by the students on particular topics, such as use of technology.

Pressing students to develop deep understandings of difficult concepts such as division of fractions or subtraction of integers may also require extensive reflection in their learning. The use of journal entries may be helpful in the self-reflection process. Grading can be facilitated for large classes by asking students to choose their best entry (or two) to be marked.

Conclusion

Preservice mathematics teachers, as is the case with students in general, construct understanding based on *what* they are taught and based on *how* they are taught. We believe our mathematics teaching philosophy should be mirrored in our teacher education practice. We also believe that this is not a simple task.

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DIDACTICAL ANALYSIS AND PLANNING OF SCHOOL TASKS IN PRESERVICE MATHEMATICS TEACHER TRAINING

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In this paper, we propose the didactical analysis as a tool to assist future secondary mathematics teachers in their analysis of mathematics content in order to plan, implement, and assess teaching and learning tasks in class. We propose a conceptual structure that supports the curriculum design of a methods course and present the elements composing the didactical knowledge that we expect preservice teachers to develop in such course.

Introduction

The methods course we are in charge of is taken by last-year mathematics students. It is a theoretically oriented course that is complemented with a second course in which future teachers perform some practical activities in a high school. One of the main goals of our course is to develop preservice teachers' didactical knowledge as the basis for curriculum design at a local level (didactical units). The content of the course is therefore related to the disciplinary knowledge that serves as reference to didactical knowledge: the concept of curriculum, school mathematics, their teaching and learning, and the curriculum organizers. Preservice teachers work in groups of four to six people. Each group chooses a specific mathematical subject on which it develops its curriculum design. This work is periodically and systematically presented, shared, and discussed in class at different levels of development, until the final document is produced. This allows everybody to participate, individually and socially, in the construction of the technical meanings they are using in their own work. In what follows, we introduce the idea of didactical analysis and show its relationship with the notion of didactical knowledge. We show how the content component of the curriculum design of the course is based on these two notions.

Didactical Analysis As A Tool For Local Curriculum Design

The notion of pedagogical content knowledge (Shulman, 1987) highlights that knowledge of mathematics and pedagogy are not sufficient in order to become a mathematics teacher. Teachers need a specific knowledge for understanding "how particular topics, problems or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). Despite of its importance, this notion has not been treated with precision in the research on teachers' knowledge (Bullough, 2001, p. 657). The issue of which types of knowledge are needed by mathematics teachers in order to be efficient and how this knowledge can be developed remains open and is central in teacher training (Cooney, 1994, p. 608; Richardson, 2002).

The design of a mathematics methods course should be based on a conceptualization of the activities that the teacher has to do in order to promote students' learning and of the knowledge that is necessary to perform those activities. We call the structuring of a cycle of these activities a *didactical analysis*. It is organized around four analyses: content, cognitive, instruction, and performance. Didactical analysis allows the teacher to examine and describe the complexity and multiple meanings of the subject matter, and to design, implement, and assess teaching/learning activities.

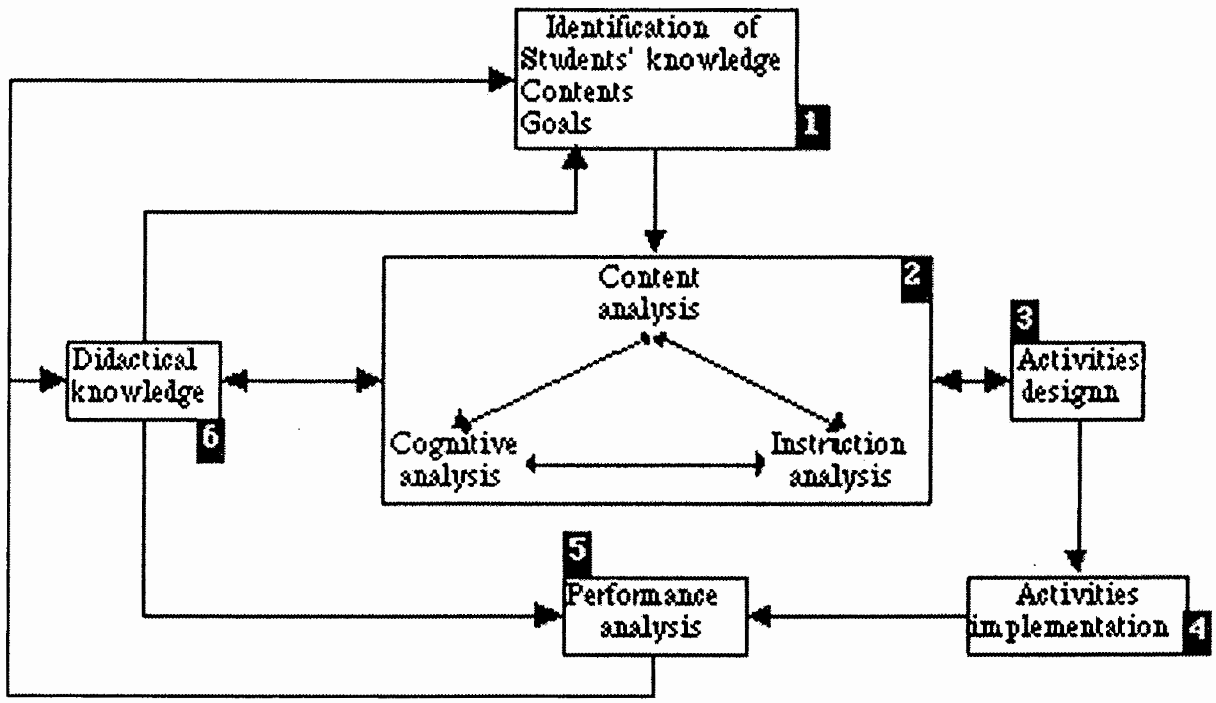

Figure 1. Diagram of a didactical analysis cycle.

Any cycle of the didactical analysis begins with the identification of the student's knowledge for the subject matter at hand (see Figure 1). We expect the teacher to use his knowledge and previous experience for establishing the tasks that the students can and cannot solve, the mistakes they can make, and the difficulties underlying those mistakes. With this information, and taking into account the global planning of her course, we expect the teacher to determine the goals she wants to achieve and the mathematics content she wants to work on (box 1).

The next step of the cycle involves the description of the mathematical content from the viewpoint of its teaching and learning in school (box 2). The *content analysis* stresses the relationship among concepts, highlights its multiple representations, and distinguishes the connections between the elements of the conceptual structure and between those elements and the phenomena from which they emerge. This information is used in the *cognitive analysis,* in which the teacher describes hypothesis about how students construct their knowledge when they face the learning activities that are proposed to them. The cognitive analysis involves the identification of the skills, reasoning, and strategies necessary to solve the tasks, of the mistakes students can make when they are solving them, and of the difficulties and obstacles they might face. The information from the content and cognitive analysis allows the teacher to carry out an *instruction analysis*: the identification and description of the tasks that can be used in the design of the teaching and learning activities that will compose the instruction in class (box 3). These tasks should mobilize students' knowledge in order to generate cognitive conflicts and promote the construction of meaning using the materials and resources available. In the *performance analysis* the teacher observes, describes, and analyzes students' performance in order to produce better descriptions of their current knowledge and review the planning in order to start a new cycle (box 5).

The *didactical knowledge* is the knowledge that the teacher enacts when she performs the didactical analysis (box 6). In other words, it is the knowledge needed for organizing teaching and learning activities in didactical units. Expert teachers perform didactical analysis based on their experience and the materials they have available. However, preservice teachers need guidelines and criteria with which to organize their activities, produce their work, and structure their future experience. The didactical analysis provides guidelines for preservice teachers to 1) explore and recognize the richness and variety of meanings of the mathematics subject matter, 2) collect, organize and select information concerning these multiple meanings, and 3) use this information to design materials and activities to promote students' mathematical learning. The knowledge underlying these guidelines can be organized in three categories: 1) the concept of curriculum as a tool for planning and global structuring, 2) the foundations of school mathematics (mathematics, learning, teaching, and assessment), and 3) mathematics education notions that can be used, as conceptual and methodological tools, for local planning, as suggested in the cycle of didactical analysis. We call these notions "curriculum organizers" (Rico et. al, 1997): conceptual structure, representation systems, didactical phenomenology, modeling, errors and difficulties, materials and resources, and problem solving. Didactical knowledge is the integration of these three types of knowledge.

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COMBINING CGI WITH WORKING WITH CHILDREN: EXAMINING REFLECTIONS OF PRESERVICE TEACHERS

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This paper examines some of the learning experiences of 43 undergraduate preservice teachers in two pilot sections of an elementary math/science methods[1] course. Vinz (1996) argues that it is worth asking: "What do teachers reflect upon and what forms does that reflection take?" (p. 84). I explore this question, focusing mainly on one learning activity used in the course. In this learning activity, preservice teachers work in pairs and are assigned a small group of four to six second-grade children with whom to interact in regular elementary school classrooms. As teaching partners, the preservice teachers are required to: (1) develop/use a set of problem solving tasks that build on children's thinking and ideas (drawing mainly from the Cognitively Guided Instruction framework (CGI) (Carpenter, Fennema, Franke, Levi, & Empson, 1999)), (2) engage the children with the problems and audio tape the interaction, (3) listen to the audio tape and write a structured reflective journal,[2] and (4) use information they learned from the reflection to develop and justify the next set of problem solving tasks to pose.

The structured reflective journals consisted of four parts,[3] which I use to report my preliminary findings below. Because the structured reflective journal defined *what* the preservice teachers focus on, I examine *how* they reflected on these areas. The corpus of data collected include weekly written structured reflective journals, audio-taped class sessions, observations made by collaborating experienced teachers and myself, transcripts from a focus group interview that took place at the end of the semester, and final case studies prepared by the preservice teachers which describe the 2nd graders' content, processes, and attitude development over the course of the semester. For this manuscript, I present preliminary findings after having analyzed the structured reflective journals and final case studies. Further analysis over the summer will produce a more complete picture of the findings reported here.

Describing the Mathematical Content in the Problems Posed

Many preservice teachers easily classified the operations (addition, subtraction, multiplication, division) in the problems they posed. In fact, they drew from the problem structures described in the CGI framework and often identified the correct operation, even though the problem could be solved with a different operation. For instance, if a problem was a Join-Change Unknown problem, subtraction could be used to solve the problem even though the structure of the problem, as written was addition. Students sometimes used the problem type to label the operations rather than stating the operation itself. Identifying when base ten was a prominent idea was more difficult for many of the preservice teachers. For example, many students did not distinguish between problems that required regrouping and those that did not. Because we had viewed an experienced CGI teacher on a videotape using groups of ten in division problems as one way to integrate base ten with division, many of my students were able to recognize problems of that type as including the concept of base ten.

Collecting Data[4] on Each Child's Content, Processes and Attitudes (Cantlon, 1994; National Council of Teachers of Mathematics [NCTM], 2000).

Preservice teachers attended carefully to the children's attitudes, and this became the most prominent piece in the case studies they wrote. Many preservice teachers began the semester by making general statements (e.g., "they all used blocks") about the children's mathematical processes, but became more attuned to them both through observing the children and through reacting to comments I wrote on their reflective journals. They often recorded the language and actions the children used when describing their solution strategies as evidence that a child was using a particular solution strategy. However, some preservice teachers used, "he said he did it in his head," as evidence that a child used "derived facts" rather than pushing the child to articulate *how* s/he did it in his/her head.

The area that was the most difficult for preservice teachers to address was the content piece. Much of what they recorded (under the "content" section) pertained to processes (e.g., communication skills) and attitudes (e.g., "he focused and tried hard"). While the processes and content are overlapping areas (and we discussed this at length in class), the connection between them was often not identified in the reflective journals, even when I asked about it in my written feedback. Some students did make this connection in the case studies they wrote; others did not.

Examining the Interaction That Too Place (e.g., the questions they asked and how they followed them up, the maintenance or decline the cognitive complexity (Stein & Smith, 1998) of the tasks they posed)

Most of the preservice teachers used the course readings and ideas to examine their interactions with the children in insightful ways. For example, some expressed dissatisfaction with the way that they stepped in and did the problems for the children if they were struggling because they realized they reduced the cognitive demand of the high level tasks that they had written (see Stein & Smith, 1998, for this reflection framework). Other preservice teachers focused on trying to show me that they were doing it "right" by defending what they did. Most of the preservice teachers did ask open-ended questions that focused on helping the children articulate and explain their thinking, rather than mainly asking "teacher questions."

Describing Other Items About Which They Noticed or Wondered

In this more open-ended section of the journal many of the preservice teachers focused on concerns about the children, e.g., how the children felt about having other children explain solutions to them if they couldn't solve the problem. They also expressed concerns about teaching, e.g., how to support the children in articulating their own thinking or how to work with such varied levels of engagement and understanding of the content.

These preliminary findings support Nicol's (1999) findings. The preservice teachers in this study were able to attend to the mathematics, instructional issues, and children's thinking in their work with second-grade children. Some of the aspects with which the preservice teachers struggled are problematic (e.g., not seeing the connection between identified processes and what that means to the children's mathematical understandings). However, as a teacher educator, I see these areas as ones that I need to address in the course, not weaknesses of the preservice teachers.

Notes

[1]The class is called a "methods" class, but that term seems to indicate that there are a series of activities that one might learn to teach math in an 'appropriate' way. I prefer to focus on teaching as a reflective activity in which one makes informed decisions about what s/he will do, drawing on his/her own experiences, experiences of others and on pertinent literature related to what is being taught.

[2]I use "structured reflective journal" to indicate that the items preservice teachers are required to reflect on are determined by the questions I pose rather than being more extemporaneous.

[3]The parts to the journal were developed based on Nicol's (1999) observations and literature related to *experienced* teachers using CGI. Nicols found that preservice teachers *were* able to consider and attend concurrently to concerns of management, student's mathematics, and instructional issues.

[4]The data collected by preservice teachers was used to write detailed case studies for the classroom teachers, focusing on each child's content, processes, and attitudes.

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THINKING ABOUT UNIVERSITY MATHEMATICS TEACHING: FORMAL ASSESSMENT AS INTERACTION

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The purpose of this paper is to examine the instruction of one university mathematics faculty member in his effort to implement a capstone mathematics course for preservice secondary mathematics teachers. The faculty member, Dr. Jeffords (a pseudonym), had as one of his overarching goals to teach a course in accordance with the recommendations of various organizations (Conference Board of the Mathematical Sciences, 2001). One of the major recommendations of these documents and programs was that preservice teachers experience mathematics in ways that are similar to the ways that they are being asked to teach: student-centered; focused on student learning and connections to classroom teaching; connected the content to the practice of teaching; and used technology. In this paper, I examine, using Simon's (1995, 1997) Mathematics Teaching Cycle, the ways in which the assessments given in this course influenced Dr. Jeffords' goals, interactions with the preservice teachers and knowledge for teaching a course of this type.

Theoretical and Analytic Framework

I used the Mathematics Teaching Cycle [MTC] (Simon, 1997) as a theoretical and analytic framework for guiding data collection and analysis. In brief, the MTC describes the iterative process that is a model for teaching. In this process, the teacher develops a *hypothetical learning trajectory*, a combination of the teacher's goals, which inform the teacher's plan for achieving these goals also informing and informed by the teacher's hypothesis about how students learn. These plans are then enacted with students to allow for interactions with students, leading to changes in the teacher's knowledge of the students, their mathematics, his own mathematics, and other types of knowledge necessary for teaching. This knowledge then cycles back to inform the teacher's goal, hypothesis for students learning and plan.

This cycle can be used describe a particular episode in a class as well as weeks of classroom instruction. For the purposes of this brief report, the cycle will be used to model a more global aspect of Dr. Jeffords' instruction, the ways in which his evaluation of assessments serve as an opportunity for interactions with students in ways that impacts both the teacher's knowledge of the students' mathematics and his goals and plans for and enactment of instruction.

Data Collection

I collected data during the Fall 2001 semester at a large midwestern university in the US. Dr. Jeffords, a Caucasian male, is a tenured mathematics educator in the mathematics department at this university. The class, a capstone course designated for senior-level preservice secondary mathematics teachers, also served as an upper-division writing course as part of the university's writing across the curriculum program. During the semester, I interviewed Dr. Jeffords both immediately before and immediately after each class period. I also attended, as an observer, each class period and took copious notes, along with a graduate assistant. Two undergraduates videotaped the class using digital video cameras, one focusing on the teacher and the other focusing on the students. When students worked in groups, the camera focusing on the teacher followed the teacher's movements through the class, the other focused on one group of students. I also collected all student work that was submitted to the teacher, after Dr. Jeffords had graded the papers and/or provided feedback.

Data Analysis and Results

Dr. Jeffords' Initial Goals

For analysis, I examined the Dr. Jeffords' plans for the class, as well the interview data to classify his goals for each class period, as well as overarching goals that were always present. A goal that Dr. Jeffords expressed as important was that the preservice teachers begin to frame their own mathematical questions, pursue these questions, and communicate their responses in ways that would be useful to other teachers. Dr. Jeffords stated in an interview that he wanted students to begin to ask themselves the questions "What is the mathematics in this situation? Where does it lead? How do we get there?" (Interview, 8/28)

Dr. Jeffords' Hypothesized Learning Path and Formal Assessment as Interaction

Dr. Jeffords initially hypothesized that by presenting students with a mathematically rich context and helping them to formulate mathematical questions to pursue, he and the preservice teachers would begin to address his goal. Difficulties arose, however, as students had difficulties formulating mathematical questions, as opposed to more science-oriented or education-oriented questions from rich contexts.

In addition to interactions with preservice teachers in the classroom, Dr. Jeffords' evaluation of the their Mathematics Investigation for Teachers Journals was a form of interaction with the preservice teachers. As Dr. Jeffords explained it, the students were to pursue a mathematical topic of interest to them and write an explanation of their findings, with other teachers as the intended audience. This assessment task was in keeping with his goal that the preservice teachers should develop their own mathematical questions and pursue them and the writing goals of the course.

From students' early journals, Dr. Jeffords felt they had difficulty in framing mathematical questions and pursuing them mathematically, particularly questions that were different from questions already pursued in classes. In the beginning of the semester, Dr. Jeffords would provide contexts for the journals. After the class' first journal entry, he engaged the students in attempting to develop a rubric for the journal that focused on asking an appropriate mathematically based question and pursuing the question sufficiently. The students also focused on the aspects of communication, such as using appropriate representations and clear writing.

Dr. Jeffords and the students recognized that they had difficulties with meeting his expectations for the journal assignments. During one interview, he explained why he was going to discuss the journals during class time. He felt that "most didn't pursue mathematics and of those interested in math, it wasn't really investigated. Or they stopped at the initial solution" (Interview, 9/11). He communicated this issue of framing mathematical questions and pursuing the question in mathematical ways to students, but they continued to have difficulties. For example, at a point at which the students had done two journals and had questions about how to determine what was a good question and how to pursue a question "deeply" one student, Susie (pseudonym) asked a question about "How deep is deep?" (Class Observation, 9/13). Dr. Jeffords focused his discussion on moving toward mathematical generalization and proof, answering in ways consistent with a senior-level mathematics major. Later in the semester, this same student, who often acted as a spokesperson for the class, asked Dr. Jeffords to give examples of generalizing a topic (Class Observation, 10/25).

Changes in Dr. Jeffords' Knowledge and its Influences

Dr. Jeffords came to recognize that his model of the preservice teachers' mathematics did take into account that the preservice teachers had difficulties with the process standards of representation, reasoning and justification, communication, and problem formulation as part of problem solving. He also came to understand that the students had constructed both mathematics and mathematical communications through previous and current interactions in other mathematics courses in the department. Thus, Dr. Jeffords' plans for attaining his mathematical goals changed. Through the MIT Journal entries, he refined the task by asking the preservice teachers to begin focusing on generalizing ideas from secondary mathematics (Interview, Class Observation 9/27). Additionally, he added a weekly assignment in which the students worked in groups to present to their classmates investigations from a reform secondary mathematics curriculum, focusing on particular connections to undergraduate mathematics (Interview, Class Observation 9/27). Dr. Jeffords created additional assessment opportunities to interact with his students and their mathematics.

Discussion

The MTC operates as a theoretical lens to observe the moves in Dr. Jeffords' teaching over the course of the semester. Dr. Jeffords reassessed his understanding of the preservice teachers' mathematical knowledge and their identities as mathematics students, thus impacting his ways of achieving his goals, illuminating a larger issue. In many classes, assessment does not inform instruction, only operating as summative evaluation of student knowledge. Often faculty do not view assessment as a lens into the students' beliefs about mathematics and themselves as students of mathematics, potentially impacting overt discussions of the assessment practices involved in the course. This analysis shows how formal assessment can be used to inform instruction as interaction with students in the MTC. Assessment, particularly in secondary and post-secondary courses where work done outside of class is expected to be a major contribution to students' learning, becomes an important site for examining instruction and impacting faculty professional development.

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PRESERVICE TEACHERS' PART-WHOLE CONCEPT UNDERSTANDING: TWO EXAMPLES

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An earlier study (Steinke, 1999) reported that, in a ten minute non-paper-and-pencil assessment, 2 of 15 preservice teachers showed lack of part-whole concept understanding in answering a question of the form: If I have 25 circles altogether, and 7 circles are on the plate (visible), how may circles are here (hidden)? This article explains how their manner of responding identified these students as lacking part-whole concept understanding and what this implies for those involved in teacher education.

Even with solid, NCTM-Standards-based instruction in their math methods and/or math content courses, some preservice teachers (PSTs) fail to achieve a level of understanding such that they feel capable of teaching mathematics in other than a procedural manner (McGowen & Davis, 2001; Raymond, 1997; Thompson, 1992). Many studies have focused on teachers' or PSTs' grasp of rational number, division, or geometry. There is evidence, however, that some PSTs lack the earlier, more fundamental concepts of place value (Ross, 2001) and the part-whole relationship (Steinke, 1999).

Steinke's 1999 study reported that two of fifteen volunteers from a math methods class for PSTs did not show part-whole concept understanding when given a non-paper-and-pencil assessment in a videotaped interview. Only the raw data, in terms of correct and incorrect responses, were given in that paper. However, it is the behavior of the interviewees as much as, or even more than, their answers that reveals their thinking about numbers.

Steffe and Cobb's (1988) 3 Stages model of number sense development in young children (Perceptual, Figurative, Abstract) is the framework for the interview questions. Not until the Abstract Stage do children understand that the parts and the whole always coexist, allowing them to solve with ease "missing addend" or "missing part" problems (P[art] + ? = W[hole]) beyond the range of memorized addition and subtraction facts.

The assessment used with the PSTs is the one originally developed for and administered more than 700 times to over 300 children (Steinke, 2001) over a period of 8 years. The test has also been administered to over 100 adults. The behaviors of the non-Abstract Stage adults are virtually identical to that of non-Abstract Stage children.

The assessment begins with Perceptual Stage questions (P+P=?) using two sets of visible objects, then proceeds to Figurative Stage questions (P+P=?) with one set of visible objects and a written numeral. Finally the questions are switched to the form P+?=W (Abstract Stage) with the Part a set of visible objects and the Whole a written numeral. To ensure that the interviewee, child or adult, catches the change in question form, the first Abstract Stage question is 3+?=7. The next is 8+?=13. This is followed by a whole of 25, then, for the adults, a whole of 42, and a whole of 34. The last three questions are not memorized facts and involve crossing the decade so that interviewees must deal with the whole and the parts as coexisting. Further, the testing rubric requires that children's responses must show evidence of having reached the Abstract Stage, and that adults must show positive evidence of NOT having reached the Abstract Stage.

All the preservice teachers answered the P+P=? questions correctly. With the P+?=W questions, the two PSTs of interest answered 3+?=7 correctly and without hesitation, showing they were aware of the change in form of the question. Subject 1 then answered "21" for the question "8+?=13"; she gave a sum of the two visible numbers. When the question was reposed, she answered "13", the visible written numeral. For the question "9+?=25", Subject 1 answered "25."

These are typical answers from Figurative Stage thinkers. At the Figurative Stage, students are only able to deal with the given quantities and, regardless of the form of the question, often take the given quantities as Parts that are to be combined to form the Whole. Alternatively, they give the larger numeral as the answer.

At the question with 42 as the whole, Subject 1 became aware that her answers of 13 and 25 were not correct. After an 11 second pause, she repeated the posed question, which the interviewer affirmed as the correct question. Subject 1 then responded with a guess of "32", then shook her head knowing that 32 was not the answer but not knowing how to find the answer. For the final question, 8+?=34, her minimal finger pressure on the interview table indicated she likely was counting by 1's eight times. Her answer was "27." Instead of "counting down" eight numerals FROM the Whole, she counted eight numerals, STARTING with the Whole, and gave the last-named numeral as her answer.

All of Subject 1's behaviors are found in Figurative Stage children. Subject 1 was unable to answer correctly a P+?=W question where the Whole was greater than 10. More importantly, she showed no confidence in the correctness of her answers, an important Abstract Stage behavioral response (Steffe & Cobb, 1988).

Subject 2 answered the first two "missing part" questions correctly (Whole is 7 and Whole is 13). The question 9+?=25 is the touchstone. Here the subject repeatedly guessed an answer. Her successive responses were 12, 13, 18, 17 and finally 16. The "16" response came after the interviewer had proceeded to the next question.

Subject 2 was using a "guess and check" strategy: keep trying answers until you get the one which gives the correct total when added to the given part in the P+?=W question. This is a Figurative Stage mechanical procedure that will eventually yield the correct answer. The "guess" and the given Part are added as in a P+P=? question. There is no sense of part-whole coexistence here. Subject 2's most revealing comment about this problem was, "I have to do it on paper." Pencil and paper mechanical skills hide her lack of the part-whole concept. Her behaviors, not her final response, show that she is NOT at the Abstract Stage.

When the instructor for these Subjects' math methods class was informed of their interview results, the instructor commented that she was sure there were several more such students in her class of 30, based on their class performance. The percentage result here (13% not-Abstract Stage) is not out of line with that of a small group of volunteers (n=11) at another two-year college (36% not-Abstract Stage) (Steinke, 1999), or the 1992 National Adult Literacy Survey, in which 25% of the sampled adult population performed at Level 2 on quantitative literacy tasks and 22% of adults were at Level 1, unable to perform some of the most basic quantitative literacy tasks.

The interviews presented here raise the concern that the conceptual level of some PSTs may lie at the K-2 grade level, well below the mid-elementary-school level studied by most content-knowledge research of PSTs. Those involved in teacher education need to be aware that such students exist, be able to quickly and easily identify such students, and to provide them with targeted instruction in the part-whole concept and other K-2 level concepts. The part-whole concept is a prime underpinning for grasping later math topics. Only with a solid base in their own mathematical thinking can PSTs become effective math teachers (Fennema & Frank, 1992).

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MATHEMATICS TEACHING: FROM FRAMEWORKS TO FLOW

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Theoretical "foundations", instructional "units", conceptual "frameworks", "networks", learning "jigsaws", cognitive "structures", "scaffolding", "building" knowledge...and the list goes on. Metaphors. We are constantly reading them, hearing them, using them, and thinking with them. As a result, they often go unnoticed or unquestioned when they are taken to be, simply, figures of speech with which to describe mathematics teaching and learning. Differently, scholars such as Sfard (1997; 1998), Lakoff & Núñez (2000), Bowers (2001), and Abram (1996) interrogate the manners in which metaphors become embodied in our forms of knowing, our actions, and our identities. In keeping with this, the following research sought to: identify the metaphors rooted within my own thinking as a second and third grade mathematics teacher, interrogate the ways they shaped my teaching of elementary school mathematics, and root more ecologically minded metaphors within my classroom practice.

Key metaphors embedded in my taken for granted ways of thinking were: the "mathematics class" as *discrete* entities of teacher, *individual* students, and the *external* environment, "mathematics" as a *connected* yet *fixed* body of knowledge, "teaching" as *facilitating* students' *movement through* an *obstacle course*, "learning" as *building* knowledge *structures*, and "curricula" as *assembling jigsaw puzzle*s. My enaction of these metaphors in the classroom were evident not only in the written, spoken, and spatial forms given above, but also in temporal and symbolic manners in which I taught graduated, grade specific mathematics programs and how my teaching of concepts progressed always, from "informal, concrete" stages to more "formal, abstract" ones. My teaching focused on enabling the conceptual, procedural, and relational understanding of the autonomous child.

What was revealed by identifying these metaphorical conceptions and the systemic patterns of actions which they engendered, was how my view of classroom mathematics had become "framed" within ways of thinking and teaching which amplified notions of: each child as an "individual", being "organized", being "sequential", and communicating mathematics as "straight-forward". As well, I had developed conceptual "blind spots". I was unaware of the mechanistic, linear, and hierarchical manners of thinking that these metaphors embodied. They were not conducive towards engendering ecological, non-linear, dynamically fluid forms of teaching mathematics. The challenge I encountered in realizing the limitations of my metaphors and wanting to enact ecologically coherent forms of teaching was that I could not simply change what I was *doing* in the classroom. Rather, I had to first critically assess, question, and provoke shifts in my thinking. Below is an excerpt from my journal in which I reflected on my metaphor of curricula as "jigsaw puzzles" in terms of its usefulness for mathematics teaching:

> So what of my mathematics curricula as jigsaw puzzles metaphor? The metaphor creates the image of a math curriculum as being a set of pieces... like concepts, skills, domains of mathematics, ...that when given to the students and the pieces are assembled correctly, reveals a coherent picture from its interlocking parts. Okay, yes, this... is a very "tidy" and "systematic" way for a teacher to think about math curricula.... but what this metaphor doesn't do is reveal the ecological qualities [i.e., an enactive view] which distinguish teaching and learning of math as fluid, co-emergent, and dynamic.... Even if a teacher did manage to design a program that *fit* mathematical topics *together* in a way which *produced* a *complete picture*, the curriculum would remain static. It would still be a set of distinct pieces. (July, 14, 2000)

By reading and working from within an enactive perspective, I developed metaphors that could give rise to forms of mathematical teaching which possessed an open, fluid integrity. The following journal entry details my conceptualization of "geographical maps" for curricula and "paths" for mathematical understanding and learning:

> An ecologically coherent metaphor of a mathematics curriculum needs to be one which creates an image of a flexible, dynamic network which evolves as a result of the interactions of the teacher, children, and the environment. A mathematics curriculum might then, [and as Bowers has described] not be thought of as a commoditized "thing"... that which "prescribes" what teachers are to teach and what children are to learn. Gregory Bateson's "map" metaphor works well in that mathematics curricula can be understood in light of an ecological... perspective. A curriculum envisioned as a map, enables teachers to locate mathematical topics, concepts, and skills that are considered to be important landmarks for the class' learning. Once these locations are marked out, the teacher can then think about how they are related to one another. This metaphor also expresses the idea that, what cannot be sketched out in advance, are the actual paths which the children will travel...to get to the mathematical locations...or the understandings they will establish when and after they come to these sites. Yes, the map locates mathematical landmarks, but it cannot possibly show the diversity of the landscape (the relational responses of teacher, students, and environment while engaged in mathematical studies)... children's mathematical paths will likely NOT be linear but instead, spread out in several directions and entail twists, turns and switch-backs. And so, it is only when children and their teacher engage in mathematical interactions that paths of learning come to be. Math curricula

> [and learning] imagined in this manner distinguishes them as co-emergent phenomena that are brought into being through children, their teacher, and mathematical settings. (August 2, 2000)

Other metaphors which I conceptualized during the study included: the "mathematics class" to be teacher, students, and the environment as *interacting, interdependent systems*, "mathematics" as a *dynamic, living* entity, as *residu*e, and as *source*, "mathematical processes" as *mathematical languaging*, and "mathematics teaching" to be more than me as the teacher but also the material and nonmaterial environment.

These metaphors highlighted for me, ecological notions of non-linearity, dynamic fluidity, co-emergence and complex circularity. They not only created critical shifts in how I thought about mathematics and mathematics teaching, but differences in my teaching also emerged. I created an integrated two year math program as opposed to two, one year programs. I no longer thought of my teaching of mathematics as moving children along sequential curricula but instead, mapped out different mathematical spaces for the class to explore, such as "Who are these things we call numbers?" and "Taking a closer look at snowflakes". As well, by engaging in mathematics for entire school days instead of 45-60 minute "blocks", I developed a greater flexibility and understanding for the unpredictability regarding the kinds of mathematics that could and did arise while the children were working.

For instance, as the class studied snowflakes in nature, on video, and in photographs, not only did they lay down learning paths that were diverse and unpredictable, the children collaborated with me in occasioning further investigations based on their learning. These included identifying and reflecting on the number patterns, arrangements that were found to be in multiples of six, and the geometric shapes and structures within snowflakes. The children were also challenged to explore different ways of thinking about and working with their mathematics. Through their poetry, symbolic notation, verbal descriptions, diagrams, and physical models, the mathematics that emerged existed in many forms--forms which were very much *alive*. For example, while making their own Koch snowflake, the children described the mathematics that they were seeing as "growing" patterns of triangles, corners, and sides. The snowflake investigations illustrate how an ecological sensibility in my thinking opened up a dynamic and fluid space for the class to interact with mathematics. Moreover, as the notion of complex circularity was integrated in the way that the class remained working within a particular mathematical space for days, weeks, months, or revisited these spaces, opportunities for recursion to occur in the children's mathematical understandings were possible. For example, after the children's initial study of the Koch snowflake, they later returned to it on three other occasions during the school year, reviewing the patterns they saw, and expressing them in descriptive and symbolic forms of addition, multiplication, and division. These revisitings were seen to provoke the children to move back and forth between their previous and present places of knowing, thereby giving them time and space to "thicken" and "extend" their mathematical understandings.

Concluding Thoughts

Researching teaching in the mathematics classroom continues to be a vast, rugged terrain that poses significant challenges for those who explore it while working in the midst of it. In taking an ecological, enactive view, it is important to understand that responsive teaching does not simply mean "moving with the flow" of the classroom or educational system. Teaching responsively involves continually questioning and responding to the ways in which one's teaching contributes to such a flow. This kind of systemic awareness focuses attention towards understanding the relationships or patterns that connect one's teaching to the kinds of mathematics and mathematical learning that emerge in the classroom. This study attempts to open a thinking space in which to examine and assess the metaphors which underpin mathematics teaching practices.

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TOOL-MEDIATED PEDAGOGY OF ELEMENTARY TEACHER EDUCATION: FROM MATHEMATICS ANXIETY TO CONFIDENT TEACHING

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This presentation focuses on the important role played by cultural tools (commonly referred to as technology in contemporary educational practices) in the development of mathematical and pedagogical content knowledge of pre-service elementary teachers. Particular attention is given to the utilization of various manipulative and computational learning environments, including Unifix® Cubes, Kid Pix Studio® Deluxe, and spreadsheets. The author argues that didactical emphasis on a tool-mediated and open-ended approach to the teaching of mathematics improves traditional pedagogy of memorizing mundane artifacts and computational routines in early grades, provides enjoyable learning experiences for the teachers, and turns one's mathematics anxiety into an enthusiasm towards teaching it.

From a theoretical perspective, a few instructional vignettes featuring the presentation are grounded in well-known action-oriented approaches to the teaching of mathematics to younger children. Furthermore, the vignettes are structured by a context-based and technology-enabled multi-step framework for concept development in a mathematics methods course that stems from the author's early experience in teaching mathematics to gifted children in Russia (Abramovich, 2001). The framework suggests that context itself does not account for a mathematical content. There is evidence, however, that the teachers believe in the robust interplay between the context and the concept and lack the experience of exploring mathematical potential of a particular context. Thus the framework points at the importance of creating situated learning environments that enable the emergence of new concepts from the variation of situated counting questions.

The presentation shows that the use of manipulative and computing technology has a great potential to enhance tool-mediated mathematics pedagogy enabling interactive experimentation with mathematical content and promoting exploratory approach to mathematical concepts. The possibility of interactive experimentation and exploration suggests that the tools of technology can serve as powerful motivational factors in learning mathematics by all students. Included in the presentation comments by the teachers indicate that the appropriate use of cultural tools in conducive to presenting mathematics as meaningful and motivated subject matter that ultimately one would enjoy teaching to younger children.

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THE DEVELOPMENT OF AN INSTRUMENT: MEASURING PRESERVICE TEACHERS' ATTITUDES ABOUT DISCOURSE IN THE MATHEMATICS CLASSROOM [PADM]

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This study explores the factors that comprise preservice teachers' (PSTs) attitudes regarding discourse in the K-12 mathematics classroom through the development of a 5-point Likert instrument. Analysis of the PADM (N = 277) resulted in three reliable factors: *Promoting Mathematical Reasoning* ($\alpha_1 = .85$), *Examining Complex Mathematical Concepts* ($\alpha_2 = .81$), and *Valuing Students' Mathematical Ideas* ($\alpha_3 = .85$). These results suggest a framework from which mathematics educators can use to prepare prospective teachers to realize recent mathematical reform efforts.

Discourse, or "the [ways] of representing, thinking, talking, agreeing and disagreeing" about mathematics (NCTM, 1991, p. 34) has been a central focus of recent NCTM *Standards* documents (1991, 2000). It allows students to become reflective and critical of their work (NCTM, 2000) while nurturing their development of conceptual mathematical ideas and in-depth knowledge (Brendefur & Frykholm, 2000).

Preservice teachers' mathematics attitudes (particularly in elementary) are generally negative, and most have been exposed to traditional lecture-style mathematics teaching. The combination of these two characteristics presents a situation where prospective teachers are at odds with reform efforts that call for discourse to be used as a pedagogical tool (NCTM, 1991). Since teachers' attitudes predispose them to particular behaviors (Gable & Wolf, 1993) and teaching strategies (Harper & Daane, 1998), we need to investigate PSTs attitudes regarding mathematical discourse. We first must identify the underlying factors contributing to the theory.

This study that was exploratory in nature, developed an instrument that utilized a 5-point Likert scale with the presumption that the items would cluster around NCTM's *Environment, Teacher's Role*, and *Tasks*. A principal axis factoring analysis of the pilot, administered in Spring 2001 to a sample of 179 PSTs at a major New England university, derived 6 factors, 3 of which were reliable: *Classroom Environment* ($\alpha_1 = .80$), *Representations of Conceptual Reasoning* ($\alpha_2 = .73$), and *Verbal Communication* ($\alpha_6 = .77$). Items that did not load on a factor were eliminated, those that contained unclear language as indicated by either a low loading or loaded across factors were rewritten, and items that more comprehensively represented the literature and PSTs written responses to questions regarding discourse were added.

The current 26-item version of the instrument was administered in Fall 2001 to a sample of 274 PSTs at the same university. A principal component analysis was conducted, resulting in a three-component solution detailing more explicit factors than the *Environment, Teacher's Role*, and *Task*. Specifically, items clustered around *Valuing Students' Mathematical Ideas* ($\alpha_1 = .85$), *Promoting Mathematical Reasoning* ($\alpha_2 = .81$), and *Examining Complex Mathematical Concepts* ($\alpha_3 = .85$). These factors suggest a framework that can be incorporated into courses preparing PSTs to use mathematical discourse as a pedagogical tool that integrates the three hypothesized factors.

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LEARNING TO TEACH THROUGH MICRO-TEACHING LESSON STUDY

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In recent years, documents such as *Tomorrow's Teachers* (Holmes Group, 1986), *A Nation Prepared: Teachers for the 21st Century* (Carnegie Task Force, 1986) and *What Matters Most: Teaching for America's Future* (NCTAF, 1996) have provided an impetus for increased attention to teacher education programs. These documents indicate that teacher education programs are not preparing teachers adequately to educate future generations of students. In conjunction with these documents and pertaining specifically to mathematics, documents published by the National Council of Teachers of Mathematics [NCTM] (1991, 2000) have provided a vision of mathematics teaching and learning that calls for a revision of mathematics teacher education. Many prospective secondary mathematics teachers continue to lack images of "reformed" teaching, teaching that engages students in experimenting, analyzing, conjecturing, justifying, and making connections. Ball and Cohen (1999) claim that teacher education needs a "fundamentally different content and character than it has now" (p.26). They surmise that much of teachers' learning is based on their own observations of practices from their own learning experiences and their individual observations of teachers in placement schools. For preparing secondary mathematics teachers, we need to engage them in more collaborative tasks, discourse and environments that will help them more systematically develop images of reformed mathematics teaching and the capabilities to reflect on, reason about and engage in change.

For this investigation, a micro-teaching lesson study task was developed to help prospective teachers begin to plan, implement and analyze their own teaching with respect to the *Teaching Standards* (NCTM, 1991). The lesson study task consisted of prospective teachers working in groups of three through two cycles of planning, teaching, and analysis of peer micro-teaching lessons. Working in groups of three, the preservice teachers planned 20 minute lessons to teach to a small group of their peers. The topics for the lessons were selected randomly from a hat; topics included mathematics concepts and relationships that were somewhat familiar to the participants (e.g., types of polyhedra, limit of a function, distance formula) to mathematics that was unfamiliar (e.g., fractals, traceable paths, Euler's Formula). After the planning, each group member was video-taped teaching their lesson to a small group of their peers. After analyzing their own lesson, the participants again in their groups of three analyzed the videotapes of and gave feedback to their two group members. During this time, I, as the course instructor, was reading the lessons and watching the videos to also provide individual and group feedback to the class. Later the groups of three discussed what they had learned from the experience and began a second cycle of the micro-teaching lesson study.

The central question under investigation was 'How does this lesson study influence prospective teachers' development and implementation of reform-oriented teaching and self-analysis of their teaching?' The data collected included group notebooks compiling their lesson plans and reflections, video tapes of their teaching, field observations, and written surveys. Overall, from the first to the second lesson, the prospective teachers' teaching became less teacher centered and incorporated student experimentation, analysis and reasoning to a greater extent. They also enhanced their understanding of the *Professional Teaching Standards.* All eighteen prospective teachers felt the micro-teaching lesson study task was valuable, particularly, the teaching and receiving feedback from others. The majority felt that analyzing others lessons and using the *Professional Teaching Standards* as a framework was of value; however, a few of the prospective teachers felt that watching and analyzing others lessons was not helpful in developing their own knowledge and that using the *Professional Teaching Standards* as a framework to analyze the lessons was somewhat tedious.

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ALLOWING TEACHERS TIME TO LEARN: STAGE ONE

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The Problem and the Questions

Improving the mathematical preparation of elementary teachers is a crucial element in improving K-8 students' understanding of mathematics. At Eastern Washington University (EWU), pre-service teachers' limited understanding of mathematics is evident in a methods course with two quarters of mathematics as a prerequisite. In a recent section of the methods course when asked to explain which is larger 0.41 or 0.411, 18 of 26 students explained why 0.41 was larger.

Question 1--Will students enroll in a three-quarter sequence of mathematics courses offered as an alternative to the traditional two-quarter sequence?

Question 2—Will the three-quarter sequence of mathematics improve students' retention of mathematical understanding?

The Experimental Sequence and the Students

The experimental sequence used the same text as the traditional sequence. The additional time in the experimental sequence was used to develop study skills, address students' lack of prerequisite knowledge, have guest speakers, and show videos of K-8 students and classrooms to place learning in context.

The SAT mathematics scores for the 20 students enrolled in the experimental sequence ranged from 330 to 500. Seven of the students had failed the initial course in the traditional sequence. Thirteen of the students had taken remedial mathematics courses at EWU. Five of the students were adult returning students.

Preliminary Results and Stage Two

The initial analysis of data suggests that students a) will enroll in an additional quarter of mathematics and b) are developing an understanding of mathematics. Twelve of the students enrolled in the experimental sequence answered the following question correctly compared to only 2 of 7 students with a concentration in mathematics enrolled in a methods course.

If C equals the height of a person in centimeters and M is the height of a person in meters, write an equation that will give the height of a person in M meters if the person is C centimeters tall.

During the 2002-2003 academic year, a) a non-tenure track instructor will teach the experimental sequence and b) in the methods course, the mathematical understanding of students who completed the experimental sequence will be compared to the mathematical understanding of students who completed the traditional sequence.

THE IMPACT OF EXPERIENCES WITH MATHEMATICS TEXTBOOKS AND CURRICULUM MATERIALS ON PRESERVICE ELEMENTARY TEACHERS' BELIEFS

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Numerous empirical studies over the past decade have indicated that teachers' attempts to enact reform visions in their classrooms are fraught with conceptual and practical challenges, many of which relate to teachers' lack of personal familiarity with reform-oriented instructional practices and representations of mathematics. In light of this situation, it is important for teacher educators to challenge teachers' conceptions of mathematics and pedagogy to promote meaningful, long-term changes in classroom practices.

Our ongoing research project examines the impact of different text materials on the professional development of preservice elementary teachers. In two sections of a *Mathematics for Elementary Teachers* course, different text materials were implemented. In one section [Section A], a widely-used college textbook was implemented (*A Problem-Solving Approach to Mathematics for Elementary School Teachers* by Billstein, Libeskind, & Lott). In the other section [Section B], selected units from reform-oriented middle school curriculum materials were implemented (*Mathematics in Context [MIC]* and *Connected Mathematics Project [CMP]*). The MIC and CMP units were chosen to correspond with the mathematical emphases of the traditional college textbook. Results of mathematical pre- and post-tests indicate that the teachers in the two sections had similar mathematical preparation and gained comparable mathematical knowledge in the course.

Because the text materials are based on different assumptions about mathematics teaching, teachers in the two sections of the course had different learning experiences. For instance, when using the traditional college textbook, teachers are expected to first gain exposure to important information (via the book, the instructor, or both) and then to complete a collection of practice problems for further practice and understanding. In contrast, the reform-oriented middle school units require teachers to engage in investigative activities to develop general rules and concepts *while* they solve problems. Although *mathematically* the two sections of the course were comparable (both sections reached the same mathematical conclusions from their work), *pedagogically* the sections differed in ways that appear to have significantly impacted the teachers' views of teaching and learning, including their views of appropriate curricular materials for teaching and learning. Our poster focuses on differences in these views.

At the beginning of the course, most teachers portrayed a good mathematics textbook as one that contains example problems (solved for students), clear explanations, practice problems for students to solve, problems representing real-world situations, easy to understand steps for solving problems, and helpful visual aids. Implicit in these beliefs is the notion that students first *learn mathematical material* via clear explanations, examples, and visual displays, and then *practice what they have learned* by solving problems similar to the textbook's examples.

At the end of the semester, teachers in Section A maintained the same chiefly traditional views, and in some cases, *strengthened* their adherence to traditional textbook formats. In contrast, at the end of the semester, teachers in Section B, who worked with middle school materials, appear to have developed several exciting new perspectives including the notion that students can generate ideas and determine mathematical correctness for themselves. This notion, coupled with the suggestion that mathematical understanding includes more than just procedural competence, represents a significant shift from traditional views of mathematics teaching. Section B teachers also identified the idea that teachers can learn about both mathematics and teaching from students' textbooks.

These results, illustrated in greater detail on our poster, suggest that the text materials with which preservice teachers engage impact their conceptions of textbooks as well as their more general conceptions of mathematics teaching and learning. Our results also give rise to numerous questions for further consideration. What differences might we observe in Section A and Section B teachers as they progress through their teacher education programs and early teaching experiences? How might the teachers' classroom practices with mathematics relate to their course experiences as preservice teachers? Investigation of such questions will bring us a deeper sense of what it means for teachers to learn in reform-oriented ways and how such learning much benefit teachers as they interact with children in elementary classrooms.

ENHANCING STUDENTS' MATHEMATICAL UNDERSTANDING BY DEVELOPING TEACHER CONTENT KNOWLEDGE

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Professional development can empower teachers to be better prepared to address mathematics education challenges to enhance student achievement. A cohort of middle school teachers participated in a funded grant program in Texas focusing on strengthening content knowledge, improving instructional strategies, and implementing curricula. Gains were reported in teacher content knowledge, understanding, and skill that may have promoted gains in students' mathematics understanding as reflected on test score results. This discussion highlights the importance of strengthening teachers' mathematical content knowledge (rational numbers concepts and proportionality), strategies, and techniques to promote greater student achievement.

Reform efforts in mathematics education challenge teachers to explore and implement alternative approaches to teaching and learning mathematics. There must be support offered to develop and sustain all teachers. "Teachers of elementary, middle, and high school mathematics need broad and deep knowledge in three fundamental domains: mathematics, mathematics teaching, and students" (National Council of Teachers of Mathematics (NCTM), 1991, p. 191). Through teacher professional development, there are opportunities to enhance student achievement (Loucks-Horsley & Matsumoto, 1999). The NCTM (2000) has identified rational numbers and proportionality as areas needing urgent attention in the middle school. It is crucial that teachers have a solid understanding of the content that they are teaching and have the pedagogical expertise to provide rich learning experiences.

Data were collected through quantitative and qualitative methodologies (pretest/posttests, observations, analysis of student state assessment scores, surveys and questionnaires, and open-ended evaluations). Teachers of grades 6-8 participated in a standards-based, constructivist program that integrated knowledge components (mathematics content, school curriculum, and concept development) with instructional components (student-centered experiences, activity-based learning, use of manipulatives, and technology). Goals were to strengthen teachers' content knowledge in low test performance areas of their students and to implement problem solving strategies through learner-centered activities using manipulatives and technology. Teachers developed confidence in teaching rational numbers and proportionality through cooperative learning, and questioning and assessment strategies. Teachers enhanced personal knowledge of rational numbers and proportionality and began to implement better classroom lessons. Increases in student test scores (ranging from 4.1 to 8.2 percentage points) indicate the teachers may have enhanced student achievement in mathematics by deepening their own conceptual understanding. There must be opportunities for professional development in content, process, and methodology. It is crucial that a support system in the mathematics learning community be developed along with any efforts to change, alter, and improve the instruction in the classroom (Chapin, 1996). Research in the areas of teacher content knowledge and professional development helps to further a deeper insight into the psychological aspects of teaching and learning mathematics.

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PRESERVICE TEACHERS' UNDERSTANDING OF WHOLE NUMBER OPERATIONS

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Our primary research goal was to investigate the development of elementary preservice teachers' understanding of arithmetic operations over a semester-long course in number systems, the first of a three-semester mathematics sequence for elementary teachers. Our secondary goal was to use this data to create and implement curricular adaptations. This poster will 1) describe preservice teachers' initial understandings of arithmetic operations; 2) describe changes that were implemented in our mathematics sequence for elementary teachers; and 3) discuss changes in these understandings.

Perspective

Carpenter, Fennema, Franke, Levi and Empson (1999), in their research on cognitively guided instruction (CGI) have developed a framework for analyzing conceptions of addition, subtraction, multiplication, and division. Their work provided a description of the different problem types related to the four arithmetic operations. Specifically, they described in detail the four basic classes of addition and subtraction problems: *Join, Separate, Part-Part-Whole,* and *Compare*. Carpenter et al. also discussed three problem types for multiplication and division: *Multiplication, Measurement Division*, and *Partitive Division*. These problem types encompass the range of grouping and partitioning problems, and each one reflects one of the three quantities that can be unknown. These problem classifications were used in the analysis of responses given in the study that will be discussed in this presentation.

Methodology

The participants in this study are preservice elementary teachers who were enrolled in the first of a three-course sequence in mathematics (Number Systems) designed for elementary teachers. Numerous changes were made to incorporate pedagogical revisions that would positively impact students' understanding of mathematics. Revisions were made according to the new National Council of Teachers of Mathematics Standards (2000); these changes include an increased emphasis on: 1) multiple concrete representations; 2) cooperative learning and mathematical discourse; and 3) technology.

During the course, preservice teachers studied different models used to solve word problems involving whole numbers, and investigated the connections among the operations. Data were collected on these understandings in all of the number systems courses in Fall 2001. An instrument was administered at the beginning and end of the semester to obtain data on preservice teachers' knowledge of whole number operations. Participants were asked to create two different types of subtraction and division problems, and model their solutions. Six participants were then selected for interviews during the second half of the semester, in order to further investigate understanding of connections among whole number operations.

Results

Results demonstrated that the dominant models at the beginning of the course were the Separate (Take away) model for subtraction and the Partitive (Sharing) model of division. These models remained dominant throughout the semester, but end-of-semester data revealed that other models were used more frequently than before the course.

Responses from the interviews revealed a broader interpretation of the use of subtraction. Participants were able to discuss the different types of subtraction and division, to varying degrees, as well as model their solutions with some success. What is interesting is that, although none of the students categorized the problems according to the way Carpenter et al. (1999) would, several of them used a consistent and coherent organization scheme, and were able to discuss and defend their decisions.

Implications

Preservices teachers need to be knowledgeable about the different models related to arithmetic operations, so that they may supply a rich environment for their students in which to investigate such problems. Their future students may also use some of these models in their own efforts to make sense of the mathematics they are learning, and teachers need to be well informed in order to appropriately guide their students.

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MATHEMATICS CURRICULUM REFORM AND THE ROOTED CULTURE OF TEACHING AND LEARNING: THE CASE OF MALAWI

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The purpose of this study is to describe mathematics education reform in Malawi. Malawi, once a British colony, gained independence in 1964. The country lies east of Zambia and shares borders with Tanzania in the north and Mozambique in the west, south and southwest. Malawi is a landlocked country and covers about 118,484 sq km (46,208.8 sq. miles). The population is approximately 11 million people with a population growth of 3.5%. Total literacy level is approximately 56%. Current primary school enrollment is approximately 3.1 million children and about one third are girls.

Malawi is striving to improve the quality of education through curriculum reform, improvement of teacher education programs, physical infrastructure and strengthening of educational management systems. Due to free primary education, which was introduced in 1994, the education system has experienced an influx of students resulting in high teacher-student ratios, inadequate textbooks, inadequate classrooms and recruitment of untrained teachers. The purpose of this study is to investigate mathematics education reform in Malawi in light of these challenges in addition to a rooted culture of teaching and learning which equates teaching to telling and learning to listening.

Growing evidence suggests that students' learning is more meaningful, and that they feel empowered and more motivated to learn if they are given the opportunity to be active and investigative and take charge of their own learning (e.g., Cobb, Wood, & Yackel, 1991; National Council of Teachers of Mathematics [NCTM], 1989, 2000). Such a vision of learning suggests a different role for the classroom teacher (NCTM, 1989, 1991, 2000; Simon, 1995) and has implications for the preparation of new teachers as well as the ongoing professional development of in-service teachers (Brown & Borko, 1992; Kilpatrick, Swafford, & Findell, 2001). The challenge for mathematics educators in general, and Malawi in particular, is to prepare new teachers that can initiate reform, while at the same time encourage growth and change for in-service teachers. This struggle is not a new one, but is especially challenging in Malawi amidst the many other limitations with which educators must deal.

Based on a descriptive analysis of teacher interviews, observations of school settings, and printed documents, this study provides a snapshot of the present state of mathematics education reform in Malawi. Data depicts efforts being made and challenges being experienced by primary (grades 1-8) school teachers and teacher educators in Malawi as they try to implement investigative and inquiry-based approaches to the teaching of mathematics (Baroody, 1998; NCTM, 1989, 2000). Data also includes pictures of classrooms, classes in session and school infrastructure. In addition, other materials will depict the evolution of teacher education programs over the past thirty years, both for pre-service and in-service teachers, with particular attention to mathematics education.

Apart from depicting what Malawi is doing to overcome the challenges, this study offers recommendations as well as raises further questions to solicit ideas on how Malawi may forge ahead to improve the quality of mathematics education amidst the challenges being experienced.

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EXTENDING CLASSROOM BOUNDARIES AND SUPPORTING INSTRUCTIONAL GOALS USING A WEB-BASED COMMUNICATION TOOL

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The Dialogue application is a web-based communication tool that extends classroom boundaries and supports collaboration among community members. The program's features provide environments where professional dialogue may be modeled and nurtured. The Dialogue contains a hierarchy feature that sustains communication among all members of a learning community within and across courses and over semester and years of a program. Graduates of the teacher preparation program remain in their communities for two years after graduation. The Dialogue application has been most useful to those communities based on the belief that students as a community create a knowledge base for the community and become resources for each other.

The Dialogue Project is the product of education faculty and students working together with a technology research and development group to develop software that responds to the needs of the education community. The focus of the Dialogue project is on both the technology integration issues and the pedagogical issues embedded in the implementation of on-line resource to enhance teaching and learning. It targets issues that arise when producing an application that meets the challenges present in various educational environments. These environments include single courses, learning communities for specific cohorts of students and an entire School of Education program.

The Dialogue application supports multiple aspects of university and schoolwork by providing a variety of workspaces that support in-depth reflection and conversations by students and faculty as well as other members of specific learning communities. The features for this application have emerged from the classrooms and needs of K-16 educators. The metaphor used in the development interface is based on the building of community.

The Dialogue project has three discussion forums called conversations: community wide, small group and one to one. The community-wide conversations are shared with all members of the community. These conversations give each class member the opportunity to share products such as a lesson concept map, a photograph or an original historic document. Small group conversations are among community members but private to the members of the small group community. Small groups have been used to build understandings around observed events, readings, or shared experiences. One-to-one conversations are for those events that are to be shared between a community member and the administrator, or between student and instructor. The one-to-one conversation area supports the reflective needs of an educational community. An outcome of both the collaborative and reflective experiences is the creation of a community knowledge base.

Extending this community created knowledge base are resources and a toolbox. Resources are often items such as a syllabus, videos, original documents, and other informative pieces. A community directory holds each member's profile of information to be shared with other members The toolbox includes evaluation templates, the cumulative record of comments a member has received, as well as the ability to search the community conversations for specific data. Members have the opportunity to create a personal portfolio. A community instructor may create a community portfolio as well as a portfolio of documents for research purposes.

Questions that arise when infusing technology into a course are: what kinds of conversation do instructors want to support; what kinds of conversations and use of resources will help students view an event through more than one perspective or lens; what kinds of conversations and resources will form a basis for students to construct their understanding of the content of the course; and what are the most appropriate technological tools to achieve these goals? This presentation will use data collected during the project to begin to address the above questions.

Note

The development and the implementation of the Dialogue application is supported in part through Preparing Tomorrow's Teachers to Use Technology (PT3) funding.

THE IMPORTANCE OF FLEXIBLE MATHEMATICAL AND PEDAGOGICAL CONCEPTIONS FOR SECONDARY TEACHERS

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The poster will focus on secondary teachers' knowledge. Teachers must understand mathematical ideas in different ways than other mathematically trained individuals. For example, it is more important for teachers to understand mathematical concepts in multiple ways. Flexible mathematical understandings not only help teachers to help their students develop deep understandings, flexibility helps teachers understand a variety of possible ways their students might interpret important ideas (e.g., Even & Tirosh, 1995). The poster will provide examples about two theoretical ideas: *mathematical authority* and *pedagogical authority* (Wilson & Lloyd, 2000).

The poster will focus on the mathematical concept of function. To illustrate how college courses might help secondary mathematics teachers develop flexible understandings of important mathematical and pedagogical ideas, the poster will show an activity emphasizing multiple representation of functions. The activity requires teachers to sort collections of functions represented in a variety of ways (Cooney, Brown, Dossey, Schrage, & Wittman, 1996). Cards containing functions will be displayed, as well as descriptions of required sorting tasks and reflections about the pedagogical benefits of sorting activities.

It is important for mathematical ideas to be correct because they make sense and work, not solely because an external authority (such as the textbook or teacher) declares that they are right. This issue exemplifies the importance of reasoning. The poster will show an activity designed to help teachers learn how to share mathematical authority with their students. This example involves teachers writing about and then discussing issues related to historical developments of functions (e.g., Cooney & Wilson, 1992). In addition to communicating the human and changing nature of mathematics, this activity helps teachers understand that most of the common definitions and conventions associated with functions developed out of need (e.g., to solve problems). As with other important mathematical ideas, the definitions and conventions are right because they make sense and work, not because of someone's declaration.

The poster will also illustrate how one high school teacher's flexible understanding of functional relationships contributed to his teaching effectiveness (Lloyd & Wilson, 1998). Although this teacher preferred graphical displays of functions, he was able to attend to some of his students' preferences for tabular representations because he understood functional relationships in multiple ways. It can be argued that everyone should understand functions as relationships between varying quantities, as sets of ordered pairs, as graphs, as variable expressions, and as tables, but it is even more important for secondary teachers to think flexibly about functions (and other secondary topics).

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Probability and Statistics

SOME CHALLENGES FOR THE USE OF COMPUTER SIMULATION FOR SOLVING CONDITIONAL PROBABILITY PROBLEMS

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We present some of the results of an exploratory study with some university students, that besides looking students' intuitive ideas about probability, looking for the spontaneous ideas they have about the frequentist probability and, in particular the difficulties they face solving conditional probability problems using computer simulation. Furthermore, we tried to identify the strategies they used estimating the values of the requested probabilities.

The conceptions and the difficulties that the students have with the conditional probabilities have been widely studied for a long time (Kahneman & Tversky, 1982, Falk, 1982; Gras & Totohasina, 1995; Ojeda, 1994; Sánchez, 1996).

One deeply-rooted idea in the students is that the conditional probability is a measurement of the causal relation that exists between two events, in such a way that the values that the people assign to the conditional probabilities depend on the causal connotation of the conditioning event in relation to conditioned event (Kahneman & Tversky, 1982).

Another deeply-rooted conception in students and related to this causal aspect, is the chronological conception by which is thought that the conditioner event cannot be later in the time than the conditional event. Concretely, Falk (1986) analyzes the following problem:

> *Urn Problem.* An urn contains two white balls and two black balls. We shake the urn thoroughly, and blindly draw out two balls, one after the other, without replacement. (i) Suppose we know that the first ball that was drawn was white. What is the probability that the second ball is also white? (ii) Suppose we know that the second ball that was drawn is white. What is the probability that the first ball is also white?

Falk reports that out of the 80 students who responded the first question well, 41 (51.2%) responded ½ for second, which could be explained by the fact that for the students the first drawing, being causal independent from the result of the second, would have to retain its original probability without considering the result of the second drawing. Falk suggests a physical simulation of the problem, so that the students grasp the idea that the second event can be the conditioning one.

In this same direction is the work of Gras and Totohasina (1995) who ratify the existence of the chronologist and causal conceptions and propose the hypothesis that these conceptions are not correlated, i.e., that they are constructed separately in the students. In addition, they conjecture, that the used notation of "A given B" can induce a chronological conception as well as "A if B" can be indicative of a causal conception. Both notations make the students think that they deal with a new event. The students who begin the learning of this notion seem develop a certain confusion between the variation of the probability and the variation of the information. This confusion is expressed by thinking that when the information increases the probability increases too, which implies adopting the causal conception of the conditional probability.

Falk (1986) shows how the difficulties in selecting the event to be the conditioning event can lead to misconceptions of conditional probabilities. This is a nice example:

> *Chip Problem.* Three chips are in a hat. One is blue on both sides, one is red on both sides, and one is blue on one side and red on the other. We draw one chip blindly and put it on the table as it comes out. It shows a red face up. What is the probability that the hidden side is also red?

Falk relates that the students often mistakenly believe that the probability that the second side is red is ½ . They argue that there are two possible chips to consider after the first one is shown to have a red side, the RR chip and the RB chip. However, the conditioning event is not the chip, is the side of the chip. Thus, the RR chip has two ways to give us a second side that is red. The probability that the other side is red is thus 2/3 .

Confusing the conditional probabilities and the joint probabilities is another one of the common errors among the students when solving problems of conditional probability (Pollatsek and cols. 1987; Ojeda, 1994; Yáñez, 2002).

This confusion takes to Pollatsek and cols. to speculate that , "instead of separate, differentiated of joint and conditional probability, some subjects may have available a concept that is some amalgam of the two." (p. 269).

From another point of view, Gigerenzer and Hoffrage (1995) reported the result of a theoretical and empirical analysis that allows them to conclude that the Bayesian reasoning in the people improves when the information is given in formats of frequencies, this means, when instead of talking about 5%, it is said that from 100 people, 5 have the disease.

All these exposed difficulties resist to the educative process and the passage of time, as show the researches of Fischbein and Gazit (1984), Fischbein and Schnarch (1997), Lecoutre and Fischbein (1998). For all these reasons, for some time the researchers in these subjects have been proposing that an answer to these difficulties could be to assume the frequentist approach of the probability (Shaughnessy, 1992).

The frequentist approach of the probability responds to an experimental conception of the probability. It is only after having made the experiment several times that the obtained results allow us to make an estimation of the probability of the event at issue. This empirical approach is expressed mathematically in the form:

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N} \quad (1)$$

where N is the number of times that the experiment is made and N_A is the number of times that the experiment was favorable to the event, and thus (1), the limit of the relative frequencies, is the value of the probability of the event A. Naturally, the use of this approach does not allow to calculate the exact value of the probability but only one estimation that depends on the number of experiments, what in principle adds certain difficulty to the use of this approach in the school. In order to overcome this limitation, the computational simulation allows to generate a great number of repetitions of the experiment and, with graphical aids, to analyze the stability of the relative frequencies and obtain a quite acceptable estimation in most of the cases. However, this idea of "discovering" the trend to the infinite of the relative frequencies is a demanding process that can not support on algorithmic ways, which causes that often the students try to avoid it and resort to strategies that satisfy them and that are not very demanding. The probabilistic process of estimation is the front door to the statistical processes of estimation where the estimations always entail an error what shocks with the mathematical exactitude to which the students are accustomed.

One of the strategies we will show the students use, is *the last generated value* of the relative frequencies, i.e. if they generate 1000 cases, (repetitions of the experiment) the value of the quotient of the number of favorable cases to the event divided between 1000 is the value adopted like estimation of the probability requested.

Besides being a universal problem solver (Batanero 2001, p. 136) the computer simulation has some interesting features for its implementation in the classroom. Some of them are:

- Its close contact with the reality of the random experiments can allow to overcome the people's misconception about on the chance that oscillates between the idea of a disordered and irregular process and a process where anything can happen with the same likelihood. (Serrano, 1996; Yáñez, 2002).
- It allows to overcome, as Biehler (1991, p. 170) says, the "lack of experience" that the students have about random experiments, providing representations of the real phenomena and increasing the number of repetitions in short time.
- Solving problems by to computer simulation increases the range of problems and realistic situations accessible to students' activities overcoming the "concept-tool gap" as Biehler says (p. 170), i.e. the gap between the intended generality of the probability concept and the system of operations and tools students actually use.
- Acting as a laboratory, checking the results obtained theoretically. In this sense,

allows an exploration, changing the assumptions of the model, making further experiments, etc.

- The inaccuracy of the produced estimations becomes a trigger to construct a mathematical model that under certain considerations produces exact explanations and results.

With these perspectives, it raises the question about the effects that the computer simulation might have on the understanding of the probabilistic concepts. There are studies that document that even though the simulation as a method for modeling probability problems produces good results, it exists evidence that points out that the random experiences are not enough, neither for improving the probabilistic methods of the students nor for helping to develop the inductive methods. However, it's useful for highlighting their misconceptions and giving the students better explanations about the reality (Zaki, Pluvinage, 1991, Bordier & Bergeron, 1998).

From another point of view, the use of the computer in order to adopt the frequentist approach in probability raises questions about its use that are worthy to study. In this sense, Coutinho (2001) reports some difficulties the students had when several situations were given for building urn models for simulating some probabilistic games. These difficulties are associated with the use of the software, the resistance to use the simulation for solving a problem that can be solved directly by calculation and the difficulty of accepting simulation data that have not been obtained by themselves in order to estimate probabilities.

With these ideas in mind, we are developing a research project to study the effects that produces on the understanding on probability and conditional probability an instruction supported by the use of computer simulation. We are particularly interested in knowing the effects of this instruction on the misunderstanding about probability and conditional probability; in the development of skills to solve problems; in identifying the difficulties that the students have with the use of the computer simulation and knowing the relationship produced between mathematical theory and the reality with the use of the computer simulation.

The search, that has a qualitative character, carrying out an extensive course with the basic subjects of the theory of the probability. Before the course we made an exploratory study to know the difficulties that could be presented with the use of the statistical package Fathom (2000) that demands certain programming to define the attributes that allows to solve problems of conditional probability using simulation. In this exploratory study, besides looking students' intuitive ideas about probability, we look for the spontaneous ideas they have about the frequentist probability and, in particular the difficulties they face solving probability problems using computer simulation. Furthermore, we tried to identify the strategies they used estimating the values of the requested probabilities. Some of the results obtained are the content of this paper (Yáñez, 2002).

Description of the Students and the Activity

Twelve university engineering students in Mexico City participated in this exploratory study. The activity was carried out in four sessions of three hours each. In the first two sessions besides a diagnostic test for knowing their probabilistic intuition and after a short introduction to the frequentist approach, we started to work with Fathom (Finzer et al., 2000), solving some problems with computer simulation. In the third session the students worked in pairs solved three problems of conditional probability; in the fourth session the students individually solved three problems of conditional probability. The idea was to detect the spontaneous work and conceptions that the students had with respect to simulation and the frequentist probability, and find out the strategies they used in solving problems.

Presentation and Analysis of the Obtained Results

In the following, we present the solutions given by some of the students to the problems of urn and the chips mentioned in the introduction, and that were proposed to the students in the last session.

The Urn Problem

In the diagnostic test, half of the students who participated in the study answered ½ to the second question showing that they had the chronological conception of the conditional probability, in the sense that they believed the "time axis" prevents an event from being a conditioning event, as had been reported by Falk (1986).

When the students were asked about this problem in the last session to be answered with the aid of Fathom, the answers of some of the students were the following ones:

> Carlos, to solve the urn problem simulated the drawings of the two balls with substitution. To answer the first question he produced the graphic in Figure 1.
>
> He wrote:
>
> "From this exercise I can say that the stability is between 0.2 and 0.3 and its frequency is 0.23 with 500 cases. This for question 1, and for question 2 with 500 cases, its stability hardly change, staying between 0.2 and 0.3 and a frequency of 0.274"

Carlos proposes an estimation by intervals that he identifies with the interval of stability of the relative frequencies in the sense that their values from certain moment are included between the given values. In addition, he proposes an estimation that he calls "frequency", and that corresponds with the last value obtained for the relative frequencies.

However, Carlos's answer on paper was the following:

(i) Probability 2nd ball is white = 1/3 because one ball has been drawn.

(ii) Probability 1st ball was white = 1/3 counting all the possible balls and knowing that the second drawn ball was white.

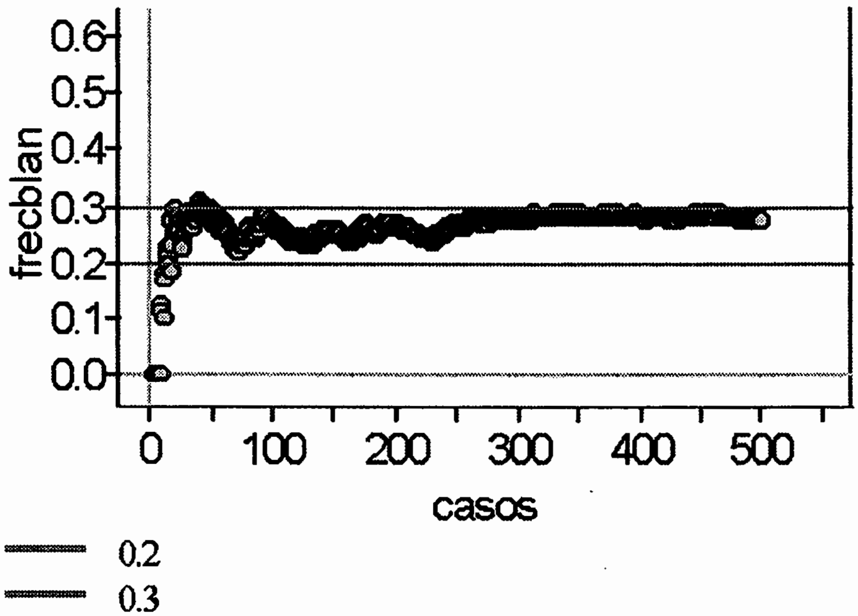

Figure 1. Carlos's graphic for the first problem.

Carlos' analysis and answer on paper and pencil were right, specially in the second question which responds in a similar way to the first one emphasizing the equivalence that exists between the two questions.

As evidence of the transformation that Carlos lived in the course of the activity we showed the answers that he had given when the same problem was asked in the diagnostic test":

(i) P(2nd white ball) = 1/2 , i.e. one white and two blacks.

(ii) P(1st white) = 2/2. I say this because it is possible that the two black balls could have been drawn, because the number of balls are the same.

To what extent their new answers are the product of their work with the simulation is an answer that it is still to be unanswered.

Cesar simulates the experiment well and answers both questions correctly. However when he estimates the probability of the second question, which he had not answered in the diagnostic test, with the value 0,33 that he fixes to 1/3, adds: "*I got the same value as the first question but I am not very convinced.*" In spite of his confidence in the simulation process he made, he does not accept the answer because it is against his convictions. Will it be that the single computational simulation is not enough to overcome the misunderstanding of the students and it is necessary to resort to the physical experimentation?

Jose does not simulate the experiment, but as a white ball had already been drawn he simulates the drawing of one second ball of a bag with two black balls and one white, and to estimate the probability that the ball was white. His programming makes us believe that the students can be thinking not about simulating the random experiment but about making a directed programming to respond the questions asked.

The Chips Problem

Cesar makes the following programming, where the created attributes appear before the points and the expression that defines them appears after the points. The randomPick function allows to randomly select the characters that appear in the parenthesis.

Bolsa: Randompick("f1","f2","f3")
f1: Randompick("R","R")
f2: RandomPick("A","A")
f3: RandomPick ("R","A")

numf1: if((bolsa = "f1") and (f1 = "R")) $\begin{cases} 1 \\ 0 \end{cases}$

numf3: if((bolsa = "f3") and (f3 = "R")) $\begin{cases} 1 \\ 0 \end{cases}$

cantidad: prev(cantidad) + numf1 + numf3
cantf1: prev(cantf1) + numf1

frecuencia: $\dfrac{cantf1}{cantidad}$

As it is observed, Cesar begins creating an attribute to choose the chip, and although later he defines redundant attributes f1 and f2, his analysis indicates that he assumes that the conditioning event is the sides of the chips.

Rafael identifies the problem with a urn problem where he assimilates each side of chips with a ball that has the same color. His programming was the following:
Bolsa: randomPick (A, A, R, R, A, R)

CaraRoja: if (Bolsa = R) $\begin{cases} RP(R,R,A,A,A) \\ RP(R,R,R,A,A) \end{cases}$

CaraR: if (CaraRoja = R) $\begin{cases} 1 \\ 0 \end{cases}$

Sumnacar: prev (Sumnacar) + CaraR

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RojayRoja: if ((Bolsa = R) and (CaraRoja = R)) $\begin{cases} 1 \\ 0 \end{cases}$

SumaRyR: prev (SumaRyR) + RojayRoja

FrecRyR = $\dfrac{\text{SumaRyR}}{\text{Sumacar}}$

Rafael's design, emphasizes how difficult it can be to define the model of urn equivalent to the problem that is tried to simulate.

The programming made by Alejo shows that he defined the chips as the conditioning event, falling in the mistake related by Falk (1986) and that we previously described in the introduction:

Bolsa: RP(R, A, RA)

CondR: if (Bolsa = R) $\begin{cases} 1 \\ 0 \end{cases}$

SumaR: prev (SumaR) + CondR

CondRR: if ((Bolsa = R) or (Bolsa = RA)) $\begin{cases} 1 \\ 0 \end{cases}$

SumRR: prev (SumRR) + ConRR

FrecR: $\dfrac{\text{SumaR}}{\text{SumRR}}$

His previous conception about the problem prevailing on his programming that does not allow him to observe that when considering the cases where chips R or RA are obtained this includes the cases where the shown face of the second chip was blue.

For solving the problem of the chips Carlos produced a program that allows to calculate the probability of drawing the red side of the chip with two colors. He drew the graph in Figure 2 and wrote: "*I generated 210 cases and with the help of the graph I can say that its stability is between 0.4 and 0.6 and a frequency of 0.46667*"

In the paper he wrote: "*1/2 because one chip is red on both sides and another one is red on one side and blue on the other*".

Again Carlos, besides talking about stability and the value of frequency calculated with the strategy of the last value, shows that for him there are two different answers: the answer that the computer offers and another one, very different, the one obtains with his analyses in paper and pencil.

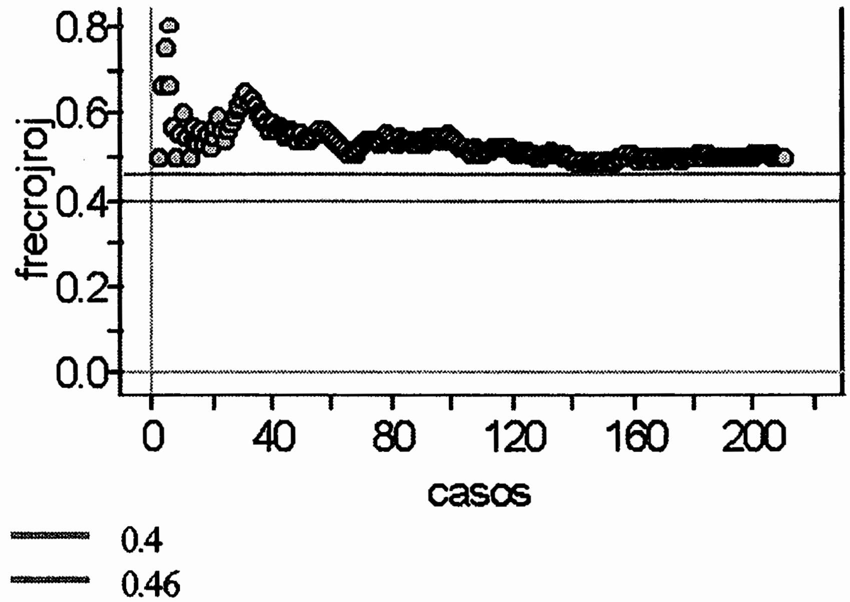

Figure 2. Carlos's graphic for the chips question of the urn problem.

Luis and Valentín, in the chips problem, as Carlos, estimated the probability of drawing the red side of the chip with different colors. They generated 170 cases and the estimated value was 0.51 using the mode of the results:

> "We concluded that the probability of getting the red-red chip is of 0,51 since in the most of the cases the previous result was obtained. In the 7 first cases we obtained a probability of 1 because there is a possibility that same event the chip is red-red."

They made the graph shown in Figure 3.

Conclusions

From the results obtained, it is seen that the students have difficulties with the modeling of the random experiment and with the programming in the simulation computer language. Confidence in the simulation process may need much more instruction time that the one we used.

Another difficulty deals with the interpretation of the graphs of the relative frequencies for estimating the probabilities. Indeed the necessity to obtain a single value of the probability as a result of a process to the limit (infinite) disturbs the students who resigned themselves to use the last value strategy that identifies the last value of the relative frequencies with the requested probability.

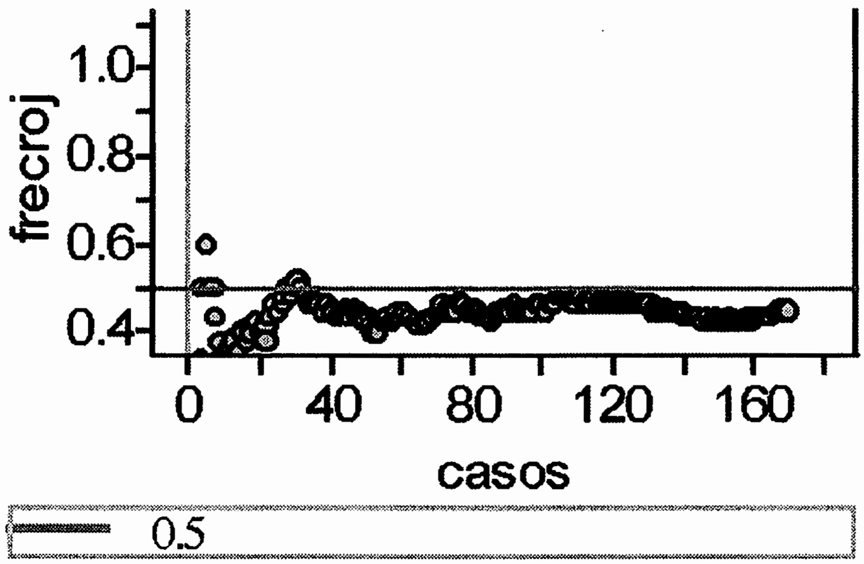

Figure 3. Graphic made by Luis and Valentín in the chips problem.

From the point of view of the programming it also appears a dependency with respect to the questions asked. That means, program according to which it is asked, instead of simulating all the random experiment. It is like they were trying to transfer the theoretical analysis to the simulation computer language. Anyway it is necessary to recognize that the translation of a random experiment to a equivalent urn model not always is evident, which causes that this frequentist approach demands requirements that are not often considered in a traditional education.

It might be that the students make a distinction between a theoretical probability and a simulated one. Some students (Carlos, for example) constructed two types of solutions, one given by the simulation and another one given by a reasoning in pencil and the paper. To what extent one influences on another is a question that still remains open.

When the results in the computer and in the paper were different, they chose the paper answer, maybe because of their lack of confidence in the simulation method and because the estimation by simulation is based on the reading of the trend in a graph in an infinite process.

On the other hand, the problem implied by the estimation of the probability through repetitions of the experiment was not thoroughly accepted by students who had previous experience in the classic probability. So, for some of them, (Luis and Valentin, for example) there are as many probabilities as generated values of the relatives frequencies. For some students, even, the requested probability is the mode of the values of the generated relative frequencies. That means, this problem of the estimation involves problems of convergence not easy at all for the students.

Another thing to mention is the small quantity of cases generated by students showing their belief in "the law of small numbers" and therefore a lack of understanding about the necessity for a large number to achieve the stability of the relative frequencies.

The results of this exploratory study are clear in the sense that when software that demands a certain programming is used, it is not possible to observe effects of the use of the computational simulation on the understanding of the probabilistic concepts without giving a previous and extensive training on the use of the software. We believe that the first approaches to the software must deal with nonrandom situations, for instance, finding the limit of successions defined algebraically. Working with successions is particularly interesting because in addition to allowing the students to develop a "feeling" in the estimation of the limits, they can confront the computational results with the ones obtained by mathematical handlings on the expressions that define the analyzed successions.

Definitively the previous experience with physical devices to represent random experiments becomes a referent to which the student can go in the phase of computational programming. Moreover, it allows him to realize of the necessity of simulating the random experiment more than to give answers to the questions asked.

A direct form to gain confidence with respect to the results of the simulation method is by solving "difficult" problems where the combinatorial that justifies them is not easy to get. Fathom, has even, the option to work with "sampling" which allows to consider more interesting situations, like those that imply the calculation of the expectancy of certain variables.

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DEVELOPING SECONDARY TEACHERS' STATISTICAL INQUIRY THROUGH IMMERSION IN HIGH-STAKES ACCOUNTABILITY DATA

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To support the development of statistical inquiry, teachers must know more than statistical procedures; they must learn to investigate, reason, and argue with statistical concepts and techniques. For mathematics teachers, this is often unfamiliar territory. This study investigates the process of teachers' statistical inquiry and documents its development during a 6-month professional development sequence that immerses teachers in learning statistics while analyzing their students' high-stakes state assessment results. We will report, based on empirical research, how an immersion model in statistics can influence that development.

Increasingly, teachers of mathematics are expected to include uses of data and data analysis in their instruction: Computational power makes the use of data common in today's society and contextual problems stimulate student engagement. At the heart of statistical reasoning is the ability to investigate: to formulate and test hunches, justify conjectures with evidence, and communicate findings, if any, with a convincing argument. Often, an investigation leads to further questions and only tentative answers. In order to support the development of statistical inquiry, teachers must know more than statistical procedures; they must learn to investigate, reason, and argue with statistical concepts and techniques. "The importance of attending to process as well as to content in statistics becomes apparent once we acknowledge that anticipation is at the heart of data analysis" (p. 11, Cobb, 1999). Teachers who expect consistency and predictability will most certainly perpetuate the belief to their pupils that mathematics is about right and wrong answers, faultless algorithms, and certainty. Ambiguity is not usually associated with the mathematical sciences and many traditionally trained teachers see their expertise as resting on facile, quick and accurate recall of a range of mathematical procedures. If students are to become investigators, anticipating and tolerating ambiguity and uncertainty surrounding authentic problems, we must help teachers to develop an investigative mindset. This study investigates the process of teacher inquiry and reasoning and begins to document its development over time. We report how an immersion experience in statistics can influence that development.

Theoretical Framework

Generating inquiry is a skill that requires breaking through a mindset of having to find an answer to every question. Wild and Pfannkuch (1998) argue that a mindset of probing, evaluating, and describing, an awareness of contextual constraints involved, and a balance of curiosity and skepticism are critical to applying statistics. Koedinger

(1998), working in the domain of geometry, argues that the ability to investigate, generate and test "interesting" conjectures, and produce a convincing argument, can in fact be learned by anyone, but "requires self-reflection and extra cognitive resources that come only from practice" (p. 333). Investigation as learners is an area in which teachers have little experience, and for them, the process of change is difficult (Cuban, 1990).

Initial experiences with generating conjectures are frequently stifled or overly simplistic. Student conjectures often focus on one characteristic of the data (Hancock, Kaput, & Goldsmith, 1992; Konold & Higgins, in press). In addition, Cobb (1999) notes the difficulty students have with being autonomous thinkers: "as a consequence of their prior experiences in school, the students assumed that their role was to infer the responses that the teacher had in mind all along rather than to articulate their own interpretations" (p. 7). Trigger questions, used in the quality control industry, can help to stimulate thinking (Wild & Pfannkuch, 1998). The first step in producing trigger questions is learning how to brainstorm ideas without criticism. "Criticism is forbidden in the idea-generation phase, in recognition of the fact that criticism and the urge to censor can stifle the generation of new ideas and erode confidence to suggest them" (p. 336). Motivation to solve problems is also a factor in generating conjectures. Peirce (1923) recognized that inquiry is embedded in a need to resolve doubt. Dewey concurred, and stated the importance of creating a problematic in his statements about inquiry; in investigating, the inquirer needs to feel disequilibrium in order to push ahead to reach a resolution and that an internal cognitive motivator kicks in (Hickman, 1990).

Often, learning in statistics is steeped in rule-based outcomes and statistical formulas that can often discourage inquiry, doing more harm than good (Shaughnessy, 1992). At the elementary and middle school level an early focus on calculations can hinder later statistical thinking; without sufficient conceptual development, the premature use of hypothesis tests at the high school and undergraduate level can aggravate the black and white misuse of the accept-reject dichotomy of statistical tests (Reichardt & Gollob, 1997; Abelson, 1995). Yet content knowledge *is* important for investigation. As study of experts' approach to ill-structured problems indicated, inquiry "will lead to inadequate solutions unless the individual has and employs substantial knowledge of the domain" (p. 283, Voss & Post, 1988).

Research Design & Implementation

The Systemic Research Collaborative for Education in Mathematics, Science and Technology (SYRCE) at the University of Texas at Austin is conducting a series of studies of methods of research partnership with schools in which a model of systemic reform is investigated. That model (Figure 1) links data on student outcomes to professional development to the building of teacher knowledge and community to the implementation of standards-based, technology-rich curricula and back to student outcomes. SYRCE is studying how to partner with schools, in order to undertake effective reform, through an approach called implementation research (Confrey, Castro, and Wilhelm, 2000; Confrey, Bell, and Carrejo, 2001). The project was conceived as a mathematical parallel of the Writers Workshop from the National Writing Project, where teachers learn to write rather than to be teachers of writing, and had a set of five related objectives:

1. Strengthen *teacher content knowledge in probability and statistics* by giving them the opportunity to learn statistics well beyond their curriculum, rather than learn how to teach statistics
2. Immerse teachers in *focused investigation and chains of reasoning* about student performance data at an urgent time in a high-stakes accountability environment
3. Build *teacher confidence and facility in using powerful dynamic statistical software* (*Fathom*)
4. Orient teachers with a *healthy mindset about data and inquiry*: the acceptance of uncertainty when searching for solutions, and the limitations and misuses of statistics and inferential reasoning
5. Provide teachers with an opportunity to be *learners* in an environment that models Standards-based teaching

The long-term conjecture for the project is that when teachers are immersed in content beyond their curriculum in a context they find compelling and useful, this

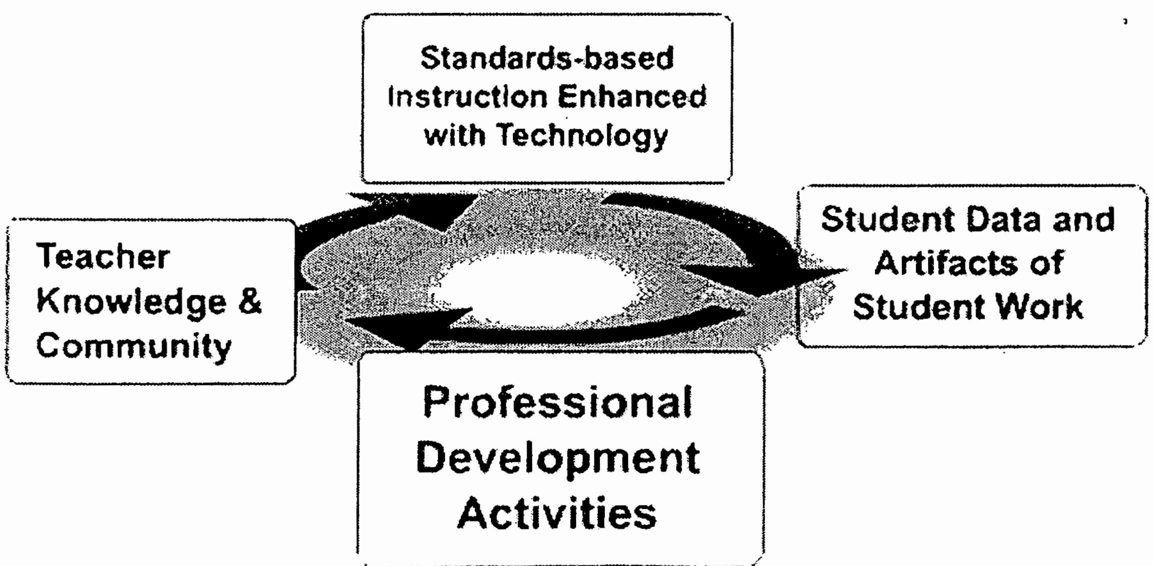

Figure 1. SYRCE model of implementation research in systemic reform.

experience will transfer into improved classroom practice. Specifically, teachers will teach statistics more authentically if their understanding of statistics and how it can be used is developed through their own investigations as statisticians. Another longer-term conjecture is that as a result of these experiences, we should see teachers more willing and able to use technology in their classes, particularly when teaching statistics. In this paper, we concentrate on the fourth objective, describing how teachers' statistical inquiry changes as their understanding of statistics concepts matures in the context of data investigation.

Texas, where the study took place, has a high-stakes accountability system. Students are tested in grades 3-8 and 10 on the Texas Assessment of Academic Skills (TAAS); high school graduation depends on passing in grade 10. Schools and teachers are held accountable for their students' performances. In this climate, urban schools that serve less academically advantaged children are under increased scrutiny and pressure to ensure they do not receive unacceptable ratings. As a result, professional development time is spent focusing on their TAAS results. In general, TAAS results are shared with teachers and schools in the form of summary data and hardcopies of individual student performance. Data are also reported by content and process objectives. Teachers feel that the accountability system creates a context in which they are under the gun and over which they feel very little power. Their stance is reactive, rather than proactive. This context seemed ripe to invite teachers to examine the statistical data as investigators.

Our focus in the project was the mathematics department at our partner school, an urban middle school that feeds into a low-performing high school. The department had five Hispanic women, one African-American woman, and one white male with experience ranging from none to over twenty-five years (median 8 years). Only three of the teachers were certified in secondary mathematics. The demographics of the school are 72% Hispanic, 21% white, and 7% African American. Sixty-two percent of the students are classified as Economically Disadvantaged.

The research conjecture was measured with a pre-post test of statistical content. Additionally, classroom observations were made before teachers began the study providing a baseline of their teaching practice of statistics and mathematics. At the end of the summer institute, teachers presented their findings to their colleagues and an audience of researchers. Clinical interviews at the conclusion of Phase II provided important triangulation of the data collected from the session videos, observations, pre-post tests, and videoed final presentations.

The implementation of the project consists of two phases, which occurred over the course of 6 months. Phase I was carried out from January to May 2001 with two full day and three 2-hour after-school sessions immersing teachers in exploratory data analysis, examination of accountability data, and time for orientation of the dynamic statistical software, *Fathom*. Approximately half of the contact time during the project was spent in front of the computer: learning the software, mirroring hands-on activities, creating simulations, testing conjectures, and searching for and investigating data. The software that we chose to use, *Fathom*™ (Finzer, 2001), is unique in its application as a teaching and inquiry tool. Whereas most statistical software tends to be like a "black box" (data in, answers out), or designed for very specific kinds of tasks, *Fathom* can be used to investigate a broad range of tasks at both an elementary and intermediate level. In addition, many schools in the district had already been purchased *Fathom* (although it was not yet widely used). During sessions, teachers worked to create increasingly more robust formulations and investigations of conjectures about their students' data while simultaneously building on their statistical thinking, reasoning, and content knowledge. Phase II of the project, a two-week summer institute, provided teachers time and support to further probe specific areas of inquiry of their own choice, examine deeper statistical concepts and tools as they were needed, and present their findings to their colleagues and a group of researchers. Phase I was repeated, by request, as a stand-alone professional development workshop for two intensive days just prior to the beginning of Phase II and included nine secondary math and science teachers from surrounding schools. A high school math teacher and preservice math teacher stayed on for Phase II; only two teachers in the partner school were able and obligated to attend the summer institute (due in part to a change in school leadership).

Typically work sessions began with some limited exploration of a data set as an impetus for continuing with training in the use of the software. A hands-on activity of a statistical concept followed, which was then continued on the computer with the assistance of the newly acquired skill in the software. Teachers were initially introduced to and explored the ideas of central tendency, distribution, variation (particularly as it related to small sample size) and graphical representations of data: boxplots, histograms, dotplots, and scatterplots. During the summer institute, statistical concepts were expanded to include correlation and least squares regression, influential points and outliers, standard deviation and variance, the central limit theorem, confidence intervals, sampling distributions, the null hypothesis, z-scores, and t-tests. We delayed introducing statistical tests until nearly the end of the summer institute, as we feared that the premature use of significance tests would aggravate a mindset of the accept-reject dichotomy of statistical tests (Reichardt & Gollob, 1997; Abelson, 1995). Sampling distributions were used frequently in class problems to imbue a tolerance for variation and to provide a conceptual foundation for confidence intervals, p-values, and t-tests without focusing on formulas and rules. Software instruction was provided on graphs and statistical summaries, importing data, least squares regression, relational graphing, sampling, simulations, hypothesis testing, and more advanced software features (collecting measures, scrambling, and stacking). At an informal level, teachers were able to test simple conjectures within the first few minutes of use with the software. *Fathom*'s ability to easily "drag and drop" variables onto graphs and to be able to link relationships in several graphs made it easy for us to begin using inferential language with teachers from the very beginning.

We pushed to enlarge teachers' view of data to broaden their perspective, encourage more robust conjectures, expect variation and ambiguity, and look for relationships in the data. As a result, every activity we planned worked towards developing a mindset that would allow for a richer experience with the data. Time was regularly used to discuss issues of accountability, reactions to assigned readings, and general issues of data. As the study progressed, increasing time at the end of the day was dedicated to the teachers' own explorations. The study wove the development of teacher knowledge around four strands: statistical reasoning, investigation into student data, the use of the software, and the process of inquiry. This paper focuses on the final strand.

Results and Discussion

At our first meeting with the teachers, we asked them to come up with conjectures about the data that they might want to investigate – their private "theories" about what they expected to find. Some initial conjectures were framed as topics they wanted to investigate from the data as they struggled for wording questions in the form of a conjecture:

Brenda: What about previous scores?

KM: So the connection-

Brenda: -between previous scores and seventh grade scores.

KM: Ok.

Leesa: I want to know about the correlation between failing the reading test and failing the math test.

Larry: Well, even a conjecture would be: If they fail the language arts, they-

Leesa and *Larry* (together): -fail the math.

Initially, teachers expected that relationships in the data would always be biconditional. For example, if they found that "Students who did well on the TAAS test in grade 7 would tend to do well on the test in grade 8", they assumed the inverse would also be true: If students did not do well on the TAAS test in grade 7, they would not tend to do well on the TAAS test in grade 8. This conditional conjecturing became useful in breaking through misconceptions teachers had about both data in general and the TAAS exam itself.

When provided the data for their school, the teachers' first reaction was to look up specific students and try to figure out if they could pick out the students who were represented by outliers. At this point, many of their conjectures involved a view of the data focusing on *their* students, *their* classes, or a generic student.

Larry: If they score a 3 or better on the writing sample, they should pass the math, I would think.

Selma: I have some, purely, they're purely, uh, in my field, ESL. If they came from a non-educational background, and, well—wherever they came from—they're not going to succeed very well here.

Commonly, conjectures focused on questions that could be answered with a single number rather than richer outlook of the data that reflects relationships, trends, or a more holistic perspective. Examples such as "It's been my experience that the seventh grade scores are usually the lowest" reflect this simplistic use of data focusing on a single descriptive sentence. Teachers also struggled with developing an appropriate discourse about statistics. Often they viewed statistical investigations as tasks - wanting explicit instructions on what and how to explore the data - and did not see a need to describe or justify the path they took to reach their findings. When first given open investigation time, they relied on the researchers to give them specific questions to investigate (expecting to find a single, "correct" answer). Just as Cobb (1999) describes the difficulty students have with being autonomous thinkers, at times it was even more challenging for the teachers. Toby, an experienced but fairly traditional high school math teacher found voicing his own ideas uncomfortable. If his thinking was probed in discussions, he assumed he had responded incorrectly. As he ventured out into creating questions, he looked to us for reassurance – Did I say that right? *Can* I ask that? As their tolerance for living with unanswered questions improved, teachers' conjectures were less stifled and began to reflect true curiosity and involvement in the data.

Leesa was more comfortable than most teachers asking her students to create their own questions rather than answer hers. As she first explored data about districts across the state that the Texas Education Agency (2001) posted on the Internet, Leesa wondered out loud if there was a relationship between teacher turnover and salary. An experienced middle school teacher and department head, she found generating inquiries less intimidating than any of our other participants. She kept her notebook close at hand while she investigated and explored new data sets. Ideas that were not written down were soon lost to new questions. Creating an environment, even within oneself, comfortable with unanticipated and unresolved questions is key to building a momentum of 'I wonder's'. Although most of the teachers had taken at least one statistics class, usually many years previously, the experience they remembered in statistics was not about asking questions, it was about answering them, and answering them using just-the-right-combination of a long list of procedures, tests, and monstrous-looking formulas. Once again, re-creating a mindset geared to inquiry was the greatest challenge we encountered.

The professional development workshops followed an immersion model. One of the main objectives of the workshop was to allow teachers to *do* statistics by investigating their own questions. As statistical tools were needed, they were developed through a hands-on activity and/or computer simulation. The problematic was key here – the teachers investigated data, often from their own students, and learned the necessary statistics, *as it was needed*, to support their investigations. For us, providing a relevant context was critical to getting teachers in the study to look at data with meaning.

In an early session, teachers were shown four pairs of histograms (delMas, Garfield, & Chance, 2001) with no context provided and asked to compare the similarities and differences they saw in each pair (Figure 2). The descriptions teachers gave ranged from bar-to-bar comparisons to reports of means and ranges, to claims of missing data. Many of their descriptions included the shapes of the distributions since they had categorized and named the common distribution shapes in our previous meeting. Overall, the conversation lasted just over 5 minutes. Brenda begins the discussion:

> *Brenda*: ...Evidently there's something I'm missing 'cause I keep staring at this ... Um, the first one is just uniform, it's, the frequency stops right, pretty much in the middle. There's nothing going on, I mean ... It just stops. The second one, is it bimodal? It's just going up and down. Uh, there's frequency, and then the zero, I mean there's missing scores. I mean it doesn't start at zero and end at six.

Selma, her partner in this activity, clarified:

> *Selma*: Well, she means because zero through one, there's nothing there. ... And five through six there's nothing there. So that's why she was saying some scores seem to be missing.

Following the initial iscussion of the pairs of histograms, we added a context to the data and asked teachers to compare the graphs again if each pair represented the quiz scores from two of their classes. Set in a meaningful context, the discussion became richer, and lasted over 40 minutes. Comparisons of the distributions gradually developed into qualitative and more holistic descriptions with emerging observations

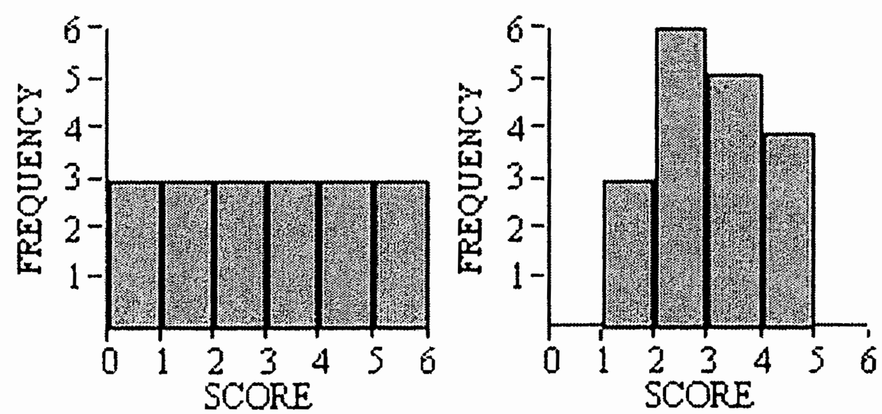

Figure 2. Pair of histograms used to compare two groups – with and without context.

(their first) about the spread and "closeness" of the data, theories about how to teach each class, and a discussion of which class would be more challenging to teach. The teachers determined which classes would be more challenging to teach, not based on central tendency, but on something they hadn't yet articulated.

Larry: But you can adjust for that [a difference in central tendency]. ... As long as the class is,

KM: The class is what?

Larry: More together.

This comparison of the two distributions was the first time one of the teachers had expressed some quality of the spread of the data, beyond just the range. Later, when asked to quantify "more together", Larry came up with his own rule: "Well, ok, on this one I checked as a matter of fact and it turns out that five-sevenths of the class on each of these is within one score, plus or minus, of the mean." In later sessions, we continued to provide situations that would help them become aware of the importance of having a robust measure of spread, to look for sources of variation, and to recognize when variation seems extreme. As the expectation for variation in data gradually increased, they slowly lost confidence in descriptions of data solely based on measures of center. As their skepticism increased, they felt more compelled to dig deeper into the data.

As teachers created and investigated conjectures, they were constantly surprised by what they found buried in data that they thought would speak for itself. Statistics is a unique domain in that in doing data analysis, you often come up with more questions than answers. This is rather uncomfortable for novices, especially those trained in mathematics, where answers (correct ones at that!) are expected. In one of our sessions aimed at introducing sampling distributions, each participant was given a die "loaded" with eight small stickers on a single face and asked if they felt the die was fair. Most agreed it was not. They collected data on 30 rolls of their loaded die and decided on how they might argue that the die was not fair. In doing so, they came up with several ideas for creating measures to quantify the discrepancy they saw in their data from what they expected of a fair die: five occurrences of each face of the die. Then they rolled a fair die 30 times to compare the count of each face with that of their "loaded" die and were surprised to find very little perceived difference between the rolls of the loaded and fair dice. There was an air of confusion and disappointment: the results weren't as they expected, but they were still unconvinced there was no difference between the fair and loaded dies. Following this, we worked with the teachers to create a computer simulation of 100 trials of 30 rolls of a fair die and then graphed a distribution of the measures they had devised. They still found that their "loaded" die was within a comfortable region in the distribution of rolls of the fair die. This provided the opportunity to talk about the null hypothesis and the informal theory behind significance testing. Intrigued, most of the participants refused work on anything else until they could create a loaded die simulation and see how loaded it needed to be in order to "detect" a significant difference. Discussions about z-scores and the concept of t-tests ensued. One participant decided the problem was the sample size and rolled his loaded die 60 times to try and detect a difference. What was meant to be a 45-minute activity turned into almost the entire morning with unexpected opportunities to relate difficult statistical concepts to what they were investigating. The disequilibrium they felt in Dewey's process of inquiry pushed them ahead - beyond what we had anticipated discussing. This investigation became a key turning point in our work with the teachers. For the first time, they experienced control over an investigation. Not only had they built up enough experience with the software that it was now transparent, they felt power in their ability to create and investigate their own questions. This was the first time that they felt enough confidence in their statistical knowledge, mastery of the software, and comfort with experimentation to weave them together into a powerful tool for inquiry. The loaded die activity had also provided an unexpected result: no apparent difference between the loaded and fair dies. The problematic paved the way for them to desire more powerful statistical methods to test conjectures. The very difficult concept of a sampling distribution became a new, powerful tool; z-scores were relevant and an important measure to justify differences between two groups.

At the end of the institute, each participant gave a 30-minute presentation of a conjecture they investigated. Several of the conjectures teachers presented are given below. Although we made it a point to give teachers experience thinking inferentially from the outset, it was much harder than we expected to get teachers to think about providing *statistical* evidence for their conjectures. More often than not, they chose to support their hypotheses with proportional reasoning.

- Practice TAAS tests are not taken as seriously as the real TAAS test and are of limited value, even for those students who need it most (remedial students)
- The schools in our district remediate students at a much higher rate than those in other districts similar in size.
- Test objectives that are problematic at the Exit Level (Grade 10) TAAS are also problematic at lower grades. Furthermore, if students do well on the problem-solving objectives, they are highly likely to pass the overall test
- There is a strong relationship for "typical" (middle 50%) students between the reading and math portions of the TAAS exam.
- There is not a strong relationship between student performance on the TAAS and their performance on the Iowa Test of Basic Skills (ITBS); in addition, the passing level for TAAS is well below grade level on the ITBS. Also, students who do well on ITBS almost certainly pass TAAS, but the converse is not true.

These conjectures reflected a more probabilistic and deeper view of data inquiry than the initial conjectures that focused more on simple answers.

Conclusions and Implications

Statistical literacy has been heralded as a vital skill for citizens in both the developed and developing world. At a time when misused, high-stakes accountability systems have the potential of retarding the growth of reform-based instruction, it is negligent to ignore their impact and naïve to think they will be short-lived. Rather than ignore what is clearly a system that is here to stay, it is imperative to work with the system to better inform it. This serves two purposes: students will be better served by teachers with stronger content knowledge and support for continuing to implement standards-based instruction; secondly, teachers and administrators will be able to make informed choices based on evaluation of data when implementing programs crucial to student learning. This study is the beginning of a viable model for working with schools to develop their own literacy while submerged in data of their own students. In order to improve statistical literacy in our society, working with schools is an obvious starting place.

As regards our conjecture on immersion, we found that most of the teachers we worked with were reluctant and anxious about undertaking statistical study. They frequently masked their own levels of discomfort with conversation about children's understanding, feeling on safer ground. It became apparent that fundamental statistical concepts were weak, including measures of central tendency as conceptual ideas, recognition of the importance of distribution and variation and simple kinds of issues of display such as how intervals affect shape and what a box plot represents. Nonetheless, we found that as they recognized that they would not be ostracized for a lack of knowledge and as they became more intrigued and competent with the software and statistical concepts, they became more willing to focus on their own learning.

In the arena of inquiry, we believe this is a fruitful area to explore conceptual development. We found that novices start conjecturing in relatively simple ways, seeking single number descriptors, direct comparisons and proportional reasoning. Over time, our teachers were increasingly confident to share their private conjectures and to produce more sophisticated claims and evidence. We believe that when inquiry is developed simultaneously to learning or relearning the content and software tools and in the company of other professionals, the integration of the parts seems to develop more seamlessly. Through an authentic statistical experience in a context that is urgent and compelling to teachers, the research study has shown evidence of change in teachers' understanding of data analysis as a tool for inquiry. In future work, we plan to revise and reimplement the materials and undertake further study of the teachers' inquiry activities. We will also follow them into the classroom to see evidence of impact of their experience on instruction.

Acknowledgement

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DESCRIBING MIDDLE SCHOOL STUDENTS' ORGANIZATION OF STATISTICAL DATA

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The purpose of this study was to describe how middle school students physically arrange and organize statistical data. A case-study analysis was used to define and characterize the styles in which students handle, organize, and group statistical data. A series of four statistical tasks (Mooney,, Langrall, Hofbauer, & Johnson, 2001) were given to twelve students, 4 from each of grades 6–8, in an interview format. Based on an analysis of the interview data, five categories for students' arrangement strategies emerged: *No Arrangement* (students made no attempt to arrange the data); *Clustered Arrangement* (students sorted the given data in groups with no totals); *Sequential Arrangement* (students listed data in least to greatest order or alphabetical order); *Summative Arrangement* (students provided totals for categories or groups of data); and *Regrouped-Summative Arrangement* (students re-grouped the data into new categories and gave totals). This study implies that students need to be presented with raw, unorganized data sets in addition to organized data sets and ask students to arrange or rearrange data in a meaningful way.

As part of the school curriculum, statistics has been a topic of research for several decades (Jones & Coxford, 1970; National Council of Teachers of Mathematics [NCTM], 2000; Pieters & Kinsella, 1959; Shulte & Smart, 1981). Paulos (1988) mentions that graphical representation is important for every day matters; while Steen (1997) points out that the centerpiece of middle school mathematics includes topics in exploratory data analysis and numerical reasoning. It is important for students to study statistics in the middle school, if not earlier, because the numbers that surround our daily lives are staggering. These numbers can be found on the Internet and in newspapers, television, movie ratings, food labels, consumer reports, and census reports to name a few. Making intellectual decisions in the global society in which we live is a task that every person needs to be able to perform accurately in order to be a better informed citizen and consumer.

Students come to understand the fundamentals of statistical ideas by being engaged in data handling processes such as collecting, organizing, and representing data (NCTM, 2000). According to NCTM, middle school students should begin to compare the effectiveness of various ways of organizing data for analysis or presentation (p. 49). It is also important to remember that in real-life situations, data are not always presented in an organized fashion; in fact, data can be messy. In order for students to make a connection between school and out-of-school mathematics, they should be encouraged to solve problem situations within a meaningful statistical context. However, many of the current middle school mathematics textbooks do not give students the opportunity to handle raw data (e.g., Charles, et al., 2000). Students are typically given organized data within a real-world situation then told to construct a particular graph. While much of the statistics research at the middle school level has looked at how students deal with reducing, representing, and analyzing data (e.g., Curcio, 1987; Mokros & Russell, 1995; Strauss & Bichler, 1988), virtually no research has examined how students organize or reorganize data.

We have taken the view that the first step in building a knowledge base in this area involves describing or characterizing the general strategies that students use when working with data. Thus, this study was exploratory in nature and did not interpret the data through the lens of any particular theoretical perspective. The intent of this study was to describe the strategies middle school students use to organize data.

Method

Participants

Students in grades six through eight from four mid-western schools formed the population for this study. Twelve students, four from each grade level, were selected for case-study analysis based on levels of performance in mathematics: one high, two middle, and one low.

Instrument

The interview protocol (Mooney et al., 2001) was comprised of four tasks, each requiring students to arrange data to complete the task. For each task, the students were given the complete data set on paper and the same data with each data value on a separate card. Questions were designed so students could respond orally or by generating tables or graphical representations. At the end of each task, students were given the opportunity to describe an alternate method of reorganizing the data.

A description of each task is provided in Table 1. For each task, students were asked to arrange the data in a suitable format to be presented in the school newspaper. For Task 1, *Shoe Size,* the students were asked to arrange a set of 50 shoe sizes; for Task 2, *Teachers' Pets,* the students were given a set of 39 pets to organize. In Task 3, *Academy Awards*, the students were given the ages of each year's Best Actor and Best Actress Academy Award winners for 30 years and asked to rearrange the data to choose an appropriate headline. Finally, in Task 4, *Classical Music*, the students were given test scores for students in two math classes. The students were asked to reorganize the data to determine if the test-takers who listened to classical music while studying performed better than those who did not. It should be noted that the data for the first two tasks was unorganized. Data for the last two tasks were organized, but potentially needed to be reorganized to complete the tasks.

Table 1. Task Interview Protocol

Task 1: Shoe Sizes 50 eighth-grade students were surveyed about their shoe size. This list shows the data collected. Your job is to arrange the data to be presented in the school newspaper.
Task 2: Teachers' Pets The teachers at your school were asked what kinds of pets they have at home. In all, the teachers had 39 pets. A list of these pets is shown on this page. Your job is to arrange the data to be presented in the school newspaper.
Task 3: Oscar Winners This table shows the ages of the last 30 winners for the Best Actor and Best Actress in a movie. The editor of the school newspaper wants you to arrange the data to be presented in the school newspaper and to determine which of these 3 headlines should go with the data: Headline 1: Academy Likes Actors Older Than Actresses Headline 2: No Age Bias in Best Actor and Best Actress Winners Headline 3: Academy Likes Actresses Older Than Actors
Task 4: Study Habits Mrs. Jones talked to the students in her mathematics classes one day about an article she read. It said that children who listened to classical music while studying performed better on tests than children who did not listen to classical music while studying. Some of her students planned to listen to classical music while studying for the next math exam. The results of the 80-point test are listed on this table. The students who listened to classical music have an "X" marked next to their name. Your job is to arrange the data to see if students who listened to classical music while studying performed better on the math test than students who did not listen to classical music while studying. The editor of the school newspaper wants you to present the data along with a headline about the comparison

Procedure

Using the interview protocol, each student was individually interviewed during a 60-minute session. All interviews were audio taped and all student-generated work was collected. The interviews were transcribed for analysis.

Data Sources and Analysis

Data sources consisted of the transcribed interviews, students' written work and data displays, researcher field notes, and summaries generated during the analysis. Using a grounded theory approach, we examined students' responses to discern general patterns of arranging data. A double coding procedure (Miles & Huberman, 1994) was used to analyze student responses. We individually coded each student's response based on the type of arrangement the student produced. Then, we compared the coded responses within a category to discern the characteristics of that type of arrangement. Throughout this process differences in categorization and coding were discussed and agreement was negotiated.

Results

We found that students' arrangement strategies fit into five categories, which will be described in the following paragraphs. The resulting categories were as follows: No Arrangement, Clustered Arrangement, Sequential Arrangement, Summative Arrangement, or Regrouped-Summative Arrangement.

In *No Arrangement,* students made no attempt to arrange the data, leaving it as raw data, or they arranged it inappropriately (see Figure 1). The most obvious way to fit in this category was if no arrangement was provided. Two students (6B, 6C) in the Academy Awards task and two in the Classical Music task (6A, 6B) did not attempt to rearrange the data. The Classical Music task generated the most varied types of responses that we categorized as No Arrangement. Some students dealt with the unequal ends data in an inappropriate manner. Unequal ends are defined here as meaning the data did not include the same number of students in each group. As a result, they forced the data into data pairs, using only 13 of the non-classical music scores to match the 13 test-takers who listened to classical music (7A, 7D). Student 7A chose the top 13 scores of those who did not listen to classical music, whereas student 7D chose the first 13 scores he saw to compare to the test-takers who listened to classical music. The most common example of an inappropriate arrangement was when students found averages that either did not represent the data or did not solve the task. This occurred in the Shoe Size task with four students (6A, 7B, 8A, 8C), in the Academy Awards task with five students (7D, 8A, 8B, 8C, 8D) and in Classical Music task with five students (7B, 8A, 8B, 8C, 8D) who found the average shoe size, average ages of the actors/ actresses, and average test scores, respectively. Three other inappropriate arrangements were presented by students (6D, 7B, and 8C) solving the Classical Music task. Student 6D found the average of the top three scores and lowest three scores of students who listened and did not listen to classical music and compared those averages. The test in the task was out of 80 possible points, so student 7B figured out what score was a 70% (which he figured, incorrectly, to be 57 out of 80) and found how many students scored above this score. One student (8C) stated that a graph would be inappropriate and that a description of the results would suffice in the school newspaper.

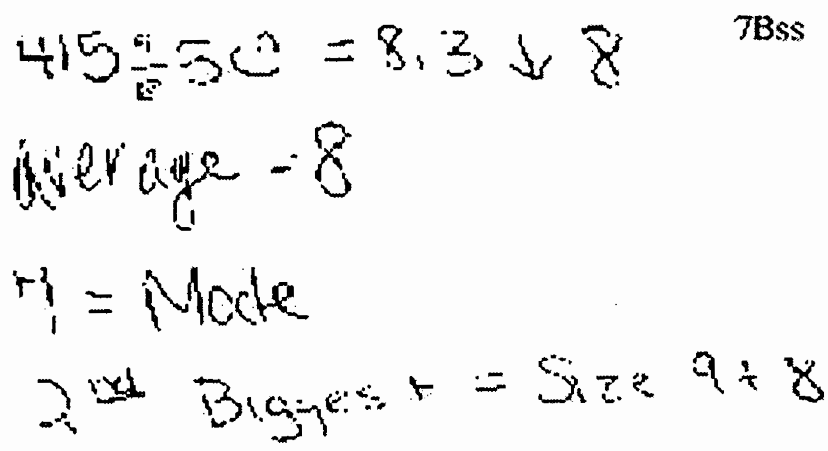

Figure 1. No Arrangement.

With *Clustered Arrangement*, students sorted the given data in groups with no totals (see Figure 2). This arrangement appeared only in the Teachers' Pets and Academy Awards tasks. For example, with Teachers' Pets, some students (6B, 6D, 7C, 8D) grouped the pets by the breed of animal or made a new group, but they did not provide a total amount. The students who provided graphical representations (6B and 6D) used a line plot with an 'X' representing each animal. In the Academy Awards task, student 6D placed a '+' beside actors and actresses who were 40 or over and a '–'beside those who were under 40. At the end she saw that she had more '+' for actors than actresses and chose Headline 1. One student (7A) subtracted the actor's age from the actress's each year to see if the difference was more than 10 years (Headline 3), less than 10 years (Headline 2), or negative (Headline 1). Still another student (8B) who provided a Clustered Arrangement looked at the years that the actresses were older to discover this occurred "in about one-third of the years".

In *Sequential Arrangement*, students listed data either in least to greatest or in alphabetical order (see Figure 3). For instance, with Shoe Size, six students (6A, 6B, 6D, 7C, 7D, 8C) decided to list the sizes in order from least to greatest. In the Teachers' Pets task, one student (6C) placed the animals in alphabetical order. For the Academy Awards task, four students (6A, 7C, 7D, and 8C) listed the data in numerical order. In addition, student 7C noticed that there were more shaded ages (actors) close to the

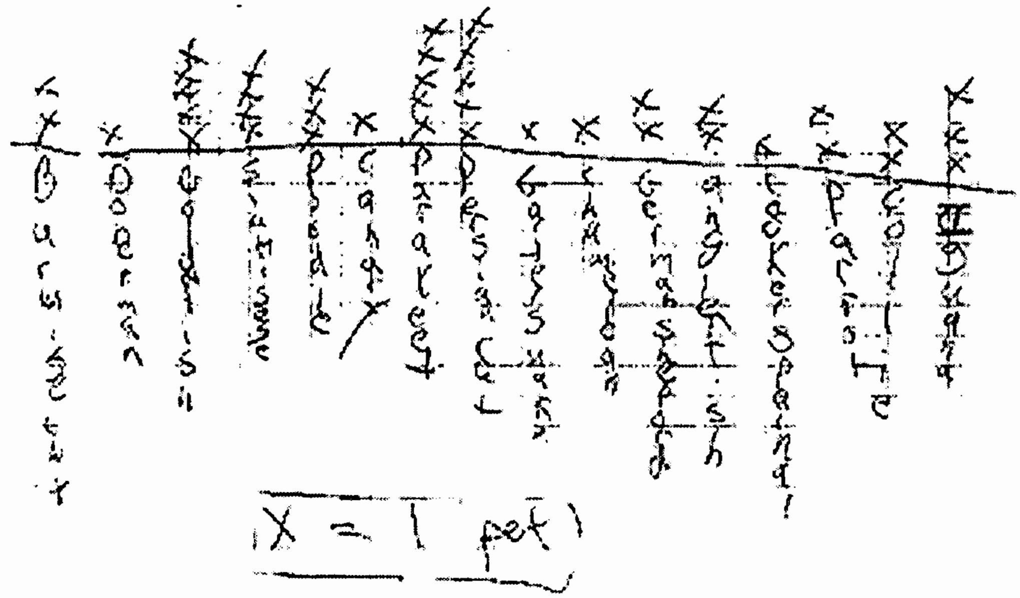

Figure 2. Clustered arrangement.

bottom, where the older ages were and less at the top, where the younger ages were, to reach the conclusion that Headline 1 would fit the data. Student 7D suggested to put the data in a graph with age on one axis and year on another but did not suggest the exact type of graph to represent this data. Four students (6C, 6D, 7C, 8C) arranged the cards in least to greatest order for the Classical Music task and expressed that arrangement of the cards was the best way to present the information in the school newspaper.

With *Summative Arrangement*, students provided totals for categories or groups of data (see Figure 4). For instance, with Teachers' Pets, a student sorted the pets by breed and then provided the total number of each breed in the data set. For the Shoe Size task, in particular, the students mostly arranged the data in a Summative Arrangement (6C, 7A, 7B, 8B, 8D), which included some variation of finding how many students wore certain shoe sizes and representing the data in a bar graph or line plot. One student (6C) placed the animals in alphabetical order, giving a Sequential Arrangement. When asked to provide an alternate arrangement, she joined four other students (7B, 7D, 8A, and 8B), who organized the data in a Summative Arrangement according to the breeds of animals (i.e., Doberman, Cocker Spaniel, German Shepherd, etc.). Two students (8A and 8D) used a Summative Arrangement, comparing age totals between the actors and actresses in the Academy Awards tasks.

Throughout the coding process, we found that some students regrouped the data into new categories and gave totals, which was characteristic of the *Regrouped-Summative Arrangement* (see Figure 5). For example, with Academy Awards, student 7B placed a mark next to winners who were 40 or older and then noted that 10 out of 30

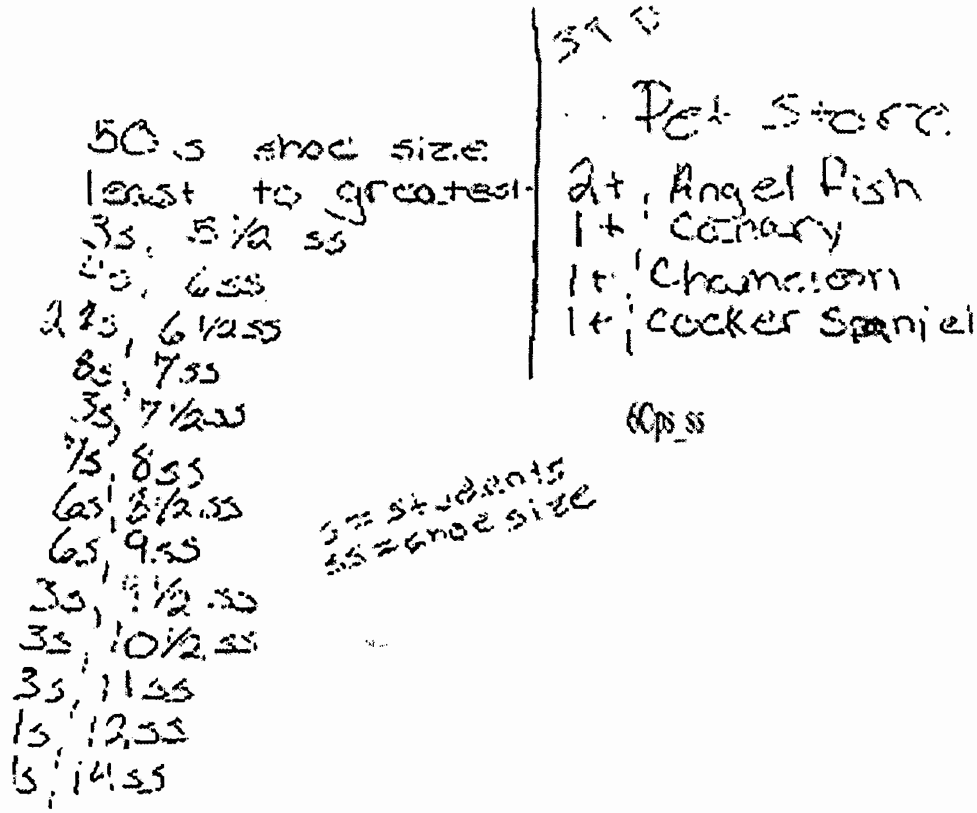

Figure 3. Sequential Arrangement

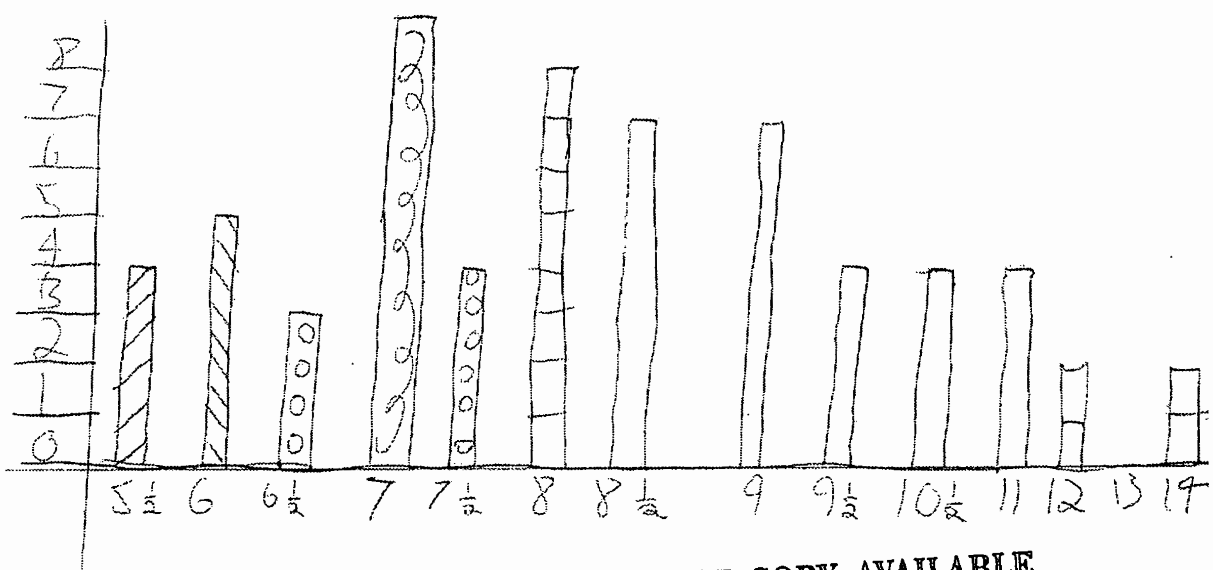

Figure 4. Summative arrangement.

actresses had marks whereas 19 out of 30 actors had marks leading to a Regrouped-Summative Arrangement. In the Teachers' Pets task, six students (6A, 7A, 7B, 8A, 8B, 8C) grouped the data into categories not given in the data set. For instance, instead of grouping all Dobermans or all Persian cats together, these students formed new groups: Dogs, Cats, Birds, and Reptiles (or Lizards). They went on to sum each new animal group and represent their data in a bar graph, line plot or circle graph.

Table 2 shows how students arranged the data by task, including initial and alternate arrangements. The use of an arrangement was task specific. For Shoe Size, the majority of students' ways of handling the data was split between Sequential and Summative Arrangements. With Teachers' Pets a majority of the students used either the Regrouped-Summative or Summative Arrangement. For the Academy Awards and Classical Music tasks, a majority of the students used No Arrangement as their choice of handling the data. We attribute this to a number of reasons. Both tasks included two data sets presented in an organized table. Some students were satisfied with the presentation and felt no need to rearrange the data while several students expressed their inability to rearrange two data sets within a single task. Still other students found the averages of the data sets, which did not involve reorganizing the original data sets.

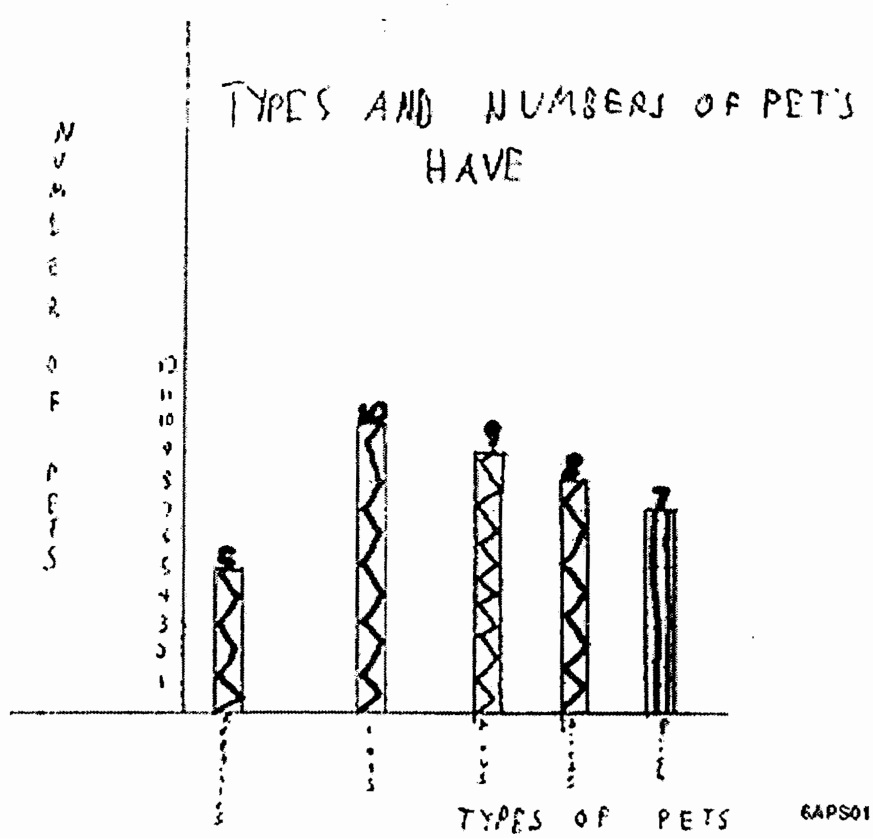

Figure 5. Regrouped-summative arrangement.

Table 2. Student Arrangements by Task

	Shoe Size (Task 1)	Teachers' Pets (Task 2)	Academy Awards (Task 3)	Classical Music (Task 4)
No Arrangement	4 27%	0 0%	7 41%	11 73%
Clustered Arrangement	0 0%	4 25%	3 18%	0 0%
Sequential Arrangement	6 40%	1 6%	4 24%	4 27%
Summative Arrangement	5 33%	5 31%	2 12%	0 0%
Regrouped-Summative Arrangement	0 0%	6 38%	1 5%	0 0%

***Note.* The percentage values represent the number of students using each arrangement per task. There were a total of 15 Shoe Size, 16 Teacher Pets, 17 Academy Awards, and 15 Classical Music arrangements.**

When viewing the results of student arrangements by grade levels (see Table 3), we noticed that the highest percentages of students' arrangements found across all grades levels was No Arrangement. Recall, arrangements in this category include inappropriate arrangements as well as no attempt to handle the data. When the students selected an appropriate arrangement, both the sixth and seventh graders' highest percentage of arrangements was Sequential. The eighth grade students' highest percentage was Summative Arrangement. This grades' lowest percentage was Clustered Arrangement, which was the same for the seventh graders. The lowest arrangement percentage for the sixth graders was Regrouped-Summative.

Table 2. Student Arrangements by Grade Level

	6^{th} Grade	7^{th} Grade	8^{th} Grade
No Arrangement	7 35%	5 26%	10 42%
Clustered Arrangement	3 15%	2 11%	2 8%
Sequential Arrangement	7 35%	5 26%	3 13%
Summative Arrangement	2 10%	4 21%	6 24%
Regrouped-Summative Arrangement	1 5%	3 16%	3 13%

***Note.* The percentage values represent the number of students using each arrangement out of a total of 20 sixth grade, 19 seventh grade, and 24 eighth grade arrangements.**

Discussion

The results of this study show that a majority of the middle school students interviewed were able to reorganize, rename, and sum single sets of unorganized data and give alternate arrangements of data. However, when organized data sets were presented to the students, several were unable to interpret or rearrange the data, possibly due to the fact that they had to handle more than one data set within a single task. Many students manipulated the numbers in the data set(s) and found the average value of the data as opposed to organizing or reorganizing the data. In this study, sixth and seventh graders were more likely than eighth graders to arrange data in an appropriate manner. The eighth grade students mostly tried to find averages for every task or they felt the organized data was in the appropriate format for the school newspaper. When probed to provide a graphical representation, these students were more likely to draw a box-and-whiskers graph.

Conclusion and Recommendations

Though this research was conducted using a small sample size, it does show that students can handle statistical data when given the opportunity. The sixth grade students we interviewed had recently completed the unit on statistics and probability which would explain the percentage of students who were able to arrange the data appropriately. Retention and additional practice would explain the increase in the percentage for the seventh grade students. However, since eighth grade textbooks generally extend the curriculum to include box-and-whisker, stem and leaf graphs, and line plots, they have an extended repertoire of graphical representations. This additional information with the lack of practice could explain why the eighth grade students had a high percentage of inappropriate graphs. We conclude that students need to be presented with raw, unorganized data sets in addition to organized data sets and asked to arrange or rearrange data in a meaningful way. This practice will allow students' statistical reasoning to evolve and develop which should lead them to effectively organize, analyze, and present data as suggested by NCTM (2000).

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YOUNG URBAN STUDENTS' CONCEPTIONS OF DATA USES, REPRESENTATION, AND ANALYSIS

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This paper reports preliminary results of an investigation of urban elementary students' thinking on data uses, representation, and interpretation. These results indicate that students generally entered the study with very local, pre-causal views of data, and a predisposition (at least among third and fourth graders) to respond to any data set by simply adding up the numbers. During the course of a five-week mathematics enrichment program, they created their own representations of data that they had gathered, investigated questions they had generated, and explored probability through games based on chance. We saw some students begin to develop a more global view of data and an ability to identify trends in data, while the tendency toward addition diminished. Further, we have identified a natural tendency in some students to create what we have labeled 'bin plots'; such bin plots could form the initial step in a hypothetical learning trajectory beginning with students' native thinking and moving toward standard, conventional representations of data.

Data interpretation is increasingly a focal point in pre-college mathematics curricula. However, urban children may not have sufficient access to Standards-based instruction on this topic. For example, the administrators of an after-school program in an economically disadvantaged urban area in Texas perceived that students in their traditional, skill-based program were not receiving authentic, challenging instruction in mathematics. They negotiated with our research group to develop mathematics enrichment components for the program. One of our objectives in undertaking the task was to investigate elementary students' thinking on data uses, representation, and interpretation. We report our preliminary findings here.

Theoretical Framework

Our framework for this study presupposes that students' understanding of data and their ability to use data in meaningful ways develop over time (Figure 1). Their early understanding, characterized by a local, concrete, and pre-causal (Piaget, 1930/2001) perspective, expands with experience to include global, abstract and logic or data-based thinking. Young students focus on individual cases and are not able to abstract from the specific to the general. Konold and Higgins (in press) document that "younger students tend to make plots that allow them to identify and answer questions about individual data points" (p. 8). Hancock, Kaput, and Goldsmith (1992) concur that students continued to focus on individual cases even after instructional efforts to move them to more group-oriented views of data. Lack of a causal perspective can lead students to focus on appearance and decorative aspects that do not add information as they organize data. Lehrer and Schauble (2000) describe the evolution of student-generated representations of data and argue that children often do not focus on theoretically important elements. Lacking an understanding of physical causality, young children also predict and explain outcomes based on "morality": What happens is what should happen, because it is right or to their benefit (Piaget, 1930/2001).

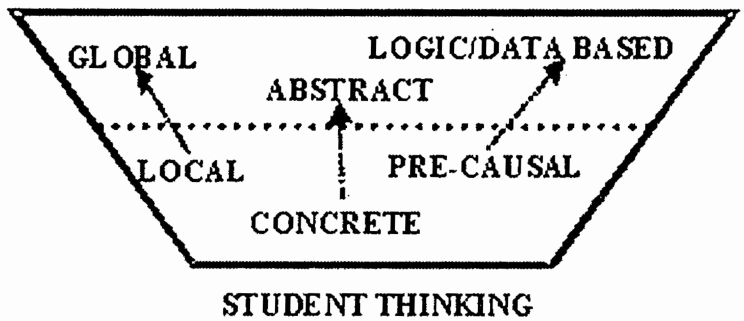


Figure 1. Development of student thinking about data

Our framework also comprises the contention that instruction affects students' thinking on these issues in both positive and negative ways. Schools now incorporate statistics and data analysis into their curriculum as early as Kindergarten (Economopoulos & Russell, 1998). Ideally such instruction would be designed to provide students with a more global perspective, allowing them to use representations of data to compare populations, make generalizations based on subsets, and make predictions based on trends (Friel, Curcio, & Bright, 2001). McClain, Cobb, and Gravemeijer (2000) assert that the ability to identify such "propensities" in the data is central to a "statistical perspective" (p.185). Lehrer and Schauble (2000) argue the importance of meaningful instruction in developing this mindset saying "reasoning about aggregates is necessary for developing and sustaining explanation in many domains, yet this form of reasoning seems to be mastered only over an extended period and depends on thoughtful instructional support and repeated opportunities for practice and use" (p.114).

Many elementary students, however, do not have access to such support and do not have opportunities to develop an understanding of statistics as a natural outcome of interpreting data in a meaningful way as part of an investigation. Statistics instruction is often limited to measures of central tendency and conventional graphing. Shaughnessy (1992) reports that such traditional rule-based instruction in statistics "may have negative effects on students' understanding" (p. 482). Further, some authors argue that an elementary curriculum emphasizing addition and counting to the exclusion of other mathematical concepts may lead students to view these operations as the only legitimate approaches to dealing with numerical data (Confrey & Smith, 1995).

Instruction can also cause students to abandon rich, personal conjectures in favor of more tractable ones. Hancock et al. (1992) report that when using *data modeling* to answer authentic questions, students must be involved in the *creation* as well as the *analysis* of the data: deciding what data to collect and developing organization schemes and systematic methods of measuring and categorizing data. Konold and Higgins (in press) describe the difficulty students had in transforming their original question into one that can be interpreted uniformly and for which appropriate data can be collected.

Recently, attention has focused on construction of representations to help students make sense of graphs and data (Friel et al., 2001; Lehrer & Schauble, 2000). Allowing students to create their own representations and describe their own ideas permits instructors to identify innate tendencies, which can be employed as building blocks in a guided process of constructing knowledge. Bakker (in press) discusses a hypothetical learning trajectory in graphical representation. He highlights that using value bar graphs guides students more intuitively toward the development of conventional plots. In particular, he argues that finding the mean is more intuitive with value bars (than with dots, for example) and identifies the ease with which horizontal value bar graphs can be collapsed vertically to create histograms. Bakker, Friel et al., and Lehrer and Schauble all argue that allowing students to compare and transition between different types of graphs is key to developing their understanding.

Methodology/Evidence

Our approach consisted of developing a Standards-based enrichment unit on data uses, representation and analysis, and implementing the unit in an after-school program. Our study population consisted of 38 students (37 African American and 1 Hispanic, 24 female and 14 male) in grades 3 through 6 in an economically disadvantaged urban area of Texas. These students were a volunteer subset of the students enrolled in the regular after-school program. The data unit was the second in a series of five-week stand-alone units. Students met with instructors four days per week, approximately 45 minutes each day, as a pull-out program from the regular after-school tutoring activities. They were grouped into two classrooms, grades 3-4 (19 students) and grades 5-6 (19 students) at the after-school site. The unit was taught by a team of researchers at the Systemic Research Collaborative for Education in Mathematics, Science and Technology (SYRCE) at the University of Texas at Austin.

During the first three weeks of the data unit, students made representations of data that they had gathered about the weights of their backpacks and derived conclusions from the representations. Students first measured the weights of their own backpacks in groups. Following the approach described in Lehrer and Schauble (2000), we did not prescribe standard forms of data description and display but rather relied on "children's progressive invention of schemes for inscribing data in the course of an inquiry" (p.115). Students were given paper and markers and asked to show how they would organize the backpack data either from their group or the entire class. In another activity, students investigated and answered questions they had generated about boomerangs. Later, they used Tinkerplots software (Konold & Miller, 2001) to create graphical representations of the both the backpack and the boomerang data. Finally, students investigated fair games during the last two weeks of the data unit where they obtained data on the outcomes of coin, dice, and spinner games. Students were asked to make predictions about the outcome of the games, and to determine whether the games were fair, that is, whether all players had an equal advantage.

We used both quantitative and qualitative methodologies to investigate our students' thinking. Quantitative assessments included a pre-test at the start of the program, a mid-program post-test at the end of the data unit, and a final post-test at the end of the program. Here we report primarily on the qualitative components. These included analyses of observations and videotapes of classroom interactions, interviews, open-ended responses on the pre and post assessments, and student-produced artifacts, including representations created both by hand and those generated in Tinkerplots. Observations, videos, and interviews were analyzed using standard methods in discourse analysis to describe students' states of understanding and to seek evidence of a growing understanding of data representation and interpretation. Student-produced artifacts were analyzed using project-developed rubrics.

Results

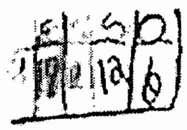
Our qualitative analyses identified the following tendencies in students' conceptions of data use and representation.

Natural or Moral Law (as opposed to causality) to Explain or Predict Outcomes

We found children in our study group who clearly fall into Piaget's (1930/2001) pre-causal category. For example, students were given spinners colored one-third red, one-third yellow, and one-third blue. When asked why the pointer had not landed on red as often as blue in a series of trials, Sonny responded that it should land on blue more because "red is a part of the Devil". When students were asked to predict the outcome of a game of chance, they most often predicted that the winner would be *their candidate*, either a personal favorite or the number or color that had won *for them* before (i.e., the result when they themselves had last played the game). For example, a student predicted seven would 'win' (come up most often) in a game where two dice are rolled and the sum added, not because there were more combinations of individual die faces that would add up to seven, but because it had won for her in a previous game.

Focus on Superficial Attributes

The younger children in the project (especially the girls) tended to focus on aesthetic elements of their representation, spending a great deal of time adding decoration rather than focusing on elements that would assist in communicating their ideas or answering their questions. Students collected data on the weights of their backpacks and created a representation that would allow them to answer questions about their backpack weights. When Tasha and Hope organized data about backpack weights of students, most of their time was spent decorating. In the representation in Figure 2, Chloe chose a different color for each different bin of tally marks, even though the colors carried no meaning. Even when organizing their data in Tinkerplots, the majority of the third and fourth graders were more interested in creating colorful representations than ones that produced meaningful information.



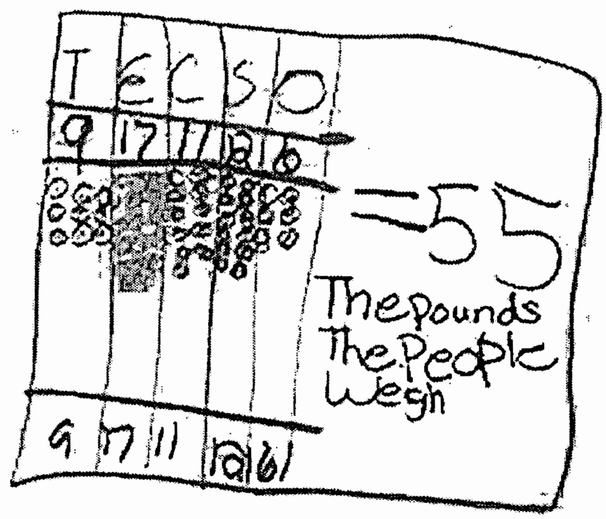


Figure 2. Chloe's representation of her group's backpack weights. Each letter represents a child in the group.

Tendency to Focus on Individual Cases

Confirming that young children focus on individual data cases rather than group characteristics (Bakker, 2001; Konold & Higgins, in press), our students sometimes used an individual case to answer a question about an entire group. For example, our students were quick to identify the extreme cases when asked to make true statements about the weights of their backpacks, but not to identify global trends in the data. The transcript below records a typical response.

RN: If this was the data that we collected, what could you tell me about the weight of the backpacks for this group of people?

Cherise: The highest one was this one (pointing to bar with highest value).

RN: That's Kendra, and hers was 20 pounds.

Cherise: And the lowest one was Sonny.

This tendency persisted for some students, as evidenced by their responses on the post-test at the end of the unit, in which they were shown the value bar graph in Figure 3 and asked, based on the graph, who reads more, boys or girls. This graph was generated using Tinkerplots, and the length of each bar represents the number of books read by an individual student. The bars in the graph that students saw were color-coded, yellow for girls and pink for boys. In the reproduction here, the girls are the lighter gray and the boys are the darker gray.

Five out of 30 students answered based on a single student who had read the most books (boys read more because "Roberto read more books than anybody" or "a boy read 14 books"), ignoring the general trend. On the other hand, eight students responded that girls read more, giving reasons like "more girls are at the top", "more girls higher up."

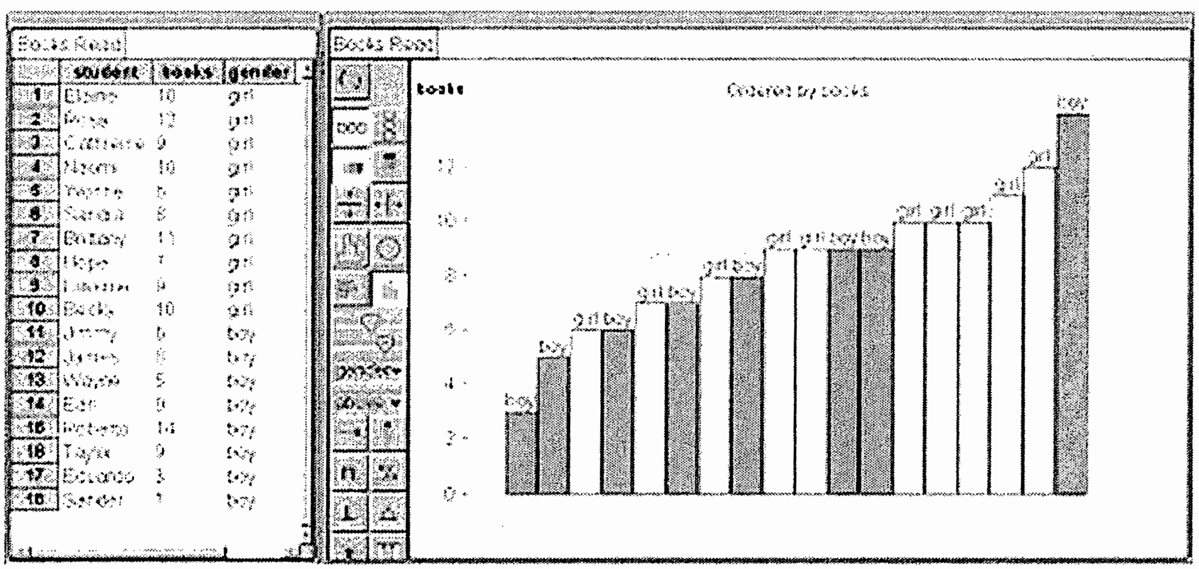


Figure 3. Tinkerplots representation of books read.

Personal Identification with Data Representations

Even as young children create models that resemble their referents (Lehrer & Schauble, 2000), we found that children continued to identify classmates with data points long after representations became more abstract. For example, when students saw the weights of their backpacks plotted in Tinkerplots, they took note that the value bar corresponding to Leon, an African-American boy with the lightest backpack, was white. Even later, when students were creating their own representations using Tinkerplots, one student pointed to the white segment in a pie chart that he had created to represent the backpack data and turned to his neighbor and exclaimed, "Look, who is this? Leon is white!"

Preference for "Bin Plots"

Our fifth- and sixth-grade students showed a clear preference for value bar graphs, as opposed to dot plot or frequency representations such as histograms when they were instructed to create a chart or graph. This is very likely due to their school curriculum, which includes bar graphs at this grade level, and to the tendency of teachers to emphasize correct procedures for constructing bar graphs over opportunities for construction of data inscriptions of the students' own creation. Our study also confirms Bakker's (in press) work identifying student preference for value bar graphs over dot plots when landscape-type statistical tools such as Tinkerplots permit them a choice of representation. We see, however, a more primitive stage of development in a preference for "bin plots" among our third and fourth graders. These are data representations in which the number of markers within a bin represents the value of one case, in the same way that the length of the bar does in a value bar graph. An example is Chloe's representation of her group's backpack weights, shown in Figure 2. Here she uses individual pound markers, arranged in rows and columns within a separate bin, as specifiers to represent the weight of each student's backpack.

"Addition Syndrome"

When given open opportunities to respond to data, our students often reverted to adding up the numbers, thus reducing the distribution to a single (and in many cases meaningless) number. When asked to organize data on backpack weights for their groups, for example, many students responded by adding the totals rather than comparing distributions. Note that, in Figure 2, Chloe has added up the individual weighs and presents the sum (55) as part of her representation. The tendency was extremely common among third and fourth graders, the majority of whom added the weights of the backpacks as part or all of their representation of the data. In another activity, students were asked what they could say based on a value bar graph showing backpack weights colored differently for girls and boys in their class. Note that two of the three students whose responses are transcribed here made a response based on addition, while only one commented on a trend that is visible in the data.

Jonah: The girls altogether weigh 53 pounds. And the boys weigh 30 pounds.

Marcus: The blue is the, is kind of to the lighter side and the girls are more… The girls weigh more than the boys.

Diana: I think the girls weigh more than the boys because there is one young boy that weighs 19 pounds and there's another [girl] that

> weighs 10 pounds and if you like add all the girls, all the three girls' pounds, for their backpacks together then it comes to the biggest number and if you add the two boys' backpacks together then it comes to a smallest number.

We saw this tendency diminish as the unit progressed. When asked to make a true statement based on the graph shown in Figure 3, taken from the post-test at the end of the unit, no student responded by adding up the total number of books read. On a second post-test at the end of the academic year, only one student out of 11 who took the test added up the data in response to a question about whether a third grader or a fourth grader was likely to sell the most candy based on data from a graph.

Loss of Confidence in Self-generated Conjectures

Our students tended to modify their originally rich, meaningful questions into ones for which data could be collected and analyzed. Questions that students wanted to answer at the beginning of an investigation showed a great deal of interest in the flight of boomerangs. For example, students were interested in finding out both how far each of two different styles of boomerang flew and how close each one landed to its thrower. However, after collecting and representing their data, most students found that they had not recorded the data needed to answer their questions and found themselves answering questions about the data, rather than the boomerangs, or answering questions about the boomerangs that did not at all resemble the questions originally intended.

Discussion

Our results indicate that, in general, our students entered the study with a local, concrete and pre-causal perspective in thinking about data. The tendency to respond to a general question about a data set based on an individual point is a sign that students had not yet developed a global perspective. They did not, in general, "think about data sets as entities that have properties in their own right rather than as collections of points" (McClain et al., 2000, p.175), which is a desired outcome of instruction in statistics. Their focus on decorative aspects of data representation indicates that they did not see data primarily as a source of information, from which generalizations could be drawn and predictions could be made. This attitude that data are not useful in creating logical inferences and explaining outcomes is a sign of pre-causal thinking. Children, for example, might believe that boomerangs fly well simply because they want them to, or because "that's just the way it is," rather than because of their shape or some other characteristic whose relation to flying well could be measured. Such children may not see any reason to seek out trends in the data. For them, the salient feature of the graph may be its appearance, making it more likely that they will spend their time predominantly on decorating the graph. In the same manner, the expectation that the outcome of a game was likely to be similar to one that they had experienced personally, or to be what they wanted it to be, is also a sign of pre-causal thinking. Adults might expect natural variation in outcomes; children typically believe that any given outcome "admits of justification or of motivation as everything in nature has been willed" (Piaget, 1930/2000, p. 275).

Likewise, our students' personal identification with data points is evidence of concrete thinking. To them, the data point representing Leon's backpack was about *Leon*, and not an abstract element of a set of data from which generalizations could be made. Hancock et al. (1992) report a similar tendency in the children they worked with.

It could also be argued that the tendency to associate the data with its referent represents an awareness of *context*, which is considered a key component of statistical thinking and also a desirable outcome of instruction in statistics. For example, our students were shown data representing the amounts of candy sold by students in one third-grade class and one fourth-grade class, clearly showing that in general the fourth graders sold more candy. Students were told that another third-grade and another fourth-grade class sold candy, and asked whether they would guess that the student who sold the most out of those two classes would be probably in the third grade, or probably be in the fourth grade. One of the fourth graders in our study based her response on her personal knowledge of third and fourth graders (A fourth grader would sell more because fourth graders have the "props [signs, displays] and ways" to sell more), rather than on an argument from the data. Hancock et al. describe this as "the dominance of personal knowledge" (p.387) and argue that it limits a student's inclination to use data to answer questions. It is possible, for example, that had the data on candy selling indicated that, in general, third graders sold more candy, our fourth grader would have trusted her personal knowledge and discounted the data as not representative. On the other hand, a more generalized awareness of context might serve to make students aware of cases in which a data sample might actually *not* be representative, limiting its effectiveness in extrapolating a general answer.

In the course of our work, we found evidence that instruction can affect students' thinking in both positive and negative ways. A case in point of a negative effect is the "addition syndrome" we observed among third and fourth graders. We assert that this tendency to reduce all data sets to a single number by adding up the values, regardless of whether the sum had any meaning with regard to the questions under consideration, is the result of an elementary curriculum with a strong focus on addition, to the exclusion of other ways of looking at data. Likewise, by exhorting our students to refine their original questions to ones whose answers could be determined based on data readily measured within the time allotted and with the equipment available, we may have led them to believe that, in general, real questions do not yield to data-based analysis, and reinforced their teleological perspective. The answer to "Why is a boomerang made of this foam material?" might just as well be "Because that's just the way it is" if we give students the impression that there is no way to find an answer to authentic questions through investigation.

On the other hand, we believe that our students did make some progress toward a more global perspective on data as a result of the program of instruction they experienced. By the end of the unit, while some students continued to focus on outlier points, others had begun to base responses on group propensities, as seen in their responses to the post-test question. We assert that this change was due to their experiences with creating and discussing data representations, in particular their experiences with Tinkerplots. It is much easier to see at a glance that "more girls are at the top" from a Tinkerplots graph than from a list, as can be seen in Figure 3. Students provided justification such as "because there is more yellow and that stands for girls" in deriving a result from the data in Figure 3. This clearly indicates a visual assessment of the data, facilitated by the color-coding made possible by Tinkerplots.

Tinkerplots may also have helped to curb the 'addition syndrome' in our third and fourth graders. When these students were first introduced to Tinkerplots, they were disappointed that it would not allow them to add up the data values they had entered. In fact one student went so far as to suggest that we needed to access other software ("You could go to 'addition.com'") to make this possible. However, as our students were able to discover other ways of organizing and analyzing data with Tinkerplots, they seemed to think less in terms of addition as a response to data, as seen in their post-test responses. These exploratory results, if confirmed, bode well for the possibility that elementary students can develop a statistical perspective through investigation-based instruction using software that allows them to display and manipulate data in a variety of ways with ease.

Finally, we have identified a native tendency among students, the bin plot, which could serve as a building block for the construction of understanding of standard forms of representation. According to Friel et al. (2001), "Unlike plans, maps, or geometric drawings that use spatial characteristics (e.g., shape or distance) to represent spatial relations, graphs use spatial characteristics (e.g., height or length) to represent quantity" (p.126). By this definition, a bin plot qualifies as a primitive form of graph, in that the *area* of the array of specifiers within each bin corresponds to a quantity (in Figure 2, the weight of an individual backpack), but the discrete specifiers are still retained. In that sense, a bin plot represents a transitional state between an iconic representation in which one "sees" 12 individual tally marks to represent 12 pounds, and a more standardized value bar graph, in which the length of the bar represents the quantity.

Note that these bin plots invert the standard "line plot" construction (see Figure 1 from Friel et al., 2001, p.127). In Chloe's bin plot, the horizontal dimension is demarcated into bins, which represent individual students' backpacks (Figure 2), while the markers represent units of a quantity (pounds). Conversely, in a traditional line plot, horizontal positions along the line are labeled to represent units of a quantity (pounds, for example), while individual cases show up as markers (x's, for example) stacked in the vertical direction. As such, one might argue that allowing students to focus on bin plots might impede progress toward understanding of representations based on frequency (histograms). However, it is easy to see how a bin plot that was constructed using graphing software with sufficient flexibility could be reshaped so that the array in each bin had the same width or the same length, allowing a smooth transition to a value bar graph. As mentioned above, Bakker (in press) presents a similar natural transition that allows line plots or stacked dot plots (precursors of histograms) to be created in the collapse of horizontal value bar graphs. Thus we have a possible learning trajectory beginning with iconic representations, then moving to bin plots, value bar graphs, frequency-based line plots and finally standard histograms. With appropriate software, these transitions could be accomplished through a series of visible manipulations of the data, which could serve to heighten understanding.

Conclusions and Implications

Our results indicate a strong need for urban elementary students to have ample opportunities to carry out the full spectrum of an investigation: making a conjecture, obtaining data, organizing and creating representations of data, and synthesizing the propensities of the data they have gathered to reach conclusions. Our results imply that urban elementary students will benefit from viewing and representing data in a variety of forms, especially those that exploit their native tendencies. Allowing students to create representations and ways to quantify data that make sense to them will counter the tendency to simply add up numbers, or produce standard plots that hold no meaning. This implies that software and other tools that allow students maximum flexibility in representing data should be developed. If students' initial tendency is to organize data into bin plots, then it is important that software allow them to enter data without regard to order or frequency, and to display values in terms of discrete units (pound markers) rather than simply as the length of a bar or a position in Cartesian coordinates. Further, such software should facilitate movement through a hypothetical learning trajectory, for example, by readily converting bin plots to value bars, and then collapsing value bars to a standard histogram plot. Clearly further study is called for to verify this conceptual path and develop and assess curricular sequences to exploit it.

Understanding how urban children represent, analyze, and use data is a comparatively new area of study. Further research is needed on all aspects, from cognition to classroom implementation. Our work in an urban after-school program showed that such strategies can be successful. However, keeping in mind that teachers tend to emphasize correct procedures for producing graphs rather than encouraging students to generate their own representations and allowing those representations to evolve over the course of inquiry (Lehrer & Schauble, 2000), implementation research in actual urban classrooms will be necessary to determine the conditions and supports necessary for working teachers to adapt these strategies.

Acknowledgement

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STUDENTS' SCHEME-BASED CONCEPTIONS OF SAMPLING AND ITS RELATIONSHIP TO STATISTICAL INFERENCE

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We distinguish two conceptions of sample and sampling that emerged in the context of a teaching experiment conducted in a high school statistics class. In one conception "sample as a quasi-proportional, small-scale version of the population" is the encompassing image. This conception entails images of repeating the sampling process and an image of variability among its outcomes that supports reasoning about distributions. In contrast, a sample may be viewed simply as "a subset of a population"- an encompassing image devoid of repeated sampling, and of ideas of variability that extend to distribution. We argue that the former conception is a powerful one to target for instruction.

Background

On the basis of empirical evidence Kahneman and Tversky (1972) hypothesized that people often base judgments of the probability that a sample will occur on the degree to which they think the sample "(i) is similar in essential characteristics to its parent population; and (ii) reflects the salient features of the process by which it is generated" (p. 430). This hypothesis suggests that Kahneman and Tversky's subjects focused their attention on *individual* samples. In later research, Kahneman and Tversky (1982) conjectured that people, indeed, tend to take a *singular* rather than a *distributional* perspective when making judgments under uncertainty. In the former, one focuses on the causal system that produced the particular outcome and assesses probabilities "by the propensities of the particular case at hand" (p. 517). In contrast, the distributional perspective relates the case at hand to a sampling schema and views an individual case as "an instance of a class of similar cases, for which relative frequencies of outcomes are known or can be estimated" (p. 518).

Konold (1989) found strong empirical support for Kahneman and Tversky's (1982) conjecture. He presented compelling evidence that people, when asked questions that are ostensibly about probability, instead think they are being asked to predict *with certainty* the outcome of an *individual* trial of an experiment. Konold characterized this orientation, which he referred to as the *outcome approach*, as entailing a tendency to base predictions of uncertain outcomes on causal explanations instead of on information obtained from repeating an experiment.

Sedlmeier and Gigerenzer (1997) analyzed several decades of research on understanding the effects of sample size in statistical prediction. They argued compellingly that subjects across a diverse spectrum of studies who incorrectly answered tasks involving a distribution of sample statistics may have interpreted task situations and questions as being about individual samples.

Recent instructional studies (delMas, Garfield, & Chance, 1999; Sedlmeier, 1999) indicated that engagement in carefully designed instructional activities using computer simulations of drawing many samples can help orient students' attention to collections of sample statistics when making judgments involving samples. However, analyses in these studies did not focus on characterizing students' evolving conceptions and imagery in relation to their engagement in instruction.

Despite the centrality of variability in statistics, students' understanding of sampling variability and our comprehension of variability's role as a central organizing idea in statistics instruction has received little research attention (Shaughnessy, Watson, Moritz, & Reading, 1999). Rubin, Bruce, and Tenney (1991) proposed that a coherent understanding of sampling and inference entails integrating ideas of sample representativeness and sampling variability to reason about distributions. Images of the re-sampling process, however, were not at the foreground of their conceptual analysis. Other conceptual analyses of sampling (Schwartz, Goldman, Vye, & Barron, 1998; Watson & Moritz, 2000) characterized the relationship between population and a randomly selected subset of it in a way that did not entail images of the repeatability of the sampling process nor of the variability that we can expect among sample outcomes.

In sum, substantial evidence from research on understanding samples and sampling suggests that students tend to focus on individual samples and statistical summaries of them instead of on how collections of sample statistics are distributed. Furthermore, students may tend to predict a sample's outcome on the basis of causal analyses instead of statistical patterns in a collection of sample outcomes. These orientations are problematic for learning statistical inference because they disable students from considering the relative unusualness of a sampling process' outcome. Finally, sampling has not been characterized in the literature as an interrelated scheme of ideas entailing repeated random selection, variability, and distribution.

Purpose and Methods

This study investigated the development of students' thinking as they participated in instruction designed to support their conceiving sampling as a scheme of interrelated ideas including repeated random selection, variability among sample statistics, and distribution.

Twenty-seven 11th- and 12th-grade students, enrolled in a non-AP semester-long statistics course, participated in a 9-session whole-class teaching experiment (TE) addressing ideas of sample, sampling distributions, and margins of error. Our aim was to develop epistemological analyses of these ideas (von Glasersfeld, 1995; Steffe & Thompson, 2000; Thompson & Saldanha, 2000) – ways of thinking about them that are schematic, imagistic, and dynamic – and hypotheses about their development in relation to students' engagement in classroom instruction.

Three research team members were present in the classroom during all lessons: one author designed and conducted the instruction; the other author observed the instructional sessions and took field notes; a third member operated the video cameras. Students' understandings were investigated in three ways: by tracing their participation in classroom discussions (all instruction was videotaped), by examining their written work, and by conducting post-experiment individual interviews.

Instruction stressed two overarching and related themes: 1) the random selection process can be repeated, and 2) judgments about sampling outcomes can be made on the basis of relative frequency patterns that emerge in collections of outcomes of similar samples.[1] These themes were intended to support students' developing a distributional interpretation of sampling and likelihood. Though an a priori outline of the intended teaching and learning trajectories (Simon, 1995) guided the progress of the teaching experiment, the research team made on-line adjustments to instruction according to what they perceived as important issues that arose for students in each session.

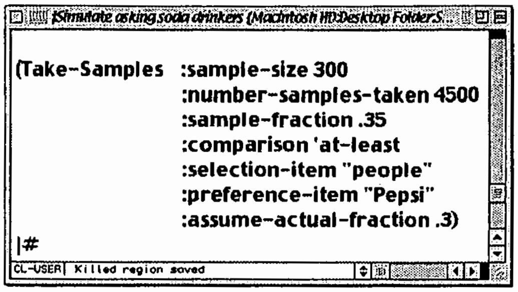
The teaching experiment unfolded in three interrelated phases: it began with directed discussions centered on news reports that mentioned data about sampled populations and news reports about populations per se (raising the issue of sampling variability). The experiment then progressed to questions of "what fraction of the time would you expect results like these?" This entailed having students employ, describe the operation of, and explain the results of computer simulations of taking large numbers of samples from various populations with known parameters (see Figure 1).

[insert Figure 1 here]

The experiment ended by examining simulation results systematically, with the aim that students see that distributions of sample proportions are largely unaffected by underlying population proportions (see Figure 2), but are affected in important ways by sample size.

Results and Discussion

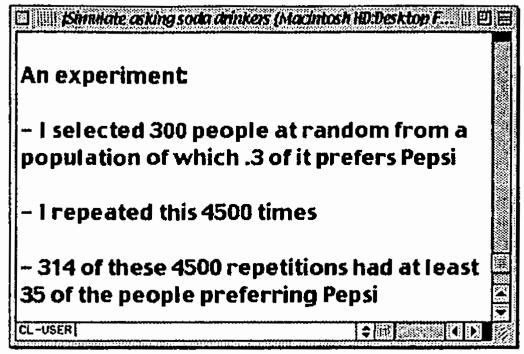
In this report we move toward elaborating an important distinction between two conceptions of samples and sampling that emerged in the teaching experiment. Our analyses revealed that some students – generally those who performed better on the instructional activities and those who were able to hold coherent discourse about the mathematical ideas highlighted in instruction – had developed a multi-tiered scheme of conceptual operations centered around the images of repeatedly sampling from a population, recording a statistic, and tracking the accumulation of statistics as they distribute themselves along a range of possibilities. These images and operations were tightly aligned with those promoted in classroom instructional tasks and discussions.



Explain what each number stands for in the command we have been using to instruct the computer to simulate drawing random samples from a population of soda drinkers.

Explain what information we will get after having run the simulation (with the values provided above).

What result do you expect the simulation will produce (with these values provided above)? Please justify your answer.



Interpret the simulation's output above.
How does your prediction compareto the result produced by the simulation?
Are they significantly different?
Are you surprised by this difference?
What might account for the difference?
What fraction of the time would you expect results like these?

Figure 1. Part of an instructional activity designed to help students make sense of computer simulations of drawing many random samples from a population. Simulation input (left) and output (right) windows were displayed in the classroom and the instructor posed questions designed to orchestrate reflective discussions about the simulations.

As such, we conjecture that these students' engagement in the instructional activities played an important role in their developing such a scheme. For instance, we had students practice imagining and describing a coordinated multi-level process that gives rise to sampling distributions (and to the simulations' results):

Level 1: Randomly select items to accumulate a sample of a given size from a population. Record a samplestatistic of interest.

Level 2: Repeat Level 1 process a large number of times and accumulate a collection of statistics.

Level 3: Partition the collection in Level 2 to determine what proportion of statistics lie beyond (below) a given threshold value.

In classroom discussions the instructor employed a metaphor designed to help students distinguish and coordinate these different levels. The metaphor entails imag-

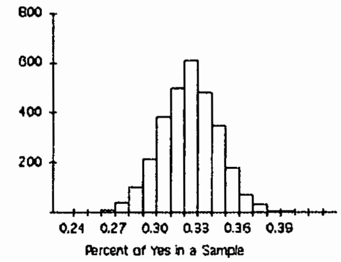
Here is a table similar to one we filled out in an in-class activity:

Percent of Yes in Population	Number of People in a Sample	Number of Samples Drawn	% of Sample Percents within 1 Percentage Point of Population %	% of Sample Percents within 2 Percentage Points of Population %	% of Sample Percents within 3 Percentage Points of Population %	% of Sample Percents within 4 Percentage Points of Population %
65%	500	2500	36.7%	64.5%	34.8%	91.5%
32%	500	8000	37.1%	65.8%	89.9%	91.1%
57%	500	6800	35.2%	64.9%	84.2%	91.8
60%	500	5500	36.1%	65.2%	84.2%	91.4%

a) Explain what "% of Sample Percents within 2 Percentage Points of Population %" means.

b) What do you infer from this table that is true about the process of taking a random sample of 500 people?

c) What does this suggest about a relationship between sample size and the percent of samples within certain percentage points of a population percent?



Levels	Counts
0.24000000	3
0.25000000	2
0.26000000	9
0.27000000	42
0.28000000	103
0.29000000	214
0.30000000	386
0.31000000	502
0.32000000	612
0.33000000	478
0.34000000	348
0.35000000	179
0.36000000	76
0.37000000	33
0.38000000	7
0.39000000	6

3000 Samples of 500 each taken from a population having .32 of it believing "yes"

Figure 2. Part of an instructional activity designed to structure students' investigation of the relationship between sampling distributions and underlying population proportions. Students filled out the table on the left by organizing information (like that shown on the right) generated by computer simulations of drawing many random samples from populations with given proportions.

ining a collected sample of dichotomous opinions ("yes" or "no") in Level 1 as a box containing "1"s (for "yes") and "0"s (for "no"). It then entails labeling the box with a "1" (or a "0") if the proportion of its contents is greater (or less) than a given threshold value. In this metaphor, what accumulates in Level 2 is a collection of "1"s and "0"s (or boxes/slips of paper labeled "1" or "0"), each of which represents a sample whose statistic is greater (less) than the threshold value. At Level 3, the metaphor entails calculating the percent of the collection of "1"s and "0"s in Level 2 that are "1" or that are "0", depending on the required comparison.

The following excerpt illustrates one student's coherent image of the multi-tiered sampling process, the development of which appeared to have been facilitated by his use of this metaphor. We take this student's coherent image as an expression of the stable scheme of conceptual operations characterized above. In the excerpt, the student (D) interpreted a sampling simulation's command and the result of running it as he viewed familiar simulation windows on a computer screen (see Figure 1)[2]:

D: Ok. It's asking...the question is...like "do you like Garth Brooks". You're gonna go out and ask 30 people, it's gonna ask 30 people 4500 times if they like Garth Brooks. The uh...(talks to himself) what's this? let's see...the actual...like the amount of people who actually like Garth Brooks are...or 3 out of 10 people actually prefer like Garth Brooks' music. And uh...for the 30...when you go out and take one sample of 30 people, the cut off fraction means that if you're gonna count, you're gonna count that sample, if like 37% of the 30 people preferred Garth Brooks. And then it's going to tally up how many of the samples had 37% people that preferred Garth Brooks. So like the answer would be I don't know, like whatever, 2000 out of 4500 samples had at least 37% of people preferring Garth Brook.

[...]

I: How was it that you thought about it that allowed you to keep things straight?
[...]

D: I just thought of it like ... I don't know, I sort of thought of it like how you were saying. Like...if the like 1s and the 0s if you ask 30 uh if like 10 of them say they like Garth Brooks-- or for every person who likes Garth Brooks you put a 1 down, if they don't you put a zero. You do that 30 times and you're gonna get like I don't know, 15 ones and 15 zeros you add up, you add them up. Then it says the cutoff fraction for each sample is 37% so you have like at least 37% of the...like those or...30-- if you add it up and divided it by the 30 and it's at least 37% then you have like another pile of like little papers and you put a one on like the big, the big one for the sample or a zero if it's less than-- if the whole sample is less than 37%. The 1s and 0s I don't know...you said something about like...that sort of helped.

A significant feature of student D's thinking was his ability to clearly distinguish different levels of the resampling processes — never confounding the number of people in a sample with the number of samples taken — while coordinating the various levels into a structured whole. Additionally, and relatedly, student D interpreted the result of the simulation as an amount (percentage) of *sample proportions*, thus suggesting that he understood that the multi-level process generated a *collection* of sample proportions.[3]

Student D's coherent image contrasts sharply with that of many poorer-performing students who persistently confounded numbers of people in a sample with numbers of samples drawn. The following interview excerpt illustrates one such student's (M) difficulties in the context of explaining similar computer simulations:

Segment 1

I: Ok, Suppose that, here's what I'm gonna do, uhh instead of 4500 samples I'm gonna take uhh, 1000 samples. Everything's gonna stay the same – sample size is 30, population fraction is 3/10ths, but now were' just taking 1000 samples. What would you expect the results to be?

[...]

M: Uhh, somewhere around like (short silence), hmm around like 25-30% of those 1000 samples.

I: Why 25-30%?

M: Because it's uhh...easier to uhh, I mean

I: What are you basing that judgment on?

M: Uhh, the actual population percentage, of 30

I: Ok, so you figure it'll be about 30%, 25 to 30, because the population fraction is 30%?

M: Yeah, somewhere close to that.

[...]

Segment 2

I: Alright (runs simulation, result displayed on output screen is "189 of these 1000 repetitions ...")

M: 2/10ths, 20%. Hmm, it's still a little less

I: So it's a little less than 20%, right?

M: Hmm hmm, huh (seems surprised)

[...]

Segment 3

I: Alright. Suppose that now we, let's do this, let's make 2500 samples (changes parameter value in command window). What fraction of those samples, I mean what result would you now expect, for the number of samples that we're going to get that exceed 37% preferring Garth Brooks?

M: About 1/5 of those.

[...]

I: Now, before you would have said "well, 3/10ths of the 2500 samples, the 2500 repetitions"

M: Hmm hmm

I: Do you still sort of lean that way, that you should get around 3/10ths of the --?

M: I think it should, but I don't understand why it's not, why it keeps coming out with 1/5$^{\text{th}}$

rather than 1/3$^{\text{rd}}$.

I: Alright, what is that "3/10ths" 3/10ths of?

M: Uhh, hmm 3/10ths of the entire population

I: Alright, and those are people, right?

M: Hmm hmm (nods)

I: Now, if you took 3/10ths of the 2500 repetitions you're taking 3/10ths of what?

M: Of the uhh...people sampled (chuckles)

I: No, 3/10ths of the *samples*.

M: Oh. Hmm hmm

Segment 1 of the excerpt suggests that student M expected the simulation to produce an amount (number) of samples and that he expected the percentage of that amount to hover around the sampled population percent (30%). Segment 2 illustrates his surprise at finding the actual percent being 20% of the 1000 samples generated. In segment 3 the student anticipated the same (20%) result for a simulation involving a larger number of samples, but he did not understand why this should be so because his conviction was that the simulation should produce a numerical value close to the sampled population percentage. The remainder of the segment reveals that student M had been interpreting the simulation's result as a percentage of people sampled rather than as a percentage of samples.

During such instructional activities most students experienced great difficulty conceiving the re-sampling process in terms of distinct levels. They would often unwittingly shift from speaking and thinking of a number of people in a sample to a number of samples selected. Their control of the coordination between the various levels of imagery was unstable; from one moment to the next their image of a number of samples (of people) seemed to easily dissolve into an image of a total number of people. These difficulties led many students to misinterpret a simulation's result as being about a percent of people rather than about a percent of sample proportions. This muddling of the different levels of the resampling process, in turn, obstructed their ability to imagine how sample proportions might distribute themselves around the underlying population proportion.

A salient consequence of these students' difficulty in imagining a sampling distribution was their tendency to judge a sample's representativeness only in relation to the underlying population proportion. Their image of sampling did not entail a sense of variability that extended to ideas of distribution: they understood that sample statistics vary, but only to the extent that if we were to draw more samples and compute statistics from them, those statistics would differ from the ones for the samples already drawn. Thus, judgments of a particular sampling outcome's unusualness were based largely on how they thought the outcome compared to the underlying population parameter per se, instead of on how it might compare to the way similar sample statistics were clustered around the parameter.

On the basis of such characteristics, we conjecture that these students' encompassing image of sample was *additive* — that is, in these instructional settings they tended to view a sample simply as a subset of a population and to view multiple samples as multiple subsets.

A contrasting image of sample is suggested in the following excerpt of student D explaining the purpose of simulating resampling:

D: If like...if you represent-- if you give it like the split of the population and then you run it through the how-- number of samples or whatever it'll give you the same results as if-- because in real life the population like of America actually has a split on whatever, on Pepsi, so it'll give you the same results as if you actually went out, did a survey with people of that split.

I: Ok, now. What do you mean by "same results"? On any particular survey at all--you'll get exactly what it-?

D: No, no. Each sample won't be the same but it's a...it'd be...could be close, closer...

I: What's the "it" that would be close?

D: If you get...if you take a sample...then the uh...the number of like whatever, the number of "yes"s would be close to the actual population split of what it should be.

I: Are you guaranteed that?

D: You're not guaranteed, but if you do it enough times you can say it's within like...1 or 2% of error depending upon uh how many times-- I think-- how many times you did it.

The resemblance between sample and population was clearly foremost in student D's mind, but his image was of a *fuzzy* resemblance bound up with ideas of variability and proto-distributional images of a collection of sample proportions. He did not expect a sample to be an exact replica of the sampled population, instead he anticipated that in repeating the sampling process many sample proportions would be "more or less" close to the population proportion. Moreover, student D's confidence in a sample's representativeness was based on this anticipated image of how a collection of similar sample proportions might be distributed around the population proportion.

We put that student D's description is consistent with his having conceived a sample as a quasi-proportional mini version of the sampled population, where the "quasi-proportionality" image comes from anticipating a bounded variety of outcomes, were one to repeat the sampling process.

It is often useful to refer to a germinating idea with suggestive terminology; we call this image of sample a *multiplicative conception of sample* (MCS) because its constitution entails conceptual operations of multiplicative reasoning. An elaboration of multiplicative reasoning (Harel & Confrey, 1994) is beyond the scope of this paper. For the present discussion we draw on Inhelder and Piaget's (1964) broad characterization of multiplicative reasoning as conceiving an object (quantity) as *simultaneously* composed of multiple attributes (quantities). For instance, conceiving a proportion involves multiplicative reasoning when it entails comparing two quantities in such a way as to think of the measure of one in terms of the measure of the other (Thompson

& Saldanha, in press). An example is when one thinks of percentage as quantifying a part of a whole *in terms of the whole*. This conception entails keeping both the part and the whole simultaneously in mind and the ability to reciprocally relate and express one in terms of the other. This is different from thinking of measuring a subpart of a whole only in absolute terms.

We hypothesize that MCS entails multiplicative operations on several levels: on one level it entails conceiving a relationship of proportionality between a sample and a population. On another level, imagining the emergence of a *proto*-distribution of sample statistics entails structuring statistics as subclusters of the range of an entire collection of statistics. This involves fractional reasoning. Finally, a mature and well articulated image of distribution supports quantifying the expectation of a particular kind of sampling outcome and thus quantifying one's confidence in a sampling outcome's representativeness. This entails the operation of juxtaposing the individual sample result against an aggregate of similar sample results to compare the one against the many – an image of simultaneity that is central to multiplicative reasoning.

Conclusion

Though our elaboration of these two images of samples and sampling is empirically grounded, our point in presenting it is not to imply that students in our experiment fell into one or the other camp. Rather, our point is to highlight two significantly different conceptions and images of samples and sampling — perhaps exemplary of extremes in a continuum of students' conceptions — that provide insight into what may be more or less powerful conceptions to target for instruction.

From our perspective, there are two reasons why the distinction between the additive and multiplicative conceptions of sample is significant. First, in contrast to the additive conception, MCS entails a rich network of interrelated images that supports a deep understanding of statistical inference. In practice, statistical inferences about a population are typically made on the basis of information obtained from a *single* sample randomly drawn from the population. This practice is common among statisticians despite expectations of variability among sampling outcomes. In statistics instruction, however, it is uncommon to help students conceive of samples and sampling in ways that support their developing coherent understandings of *why* statisticians have confidence in this practice. We claim that MCS empowers students to understand the *why* by orienting them to relate individual sample outcomes to distributions of a class of similar outcomes. In the same way, MCS enables students to consider a sampling outcome's relative unusualness. As such, we propose that MCS characterizes a powerful "target" conception that can guide efforts to design instructional activities and student engagements intended to support their developing a deep understanding of sampling and inference.

The second reason why we consider the distinction between these two conceptions of samples to be significant is that few of our students developed MCS. Instead, most students seemed to tend toward an additive image of sample. To us, this state of affairs suggests that developing MCS is non-trivial. The reasons for students' difficulties in this regard are currently unclear to us. However, one plausible hypothesis grounded in our data is that for many students the simulation and sampling distribution activities were of such a complexity so as to essentially overshadow ideas of sampling variability highlighted in the first phase of the teaching experiment. In a subsequent teaching experiment (Saldanha & Thompson, 2001) we took this hypothesis seriously and engaged students in instructional activities designed to support their developing a MCS.

Notes

[1]Research reported in this paper was supported by National Science Foundation Grant No. REC-9811879. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of NSF

[23]Similar samples share a common size, selection method, and parent population. Furthermore, they are selected to obtain information about a common population characteristic.

[4]The simulation was of sampling people's preference for a particular musician from a hypothetical population having a known proportion of it preferring the musician

[5]We note that student D's prediction of the simulation result was highly inaccurate in this excerpt. Shortly thereafter, however, he quickly revised his prediction with a highly accurate one and continued to make such accurate predictions throughout the rest of the interview. We thus believe that his initial prediction was not an indication of a poor sense of how the sample proportions were distributed, rather it was merely the result of his focus, in the moment, on explaining *how* the simulation worked and *what* it generated.

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USING MULTI-REPRESENTATIONAL COMPUTER TOOLS TO MAKE SENSE OF INFERENCE

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We report how two sixth-grade students used software tools to formulate and evaluate inferences based on simulation data. Students' interactions were analyzed during a 12-day instructional program that utilized *Probability Explorer* software as a primary investigation tool. A variety of microworld tools enabled students to recognize the importance of using larger samples to draw valid inferences. Vignettes from three instructional tasks are included.

In recent years, the National Council of Teachers of Mathematics (NCTM) (2000) has advocated that *all students* develop and evaluate inferences that are based on data. Specifically it is recommended that "upper-elementary and early middle grades students... begin to develop notions about statistical inference" (p. 50) and suggest the use of computer simulations to facilitate student learning in probability. Investigating students' probabilistic reasoning with technology tools, however, is a recent endeavor (e.g., Pratt, 2000; Drier, 2000a, 2000b). In fact, there exists a gap in knowledge of early adolescent learners' development of probabilistic reasoning with open-ended multi-representational software. Our research sought to determine the role technology tools plays in students' development of the notion of statistical inference. In this paper, we report how two students used software tools to formulate and evaluate inferences based on data from simulations.

Theoretical Perspective

The framework for our research is based on a coordinated perspective of learning. In particular, we believe that learning is a constructive process of resolving perturbations through reflection and abstraction (von Glasersfeld, 1995), occurs within a social context in which meanings are negotiated through interactions with other cognizing individuals (Voigt, 1996), and is enabled and constrained by tools and resources within the environment (Graue & Walsh, 1998). We believe that the intended and enacted mathematical tasks, students' interactions with microworld tools, and the social interaction among students and between students and teachers, all operate as potential meaning-making agents for the development of probabilistic reasoning. Thus, this constructivist framework guided the design of the microworld tools, instructional tasks, and classroom teaching throughout our research.

Several researchers (e.g., Battista, 1998; Biddlecomb, 1994; Land & Hannafin, 1996) have worked with children and developed principles for designing and using open-ended computer microworlds to facilitate students' construction of knowledge.

Biddlecomb (1994) emphasizes that "computer environments must be very flexible in order to make them as open as possible for the teacher and students to construct their own individual and shared mathematical environments" (p. 91). A well-designed open-ended learning environment (OELE) enables learners to build and test their intuitive notions in an exploratory manner such that their understanding evolves as a continuous and dynamic process through observation, reflection, and experimentation (Land & Hannafin, 1996). *Probability Explorer* (PE) (Stohl, 1999-2002) is purposefully designed as an OELE with multiple ways to represent data that engage students in designing, simulating, and analyzing results of probability experiments. At a fundamental level, data is represented in *PE* with randomly generated icons that can be sorted, stacked (in a pictograph) or lined up in the sequence in which they occurred. A Pie Graph (relative frequency), Bar Graph (frequency), and Data Table (counts, fractions, decimals, and percents) are also available to display results dynamically changing during a simulation (see Figure 1).

Students can use *PE* to develop their abilities to conjecture and evaluate inferences when they have opportunities to explore a variety of probability situations, determine

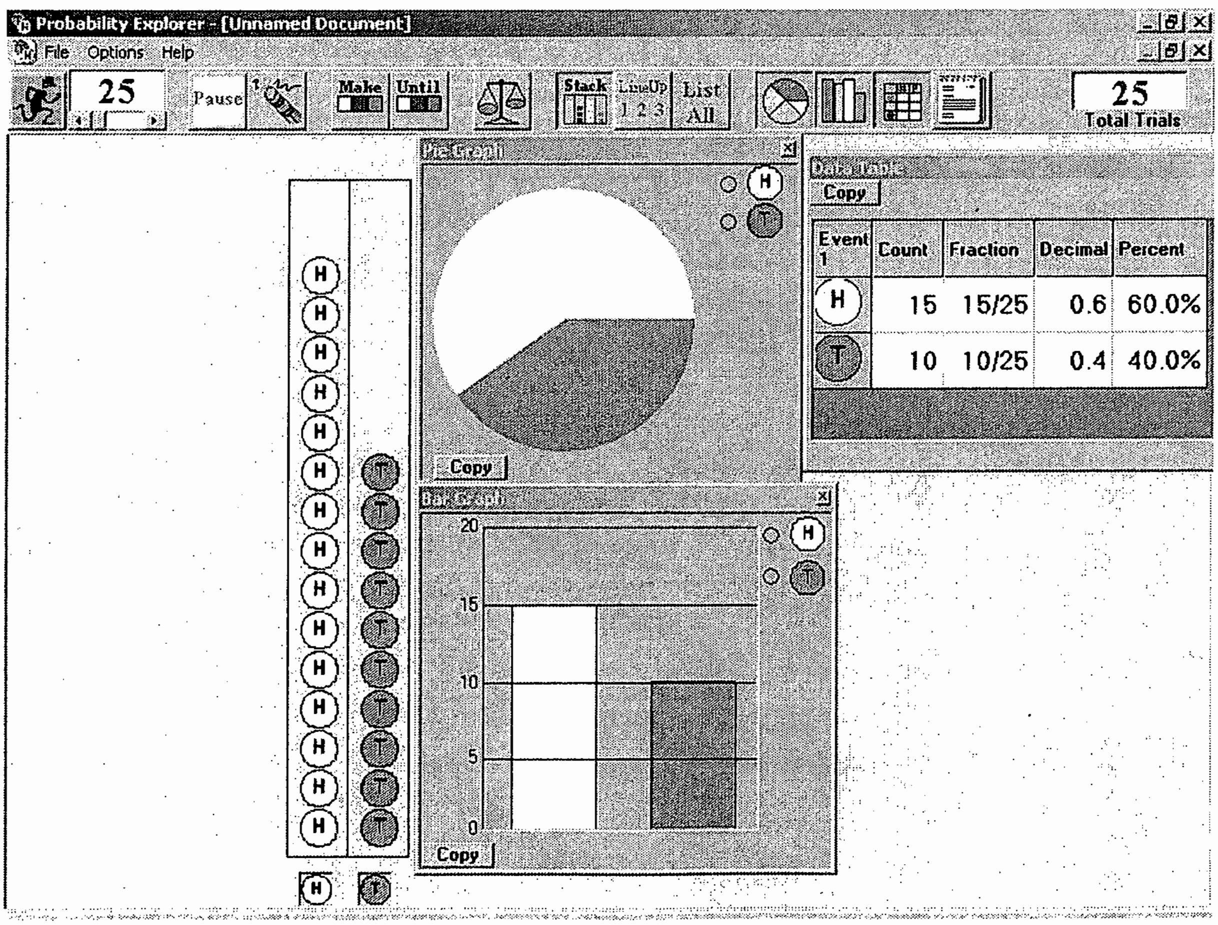


Figure 1. Data displays of 25 tosses of a fair coin.

how many trials to run, and choose which tools to use to analyze data. One of the most important aspects of formulating and evaluating inferences is understanding the *unpredictability* of random phenomenon in the short-run but *predictability* in the long-run trends in data (i.e., the law of large numbers). Thus, the sample size in a simulation becomes an important factor for students to consider when making inferences from a sample distribution to the population and making connections between empirical and theoretical probability. Prior research has shown the difficulty students experience in reasoning consistently about the effect of sample size. Fischbein and Schnarch (1997) found that students' acceptance of the use of small sample sizes increased dramatically from 5^{th} to 11^{th} grade. They hypothesize that "this misconception is based on the idea that a ratio is representative of an indefinite pair of numbers" (p. 101). Stavy and Tirosh (2000) attribute this erroneous conception to an intuitive rule based on "Same A - Same B" by which students assume that if the results are in the same proportion then the chance of each result occurring is the same. Stavy and Tirosh also found that students' use of this intuitive rule seems to become more prominent with age and as their ability to recognize proportionality stabilizes.

Research on instruction aimed at helping students develop intuitive notions about sample size has shown to positively effect students' understanding about the power of sampling in making inferences. Metz (1999) found that, after the provision of extensive experiences in designing, implementing, and critiquing investigations within a science context, 41% of second, fourth, and fifth grade students made arguments for the use of large samples to make inferences about a population. In a study with sixth grade students, Aspinwall and Tarr (2001) found that prior to instruction, 5 of their 6 case study students showed no or little awareness of the relationship between experimental probability and sample size, and typically thought that any size sample should reflect the parent distribution. Their analyses indicated that game-like situations in which the chance of winning is unequally distributed between players helped students to understand the role sample size plays in making probability judgments. Students learned that outcomes with smaller probabilities are more likely to win with a small number of trials while outcomes with higher probabilities are increasingly favored as the number of trials increases.

Using computer tools and representations provides students with different ways to simulate and interact with data to make inferences. In prior research with *PE*, Drier (2000a, 2000b) found that fourth-grade students used the representations in *PE* as both objects to display and interpret data, and as dynamic objects of analysis during experimentation to develop a notion of the "evening-out phenomenon." These students recognized that larger number of trials resulted in distributions that closely resembled what they expected from the theoretical probabilities based on how they had designed an experiment. Pratt (2000) also found that 10-year old students working with a *Chance-Maker* microworld made a connection between the number of trials and the distribution of data (viewed in pictographs and pie graphs) and that they could use a “workings box” to control the theoretical probability, which also affected the distribution of data. The findings of both Pratt and Drier suggest that simulation tools that give students control over designing experiments, running as many trials as they desire, and viewing graphical representations of results may help in the development of deeper understandings of how theoretical probability, empirical probability and sample size can be used to make inferences.

Considering our theoretical perspective on learning in a technological environment and the implications from prior research, we sought to explore the following questions in our focused analysis of two case students:

- How do these students use *Probability Explorer* tools to simulate experiments and analyze data as part of their meaning-making processes in solving a variety of probability tasks?
- How do these students socially negotiate an understanding of the interplay of theoretical probability, empirical probability and sample size, and use these understandings and computer-based tools to conjecture and justify inferences based on data?

Methods of Inquiry and Data Sources

This research report focuses on two average-level sixth-grade students who participated in a larger study of 23 sixth-grade students in an average-level mathematics class in an urban, southern public middle school. Prior to instruction, in collaboration with the regular classroom teacher, we purposively selected (Lincoln & Guba, 1985) Manuel (Hispanic male) and Brandon (Caucasian male) to serve as one of three case study pairs. These students were chosen to reflect the ethnic and socioeconomic diversity within the class and were representative of average-level mathematical achievement based upon scores on standardized tests in mathematics as well as a pre-instructional test in probability.

Research Setting and Data Sources

All students were seated in pairs at tables in their regular classroom with a PC laptop, calculators, and manipulative materials (e.g., dice, spinners) readily available. Manuel and Brandon’s laptop computer was connected to a PC-to-TV converter to internally video-record their computer interactions while microphones captured their conversations. We used video recording because it is particularly useful in trying to access students’ construction of mathematical understandings (Pirie et al., 2001) and gave us a direct record of how they used the computer tools. For this paper, additional data from the instructional sequence was used in the analysis. In particular, the whole-class video, students’ written class work and homework assignments comprised the data corpus.

Instructional Program

We developed and co-taught a 12-day probability unit utilizing *PE* as a primary investigation tool. Several characteristics were common to the instructional tasks. Problem tasks represented random phenomena that occur in a variety of contexts (e.g., pulling fish from a lake, selecting marbles from a bag). Typically instructional tasks required students to model the phenomena, carry out simulations using *PE*, and collect, display and analyze data in order to draw appropriate inferences and formulate convincing arguments based on data.

The design and implementation of our instructional program were influenced by the Mathematics Teaching Cycle (Simon, 1995). We utilized this conceptual framework to create purposeful pedagogy that reflected our goals for student learning yet was responsive to students' ideas and practices. More specifically, we generated hypothetical learning trajectories prior to instruction and made adaptations based on our ongoing assessment of students' probabilistic reasoning. In this regard the design of our curriculum was an iterative process based on student interaction with the instructional tasks, microworld tools, and the teacher-researchers.

Methods of Analysis

To understand the interactions that comprise students' meaning-making processes, our research draws upon an interpretivist approach to inquiry (Graue & Walsh, 1998) using a case study method (Stake, 1995). In particular, we used qualitative research methods to observe and critically analyze students' probabilistic reasoning, meaning-making processes, and social and computer interactions while using *PE*. An analytic model for examining videotape data (Powell, 2001) was used to study the growth of students' probabilistic reasoning and their use of computer tools. More specifically, this method was used to identify critical events (or conceptual leaps) and the interactions that elicited such critical events.

Results

For the purpose of this paper, we included an analysis of Brandon and Manuel's use of *PE* tools on three instructional tasks. In the first task, students used features of *PE* to estimate the distribution of marbles in a mystery bag. In the second task, they used *PE*'s Weight Tool feature to design a simulation model for a given spinner and to validate their model. In the third task, they used repeated simulations of various sample sizes to determine whether a die was fair and to estimate the probability of each number on the die occurring. Throughout these tasks, Brandon and Manuel used several *PE* tools in making informal inferences based on their developing understanding that simulations and data analysis should inform their judgments. The results presented below illustrate how their ability to design and simulate experiments, use various microworld tools, support inferences with data, and formulate convincing arguments developed across the tasks.

Mystery Marble Bag Task

On the 4th day of instruction, students used *PE* to collect and analyze data in order to draw inferences regarding the composition of a bag containing 10 marbles. Brandon and Manuel chose the Run Until tool as a novel heuristic in their problem solving. (The Run Until tool allows users to select an outcome and run a simulation until that outcome occurs). They effectively used Run Until in two distinct ways. First, by repeatedly sampling one marble with replacement they inferred which colors of marbles were most likely *not* in the bag; and second, in deciding which colors were certainly present. The following excerpt reveals both uses of Run Until.

Brandon: Wait. Wait. Wait. This is what you do: Run experiment until you get an outcome of white. Do Run Until so you can see if there are any whites. [After 17 trials with no white outcomes] I don't think there are any whites in the bag -- zippo!

Teacher: Why do you say that?

Brandon: [As marbles continue to be sampled with replacement] Because we're running 30 trials and there's no whites!

Manuel: So far and we've run it until there's a white. [After Run Until yields no green outcomes in 76 trials, they use Run Until black] Well what do you know? No black!

Teacher: Why do you say that? You've done it (only) 14 times.

Manuel: [As marbles continue to be sampled with replacement] Make that 25 -- No, 30!

Teacher: At what point... do you have enough evidence to say, "There's no black"?

Both: Because you've run it 50 times!

Teacher: But you were saying that back at 10 (trials).

Brandon: Well, that was our guess. Now we're for sure (right). Now we're very confident and that was the goal, to be very confident.

Manuel: I know what the colors are: Red, yellow and blue!

On the subsequent day, Brandon preselected a distribution of 12 marbles. Manuel's goal was to estimate the composition of the bag. He made efficient use of Run Until since he only needed to use Run Until once to decide which colors were and were not in the bag. After 140 trials yielded no white outcomes, the sample continued to grow to 500 after which Manuel observed that it contained no green or yellow marbles. Moreover, he remarked that only red, yellow and blue marbles comprised the set of 500 outcomes. After concluding that only these three colors were present in the bag, Manuel subsequently ran several sets of 10 trials and used an informal approach for inferring the number of marbles of each color by estimating a mean of the most commonly observed frequencies.

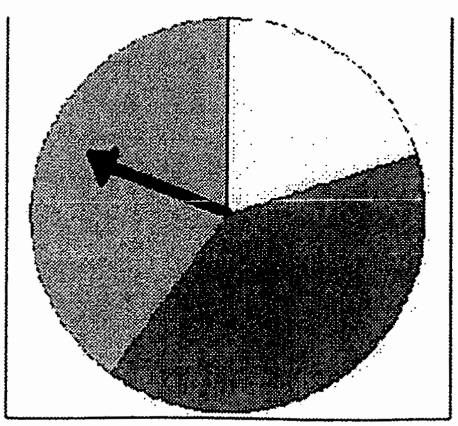
In this task, Brandon introduced the use of the Run Until tool as a strategy, but Manuel quickly realized how the data was providing relevant information. It is interesting that Manuel only used a large number of trials with the Run Until tool as evidence to infer which colors were in the bag, but repeatedly used a small sample size (10) to make his inference about the distribution of colors of marbles. Since he knew that the bag only contained 12 marbles, the use of a large number of trials may not have helped him unless he could use proportional reasoning to infer the distribution based on percentages or the empirical distributions of colors. Although Manuel used a larger number of trials to infer which of the six colors were present in the bag, large numbers of trials would only be useful to find the distribution if he was able to use proportional reasoning which may not have been intuitive or accessible to him.

Spinner Simulation Task

On the 9th day of instruction, Brandon and Manuel used the Weight Tool to create a model for a spinner (Figure 2). They used the Pie Graph as a primary representation to analyze data and test the "goodness" of their model and used decimals and percents in the Data Table as secondary representations. They mostly ran multiple sets of 100 trials and occasionally a larger number of trials. Consider the following two episodes that illustrate their meaning-making about the connection between theoretical and empirical probability using several *PE* tools.

Episode 1

A teacher-researcher challenged Brandon and Manuel to design a model of the spinner shown in Figure 2 using a total weight of 50. Manuel typed in 20:10:20 in the Weight Tool. Brandon claimed "that's not right" and Manuel said, "I bet you a billion dollars it is." The teacher-researcher asked Manuel to convince Brandon that



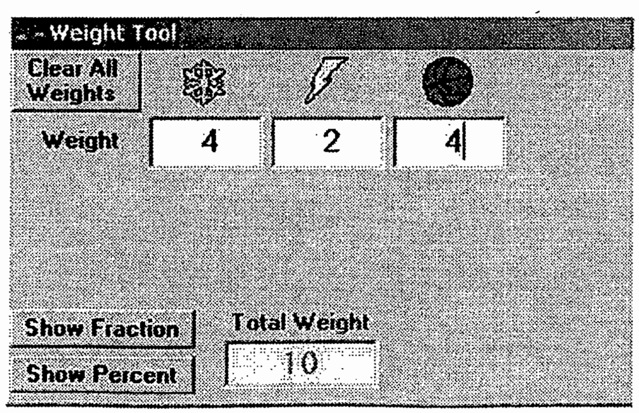


Figure 2. Spinner and weights used in *PE.*

20:10:20 could be used to model the spinner. Manuel struggled to explain how the weight model was in proportion to the original weights of 4:2:4 or the spinner regions. Brandon decided to run simulations in *PE* to "see if it still comes close, as long as we have the same percentages." He ran 100 trials with the Pie Graph and Data Table open and after 60 trials said, "That looks pretty right." When the 100 trials were complete [showing a 34:23:43 distribution] he said, "Okay, that's right." Manuel interjected, "Yeah, I just don't know how to explain it." Brandon continued to run sets of 50 and 100 trials. After several sets of trials Brandon claimed, "Well, that is pretty close... Well, he's right cause I see the pie graph and I agree with him." By comparing percentages of the theoretical probability in the Weight Tool with the empirical data shown in the Pie Graph and Data Table, Brandon seemed to understand that empirical data could be used to support or confute the notion that weights of 20:10:20 appropriately modeled the spinner. His use of 50 and 100 trials were not particularly large, but he *may* have been using these numbers since 50 is compatible with the total weight and 100 is convenient in reasoning about part-whole relationships in the Pie Graph. It is important to note that Brandon needed to run several sets of 50 and 100 trials before he was convinced that the empirical data supported that weights of 20:10:20 accurately modeled the spinner.

Episode 2

The software application froze while they modeled a second spinner. The spinner and their initial weights are shown in Figure 3. Brandon restarted the application and redesigned the weights but mistakenly inserted a 5 (rather than 3) for the "rock" representing the green sector.

Brandon ran 100 trials and Manuel commented that the Pie Graph looked "way off" (sectors representing the Smiley Face and Rock were each about 40%). Brandon responded, "No sir!" and Manuel argued, "Yes, that is <u>so</u> off, let me look at the Weight Tool." He opened the Weight Tool and remarked, "See? You did the wrong weights.

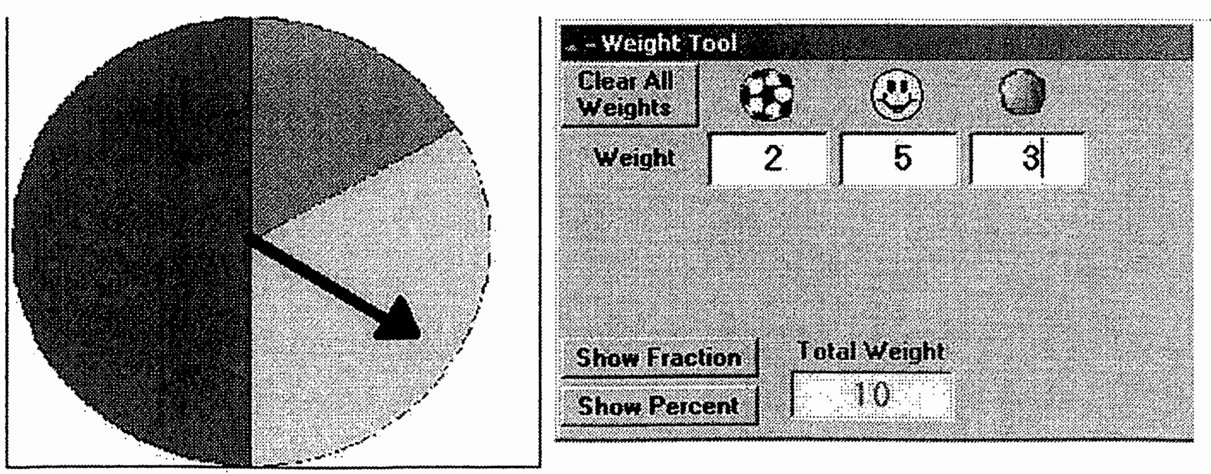


Figure 3. Second spinner and weights in *PE*.

Rock was supposed to be 3, that is why the pie (graph) looked funny." Although Brandon used empirical data to test the weights used for the first spinner, it was Manuel who seemed to make the connection between the empirical data and the weights and was able to apply his understanding when the Pie Graph did not appear as he expected. In this task, both students made sense of how empirical data could be used to test a conjecture. This type of meaning-making is an important component to being able to use empirical data from a large number of trials to make inferences. The subsequent task in the instructional sequence (*Schoolopoly*) required students to make inferences regarding the fairness of a die and support their claim with evidence (i.e., data).

Schoolopoly Task

On the 10th and 11th days of instruction, Brandon and Manuel sought to investigate claims that a company manufactured faulty (biased) dice. Unbeknownst to them, weights of 2-3-2-3-2-3 were preloaded into the Weight Tool for the outcomes, 1-6, respectively. Their assignment was to collect evidence to support or reject claims that the die is biased and to estimate the theoretical probability of each outcome, 1-6. On Day 10, they began by running a total of 51 trials; they noticed that 5 occurred only three times and this result focused their attention on the number of 5s in a subsequent set of 500 trials. While the simulation ran, they repeatedly commented on the low number of 5s and 1s in relation to the other frequencies and hypothesized that these outcomes were relatively difficult to obtain. Manuel decided to run 50 (new) trials and used the Data Table and Stack column to analyze data. This set of data yielded many 5s leading Brandon to argue that the die was indeed fair. Manuel ran a second set of 50 trials with a 9-6-4-11-9-11 distribution which led Brandon to announce "dang this is fair." His notion of fairness was grounded in the observations of indiscernible patterns in the sample distributions from relatively small sets of data. Brandon and Manuel concluded Day 10 by writing that they believed their die was fair.

At the start of Day 11, Brandon had control of the mouse and set the number of trials to 300 because it represented a large multiple of 6. As the simulation ran, they monitored the relative frequencies displayed in the Data Table, Pie Graph and Bar Graph. At 60 trials, with few 5s occurring (represented in red on the Pie Graph and Bar Graphs), Brandon cheered, "Come on! Get even red!" then remarked, "I think it's actually fair" and dismissed the non-uniform sample distribution by arguing, "5 just got off to a bad start." As the simulation of 300 trials neared completion, Brandon concluded, "It's pretty fair... It's only 5 and 3 and 1 are a bit behind" (pointing to sectors in Pie Graph). Upon reflection, Brandon altered his belief about fairness and stimulated the following discourse:

Brandon: I really don't think it's fair.

Manuel: [In disbelief] Why?

Brandon: Just because, I --

Manuel: Every single thing doesn't have to be even, man, it's the luck. They are pretty much close.

Brandon: Yeah, you're right. Let's just do another 500 (trials).

Manuel: Let's do one million.

Brandon: No, let's do 6000. [Manuel sets the number of trials to 500] Now run it once. [Manuel hits Run] Now click 500 one more time.

Manuel: I want to do the Pie Graph [opens Pie Graph, then Bar Graph, and Data Table at about 400 trials].

Brandon: I still think that 1… 5 is continuously behind.

Manuel: If you don't think this is fair… It's fair, man.

Brandon: But look at the 5 [There are 580 trials with distribution of 88, 115, 80, 108, 79, and 111]

Manuel: It doesn't all have to be perfect, man! No one is going to get theirs this way that much.

Brandon: [At 650 trials] Look at the percents: 13 (percent for 5), 13 (percent for 3)… I bet you that's (the weights of) 3, 3, 3, 2, 2, and 1.

Manuel: I bet you're wrong. I bet we're fair.

Brandon: I bet we aren't fair.

Manuel: Well, I don't care. We are fair. Just because it's not all even doesn't mean we're not fair. Dude, we're already up to 1000 (trials).

Brandon: But still, I really don't think… it's only beating it by about a hundred.

Manuel: It's not that unfair. [At 1300 trials] See 3 and 5 and 1 are practically the same.

Brandon: So they must have the same probability but that might have been more because (inaudible).

Manuel: [At approximately 1500 trials -- See Figure 4] Wait a second. Wait a second. We *are* unfair. These two… all of these [pointing to 1, 3 and 5] are (weighted) 1 and all of these [referring to 2, 4 and 6] are (weighted) 2. So 2, 4, 6. We're unfair.

Brandon: Thank you. I told you!

They continued to run sets of large trials (1500 or more) and determined that "6, 4, and 2 had higher probability than 5, 3, and 1" and estimated the probability for 1-6 occurring, as 14%, 20%, 13%, 20%, 13%, and 20%, respectively.

Students' use of software features (most notably displays of data) challenged their beliefs regarding fairness. More specifically, they learned to place value in sample distributions generated from larger sets of data. Specifically, Brandon monitored the sample distribution as the number of trials grew large; in doing so, he was able to reconsider the notion that "5 just got off to a bad start" and instead inferred that the die was biased. Similarly, Manuel used data to reject his initial belief that variation among a sample is attributed exclusively to randomness; he detected bias using patterns in data as the number of trials grew large. Together, they negotiated the inference about the die based on data collection, lengthy discussions, and analyzing patterns in data from increasingly larger samples of data.

Discussion

Our research indicates that the instructional sequence and microworld tools successfully fostered students' ability to make appropriate inferences based on data. In

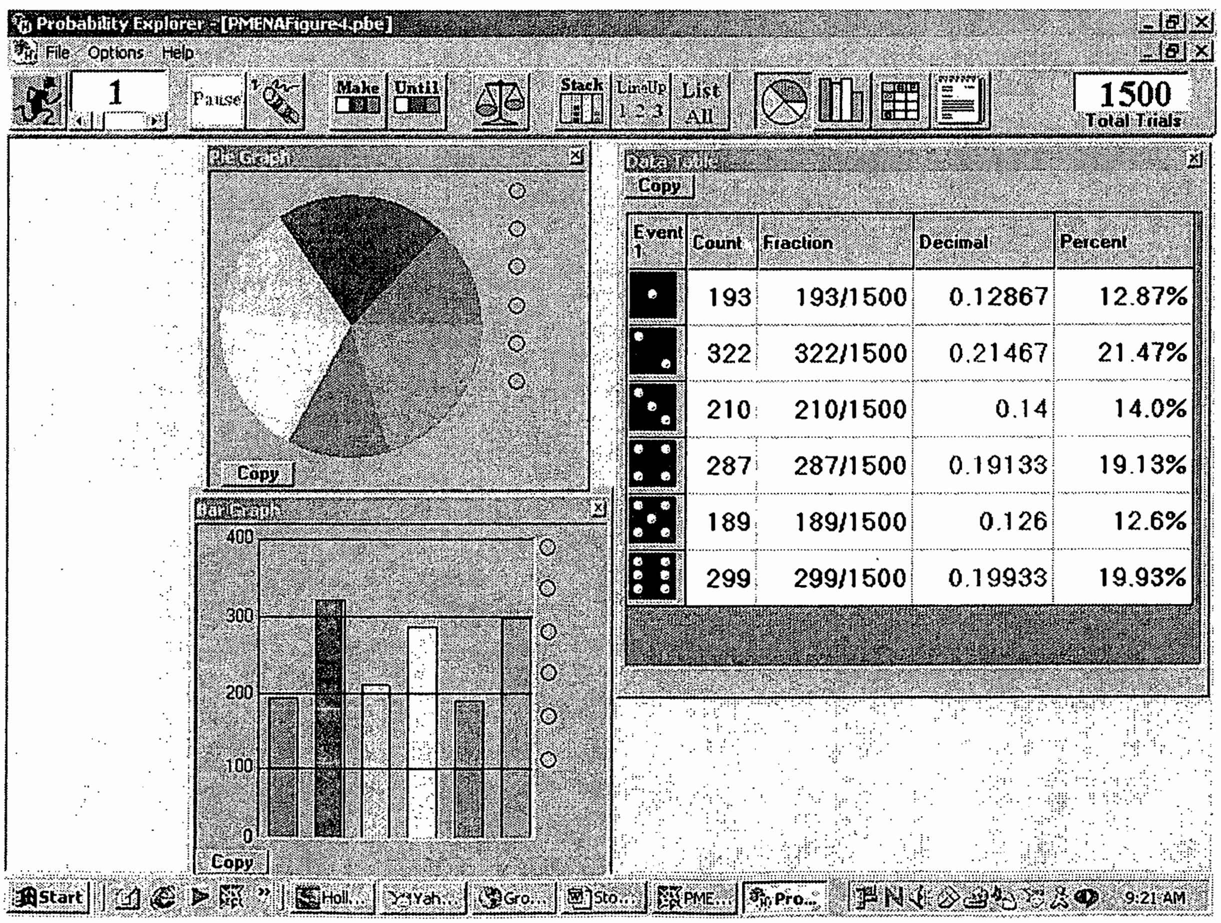


Figure 4. Data from simulation of *Schoolopoly* task.

particular, Brandon and Manuel's use of computer tools, coupled with social interaction, enabled them to make connections between simulation data (empirical probabilities) and weights in the Weight Tool (theoretical probabilities). Since the tasks were purposely designed to build toward a more powerful form of inference, their use of data to make inferences in each task was slightly different. The *Schoolopoly* task intentionally required students to make inferences about a population, which is more sophisticated than drawing inferences regarding the part-part distribution present within a bag of marbles. The intermediate task, *Spinner Simulation*, represented a purposeful attempt to transition students into part-whole reasoning through use of a pie graph representation. In addition, by grouping students in pairs or small groups, they were able to negotiate meaning of what constitutes evidence and, in particular, the role data plays in supporting arguments. Brandon's use of software tools (Pie Graph, percents in Data Table) to reason proportionally when making inferences enabled Manuel to make a transition from primitive reasoning into informal proportional reasoning.

Previous research (Fischbein & Schnarch, 1997; Stavy & Tirosh, 2000) indicates that students may not conceive the power of sample size because they invoke proportional reasoning and assume that every sample should be in proportion to (or reflect) the parent population. Such research, however, was based on written tasks that denied student access to simulation tools. Consistent with Pratt (2000) and Drier (2000a, 2000b), our results suggest that young adolescents can develop powerful notions about statistical inference when using simulation tools, recognize the importance of using larger samples in drawing valid inferences, and use data displays to make connections between theoretical and empirical probabilities. Further research needs to investigate whether sustained access to simulation tools and engagement in tasks about inference can help young students maintain notions of the effect of sample size even when their proportional reasoning stabilize at a later age.

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RANDOMNESS: RETHINKING THE FOUNDATION OF PROBABILITY

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In this article, we highlight a series of tensions inherent to understanding randomness. In doing so, we locate discussions of randomness at the intersections of a broad range of literatures concerned with the ontological nature of probability theory, and epistemological conceptions of probabilistic ideas held by people. Locating the discussion thus has the advantage of emphasizing the growth of probabilistic reasoning and deep connections among its aspects.

Although we find no explicit definition of *randomness*, we might catch a glimpse of its meaning by looking at what texts say are the meanings of *random phenomenon*, and *random sampling*. Yates, Moore and McCabe (1998, p. 314) define a *random phenomenon* as one in which "individual outcomes are uncertain but there is nonetheless a regular distribution of outcomes in a large number of repetitions." Bluman (2000, p. 632) defines a *random sampling* as one in which all possible samples of a certain size "must have an equal chance of being selected from the population." As we shall elaborate below, these definitions are circular. In order for a student to understand them, they must already understand the idea of random variable and the distribution of its values, which itself entails an understanding of randomness. The problem, then, is that instruction is often designed to introduce students to ideas in ways that presume, essentially, that students already understand them.

Yet, surprisingly, we found little attention paid in statistics textbooks and research studies to what it means to understand randomness. Our analysis of secondary statistics textbooks (Thompson & Liu, 2002) revealed that 45% of questions concerning probability are stated, grammatically speaking, as if about a single, unrepeated (and hence non-probabilistic) event. Moreover, if students understand the phrase "at random" simply as "cannot predict ahead of time" or "unexpected", then 98% of the questions can be interpreted as if about a single, unrepeated (and hence non-probabilistic) event. Yet, helping students understand "at random" productively is not explored. As Falk and Konold (1994) state, the complex nature of randomness poses an educational dilemma: "Shall we forgo discussing the meaning of the concept in our teaching (relying on students' existing intuitions)? Or shall we endeavor to find a satisfactory way of presenting randomness, undertaking the challenge of bringing up the doubts and difficulties students will predictably have?" (Falk & Konold, 1994, p. 2).

Randomness and Determinism

Ideas of randomness range along an ontological and epistemological spectrum. On one side of the spectrum, probabilities are regarded as not being inherent in objective nature but as reflecting human ignorance of a true determinist course of events. This view of probability was a product of the thinking of the European Enlightenment, the admiration for Newtonian mechanics, and the consequent belief in universal determinism. As Laplace (1814) remarked, "We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow."(p. 4). It is only a short step from this to the conclusion that absolute randomness does not exist, and therefore that all probabilities will be 0 or 1. The other end of the spectrum comes from the renunciation of determinism that was brought by quantum mechanics in the twentieth century. The understanding of Brownian motion as an expression of constant internal agitation of microscopic particles without external causes (von Plato, 1994), was taken as proving the existence of "irreducible chance" and therefore that randomness is an inherent feature of nature. "Causality, long the bastion of metaphysics, was toppled, or at least tilted" (Hacking, 1990, p. 1). von Mises concluded, "there exist genuinely statistical sequences in nature" (von Mises, 1957). However, controversies still exists as to whether randomness is a characteristic of a sequence itself or the process from which the sequence was generated. In *The Logic of Chance*, Venn said that it is 'the nature of a certain arrangement," but not 'the particular way in which it is brought about," that should be considered when judging a random arrangement, and the arrangement must be judged by what would be observed in the long run (Bennett, 1998, p. 166). Venn hoped to illustrate randomness by building a graph using the decimal expansion of π. From a deterministic perspective, however, even though the decimal expansion of numbers such as π, e, and $\sqrt{2}$ might exhibit disorderliness, the process from which each is generated is completely predetermined. Hence, for those who believe that randomness is a property of the process there exists no randomness in this situation. On the other hand, a sequence obtained from a random process may exhibit certain visible pattern in appearance. Thus, there is a necessary distinction between the idea of a random sequence and the idea of a sequence whose elements are chosen by a random process.

Formalization of Randomness

von Mises' formulation of the notion of randomness was based on the intuition of "the impossibility of a gambling system". The randomness in a sequence lies in the "impossibility of devising a method of selecting the elements so as to produce a fundamental change in the relative frequencies"(von Mises, 1957). This definition of randomness had long been criticized as being "not mathematically expressible" and being "too inexact to serve satisfactorily as the basis of a mathematical theory" (McShane, 1970; Gillies, 2000). Kolmogorov and others later proved that von Mises-type sequences would exist if only simple formulas, rules, or laws of prediction are allowed. In other words, rather than requiring the randomness of a sequence to be judged by absolute unpredictability, Kolmogorov would require only unpredictability by a small set of simple rules. Kolmogorov further constructed a definition of random sequence based on the notion of complexity in information theory: a random sequence is one with maximal complexity, in other words, a sequence is random if the shortest formula which computes it is extremely long (von Plato, 1994). The modern mathematical treatment of random process is based on Kolmogorov's measure theoretical probability. In his approach a random process (or stochastic process) is a family of random variables having time t as its independent variable. Kolmogorov distinguished random process from determined process by considering time and distribution of outcomes simultaneously: "If the state y of a system at time t is uniquely defined by its state x at an arbitrary moment t_0 through a unique function f such that $y=f(x, t_0, t)$, situations of this general type are called *schemes of a well-determined process*. On the contrary, if the state x at time t_0 only determines a probability distribution for the possible future state y, these are called *schemes of a stochastically definite process*" (von Plato 1994). Kolmogorov's solution to distinguishing between randomness and determinacy rests, however, on a troubling assumption: The function $f(x, t_0, t)$ exists in its own right, which means that processes *have* certain properties timelessly, even if, over time, they vary.

Implications for Teaching and Learning

Philosophical and mathematical debates on what "randomness" means highlight its deeply problematic nature and therefore highlight an equally problematic question of what "probability" means. Two approaches to addressing the issue are (1) ignore it and (2) try to bring the debate to students' levels. We propose a third approach, which addresses the problem by cutting the Gordian knot. We note that both ends of the spectrum presume that the question is ontological, that it is about what randomness *is*, and approach it by speaking of it as a property of something that exists, be it sequence, process, or distribution. We propose that instructional design finesse the question, focusing instead on what we might *mean* by randomness. That is, focus on the imagery and operations that support coherent thinking and reasoning about situations that we see as entailing randomness and that we hope students come to see as entailing randomness. One scheme of images and operations that we will describe in our presentation is that of students building an image of a loosely-coupled process with imprecisely-determined inputs that generate collections of outcomes that have predictable distributions in the long run but unpredictable distributions in the short run. We hasten to point out that we are not talking about adopting a perspective on the ontology of processes or collections. Rather, we are talking about helping students *imagine* processes and collections in a way that supports coherent stochastic reasoning.

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CONSTRUCTING GENERALIZATIONS: AN ANALYSIS OF ONE STUDENT'S PROGRESS TOWARD A GENERALIZED UNDERSTANDING OF THE LAW OF LARGE NUMBERS

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Knowledge transfer, traditionally defined as the successful use of knowledge learned in one situation in a different situation, has a long and controversial history. While in pragmatic terms the bulk of formal education in the U.S. is predicated on the assumption that transfer takes place, it has been notoriously difficult to demonstrate in research studies, particularly those in the cognitive tradition (Bransford & Schwartz, 1999). One of the reasons underlying the controversy and mystery of transfer is the lack of specificity with which the word has been used. The present research considers transfer of one particular kind: the ability to understand and interpret a variety of problem situations as supporting the application of a single mathematical principle. I use the techniques of microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1993) to examine how one undergraduate student came to see a series of mathematics problems as "alike" as she constructed an understanding of the law of large numbers.

Many theories of transfer and analogical reasoning are grounded in assumptions involving structured mental representations. Specifically, most cognitive theories assume that individuals construct mental representations or schemata structured at an appropriate level of abstraction so as to be recalled and applied across relevant situations and contexts (see, for example, Gentner & Markman, 1997; Gick & Holyoak, 1980; Reed, 1993). Research in situated cognition offers an alternate approach by interpreting transfer as the result of an individual's attunement to affordances that support invariant activities across situations (Greeno, Moore, & Smith, 1993). The present research examines the kind of fine-grained data necessary to support the analysis and critique of these theories. I offer the beginning of an understanding of transfer that is consistent with the epistemological perspective taken by diSessa (1993) and diSessa and Sherin (1998). In this analysis, the development of scientific and mathematical knowledge takes place through systematization, reorganization, and integration of often context-sensitive, intuitive knowledge. The resulting knowledge systems permit greater flexibility than the more structured mental representations posited by traditional theories. This approach complements the perspective taken by Greeno et al. (1993).

Data for this research were taken from audio- and video-taped sessions recorded over a period of eight weeks as Maria, an undergraduate student, was enrolled in an introductory course in statistics. Maria was one of six students who agreed to meet with me one-on-one for a series of two-hour, weekly sessions that were divided into two parts. During the first hour, students were invited to use me as a resource or tutor as they worked on the material of the course; during the second hour, students agreed to take part in other course-related activities that I provided for them. These activities included interviews, think-aloud problem-solving sessions, explorations with computer simulations, and simple empirical experiments using dice or other random devices. Thus, aspects of the sessions resembled classical clinical interviews. At the same time, students had significant motivation to learn during the interviews, since they would be held responsible in their coursework for the concepts we discussed.

The data considered here were taken from three sessions during which Maria worked on a variety of problems involving the law of large numbers and the central limit theorem. The Coin Problem, for example, asked if it would be to her advantage to flip more times or fewer times if she wanted a series of flips to land "more than 70% heads," or "between 40% and 60% heads." The Post-Office Problem asked if it was more likely for a post office with more or with fewer customers to find that the average height of the men arriving on a particular day to register for the draft was significantly above the national average height. Maria considered five such problems in all, involving different cover stories and contextual details, different problem types (some involving proportions, others involving means), and different aspects of the central limit theorem (some desiring nearness of a sample statistic to a population parameter, others desiring a deviation from it). Upon her first exposure to the problems, Maria approached them all differently, answering most of them non-normatively. During the course of the three sessions, however, she eventually came to see them as "alike" as she constructed an understanding of the law of large numbers that gradually came to encompass all of them.

Despite the fact that Maria could state the law of large numbers informally in what she called her *rule* ("The more you flip, the closer you get to what's expected."), a fine-grained analysis of Maria's work on the Coin, Post Office, and other problems reveals that different problem contexts, problem types, and problem aspects consistently cued different language and different explanatory and descriptive ideas. The processes by which Maria came to see the problems as alike and their solutions as guided by a single principle took place only gradually and with the assistance of computer simulations, some questioning by the instructor, and explicit problem comparisons. However, these processes are not readily explained by theories that suggest Maria merely acquired a particular schema or applied an appropriately abstracted rule to relevant problems. Two points are crucial to this claim.

First, Maria's developing understanding of the law of large numbers was inseparable from her knowledge of how and where it could be applied. Learning it required her to learn that different circumstances required attunement to different kinds of information and different ways of coordinating it. For example, Maria's struggles to reason consistently about situations requiring sample outcomes near a population parameter and those requiring outcomes distant from it revealed her need to acquire means of selecting and coordinating different types of information in different circumstances. According to the demands of each problem, Maria selectively attended to and coordinated individual outcomes of random experiments, expected values of outcomes, ranges of outcomes, and entire sampling distributions. Additionally, Maria initially and spontaneously insisted that she could not apply her rule to a series of spins of a spinner colored 70% blue and 30% green. ("It's not like the coins.") Her ability to see the Spinner Problem as analogous to the Coin Problem depended not only on an adjustment to her understanding of the law of large numbers, but also on a change in her understanding of how spinners behave. Empirical experience with a spinner simulation enabled Maria to recognize that the long-run percentage of blue outcomes on the spinner does indeed *have* an expected value. This analysis calls into question theories suggesting that individuals depend largely on acquiring generalized rules that either are or are not used in objectively relevant situations. In at least some cases, learning a concept, rule, or principle includes learning what kinds of situations are relevant, how such relevance is determined, and how information is differentially selected according to circumstances.

Second, although Maria's ability to state her informal "rule" corresponding to the law of large numbers changed virtually not at all throughout the three sessions, her understanding and use of it changed significantly and depended on an underlying, growing compilation of explanatory concepts and ideas initially evoked only selectively according to problem circumstances. The most striking example of this resulted from the positive role that different problem contexts played in Maria's developing reasoning. The problem involving heights of men walking into a post office to register for the draft elicited only a limited use of statistical reasoning. However, the suggestion to consider the men's heights as being acquired in the context of a campus poll or survey immediately resulted in Maria's description of the relevance of sample size in terms of her ideas of *representativeness* and *accuracy*—ideas that had nowhere before been used by her in any of the five problems she considered. Consequently, through further problem comparison, Maria came to explain the similarity of the problems through a flexible use of ideas revealed through the language of *representativeness*, *accuracy*, *expected value*, and *samples*, which fleshed out her stated *rule*, "the more you do things the closer you get to what's expected." These ideas, once used only selectively according to different circumstances, were ultimately used simultaneously and interchangeably as her understanding grew.

The result of Maria's activities appears not to be a structured, generalized abstraction that she applied to problems having some particular structure, but an increased association and systematization of a set of complementary descriptive and explanatory ideas that could be used flexibly across even non-isomorphic problems. Data from Maria and other students are being used in the construction of a more complete and substantiated account of these processes, which seem consistent with the "knowledge in pieces" perspective suggested by diSessa (1993) and diSessa and Sherin (1998). In addition to helping to elaborate the notion of transfer, a clearer understanding of what students actually learn in processes of generalization will help educators select learning activities that encourage students to interpret familiar situations in new ways.

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THE ORIGIN AND PERSISTENCE OF MISCONCEPTIONS IN STATISTICAL THINKING

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The importance of accounting for students' ideas has long been recognized. Ausubel (1968) noted that ". . . the most important single factor influencing learning is what the learner already knows."(p. 68) Curricula in American schools is often broad but shallow (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999; Stigler & Hiebert, 1997), creating an environment that is rife with opportunity for misconceptions to develop and perpetuate. In particular, nearly all of the more traditional textbooks provide little support for building on students' previous knowledge, including identifying and addressing naïve or misconceptions. In particular, the problems with learning statistical concepts have been well-documented from elementary grades through undergraduate statistics classes (Conners, McCown, & Roskos-Ewoldsen, 1998; Fast, 1997; Jones, Langrall, Thorton, & Mongill, 1999; Shaughnessy, 1992). Misconceptions at the middle grades focuses on interpretation and representation of data whereas, at the post secondary level misconceptions have been more prevalent in measures of central tendency and its interpretation. In the present study, we address specifically the ideas related to producing bar graphs from data and using these representations to make conclusions and interpretations from the data.

Methodology and Results

A sample of 44 preservice elementary teachers and 134 sixth-grade *Connected Mathematics* (Fey, et al.) students were administered an open-ended item from the *Balanced Assessment* (2000). The item, *Vet Club,* was selected as a tool to determine the pervasiveness and persistence of misconceptions in statistical thinking. A rubric for scoring the items was designed to discern students' correct strategies as well as identify specific misconception in graphing and using data to solve a problem. The data are summarized in Table 1.

Discussion

Vet Club elicited several difficulties and misconceptions in statistical thinking. The notion that a particular type of graph was needed to best answer a question about typical data was poorly understood. Interviews indicated that most students constructed

Table 1. Data Summary Table

	Graphed Zero		Misrepresented Axes		Type of Graph	
	6th	UG	6th	UG	6th	UG
Bar Graph	46%	31%	47%	53%	84	32
Circle Graph	35%	33%	NA	NA	21	3
Line Graph	31%	33%	34%	33%	29	9
*Histogram					67%	62%

Note. * Bar graphs drawn or referred to as histograms.

their favorite graph with little notion of how to use and interpret it. Student's attributed this naïve conception to previously having constructed graphs as the end-product and not as a means for interpreting data. Similar to the K-12 curriculum, elementary pre-service teachers are seldom provided direct experiences determining the most appropriate graph for data and then interpreting the representation of the data.

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HOW EXPERTS AND NOVICES VISUALLY INSPECT AND THEN INTERPRET GRAPHS

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This study examines differences in how expert and novice graph-readers explore, compare, and consequently interpret data in a graph comparison task using pairs of stacked-dotplots. Several studies have shown that novices tend not to examine data in graphs as a whole, but view data in smaller sections. This sometimes leads them to draw conclusions about groups that are at odds with what experts might do. Few studies have directly examined what these differences are and how they might result. In this study, we use the Restricted Focus Viewer program to examine whether the way experts vs. novices visually scan graphs are predictive of the types of interpretations they give.

When drawing conclusions about two groups of data, one can view the data in its entirety or concentrate on subsets of the data. Experts tend to view datasets as an entire collection, or aggregate, while novices tend to focus on individual data points or clusters of data with the same values (e.g., Ben-Zvi & Arcavi, 2001; Konold, Robinson, Khalil, Pollatsek, Well, Wing, & Mayr, 2002). From this perspective, an important part of the process of educating students in data analysis involves helping them come to see data in holistic terms. To guide these pedagogical interventions, it would help to know more about how experts vs. novices visually explore data to draw their conclusions. Existing research on this issue is sparse. An eye-tracking study by Embse (1987) using graphs with mathematical functions found that experts (i.e., graduate students and faculty in the mathematics and mathematics education departments) were better at recognizing a target graph compared to novices (i.e., college freshmen in remedial mathematics courses). Embse's study only addressed how well a graph is recognized but not how data in the graph are interpreted.

In this study, we compare how experts and novices behave in a task that requires them to compare data in a pair of stacked-dotplots (a.k.a. frequency bar graphs). Experts were faculty members in the department of psychology while novices were students taking undergraduate psychology classes. Participants examined a variety of graphs showing, for example, daily temperature in two different towns measured on 50 days. The data for the daily temperatures in the towns (Town A and Town B) were plotted in two stacked-dotplots, with the temperature on the x-axis. The participants' task was to determine whether the daily temperature in one town tends to be more or less than the other. Participants visually inspected the stacked-dotplots on a computer monitor using the Restricted Focus Viewer (RFV), a program recently developed as an alternative to conventional eye-tracking apparatus (Blackwell, Jansen, & Marriott, 2000). This program records where and when the participant moves the mouse as they visually inspect the graph. The participant sees only the data located inside a small view window, centered around the cursor of the mouse. This enables us to determine where the participant is looking at any given moment.

We hypothesize that the RFV data will show that novices tend to visually inspect and compare the data using selected vertical slices of the data. For example, they might visually inspect the graphs by drawing the mouse up and down over vertical slices of the graphs and then decide that one town is warmer by comparing the number of days in the two towns in one of these slices (e.g., temperatures above 90 degrees). On the other hand, we expect experts to inspect both distributions systematically by visually scanning each stacked-dotplot horizontally trying to estimate the approximate averages and spreads of each distribution. They will then justify (and perhaps quantify) group differences in terms of these group indices. If it turns out that differences in interpretations are related to different patterns of visual inspection, we would then explore whether structuring how novices visually inspect graphs might help them come to see data more holistically and suggest possible teaching strategies.

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PAIR PROBLEM SOLVING IN PROBABILISTIC SITUATIONS

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Fairly large amount of research on collaborative and cooperative learning has been reported in the literature (Kieran & Dreyfus, 1998; Mueller & Fleming, 2001). Much of this research has focused on children working on pairs or small groups and their learning of content and social behaviors. This poster focuses on mathematical work of college graduates when they were asked to solve probability problems in pairs. Two probability problems, transcripts of student discussion, and researcher's interpretation will be included in the poster.

The main task posed in the pair problem solving context was called the head-tail sequence task, wherein participants were asked to determine the most likely sequence of alternatives that might result from flipping a fair coin six times. This was followed by the five-boy task, which asked the participants to determine if a couple has more chance of having a girl if they already have five boys.

Twelve secondary school preservice teachers, who had taken at least 10 mathematics undergraduate courses, including a probability and statistics course, participated in this study. These preservice teachers were paired into six groups based on their responses to a written questionnaire. Each pair consisted of preservice teachers with diverse backgrounds and opinions so as to produce interactions that would help broaden the exploration of students' conceptualization of probability.

In the pair-problem solving, participants were asked to discuss possible solutions to tasks. That is, pairs discussed and attempted to solve tasks together. The investigator observed the pairs' work and made notes. Only in certain circumstances, for example if pairs stopped talking or reached an agreement very quickly, did the investigator provide prompts. The discussions were audiotaped.

Two major findings were noted. First, the participants reported that they felt comfortable in the pair setting because solving the given problems was a shared responsibility of both participants. The preservice teachers were asking questions and providing answers to each other without hesitation, even though the investigator was present during the pair problem solving. Because the participants were sharing their thoughts with each other, the researcher was able to understand their probabilistic thinking without any difficulty.

While attempting to solve probability problems, the preservice teachers used two types of knowledge: Formal probability based on university courses and informal probability based on their everyday intuitions and experiences. Some participants had taken more mathematics and probability courses than the other. However, the number of mathematics courses taken was not necessarily a prominent factor in preservice teachers' attempts to use formal probability during the pair problem solving.

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ANALYZING WRITTEN EXPRESSIONS OF STUDENTS' PROBABILISTIC REASONING: DEVELOPMENT AND USE OF SCORING RUBRICS

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Assessing students' probabilistic reasoning can be a daunting task. In his wish list for future research in stochastics education, Shaughnessy (1992) recognized the lack of reliable assessment tools and called for development of research-based tools to assess students' understanding. This poster will highlight the test items and scoring rubrics that were part of a research study with sixth grade students, the establishment of inter-rater reliability, and analysis of the test scores on pre/post/retention forms.

Stohl and Tarr created the tests using newly developed assessment items, and modified items from prior research (e.g., Piaget & Inhelder, 1975; Fischbein & Schnarch, 1997). The rubrics were based on a synthesis of research in probabilistic reasoning (e.g., Jones, Langrall, Thornton, & Mogill, 1999). The rubrics were developed such that a score on any given item would reflect the student's level of understanding rather than a simple right or wrong evaluation. Such robust descriptions of students' reasoning, coupled with a multi-point scoring system, allows for a deeper and more meaningful analysis of changes in students' thinking over time.

Data Sources

The test data represented one aspect of a larger study (see Stohl & Tarr, this volume). The 23 sixth grade students (14 boys, 9 girls) were in an average-level mathematics class in an urban, southern public middle school. Of the 23 students, six are African-American, two are Hispanic (one ESL), and 15 are Caucasian. Students participated in a 12-day instructional unit on probability that utilized Probability Explorer (Stohl, 1999-2002) simulation software as a primary investigation tool. All students took a pre- and post-test, as well as a retention test given 10 weeks after the completion of the unit.

Results

To establish inter-rater reliability, Pusey and Townsend used scoring rubrics to assign item-level scores on pre- and post-tests (35 items/test) for four students with 97% agreement on the pretests and 91% agreement on the post-tests. The analysis of class mean scores indicates a significant increase from the pre-test mean score to the post-test mean score as evidenced by a paired t-test ($p=0.00023$, $\alpha=0.05$). Duplicate methods elucidated a significant increase in mean scores from the pre-test to the retention test ($p=0.01666$, $\alpha=0.05$). Item-level mean scores increased significantly from the pre-test to the post-test on 11 of the 35 items and significantly decreased on one question. Between the pre-test and the retention test, nine of the item-level mean scores significantly increased while three items showed a significant decrease. From the post to the retention test, no overall significant difference was found; however, two item-level mean scores significantly increased, while five scores significantly decreased.

Of notable interest to our research group were the significant increases shown on four test items assessing use and understanding of the law of large numbers. A significant increase was demonstrated between pre- and post-test mean scores and the pre- and retention test mean scores on all four items. Since specific interventions using *Probability Explorer* targeted increasing students' understanding of the law of large numbers, the significant test results tend to support the effectiveness of this intervention. Specific items, scoring rubrics, and more detailed analysis will be shared during the poster session.

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Problem Solving

TEACHING WORD PROBLEMS: WHAT HIGH SCHOOL MATHEMATICS TEACHERS VALUE

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Word problems form a central component in the learning and teaching of mathematics. This paper reports on a study of mathematics teachers' thinking in the teaching of word problems with particular focus on what exemplary high school mathematics teachers value in framing their teaching of word problems and what characterizes their teaching of word problems. Case studies of six teachers were conducted. Data collected through interviews and classroom observations were analyzed for attributes of the participants' thinking and actions that were characteristic of their perspective of teaching word problems. The findings provide insights of teachers' way of thinking and classroom behaviors that are important to motivate and engage students meaningfully in doing word problems. There are five student characteristics the teachers intentionally incorporate in their teaching and eight teaching characteristics they value that form the essence of their teaching of word problems. The findings have implications for what we consider and how we work with prospective and experienced teachers to help them to develop/change/enhance their teaching of word problems.

This paper is based on a larger study that is investigating the teaching of mathematical word problems from the perspective of the teacher. The focus here is on the perspective of exemplary high school mathematics teachers, in particular, what they value in framing their teaching of word problems in the classroom, and what characterizes their teaching of word problems.

Background and Theoretical Perspective

Recent reform recommendations in mathematics education, for example, the National Council of Teachers of Mathematics [NCTM] standards and principles (NCTM 1989, 1991, 2000) assign a significant role to problem contexts in developing meaning for mathematics and a problem-solving perspective of teaching and learning mathematics. Implementing such recommendations will necessitate an increase in importance in the use of a range of mathematical word problems, routine or non-routine, in the learning and teaching of mathematics. This is likely to pose increase challenges for both teachers and students. In spite of several studies on problem solving and several theoretical prescriptions on how to become a good problem solver and how to teach problem solving, word problems remain a component of mathematics many students fear and experience increasing difficulty with as they move up the Grades.

Studies on mathematical word problems and problem solving have traditionally focused on the learner to study cognitive and affective factors that aid or hinder his/her performance as a problem solver (e.g., Carpenter, Moser & Debout, 1988; Silver, 1985). Such studies generally identified determinants of problem difficulty, characteristics of successful and unsuccessful problem solvers, and the relation of meta-cognitive factors to problem solving. They were then used as a basis to design, test and prescribe instruction. In particular, studies on arithmetic and algebraic word problems continue to reflect this trend of focusing on learner-related variables. For example, studies on arithmetic word problems, involving elementary students, (Carey, 1991; Cummings, 1991; Debout, 1990; Feinberg, 1988; Fuson & Willis, 1989; Lewis & Mayer, 1987; Sowder, 1988) have looked at the mathematical and linguistic structure of these problems in relation to the children's performance; factors that affect the difficulty of the problem for children; strategies and methods children use; and the errors children make in their solutions. More recently, Verschaffel, Greer and DeCorte (2000) have looked at students' suspension of sense making in doing word problems. They pointed out that after several years of traditional mathematics instruction, students have developed a tendency to reduce word-problem solving to selecting what they take to be the correct arithmetic operation with the numbers given in the problem, without seriously taking into account their common-sense knowledge and realistic considerations about the problem context. A similar situation has existed for studies on algebraic word problems, involving high school students, where, for example, the focus has been on students' errors and methods in the translation process (Clement, 1982; Crowley, Thomas & Tall, 1994; Kaput & Sims-Knight, 1983; MacGregor & Stacey, 1993; Wollman, 1983). Reed (1999) provides a current overview of studies of both arithmetic and algebraic word problems.

Given the focus on the learner, studies on mathematical problem solving have generally ignored or trivialized the classroom teacher. Even in cases where instruction was considered, the studies were still biased towards the learner in that they generally focused on the effectiveness (in terms of students' performance) of instructional methods designed by the researcher and administered under experimental conditions. Thus, while these studies have enhanced our understanding of important issues associated with the learning of word problems, they offer very little on the teacher. But this situation was eventually considered by some researchers to be a significant limitation in understanding and improving the teaching, and consequently, the learning of problem solving. For example, such lack of research on the teacher in problem-solving instruction has been proposed as a significant limitation in understanding and improving the teaching of problem solving (Lester, 1994; 1985). The initial concern was that "we will make no progress toward developing practical guidelines for mathematics teachers to follow until we are able to make explicit that knowledge about teaching that good teachers have internalized and made 'second nature'" (Lester, 1985, p.56). The study in this paper is intended to make explicit aspects of teacher thinking and classroom behavior that exemplary high school teachers consider important in framing their teaching of word problems in the classroom.

The study, then, is framed in the theoretical perspective of teacher thinking in which teachers are viewed as creating their own meaning to make sense of their teaching, i.e., a constructivist orientation of knowledge construction. Research on teacher thinking is well established (Elbaz, 1991; Fenstermacher, 1994; Grossman, 1990). The importance of researching the teacher is associated with the view that teachers are the determining factor of how the curriculum, mathematics in this case, is taught. Within this humanistic view of the teacher, the teacher's thinking is valued as a basis for understanding her/his classroom behaviors, in particular, and teaching, in general. This validates the importance to learn from teachers what they do and how they make sense of what they do (i.e., their perspective) in the context of the classroom.

In recent years, there has been increased focus on researching the mathematics teacher. Thus, there is a growing body of literature on mathematics teachers' content knowledge, beliefs, conceptions, classroom practices, learning, professional development and change (e.g., Lampert & Ball, 1998; Chapman, 1997; Fennema & Nelson, 1997; Lloyd & Wilson, 1998; Schifter, 1998; Thompson, 1992; Tzur, Simon, Heinz & Kinzel, 2001). These studies have provided us with insights on, for example, the relationship between beliefs and teaching, deficiencies in teachers' content knowledge, and the challenges of teacher education and change. The literature also suggests that mathematics teachers' beliefs and conceptions about mathematics and its teaching and learning are significantly related to their teaching. As Ernest (1989) explained, in relation to mathematics teachers, it is the teachers' mental or espoused models of teaching and learning that are transformed into classroom practices. This view of the teacher suggests that the teaching of word problems in a classroom context could depend on the teachers' underlying beliefs and conceptions framing it, or, more generally, on the teacher's thinking. However, ongoing research to understand the high school mathematics classroom from the teacher's perspective, in general, and explicitly dealing with the teaching of word problems, in particular, is important as we try to reform the teaching of mathematics. In this study, the teachers' perspective includes their conceptions, their intentions, what they value and personal meaning.

Research Method

The study is framed in a naturalistic inquiry perspective that facilitates a phenomenological context to interpret one's thinking and behavior (Taylor & Bogdan, 1998). The participants were six experienced high school teachers (4 females, 2 males, 16-30 years experience), considered to be excellent teachers in their school system. Their students have consistently scored very high on their provincial standardized examination that has a growing emphasis on problem solving following the NCTM standards. They had positive attitudes to teaching mathematics, in general, and word problems, in particular. They have been involved in making presentations at mathematics teachers' conferences and/or leading professional development workshops, although not specifically related to teaching word problems.

The main sources of data for the study were interviews and classroom observations. The interviews were open-ended to give the participants an opportunity to talk freely and in an unprompted way about their own understandings of teaching word problems. They were also partly framed in a narrative perspective to allow the teachers to describe their experiences as actual phenomena. This produced stories of actual events that embodied the meaning of the participants' behavior. The interviews dealt with the participants' thinking in three contexts: past, present and future. The past dealt with their past experiences with word problems as both students and teachers focusing on teacher and student presage characteristics, task features, and contextual conditions The present dealt with their current practice with particular emphasis on classroom processes, planning and intentions. The future dealt with expectations, e.g., possible changes due to personal or external factors, and, "if ... what" scenarios.

Classroom observations focused on the participants' actual instructional behaviors during lessons involving word problems. Special attention was given to what the teachers and students did during instruction and how their actions interacted. Complete units involving word problems were observed. Post-observation discussions with the participants focused on clarifying their thinking in relation to their actions. All interviews and discussions were audio taped and transcribed. The data were thoroughly reviewed by the researcher and two research assistants working independently to identify attributes (e.g., recurring conceptions/beliefs, intentions, judgements) of the participants' thinking and actions that were characteristic of their perspective of teaching word problems. These attributes were grouped into themes. Both attributes and themes were validated by comparison of findings by the three reviewers and triangulation of findings from interviews and classroom observations.

Findings

The teachers' thinking about mathematics, word problems, problem solving and the teaching and learning of word problems played a significant role in framing their teaching of word problems in a classroom context. For all of the participants, for the most part, there was consistency between their thinking about teaching word problems and what they actually did in the classroom. There was evidence of some of them, over their careers, having consistently worked at resolving significant conflicts they were aware of between their beliefs about mathematics and their teaching that could help to explain this consistency (Chapman, in press).

The thinking of the participants identified as exemplary teachers reflected a broader range of characteristics and more depth in conceptualizing word problems and teaching word problems than that of the traditional teachers who were participants in the larger study. Each participant was unique in her/his thinking and behavior and there were differences in how each characterized her/his conceptions and in her/his classroom actions. However, as exemplary teachers, there were many similarities conceptually resulting from a common underlying theme in their thinking and behavior that reflected a humanistic view of mathematics, word problems and problem solving and a student-centered view of learning and teaching. All of the teachers thought that, and a student-centered view of learning and teaching. All of the teachers thought that, as one of them put it, "it does not work to just stand up in front and say, okay, if you see a question like this you do this, this, and this, and this is how you solve it." They also acknowledged that the students who came into their classes generally had a fear of word problems, but they believed that this fear could be significantly reduced if the situation was set up to facilitate this. Thus their teaching was characterized by several factors that were intended to enhance students' affective relationship to word problems and cognitive perspective of dealing with word problems. A detailed discussion of all the key aspects of the teachers' thinking and behaviors cannot be adequately provided here, so the focus will be on outlining some of the key factors which reflect what the teachers valued in framing how they created and facilitated effective/meaningful instruction in the teaching of word problems.

Although other factors (e.g., conceptions of mathematics, word problems, and problem solving) played a role, the teachers considered the students as the primary basis of determining what their teaching of word problems should look like. Their student-centered approach to teaching was directly related to the importance they placed on the following five student characteristics they intentionally incorporated or tried to achieve in their teaching: (i) Real life activities/experiences: This was used to make the word problems relevant or interesting for students. (ii) Ability: This was used to determine how and when to integrate difficult problems with easier ones. (iii) Affective factors, e.g., fear, motivation, sense of success, confidence: This formed a basis for the teachers' goal to get students motivated to move from fear to confidence and success. As one teacher explained: "The teacher should also help students develop a feeling of success, to experience success and to be confident in their ability to do word problems and to see them as fun." (iv) Autonomy: This was used to give students ownership of their learning, e.g., selecting word problems and solution processes that they found to be meaningful. (v) Understanding: This was interpreted as allowing students to construct their own meaning, e.g., their interpretations of word problems and solution processes. These characteristics are not elaborated on here, but are reflected in the description of the teaching characteristics that follow.

The exemplary teachers' thinking and classroom behaviors revealed eight characteristics they valued that formed the essence of their teaching of word problems. While they are all listed here, they are described with varying depth that is reflective of the constraint on space and not importance of each characteristic. All quotes in the description of each characteristic are from the data.

(i) Trialogue Relationship: The general structure of the teachers' teaching was a triangular relationship between the teacher, students and word problem, i.e., there was a trialogue in which both teacher and students could learn something from each other. In this model (Chapman, forthcoming), the teacher specifies the problem for him/herself based on his/her interpretation of it and is speciimposing this interpretation on students, the teacher allows them, individually and/or as small groups, to specify the problem for themselves based on their interpretations of it and to be specified by what is allowed by the context of the problem. Finally, by listening to, observing and strategically intervening in the students' relationship with the problem, the teacher allows him/herself to be specified by the students and the students to be specified by the teacher in an open-ended way based on the teacher being flexible and open to a range of justifiable possibilities for solution to the problem. *Specified* is being used in a phenomenological context and relates to how a phenomena is experienced or provides options for it to be experienced.

(ii) Connected-Separate Knowing: The teachers often supported students working from a form of *connected-knowing perspective* to a *separate-knowing perspective* (Chapman, forthcoming). In the former, students were allowed to focus on the social context of the problem by trying to experience the problem, to relate it to their personal world. The teachers viewed this as being motivational, so to facilitate it, they preferred to work with problem situations that had "realistic value for the students." When such problems were not readily available, e.g., in case of traditional textbook problems, the teachers encouraged students to make fun of "stupid problems" and joke about the cover story of the problem, to revise problem situations/context, and to construct problems about things/situations that they found interesting to share with each other. For example, as one teacher explained: "If they say 'it's a stupid problem', I say, 'Ok, so how would you write this problem? You write the problem so it has more meaning for you instead of using some name that you don't know in the problem.'" The teachers treated this process of personalizing the problem context as both a way of knowing and a means of motivating students. They viewed it as paving the way for students to move beyond surface features of the problem and to engage in separate knowing, i.e., focusing on the mathematical context of the problem by suppressing the self in relation to the social context of the problem and taking an impersonal stance toward the problem. For example, students were required to reflect on the problem to see the structure of it and how they could generalize it. Students were also required to reflect on the solution to allow them to see the structure of it and the circumstances under which it would work and not work. The goal during separate knowing was on developing and applying some model for dealing with problems of similar mathematical context.

(iii) Integration: The teachers preferred "integrating the word problems throughout the course and not treating them as something separate or as an isolated topic of mathematics" … "[treating them] as something that is ongoing and not a particular unit in mathematics." For e.g., the teachers would use word probparticular unit in mathematics." For e.g., the teachers would use word problems to start, develop and/or end each topic.

(iv) Collaboration: The teachers emphasized collaboration in developing the solution process when word problems/new topics were introduced. This was done through whole-class discussions and small-group work. Whole-class discussion was always accompanied with small-group work, which could occur before and/or after it. During the whole-class discussions, the teachers' focused on "drawing on the students' experiences and perspectives". There were various ways in which the group-whole-class combination unfolded depending on the teacher, students and topic. In one scenario, the teacher would put students in groups at the beginning of the year and give them some word problems "to look at and to see what they come up with on their own". This is followed by a whole-class discussion where the students have to explain and discuss their interpretation of the problem and their solutions, whether completed or not, and reflecting on their approach. As this teacher explained, they have to address, "What did we [the students] do? What tools did we use? Why did it work or not work? Is there a better way?"

Collaboration among students was also encouraged, with individual work, for practicing problems. The teachers viewed collaboration among students as being important in helping them "to understand that there are others who are experiencing the same difficulties as themselves, who can have a different perspective that they can learn from, and they can bring something to the problem situation that others may not have thought of." Thus, in general, the teachers viewed this shared experience as playing an important role in motivating students and giving them confidence to work individually.

(v) Questioning: The teachers used a questioning approach to facilitate whole-class discussions or to intervene in the groups. Questioning was in the form of questions and prompts that required students to engage in mathematical thinking and to share their thinking – "their experiences and perspectives."

(vi) Writing: The teachers gave students opportunities to write as a way of learning how to approach word problems. For example, students got to construct word problems and were encouraged to write about their thinking when they solved a word problem, particularly in situations where they were encountering difficulties getting started or getting to a resolution of the problem. As one teacher explained, when they were stuck, her students were required to write about "how they started the problem and how this didn't work and that didn't work. … [This allows them]to describe their process so they can see that they are on a journey, if you will, to find the solution and sometimes the journey does not go right but at least they've started and done something. They should begin writing whatever they are thinking of immediately; so that they don't leave blank pieces of paper and they are not afraid to try different kinds of things."

(vii) Choice/ Flexibility: The teachers gave students the choice to use "whatever mathematical tools they are made aware that they have to solve the problems." As another teacher said, "it doesn't make any difference what kind of methodology they use as long as it is logical and they can explain it and let other people understand it." Another noted, "If we mark them on what they can show us and not just the right answer or the wrong answer, we can develop that confidence in students." Students were also given choices on tests and assignments. For example, with practice problems, some teachers sometimes assigned to students 5 or 6 "longer problems" during the class or for homework, but required that "they only turn in the ones that they think were their two best solutions so that they have some choice." Sometimes on tests, they were also given a choice.

(viii) Assessment Disclosure: The teachers made students "aware of how these problems will be assessed". For example, they helped students to understand how they were going to be graded, "whether it is by use of rubrics or something else". One way this was done was by "looking at examples of solutions that ranged from excellent to really terrible" in terms of how they were presented and not "the correct way" and discussing them from an assessment perspective.

The above eight teaching characteristics were reflected in all of the teachers' thinking and classrooms, but there were variations in how they were articulated and lived in the classroom. Thus, each teacher was unique in how these teaching characteristics unfolded in her/his classroom. For example, each teacher weaved his/her classroom activities in teaching algebraic word problems with systems of linear equations with his/her own personal touch. The following is an example of one of the teachers' approach for a unit on systems of equations. The whole unit, as opposed to a single class, is presented because of the way the word problems are integrated throughout the unit as an integral part of it. It is also written in a way to reflect the teacher's ongoing use of the approach.

One week prior to starting and discussing the topic in class, the teacher asks the students to collect pictures of graphs that intersect from any source other than a mathematics textbook. The graphs do not necessarily have to be lines. They can be any graphs that intersect. But they should represent actual real-life situations. These pictures are considered to be related to word problems because there is a cover story associated with them.

For the first class of the unit on systems of linear equations, students bring their pictures and share them in a whole-class setting. They spend the whole class looking at and discussing everyone's picture. Each student shows his/her picture and talks about what the graph represents, the significance of the graph, what it means when graphs intersect, what the intersection shows, why the intersection is important, why anyone would want to find the intersection in the first place. The graphs the students collect are usually about business/economics and physical science situations. Economics situations tend to be dominant because, according to the teacher, they are talking about money with which the students seem to relate more meaningfully. For homework, the teacher gives the students a word problem that requires them to determine, if they have a part-time job, whether it is better to have a fixed hourly rate or a fixed weekly salary plus commission over a specified time period. The students are to try and solve the problem in any way they could.

The second class starts with sharing and discussion of the students' solutions to the homework problem. The teacher poses questions about how the graphical approach was or can be used to solve the problem, how the point of intersection or break-even point was or is useful to the analysis and solution of the problem, and the relevance of solving systems of two linear equations to the students' real-life experiences. The teacher then points out that since they [teacher and students] have decided that it is important to look at where graphs intersect, then they are going to particularly study lines, linear equations, and their intersection. She asks students to return to the graphs they had collected and, for graphs involving two intersecting straight lines, to work in groups to set up the equations for each set of lines using data they read off the graphs. Following this, the teacher facilitates a whole-class discussion of the meaning of the algebraic representations in the context of the applications and why they are useful. In particular, the class discusses the usefulness of an algebraic approach instead of a graphical approach to find the point of intersection. The students, working in groups, then use their graphing calculators to plot examples of special cases of systems of two linear equations that the teacher writes on the chalkboard, e.g.,

$x + y = 4$ and $2x + 2y = 8$

$2x + y = 3$ and $2x + y = 5$.

These are intended for students to investigate and discuss other relationships between two equations, e.g., "what happens if they [the lines] do not intersect? What does it mean if one line is on top of another?" The investigation is followed by whole-class sharing and discussion of their findings. To end the class, the teacher requests that students form groups and each group is assigned a different method to solve systems of equations as required by the curriculum. As she explained, "One group looks at solving these systems of equations graphically, another group looks at solving them using an addition method or a subtraction method [i.e., elimination method] and the third group looks at solving them by substitution."

For the next two classes, students work in their groups investigating the particular solution pattern for the one method to solve a system of two linear equations they are assigned. The goal is for the group to study solved examples by themselves to try to understand the structure of the method and be able to teach it to others. Thus the students are developing their own versions of what the methods are about. The teacher circulates, observes and intervenes with questions or a counter example to make students think about any limitations she sees in the way they understand the method, e.g., "Would that work for ...?" Students are left to resolve all conflicts for themselves. The teacher assigns exercises to each group to use to test the understanding they construct of the method. The exercise contains equations with coefficients and constants that are all whole numbers, all integers, all rational numbers or some combination of these numbers. For homework, the teacher asks the students to make up a word problem that can be solved with the method they explored in class. The teacher rewards students for creativity, so reproducing problems from a textbook is discouraged and not valued. To end the class, the students have to plan how to teach their approach to others in a way that is interesting and will help them to understand it.

In the fifth class, each group teaches the method they explored to the other groups. This is done by representatives from the teaching group going to each of the other groups to explain to them what they had done in their own group and teach them the method. They are to also make a case that their method is the most efficient. Again, the teacher circulates, observes and intervenes with a counter example to make students think about any limitations she sees in the way they understand the method. Students are left to resolve all conflicts for themselves. For homework, the students are assigned to select and complete practice exercises from those exercises assigned to the groups for each method that have not been completed in class before it ended. The students also are assigned "to bring in real-world word problems that they find elsewhere, other than the textbook, to talk about how they use or can solve systems." For example, "one of the kids brought in internet prices, so they were looking at going with AOL or someone else, and so they brought in a graph that showed these two particular companies and where the break-even point would be so that it wouldn't make any difference which one they had." The teacher assigns 20 percent of their unit mark for "finding a pertinent problem that uses systems and solving it".

In the next two classes, each group works on word problems the teacher selects from the textbook and from what the students have developed. The teacher provides open-ended hints whenever students are stuck but never tells them explicitly how to do anything. Each group then "comes to the front of the class with a different problem that they want to share with the class." Any problems no one in the class is able to do, even after receiving the hints, are identified at the end of the class. Students are asked to continue to think about them for homework.

In the eighth class, the unresolved problems are dealt with in a whole-class discussion in which the teacher poses questions that help the students to arrive at a solution collectively. For the remainder of the class, the teacher facilitates a whole-class discussion on informal extensions of the topic posing questions like, "what if there are three equations instead of two?"

Finally, in the last class of the unit, the teacher gives the students a written test on the topic. For word problems on the test, the teacher includes the final answers. Thus, there is no mark for the final answer, reinforcing the importance of understanding and explaining [through writing] the process.

This teacher's approach is far removed from the traditional classroom approach to dealing with systems of equations, and the most innovative of all of the participants' approaches. But all of the teachers taught in a way that reflects current reform recommendations in mathematics education. These teachers, at the beginning of a semester, explicitly worked at creating the tone in their classroom to allow students to work in this way. In general, their teaching had positive effects on students' attitude and learning of word problems.

Conclusion

The study shows that for these teachers, effective teaching of word problems is more complicated than walking students through examples of solution patterns or heuristics. Instead, it is a particular interplay between teacher, student and word problem that is orchestrated by the teacher based on what the teacher values. The findings provide evidence for teachers that the teaching of high school mathematics does not have to be dominated by telling. But a relativistic view of mathematics teaching based on students' understanding and motivation is desirable and achievable. The findings also reveals specific factors, from real classrooms, of teachers' way of thinking and classroom behaviors that are important to motivate and engage students meaningfully and effectively in doing word problems. This has implications for what we consider and how we work with prospective and experienced teachers in order to help them to develop/change/enhance their teaching of word problems.

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PRESERVICE SECONDARY MATHEMATICS TEACHERS' MODELING STRATEGIES TO SOLVE PROBLEMATIC SUBTRACTION AND ADDITION WORD PROBLEMS INVOLVING ORDINAL NUMBERS AND THEIR INTERPRETATIONS OF SOLUTIONS

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In this study I examined 19 preservice secondary mathematics teachers' solution processes to word problems for which the subtraction or addition of the two given numbers yields 1 more or 1 less than the correct solution. Among the aspects of their solution processes that were examined are: the modeling strategies, the type of errors, and the interpretations of the solutions produced by the procedure. It was found that about 87% of the solution processes to such problems contained formal strategies while about 13% contained counting strategies. It was also found that about 61% of the responses contained errors of which 91% were ± 1 errors. That is, errors due to the interpretation that the answer provided by the addition or subtraction of the two given numbers is the solution to the problem. It is argued that some incorrect interpretations were due, at least in part, to a lack of understanding of the connection between the enumeration process needed to obtain the solution to a problem and the answer provided by the addition or subtraction of the two given numbers.

Current reform documents such as *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) call for students to learn to solve non-routine problems and to establish connections between mathematical ideas and real-world situations. Special types of non-routine problems include problematic story problems involving arithmetic operations. For the purpose of this article, a problematic story problem is a problem for which the result provided by the mathematical operation or procedure with the numbers given in the problem statement does not necessarily represent the solution to the problem. Research studies (e.g., Cai & Silver, 1995; Contreras, 2001; Greer, 1993, 1997; Nesher, 1980; Reusser & Stebler, 1997; Silver, Shapiro, & Deutsch, 1993; Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Lasure, 1994; Verschaffel, De Corte, & Vierstraete, 1999; Verschaffel, Greer, & De Corte, 2000) suggest that students tend to approach problematic story problems mechanically or superficially without paying attention to the realistic considerations of the situational context of the problem or to their modeling assumptions. Some examples of problematic word problems used in some of these studies are the following:

(a) What will be the temperature of water in a container if you pour 1 l of water at 80° and 1 l of water of 40° into it? (Nesher, 1980)

(b) John's best time to run 100 m is 17 sec. How long will it take to run 1 km? (Greer, 1993)

(c) The Clearview Little League is going to a Pirates game. There are 540 people, including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game? (Silver, Shapiro, & Deutsch, 1993)

(d) Lida is making muffins that require 3/8 of a cup of flour each. If she has 10 cups of flour, how many muffins can Lida make? (Contreras & Martínez, 2001)

(e) In September 1995 the city's youth orchestra had its first concert. In what year will the orchestra have its fifth concert if it holds one concert every year? (Verschaffel, De Corte, & Vierstraete, 1999)

Verschaffel, De Corte, and Lasure (1994) used, among others, items a) and b) in their study involving 75 fifth graders in Flanders. Their analysis revealed that only 7 (9%) students provided a realistic and correct response to the first problem and only 2 (3%) provided a response to the second problem that was based on realistic considerations. Silver, Shapiro, and Deutsch (1993) investigated 195 middle grade students' solution processes and their interpretations of solutions to the third problem. They reported that about 22% of the students correctly performed an appropriate procedure but provided an incorrect solution without explicit interpretation. Most of the students interpreted the result of the division (e.g., 13 or 13 with another number) as the number of needed buses. In their study, Contreras and Martínez (2001) examined 68 preservice elementary teachers' solution processes and realistic reactions to the fourth problem. They reported that only 19 (28%) of the participants' responses contained a realistic solution to the problem. They also reported that none of the participants made any comments about the problematic nature of the problem. Finally, Verschaffel, De Corte, and Vierstraete (1999) examined 199 upper elementary school pupil's difficulties in modeling and solving problematic additive word problems involving ordinal numbers. They administered the subjects a paper-and-pencil test consisting of 17 word problems, nine of which were experimental items and eight buffer items. Three of the nine experimental items can be solved by a simple addition or subtraction of the two given numbers. An example of this type of problems is "In January 1995 a youth orchestra was set up in our city. In what year will the orchestra have its fifth anniversary?" The solution to the other six items is either 1 more or 1 less than the answer provided by the addition or subtraction of the two given numbers. An example of this kind of problems is problem e) stated above. Verschaffel, De Corte, and Vierstraete found that the percentage of correct responses for each of the six problematic word problems was less than 25%. They reported that 83% of the errors made on these problems were ±1 errors. That is, most of the pupils' errors were due to their interpretation that the addition or subtraction of the two given numbers provides the solution to the problem.

As argued by Verschaffel, De Corte, and Vierstraete (1999), a problem with all of these investigations and others reported in the literature is that they have involved elementary or secondary students and, thus, a possible generalization to college students and, in particular, to prospective secondary mathematics teachers has not been established empirically. Second, since prospective secondary mathematics teachers are a more mature population from a psychological and mathematical point of view, it would be worthwhile to analyze their solution processes to examine the strategies, errors, and interpretations when solving problematic word problems. Finally, it is important to document, examine, and understand prospective secondary mathematics teachers' strategies, errors, and interpretations when modeling and solving problematic word problems because the teacher is one of the major agents in the classroom. It is the teacher who designs, adapts, or implements the instructional activities. If we want students to model and solve problematic word problems by taking into account the realistic considerations embedded in the problem situation, then it is necessary that the teachers themselves have the experience, disposition, and ability to model and solve such problems realistically. The purpose of this study is to extend Verschaffel, De Corte, and Vierstraete's (1999) investigation. First, I examine prospective secondary mathematics teachers' modeling strategies to solve problematic subtraction and addition word problems involving ordinal numbers. Second, I examine the type of errors, if any, that secondary teachers make when solving such problems. Finally, I examine their interpretations of the solutions provided by the procedure or mathematical model.

Theoretical Framework

Aspects of reality can be represented by mathematical means. This process of representation is called mathematical modeling. Some physical or real-world problems can also be solved by means of a process of mathematical modeling such as the one depicted in Figure 1 that was proposed by Silver, Shapiro, and Deutsch (1993). There are other models described in the literature (e.g., Verschaffel, Greer, & De Corte, 2000) but Silver, Shapiro and Deutsch's model is appropriate for the present study.

According to Silver, Shapiro, and Deutsch's model, the (simplified) process of mathematical modeling involves four phases. The first phase consists of understanding the structure of the mathematical problem embedded in the story text. During this phase we need to understand the given information, the unknown information, extraneous information, and realistic considerations embedded in the situational context. The second phase consists of constructing a mathematical model or selecting an appropriate procedure, operation, or algorithm whose result will lead us to the solution of the word problem. During the third phase we execute the procedure or algorithm. Finally, we interpret the result provided by the mathematical model or procedure in terms of the realistic context embedded in the story text of the word problem or in terms of the real-world story situation. It is during the fourth phase that we focus on the meaning of the answer produced by the mathematical procedure or computation. Students' responses to problematic word problems could include realistic or correct

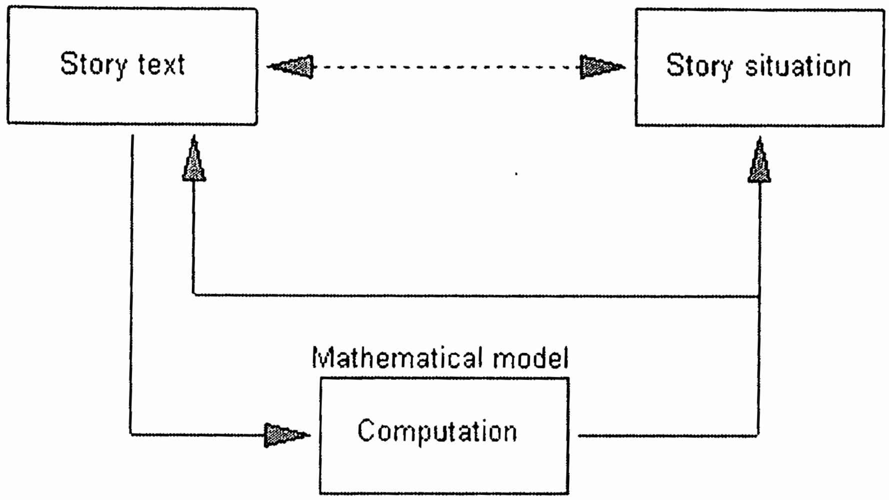

Figure 1. Silver et al.'s (1993) referential-and-semantic-processing model for successful solutions.

solutions if they select an appropriate procedure or operation and understand or pay more attention to the meaning of the result produced by the mathematical model.

Silver, Shapiro, and Deutsch's model implies that there are three main potential sources of errors when solving word problems: lack of understanding of the problem, which is suggested when an inappropriate procedure is chosen, incorrect execution of procedures, and incorrect interpretation of the answer produced by the mathematical model or procedure. In their study of the division problem involving remainders stated above, Silver, Shapiro, and Deutsch (1993) found that most of the students' responses, 91% in fact, contained an appropriate procedure (e.g., long division, repeated multiples, repeated addition, etc.) but only 61% of the students who selected an appropriate procedure performed it flawlessly (about 56% of the total number of students). These researchers reported that only 43% of the total number of students provided the correct answer of 14 to the problem but that some of them gave inappropriate interpretations. For example, one student wrote "14 buses because there's leftover people and if you add a zero you will get 130 buses so you sort of had to estimate. Are we allowed to add zeros?" (p. 124-125). About 55% of the students did not get the correct answer because they either failed to interpret the answer produced by the division computation or made computational mistakes that could have been detected if students had interpreted their solutions correctly. The researchers proposed the model exhibited in Figure 2 as a schematic representation of an unsuccessful solution. That is, some students failed to get the correct solution to the problem because they did not map the result produced by the mathematical model (in this case, a division) back to either the story text or the real-world story situation.

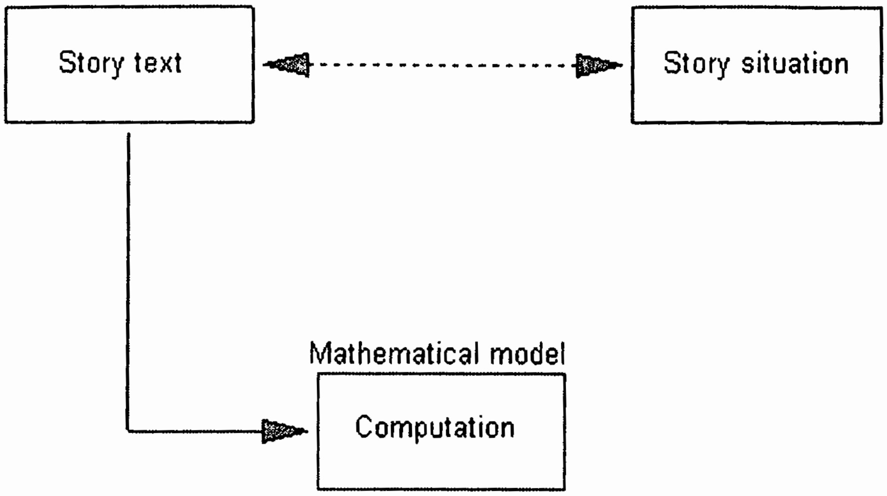

Figure 2. Silver et al.'s (1993) referential-and-semantic-processing model for unsuccessful solutions.

Methods and Sources of Evidence

A paper-and-pencil test was administered to 19 prospective secondary mathematics teachers from three required mathematics classes. Fifteen students were female and four male. All students except one were mathematics majors. At our institution, teachers seeking 7-12 certification in mathematics are required to complete a mathematics major. The non-mathematics major was seeking a supplementary endorsement in mathematics. The written directions included asking students to show work to support their responses. Calculators were not allowed. The test contained nine experimental items and some buffer items. The experimental items were adapted from Verschaffel, De Corte, and Vierstraete's (1999) test. Table 1 displays the nine experimental items. All the experimental items were addition and subtraction word problems involving ordinal numbers. Three of the nine experimental items can be solved by the straightforward addition or subtraction of the two numbers given in the problem statement. The solution of the other six items is 1 more or 1 less than the answer produced by the subtraction or addition of the given numbers.

A difference from some previous research, where the word problems have been designed in an ad hoc way, Verschaffel, De Corte, and Vierstraete's were based on a taxonomy of the possible modeling complexities. The nine experimental items differ in terms of two dimensions: (a) the nature of the underlying mathematical structure and (b) the nature of the unknown information. We can distinguish three categories of problems (Types I, II, and III) based on mathematical structure. The solution of Type I problems can be obtained by adding or subtracting the two numbers given in the problem. The solution of Type II and Type III problems is 1 more or 1 less than the answer produced by the addition or subtraction of the two given numbers. Types

Table 1. The Nine Experimental Items

Type	Item	Required operation(s)*
I-L	1. In January 1985 a youth orchestra was set up in our city. In what year will the orchestra have its twenty fifth anniversary?	S + D
I-D	2. Our youth club was set up in September 15, 1970. I became a member in September 15, 1999. How many years had the club already existed when I became a member?	L - S
I-S	3. In March 2000 it had been 34 years since our school had held its first annual school party. In what year was the school party held for the first time?	L - D
II-L	4. In September 1975 the city's youth orchestra had its first concert. In what year will the orchestra have its fiftieth concert if it holds one concert every year?	(S + D) - 1
II-D	5. Last October (2001) I participated for the first time in the great city running race that is held every year. This race was held for the first time in October 1959. How many times has the race been held?	(L - S) + 1
II-S	6. In November 1994 the twenty fifth annual school party took place. In what year was the school party held for the first time?	(L - D) + 1
III-L	7. There was a summer market in our city every summer from 1950 up through 1969. Since then the summer market was cancelled 30 consecutive times. In what year did the summer market restart?	(S + D) + 1
III-D	8. For a long time the city held a fireworks display every year on the last day of the October festival. In October 1982 we had our last fireworks, and thereafter there was no fireworks display. In October 1999 they restarted the tradition of the annual fireworks display. How many years did we miss the fireworks?	(L - S) - 1
III-S	9. In December 1999 our sports club held its annual election for its officers. Because of a lack of candidates, there had not been elections for the 23 years preceding 1999. Prior to this election, in what year did the last election occur?	(L - D) -1

* L = larger ordinal number
S = smaller ordinal number,
D = difference between the two ordinal numbers.

II and III problems differ in the nature of the enumeration process used to obtain the solution. For Type II problems, the enumeration process begins with the smaller number or the larger number. For Type III problems, the enumeration process needed to obtain the solution does not include neither the smaller nor the larger number. With respect to the nature of the unknown information, three categories of problems can be distinguished: (a) problems for which the larger number is unknown (Type L problems), (b) problems in which the smaller number is unknown (Type S problems), and (c) problems for which the difference between the two ordinal numbers is unknown (Type D problems). Combining the two dimensions in which the problems differ, we can obtain nine possible different types of addition and subtraction word problems involving ordinal numbers.

The main source of data was the written responses provided by the prospective secondary mathematics teachers. I recognize that written responses have some limitations as compared to verbal protocols. However, several researchers (e.g., Hall, Kibler, Wenger, & Truxaw, 1989) have validated the use of written responses to infer cognitive processes. In fact, I did not have any difficulty to determine the strategies that the prospective secondary mathematics teachers used to solve the problems. Nevertheless, I conducted interviews with the students to gain a deeper understanding of the thinking and reasoning that students used to solve the problematic word problems.

Analysis and Results

Students' written responses were analyzed with respect to four aspects of the process of mathematical modeling represented in Silver, Shapiro, and Deutsch's (1993) model: (a) the strategy, procedure, and operation used by the students to solve each problem, (b) the execution of procedures, (c) the solution to each problem, and (d) the (implicit or explicit) interpretation of the result produced by the mathematical model or procedure. I also conducted an error analysis to determine the type of errors that prevented students from obtaining the correct solution to each experimental item. The students produced a total of 171 responses (57 responses to the non-problematic experimental items and 114 responses to the problematic experimental items). The strategies used by the students were categorized as formal strategies (addition or subtraction of the two numbers given in the problem), or informal (e.g., counting). A total of 55 (96%) responses to the non-problematic items contained a formal strategy, one (2%) contained a counting technique, and another one (2%) contained solving a similar simpler problem. On the other hand, 99 (87%) responses to the problematic items contained a formal strategy and the remaining 15 (13%) contained counting techniques. Overall, 154 (90%) responses contained a formal strategy, 16 (9%) contained counting techniques, and only one (1%) involved solving a similar simpler problem. Students' responses were also analyzed to determine the appropriateness of the procedure, algorithm or operation used to solve the problems. A procedure was judged as appropriate if it could lead to the correct solution or as inappropriate otherwise. Not surprisingly, all students used appropriate procedures. With respect to the execution of procedures, students performed 162 (95%) procedures correctly. Regarding the solutions to the experimental problems, 85 (50%) of the responses contained correct solutions. Specifically, 41 (72%) of the 57 responses of the non-problematic word problems contained correct solutions and only 44 (39%) of the 114 solutions to the six problematic word problems were correct. Considering that all subjects but one were mathematics majors, the percentage of correct solutions was much lower than I expect it. The percentage of correct solutions to each experimental item is exhibited in Table 2.

As we can see from Table 2, the percentage of correct solutions to the problematic word problems varied from 32% (problems 4 and 6) to 47% (problem 8). Even though the participants have probably had extensive experience with routine arithmetic word problems, I was expecting that this sample of prospective secondary mathematics teachers would perform much better on the six problematic word problems because all but one were math majors. Since a high percentage of the procedures was executed correctly, I conducted an error analysis to find out what prevented the prospective teachers from getting the correct answer on the problematic word problems and to further our understanding of students' solution processes. Based on previous research, I predicted that most of the students' errors were ±1 errors. The results are displayed in Table 3.

The error analysis confirmed my prediction. As shown in Table 3, a high percentage of errors for each problematic item was ±1 errors. Overall, the percentage of ± 1 errors made on the problematic items was 91%. A total of 64 (56%) of the solution processes to the problematic items contained ±1 errors and 6 (5%) contained other kinds of errors. The error analysis indicates strongly that the errors on the problematic items resulted from students' interpretation that the addition or subtraction of the two numbers given in such problems provides the solution.

Discussion and Conclusion

The major purpose of this study was the examination of prospective secondary mathematics teachers' solution processes when solving problematic addition and

Table 2. Percentage of correct solutions for each experimental word problem

Problem	Number of correct solutions	Percentage of correct solutions
1	14	74%
2	15	79%
3	12	63%
4	6	32%
5	7	37%
6	6	32%
7	8	42%
8	9	47%
9	8	42%

Table 3. Type of ± 1 Errors for Each of the Six Problematic Word Problems

Type of problem	Required operation(s)	Type of ±1 error	Percentage of students' ± 1 errors
II-L	(S + D) - 1	+ 1 error	85%
II-D	(L - S) + 1	- 1 error	83%
II-S	(L - D) + 1	-1 error	92%
III-L	(S + D) + 1	-1 error	100%
III-D	(L - S) -1	+ 1 error	100%
III-S	(L - D) - 1	+ 1 error	91%

subtraction word problems involving ordinal numbers to determine the nature of their modeling strategies, interpretations, and errors. A paper-and-pencil test was administered to a total of 19 participants. The test included nine experimental items, three of which were non-problematic and the other six were problematic. The solution of the problematic items is 1 more or 1 less than the addition or subtraction of the two given numbers. Overall, it was found that 90% of the responses contained a formal strategy (addition and subtraction with the two given numbers). The other 10% contained informal strategies (counting techniques and solving a simpler similar problem). It is worth to notice that, although prospective secondary mathematics teachers have more familiarity with anniversaries and their mathematical knowledge is more developed than that of the fifth and sixth graders from Verschaffel, De Corte, and Vierstraete's (1999) study, some of their successful solutions to the problematic items were obtained with counting strategies. This suggests that students knew that the addition or subtraction of the two given numbers does not yield the correct answer to the problem and that some adjustment had to be made but they did not know how to make it. It could also indicate that, at least for the subjects using the counting strategies, their knowledge of addition and subtraction involving ordinal numbers was not completely developed. Both conjectures were verified with interviews conducted with the participants.

While the results for the non-problematic items were less than satisfactory (72% of the solutions were correct), the results for the problematic items were alarming (39% of the solutions were correct), especially given that all but one of the participants were majoring in math. An analysis of errors revealed that 56% of the solution processes to the problematic items used by the students contained ±1 errors. That is, it seems that students interpreted that the addition or subtraction of the two given numbers yielded the correct solution to the problematic word problems. Therefore, the model depicted in Figure 2 seems to explain, at least in part, the ±1 errors: students failed to correctly interpret the result of adding or subtracting the two given numbers. However, a deeper, perhaps more important question remains: why did some students interpret the result of the addition or subtraction with the two given numbers as the solution to the problematic items? Several hypotheses could be offered to explain this finding. First, there is the possibility that students approached the problems in a mechanical way because they are used to solve addition and subtraction problems in a straightforward manner. This hypothesis is supported, at least partially, by some students who stated during the interviews that they were not used to think when solving this type of problems because the problems involving addition and subtraction that they have encountered previously had been solved with an addition or subtraction of the two given numbers. A second hypothesis is that some students lack an awareness of informal techniques such as drawing a diagram or solving a simpler similar problem or they may have an underdeveloped repertoire or understanding of such heuristic techniques. I do not offer any direct evidence to support or refute this conjecture. The third hypothesis is that some students do not have a clear understanding of addition and subtraction involving ordinal numbers. This hypothesis is supported by some students who were aware of the problematic nature of the problem but they did not know how to adjust the result produced by the straightforward application of the addition or subtraction of the two given numbers. Some of these students obtained the correct solution by counting techniques. The interviews revealed that these students knew that an addition or subtraction was involved but not how to make the adjustment either on the solution provided by the addition or subtraction of the two given numbers or on the given numbers. It seems then that a plausible explanation for students' lack or interpretation (or misinterpretation) or their use of counting techniques is that they do no have a complete understanding of addition and subtraction involving ordinal numbers. This explanation is in contrast with the one provided by Silver, Shapiro and Deutsch (1993) to understand some middle grade students' solutions to the bus problem when their responses involved 13 or 13 with another number such as a fractional remainder. Silver, Shapiro, and Deutsch reported that about 22% of the students were able to correctly perform an appropriate procedure but did not provide an interpretation for their incorrect numerical answer. These researchers also found that nearly 24% of the students performed the computation procedure incorrectly and provided a numerical solution other than 14 with no interpretation. The researchers argue that both sets of students failed to obtain the correct answer of 14 because they failed to interpret the solution provided by the mathematical procedure. In the case of division problems involving remainders, the lack of interpretation (or misinterpretation) is rooted more deeply on the meaning of the quotient and remainder than on the understanding that the solution of the problem can be represented with a division of the two given numbers. In the present study, in contrast, the lack of interpretation, or misinterpretation, may lie more on an incomplete understanding of the connection between the nature of the enumeration process needed to obtain the solution and the answer provided by the addition and subtraction of the two given numbers. It seems that some of the middle grade students understood that a division with the two given numbers was needed to solve the bus problem but failed to interpret the remainder. In the present study, some prospective secondary mathematics teachers, in contrast, might not have understood that the addition or subtraction of the two given numbers did not provide the solution to some of the subtraction and addition word problems involving ordinal numbers. It seems then that the semantic feature of Silver, Shapiro, and Deutsch's (1993) model does not completely account for the ±1 errors made by the prospective secondary mathematics teachers who chose an appropriate procedure and executed it correctly.

Since a high percentage of errors for the problematic word problems was ±1 errors, 91% in fact, it seems that the prospective secondary mathematics teachers need at least some minimal intervention such as telling them that some of such problems are "tricky" or creating a cognitive conflict by asking them to solve simpler similar problems. The cognitive conflict technique was used during the interviews with some of the students who solved all the problematic word problems by adding or subtracting the two given numbers. Some of them immediately realized that some adjustment had to be made to solve the problems correctly.

While the sample size does not allow to generalize any of the results to larger populations of prospective secondary mathematics teachers from the USA or any other country, this study provides useful practical and theoretical information. From a practical point of view, at the very least, this study suggests that some prospective secondary mathematics teachers might approach some problematic word problems, such as the ones examined here, in a mechanical way. This suggests the introduction of problematic word problems in the school curriculum so that future teachers learn to approach word problems with a realistic perspective. Another contribution of this study is that some prospective secondary mathematics teachers might have an incomplete understanding of subtraction and addition involving ordinal numbers and these teachers will need more than a minimal intervention. Another contribution of this study is related to helping prospective secondary mathematics teachers develop a disposition to provide their students with problematic word problems so they (their students) learn to solve such problems realistically, and, as it is the case with the problems used in this study, develop a deeper understanding of addition and subtraction involving ordinal numbers. From a theoretical point of view, the findings help us to better understand some of the psychological aspects of learning mathematics within the context of problematic word problems. The results also shed some light on some aspects of Silver, Shapiro, and Deutsch's (1993) referential-and-semantic processing model as discussed above.

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A FRAMEWORK FOR POSING TECHNOLOGY-RICH MATHEMATICS PROBLEMS

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This study investigated how secondary school preservice teachers used the *Critique; Compare; Compose; Pose and Check* (C^3PC) strategy to engage in posing, sharing, and critiquing technology-rich mathematics problems. The study attempted to investigate the psychological aspects of teaching and learning mathematics in a technology-enhanced classroom. C^3PC helped students to think more deeply about mathematics, facilitated generalization, and empowered students to solve difficult problems. It encouraged reflection on promising practices; solving problems; and reconstructing knowledge of the process of growth and learning. The implications of this study serve to promote reflection by increasing preservice teachers' awareness of their own values of mathematical problems, and perspectives about the teaching and learning of mathematics, and by encouraging them to entertain perspectives different from their own.

Theoretical Framework

Many educators and education groups have called for fostering problem posing in mathematics students (Freudenthal, 1973; National Council of Teachers of Mathematics, 2000; Polya, 1954). Problem posing has been identified as of central importance in the discipline of mathematics and in the nature of mathematical thinking (Silver, Marmona-Downs, Leung, & Kenney, 1996).

The availability of technology and the innovative ways that it can be used provides an opportunity and a challenge to mathematics teachers (Kaput, 1988). According to Clarke (1999), computers used as powerful educational tools are providing the possibility of amplifying human ability, by enabling users to undertake tasks previously considered too difficult, too tedious, or too time-consuming and by providing a tool for the development of cognitive skills. However, not all types of mathematical problems call for the use of technology for effective solutions. On the contrary, most textbook problems can be solved equally as well with paper and pencil as with technology. The use of technology to enhance problem posing, combined with technology-assisted explorations and solutions of the problems, may be a powerful pedagogical alliance that may lead students to think more critically.

According to Moses, Bjork and Goldenberg (1990), the teacher is the essential ingredient in the problem posing process. They propose four main ideas to foster problem posing: (1) use problems in textbooks as a basis for problem posing, (2) avoid questions that have unique answers, (3) create a classroom climate where teachers model the problem posing process for their students, and (4) use technology to promote problem posing. In addition, English, Cudmore, and Tilley (1998) suggest that problem posing paired with problem critiquing provides an effective vehicle for fostering analytical skills in the mathematics classroom.

I propose a modification of Moses, Bjork and Goldenberg's (1990) framework that includes a five-phase model to foster the posing and checking of technology rich problems: Critique; Compare; Compose; Pose and Check (C^3PC). Technology-rich problems are mathematics problems that could be solved effectively using technology. The first phase in the C^3PC process is to *critique* school mathematics textbook problems, problems published online at the various mathematics sites, and problems written by the teacher and students. The second phase, *compare*, involves contrasting and classifying the problems examined into those that could be solved easily with paper and pencil and those where the solution can be enhanced using technology. The third phase, *compose* provides students with simpler problems or situations to create more complex problems. They may also be expected to decompose complex problems into simpler ones. The forth phase requires students to pose original problems. The last and fifth phase, *check*, requires students to reexamine the problems they posed. The check process is endless with each new problem posed, critiqued, compared, and composed.

Method and Data Sources

This exploratory investigation was conducted to describe the nature of problems posed by preservice teachers that call for the use of technology for effective solutions. The research questions were:

1. What factors motivate students to write technology-rich mathematics problems?
2. How helpful was the C^3PC strategy for posing technology-rich problems?
3. What do technology-rich problems posed by students say about their understanding of a mathematics concept?

The participants were 23 secondary school preservice teachers in their junior year enrolled in a *Technology and Secondary School Mathematics* course. This course included the use of various technology tools to investigate mathematical topics at the secondary level. Participants were given class time to examine and discuss mathematics problems. Each problem asks the students to employ the C3PC strategy to determine its technological merit. The instructor also modeled the formulation of technology-rich problems in class. An example is given below:

> You have the following students in your class: Frank, Elizabeth, Ophelia, and Felix. Frank made 98, 85, 81, and 85 on four tests all out of 100 points. Elizabeth made 88, 85, 98, and 87. Ophelia made 83, 83, 83, and 87. Felix made 73, 92, 100, and 87. The first two tests account for 20% each, and the last two tests account for 25% each of the final grade. The only other grade you have is for their notebooks. Your school sets the lowest passing grade of C at an average of 70 up to 79, B at 80-89, and A at 90 and over.

(a) Which of the students could receive an "A" grade? A "C" grade? Why?

(b) Is it possible for any of your students to fail? Explain.

(c) If Frank scored 75 on his notebook, and Ophelia scored 95, overall, which of the two students is the better student?

(d) Could Elizabeth be the best student? If not why? If yes, what minimum notebook grade should she receive to be the best student?

Even though this problem could be solved using paper and pencil, the solution could be better illustrated using spreadsheets.

Data sources for the study included: student generated problems, follow-up class and E-mail discussions, individual interviews, and a town meeting. Participants were expected to pose two problems each week. The best problem of the 46 problems posed each week was posted on the Web. Selected student-generated problems were also used on tests. It was expected that at least 50% of students' problems be the type that called for the use of technology for effective solution. This method generated over 230 technology-rich problems from the participants. A "problem of the week file" and follow up discussions, observations of students writing mathematical problems on the spot, and interviews of selected students were the main sources of data. Using "talk aloud" protocol, participants' problem posing processes were documented. E-mail and class discussions of the problems were ongoing throughout the semester. Other sources of data were through whole class "town meeting at the end of the semester." At the "town meetings" the results, as they were emerging from observations and interviews, were shared with the participants for their feedback.

The constant comparative method of Glaser and Strauss (1967) was used to understand the nature of problems posed by the preservice teachers. Efforts were made to uncover evidence of patterns, similarities, and differences in the problems posed by the participants. Participant observation and open-ended interviews were used to answer the research questions.

Results

Most preservice teachers valued the C^3PC process and thought it helped them to have a deeper understanding of the concept they were studying and was also relevant to their future teaching careers. Although most students could pose technology rich problems, their mathematical understanding varied. A synthesis of some of the results are: (1) Preservice teachers were motivated to pose technology rich problems by external rewards such as a chance to have their problems posted on the web or used on a test. (2) They viewed the C^3PC process as relevant to their future teaching careers. Students who were successful problem posers were more likely to succeed in the course and in the program. (3) The problems students posed provided a medium for analyzing their understanding.

What does the problems students pose say about their understanding of a concept? Three students were unable to pose technology-rich problems. Students with a strong foundation were able to explain and extend their problems in class discussions, however, those with limited mathematics understanding were unable to.

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IDENTIFICATION OF STRATEGIES USED BY FIFTH GRADERS TO SOLVE MATHEMATICS WORD PROBLEMS

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When students confront arithmetic or algebraic word problems, they develop ideas and notations during the processes of solving them by using various arithmetic strategies. Those ideas and notations are the basis for solving that type of problems. Is it possible to aid the development of students' algebraic thinking during their transition from arithmetic to algebra? We think that a way of doing so is by presenting them with different types of problems—as proposed by Bednarz and Janvier (1996)—and encouraging their reasoning and development of strategies linked to their arithmetic thinking. Hence, the identification of strategies used by fifth graders when solving algebraic word problems is of outstanding importance for finding effective ways to aid students during their transition from the arithmetic to the algebraic thinking. Results obtained in this research study show that with these types of mathematics word problems, students generate strategies that at certain moment become useful for their transition to the algebraic thinking.

Background

In the plan and programs for elementary school education (grades 1-6) in México, a methodological approach based on problem solving is proposed for the teaching of mathematics. Recommendations in these official documents emphasize dialogue, interaction, and confrontation of viewpoints as important activities for promoting learning and the construction of knowledge. It is argued that by implementing these activities students will view mathematics as a functional and flexible tool for solving problems posed to them (SEP, 1993, pp. 49-70).

However, elementary school teachers seldom put these recommendations into practice and, in the school environment, students perceive mathematics as complex and governed by rules under which their learning depends mainly on memorization. Students in the lower secondary level of education (grades 7-9) have a similar perspective about mathematics. When students in this level (ages 12 to14) solve algebraic problems, for instance, they usually have recourse to rules and procedures with no reflection on these; they believe that this type of activities represents the essence of algebra. Kieran (1988) reported that students in the lower secondary level of education are often unable to apply their knowledge of basic algebra when solving problems.

Solving algebraic word problems enhances students' attitudes for searching and producing conjectures that in a middle term will allow them to acquire algebraic notions such as equation, unknown, generalized number, variable, and function.

Therefore, by means of solving this type of problems, it is possible to help students make the transition from arithmetic to algebra.

The passage of students from the elementary to the lower secondary level of education underlies the development of new mathematical abilities. Several authors have reported that students encounter great difficulties during their transition from arithmetic to algebra—Filloy and Rojano (1989), Herscovics and Linchevski (1991), Kieran (1981, 1988), Kieran and Chaloug (1993), Booth (1984), and Bednarz and Janvier (1996), among others.

In the trend of research that approaches the area of transition from arithmetic to algebra, Filloy and Rojano (1989) introduced the concept of *didactic cut* to characterize the transition in the specific realm of solving equations. Herscovics and Linchevski (1991), in a similar trend of research, described that a *cognitive shift* occurs when students operate on the unknown. Bednarz and Janvier (1996) made a different contribution to this area of educational research: they investigated the conditions that allow students to construct algebraic arguments in a context of solving mathematics word problems.

In a previous research paper, Bednarz and Janvier (1994) presented an analysis of arguments produced by students when solving either arithmetic or algebraic problems. These researchers discovered that before introducing students to algebra, they put different stable profiles of arithmetical reasoning into play when confronted with different types of problems traditionally presented in algebra.

One objective in the investigation by Bednarz and Janvier (1996) was to clarify the conditions under which, in a context of solving word problems, students' algebraic thinking emerges and develops. They also studied two basic aspects for the characterization of such problems: nature of the problems—their structure—and the relative difficulty for students to solve them. Moreover, they noticed the importance of identifying the general structure of a problem according to the involved (known and unknown) quantities, the relationship between them (connection between quantities), and the type of relations implied (additive or multiplicative comparison).

Thus, according to the structure of problems, Bednarz and Janvier (1996, p. 123) classified them as arithmetic or *connected*, and algebraic or *disconnected*. With connected problems, it is possible to build bridges between known information and students can work from the known to the unknown. Disconnected problems allow no direct relation between known and unknown information to be established.

Hence, these authors could show the complexity of algebraic word problems usually posed to students during their regular courses of lower secondary education. They also emphasized that the knowledge of strategies applicable in the solution of arithmetic problems is the basis for the transition from arithmetic to algebra. In the selection of the word problems for our research, the characterization proposed by Bednarz and Janvier (1996) was of fundamental importance.

The Research Study

The purpose of our investigation was to identify arithmetic strategies that fifth graders have recourse to and the way they use them for solving mathematics word problems—above all, those problems related to algebra. The research activities for this study initiated with 35 fifth graders (ages 11 and12) of a public school in an urban area. Students in this group had an acceptable performance in mathematics, although they had little experience in problem solving.

Since working with 35 students would make it difficult to observe their interactions and their processes for solving problems, only 15 students were selected for the experimental phase: five high-achieving students, five average achievers, and five low-achieving students. We based our selection on the analyses of results obtained from a diagnostic questionnaire (DQ) applied to all students in this group. Then the selected students formed working teams, each with one student of each level of achievement.

The difficulties students had when trying to solve the selected problems we presented them were evident. Thus, we reconsidered the question posed by Bednarz and Janvier (1996): *Is it possible to help students make the transition from arithmetic to algebra?* We think that it is possible to aid the development of students' algebraic thinking during their transition from arithmetic to algebra. A way of doing so is by presenting students with problems of different nature—as proposed by Bednarz and Janvier (1996)—and encouraging their reasoning and development of strategies linked to their arithmetic thinking.

Methodology

During the first phase of the investigation, we looked for and designed problems to elaborate the DQ. As a part of the search, we revised mathematics textbooks of the last cycle of elementary education (grades 5-6) and of the first grade of lower secondary education (grade 7). The questionnaire students had to answer contained arithmetic word problems mainly. In the analyses of the responses by students, we noticed some arithmetic difficulties they had and we could identify some of their problem solving abilities as well—which were deficient.

Based on the identification of students' previous arithmetic knowledge, we designed 22 working sheets with an algebraic problem each (for examples of these problems, see Appendix) and planned the manner in which we should pose algebraic problems. The purposes for the elaboration of those working sheets were to:

- present students with problems traditionally studied in school algebra;
- observe the advantages of working in teams;
- identify arithmetic strategies students use; and
- enhance students' development of abilities to solve algebraic problems.

In the second phase of the investigation, during three weeks we had 15 working sessions of about 50 minutes each. All sessions were video-recorded and, additionally, the researcher took notes about features of each.

Students approached every proposed problem by working in a collaborative environment; that is, they worked in teams during this experimentation phase. Due to this way of working, it was easier to identify the strategies students used when solving word problems.

At the beginning of each working session, the researcher would ask students to read the statement of the problem carefully so that they could understand what was required. We discussed with them what it means (a) to understand the problem, (b) the selection and development of a strategy, and (c) the verification of a solution. Moreover, we suggested them to use a pen to do all their writing during the process of solving a problem.

Students would re-read the problem and decide the way to approach it by working in teams. The three students in each team discussed and tried strategies, checked their results, and registered their trials in the working sheets. An example of those discussions is the solution process in one of the teams for the following problem.

> The perimeter of a rectangular plot of land is 102 meters. The depth of the plot is double its width. What is its depth? What is its width? (Adapted from: Alarcón, Bonilla, Nava, Rojano, & Quintero, 1994, p. 162)

Researcher [R]: How did you solve the problem?

Student 1 [S1]: First, we divided 102 by 2 [student writes the division] and we got 51.

R: What did you do with that number?

Student 3 [S3]: Then, we divided 51 by 2 [student writes the division] and we got 25.5. This was not right because when multiplying 25.5 by 51 [student writes the multiplication] we got 1300.5 … So it was wrong.

S3: Next, we decided to divide 102 by 3 [student writes the division] and we got 34.

R: Was this result useful?

Student 2 [S2]: We wrote 34 plus 34 [student writes the addition] and we got 68, but we did not have the width yet.

S2: We thought of 34 divided by 2 and we got 17.

R: How do you know that the solution you found is correct?

S1: Because we multiplied 17 by 2 [student writes the multiplication] and we got 34.

S1: Then, we added 68 plus 34, and we finally got 102.

When necessary, the researcher participated by asking questions and giving suggestions to students with the purpose of encouraging them in their work processes of solving the problems. Moreover, the researcher aided students to overcome difficulties that appeared at any point during the solution processes. This help provided to students consisted of asking questions to promote their reasoning and lead them to analyze the problem from a different perspective to the one they had adopted. The following are some of the questions asked.

Regarding the way of posing the problem:

Have you understood the problem?

Can you explain it to your teammates?

Can you explain what the problem is about with your own words?

What do you want to find?

Do you have any difficulty in understanding any part of the problem?

What part of the problem you do not understand?

What do you know?

What you do not know?

What are the data of the problem?

Did you identify the relevant information?

Does the problem have any information that is not relevant?

Can you make a sketch or draw a picture to interpret or illustrate the problem?

Regarding the solution process:

Do you have any idea on how to solve the problem?

Have you ever solve any similar problem?

How are you going to solve it?

What strategies could be useful to solve the problem?

What are the conditions of the problem?

What are you doing now?

Are you getting somewhere with that?

How does that relate with the solution?

Could a table or a graphic be useful?

Can you think of another way or method to solve the problem?

After solving the problem:

How do you know that the solution you found is correct?

What do you do with that result?

Does your answer make sense with respect to the conditions of the problem?

What strategies did you use?

Can you check or verify your solution?

Could you have solved the problem in a different way?

How would you do that?

Can you express with words what you have checked?

What do you think about the solution methods used by your classmates?

Thus, the role played by the researcher during this phase was that of facilitator and guide. After students had finished solving a problem, we did a group revision of the solutions found. This comparison was helpful for analyzing the solution processes and strategies used. Our intention was to leave no doubts as to the solution and verification of results. We must emphasize that this group revision allowed students to identify the strategies that emerged. Then, for solving similar problems they could have recourse to those strategies they thought were more useful.

During the working sessions, a collaborative and trusting environment was created. Students could freely exchange their ideas and they rotated the responsibilities of coordinating and presenting the teamwork. At the beginning of the experimental phase, high-achieving students took the initiative for coordinating the teamwork. As the working sessions progressed, average- and low-achieving students had a relevant participation both in the teamwork and in the group revisions of the work done.

During the third phase of the investigation, students had to answer a final questionnaire (FQ) individually. (We had previously designed the FQ with problems similar to those in the DQ.) Problems in the FQ aimed at: (i) exploring the individual progress of students during the experimental phase, and (ii) identifying strategies used by students for solving algebraic problems.

Discussion of Results

By working in teams, students could discuss different ways of interpreting the same problem and come to an agreement as to the most convenient way of approaching it. Besides, due to the group revisions and the use of the different solution strategies that emerged, they constructed new knowledge. At the end of the experimental phase, we observed that students could efficiently use the strategies analyzed during the group revisions. At the beginning, non-systematized trial and error was the strategy they used the most.

Moreover, the lack of reflection by students on the initial conditions of the problems when preferring the use of the basic arithmetic operations (addition, subtraction, multiplication, and division) was remarkable. Additionally, students systematically valued the results of their own activities as well as the solution criteria of their teammates. Confronting their solution processes was also an achievement.

The strategies systematically used by students during the experimental phase were the following:

S1. Propose a number and check it to find a solution.

S2. Divide one of the quantities into parts "to be distributed," followed by the search of numbers ending in 0 or 5, and, then, approximate to the desired quantity adding one by one.

S3. Base their work on the design of a drawing to find the solution.

S4. Construct a table for comparing the data and approximate the solution.

S5. Draw a number line to compare paths covered by a series of *jumps*.

S6. Mechanical use of basic arithmetic operations (addition, subtraction, multiplication, and division), that is, without any reflection on the initial conditions of the problem.

S7. Use of the rule of 3.

S8. Preference for the use of mental arithmetic without having to write the operations used (numeric answer).

At the beginning of the experimentation, the strategy most frequently used by students was *S1*. However, they soon realized the disadvantages of having to do several operations to find the solution. Hence, thereafter they preferred to change to the use of strategy *S2*. Few students used *S1* for answering the FQ (8.6%).

When using strategy *S2*, students first approximated the desired quantity by dividing; then, they approximated it with quantities ending in 0 or 5, and, finally, they tried with adding one by one to a number until they found the solution. Students preferred this strategy both in the experimental phase and in the FQ (49.3% used it in the FQ).

The solution of the following problem illustrates what has been just described above (see Figure 1).

> Seventy-eight (78) candies are distributed among Edgar, Juan, and Saúl. Juan receives 3 times as many candies as Edgar, and Saúl receives two candies less than Edgar. How many candies does each boy receive? (Adapted from: Alarcón et al., 1994, p. 166)

Some students found strategy *S3* very useful during the experimental phase, especially for grasping the problem posed. Nevertheless, students seldom had recourse to strategy *S3* in the FQ (0.6%). Only 4.6% of students had recourse to a table *(S4)* to solve some problems. To compare traveled distances in problems of "reaching," 8.6% of students based their work on a number line, strategy *S5*, a percentage similar to the one for *S1* and *S8*.

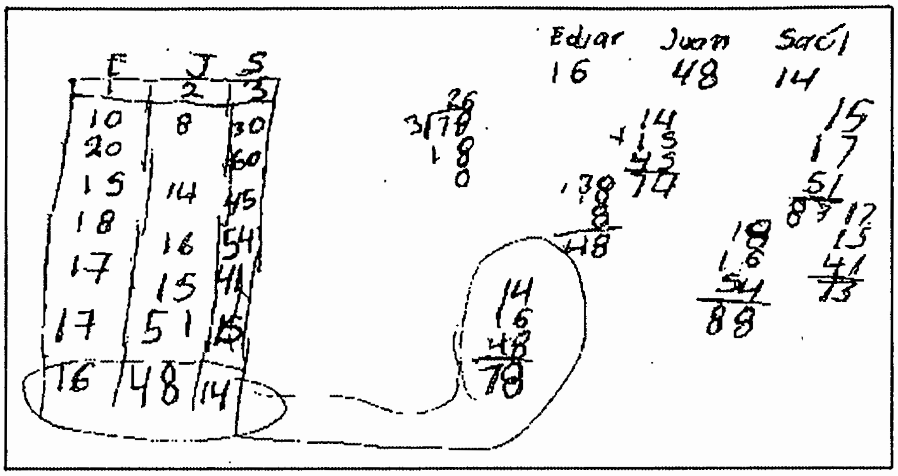

Figure 1

Strategy *S6* was used 15.3%. Generally, students did arithmetic operations with the data of the problem without any reflection on the conditions given at the beginning of it. It is evident that, in a school environment, students usually proceed so. School instruction promotes strategy *S7* to solve certain kind of problems. When the experimentation started, *S7* was preferred; however, students participating in this research study soon realized that *S7* was not useful to solve the proposed problems and its utilization was reduced (2.0% in the FQ).

The strategy of mental arithmetic, *S8*, was also important for students: once they had understood the conditions of the problem, they did operations without having to write them down. Although it is common for students to use this strategy in their daily work, generally teachers do not accept it and demand from students to write down the operations they carry out mentally. From the analyses of results of the FQ, besides observing that students had recourse to the arithmetic strategies that emerged during the experimental phase, we can claim that they did develop abilities to solve problems of different nature.

Conclusions

It is complicated for students to transit from arithmetic to algebra. The difficulties they have when solving arithmetical and algebraic word problems is a clear evidence of this. This evidence is more apparent when we realize that the strategies and reasoning of students for solving algebraic problems are fundamental for the construction of their algebraic thinking.

We think that it is possible to aid students' development of algebraic thinking during the phase of their transition from arithmetic to algebra. We think that a way of doing so is by presenting them with different types of problems—as proposed by Bednarz and Janvier (1996)—and encouraging their reasoning and development of strategies linked to their arithmetic thinking.

At the beginning of the experimental phase, we observed that students lacked concentration when they tried to solve algebraic problems. They even claimed that there was not enough information and that therefore they could not approach the proposed problems. Then, when students attempted to solve one of the problems posed to them, they tried different ways. Their main strategy was *S1*, but they soon realized that it was not necessary to do so many computations. Thus, as a first approximation, they started systematizing their strategies using quantities ending in 0 (by the easiness to operate with them, as can be seen in Figure 1).

Later, they continued with *controlled quantities*, that is, they had recourse to strategy *S2*. Eventually, this strategy became the most used. For instance, some students perceived the structure of the problem in an arithmetic context, and then they solved it by means of strategy *S2*.

Despite that in some occasions students did not take into account some portion of the information contained in the proposed problem, there was a positive evolution of most students to solve the problems as the experimental phase progressed. Moreover, students would check their answer to the problem according to the initial conditions of it. This was not easy, for students were used only to find a result without having to verify it. Thus, verification of the answer by students was an important accomplishment.

Now, with respect to the purpose of the investigation, the identification of strategies that fifth graders have recourse to for solving mathematics word problems, we can claim that students used informal arithmetic strategies—non-school strategies—to solve them. These strategies can evolve whenever students are stimulated to work with those relations and transformations immersed in the text of the problem. By means of an adequate intervention of the instructor, emphasizing correctly the arithmetic experience of the students, it is possible to make them come close to the algebraic thinking by designing ad hoc activities and problems.

By taking into account the complexity of word problems according to the characteristics proposed by Bednarz and Janvier (1996), we can select problems and rank them suitably for the design of a didactical intervention. Such a didactical intervention would support—in a controlled manner—the evolution of student' arithmetic reasoning toward their algebraic reasoning.

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Appendix

Examples of Problems Included in the Working Sheets

1. We want to distribute 100 chocolates among Saúl, Ricardo, and Nelson in such a way that Ricardo receives 4 times as many chocolates as Saúl, and Nelson receives 10 chocolates more than Ricardo. How many chocolates does each boy receive? (Adapted from: Alarcón et al., 1994, p. 165)
2. We want to distribute 91 bracelets among Rita, Luisa, and Candy in such a way that Rita receives 3 times as many bracelets as Luisa, and Luisa receives 3 times as many as Candy. How many bracelets does each girl receive? (Adapted from: Alarcón et al., 1994, p. 165)
3. An airplane flew 12 kilometers in 72 seconds. What was its average speed? How long will it take the airplane to fly a distance of 1800 kilometers? (Adapted from: Cárdenas et al., 1976, p. 199)
4. An automobile leaves from México City to Nayarit at an average speed of 40 kilometers per hour. Two hours later another automobile also leaves México City with the same destination at an average speed of 60 kilometers per hour. How long will it take the second automobile to reach the first one? (Adapted from: Preciado & Toral, 1971, p. 173)
5. There are three piles of flat bottle caps. The first pile has 5 bottle caps less than the third, and the second pile has 15 more than the third. The total number of flat bottle caps is 31. How many bottle caps are there in each pile? (Adapted from: Alarcón et al., 1994, p. 166)
6. The perimeter of a rectangular plot of land is 102 meters. The depth of the plot is double its width. What is its depth? What is its width? (Adapted from: Alarcón et al., 1994, p. 162)

PROBABILISTIC MISCONCEPTIONS: A 50/50 APPROACH AMONG MIDDLE AND HIGH SCHOOL STUDENTS

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This paper reports selected findings from a larger study on student probabilistic reasoning abilities. Students in grades 5,7,9 and 11 at a private school for boys in New York City (n = 173) completed a Probability Inventory, which required students to answer and justify their responses to ten items. Supplemental clinical interviews were conducted with 33 of the students to provide further detail about their reasoning on these items. A 50/50 approach was typical, especially among younger students, by which they over-generalized the probability of a single trial of a coin to a compound situation. A student using the 50/50 approach would indicate "correctly" that the probability of two coins resulting in one heads and one tails is 50% but also that the probability of any number of coins resulting in any outcome is 50%. The 50/50 approach was also used by students when comparing the likelihood of 7 tails on 10 tosses with 700 tails on 1000 tosses. 40% of the sample used a 50/50 approach on at least two items of the Probability Inventory.

Purpose of the Study

In articulating standards for school mathematics, the National Council of Teachers of Mathematics (NCTM) indicates that students, by graduation from high school, should be able to "develop and evaluate inferences and predictions that are based on data" (NCTM, 2000). The NCTM specifies, "Middle grades students should learn and use appropriate terminology and should be able to compute probabilities for simple and compound events...and high school students should compute probabilities of compound events and understand conditional and independent events." (NCTM, 2000) The inclusion of probability and data analysis as one of the ten NCTM standards reflects an increasing emphasis on data and the importance of the ability to infer conclusions based on data. This type of decision-making is commonplace in the course of everyday life, as well as in professional fields as diverse as psychology, political science, physical sciences, economics, engineering and medicine.

The main objectives of the larger study were to explore the following probabilistic misconceptions among people of various ages: representativeness, effect of sample size, availability, the conjunction fallacy, and assumed equiprobability in a special class of conditional probability problems. Previous studies of such misconceptions have focused on the decision-making abilities of adults, typically using college students as subjects. The focus of this study was to broaden our understanding of probabilistic misconceptions among middle school and high school students, including an analysis of the effects of age. The current report presents one theme found among the sections of the larger study dealing with coin problems.

Brief Theoretical Framework

Examples of research identifying probabilistic misconceptions from a cognitive psychological point of view abound (Konold, 1993; Nisbett & Ross, 1980; Tversky & Kahneman, 1971, 1973, 1982, 1983). In recent years, as probability and statistics receive greater attention in school curricula, mathematics education researchers have made efforts to identify misconceptions in this area, seeking also to explore how teachers and school programs can intervene and change these misconceptions. (Fischbein 1975, 1991; Fischbein & Schnarch , 1997; Konold, 1989; Konold, Pollatske, Well, Lohmeier, & Lispon; 1993; Shaughnessy, 1981). While not directly dealing with probability, other research in mathematics education focuses on students' misconceptions, intuitions, and inconsistencies in a variety of mathematical contexts, and the role the play in teaching and learning (Fischbein 1987; Tirosh & Graeber, 1990; Tirosh and Stavy, 2000; Vinner, 1990).

Fishbein and Schnarch (1997) conducted a preliminary study to determine the evolution of probabilistic misconceptions with respect to age. Their study examined samples of twenty Israeli students in each of grades 5, 7, 9, and 11, along with eighteen college students. Interestingly, their results indicate that certain probabilistic misconceptions increase with age, others decrease with age, and others are relatively stable. The current study implemented some of the questions used by Fischbein and Schnarch (1997), supplemented by additional questions relating to probabilistic misconceptions as well as traditional probability questions. The current study both replicates and extends Fischbein's work by studying samples twice as large, approximately forty students in grades 5, 7, 9, and 11 and deepens the potential to understand student responses by including student justifications and clinical interviews as part of the methodology.

Methods

The two data gathering phases of the study were conducted at a private school for boys, known for its academic excellence, in New York City. All students in grades 5, 7, 9, and 11 were selected as potential participants, resulting in a total of 173 subjects. These students worked independently on a written Probability Inventory for the duration of a class period (40 minutes). The second phase of data gathering was the clinical interviewing of thirty-three students, representing each of the age groups. The researcher conducted all of the interviews, using a standstill video camera for recording. The primary goal of each interview was to gain greater detail by using "How did you get this?" or "Why does this work?" types of questions. Borovcnik and Peard (1996) describe a process called "interview teaching," in which students are confronted with items and then given an opportunity to individually explain answers to those items. The teacher can then use these responses to "provoke insight" by "maneuvering the learners into a situation where this approach would become obviously absurd." The researcher used the "interview teaching" approach with regard to misconceptions that surfaced in the course of the clinical interviews.

Selected Results

Two Coin Item

The Two Coin Item asked students to determine the likelihood that Eminem gets one heads and one tails on a toss of two fair coins. Nearly all students in grades 5 and higher were successful in computing the probability of a simple event. Students in grades 5 and higher had difficulties, however, with determining the probability of a compound of two events. In responding to the Two Coin item, only 55% of the students, stable across all ages, responded 1/2. Some students justified this correct answer either by listing the complete, equiprobable sample space or by using conditional reasoning. However, most of the younger students who gave the correct answer, and 36% of the entire sample, used a "50/50 approach," by which they over-generalized the probability of a single event to the compound situation. The 50/50 approach perhaps explains the discrepancy reported by Carpenter, Corbitt, Kepner, Lindquist, and Reys (1981) regarding student performance on items related to compound events on the second National Assessment of Educational Progress. While more than two-thirds of seventeen year olds were able to compute the probability of one head and one tail on two fair coins as 1/2, 58% of 13 year olds and 50% of 17 year olds also indicated 1/2 as the probability of getting *two* heads on two coins.

Based on their written justifications of their answers as well as comments made in the interviews, we can see that many of the younger students arrived at the correct answer to the question by using reasoning that would also cause them to indicate that there is a 50% chance for *any* of the possible outcomes on two coins, similar to the results of Fischbein, Nello, and Marino. (1991), Lecoutre (1992), and Fischbein and Schnarch (1997). We label this the "50/50 approach" in that students over-generalize the 50/50 nature of an individual flip of a single coin to compound events. This is an extension of the equiprobability bias described by Lecoutre (1992), used to denote the intuition that a tail and a head would be as likely as two tails on two coins.

An example of the 50/50 approach and its connection to the equiprobability bias is found in the following excerpt from an interview with Max, a 9th grader.

Researcher: "Eminem tosses two fair quarters in the air. What are the chances that they land so that there is one heads and one tails?"

Max: "50% because every time there's a 50% chance."

Researcher: "What if I changed the question and asked you to find the probability that he gets two heads?"

Max: "50% because there's two sides. Again, you can never predict."

We see that Max believes both probabilities are equal, according to the equiprobability bias, and, also, that both probabilities are 50%, according to the 50/50 approach.

At this point, the researcher attempted to direct Max to think about the sample space of this experiment by using actual coins.

Researcher: "What could happen when you toss two coins?" (There are two coins on the table)

Max shows the cases HT, TT, and HH with the actual coins.

Researcher: "What's the probability of getting two heads?"

Max: "I think 50% because this coin has 50% of being a heads and this coin has 50% of being a head."

Even though he showed a sample space of size three, Max continued to use the 50/50 approach. At this point, the researcher attempted to create a conflict by having Max indicate the probabilities of two heads, two tails, and one of each, and then find the sum of those probabilities. Using the 50/50 approach, one would have the sum of the probabilities exceeding 100%, which could potentially create a conflict for the learner.

Researcher: "How about two tails?"

Max: "50% because it's like the heads again."

Researcher: "How about heads and tails, like this or like this?" (I show him with the coins)

Max: "50%."

Researcher: "So how many 50% chances do we have, can you add them up?"

Max: ".....2.5."

Researcher: "Is it ok to have probabilities add up to 2.5?"

Max: "A 2.5 probability? I mean, I guess it must be."

Another example of the 50/50 approach is demonstrated in the following excerpt from an interview with Daryl, a 7th grader. In this interview, the student responded using the 50/50 approach. The researcher attempted to create a cognitive conflict by adjusting the number of coins.

Daryl: "It's a 50% chance each so he has an even chance of getting both. Even if you have two quarters, there's still going to be a 50% chance."

Researcher: "What if we had 3 quarters? What's the probability that we get all tails?"

Daryl: "I still say 50%."

Researcher: "Why's that?"

Daryl:	"Because unless something affects the way the quarters come down, it's still going to be equal."
Researcher:	"What if we had 100 quarters? What's the probability that we get all tails?"
Daryl:	"Half-way."
Researcher:	"So if we threw up 100 quarters, you think we'd have a 50% chance that every single one of them lands on tails?"
Daryl:	"Yeah."
Researcher:	"Ok – what about 100,000 quarters?"
Daryl:	"That's a lot. But it's still 50%."

This student believes that any outcome of any number of coins is 50%, a mammoth over-generalization of the single coin model.

Other common responses to the Two Coin item were 1/3 and 1/4. In an era where increasingly great emphasis is placed on standardized testing results, these findings point to two serious difficulties. First, as we see in the results of the Two Coin item, it is possible for students to answer a question correctly, for reasons that are incorrect. Standardized test items must, therefore, be carefully studied prior to their use. Second, the results of this study show that it is possible for a wrong answer to be more mathematically sophisticated than a right answer. A student who determined a non-equiprobable sample space for the Two Coin item to arrive at an answer of 1/3 arguably knows "more" than a student who gives the correct answer using the 50/50 approach.

Four Heads Item

The Four Heads Item asked students to determine the most (and least likely) outcomes of the fifth toss of a fair coin that has just resulted in four consecutive heads. Most of the 7th, 9th, and 11th graders (89%, 70%, 83%) said that there is no most likely outcome on the fifth toss of a fair coin which has produced a streak of four consecutive heads. The correct response rate slightly *decreased* across students from grade 7 to grade 11, perhaps as a result of the stronger presence of the 50/50 approach among younger students. While 17% of all students did commit the gambler's fallacy by saying that a tails would be most likely after a string of four heads, 9%, or 15 students, said that another heads would be the most likely outcome since that follows "the pattern."

Coin Sequences Item

The Coin Sequences item asked students to choose a most (and least) likely, if any, sequence of coin tosses among HTHTHT, HHTHTT, HHHTTT, and TTTHTT. About two-thirds of the students indicated that each of the coin sequences listed in the

Coin Sequences item is equally likely. However, only 59% of those students justified this answer referring to independence of trials or the significance of order. Students in grades 5 and grade 7 used the 50/50 or outcome approaches more often than referring to independence or order, while most of the older students, in grades 9 and 11, who answered the question correctly also justified their answer correctly.

Examining the sequences HTHTHT, HHTHTT, HHHTTT, and TTTHTT, the sequence HHTHTT is the most representative, according to Tversky and Kahneman's definition of representativeness, in that it has both three heads and three tails as well as a seemingly random ordering. However, only ten students, or 6% of the sample, chose HHTHTT as the most likely sequence. On the other hand, 13%, or 22 students, said that HTHTHT was the most likely. Another 18 students, or 10%, indicated that any of the sequences with three heads and three tails would be the most likely. This means that nearly a quarter of the sample did not relate to the random process of each individual coin flip but only to the overall composition of the number of heads and tails in each sequence.

Effect of Sample Size

Two questions were asked to determine student perception of the effects of sample size. The first question had a familiar real-world context and the second related to a coin situation. All students were asked the Yankees item prior to the Coins Sample Size item. The Coins Sample Size item asked students to compare the likelihood of a result of 7 tails on 10 tosses of a fair coin with a result of 700 tails on 1000 tosses of a fair coin, with an option that they are equally likely. 72% (124 students) of all students indicated that 7 tails out of 10 tosses is as likely as 700 tails out of 1000 tosses. Most students who indicated that the events are equally likely on both items gave justifications for their choice based on equal ratios, equal fractions, or equal percentages. This is another instance of the intuitive rule, Same A Same B, described by Tirosh and Stavy (2000). While in most cases, students justified this answer by writing that 7/10 is equivalent to 700/1000, some students blended their conception of the 50/50 approach with the effect of sample size and knowledge of ratios.

Students using the 50/50 approach believe that *any* two outcomes involving coins are equally likely. In other words, for instance, in response to the Coins Sample Size item, 9th grader Niles said:

Niles: "They're equally likely because there's still a 50/50 chance."

Researcher: "What about the likelihood of getting 10 tails out of 10?"

Niles: "That's equally likely."

Niles does not seem to relating to the equal proportions whatsoever; instead, his decision is based on a misconception of the meaning of 50/50.

Consistency of 50/50 Approach

There were four items on the Probability Inventory that dealt with trials of coin flipping: the Two Coin item, the Four Heads item, the Coin Sequences item, and the Coins Sample Size item. We have described the incidence of an approach called the "50/50 approach" by which students over-generalize the probability of a single trial to compound events. How consistent is this approach? To answer this question, we can examine profiles of students on those four items. On the Four Heads item, the 50/50 approach is effective: a student reasoning with the 50/50 approach will write the correct answer (50%) and will justify this on the basis of the probability of a single trial of one coin. However, a student using the 50/50 approach on the Coin Sequences item will again write the correct answer, but justify this answer not on the basis of independence but again, by saying that any of the sequences could happen, because, for example, "there are two sides and a 50/50 chance of getting any pattern." On the Coins Sample Size item, while most students employed the Same A Same B line of thinking, there were, however, students who justified the response of "equally likely" with the 50/50 approach. In total, 58% of 5th graders, 47% of 7th graders, 44% of 9th graders, and 12% of 11th graders used the 50/50 Approach on at least two of the items. However, the results also show fluidity in students' thinking, between using ratio thinking or the representativeness heuristic as well as the 50/50 approach. Although approximately half of all 5th, 7th and 9th graders used the 50/50 approach on at least two of the four items involving coins, very few students used the 50/50 approach across all four items: one 5th graders, two 7th graders, and three 9th graders. This prevalence among younger students calls for further investigation into the factors affecting development *out* of the 50/50 approach.

Conclusions

Oftentimes in mathematics assessment, students are evaluated on the basis of their ability to provide the correct answer to a question. The findings of this study demonstrate that there are a variety of methods of arriving at such a correct answer: some methods are mathematically valid and others happen to produce the correct answer to that particular question but might become grossly inaccurate in other instances. For example, a student's response that there is a 50% likelihood for two coins to result in one tails and one heads sounds flawless, but this study has shown that this same student could have arrived at such an estimate since he or she believes that *any* outcome of *any* number of coins has a 50% likelihood. This 50/50 approach, found to be quite common among students in grades 5 and 7 of this study, clearly deserves more attention. When does it take root among children? More importantly, when, why, and how does it fade away?

In terms of in-school assessment or national tests, what about the student who believes that this particular probability is 1/3? While this is an incorrect answer to this question, this student is using the concept of a sample space, more sophisticated than the 50/50 approach. How do we as educators handle the issue of a wrong answer being more mathematically sophisticated than a right one given for incorrect reasons? Researchers and evaluators, and teachers too, need to examine the types of errors students make, which often stem from alternative student interpretations of the question itself. It is not enough for teachers to know that students perform poorly on a given task: knowledge of the common errors can help a teacher prepare activities that confront those errors or direct the students' attention toward the specificity of the language. As Konold (1993) writes, "Teachers become more effective as they increase their power to interpret student utterances, many of which may initially seem incomprehensible."

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THE PROBLEM SOLVING BEHAVIOR OF A MATHEMATICS MAJOR

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In the document *Principles and Standards for Teaching School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000), NCTM identifies Problem Solving as a major content strand for school mathematics. In addition, the Conference Board of the Mathematical Sciences (CBMS) recommends that "prospective teachers need to develop habits of the mind of a mathematical thinker" and need to be proficient problem solvers (CMBS, 2000). However, most undergraduate students have had more experience with "exercises" than they have with actual problem solving (Schoenfeld, 1985).

This study investigated the problem solving behaviors of Lee, a third year mathematics major intending to teach high school mathematics. He was enrolled in a capstone course geared toward the content areas taught in secondary schools. This course was taught from a "problem solving perspective." That is, the class was exposed to a wide variety of problem solving situations from the full range of high school mathematics curriculum. Lee participated in two task-based interviews – one at the very beginning of the course and one at the very end. During those interviews, he was asked to solve three problems requiring knowledge of basic mathematical content and concepts such as basic geometry, algebra, proportions, and percents. Each interview began with the interviewer asking him to discuss his thinking and solution attempts as he worked on the problems. The interviewer observed his behaviors and periodically reminded him to verbalize his thinking and articulate a rationale for a specific behavior. The interviewer gave no indication regarding the correctness of the solution as the subject worked on the problems; however, she occasionally posed a question to promote greater articulation of his thinking and solution approach. The interviews also included periodic probing about his perceptions of his problem solving background and success.

The problem solving sessions were analyzed using the Four Phase Cyclic Problem Solving Framework (Carlson & Bloom, 2002, under review), which draws heavily on the work of Polya (Polya, 1945/1957) and Schoenfeld (Schoenfeld, 1985). One dimension decomposes the problem solving session into four phases of cognitive and metacognitive behaviors. These phases are Orienting (initial engagement with the problem), Planning (planning, strategizing, conjecturing, etc.), Executing (actual number and symbol manipulation and computation) and Checking (reflecting and checking). This dimension is very similar to Polya's 4 steps (Polya, 1945/1957) but in our work with mathematicians, we found this to be an iterative process. Experienced problem solvers cycled through these phases until either the problem was solved or they gave up. The other dimension considers various aspects of problem solving. These aspects are Resources, Heuristics, Affect and Self-Monitoring. These categories differ slightly from Schoenfeld's framework (Schoenfeld, 1985). The heading of Affect is broader, and includes emotions and attitudes in addition to beliefs. Self-Monitoring refers only to the monitoring of products and processes during the course of the problem solving session. We include only monitoring and regulation from Schoenfeld's *Control* category since other aspects of control (e.g., regulating and planning) emerged in the global description of the problem solving process. These two broad dimensions (i.e., the cyclic phases and problem solving) interact at every level.

In this paper, I have used the framework to explore and characterize the problem solving behaviors of Lee at the beginning and end of the capstone course. The table below (see Table #1) provides a brief example of the preliminary analysis of the transcripts. In this excerpt, Lee is trying to solve the following problem: *Two numbers are "mirrors" if one can be obtained by reversing the order of the digits (i.e., 123 and 321 are mirrors). Find two mirrors whose product is 92565. Find two mirrors whose sum is 8768.*

In the initial problem solving session, Lee expressed concern about his level of content knowledge and his problem solving abilities. He worried that he was "slow" and didn't always "see" what he perceived his classmates did. But he also said that he liked math and felt satisfaction when he was able to solve a problem. His problem solving session revealed a deficiency in the use of metacognitive strategies. Lee appeared to struggle to make sense of the problems; hence much of his time was spent in the Orienting phase of the cycle. He tended to not verbalize a plan, and he was unlikely to check his progress for unreasonableness. He consistently looked to the interviewer for direction and reassurance during the actual problem solving. He revealed a low tolerance for frustration and admitted that with one problem, he would have given up if he were alone. However he was able to solve two of the three problems correctly, and the third problem would have been correct had he realized that he had not competed the problem as it was written.

Two things were particularly striking about this first interview. One was the inability to express what he was thinking. He appeared to go from organizing his information to solving the problem with no intermediate planning, yet some planning had to be taking place. It was as if he lacked the language to describe his thinking. The other interesting revelation was the fact that even though Lee was a math major and planned to teach high school math as a career, he was insecure about his abilities, both in the interview setting and during class sessions.

After a semester of extensive problem solving, explaining his reasoning both orally in class and in written assignments and being exposed to a wide variety of solution paths, he was interviewed again. He expressed and demonstrated greater confidence in both his problem solving ability and his content knowledge as well as and

Table 1.

Excerpt	Behavior	Phase
Two numbers are mirrors if one can be obtained by reversing the order of the digits. Can you find two mirrors whose product 92,565?	*Initial engagement – reads problem*	Orienting
No. [laughter]	*Anxiety*	
Two numbers whose sum is 8767.	*Engagement – reads problem*	
I've seen this before.	*Relates to previous problem*	
I don't even know where to start	*Lack of confidence – confusion*	
If I have two numbers – do the mirrors, do they have to be like 4,5,6 and then 6,5,4 or can they be like 4,7,9 and 9,7,4? They can be that way too?	*Sense making*	
Okay.... that would be equal...something additional here...[working and mumbling under his breath] number 7 is.... and then 2,5	*Executing*	Executing
They're going to be obviously 4 digit numbers	*Conjecture*	Planning
what ever my second number here is will be my third number here-so automatically the one's got to go on that side if I have one here	*Conjecture*	
one's gotta go on the other side -- which would mean that this would have to be a seven. [mumble] this would have to be 5,	*Testing conjecture*	Executing
, -- wait, that doesn't work.	*Reflects on conjecture/work*	Monitoring

greater persistence in his work in general. "I feel pretty confident when I do a problem that I can see if my answer makes sense or not." He solved all three of the problems correctly. Unlike the first interview, he was regularly observed monitoring his progress, and he tended to rely on his own mathematical abilities to guide him through the problem rather than look to the interviewer for support. Lee was more accepting of his own solution paths after witnessing the variety of ways others solve problems. In the second interview, he appeared to be more able to articulate his thinking process, especially in the planning phase of the problem solving cycle. The analysis also revealed a concern for the aesthetics of the solution (as in *nice,* as opposed to *ugly*). While this concern did not arise during the first problem solving session, Carlson noted in her work with professional mathematicians that they were regularly concerned with the aesthetics and quality of the solution – not just the correctness of it.

This study demonstrated the usefulness of the Four Phase Cyclic Problem Solving Framework as a tool for analyzing the problem solving behavior of students as well as mathematics professionals. The framework uncovered movement in Lee's use of self-monitoring during the problem solving process, his confidence in his problem solving ability, his ability to articulate his mathematical thinking and it helped to highlight an emerging concern for "nice" solution paths. Further research needs to be done to determine which aspects of the capstone course were the primary influences in the improvement he experienced, or if it was a combination of all the math courses he was taking.

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MAKING CONNECTIONS AND BUILDING ISOMORPHISMS: THE PROBLEM SOLVING OF TWO-YEAR COLLEGE STUDENTS AND YOUNG CHILDREN

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Problem solving and justification of a diversified group of two-year college students were compared with approaches of younger pre-college students working on the same task. The students in this study were engaged in thoughtful mathematics. Both groups utilized similar strategies for their methods of justification and solution. They were able to make connections and build isomorphisms among the various problems. The findings support the importance of introducing rich problems to pre-college and college students, under particular conditions.

By the time students study mathematics at the college level, it is hoped that they would have developed an ability to reason effectively. At the same time, if previous mathematical learning were durable the college students would be able to apply the specific mathematical content that they have learned up to this point to new situations that they encounter. Unfortunately, many of the students have passed through their ten years of mathematics courses primarily in settings that emphasize rote and procedural learning. From a perspective of conceptualizing reasoning in terms of solving open-ended problems, it was of interest to learn whether students in a liberal-arts college mathematics course could be successful in making connections in a problem-solving based curriculum. Specifically, this paper reports on a study with community college students enrolled in liberal arts mathematics. It will describe, in the context of combinatorics (1) What connections, if any, are made to isomorphic problems and previous mathematical knowledge and (2) how college students' representations and level of reasoning contrast with those of younger children engaged in the same investigations.

Background

Researchers at Rutgers University have documented children's thinking as they investigate problems in the area of combinatorics to determine how they think about the problems and justify their solutions (Maher & Martino, 1998; Maher & Speiser, 1997; Kiczek & Maher, 1998; Muter & Maher, 1998). One of these problems is the Towers Problem, which invites a student to determine how many different towers of a specified height can be built when selecting from two different color cubes, and to justify that all possibilities have been found. A second isomorphic problem, the Pizza Problem, invites a student to determine how many different pizzas can be created from a given number of toppings and to justify that all possibilities have been found. These tasks were given to the college students in this study during the problem-solving component of a liberal arts mathematics course.

Theoretical Framework

The growth of mathematical knowledge is the process whereby a student constructs internal representations and connects these representations to each other. Understanding is the process of making connections between different pieces of internally represented knowledge or between existing internal connections and new knowledge. (Hiebert & Carpenter, 1992) Students build their understanding of concepts by building upon previous experience, not by imitating the actions of a teacher or being told what to do. (Maher, Davis, & Alston, 1991) Learners who first learn procedures without attaching meaning to them are less likely to develop well-connected conceptual knowledge. When students encounter new problems they are more likely to retrieve knowledge that is well connected than to retrieve loosely connected information. (Hiebert & Carpenter, 1992). The students in this study were encouraged to think about their solutions to problems, develop understanding of the mathematics and justify their answers.

Data Collection and Methodology

Nine classes ranging in size from 6 to 25 students were studied from 1998 – 2000. Two groups from each class were videotaped as they worked on the Towers Problem and the Pizza Problem. Following the class sessions each student was required to submit a write-up of the problems. In addition, videotaped, tasked based, interviews of some of the students were conducted. From all the available videotaped data eleven students have been chosen for a case study analysis. Students were chosen to represent the range of the population that is included in the study. Videotapes of students selected for the case study have been transcribed, coded, and analyzed for methods of problem solving, methods of justification and connections that were made. The written work of these students has also been coded and analyzed. In addition, selected videotaped segments of other students have been analyzed.

The study was conducted at a small community college in a liberal arts mathematics course. The students spent approximately half of all class time working on various non-routine problems in a small group setting. The Towers Problem was given during the ninth week of the semester. The students began by working on towers that were four cubes tall. They then were asked to consider towers that were five cubes tall. Several groups also worked with towers where three different color cubes were available. The Pizza Problem was given during the fourteenth week of the semester. The students first worked on finding the number of pizzas that could be made if four toppings were available followed by the problem with five toppings. After solving the basic problem they were asked to consider the Pizza With Halves problem in which a topping could be placed on either a whole pizza or a half pizza.

Results

Several of the college students noticed a relationship between the Towers Problem and the Pizza Problem. Three of these students were able to explain the isomorphism during the class session. Other students explained the isomorphism during task-based interviews. While working on the Pizza Problem most of the college students noticed the doubling pattern and used it to predict how many toppings would be needed to obtain one thousand twenty-four different combinations of pizza. The majority of these students also explained the reason for the doubling pattern that they had observed. Several students noticed the relationship between the pizza problem and the rows of Pascal's triangle and about half of these students were able to explain how the addition rule of Pascal's triangle related to pizzas. One student concurrently enrolled in a statistics course correctly used combinations to predict the number of pizzas. Several other students, who had previously learned about combinations, tried to calculate the number of pizzas using combinations. However, they did not understand combinations well enough to apply them to the problem correctly. These students attempted to do a single calculation to find the answer instead of doing separate calculations for each number of toppings that they could have on the pizza (Glass, 2001).

Conclusions and Implications

The students in this study made connections and built isomorphisms among the problem solving tasks Very few connections were made with previous mathematical knowledge acquired in a traditional lecture class, and most of these were trivial connections. Several students recognized that the problems could be solved by permutations or combinations, but demonstrated that they did not understand these concepts well enough to correctly apply them to the solution of novel problems. This would indicate that learning about mathematics in an atmosphere in which students are told what to do does not enable these students to develop genuine understanding. The students in this study demonstrated a high level of reasoning as they thought about the problems, justified their solutions, and made connections. They utilized strategies for solution and methods of justification and made connections similar to those of the students in the Rutgers University longitudinal study. While it cannot be disputed that the students in the longitudinal study benefited from exposure to rich mathematical experiences over an extended period of time, the students in this study, who had previously experienced a variety of traditional mathematics instruction, demonstrated that it is not too late to introduce rich mathematical experiences in a collegiate level mathematics class. The level of reasoning that these students demonstrated provides evidence that it is possible to experience thoughtful mathematics within a traditional fifteen-week semester. The findings support the importance of introducing rich problems to college students and giving them opportunities to work together toward a solution.

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FUNCTIONS VISUALIZED BY GRAPHS AND GRAPHS INTERPRETED AS FUNCTIONS

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This research, based on the APOS Theory (Action-Process-Object-Schema) theoretical perspective, addresses the following questions: How can we describe the mental constructions that a student might make in order to develop his understanding of: (1) the concept of a Cartesian graph of a real function of one variable, given by an algebraic rule. (2) the concept of a real function defined by a vertical line test satisfying curve drawn in the Cartesian plane. During the presentation of this work we shall describe our theoretical analysis of the learning of these concepts articulating it with empirical data we have collected.

The function concept is one of the most fundamental notions developed in high school and beginning college mathematics courses. At the same time the teaching and learning of this concept have been considered most problematic. Evidence of the challenging aspects of its learning is amply supplied by the mathematics education literature. The question of how the function concept can be constructed by a student has been addressed by a few researchers, although issues related specifically to the graphical representation of functions in its cognitive construction facets have not been so much emphasized. When surveying the literature, we found that most of the work done in the area of graphical representation has focused on the consequences (performance, misconceptions) of the inner constructions already made by students. Much is still needed in the way of clarifying the nature of the mental constructions underlying the learning of the production of graphs of functions given by a formula and the inner construction of a function given graphically.

This study acquires its relevance from its connecting points with the function literature and from its focused line of interest in graphical representation. The use of this representation as a supporting intuitive foundation in the introduction of definitions, properties, theorems and algorithms in mathematics, in particular in Calculus, has been debated for efficiency. What are students really "seeing" when we rely on graphs to introduce ideas of limits, continuity, derivatives and integrals? Moreover, the mental construction of functions given only by a graph seems fundamental by its relation to frequently mentioned problems related to the development of the function concept. When dealing with functions given only by graphs students have to confront their bias toward the presence of a formula and they also confront their desire for regularity of graphs or causality of functional correspondences. We also consider this environment a fruitful grounding to develop the symbolic language of mathematical functions.

The question we address in this research is the following: How can we describe the mental constructions that a student might make in order to develop his understanding of: (1) the concept of a Cartesian graph of a real function of one variable, given by an algebraic rule. (2) the concept of a real function defined by a vertical line test satisfying curve drawn in the Cartesian plane (no given formula). Furthermore, we would like to enhance our understanding of how these constructions might be made by the learner, which might be the difficulties he finds in this process and which sort of activities might potentially engender these constructions.

This research is based on the theoretical perspective known as APOS Theory (Action-Process-Object-Schema). According to this theory, understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, processes and objects can be organized in schemas (Asiala et al., 1996).

This work was conducted in Rio de Janeiro, Brazil, and its participants comprised 115 first semester college students enrolled in a course entitled Introduction to Calculus and 15 high school mathematics teachers registered on in-service activities. The research was carried on according to the following methodology: as a starting step we made an initial theoretical analysis in order to propose a genetic decomposition, or model of cognition, of each concept under study; that is, a description of the mental constructions (actions, processes and objects) a student might make in order to develop his understanding of each one. This preliminary genetic decomposition was used to study students´ and teachers´ understanding of the same concepts through the analysis of their responses to a written inventory (set of problems) with follow-up taped interviews. The collected data was analyzed by a careful articulation with our initial genetic decomposition and relevant literature. Finally the result of this analysis led to the formulation of a revised version of our initial genetic decomposition to reflect the empirical data. This new version of our model of cognition of the two concepts will be exposed at the presentation of this work when we shall also exhibit subjects´ written and oral work. In addition, we will describe our inventory problems, discuss their adequacy to elicit appropriate data for this investigation, make suggestions for pedagogical strategies and point out future studies.

We can easily see that there is no uniformity in graph representation in books and classrooms, and although some may be more suitable to support certain reasoning, students do not seem to move easily among them. Even in relation to a simple thing as representing a point belonging to a graph, it appears that there is a cognitive difference between conceptualizing the representation of a point of a function graph as the crossing of two lines or as the end point of a vertical segment. The former seems to be more related to an action conception and the latter to a process one.

The idea that to construct a graph of a function it is enough to know some of its values is frequently observed. This misconception is firmly rooted and hard to modify. When high school students are introduced to graphs of affine and quadratic functions they are told that these graphs consist of lines and parabolas, often with no justification. And what is worse and more problematic, teachers may think that there is no need for justification of these assertions which they consider self-evident. Many teachers do not seem aware that when they draw some graphs they choose a "good" table and join the points using what they already know about the function behavior. Beginning students have no way to discriminate between bad or good tables and no reason to join the points the "right" way.

The interpretation of the construction of graphs of f(ax + b), knowing the graph of f, as a result of an action on the graph of f leads to lots of misunderstandings by the students. The necessary discrimination between "shifts – distance preserving – rigid curve" and "stretch – not distance preserving – rubber band" is hard even to learn only "how to do" those sort of tasks. The understanding of these constructions, one that goes beyond strategic memorization, seems to require a difficult coordination among actions and processes in different settings - numerical, graphical, symbolic, tabular and algebraic - and a conceptualization of the graph as a set of points.

APOS Theory was very useful to reason about, explain and synthesize the observable behaviors of the subjects involved in this investigation as they performed the proposed tasks.

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GENERALIZATION PROCESSES IN COMBINATORIAL PROBLEM-SOLVING SITUATIONS

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The process of generalization is an important component of mathematical ability, and to develop this ability is an objective of mathematics teaching and learning. In this research study problem-solving strategies and behaviors of nine high school freshmen in combinatorial problem-solving items were documented with the purpose of identifying strategies and behaviors that fostered the formation of generalizations. Frank Lester's (1985) problem-solving model was adapted in order to classify student problem-solving behaviors. The generality that characterized the problem-solving items was the pigeonhole principle. Four of the nine students were successful in discovering the pigeonhole principle. The findings indicated that students who were able to generalize focused on understanding the structure of a given problem as well as engaged in reflective abstraction, in addition to being motivated and highly curious.

Introduction

The National Council of Teachers of Mathematics (NCTM, 2000) calls for mathematics instructional programs that allow students to become sophisticated with mathematical processes like problem solving, representation and reasoning as well as opportunities to reflect on and monitor their work, which lead to greater abstraction and a capability for generalization. Mathematics education research on generalization has focused on elementary school children's abilities to generalize number concepts (Dienes, 1961; Davydov, 1990), and generalizations made by students in arithmetic and algebra (Krutetskii, 1976). There is lack of research on generalization in the context of problem solving at the high school level. As a practicing high school teacher, the focus of the researcher was on understanding the nature of generalization processes used by high school students in problem-solving situations. The particular research questions were as follows:

1. What are the mathematical experiences that enable students to generalize?
2. When engaged in problem solving, how do students discover generalizations that characterize a class of seemingly different problem situations?
3. How can high school teachers create mathematical experiences that foster abstraction and generalization?

Methodology

Students in a ninth grade algebra class taught by the researcher were assigned five non-routine combinatorial problems in their journals, at increasing level of complexity. The pigeonhole principle could be applied to solve each of the five problems. The researcher hoped that some of the students would eventually be able to discover this general principle by discerning similarities in the structure of the problems and their solutions. The data for this study was collected through students' journal writings, clinical interviews and the teachers' journal writings. The interviews were open-ended with the purpose of getting students to elicit their thought processes in solving a given problem. In all, five rounds of interviews were conducted with the nine students over a three month period. The researcher/teacher documented the evolving strategies of the students and classified them according to Lester's (1985) problem solving model, which consists of four categories. *Orientation* refers to strategic behavior to assess and understand a problem. It includes comprehension strategies, analysis of information, initial and subsequent representation, and assessment of level of difficulty and chance of success. *Organization* refers to identification of goals, global planning, and local planning. The category of *execution* refers to regulation of behavior to conform to plans. It includes performance of local actions, monitoring progress and consistency of local plans, and trade-off decisions (speed vs. accuracy). Finally, *verification* consists of evaluating decisions made and evaluating the outcomes of the executed plans.

Data gathered through journal writings, open-ended interviews and classroom observations was analyzed using techniques from grounded theory (Glaser & Strauss, 1977). Lester's (1985) model was operationalized in order to code data and to identify variables related to successful and unsuccessful generalization strategies.

Findings and Implications

Four students were successful in discovering, verbalizing, and in one case successfully applying the generality that characterized the solutions of the five problems, whereas five students were unable to discover the hidden generality. Two new categories emerged as the result of adapting Lester's (1985) model, namely *generalization*, and *reflection*. *Generalization* was characterized as the process via which students identified commonalties in the structure of the problems and their solutions. It included making analogies as well as specializing from a given set of objects to a smaller one. *Reflection* in this study was characterized as the process by which the student abstracted knowledge from actions performed on the problems. In other words reflection consisted of thinking about similarities in the problems and solutions, and abstracting these similarities over an extended time period.

The results indicate that generalization is a complex process and not very many students are able to discover and formulate them on their own. The students that were successful showed a high level of reflection in addition to concern with orientation and organization in a problem-solving situation. They displayed conceptual understanding because they were able to abstract similarities through reflection (Dubinsky, 1991) and form valid conceptual links (Hiebert, 1986; Skemp,1986). Affect played a role in how they approached a problem situation. In particular their beliefs about what constituted mathematics influenced how they tackled a given problem. This leads to the question of whether generalizations can be taught? Can beliefs be changed? Can reflection be taught?

The first and foremost challenge for high school teachers is to find various classes of problems which have general solutions, that are accessible to the students and which capture their interest. Hence, the pedagogical focus of the teacher should be on understanding the mathematical structure of the problem as opposed to simply solving it. The study indicated that one of the variables that contributed to students being unsuccessful in forming generalizations was the obsession to execute, verify (often incorrectly) without having an understanding of the underlying assumptions (orientation) and poor global planning (organization). The point here is not to simply pose problems for problem solving, instead the problems should be used to lead students to think mathematically, to engage all of the students in the class in making and testing mathematical hypotheses similar to what mathematicians do.

The results indicated that reflective behavior on the part of the student contributed to forming generalizations. Many of the students who were unsuccessful in forming generalizations claimed to have "looked back (reflected)" by checking that they had utilized all the information given in a problem. In spite of this they either formed false or no generalizations. Many of these students had been "taught" Polya's (1945) four-step approach to problem solving at some point in the middle school curriculum in a manner akin to learning a procedure. However there is a big distinction between the "looking back (reflecting)" to make sure that the given information was somehow utilized in the solution, as opposed to reflecting to abstract similarities in the underlying structure of the problems and solutions. Such reflective behavior might be very hard to teach. Interestingly enough the students who engaged in a great deal of reflective behavior often characterized mathematics as "a way of thinking." So one possible way of making students engage in reflective behavior is by first trying to impact student beliefs about mathematics.

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Professional Development/ Inservice Teacher Education

UNDERSTANDING NOVICE TEACHERS: CONTRASTING CASES

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This study examined the classroom practice and beliefs of two novice teachers, Anne and Rachel, during the first year of their involvement in Project PRIME, a district-wide development program. Using accounts of practice (Simon & Tzur, 1999), the professional developer interviewed and observed the two novice teachers throughout the school year and established a hypothetical learning trajectory as part of their professional development. By the end of the first year, neither teacher's classroom practice reflected the goals of PRIME: using worthwhile mathematical tasks, questioning and promoting student's thinking. However, their practice was observably different and so were their beliefs about teaching. Anne's practice was consistent with the literature's characterization of a novice teacher, while Rachel's aligned more with that of a veteran teacher. These differences between these novice teachers proved to be an-going challenge for professional development.

A number of studies have addressed the beliefs and classroom practices of novice teachers (Borko et al., 1992; Leinhardt, 1989; Leinhardt & Greeno, 1986; Shealy, 1995; Shealy, 1994; Sherin & Drake, 2000). These studies define *novice* teacher as one with less than 3 years of teaching experience and one whose teaching tends to focus on "survival" (Miles & Huberman, 1994) and establishing basic classroom routines (Sherin & Drake, 2000). Most of the abovementioned studies have focused on novice teachers who have come directly from high school into pre-service teacher education programs. Hence, these studies may have produced characterizations of novice teachers that do not necessarily hold up for more mature-aged novice teachers. In this research, we bring a different perspective by comparing two novice teachers who brought contrasting experiences to their early years of teaching. More specifically, our study asked the following question: What were the beliefs and classroom mathematical practices of two novice elementary teachers during the first year of a professional development program?

Theoretical Orientation

The professional development program that served as the context for this study was Project PRIME (Thornton & Barrett, 2000), a systemic effort that focuses on enhancing the practice of elementary teachers working in a mid-size urban school district with a high minority student population. The key elements of PRIME are improving teachers' pedagogy and providing classroom-based support to facilitate implementation of a reform-based curriculum (TERC, 1998). More specifically we aim at enhancing teachers' development of three integrated instructional strategies: (a) posing worthwhile mathematical tasks, (b) asking responsive questions, and (c) listening to students' responses and promoting their thinking and engagement.

Accounts of practice (Simon & Tzur, 1999) provided the theoretical infrastructure for the case-study analysis of the two novice teachers in this study. Simon and Tzur describe accounts of practice as an approach for understanding teachers' current practice and as a means of viewing their current practice in the context of professional development programs that embrace envisioned reforms. Research that uses accounts of practice has two key elements: (a) the development of a conceptual frame, and (b) the use of the conceptual frame to trace teachers' classroom practice. The conceptual frame should be based on research in mathematics education and the particular perspectives and concerns of the researchers as they relate to the professional development project. The conceptual frame or lens in our study was based on PRIME'S three instructional strategies. We used these strategies as a lens to observe teachers' practice and beliefs. Using accounts of practice also provided an opportunity to ascertain why the teachers taught as they did.

The multifaceted nature of the accounts of practice methodology made it very appropriate for driving the collection of data to address the research question for this study. More specifically, we were able to use the conceptual frame to trace and interpret the teachers' beliefs and classroom mathematical practices captured through videotape and interview.

Method and Setting

The design for this study was in accord with the "accounts of practice" methodology (Simon & Tzur, 1999). We used case-study analysis to examine the beliefs and classroom practices of two novice teachers during the first year of PRIME.

Participants

The two novice teachers in this study were part of a pool of 337 teachers who agreed to participate in the 3 years of PRIME. These novice teachers, identified by pseudonyms, were also part of a random sample of 16 teachers selected for detailed case-study analysis. Both novice teachers were teaching Grade 1 classes. *Rachel,* a novice teacher in her first year of teaching, came into teacher education after having 20 years in the workforce. She taught for 1 year as a full-time teacher-aide prior to obtaining a teaching position. *Anne* had entered her teacher education program immediately following high school and was in her first year of teaching.

Procedure

During the PRIME summer workshop, prior to the first year of the program, the two teachers undertook a 1-week program that focused on ways that geometric concepts can be built using tasks from *Investigations in Number, Data, and Space* (TERC, 1998). These TERC materials used by these two teachers were intended to support and enhance their mathematics curriculum in the coming years.

Working in collaboration with the second-named author, the first-named author wrote accounts of the teachers' classroom practice and beliefs during the fall and spring semesters. These accounts served as the basis for generating hypothetical learning trajectories (HLTs); that is, determining a plan, identifying activities, and conjecturing the way the teacher development process might go. We engaged teachers in a sequence of activities intended to highlight the relationship between mathematical tasks (like those contained in the TERC materials) and children's subsequent understandings.

Data Sources and Analysis

Data were gathered from four sources: (a) video tapes of two teaching sessions for each teacher, one in the fall and one in the spring; (b) detailed field notes of six teaching sessions during the intervention; (c) field notes of interviews associated with the six teaching sessions in each semester (two groups of three consecutive days); and (d) samples of students' work. Using a modification of Miles and Huberman' three-part analysis (1994), we used a double-coding procedure to analyze the video and field data in terms of the conceptual frame: PRIME's three integrated instructional strategies. The use of four sources of data and the generation of independent summaries and codes allowed for triangulation of the data in the sense that it enabled both confirming and alternative interpretations.

Results

Classroom Practices

Using the PRIME instructional strategies as a lens, the classroom practices of the two target teachers were analyzed during the first year of the program. The thick descriptors of Anne and Rachel's classroom practice are presented in Table 1.

With respect to *worthwhile mathematical tasks*, the outward manifestation of these two teachers' practice was similar: their students did not experience the TERC tasks in the way they were intended. Both teachers reduced the tasks to a sequence of simpler tasks. However, the teachers worked in different ways.

Anne gave her students' the opportunity to solve the problem but when the children reported she either left them hanging without feedback or led them inextricably to her solution. For example, near the end of the first year of the project, she posed this question: "Draw a cake in the shape of a rectangle and indicate how you and I could share it." The task was meaningful to the children, they seemed to understand that share implied *divide equally*, and the task had the potential to build understanding of mathematical concepts as well as to make connections between number and space. However, as we will see below, Anne did not use the children's responses to optimize the potential for the task.

Table 1. Classroom Practices

PRIME Strategies	Anne	Rachel
Worthwhile mathematical tasks	She is willing to let the children engage in thinking about worthwhile tasks, but lacks the confidence to implement her intent.	She demonstrates a steadfast resolve to reduce any worthwhile task to a procedural task that she can model for the children.
Questioning	Asks children to describe their solutions but rather than delve into their thinking leads them to her predetermined answer.	Her questions focus on checking that children use the correct procedure; she often seeks a response from the whole class to reinforce the procedure.
Listening and promoting students' thinking	She allows them to express their ideas; however, she does not feel comfortable analyzing and extending children's thinking	She assesses responses according to whether or not they follow her model; seldom seeks explanations as she sees the role of elaborating responses as her prerogative.

By way of contrast, Rachel broke the task into subtasks or more accurately subroutines that reduced the cognitive load of the problem. She often reminded the children of some of the procedures that had previously been established for solving problems like the one she was about to give. For example, near the end of the first year of the project she asked the students, "Which is longer than 2 inches, your fingernail or your hand?" Rather than let the children explore or even think about the task, Rachel immediately told them to look at their ruler, find 2 inches, and compare. Consequently the task was immediately reduced to a set of predetermined procedures that were not necessarily part of the children's mathematics.

With respect to *questioning*, Anne reflected what Brousseau (1992) calls the *Jourdain effect* (p. 20). By the Jourdain effect, Brousseau refers to a situation where the teacher avoids having a discussion about knowledge with the student and instead recognizes a response of the students that has been stripped of meaning. In our situation, Anne started with open questions, encouraged students to convey their own meaning, but then avoided any follow-up discussion on the children's knowledge. The following dialogue in the cake-sharing example typifies Anne's Jourdain approach.

Anne: How could the cake be shared between you and me? Do it in you own way and be ready to tell me about it.

[Anne walked around while the students solved the problem, but did not question any students. Her only interaction was to remind one student that a rectangle was long "like a square but long."]

Anne [After about 3 minutes]: What are you looking for in this problem?

Children [In unison]: Two equal parts

Anne: And equal parts mean they're the...

Children [In unison]: Same

Anne: So, you cut it down the middle.

In spite of the fact that students had valid solutions with horizontal cuts, vertical cuts, and diagonal cuts, Anne did not pick up on these solutions in the dialogue. Even though she must have seen them she simply asked the students questions about equal parts when it was obvious that many of them had gone well beyond that level of understanding and had rich solutions to share.

By way of contrast, Rachel epitomized Brousseau's (1992) *Topaze-style*. That is, she used an instructional approach in which the students' response to a question is determined in advance and the teacher negotiates the conditions under which the response is produced. The following dialogue for the fingernail–hand problem was typical of her questioning:

Rachel: Who can tell me? Which is longer than 2 inches, your fingernail or your hand?

Student1: Your fingernail.

Rachel: What do you think? [pointing to another child]

Student 2: Your hand.

Rachel: Your hand, that is correct. Why?

Rachel [without waiting for the children]: Because how big is an inch? Who can show me an inch?

Many children held their hands a foot apart, and were individually corrected. However, by now the original question had become submerged, marooned by a sea of subroutines. In essence, Rachel's questions were not predicated on doing problem solving but on checking the various subroutines she had prescribed in advance.

With respect to *listening to and promoting students' thinking*, both teachers ultimately assessed children's responses against their own established standard. Using a Jourdain approach, Anne encouraged children to generate different representations but did not feel comfortable asking the children for explanations especially when a child was going in a direction different from the teachers' own solution. Rachel's Topaze-like strategy was clear from the outset: if the child's solution was correct (fitted her solution), she elaborated and explicated the student's solution; if the child's solution was incorrect, Rachel asked another child or represented her model without further discourse with the child.

In an overall sense Anne's classroom practice was often different from what she intended; that is, it was different from her professed intent to meet PRIME goals. By way of contrast Rachel's practice was consistent with "her" intent and she had already established the routines to meet her goals. Notwithstanding these differences, the end product of their classroom practice was much the same and was clearly inconsistent with the direction of the PRIME Project and the intent of the TERC (1998) resources.

Beliefs

The teachers' beliefs were identified during interviews with the first researcher. These interviews were undertaken as part of the accounts of practice methodology. Some of these beliefs arose in response to questions; others were volunteered by the teachers during discussion. The key beliefs identified for Anne and Rachel are listed in Table 2.

The beliefs of Anne and Rachel provide the essence of their classroom practice. Anne understands the key tenets of PRIME and is able to articulate them. She believes that children will produce different mathematical representations for problems and she thinks these representations may be helpful for instruction. However, she professes a lack of confidence in her own mathematics and this is reflected in her beliefs about children's mathematical thinking and her reluctance to use children's solutions to enable sense-making and the building of mathematical knowledge. By way of contrast, Rachel's beliefs seem to be more firmly held and her beliefs are consonant with her classroom practice. She is quite definite that children can't act mathematically on their own and as a consequence teachers must demonstrate the correct mathematical procedure and then ensure that children adopt the teacher's procedure. Rachel is steadfast in her belief that learning for "important" written tests and chapter tests will not occur through any other mechanism.

Discussion and Conclusions

This study examined the classroom practice and beliefs of two novice teachers during their first year of teaching and the first year of a professional development

Table 2. Beliefs

Anne	Rachel
Children will use different representations that may be helpful for instruction.	Children are unable to do mathematics independently of the teacher.
I have trouble constructing mathematical ideas; hence, my students will not be able to construct mathematical ideas.	Children learn math when I demonstrate the procedure that are to imitate.
Children will produce different solutions and representations but the textbook solution is the ultimate authority.	Children should produce the teachers' model solution; multiple solutions are likely to confuse the children.

program that had been adopted by their school district. The end product of each novice teacher's classroom practice was similar but they reached these states in different ways. From the perspective of the first two PRIME goals, Anne's classroom practice showed potential in that her mathematical tasks were often worthwhile and her initial questions opened up children's thinking. However, she lacked the confidence to use students' thinking to engage them in mathematical sense-making. In essence, Anne's practice was in direct contrast with her espoused beliefs that were consistent with PRIME instructional goals. In an overall sense, Anne's classroom teaching reflected that of a *novice* teacher in that her actions were consistent with the literature on novice teachers: specific goals not carried through (Sherin & Drake, 2000); frequent confusion caused by mis-sent signals (Leinhardt, 1989); struggles to listen to children's thinking (Fennema & Franke, 1992); dissonance between beliefs and practice (Cooney & Shealy, 1997); and lacking confidence in her own mathematics (Ball, 1990).

By way of contrast, Rachel's practice was deliberate and she was completely in control of an approach that seemed to be built on well-established routines, albeit routines that were not consistent with PRIME or reform directions in mathematics teaching. Rachel consistently reduced the cognitive load of worthwhile mathematical tasks, used Topaze-leading (Brousseau, 1992) when she questioned students, and assessed students' responses according to their fit with her own predetermined norm. Nevertheless, Rachel's classroom teaching tended to reflect that of a *veteran* teacher in that many of her actions were consistent with the veteran teachers literature: transparent system of goals (Leinhardt, 1989); detailed agenda and well established routines (Leinhardt & Greeno, 1986); minimal student confusion about what was required of them (Leinhardt, 1989); and beliefs that were consonant with practice (Cooney & Shealy, 1997). Consequently, even though Anne and Rachel were both *novice* teachers they brought very different beliefs and actions to instruction, and these differences proved to be an-going challenge for the professional developer.

In accord with the ongoing cycle associated with *accounts of practice* (Simon & Tzur, 1999), we are refining the hypothetical learning trajectory for both teachers as we move into the second year of PRIME. Although we still intend to have both teachers focus on children's learning as their key goal, the activities and conjectured learning processes for each will be different. For Anne, the thinking of the students was distracting, confusing, and even anxiety building. Hence, we believe that she needs to be guided to study and discuss pertinent examples from her students' solutions, statements and questions. For Rachel, there is no evidence that she believes children are capable of building their own mathematical solutions. Hence we are intending to video tape the PD working with her class on open-ended problems that produce samples of children's work. Rachel and the PD will observe these videotapes together and analyze the children's work before any further attempt is made to have her adopt an approach to teaching that promotes mathematical sense making.

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LESSON STUDY IN SECONDARY MATHEMATICS

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In this research report we share results of a study of the use of an adaptation of Lesson Study, a form of professional development typically used in Japanese elementary schools, with secondary mathematics teachers in the United States. We draw on Tharp and Gallimore's (1998) framework of "assisted performance" to present our analysis of the activities of a lesson study group (LSG). We describe the activities of the LSG and then use five characteristics of good assistance activities as an organizational frame for an analysis of the quality of the activities: mutual respect and trust; inter-subjectivity; responsiveness; joint projective activity; and reciprocity (Brown, Stein, & Forman, 1996). We found that these five characteristics were present in the activities of the LSG as participants planned, taught, and reflected on lessons intended to help students understand the process of finding sums of infinite geometric series. We provide examples of how the five characteristics interacted to make this a powerful professional development activity for not only the two teachers but also other participants in the LSG.

Many teachers and providers of professional development in the U. S. have become interested in Lesson Study, a form of professional development typically used in Japanese elementary schools (Stigler & Hiebert, 1998). In Japan, *research lesson* or *study lesson* refers to lessons that teachers jointly plan, observe and discuss. The same two words in the reverse order, *lesson research* or *lesson study*, refer to an instructional improvement process of which the research lesson is the heart (Lewis, 2000). When a Japanese school engages in *Lesson Study*, teachers form *Lesson Study Groups* (LSGs). (See Lewis, 2000 for an extended discussion of LSGs and the process of Lesson Study.)

In this research report, we share some results of the use of an adaptation of Lesson Study with secondary mathematics teachers. The research was conducted as part of a professional development project entitled Collaboration for Enhancing Mathematics Instruction (CEMI), funded by Lucent Technologies Foundation. CEMI lesson studies are conducted in LSGs consisting of middle and high school mathematics teachers, university mathematicians, university mathematics educators, and pre-service secondary mathematics teachers. The CEMI participants have met as a large group and in their smaller LSGs since all 2000. We report on the activities of one LSG during the spring of 2001. The CEMI project is not trying simply to engage U. S. secondary mathematics teachers in Japanese lesson studies, but rather to adapt this model of professional development for several purposes. These include providing professional development for all of the participants in the project and creating an extensive community of people with diverse perspectives but the common goal of providing secondary students with quality mathematics education. The research component of the project seeks to understand these activities and their impact on the participants and the participants' classroom teaching.

This research report presents an analysis of the activities of one lesson study group in which two experienced teachers, Ms. JP and Mr. KB participated. Using aspects of Tharp and Gallimore's (1998) model of learning as movement from assisted performance to unassisted performance through a Zone of Proximal Development (ZPD), we analyze the activity settings of the lesson study group in which the two teachers were engaged and the characteristics of those activity settings.

The Framework of Assistance

In *Rousing Minds to Life: Teaching, Learning, and Schooling in Social Context*, Tharp and Gallimore (1998) propose a theory of teaching that they characterize as "teaching as assisted performance." Anchored in Vygotsky's theory of the ZPD and also in the concept that formal learning/teaching should resemble informal learning/teaching, Tharp and Gallimore's (re)definition of teaching offers teachers and educational researchers a lens through which to view teaching and teacher learning. As part of the framework, Tharp and Gallimore propose that learning occurs in *activity settings* and that those settings can form units of analysis that support deep understanding of the social context of assisted performance. They theorize that assisted performance does not occur randomly, nor do the activity settings that support assisted performance. Instead, they argue, assisted performance only occurs in the context of goal-directed action.

The Five W's

Attending to the "famous five W's can assist us here as an outline for considering the interlocked dimensions of activity settings" (Tharp & Gallimore, p. 74). A brief discussion of the 'who,' 'what,' 'when,' 'where,' and 'why' of activity settings is presented here.

In analysis of activity settings, the 'who' refers to those people who are present. Tharp and Gallimore propose that those people are not there by accident, and that each person contributes uniquely to the activity setting. Analysis of the 'who' of an activity setting serves to help define the setting as well as assisting in an explanation of possible differences between settings.

Similarly, the 'what' of activity settings further helps to define the setting. Collecting data pertaining to the 'what' of activity settings and the resulting analysis of those data should include two dimensions: operations (what is done) and scripts (how the operations are done).

Because "activity settings cannot exist without time" (p. 76), the 'when' of those settings is important to include in any analysis. Are the activity settings scheduled at a time conducive to assisting performance? Is enough time allotted for assisting performance? How do the participants utilize the time they have in activity settings?

The 'where' of a particular activity setting deepens our understanding of a specific activity setting on a holistic level, as well as providing data that could be used to support differences between settings. Is the space appropriate and conducive to learning? Are there sufficient resources available?

While the first four of the "famous five Ws" provide contextual data about activity settings, the 'why' of the activity setting is perhaps the most important aspect of analysis as it is the least controllable aspect of any activity setting. Tharp and Gallimore argue that there are two facets of 'why' that must be attended to: the *motivation* and the *meaning*. What motivates the 'who' to be involved? What meaning and understanding of the activity do the participants hold?

Tharp and Gallimore argue that utilizing a combination of these five components during analysis allows us to 'know' an activity setting on a deeper level. Further, they argue, if we wish to design "a school organization in which assisted performance occurs at all levels, *the task is to create activity settings*" (p. 80; italics in original). We argue that the same condition holds for organizing professional development experiences.

Characteristics of Assistance

The assistance provided during activity settings can be analyzed further. The following five characteristics of good assistance activities are suggested by Tharp and Gallimore's work and provide an organizational frame for our analysis: mutual respect and trust; intersubjectivity; responsiveness; joint projective activity; and reciprocity (Brown, Stein, & Forman, 1996). These are briefly described below.

Mutual Respect and Trust

The establishment of mutual respect and trust is an essential component of any assistance relationship (Hamann, 1992). That is, both the assistor and the assisted must respect each other and trust that each has good intentions. For teachers to make their teaching public for comment, as happens in a lesson study, respect and trust are paramount.

Intersubjectivity

Equally important is that the members of the LSG achieve some measure of intersubjectivity. That is, the development of common understanding of the purposes and meanings of the activity and the language to use in it are critical. Also, the joint engagement in cognitive strategies and problem solving are aspects of interaction that influence each participant in an assistance activity.

Responsiveness

"Assistance is best when it is contingent on and responsive to learners' level of proficiency and perceived need" (Brown et al., 1996, p. 70). Within the activities of an LSG, both the topic of the work must be seen as important and the assistance provided must take into account both the mathematical and pedagogical proficiencies of the participants. Because of the diversity of the LSG members' experiences and needs, this characteristic may be particularly difficult to achieve.

Joint Productive Activity

"It is also important that assistance be provided in the context of joint productive activity" (Brown et al., 1996, p. 71) which involves working together on an authentic task that is directed toward the accomplishment of a clear, jointly constructed and recognized outcome. The characteristic of having a specified and valued product toward which the members collaboratively work seems to be somewhat rare in the work of educators.

Reciprocity

"The joint productive nature of good assistance increases the possibility that the assistance activity might have another necessary characteristic, reciprocity" (Brown et al., 1996, p. 71). Each member of an LSG should benefit from the group's activity in some way and no member should feel they are working but not getting value from the work. This is in contrast to many professional development activities in which the teachers are the receivers of something they don't value from an outside expert who does not expect to benefit from the activities (other than monetarily).

Context and Methodology

The data used for this report were collected in the spring semester of 2001 and are data on one of three Lesson Study Groups active during that semester. We will call this LSG1. In February 2001, all the teachers and mathematics educators participating in the CEMI project met to decide on an overarching theme for the semester's lesson studies. It was determined that every group would try to pay particular attention to those aspects of teaching and learning emphasized in the Connection and Communication standards of the National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics* (NCTM, 2000). Each LSG was to determine the topic of its lesson study and the scheduling of both LSG meetings and the teaching. The members of LSG1 were two in-service secondary mathematics teachers, Mr. KB and Ms JP; two pre-service secondary mathematics teachers, Ms EB and Ms JD; one university mathematician, Dr. CY; one mathematics education profes-

Table 1

Activity	Date	Data
1st LSG1 planning meeting	March 21, 2001	Audio tape, artifacts
2nd LSG1 planning meeting	March 28, 2001	Audio tape, artifacts
3rd LSG1 planning meeting	April 2, 2001	Audio tape, artifacts
4th LSG1 planning meeting	April 11, 2001	none
1st lesson implementation by Mr. KB	April 18, 2001	Audio and video tape, artifacts
Reflection/revision of the first implementation	April 18, 2001	Audio and video tape, artifacts
2nd lesson implementation by Mr. KB	April 19, 2001	Audio and video tape, artifacts
Reflection/revision of the second implementation	April 19, 2001	Audio and video tape, artifacts
Lesson implementation by Ms JP	April 20, 2001	Audio and video tape, artifacts
Reflection/revision of the implementation	April 20, 2001	Audio and video tape, artifacts
Sharing/Debriefing with other CEMI LSGs	May 7, 2001	Audio and video tape, artifacts
Planning for and Presenting at a state conference	October, 2001	PowerPoint presentation, artifacts
Online communication between members using the Inquiry Learning Forum (ILF) website	Throughout	Electronic record

sor, Dr. PK; and two mathematics education doctoral students, Ms AK and Mr. YK. LSG1 members engaged in the following activities:

Audiotapes were transcribed and transcripts were verified for accuracy, often by using the related video tapes. Data were divided into four subsets corresponding to four subsets of activities that we identified: planning (first four activities); implementation and reflection in Mr. KB's classes (next four activities); implementation and reflection in Ms. JP's class (next two activities); and sharing LSG1 work with others (next two activities). The ILF data contributed to each of the four subsets. One level of analysis consisted of coding each of the four subsets of data for information that would allow us to describe each activity using the "five W's" as Tharp and Gallimore suggest. Another level of analysis involved coding each subset of data using the five characteristics of good assistance activities: mutual respect and trust; intersubjectivity; responsiveness; joint projective activity; and reciprocity to determine the extent to which each of these were present in the activities of LSG1. This coding work was divided among the authors. After the initial coding was completed, the authors met to discuss the results of the coding and to reach consensus on any pieces of data that seemed problematic. We then discussed the patterns we found in the data coded and agreed on general themes that are reported in this paper. There is much more that could be said about the activities of LSG1 during this brief period, but we lack the space here to report that.

The LSG Activities

We first use the "five W's" to describe the activities of LSG1. Our purpose here is to provide the reader with some understanding of the LSG's activities, but we can not be comprehensive in our description here. We begin by providing an overview of the lesson study cycle in which the group engaged and then proceed to describe each subset of activities in more detail. The members of LSG1 decided in their first planning meeting that they would develop a lesson or lessons that could be taught in the Algebra II classes of both Mr. KB and Ms. JP's. They met four times to plan these lessons, meeting in the classroom of either Mr. KB or Ms. JP so that they would have access to the materials available in those schools. Between planning meetings, members worked alone or in smaller groups and there was some correspondence and lesson plan writing done using a web site with space designed for that purpose, the Inquiry Learning Forum (www.ilf.crlt.indiana.edu). The lesson was first taught in Mr. KB's class and required two class 48-minute periods. Mr. KB taught and most of the group members observed. Immediately following the instruction, the group met to discuss the lesson and revise their plans. Ms. JP then taught the lesson in one 85-minute class period and the group met immediately after to discuss and revise the lesson. About two weeks after the lesson study concluded, all participants in the three LSGs met together in two 2-hour sessions to share the work of all three LSGs. About five months after teaching the lesson, Mr. KB and Ms. JP presented their lesson at a state meeting of mathematics teachers.

Planning

As noted earlier, the members of LSG1 were two in-service secondary mathematics teachers, Mr. KB and Ms. JP; two pre-service secondary mathematics teachers, Ms. EB and Ms. JD; one university mathematician, Dr. CY; one mathematics education professor, Dr. PK; and two mathematics education doctoral students, Ms. AK and Mr. YK. All were involved in the lesson planning process. The in-service teachers were teaching mathematics in different high schools. The pre-service teachers were enrolled in the secondary mathematics teacher education program at the local university. The mathematics educators and the university mathematician were from the same university. All the participants voluntarily participated in the lesson planning process. Although one of the mathematics educators and the mathematician were faculty members at the university, neither was an instructor of the two pre-service teachers at the time of the study. The LSG1 members met face-to-face four times to plan an Algebra 2 lesson on series and sequences in the spring semester of 2001. The planning meetings took place in the classrooms of the two in-service teachers so that all necessary materials were available. The face-to-face planning meetings were scheduled when most members could attend, at the end of the school day, around 3:30 pm. Each meeting took approximately one and a half hour. Between face-to-face meetings, members also communicated on-line through the Inquiry Learning Forum (ILF). The primary goal of the face-to-face and on line activities was to collaboratively plan classroom activities that would help students understand the formula for finding the sum of an infinite geometric series and be able to use it to solve problems. Although the LSG1 members brought very different experiences and knowledge to the effort, all were interested in improving mathematics teaching and learning.

Teaching in Mr. KB's and Ms. JP's Classes

Mr. KB, one of the in-service teachers, taught the planned lesson in two 48-minute class periods on two consecutive days. However, Ms JP, the other in-service teacher, implemented the planned lesson in one 85-minute class period. The three classes were taught on consecutive days of the same week, in the teachers' own classrooms. When Mr. KB and Ms. JP taught, most other members of LSG1 observed the class, but did not assist the teachers during the lesson. Not all LSG members could observe both teachers' lessons because other commitments. Most importantly however, Ms. JP and Mr. KB were able to observe each other. The motivation to teach the lessons in both teachers' classes was a desire to refine the activities as much as possible and to test the viability of the plan in classes on both traditional and block schedules. The teaching was viewed as an opportunity to see if the planned instruction worked as intended.

Reflections on the Lesson Implementations

The teacher who taught the lesson and the LSG1 members who observed met together immediately following each class period to reflect on the lesson and to discuss possible revisions to the plan. Much of the discussion focused on the accuracy of the mathematics and the understandings of the students. Reflection sessions took place in the teachers' classrooms and often the boards still contained work from the lesson and informed the discussions. The lesson plan was revised in ways agreed upon by the teacher and observers. LSG1 members wanted to use reflections on the experience of the taught/observed lesson to further refine their lesson plans and to better the chances of improved student learning.

Sharing with Other LSGs

When all three LSGs had completed the lesson study cycle of teaching, refining, re-teaching and refining again, all the members met together to share and discuss their LSG experiences. The meeting took place in a classroom in the School of Education at the university. Participants were seated at tables in a semi circle to facilitate discussion. Ms. JP and Mr. KB shared a short video segment of one of their lessons with entire group and reflected on the clip and the lesson as planned and as actually taught. During this meeting, others asked questions and provided feedback to the members of LSG1. The primary motivation for Ms. JP and Mr. KB seemed to be that they were quite proud of the lesson and eager to share it with their colleagues.

Presenting at a Professional Meeting

Mr. KB and Ms. JP decided to present the lesson developed by LSG1 at a state meeting of mathematics teachers (ICTM) that was held approximately five months after they taught the lesson. This was not an activity of all members of LSG1, but rather of the two in-service teachers alone. (We do not have data on their planning process.) The two teachers presented in a 1-hour session and one of the mathematics educators in LSG1 attended the presentation. There were approximately 50 others attending the session. The two teachers had felt very satisfied with the lesson they had taught in the lesson study cycle, particularly with the revisions made after the instruction, and had wanted to share it with other teachers in the state.

Characteristics of the Assistance Provided

Our examination of the data for evidence of characteristics of good assistance activities revealed considerable information about the activities in which members of LSG1participated and the potential for these activities to provide assistance to all of the members of the group. We paid attention to the fact that although the primary goal of the group's activities was to develop a lesson that could be used in the future by any teacher who might teach the topic, each member of LSG1 also came to the activities with another goal - to gain a deeper understanding of mathematics instruction that truly was planned with connections and communication in mind.

We now report on what we learned about the five characteristics: mutual respect and trust; intersubjectivity; responsiveness; joint projective activity; and reciprocity. Obviously again we cannot provide the depth of description that our analyses allow.

Mutual Respect and Trust

All the members of LSG1 except Ms. EB had participated in the CEMI project the previous semester and thus known each other before joining the group. All participants were permitted to choose in which LSG they would work, so the member of LSG1 were there by their choice, indicating some initial level of respect and trust. In their planning process, members determined that individuals or subgroups would complete various tasks and then the results would be brought back to the whole group. For example, in the first planning session, once a topic was decided, members agreed that they would work in groups of two or three to look for appropriate instructional tasks. Later, individuals were asked to work on the details of a particular part of the lesson plan.

At another level, there is evidence in the discourse that participants were carefully listening to each other and respecting the contributions of each member to the discussions that took place. Members would use each other's words and probe to be sure they understood the contribution a participant was attempting to make.

Intersubjectivity

Some of the time the LSG members spent together, particularly early in the planning, was used to create intersubjectivity, developing a common understanding of the purposes and meanings of their activities as a group and beginning to develop a language to use to talk together about their lesson. In the early planning sessions, for example, each member of the group described his or her background to help other group members understand what he or she might be expected to contribute and what informed that contribution. There is evidence that group members all felt that they were working toward a commonly understood goal and that the uses of terms such as "connections", "communication", "understanding" were common to all of them.

Responsiveness

Assistance activities are responsive to the extent they meet the learners' needs and provide assistance at the learners' level of proficiency. At a global level, the entire work of the lesson study group was a response by the group members' to their desire to learn to create lessons that are reflective of the NCTM's (2000) principles and standards. At a more individual level, we find examples of group members using another member's actions or words as a basis for some comment or suggestion. In the following example, Ms. AK and Mr. KB's assistance is responsive because it is based in Ms. JP's actions in class and their perceptions of possible problems with the mathematics.

Ms. AK: I have a question. These examples, you called them infinite series but they are finite series because you end up with an end, if you put dots after that, you might say "this is infinite series"

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+K+\left(\frac{1}{4}\right)^n=\frac{1}{3}$$

But we can say, "this is not actually equal to 1/3, it's getting closer to 1/3".

Mr. KB: As n approaches infinity.

Ms. JP: And that's another thing, . . . I haven't even put the infinite, I haven't tried, there's no infinity up there because we haven't used the symbol yet.

Ms. AK: And maybe 1/4, the first four terms and then you say it equals 1/3, maybe you could say "it approaches".

Ms. JP: It approaches

Mr. YK: Eventually it would be.

Mr. KB: You might ask "where's the missing part?"

Ms. JP: The missing part, I was wondering is it okay to say "the missing part is in that square"?

Ms AK: Yes.

Joint Productive Activity

There is considerable evidence that LSG1members felt and acted as though they were working together on an authentic task. Also there is evidence that their activities were directed toward the accomplishment of a clear, jointly constructed and recognized outcome, a good lesson and student learning. It was somewhat surprising to find the group members also had a secondary goal and that this influenced their planning and revising of the lesson. Members mentioned that they were working toward the "publication" of this lesson on the Inquiry Learning Forum website. This would involve sharing video segments, the lesson plan, student work, teacher reflections, and indicating the state and national standards the lesson addressed. This goal seemed to motivate the group members to even discuss anticipating student responses that did not occur, but might. The members seemed to be motivated by the task of creating a package of materials that would allow other teachers, who had less time to spend planning, to successfully use the lesson plan of the group.

Reciprocity

Our analysis reveals that at least some participants in LSG1 were both assisted and assisting. Clearly the pre-service teachers were assisted in developing their understanding of mathematics teaching and learning by engaging in discussions with Mr. KB and Ms. JP. When sharing their lesson with the entire group of LSG participants at the end of the semester, both in-service teachers in the LSG reported that they believed they would not have attempted the lesson they taught without the assistance of their LSG activities.

On a more micro level, in the following example from a discussion, Ms. JP assists Mr. KB (and those who may use the lesson plan) by suggesting a caution. Then Mr. KB becomes the assistor when he explains his method.

Mr. KB: If there is anything you could add, a working circle, an e-mail, we can write some down, that would be great. Because you know if somebody ends up using this, it is nice for us to provide them with anticipated questions or comments even.

Ms. JP: be careful when you go from $\frac{1}{r-1}$ to $\frac{a}{1-r}$.

Mr. KB: I did the a over 1-r first.

Ms. JP: I still, I want to stop and think about where I lost my flow, that, right there, stumped them.

Mr. KB: Yeah.

Ms. JP: is that the proof you are talking about?

Mr. KB: . . . No that's a substitution, that is not a proof (Mr. KB shows on the chalkboard). I substituted r right here, and right here, . . . and then I reduced that to a over $\frac{a}{1-r}$.

Reciprocity allows all participants in an activity to feel valuable and also supported.

Conclusions

The descriptions we provide here only hint at the richness of the activities in which the members of this lesson study group participated. For example, an analysis of the roles of the participants and the content of their contributions to the group will add important additional dimensions to our understanding of the lesson study activities as assistance activities. Also understanding the relationship of the lesson study activities to the culture of the schools and the university is critical. We are just beginning to unpack the wealth of information in our data and will continue the process of analyzing the activities of this and other lesson study groups in the CEMI project. Although in Japan lesson studies are primarily used in elementary schools as professional development activities, we argue that the process of a lesson study – planning, teaching, reflecting, revising, re-teaching, reflecting and revising – when engaged in seriously and over time, is a process that can assist any group of teachers to improve their instruction one or two lessons at a time.

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TEACHER LEARNING IN MATHEMATICS TEACHER STUDY GROUPS

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In this paper I explore the challenges and possibilities for teacher learning in study groups by examining the typical discourse generated in one particular group. Special attention was paid to the teachers' talk when doing mathematics together and when talking about their teaching practices and students' work. When doing mathematics together, the examined group's talk can be described as exploratory talk, that is talk characterized by speakers seeking and showing intellectual involvement; public disclosure of disagreements and confusion; and talk that is generative and collaborative. In contrast, the teachers' talk about their teaching and students' work can be categorized as expository talk, which can be characterized by the use of monologues; speakers seeking and giving approval; and the non-analytical or unproblematic narration of events. These are features of study group talk that seem important to attend to and study as these affect the participants' learning opportunities and impact the design and leadership of teacher groups.

Introduction

The notion of school-based study groups is a departure from traditional approaches to the professional development of teachers. Rather than attempting to "sell" the reform-minded ideas about mathematics teaching and learning, teacher study groups are designed as forums for critically examining and experimenting with standards-based mathematics teaching. This departure from traditional professional development activities stems from the negligible impact they have had on helping teachers realize the more ambitious goals of teaching and learning embedded in the mathematics reform documents. It has also grown out of the realization that professional development that seeks to engage teachers with "what is hard" about teaching and learning needs to happen in critical and reflective learning communities (Ball & Cohen, 1999) and that teacher learning needs to be "activated" rather than delivered (Wilson & Berne, 1999).

Recent years have seen an emerging consensus in the research literature on teacher learning about the "essential principles" of effective professional development. Putnam and Borko (2000), for example, have named the following as essential features to include in professional development opportunities for teachers: 1. Teachers should be treated as active learners who construct their own understanding. 2. Teachers should be empowered and treated as professionals. 3. Teacher education must be situated in classroom practice. 4. Teacher educators should treat teachers as they expect teachers to treat students. Wilson and Berne (1999) generated a similar list stating the typical features of successful professional development as work that: 1. Is ongoing; 2.

Is school based and embedded in teacher work; 3. Is collaborative; 4. Focuses on students' learning; 5. Recognizes teachers as professionals and adult learners; 6. Provides adequate time and follow-up support; 7. Is accessible and inclusive. It is however important to realize that "we know as little about what teachers learn in these kinds of forums as we do about what teachers learn in traditional staff development and in-service" (Wilson & Berne, 1999, p. 176), And, as Ball (1996) notes, "determining how to design provocative experiences for teacher learning and for engagement with what is hard about the reforms, while still honoring teachers as professionals, is a more complex matter than many may recognize." (p. 502).

Following on the reform-minded ideas for professional learning my colleagues and I have been working with four elementary schools that have close ties to our institution's elementary teacher preparation program. Each of us is both an instructor in the teacher preparation program and a study group leader in one of the participating schools. The idea is that by working with the teachers that mentor our students, prospective teachers will have access to teachers who are reflecting, studying, and innovating their teaching practice. The aim of our project, therefore, is twofold: (1) to strengthen our partner schools' teaching of mathematics; and (2) to strengthen the field-based component of our teacher preparation program. In this paper, however, I will focus on the insights, questions, and tentative conclusions I have drawn from my work as a study group leader in one of the teacher study groups of this project. More specifically, my aim is to address the following questions: (1) What opportunities for learning do teacher study groups afford the participants, and (2) How do different study group activities affect the nature and substance of these learning opportunities?

Data Sources and Analysis

The study group I will focus on is composed of seven K-4 teachers. The group meets once every 3 weeks for 2.5 hours during school hours. Meetings began in the spring of 2001 and will continue until the spring of 2003. Six of the teachers are female, and only one of them is male. Two of the teachers have been teaching for over 20 years, two of the teachers have 7-10 years of teaching experience, and three of them are in their 2nd or 3rd year of teaching. All of the grade levels are represented in the group. During these meetings we have engaged in several professional development activities including (a) discussions of selected readings, (b) engage in mathematical problem solving ourselves, (c) analyze students' work, (d) observe and discuss someone's teaching, etc. These activities have generated multiple sets of data, which have been recorded and collected. Teacher educators' project and planning meetings, for instance, have been audio taped. The teacher study group sessions have also been recorded, with both audio and videotapes. Teachers have been observed and video taped teaching in their classrooms. Written data from the students, from the teachers' notebooks, and the teacher educators' journals are also available for analysis.

One of the main foci of our teacher study groups is to engage teachers in the exploration of mathematics through problems that challenge and extend their own understanding. There are obvious reasons for focusing teachers' professional development on doing mathematics. Teachers need to understand the subject matter they teach (Ball & Cohen, 1999; Ma, 1999). Otherwise, they are confined to a practice of showing and telling that does not allow students to figure things out on their own. The analysis of students' mathematical work is also another important focus of our teacher study groups. Understanding what students say, do, and write is an important aspect of teaching that values, respects, and encourages students' thinking. We draw from the professional development ideas and materials of CGI (Fennema et al, 1996) and DMI (Schifter et al., 1999a, b) to structure the study groups' discussions about students' thinking. We also structure conversations around the participants' own students' work on problems the teachers have first worked and discussed in their respective study groups and then tried out with their students.

In this paper I will focus the analysis on the kinds of conversation that take place in a typical study group meeting. In a typical meeting, the study group leader poses a rich mathematical problem to the teachers. These problems are chosen with many of the NCTM's (1991) criteria for worthwhile tasks, such as problems that generate opportunities for mathematical discussion and exploration of mathematical ideas. In addition, the study group math problems are chosen or constructed so that they are accessible for different levels of mathematical sophistication and that are adaptable to different elementary grade levels (K-5). The structure of a typical study group meeting is akin to Simon's (1994) "learning cycles" for mathematics teacher education. In our case, the study groups have four learning cycles—Solving, Posing, Interpreting, and Reflecting. The solving and posing phases occur when the teachers solve a particular mathematics problem together and discuss and analyze their solutions and ways of posing the problem to their students. The interpreting and reflecting phases, in turn, occur after the teachers have tried their version of the problem in their own classrooms and come to the study group meeting prepared to share classroom stories along with insights into their students' thinking and their teaching practice.

A major assumption driving the analysis of the data is that opportunities for teacher learning and change in a community of learners, such as in teacher study groups, depend a great deal on the participants' willingness to share their ideas and to examine their own and their peers' ideas critically. Hence, particular attention is paid to participants' disagreements and challenges of each other's ideas, as well as to the participants' talk that reveal uncertainty, surprise, and confusion. These elements have long been considered instrumental to the personal construction of knowledge and to a community's generation of knowledge. This kind of professional talk, however, is not without some serious challenges. Chazan and Ball (1999), for example, have suggested that most discussions about teaching in all circles tend to be too judgmental to support analytical and constructive conversations about practice. Similarly, Pfeiffer and Featherstone (1995) have noted: "when teachers talk about their work, most are quite facile in talking about teaching without revealing the struggles and uncertainties inherent to the practice." Lord (1994) also contends that in order for teacher groups to become a setting for teacher learning, teachers must be prepared to openly and publicly disagree with and about practice and to engage in a "critical colleagueship" which he defines as "an alternative professional stance where teachers move beyond sharing and supporting one another through the change process to confronting practice—the teachers' own and that of his or her colleagues."

In order to make sense of this data I have drawn from the situated learning perspectives, such as Wenger's (1998) "communities of practice," and the emergent "situative perspective" in cognitive science (Putnam & Borko, 2000), which also emphasizes the social nature of cognition. In addition, because the nature of learning in study groups is discourse-based, several discourse analysis frames such as, "turn-taking", "participant's involvement," "politeness," "disagreements" and "concensus" (Cazden, 1988; Tannen, 1989; 1993) are used to uncover conversational patterns and explore Lord's notion of "critical colleagueship" or whether and when do teachers openly and publicly disagree with one another.

Results

A survey of the participants' assessment of the study group meetings revealed that at least in the teachers' minds, teacher study groups provide them with multiple opportunities for professional learning. Teachers have reported learning and enjoying the experience. The main attraction and learning opportunity, they have said, comes from being able to work across grade levels, which they say has given them opportunities to learn about curricular issues, students' understandings, and teaching struggles at the different grade levels. They have also reported that working on mathematics problems together and then trying them with their students to later report back to the group on what happened and what they learned from the experience has not only further their own understanding, but also has affected their teaching practices. When asked how the study group work were helping them, teachers for example wrote:

> I've really enjoyed the opportunity to get together and talk/share with other teachers about mathematics. I've seen myself look more critically at how I talk or ask questions to the class during a math lesson. I also have seen myself grow in the area of challenging my students... not being afraid to give them an open-ended story problem *(Special Education Teacher).*

> I find myself pushing my students to think beyond more. Instead of stopping at solving a problem one way, we talk about alternatives. We also do more thinking/talking about connections to other problems *(Second Grade Teacher).*

Features of Study Group Talk

A closer look at the discourse generated in the study groups have revealed important differences in the pattern of the group's talk when teachers are engaged in the doing of mathematics and when they talk about their own teaching and students' work. Each of these activities generates distinct conversational patterns of talk and involvement by the participants. These unique forms of talk and participation provide both challenges and possibilities for teacher learning. When doing mathematics, for instance, the teacher group's talk in this study tends to be very interactive and collaborative with participants interrupting and disagreeing with one another, freely and unprompted. Yet, when the focus of the conversation shifts towards their practice and students' work, the conversation tends to take on a different character. This talk is less interactive and collaborative, as one speaker would speak at a time uninterrupted and in a monologue style. These findings are elaborated next.

In terms of the structural features of the teachers' discourse around their own doing of mathematics, it is interesting to note the very public and explicit disagreements that are uttered. The speakers, for instance, explicitly object to another person's solution using "but," "no," and "I don't think so." In addition, the teachers' talk seems more like exploratory talk rather than a polished exposition of ideas. That is to say that the group seems to be figuring things out together and extending perhaps even revising their ideas as they talk. The focus of such conversations is on the analysis of everyone's solutions rather than on simply reaching or agreeing on the right answer. Furthermore, teachers use phrases that indicate uncertainty and exploration of ideas such as "what if," "maybe," "what about" and "I'm not sure."

The group's involvement is another prominent characteristic of this talk. The number of speaking turns and the multiple speakers on the same topic are some evidence of this. In addition, there are several instances in the transcripts where the speaker and listeners show involvement (Tannen, 1989). There are multiple instances of speakers using phrases that suggest they were seeking or showing involvement and understanding such as: "does that make sense?," "you know what I mean?," "what did you say?, "okay I see what you're saying." Another example of involvement is when the participants finish each other's sentences. In addition multiple instances of "repetition" were uncovered in the transcripts, which is another indicator of involvement. Repetition is a conversational strategy used by speakers in order to check for understanding or better understand what has been said.

In contrast, when the conversation is focused on the teachers' teaching practice and students' work, the structure of the conversation changes. The turn taking pattern is different. When sharing their practice, each teacher takes uninterrupted turns and presents with varying levels of detail and analysis what they had done and what their students thought and did in relation to the math problem everyone has tried. Also, interesting is the fact that no one except for the group leader actually asks questions to the speaker. It is also important to note that the teachers in the focus study group have shown a genuine excitement and interest in their students' work regardless of whether the work is correct or incorrect. Their talk therefore tends to be non-evaluative and somewhat celebratory of the students' work, but it also tends to be more descriptive rather than analytical of the mathematical work. Another interesting feature of the conversation is a lack of explicit disagreements; there are no "but" or "no" spoken, even when differences of opinion are expressed. Very explicitly spoken, however, are phrases that imply certainty such as: "probably," "definitely," and "must have." Teachers also use repetition of certain key phrases such as "I think," "in my opinion," and "my students." For instance, when a teacher was asked what might happen in her classroom, she responded by referencing the fact that she had taught both first and second graders before she stated her response. With such conclusive statements that make reference to their expertise, number of years teaching, or their opinion, teachers seem to position themselves as the ultimate authority on what they think and on their students. It is therefore hard to imagine anyone disagreeing or arguing with anything that anyone says.

Another interesting aspect of the teachers' talk about their practice and students' work is the amount of details that the teachers are able to recall and report. Some teachers, for instance, have been able to report the chronology of students' contributions or reproduce what the students say with great accuracy. Still others use "reconstructed dialogue" (Tannen, 1989) in their narrations. The reconstructed dialogues may or may not represent actual dialogue but serve to involve and engage the audience in one's story. It is therefore interesting and puzzling that although some elements for potentially collaborative conversations are present, the teachers' talk about their practice have tended to be anything but collaborative. This is surprising because as Tannen suggests "details create images, images create scenes, and scenes spark emotions, making possible both understanding and involvement" (p. 135). However, Tannen also suggests that a speaker or writer may use details and images to lead hearers and readers to draw the conclusion favored by the speaker or writer. She also cautions that, "one can fabricate details to make a false story sound true, or pile on details about irrelevant topics to de-fuse, diffuse, or avoid a relevant topic" (p. 161). In these teachers' case, then, the provision of detail and images can be thought of as a strategy to involve the listeners in order to help them draw the same conclusions the speaker reached and not necessarily as a way to engage listeners in further analysis of the described experience.

Opportunities for Learning

The two kinds of group talk I have discussed also provide multiple learning opportunities to the participants. When doing mathematics together, the participants have opportunities to: explain their thinking, make sense of someone else's explanation, offer counter-explanations of problems, and participate with others in discus-

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sions about the meaning of problems or mathematical ideas. During the sharing of their students' work, in turn, the participants have opportunities to: make observations about children's work and ideas; contrast what children said and did with what participants anticipated they would do or say; consider differences between children's and adults perceptions of the problem; develop conjectures about alternate approaches to presenting problems or about follow-up activities or problems. These are aspects of professional learning that are considered to be important to the teaching of mathematics as discussed by the National Council of Teachers of Mathematics (2000; 1991); the National Research Council (2000); and the Conference Board of the Mathematical Sciences (2001).

These opportunities for learning depend a great deal on the selection of tasks used in the study groups as they allow different kinds of content and processes to be explored and yield different possible mathematical learning outcomes. In all of the examined transcripts, the problems used have engaged the teachers in explaining and clarifying their own mathematical thinking, and in listening to others' explanations and ways of solving. Participating in these two activities have afforded teachers the opportunity to expand their repertoire of possible solutions or approaches to particular problems. This can be inferred from the analysis of the group's discourse when teachers use expressions of surprise after hearing somebody else's solution, indicating that a particular approach or solution is new to them. Another indicator or source of evidence can be found when teachers are compelled to repeat somebody else's solution word by word, which is a conversational strategy many use in order to better understand what has been said. One other example that supports the notion that conversations around their own doing of mathematics can serve to expand teachers' repertoire of solutions to problems can be seen when teachers disagree with another's solution and try to convince each other that their solutions is or is not mathematically sound.

When discussing ways of posing the same problem to their K-4 students and what might happen in their classrooms, the teachers' conversations have revealed multiple opportunities for professional learning. The teachers' attempts to anticipate what their students might think and how they would "nudge" them towards seeing other solutions beyond their own often led to further analysis of the problem and how it is different from the "typical" math problems they would pose in their classrooms or previous problems the group had tried. In one such conversation, for instance, the first grade teacher said that "unless I really read this over" pointing to the last sentence in the problem, her students would automatically want to prove one of the two kids in the problem right, because that is what math problems tend to ask them to do. This led to a discussion about: "do you want it to even say that (prove that both kids could be right) or would you want it have it say "which one is right?" This suggests that engaging teachers in discussions about how they might pose problems to students can provide opportunities to analyze math problems from different vantage points—the wording, structure, demands, challenging aspects, etc. across the different grade levels.

In terms of opportunities for learning when sharing and listening to each other's accounts, teachers have multiple opportunities to extend their pedagogical content knowledge, and their knowledge of students' mathematical thinking. By listening to others' accounts, the listening teachers have opportunities to learn about the speaking teachers' practice. This opportunity depends a great deal on how much of their practice the reporting teacher is willing or able to reveal. From listening to others' accounts, participants can learn that the reporting teacher, for example, had "read the problem" to her students, or that she decided against bringing actual manipulatives because she "didn't want to have predetermined solutions for them." Teachers' report on what happened in their classroom also provided opportunities for learning about the different solutions and ways of thinking from students in classes that were different from their own and how the reporting teacher had made sense of them. It is, however, unclear and harder to determine how the reporting of their insights, in turn, helped extend of challenge anything that the reporting teachers may have already learned. The opportunity for extended learning in this situation, therefore, depends a great deal on the reporter's willingness to publicly problematize their teaching practice and their students' work. It also depends a great deal on the availability of records of practice for everyone to examine and also on the study group leader's mediation and interjections.

One of such interjections that have proven very fruitful often happens when the group leader moves the conversation from a reporting to a reflective mode by asking the participants, for example, to talk about "what was hard about doing this problem with students?" In this reflection mode the participants would often reveal more about their pedagogical thinking and the problematic aspects of their teaching that were not revealed when they reported on their teaching and their students' work. One teacher, for instance, who in the reporting/analysis phase sounded self-assured and her practice sounded straightforward, revealed the tensions she experienced when deciding when it was time to move on.

> I struggled with the chunk of time that you talked about too. We had 6 different pictures and solutions all over the board and when I finally cut it off I still had 4 or 5 kids with their hands up wanting to say something more and come up to the board. I mean I wanted to get some other things done in math and we had already spent I think 25 minutes on it. You know between the pre-discussion and giving them the problem and having them start talking about it, and giving them some paper to draw on and then all the different answers and getting them to the board, it was really hard. Although I don't' think I would have gotten some of the interesting thoughts in the end if I didn't give them extra time to process but it was hard to give up the time and keep going and still wonder if I should have continued on going. (Second Grade Teacher)

To summarize, in the *solving* cycle teachers have opportunities to engage in the doing of mathematics. This work affords them opportunities to (a) explore and discuss their own mathematical solutions and approaches to proving or refuting (b) identify and expand their knowledge and beliefs about particular mathematical content and approaches to learning it, and (c) evaluate the problem's potential for supporting students' learning. The *posing* cycle provides teachers the opportunity to: (a) explore the contextual, mathematical, and linguistic demands of the problems, (b) anticipate and prepare for students' difficulties, and (c) design versions of the same problem to try out with students in different grade levels. In the *interpreting* cycle teachers: (a) explore students' mathematical work across grade levels, (b) identify insights they have gained about teaching and about students' mathematical work and understanding, and (c) develop analytical frames to interpret teaching and students' learning. The *reflecting* cycle, in turn, provided teachers the opportunity to (a) explore what they and their students might have learned, (b) identify effective practices and aspects of their current practice that they may wish to examine further or change, and (c) map possible plan for follow up and investigation into their teaching and students' learning.

Insights, Questions, and Issues

An important insight from this study is the differences between the two types of talk explored in this study and the learning that each can afford. Teachers' problem solving discourse can be categorized as exploratory talk, that is talk that is characterized by speakers seeking and showing intellectual involvement; explicit disagreements and public disclosure of uncertainties and confusion; and talk that is generative, interactive, and collaborative. In contrast, the teachers' talk about their teaching and students' work can be categorized as expository talk, which can be characterized by the use of monologues; speakers seeking and giving approval; and non-analytical or unproblematic narration of events. These are features of study group talk that seem important to pay attention to and study. Another important insight relates to the opportunities for learning that these two forms of talk afford the participants. When the conversation is focused on their own mathematical problem solving and the structure of the talk is exploratory the participants have opportunities to explore and analyze mathematical ideas and expand their repertoire of solutions and approaches to particular problems. When the discussion focuses on their teaching practice or students' work and the talk is expository, the participants have opportunities to expand their repertoire of students' solutions and strategies and to glance at each other's teaching moves and pedagogical reasoning.

After this analysis it has become clear that engaging teachers in discussions about their own mathematical ideas is likely to provoke a more generative and collaborative talk than conversations that focus on teachers' practice or their students' work. The more collaborative form of teacher talk happened more "naturally" and without much intervention from the study group leader when participants engaged in, and discussed their own, problem solving activity. This has turned out to be somewhat surprising. I, for one, would have thought that talking about mathematics would be more intimidating to elementary school teachers who are known to have a weak mathematical background and sometimes negative dispositions towards the subject. There are, however, different plausible explanations of what I have described. It is possible that for elementary school teachers admitting mathematical ignorance is not as threatening as admitting dissatisfaction or problems with one's teaching (whereas the contrary might be true for secondary school teachers). They may not find it threatening to their professional status or self-worth. Paulos (1988) has noted that the great majority of Americans openly say they are not a "math person" or were "never good at math."

Another plausible reason is that teachers are more likely to have been engaged in mathematical discourse than they would be used to talking about their teaching practice with colleagues. It is therefore a resource that they bring with them to the teacher study groups. Yet another explanation is the power of a "shared experience" in promoting and sustaining collaborative discussions. Having just solved a problem and talking about it is much easier to handle as a participant and as a facilitator than a conversation that focuses on each participant's individualized experience from a few days ago. The point here is that for the uninitiated participant (and leader) of study groups, doing problem solving seems to be a fruitful setting to begin to engage in the kinds of discussions that might lead to the "critical colleagueship" orientation Lord advocates, and to the adoption of patterns of discourse that are more analytical rather than simply descriptive or evaluative of one's and others' teaching and students' work.

By the same token, the analysis of the conversations centered in and around practice and students' work seemed to be less amenable and open to collaborative talk. The form of the teachers' talk I have called "expository" tended to keep participants away from becoming involved and from asking questions or challenging the speakers' interpretations. The lengthy and intricate stories the participants told while vivid and memorable in many respects were not conducive to interactive and analytical discussion. The study group leader's interventions then are the most crucial in this setting. Modeling the kinds of questions that would further the group's collective insight, asking participants to comment on each others' accounts, and asking participants to comment on analytical questions such as what sense or insights they gain from listening to each other or what they found hard or problematic were some of the interventions that seemed to generate more revealing and collaborative discussion. It is however important to point out that teachers' expository talk also provided multiple opportunities for learning to the participants. Absent from that kind of talk, however, were opportunities for the teacher reporter to hear others' perspectives, challenges, and questions about their interpretations and narrations of their classrooms and students' work.

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TEACHERS RESPOND TO MATHEMATICS REFORM PROFESSIONAL DEVELOPMENT

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Do particular professional development activities supporting teachers' implementation of new mathematics curricula encourage their attendance to reform-minded features of classroom practice? In particular, this study investigated middle school mathematics teachers' reports of the effectiveness of various professional development activities influencing their attention to students' thinking. Teachers' entry characteristics, nature of the activities themselves, and teachers' ratings of the activities were the focus of this study. Teachers were surveyed (n = 110) to rate each activity's effectiveness in prompting their attention to students' thinking. Teachers' years of experience, views of learning, and mathematics background correlated with their ratings of various activities. Teachers (n = 13) were interviewed to extend the quantitative results and illuminate the reasons individual teachers rated particular activities as very effective to not effective at all.

The National Council of Teachers of Mathematics (NCTM) developed a set of Standards for K-12 mathematics curriculum, teaching, and assessment. The Standards build on (a) a consensus emerging in the empirical evidence in the literature describing effective practice and (b) the learning goals most valued by the mathematics education community (Hiebert, 1999). Accompanying the release of the Standards has been a surge of new mathematics curricula designed to align to the NCTM Standards. When teachers are working with materials aligned with the NCTM Standards, it is reasonable to suspect that teachers might become more attentive to the ideas central to the mathematical reform agenda by using and learning about the curriculum.

One of the NCTM Standards (2000) claims that "effective mathematics teaching requires understanding what students know and need to learn, and then challenging and supporting them to learn it well" (p.16). To follow the above NCTM recommendation it is necessary for teachers to investigate their students' mathematical thinking and learning. It has been documented that when professional development activities focus on students' thinking, they can have a positive effect on teachers' instructional practices and provide support for teachers' continued growth (Franke, Fennema, Carpenter, Ansell, & Behrend, 1998; Simon & Schifter, 1991). Therefore, might teachers become more aware of students' thinking by using and studying these NSF curricula? In this study, I investigate two factors: (a) the nature of the professional activities themselves and (b) the entry characteristics of the teachers. The key is to find a way of classifying professional development activities and teacher characteristics that might relate to changes in teachers' attention to students' thinking. The literature provides some useful direction.

Background

Nature of Professional Development Activities

One way to summarize the diverse literature on professional development is to tease out the factors found throughout the literature that seem to characterize effective development programs. Although confirming empirical data are sketchy, many researchers are hypothesizing that effective professional development is based on various principles or frameworks of design (e.g., AFT, 1995; Guskey, 1997; Hawley & Valli, 1999; Lampert & Ball, 1999; Sparks & Loucks-Horsley, 1989; Sykes, 1996). There is an emerging and rather striking consensus regarding the core mechanisms of effective professional development. These mechanisms are viewed best as *hypotheses* (described below) that should be investigated empirically. The four mechanisms were chosen because of their empirical support in the literature. Aside from several detailed research studies, such as the Cognitively Guided Instruction (CGI) (e.g., Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996) and Summer Math programs (Schifter & Fosnot, 1993; Simon & Schifter, 1991), relatively little data have been reported that address these hypotheses directly in mathematics so it is difficult to know exactly how these emerging mechanisms facilitate teachers' learning and changes in practice.

Professional Mechanism 1: Instructional improvement is facilitated when teachers' thinking and learning is focused on students' thinking and learning (M1). Empirical evidence suggests that professional development activities should be grounded in the fundamentals of student cognition (Carpenter, Fennema, Peterson, Chiang, & Loef; 1989; Fennema, et al., 1996; Schifter & Fosnot, 1993). Students' thinking and learning are the products of teaching, and teachers might benefit if these products are a driving mechanism of their professional activities.

Professional Mechanism 2: Instructional improvement is facilitated by teacher collaboration (M2). Little (1982) collected interview and observation data from six schools and posited that professional development has the greatest influence when it occurs in a collegial environment where teachers believe they can learn from one another. A collaborative teacher culture that encourages consistent improvement and supports teacher development might be beneficial for reform (Ball & Cohen, 1999; Schifter & Fosnot, 1993).

Professional Mechanism 3: Instructional improvement is facilitated by small steady change (M3). Research studies suggest that the focus of improvement should be small and intense for thorough progress in instruction (Franke, Carpenter, Levi, & Fennema, 2001; Schifter & Fosnot, 1993). Gallimore (1996) describes classrooms as cultural routines. One facet of the routine cannot be changed without influencing other cultural routines, shaking the classroom equilibrium creating discomfort and confusion. Changes in the learning environment might last longer and be more effective when teachers focus on small changes over time.

Professional Mechanism 4: Instructional improvement is facilitated by teachers' experimentation and inquiry (M4). Encouraging teachers to share and conduct structured learning experiments in their classroom instruction might support teacher learning (Schifter & Fosnot, 1993). Few teachers are given the opportunity to observe one another but watching and learning from peers might be a useful mechanism for professional development (Little, 1999). In addition to a collaborative environment, Little's (1982) study also indicated that schools that supported a norm of "analysis, evaluation, and experimentation" were relatively more successful (p. 339).

These mechanisms are not discrete entities or requirements on a checklist but are overlapping recommendations from the literature that might foster continuous teacher development and growth. These Professional Mechanisms were used to classify various professional development activities for analyses.

Teachers' Entry Characteristics

This study investigated characteristics that might be associated with teachers' attention to student thinking. Three themes emerged from the literature to guide hypothesis development: teaching experience, views of learning, and mathematics background.

Teaching Experience

Huberman (1995) presents a summary of the literature and a model to generate hypotheses about how years of teaching experience are associated with teacher responses to professional development. Two of the five phases are described in ways that suggest links between a teacher's career phase and the nature of particular development activities. This study investigated two phases. A teaching career typically begins with two years in a "survival and discovery" phase. Teachers are confronted with the initial complexity of teaching, yet are generally enthusiastic, experience a sharp learning curve, and develop pride in belonging to the teaching profession (Huberman, 1995). The less experienced teachers have not been socialized into the typically individualistic and isolated teaching environment, which lacks meaningful interaction (Ball & Cohen, 1999; Lieberman, 1988; Little, 1999).

> H1: Teachers with less than two years of experience perceive professional development activities involving Professional Mechanism 2 (Collaboration) as a productive way to prompt their attention to students' mathematical thinking. They will rate these activities higher than 2.5 on a 1-4 Likert-type effectiveness scale.

Later in the teacher's career, Huberman (1995) identified an "experimentation and diversification" phase that might emerge between seven and twelve years of experience. Teachers have come through the "survival and discovery" phase as well as a "stabilization" period and begin exploration in their teaching. This phase is characterized by a desire to increase teaching effectiveness and is enacted through "largely private experiments" in their own classroom (Huberman, 1995; p. 198).

H2: Teachers with seven to twelve years of experience will rate activities involving Professional Mechanism 4 (Experimentation) higher than teachers with either less or more years of teaching experience.

View of Learning

A second lens through which to examine teachers' interactions with professional development is their view of student learning. Knapp and Peterson's (1995) findings suggest that teachers who view mathematics learning as knowledge construction are likely to give attention to student thinking, whereas teachers who maintain that knowledge is transmitted give little credence to the diversity of student thinking in their classroom.

H3: Teachers who hold a student-centered view of mathematics teaching will rate all the professional development activities higher than teachers with a curriculum-centered view of teaching in terms of prompting their attention to student thinking.

Mathematics Background

Middle school teachers might vary greatly in their mathematics education. In general, middle school teachers range from mathematics and mathematics education majors to uncertified teachers. Teachers' mathematical background might influence the manner in which they structure their classroom (Brophy, 1991; Fennema & Franke, 1992) and impact their attention to student thinking. It is suspected that the advanced mathematical teachers might not feel they are challenged and the teachers with a weak background might be confused or feel lost.

H4: Teachers with an average number of college mathematics credits within the group will rate the sessions focused on student strategies or development of the content, higher than the teachers with more or less mathematics credits. These teachers might be most interested in activities that embody Professional Mechanism 1 because their mathematics foundation allows these activities to be accessible but still leaves room for learning.

There might be other explanations for the differences in teachers' responses to various types of professional development activities, but this study focused only on these three specific teacher characteristics.

Method

Participants

The participants were a voluntary sample of 110 teachers from a population of approximately 200 middle school teachers in a mid-Atlantic state in the U.S. All of these teachers were participating in a statewide mathematics professional development program with a two-week summer program for first year teachers and a one-week program for second and third year teachers. All teachers also participated in on-site professional development activities throughout the school year that blend all teachers. Most public schools in the state had chosen to use one of two reform driven curricula at the middle school level. Teachers in these schools were required to attend the professional development activities. This mandate increased the likelihood of a well-stratified sample of teachers from across the state, not limited to the usual set of volunteers and self-motivated teachers.

Data Collection

Survey Constructs

The teachers filled out a written survey after they had experienced a variety of professional development activities halfway through an academic year. The survey included items that investigated teachers' reactions to the various forms of professional development and activities that prompted teachers' attention to student thinking. The survey consisted of three sections. (a) The first measured teachers' views of mathematics teaching. (b) The second explored teachers' reactions to specific forms of professional development in prompting their attention to students' mathematical thinking. (c) The third portion collected demographic information. Prior to survey administration, I coded each of the activities by the four Professional Mechanisms. Inter-rater reliability was achieved with 100% agreement on the mechanism classification of each activity.

Follow-up Investigation

Each of the hypotheses guided the analyses of the survey data. To gain insight into the reasons for teachers' survey responses and why the hypotheses were confirmed or not confirmed the investigation shifted to focus on individual teachers. Eighteen teachers were contacted by telephone to verify their survey responses regarding demographic information and professional development ratings. An attempt was made to select teachers that were stratified along the other variables in this study to obtain a diverse sample.

Teacher Interviews

Thirteen teachers were interviewed in-person after one teacher declined an interview, two did not return multiple phone calls, and two others were rejected because the researcher judged the teachers not to be a productive source of information by their answers in the phone follow-up. The thirteen interviews were recorded and transcribed. For each hypothesis three teachers were interviewed except for the third hypothesis that investigated teachers' views about learning, and four teachers were interviewed. Of the four, two teachers were chosen from each of the extreme ends based on their survey responses. In-depth interviews were conducted to inform the affirmation or refutation of each of the primary hypotheses. The in-person interview addressed the survey questions and asked teachers for specific examples and details about their professional development experiences and how the activities impacted their understanding of students' thinking.

Results

The results section reports the hypotheses testing using the survey data. The qualitative data collected during the teacher interviews will be reviewed in the discussion section to inform the results of the hypothesis tests. The goal is to provide both a quantitative and qualitative description of teachers' reports about their attentiveness to students' thinking and the professional development activities that influence that attention.

An alpha level of .05 was used to determine significance for all statistical tests. Hypothesis 1 was tested using the professional development ratings of the 20 teachers in the sample with less than two years of teaching experience. The teachers in Huberman's (1995) "survival and discovery" phase rated all four activities that incorporated Professional Mechanism 2 (Collaboration) higher than 2.5 on a 1-4 Likert-type scale (See Table 1). A t-test confirmed that all of the means are significantly greater than 2.5, rejecting the null hypothesis.

Hypothesis 2 was tested using a multivariate analysis of variance (MANOVA). There were four dependent variables, activities 3, 4, 6, and 9; and three levels associated with teachers' years of experience, 0-6 years, 7-12 years, and more than 12 years. Although the means were in the direction hypothesized, no significant differences were indicated by Wilk's criterion across the teachers' years of experience and their ratings of activities exhibiting M4 (Λ=. 662). A power analysis indicated a relatively low level of power (Power = .32), possibly the result of the small sample in the 7-12 years of experience group. The MANOVA, which analyzes only the cases that rated all four activities, did not produce results significant to reject the null hypothesis. The descriptive statistics for each of the teacher groups are detailed in Table 2.

Table 1. Teachers' Ratings With Less Than Two Years of Experience

		SD	t-test
Activity 2	3.13	.83	15.19
Activity 3	3.23	.75	19.54
Activity 7	3.09	.94	13.14
Activity 8	3.05	.70	16.46

Table 2. Teachers and their Mean Ratings & SD of M4 Activities

Experience	N	Activity 3	Activity 4	Activity 6	Activity 9
0-6 Years	44	3.20 (.73)	2.90 (.78)	2.75 (.71)	3.06 (.81)
7-12 Years	17	3.29 (.62)	3.20 (.86)	3.16 (.98)	3.30 (.67)
12+ Years	34	3.03 (.82)	2.91 (.83)	2.80 (.95)	2.60 (.83)

Hypothesis 3 was also evaluated using a MANOVA. The 110 teachers' views of teaching spread across a continuum. To form two distinct groups, twenty teachers were chosen who held views that were most different. Wilk's criterion confirmed that teachers who hold a view of teaching as best when student centered rate *all* professional activities higher with regard to prompting their attention to students' thinking ($\Lambda = .007$, Power = .884). Descriptive statistics are summarized in Table 3.

Table 3. Teachers' Views of Teaching, their Average Ratings, and SD

	N	M1	M2	M3	M4
More Constructivist	20	3.33 (.51)	2.95 (.66)	3.20 (.57)	3.40 (.49)
Less Constructivist	20	2.75 (.48)	2.49 (.38)	2.59 (.45)	2.76 (.41)

The univariate results also provided significant results indicating that the "student-centered" teachers rated each of the professional activities higher than the "curriculum-centered" teachers did (See Table 4).

Table 4. Univariate Results for each Professional Principle

Dependent	F	Significance
M1	11.545	.002
M2	6.846	.013
M3	12.252	.001

Hypothesis 4 was investigated using a MANOVA. The Wilk's criterion (Λ=. 041, Power = .821) indicated significant results for the interaction between teachers' number of university level mathematics courses and their ratings of activities exhibiting M1. The average number of mathematics courses for the entire sample of middle school teachers surveyed was 6.98 and teachers were included in the "average" group if they were within ½ a standard deviation on either side of the mean. Examining the univariate statistics shows significant differences for two of the four activities (See table 5).

Table 5. Univariate Results for each M1 Activity

Dependent	F	Significance	Power
Activity 3	3.655	.032	
Activity 5	.8310	.418	.19
Activity 6	3.618	.034	
Activity 11	.147	.864	.07

Significant differences across teacher groups for Activities 3 and 6 were detected. But the power of this test was especially low for activities 5 and 11 and might account for the lack of significant differences in these cases. However, the means for activity 11 show very little between-group difference. Many of the teachers reported that they could not remember how many mathematics courses they had taken and left the item blank on the survey. The descriptive statistics are reported in Table 6.

Discussion

The discussion section reviews the results of the hypothesis tests and augments the findings with excerpts from the teacher interviews. The goal of the teacher inter-

Table 6. Teachers and their Mean Ratings & SD of M1 Activities

No. Mathematics Courses	N	Act. 3	Act. 5	Act. 6	Act. 11
Low	15	3.29 (.71)	2.57 (.88)	3.09 (.90)	2.91 (.79)
Average	31	3.30 (.74)	2.80 (.95)	3.14 (.91)	2.92 (.76)
High	11	2.75 (.77)	2.66 (.84)	2.38 (.65)	2.88 (.78)

views was to inform teachers' ratings of professional activities on the survey. When the teachers were interviewed regardless of the hypothesis being investigated, the teachers did not keep their responses focused only on students' thinking. It seemed teachers were rating the activities on a range of goals they had in mind instead of restricting themselves to only prompting their attention to students' thinking.

Hypothesis 1

Teachers with less than two years of experience rated all collaborative activities (M2) as "effective" or "very effective" in impacting their attention to students' thinking. However, during the interviews the teachers conveyed both positive and negative collaborative experiences. The inexperienced teachers reported that collaboration with their peers working through mathematical tasks together was helpful in understanding students' thinking. This is illustrated in the following excerpt from Mark a first year teacher.

> When you see how other peers' minds are thinking. You know, 'Oh, I didn't think about that one.' . . . And they can actually explain it to you. . . because they are math teachers also. . . As opposed to, if you were to hear a student form a question for the first time. . . Whereas a peer teacher was able to explain the whole thinking process. (Mark, March 2001; p.3)

This collaborative activity and others the teachers discussed were part of the formal training provided through the local systemic change initiative in district wide or Statewide programs. The teachers had not experienced such positive interactions in their school on a daily, weekly, or monthly basis in general. Eileen expressed a frustration with the peers in her building during their after school meetings to discuss the reform curriculum. "I don't know if they just think that I'm too optimistic. I'm not sure. Because, when we do have meetings like I said, they're so negative. And it's frustrating to me, because I'm so positive about everything. I know the kids can do it. I've seen them do it" (Eileen, March 2001; p. 9). The interviews suggest that the three inexperienced teachers had experienced effective forms of collaboration with their wider peer group, but each also recalled negative instances of building level interactions. Despite varied levels of effectiveness reported during their interviews, they retained the view that collaboration could be an important mechanism for their own learning about students' thinking.

Hypothesis 2

The statistical results from the test of Hypothesis 2 were inconclusive. There were no significant differences between the groups with different years of experience on their ratings of activities that involved experimentation and inquiry. However, the average ratings for the groups of teachers were in the direction of the hypothesis and three teachers in the "experimentation and diversification" phase with 7-12 years of experience were interviewed. The interviews provide some specific details about what these teachers found beneficial to enhance their understanding of students' mathematical thinking, within the professional activities involving experimentation (M4).

Maria, a teacher with seven years experience reported the benefits of journal reading to improve her instruction and student learning. "Just finding other strategies, sample problems, the way that teachers kind of change and tweak different parts of the books, and curriculum they're using. Just hearing about other things, how other teachers are doing, as well as giving me ideas of how I can change my teaching to be more effective for the kids" (Maria, April 2001; p.6). Leslie, another "experimentation" teacher described a teacher support group that has been established at her school. This group meets weekly after school to explore the curriculum, the difficulties teachers are encountering, and has some similar underpinnings to Japanese lesson study. "The support here at the school that we've established, . . . being able to also use somebody else as a sounding board. 'Here's a lesson I'm really struggling with. Here's a concept I'm really struggling with, can you help me? Here's something my kids are really struggling with. Any ideas? Any suggestions?'" (Leslie, April 2001; p.14).

An understanding of students' thinking is embedded in the conversations Leslie experiences in her school.

The three experimentation teachers interviewed conveyed a sense of confidence in their teaching. They were also explicit about the methods they use to improve their practice. The professional development experiences they reported as beneficial range from observers in their classroom and journal reading, to lesson study type activities and watching classroom videos, yet their comments were not solely focused on understanding students' thinking.

Hypothesis 3

Teachers with a student-centered view of teaching rated *all* the professional development activities more effective in impacting their understanding of students' mathematical thinking compared to teachers with a curriculum-centered view of teaching. Joanne classified as a student-centered teacher commented about the professional activities that modeled the mathematics activities. "The only way we're going to get teachers to better do this is if the teachers are put through the exact thing their kids are put through. So that they can develop the concept too. And when they get that kick then they're going to understand what that kick does to the kid" (Joanne, March 2001; p. 9). Joanne reported the benefit of actually experiencing the exploration, whereas Sally, classified as a curriculum-centered teacher, countered that opinion and sees no value for teachers working through the tasks to analyze their own or their students' thinking. "I don't know that all the time we should have to do the problem . . . I mean the kids actually have to do it. But for all of us to sit there and do that, I think it's kind of a waste of time" (Sally, April 2001; p.2). The student and curriculum-centered teachers have different opinions about the effectiveness of some of the professional activities and the importance of understanding students' thinking. However, observing other teachers was an activity both groups agreed would help to improve their practice.

Hypothesis 4

Teachers with an average number of mathematics courses responded significantly more positive to two of the four M1 activities than teachers with less or more university mathematics courses. The M1 activities are focused on students' mathematical thinking and strategies, and are more content focused than the other activities. Ann, a teacher who has completed less than the average number of university mathematics courses within the group said,

> People who are really comfortable . . . assume that you have a higher level of knowledge than you have. So as much as it's been explained to me a little bit, I don't feel that I have all the pieces. At least not so I feel confident . . . the gap is, in teaching it to the kids . . . (I) don't have the repertoire of different angles to come at it. (Ann, April 2001; p.9)

Leslie, a sixth grade teacher with an average mathematics background shared how teachers with an average mathematical background can feel comfortable learning from those with more content expertise. During this activity, the curriculum was modeled for the teachers as students.

> We as 6th grade teachers . . . would solve something totally different, very visual. We were drawing those little pictures and things like that. And some of the 7th and 8th grade teachers were, like, not laughing at us, but laughing because of the different way(s). Where they approached a problem, maybe algebraically, with a formula, we approached it totally different. Probably how our students would. Because we weren't as familiar with the math behind it as they were. So we were trying to approach it in a totally different way. And we learned a lot from each other in doing that. (Leslie, April 2001; p.2)

The teacher who had completed more university mathematics courses was not as inspired by the sessions focused on strategies and content development. Eileen reported, "There was a lot more time spent doing, you know, catching up the non math people and telling them "This is what we're doing, this is why we're doing it" (Eileen, March 2001; p.12). The interviewed teachers' mathematics background seemed to be a significant determinant of their responses to various activities with student work, student strategies, and working through mathematical tasks.

Conclusions

Teachers' entry characteristics do seem to influence their response to professional development activities. This study found that inexperienced teachers have a great faith in teacher collaboration, despite some negative experiences. In addition, teachers' views of mathematics teaching are predictive of their responses to professional development activities focused on reform-minded curriculum. The teachers who view teaching as student-centered consistently rated activities more effective than teachers with a curriculum-centered view of learning. Finally, this study determined that teachers' mathematics background is a significant determinant of their reactions to particular activities focused on students' mathematical thinking and development of the curriculum.

Although the findings suggest some specific relationships, the interview responses indicate that the teachers might not hold these relationships in such a specific way. Their comments often drifted away from the goal of attention to students' thinking and addressed more general goals, such as "improved instruction." This suggests that teachers also might have responded on the survey in terms of an activity's general benefit rather than its influence on their attention to students' thinking.

During the teacher interviews it became apparent that the teachers were not focused on understanding students' thinking and did not view it as a primary goal of the professional development program. Despite the reform-minded curriculum and the emphasis of the NCTM Standards during their training this goal was not explicit for the teachers as they came to understand the curricula and their shifting teacher role. The teachers focused their responses on other more general instructional concerns. I have argued elsewhere that teacher learning goals need to be made the focus of professional development for the developers and for teachers as learners (Cwikla, 2002). Otherwise the purpose of the professional activities remains at the "improve student learning" level and ongoing teacher learning remains loosely defined and it is difficult to assess a professional program's effectiveness.

Next steps for investigation of the four mechanisms might move beyond the teacher self-report data and into the classroom to investigate teachers' use of students' thinking in their practice. The growing shortage of properly trained teachers, efforts to reform mathematics education, accountability mandates in the United States, and impoverished teaching compel intense examination of professional development and the ways to enhance existing teaching. The ultimate goal is to improve the learning environment for students, and this should begin by improving the learning environment for teachers.

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TEACHER LEARNING IN THE PROFESSIONAL TEACHING COMMUNITY CONTEXT

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Current efforts at teacher professional growth focus on the need to support the development of professional teaching communities that can both nurture and sustain generative growth (Franke & Kazemi, 2001; Nelson & Hammerman, 1996; Wilson & Berne, 1999). Secada and Adajian (1997) assert that mathematics teachers' professional communities provide an important context for understanding the nature of teachers' practices, change, and learning. This creates a need for mathematics educators to better understand the nature of professional teaching communities in order to understand the process of mathematics teachers' learning and how to support it.

The purpose of this paper is to explore the usefulness of the professional teaching community as a context in which to support teacher learning. In doing so we will explicate ways to cultivate teacher learning in the context of professional teaching communities though the use of a conjectured learning trajectory. The usefulness of a conjectured learning trajectory in supporting and organizing students' learning of significant mathematical ideas is made evident in the literature (Cobb & McClain, 2001; Gravemeijer, 1994; Simon, 1995). We maintain that this construct is useful for supporting teachers' learning in the context of a professional teaching community. As with the use of conjectured learning trajectories in conjunction with classroom teaching experiments, the benefit is the ability to test and revise a conjectured learning trajectory for teachers within a professional teaching community to learn how to best support and organize teachers' learning.

In the following sections of this paper, we start by clarifying our assumptions about what constitutes a professional teaching community. We then propose a conjectured learning trajectory for teachers in a professional teaching community and delineate the means of supporting the learning that emerges along that trajectory. We conclude by discussing how a focus on student thinking permeates the conjectured learning trajectory and the means of support.

Defining a Professional Teaching Community

In order to clarify our assumptions about what distinguishes a professional teaching community from a cohort of teachers, this section provides a brief synopsis of the literature on community, professional community, and professional teaching community. Using the salient features from this literature, we frame four characteristics of a professional teaching community that we find most useful for describing what constitutes a professional community.

Bellah, Madsen, Sullivan, Swidler, and Tipton (1985) specify that a community is "a group of people who are socially interdependent, who participate together in discussion and decision making, and who share certain practices that both define the community and are nurtured by it" (p. 333). Secada and Adajian (1997) define community as a group of people who have organized themselves for a shared purpose. "They adopt or are assigned formal and informal roles, they organize additional structures (such as times for meeting and planning) as needed, and they take actions—all in order to achieve their purposes" (p. 194). This shared purpose aligns with Wenger's (1998) concept of a joint enterprise in that it is a negotiated venture "produced by participants within the resources and constraints of their situations" (p. 79). More than merely a stated goal, this joint enterprise creates a sense of mutual accountability that becomes an integral aspect of the practice of the community. Besides group members being accountable to each other for working toward their common purpose, what we find most useful about these definitions is the idea of the practices of the members as a central part of the community. This is supported by Wenger's (1998) notion that practice not only determines the formation of community, but also learning of the participants within that community.

> As we define these enterprises and engage in their pursuit together, we interact with each other and with the world and we tune our relations with each other and with the world accordingly. In other words, we learn. Over time, this collective learning results in practices that reflect both the pursuit of our enterprises and the attendant social relations. These practices are thus the property of a kind of community created over time by the sustained pursuit of a shared enterprise (p. 45).

In order to differentiate between a professional community and other forms of communities with a shared purpose, Secada and Adajian (1997) claim that a professional community's members share a common base of specialized knowledge. This specialized knowledge is what sets the professional community apart within larger society. The members of a professional community have the autonomy to make decisions that fall within their specialized knowledge that outsiders do not have the power to make. For example, acceptable decorum inside a courtroom is established and enforced by members of the law profession, not by outsiders with no knowledge of the practices of the law community. Senior members of a professional community mentor and evaluate the practices of new members. A shared sense of identity, common values, collective role definitions, and a communal language stem from the practices that result from the shared specialized knowledge specific to a professional community (Grossman, Wineburg, & Woolworth, 2000).

According to Grossman, et al., a professional *teaching* community is set apart by the following features: (1) teachers acknowledging and understanding differences among members and using this recognition of diversity to cultivate the understanding of the group as a whole, (2) teachers coming to recognize the interrelationships of teacher and student learning and the ability to use their own learning as a resource to explore more deeply issues of student learning, curriculum, and teaching, and (3) teachers willingness to take on the responsibility for their colleagues' growth and development.

Using a compilation of the aspects from the literature on community, professional community, and professional teaching community we propose four characteristics that we find salient for the purpose of distinguishing a professional teaching community from a cohort of teachers. The first characteristic of a professional teaching community is a shared sense of purpose specific to what students are to learn. Similar to Wenger's (1998) joint enterprise and Secada and Adajian's (1997) shared purpose, this is a negotiated venture "defined by the participants in the very process of pursuing it" (Wenger, 1998, p. 77). The goal would be that a professional teaching community is centered on a common enterprise dealing specifically with student learning. For example, this shared purpose may be supporting students' learning of important mathematical concepts as opposed to solely memorizing procedures (cf. Thompson, Philipp, Thompson, and Boyd, 1994).

The second characteristic of a professional teaching community is teachers using their own learning as a resource in a concentrated teamwork endeavor to support students' learning. This characteristic follows Secada and Adajian's (1997) concept of teachers' coordinated effort to improve student learning where teachers are motivated to work together and set aside personal prerogatives in favor of the shared purpose. However, what we think is an important addition is Grossman et al.'s (2000) idea of teachers coming to recognize the interrelationships of teacher and student learning and the ability to use their own learning as a resource to explore more deeply issues of student learning, curriculum, and teaching.

We take the third characteristic of a professional teaching community from Secada and Adajian's (1997) notion of collaborative professional learning where teachers work together to learn about and improve their practices. Teachers challenge and support each other to clarify thinking and justify their pedagogical arguments in terms of students' reasoning. This is similar to Grossman et al.'s (2000) characterization of teachers' willingness to take on responsibility for their colleagues' growth and development. Another useful concept in collaborative professional learning is what Wenger (1998) calls mutual engagement where interactions build off of the engaged diversity of the members of the community. Grossman et al. expand on this stating participants acknowledge and understand differences among members and use this recognition of diversity to cultivate the understanding of the group as a whole. Because of the diverse knowledge of the participants of the community, it becomes important to know how to give and receive help, instead of individuals trying to know everything.

The fourth characteristic of a professional teaching community is a developed and communal repertoire of resources that is specific to the practice of the community.

This characteristic is developed from Wenger's (1998) concept of a shared repertoire that is produced or adopted by the community in the course of its existence and becomes a part of its practice. The elements of a repertoire "gain their coherence not in and of themselves as specific activities, symbols, or artifacts, but from the fact that they belong to the practice of a community pursing an enterprise" (p. 82). Since this communal repertoire is developed during the process of the collaborative, coordinated effort to pursue a shared purpose, it is specific to the professional teaching community and this shared purpose. For example, if the shared purpose is supporting of students' learning of mathematical concepts the repertoire may include, but not be limited to: knowledge about student reasoning within a certain content domain, discourses that support this student reasoning, and tools developed to support and organize student learning.

Conjectured Learning Trajectory and Means of Support for a Professional Teaching Community

Simon (1995) describes a reflexive relationship between a teacher's task selection and consideration of students' thinking that might emerge as they participate in those tasks. "The consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage make up the *hypothetical learning trajectory*" (p. 133). The teacher creates an initial learning goal and plan for instruction, which must be continually modified as students engage in the planned activities. This notion of a hypothetical learning trajectory is similar to Gravemeijer's (1994) description of developmental research in mathematics education. According to Gravemeijer, the researcher starts with an anticipatory thought experiment where he or she tries to envision how the proposed teaching-learning process might be realized. During this thought experiment, the researcher develops conjectures about the course of students' development as wells as the means of supporting it. Cobb and McClain (2001) use the notion of a conjectured learning trajectory when conducting classroom teaching experiments using coherent instructional sequences. They note that

> conjectures about both a learning route and the means of supporting development along it are provisional and are tested and modified on a daily basis as we make local pedagogical judgments in the classroom. At the same time, the conjectured trajectory serves to guide the local decisions that lead to its revision. (p. 213)

Simon, Gravemeijer, and Cobb and McClain each discuss the merits of using a conjectured learning trajectory to support and organize the learning of students. In contrast, Cobb and McClain note that the notion of a conjectured learning trajectory is a very useful construct when dealing with the learning of teachers and the means of supporting that learning. We concur with their hypotheses and propose the development of a conjectured learning trajectory for a professional teaching community with a learning goal of teachers justifying their pedagogical decisions in terms of students' thinking.

Conjectured Learning Trajectory

In proposing a conjectured learning trajectory for a professional teaching community we take as a starting point *making current teaching practices problematic.* Teachers need a reason to want to change the current way they teach mathematics. Teachers can be motivated to transform their practice by seeing that their current form of practice is not the optimal choice for supporting the learning of their students (Cobb & McClain, 2001).

Once teachers acknowledge the problematic nature of their practice, they are motivated to learn more in order to assist their students' learning. For teachers to understand student thinking and the best way to support student learning of significant mathematical ideas they must have a deep understanding of the mathematics they teach (Bransford, Brown, & Cocking, 2000; Ma, 1999; Shulman, 1986). This knowledge of mathematical content is a necessary but not sufficient aspect of teaching math for understanding. Teachers must also have knowledge of students' diverse ways of reasoning about the math content and the pedagogical methods for supporting the students' learning of the mathematics. Therefore, the next step in the conjectured learning trajectory is *cultivation of pedagogical content knowledge* (Shulman, 1986, 1987). Pedagogical content knowledge entails the merging of content and pedagogy, as well as teacher's knowledge of students' cognitions in specific content areas (Bransford, et al., 2000).

As teachers in the professional teaching community work together, they can use each other as a resource to further develop their pedagogical content knowledge. This *collaboration* also allows teachers to build from the diversity of experiences and levels of expertise of members of the professional teaching community, which in turn can promote the de-privatization of practice. Instead of working in isolation, teachers can confer with colleagues when making pedagogical decisions.

In order to support student learning, teachers are motivated to change their practice, develop pedagogical content knowledge, and collaborate with peers. This builds to the conjectured end point of the learning trajectory: *student thinking to be used as the justification for pedagogical decisions* made by members of the professional teaching community. Meaning, as teachers plan for lessons, design tasks and assessments, and facilitate classroom discourse, they make and justify their decisions based on student thinking.

Means of Supporting Emergence of Conjectured Learning Trajectory

Part of the process of developing a conjectured learning trajectory for a professional teaching community is speculating about the means of supporting the teachers' learning along the conjectured learning route. Therefore, in this section of the paper we outline possible means of support for making teachers' current practices problematic, cultivating teachers' pedagogical content knowledge, promoting teacher collaboration, and justifying pedagogical decisions in terms of student thinking.

Making Current Teaching Practices Problematic

Cobb and McClain (2001) describe one possible method for achieving a state of disequilibrium where teachers see their current practices as problematic in the use of video-recordings of interviews with students. Two sets of parallel tasks were presented to first-grade students being interviewed. One set of tasks followed the format of the students' traditional math textbooks with problems presented such as $5 + 7 =$ or $14 - 9 =$. The other set was "designed to draw on the students' pragmatic, out-of-school reasoning" (p. 209). For example, students were asked to answer the following question: *If there are fourteen horses in a barn and nine of them come outside, how many will be left in the barn?* What made these interviews so powerful was that students solved the two sets of tasks using radically different methods often resulting in different answers although the tasks presented used the same number combinations. Consequently teachers focused on the beliefs the students had developed as a result of their traditional mathematics instruction, perceived those beliefs as detrimental, and therefore had a motivation for contemplating the revision of their instructional practices.

The point here is not the promotion of the use of interviews in teacher learning per se, but the use of a catalyst that highlights student thinking therefore creating a conflict that motivates teachers to want to change their current practices. We would conjecture that the most powerful of these catalysts would involve interactions with teachers' own students, since teachers are intimately familiar with their own instructional activities and norms established in their classrooms. If a state of disequilibrium can be established as a result of working with students with which they are familiar, the teachers will be more likely to equate this to consequences of their personal teaching practices instead of student deficiencies.

Cultivating Pedagogical Content Knowledge

Pedagogical content knowledge is more than knowledge of general teaching methods or knowledge of a particular subject matter. Teachers with strong pedagogical content knowledge know the structure of their disciplines. This knowledge "provides them with cognitive roadmaps that guide the assignments they give students, the assessments they use to gauge students' progress, and the questions to ask in the give and take of classroom life" (Bransford et al., 2000, p. 155).

Within the professional teaching community, teachers' comparing with other teachers what they notice about student work, how they interpret it, and how they evaluate the quality of the work creates opportunities to support the cultivation of pedagogical content knowledge. For example, Franke and Kazemi (2001) collaborated with a teacher workgroup community where the main goal was teachers coming together to share and attempt to make sense of their students' mathematical thinking. They described student work as the artifact used to "focus the conversation, bring teachers' rich experiences and histories into the conversations, and create connections between professional development and other aspects of teachers' work" (p. 13-14). More specifically, they chose to use student work from the teachers' own classes. The teachers posed an agreed upon mathematical problem to the students in their classes. The intent was to choose a problem "that allowed conversation to develop around similar mathematical ideas and permitted a range of entry points to accommodate different mathematical understandings" (p. 14). The teachers' task at the next workgroup session was to share their students' work, generate a group list of strategies used by their students, and then rank the list according to sophistication of students' mathematical understanding. "Teachers detailed the thinking of a number of students in relation to each other, they described the circumstances within which student mathematical thinking occurred and they used the details of student thinking to support their conversations about the issues surrounding the teaching and learning of mathematics" (p. 15).

This use of student work fits with Wenger's (1998) argument that an artifact can provide a focus for the negotiation of meaning, which in turn supports the learning of the community. In other words, concentrating on student work provides a tool the teachers can use to justify conjectures about the teaching and learning of mathematics, which subsequently supports the learning of the professional teaching community. It is important to note here that this use of student work was a catalyst for focusing on student reasoning, probing the mathematics of the problem itself, and delineating ways to support the students in moving forward with their mathematical thinking. We agree with Ball and Cohen (1999) that simply looking at students' work will not guarantee that improved ways of examining and interpreting such work will automatically result. It would be essential for teachers to develop and debate conjectures about what to observe, ways to describe what is observed, and formulations of what is sufficient evidence for any given claim.

> Teachers could explore the thinking that the assignments called for by doing it themselves and then comparing their work. Both would create opportunities for teachers to discern the content entailments of assignments and to learn some of the content. They could learn how unpacking a student task can help teachers delve into the associated or underlying ideas. Discussing the assignment with one another would enable them to see others' paths and connections and others' ways of working and solutions. Such discussion would both expand teachers' own understanding and extend their view of the terrain. (p. 26).

In order for teachers to focus on and debate student work, they must first investigate and study the mathematics required in the work. This allows them to better understand students' diverse reasoning and solutions. Armed with the knowledge of the math content and a better understanding of student thinking, teachers can concentrate on pedagogy that supports students' learning of the mathematics. Therefore, student work can serve as a catalyst to support the cultivation of pedagogical content knowledge within a professional teaching community by pushing teachers to discuss the mathematical content involved, attempting to understand students' thinking in their diverse solutions, and conjecturing about teaching practices that support students' learning.

Teachers' looking at the student work together hints at the benefits of having teachers collaborate. A goal would then be to further develop collaboration in the professional teaching community so that it becomes more than just working with another person. Instead collaboration becomes teachers engaging in precise talk and debate about teaching practices where they rely on each other as learning resources. In the following section, we will outline one way to support teacher collaboration through joint lesson plan development.

Promoting Teacher Collaboration

Another opportunity for teacher learning within the professional teaching community is teacher collaboration in pursuit of a shared goal with a focus on student thinking. Working together towards a shared purpose gives members of the professional teaching community a collective responsibility for each other's learning as well as student learning. "This responsibility might include making contributions to group discussions, pressing others to clarify their thoughts, engaging in intellectual midwifery for the ideas of others, and providing resources for others' learning" (p. 55). Instead of teachers working through pedagogical issues in private, they would have the opportunity to encounter the diversity of other teachers' views and interpretations. Not only could teachers learn about teaching, learning, and mathematics, they would develop "new ways of thinking and reasoning collectively as well as new forms of interacting interpersonally" (Grossman et al., 2001, p. 46).

Promoting the collaboration of teachers can be supported as teachers plan lessons together in a fashion similar to the notion of the Japanese lesson study as described by Stigler and Hiebert (1999). According to Stigler and Hiebert, groups of teachers meet regularly over a period of time ranging from several months to a year to work on designing, implementing, testing, and improving one or several lessons.

The first step of a lesson study group is to define a math problem that will motivate and direct the work of the group. This problem usually comes from issues the teachers have identified from their interactions in their own classrooms that have been a challenge to them and/or their students. The second step of the Japanese lesson study is actually planning the lesson, which can take as long as several months. The third step involves one teacher teaching the lesson with the other members of the group leaving their own classrooms in order to observe the lesson being taught. Step four is the evaluation of the lesson and reflection on its effect. Step five is the revision of the lesson. Based on the teachers' observations and reflections of student understandings (or misunderstandings), the lesson is revised by the group.

This is not to say that an exact replica of the Japanese lesson study process will work in the United States as our education system differs from the Japanese system in significant ways, but there is much to be learned from the lesson study concept. As the teachers collaborate "to improve instruction, teachers are able to develop a shared language for describing, and analyzing classroom teaching, and to teach each other about teaching" (p. 123), thus supporting the learning of the professional teaching community.

As all members of the professional teaching community are invested in the outcome of the lesson, the collaboration during the lesson study creates an avenue for teachers to begin justifying the decisions they make about the lesson in terms of student thinking. The culminating goal of the conjectured learning trajectory we have proposed is teachers justifying decisions based on student thinking, not just decisions when planning the lesson, but all decisions such as task selection, classroom environment, discourse, and all forms of student assessment. Next we will discuss means of support specifically designed to promote teachers justifying all decisions based on student thinking.

Justifying Pedagogical Decisions in Terms of Student Thinking

The means of support discussed previously in pursuing a conjectured learning trajectory within a professional teaching community that culminates with teachers justifying pedagogical decisions based on student thinking can move the professional teaching community towards this endpoint. Two constructs that support the further development of teachers in the professional teaching community making and justifying their decisions based on student thinking are Simon's (1995) Mathematics Teaching Cycle and Cobb and McClain's (2001) enacting of an instructional sequence.

Previously in this paper we discussed Simon's (1995) notion of a hypothetical learning trajectory where teachers plan tasks based on the their knowledge of content and student thinking and then make modifications to that task as a result of student interpretation of and interaction with the task. Simon claims that this modification happens as the teacher develops new understandings of the students' knowledge constructions by observing and communicating with the students while they are engaged in the activity. The hypothetical learning trajectory gives rational to the teacher's choice of a particular instructional design based on how the teacher perceives that learning might proceed. There is a difference between the hypothetical trajectory and the actual path taken.

As the teacher interacts with and observes the students, the teacher and students collectively constitute an experience. This experience by the nature of its social constitution is different from the one anticipated by the teacher. Simultaneous with and in interaction with the social constitution of classroom activity is a modification in the teacher's ideas and knowledge as he makes sense of what is happening and what has happened in the classroom. (Simon, 1995, p. 137)

Simon expands on this idea by commenting on the continuous engagement of the teacher in adjusting the learning trajectory to better reflect what he or she has learned about the students. Sometimes this calls for a minor adjustment, while at other times the entire focus of the lesson might be changed. Simon presents a schematic model for this decision making process called the Mathematics Teaching Cycle as shown in Figure 1 below.

Cobb and McClain (2001) identify with this model calling it "enacting an instructional sequence" (p. 215). Cobb and McClain in collaboration with Koeno Gravemeijer have developed coherent instructional sequences aimed at significant mathematical concepts. Cobb and McClain point out that when pursuing an instructional agenda aimed at mathematical understanding, one must be explicit about the overarching mathematical ideas of the instructional sequence. Once the overall instructional goals are delineated, they orient the formulation of a conjectured learning trajectory that itself serves as the backdrop against which local pedagogical judgments are made. The teacher and students create the *actual* learning trajectory as they interact in the classroom. The teacher has a pedagogical schema, "an overall instructional intent and an envisioned means of achieving it" (p. 215), which provides a sense of direction that is continually modified during the act of teaching. Cobb and McClain assert that teaching must be informed by a relatively deep understanding of students' mathematical thinking and "classroom discussions should be justified in terms of their contributions to the fulfillment of an evolving instructional agenda as indicated by the mathematical significance of the issues that emerge as topics of conversation" (p. 214).

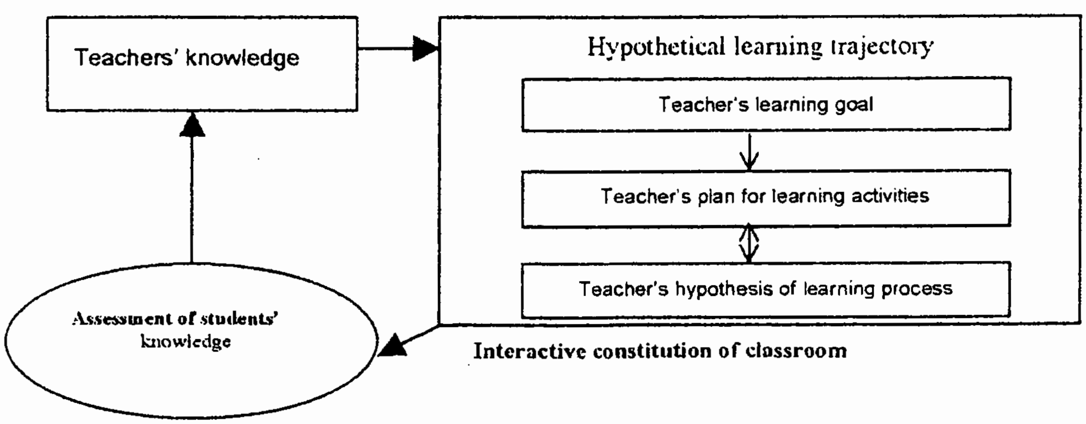

Figure 1. Mathematics teaching cycle (abbreviated).

Simon's (1995) mathematics teaching cycle and Cobb and McClain's (2001) enacting instructional sequences highlight the intersection of teachers' knowledge of subject matter, pedagogy, and student thinking. Teachers are not carrying out the plans and intentions of others. Rather, teaching is viewed as a generative process of idea-driven adaptation where teaching becomes an occasion to deepen one's understanding of the mathematical concepts that become the focus of classroom discussions, of students' reasoning, and of the means of supporting its development. Since the pedagogical decisions made by the teacher have a reflexive relationship with the contemplation of student thinking that might emerge as a result of those decisions, justifying pedagogical decisions in terms of student thinking is supported.

Conclusion

Since teaching math for understanding presumes the importance of the teachers' understanding of students as learners, permeating through this conjectured learning trajectory is the focus on student thinking. We would argue that this is paramount to the effectiveness of the conjectured learning trajectory within the professional teaching community. The underlying assumption of the support structures mentioned is a focus on students' thinking to initiate shifts. Each means of support throughout the conjectured learning trajectory builds on this idea of focusing on students' thinking. The interviews used as a catalyst to make current teaching practices problematic will not be successful unless teachers attend to the fact the students are thinking differently about the two tasks that are presented to them. This alone does not mean the teachers will attend to student thinking. Even though the teachers notice a difference in the students' thinking, they could make changes in their practice without focusing on student thinking. Therefore, it is important that this focus on student thinking permeates the conjectured learning trajectory and its means of support.

This focus on student thinking is evident when trying to cultivate pedagogical content knowledge. In fact, by definition, pedagogical content knowledge calls for teachers' knowledge about how students reason about mathematics. The conjectured means of support for cultivating pedagogical content knowledge was examining student work. As teachers examine student work, the support mentioned for cultivating pedagogical content knowledge, they must attend to more than just the mathematical content and whether or not the student produced the correct answer. The main focus of looking at student work is debating students' thinking behind the diversity of their answers. In order to support students' learning of the mathematical ideas, teachers must first attempt to understand students' reasoning. Teachers can then build from the students' thinking about the problem to support students' learning of the mathematics instead of trying to "fix" the students ways of doing the problem. Since teachers are looking at the student work together they can hold explicit conversations about their interpretations of student thinking and how to support it.

As collaboration develops within the professional teaching community, teachers can begin to hold each other accountable for focusing on student thinking. What makes the lesson study process so valuable as a support here is its "unrelenting focus on student learning. All efforts to improve lessons are evaluated with respect to clearly specified learning goals, and revisions are always justified with respect to student thinking and learning" (p. 121). As teachers plan, teach, and revise a lesson together, decisions about modifications to the lesson are made based on students' thinking.

Focusing on student thinking when planning is a fundamental shift in reasoning about teaching. However, the conjectured ending point of this conjectured learning trajectory is for all pedagogical decisions to be made in terms of students' thinking. A focus on students' thinking should permeate all aspects of teaching: task selection, planning, classroom environment, discourse, informal assessment, formal assessment, etc. One of the strengths of The Mathematics Teaching Cycle and enacting coherent instructional sequences is their focus on having a mathematical agenda while attending to students' thinking. This means decisions made are based on student thinking, but there is a mathematical agenda and thus a sense of direction.

Since the focus on students' thinking is so important to the effectiveness of the conjectured learning trajectory within the professional teaching community, it would need to be a focus of the analysis of the conjectured learning trajectory. This is not to say one would analyze whether the conjectured learning trajectory was successful based on whether or not teachers made a shift to focusing on student understanding. Instead, the analysis would focus on understanding where these shifts occurred and how they were supported. This would then be a rationale for modifications to the conjectured learning trajectory. This cyclic process would produce a modified conjectured learning trajectory that takes account of prior learning. In this way, teacher learning and learning about teacher learning feed forward to inform decisions in a new conjectured learning trajectory.

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TEACHING PLACE VALUE: SEEKING TO RECEIVE KNOWLEDGE, BUT LEARNING TO REASON

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This report is part of a study that analyzed issues in improving Sri Lankan primary teachers' learning and teaching of mathematics in the context of a distance based course for teachers (de Silva, 2001). Three questions frame the research reported in this paper: How do teachers set about learning and teaching mathematics? What aspects of the distance program facilitate their understanding and what fail? To what extent are they supported in their professional development? The questions I proposed called for qualitative research methods where one "begins with an area of study and what is relevant to that area is allowed to emerge" (Strauss & Corbin, 1990, p. 23) as theory grounded in the respondents' experiences. In particular, I used a critical ethnographic approach where the researcher raises her voice to "speak *to* an audience *on behalf* of [her] subjects as a means of empowering them by giving more authority to the subjects' voice" (Thomas, 1993, p. 4, italics in original). In this paper I analyze issues related to a second grade teacher's teaching of place value as the problem emerged and was explored in the research process.

Rationale for and Significance of the Research

Research from around the world indicates a dearth of teachers qualified to teach mathematics (Arnold, Shiu & Ellerton, 1996). The situation is crucial at the primary level where teacher education is given least priority (UNESCO, 1992). In the 1980s the situation in Sri Lanka was similar with a large number of untrained primary teachers (Tatto, Nielsen, Cummings, Kularatana & Dharmadasa, 1991; UNESCO, 1990). By the time of my study in 1998, a 3-year distance primary teacher education program begun in the mid 1980s was being phased out and short-term distance courses were being implemented to provide continuing professional development for primary teachers. Based on quantitative data and analysis of surveys, Tatto et al. (1991) found that graduates of the 3-year program showed a significant drop in both their mathematics knowledge and pedagogical skills after leaving the program.

Although the latter study does not specify the kinds of mathematics knowledge and pedagogical skills assessed, the teaching of place value concepts is fundamental to the primary mathematics curriculum. A number of studies document U.S. elementary teachers' understanding of place value as limited to procedures and sparse in connections to related concepts (Ball, 1991; Leindhart & Smith, 1985; Ma, 1999; Ross, 2001). Ma (1999) also documents Chinese elementary teachers' "profound understanding" of place value concepts and identifies aspects of their educational experience that she believes supports the development of such an understanding. On the other hand, the research reported in this paper, while analyzing one teacher's limited understanding of place value, identifies aspects of her experience that could similarly limit other teachers' understandings.

Student collaboration in planning and implementing programs is a critical in the success of distance teaching and learning of mathematics and mathematics education (Arnold et al., 1996). Given the lack of empowerment accorded to and felt by many, mainly female, primary teachers (UNESCO, 1990a), even if programs promote collaboration, due to prior experiences, teachers may not actively collaborate. Hence, I believed that research on issues faced by primary teachers who are distance learners should be geared to social change through some form of action research, where the researcher works with teachers and encourages them to reflect on and share their practice, voice their concerns, collaborate to be heard, and be active in their own learning. The above "participatory research model" has been identified as specially suitable for the way women come to know, where the validation of their experience is the basis for their making connections (Belenky, Clinchy, Goldberger & Tarule, 1986).

In particular, I believed it would be useful to delve more deeply into the reasons for the phenomenon identified by Tatto et al. (1991) using qualitative methods of data collection and analysis. Such research is best accomplished using extended participant observation as in ethnography (Eisenhart, 1988). The survey of teacher education on which the UNESCO (1990a, 1990b) reports were based recommended ethnographies as a form of research study, but found that few existed in the member countries of the Asia Pacific region surveyed and none in Sri Lanka.

Conceptual Framework

In conceptualizing my research, I used Saxe & Bermudez's (1996) emergent perspective where learning is a seen as a process of evolving goals influenced by prior experiences, social interactions, activity structures, and cultural artifacts. The sociocultural perspective offered by Crawford (1996) influenced my analysis of respondents' reflections on their teaching of mathematics:

> When activity is considered an important factor in the construction of knowledge, a number of questions are raised. How does a learner subjectively experience, or interpret, his or her own action, and the actions of others in an educational setting? What are the relationships... between the cultural context of a learning activity, the behavior of the learner, the learner's subjective experience of the action, and the qualities of the cognitive and affective activity that occur? (p. 132)

Given my questions and the relevant literature, I chose critical ethnography as an appropriate research method. Of the variants of critical ethnography discussed by Thomas (1993), I used a participatory approach where:

> researchers ... become active in confronting explicit problems that affect the lives of the subjects – as defined by the subjects – rather than remain passive

recipients of "truth" that will be used to formulate policies by and in the interests of those external to the setting. (pp. 28-29)

As I conceptualized my study, I began to reflect on issues of voice, identity, my role as a researcher, and my critical ethnographic approach. Given my questions and female respondents, Belenky et al.'s (1986) research provided a framework for analysis that complemented my theoretical perspectives and critical approach. In their research, Belenky and colleagues identified the importance of *voice* – of listening, hearing, and speaking – and of *connection* – of belief, trust, and nurture – in the way women come to know. In my analysis, I tried to hear my respondents' different "voices" and to relate their voices to how they interact with others and their ways of knowing.

Data Collection

My study investigated issues relevant to Sri Lankan primary teachers' learning and teaching of mathematics in the context of the distance-based course Continuing Education in Primary Mathematics (CEPM) offered through the National Institute of Education (NIE). As outlined in the proposal (National Institute of Education, 1997), the objective of the course was to improve primary teachers' knowledge and understanding of the content they teach and the way they teach mathematics.

The 1-year course began in November 1997. I conducted fieldwork throughout 1998 (the calendar year coincides with the academic year) at the NIE, two primary schools, and three tutorial centers. From April to June I lived "on-site" near the primary school L where the second grade teacher Thushara and two of her colleagues – Kamala and Prema, also respondents in the larger study – taught. I conducted follow up visits in October and November. This report is based on data from the following sources: classroom observations; teacher interviews; observations at the tutorial center where CEPM participants met thrice a month for discussion and assessment; interviews with tutors and officials at the NIE; and analysis of the print based course modules.

Data Analysis

A very important topic in primary mathematics, first introduced in the Year 2 curriculum, is place value. I begin by describing a lesson that Thushara taught on place value and analyzing aspects that proved problematic. Next, I discuss some issues related to Thushara's understanding of place value as they were revealed over the course of the study. Finally I discuss the extent to which the CEPM course helped Thushara improve her understanding of place value concepts and of how to teach it.

A Lesson on Place Value

During my 5th week at L, Thushara taught a 3-day lesson on place-value. Since she was teaching the topic for the second time that year, the approximately 50 students (54 on the class list) were somewhat familiar with the manipulatives and representations used.

The Importance of Instructions

Thushara began by asking students to take out their ekel bundles (10 sticks) and remove the string tying one bundle. Holding up a bundle, she asked "How many ekels?" When the children replied 10, she asked, "how many ekels must one add to it to get 11 ekels?" When a child answered one, she asked them to place a single ekel by the side of a bundle. Since Thushara did not say to use the least possible number of single ekels, it was unclear who understood the connection she eventually hoped to make with the numerical representation of numbers. For example, when asked to make 12, I observed at least four students who removed the string tying another bundle and counted out 12 single ekels. The children may have been better motivated by (a) asking whether they could show 11 using the least number of single ekels and (b) having them explain why they used one bundle of 10 ekels and one single ekel to do so.

Thushara drew the ekel representations (e.g., 1 bundle and 3 single ekels for 13) on the chalkboard after asking students to use their ekels to make each of the numbers 10, 11, 12, 13, and 14 respectively. After each was made, she went around checking whether students had placed ekels correctly. By 13, most students had caught on to the expectation of using one bundle and the necessary single ekels and, except for a few who I observed copying what another was doing, most seemed to make the expected representations. The children continued making ekel representations for 15, 16, 17, 18, and 19. Many realized adding a single ekel to the last representation led to the representation for the next number.

After the children had added a single ekel to the nine single ekels and one bundle of 10 ekels to get 20, Thushara asked, "how many bundles of 10 ekels do you have?" Most answered one, while a few children said two, despite their not having exchanged the 10 single ekels for one more bundle of ten ekels. Thushara then asked if they had any single ekels and most students replied that they had 10. Given no instructions about using the least number of single ekels or the idea of exchanging 10 single ekels for one bundle of 10 ekels, Thushara's questions were probably very confusing for the students.

The Importance of Language

In the above activity, after students had made 15, Thushara asked them to verbalize how they had made it. When some said that they had used five from the bundle that was untied, she said "no, you should say five ones". She then asked the entire class to repeat the following: "one bundle of 10 and five ones". As the children worked on 16, 17, 18, and 19, she asked them to verbalize what they used in a similar manner. Next, Thushara drew an ekel bundle, set it equal to ten (= 10) then asked the children "are there any ones in 10?" When some replied no, she pointed to the zero and said "that's why we write zero here." She then asked, "how many bundles of 10 are there?" and when some replied "one" she pointed to the numeral 1 and said "that means one bundle of 10." I noted that while she used the phrase "bundle of 10" quite often, she did not use the phrase "single ekels." Instead she referred to the latter objects as "ones." This was I believe one reason for the considerable number of incorrect responses to the questions Thushara later asked of individuals. Given the students' need to associate the concrete manipulatives with the abstract numerical representations, Thushara probably needed to be very careful and systematic in her use of language, but unfortunately she was not.

The 2nd day was spent in a similar manner representing the numbers from 1 through 20. On the 3rd day, Thushara began by writing the following on the board: Draw ekel bundles for the following: 13, 17, 10, 11, 19, and 15. While Thushara was away on an errand some students finished their work and brought it to me. Seizing the opportunity to find out how the children were thinking, I asked each to explain why they drew what they did. Many were able to explain how the numerals in the numerical representation of the number represented the number of single ekels and the bundle of ekels they had drawn. However, a few students had drawn just one bundle of ekels for each number. When questioned, their response made sense given the instructions "draw ekel bundles" for numbers that were limited to the teens. Once again, the importance of using careful language struck me as being important.

The Importance of Variation

After Thushara returned, she corrected the children's work. I observed that those who had mistakes were simply sent away without asking them to explain how they were thinking or asking questions that would help the child to understand. Later, when Thushara went over the work at the board, she asked "how many say there is one 1 in 10?" All but five children raised their hands. Although Thushara did not ask any of the children to explain their reasoning, I believe that those who replied "one" did so based on the digit 1 appearing in the numerical representation of 10. Thushara then attempted to help the children understand by showing two bundles of ekels and asking "How many bundles of 10? How many ones?" to which most children replied two and zero respectively. I suspect that since these were physical objects, the children had no trouble identifying two bundles of 10, but their response of zero for the number of ones, was very likely due to the digit 1 not appearing in the numerical representation of 20. Perhaps Thushara's lesson on place value would have been successful at least on a procedural level had she had children makes representations of 2-digit numbers other than those in the 10s where by varying the number of groups of ten in a given number she helped children make the intended connection between the concrete and numerical representations.

Discussion

The extent to which the lesson helped children understand place value appeared superficial. Their learning was limited by the lack of careful instructions, language, and variation in the activity Thushara used. As usual, Thushara based her teaching on an activity suggested in the teachers' handbook. The brief outline in the latter gave no indication of the complexity of the issues related to teaching and was thus of little pedagogical use to Thushara other than as an organizational structure. When, at the end of the lesson, students competed in two teams and were asked to identify "how many bundles of ten" and "how many ones" there were in ekel representations for different numbers between 10 and 19, the children made many errors. Although both the children and Thushara seemed to enjoy the competition very much, the focus was only on the answer being right or wrong. No explanations were required nor given. Thushara's lack of instructions about the mathematical idea behind the use of the ekel manipluatives to represent a number using the least number of single ekels and her referring to single ekels as ones were, I believe, the primary reason for most children's failure to make sense of the lesson on place value.

Thushara's Understanding of Place Value

Why was it that Thushara did not give instructions to use the least number of single ekels nor explain the idea of exchanging 10 single ekels for one bundle of 10 ekels? I believe she herself was unclear about their importance.

Seeking to Impart and Receive Knowledge

As I copied Thushara's drawings on the 1st day, I did so in a cursory manner, noting the idea of a bundle of 10 ekels and a given number of single ekels. Midway, however, Thushara said each bundle of 10 ekels should be drawn with nine vertical lines with the line going across representing the 10th ekel. Aware of the way we represent 5 using the tally mark system, I wondered whether Thushara thought this was how one should teach children to represent 10. I then looked closely at her drawings of ekel bundles and observed that they did in fact consist of nine vertical lines with one horizontal line across.

Later, I asked Thushara why she represented the bundle of 10 ekels in that manner. She replied that she used to draw 10 vertical lines and 1 horizontal line until a teacher supervisor had told her she was wrong. The supervisor had cited the tally mark representation of 5 and told Thushara that 10 should therefore be represented in a similar manner. I asked Thushara whether she thought children would understand the horizontal line to be an ekel or whether they would see it as the string. She said that the supervisor should know, but perhaps the tally style representation should be taught after the representation that best captures the physical objects. Thus although her original representation appeared to make the best sense to her, Thushara was still willing to accept the supervisor's knowledge as superior. Further, her style of learning was reflected in her style of teaching where explanations were not given and reasoning was not valued.

Learning to Question and Reason

Thushara, no doubt, continued to think about the issue of meaningful representation. During the group interview the next day, as an example of erroneous thinking on the part of an authority, she repeated what the supervisor had said. The three teachers had been discussing an issue of representation in the Year 1 textbook that their tutor had pointed out was problematic for young children:

Kamala: The thing is addition is one of the first things done in Year 1, so the error happens there. It is not us, but the big places (i.e. Ministry of Education) themselves have approved this.

Thushara: Now yesterday in the *ekel* bundle problems. When I came to Year 2 in the beginning I used to draw 10 ekels and – in the book that's what they show and then with a rubber band. Now the tally mark system is where we draw four and then indicate 5 like this. Mrs. J looked at my work and said, "first draw nine ekels like this and then put the 10th across like this". That the other one was wrong. Not to teach like that again, because that is incorrect. So I said – this was when I first came – "I didn't know, okay" and that's what I was doing since.

Prema: Yes, but then in reality we are not tying the bundle with the 10th ekel.

Thushara: We aren't, but Mrs. J said that was wrong to depict it the way it is done. She said to put nine and tie with the 10th

Kamala: But that is wrong. The children are not going to understand that.

Prema: Yes it's not going to be a bundle of 10 is it? Because you will be tying with an ekel.

As the transcript proceeds, it is evident that this particular teacher supervisor had not only given Thushara bad advice, but had also made her feel humiliated. Thushara's lack of self-confidence is obvious in contrast to the confidence of Kamala and Prema:

Thushara: I tell you, when she said that I became really upset. I thought I'd been teaching children all wrong. So from then on, I started to draw nine and put the 10th across.

Prema: No, no, no. I wouldn't have accepted that as being correct.

Rapti: I think that what you used to do, that's what the children can follow a lot more easily

Thushara: That was right. So I was really upset since she said "the way you do it there are 11 here."

Kamala: That was the string – and the children are not going to understand it as being anything other than the string. (O.C. lots of simultaneous agreement)

Prema: Then you should have asked her, "then should I get the children to have bundles of nine ekels that are tied with an ekel?"

Thushara: Yes, but see she showed me the tally mark system.

Three years later, Thushara still lacks confidence to voice her own opinion or to analyze the situation on its merits. "That was right", not "I was right", is how she responds to my comment about her prior representation making more sense to children. Further, while Thushara seems convinced that Mrs J. was incorrect, she also continues to explain the latter's reasoning. Unlike Kamala, whose focus is on what the children will understand, and Prema, whose focus is on the object being represented, Thushara's focus is on the immediacy of what the supervisor said, and how it made her feel. The tone of each teacher's voice is also different: Kamala's is calm and confident; Prema's is argumentative and confident; and Thushara's is apologetic and hesitant.

Reconstructing Her Understanding of Place Value Representation

In the October interview, held 5 months after the above lesson and group interview, I asked Thushara to describe her style of teaching mathematics. She did so, using her teaching of place value as an example:

Thushara: Often it is difficult to talk and teach them something. For example, numerical representation in order to have them understand, things like manipulatives, objects – it's not like other subjects, math isn't – when teaching math activities are essential for helping children to understand. So when teaching math specially it is through using activities and objects that I do it.

Rapti: So usually when you start a lesson you begin with an activity or use some manipulatives?

Thushara: I don't tell them we are learning this. I give something and through that I have them understand.

Rapti: After they do the activities what do you do?

Thushara: (pause) After they do it, I give exercises to see if they have understood it. I ask questions. I ask them to do something. For example, the ekels – from 21 to 99 is what they have this week – to teach place value. Today what I did was from 1 to 50 – took randomly. At the beginning I show them how it works so they understand. For example 21 is two bundles of 10 and one single ekel. This kind of thing must be done on their own and understood. If I was to write 20 and say this 2 means two bundles of 10 ekels and the zero means no single ekels, the children don't understand. But when they use ekels, the children understand very well.

Thushara speaks of "single ekels" rather than "ones," a significant difference from how she referred to the objects in the lesson I observed. Further she now has children represent random numbers from 1 to 50. Perhaps the group discussion and other intervening experiences, along with her evolving understanding as she continued to teach, helped Thushara develop a more meaningful understanding of the representations that underlie the teaching of place value. However, later in the same interview, she again speaks of the single ekels as "ones":

> Today when I asked them to make 40, they put four bundles of 10 out. When I ask them to say aloud what they have, some are unable to say they have four bundles of 10 and zero ones. But most say that. So then I show that there are no ones in what you showed. That if you add nine ones you get one bundle of 10.

Her words: "if you add nine ones you get one bundle of 10" also recall the confusion created by the supervisor.

Deepening Her Understanding of Place Value

In her final interview, Thushara identified students as very good, average, and weak in mathematics based on what each could do, rather than how they reasoned. In particular, the mathematics she referred to was their procedural abilities rather than conceptual thinking. Thus it was not surprising that, although Thushara seemed to understand the problem, she did not appreciate the significance of her student Nirmala's struggles with the concepts of money and place value being rooted in her lack of understanding the pre-number concept of the one-many relationship.

Since I sat near Nirmala's group, I was able to observe her work. From the beginning, I noticed she tended to look at another's work to check her own, especially for addition, subtraction and other activities involving number operations. During my 2nd week I observed Nirmala identify each of the rupees 1, 2, and 5 coins but hand over one of each to buy a toy priced at 3 rupees. In handing over the three coins for one toy, she appeared to understand the many-one-relationship but not the one-many-relationship. I noticed similar problems in her work with place value and measurement. She was one of the students who counted out single ekels when asked to represent a number.

Towards the end of the group interview, the conversation veered around to the project that the teachers were expected to do for the CEPM course and Thushara asked me to suggest what kind of a project would be good for her to do. I suggested that perhaps she could work with Nirmala citingthe incident with the coins.

Rapti: Nirmala ... suppose she is adding Rs 2 and Rs 5. She has yet to understand that the one coin represents 2 rupees. So though she will identify the coins as rupees 2 and rupees 5, when added she says there are 2 rupees, because she identifies each coin as one object.

Thushara: Yes, today when I asked how many ekels were in the bundle of 10 ekels, she said there was one.

Prema: Ahh, because there is just one bundle, so it is taken as one.

Her making the connection between Nirmala's problems with the value of a coin and the number of ekels in the bundle indicates that Thushara had a good understanding and an immediate grasp of the conceptual similarity between the two situations. However, Thushara's tendency to focus on surface details and procedure, and her need to have pre-structured and detailed guidelines prevents her from building on her grasp of Nirmala's problem:

Rapti: You have at least two examples where you've identified a similar abstract thinking gap in her understanding. You could figure out pre-concepts to build on to help her understand.

Thushara: Yes. (pause)

Rapti: This has to do with the value represented by one object being more than one unit. Perhaps you need to work with objects that have some value meaning for her as a child.

Thushara: How do you get a topic out of it? They say you have to focus on a topic.

Kamala: (O.C. Reads from the handout which describes the requirements of the project) "A problem, an incident, a situation." What this would be is a situation.

Rapti: Yes, you can describe how you noticed the situation—when you did this activity you noticed her doing this and then when you did another you noticed a similar thinking problem. This is what you think is the problem: one-to-one – no

Prema: (interrupting) One-to many.

Rapti: Yes, one-to-many relation seems to be the problem the pre-number concept is lacking.

Thushara: (interrupting) She copies a lot. Looks there writes or does the same. That is a big problem. So she is not willing to think. But her Sinhala is very good. Does the work on her own, is organized, reads and writes very well. (Group Interview)

Was it that Nirmala was not willing to think, or was it that Thushara did not realize the importance of spending time having her engage in activities that help her understand the one-many-relationship? Since she does not offer any thoughts with respect to the pre-number concepts of relations and representations, the extent to which Thushara realized the important developmental progression involved is not clear.

Given Thushara's emphasis on procedure in both her learning and her teaching, it was not surprising that her CEPM project focused on helping students learn a procedure rather than understand a concept. In her final interview in November, I asked Thushara to describe the ideas in her project:

> Well it was about addition where carry over was not involved. Now I was telling the sirs that when children are adding 3 + 2 is 5, some write it in a vertical line. It's the ones who can't that you take for the project. There are eight children. 3 is here, plus sign is here, 2 is here, and then the answer 5 somewhere else. Even 100s and 10s are the same. The 100s column and 10s columns are here and the answer over here. The place value idea is not there – didn't go that far, but I just did that little.

Thushara's thinking it not important to discuss concepts of place value in adding numbers where carry over was not involved is troublesome. Since she asked students to add up to 3-digit numbers, her emphasis on procedure was compounded by her lack of helping children understand that the procedure of aligning numerals enables easy recognition of "like objects" (units, tens, and hundreds). It suggests her own concept of place value was not well connected to her concept of addition. Further, during my first week, I observed Thushara begin a lesson on addition by having the children sing a song that involved vallikukulas (a kind of bird) and rambuttan (a kind of fruit) and then asking them how many vallikukulas and rambuttans there were altogether. In my field notes I noted that addition of unrelated objects did not make much sense. Given Thushara's comments in the above transcript, it is possible that her focus on procedure had resulted in a disconnected understanding of both place value and addition.

CEPM's Discussion of Ideas Related to Place Value

Despite the second module being on number and numeration systems (CEPM 102) and the third on basic operations (CEPM 103), Thushara did not develop a solid conceptual understanding of place value in the course. In CEPM 102 different numeration systems are presented separately, with brief discussions and examples of properties. Each system is further illustrated with examples of operations on numbers. On the other hand, the discussion of place value in the Hindu Arabic numeration system is extremely brief: "The value of a numeral depends on the place it holds. e.g., in the number 333, the numeral 3 represents the numbers 3, 30, and 300." A comparative discussion of the properties across the different systems would have enabled a better understanding of place value. Similarly, the discussion of basic operations in CEPM 103 focuses on set-theoretic properties and abstract definitions. Teachers like Thushara would have benefited from activities that helped them construct a concrete and meaningful understanding of the operations and how place value is implicit in why our algorithms work.

In particular, no activities that help teachers model and discuss the structure of the Hindu Arabic numeration system and our computational algorithms are presented in either of the modules. Given their use in the primary classroom, ekel based manipulatives extending beyond the bundles of 10, could have provided teachers with the tools to start understanding the concepts behind the description of place value. An activity structured with the emphasis on always exchanging groups of 10 similar smaller objects for 1 bigger object could have helped teachers to better understand that the manipulatives do not themselves have inherent mathematical meaning. Activity whose emphasis was on understanding the importance of the careful instructions that should accompany a child's use of manipulatives would have enabled these teachers. But, as with most modules, the teachers were not asked to think about teaching their students. Further, given Thushara and other teachers' reliance on the very limited activity outline in the teachers' handbook, the CEPM modules could have helped teachers to collaborate on expanding and supplementing their limited resources. However, as discussed in the larger study (de Silva, 2001), the modules rarely asked teachers questions that required them to go outside the given module let alone make connections to other modules. Overall, the course as implemented did little to help primary teachers improve their understanding of the primary mathematics content or pedagogy, despite that being its proposed objective.

Supporting the Emergence of Subjective Knowing

Thushara's narrative of learning and teaching the primary mathematics concept of place value highlighted issues that may affect other primary teachers. Given that Thushara was the one teacher in my study who diligently followed the self-study aspect of each module, her experience provided a way in which I could analyze the CEPM course by focusing on how successful it was in helping teachers improve their understanding of the concepts they teach, in the context of their professional lives. During my study, Thushara's voice and ways of knowing seemed most similar to those of women Belenky et al. (1986) describe in their chapter on "Received Knowledge: Listening to the Voices of Others" as follows:

> She "learns" the material; that is she stores a copy of it, first in her notes, and then in her head. She does not transform the material; she files it "as is". She willingly reproduces the material on demand, as on an exam; but she feels betrayed if the teacher asks her to "apply" it or to produce materials on her own. (p. 42)

However, during my observation, rather than the "subjective" knowing that follows "received" in Belenky et al.'s (1986) continuum of ways of knowing, Thushara showed signs of the kind of knowing the former call "procedural": "Unlike many of the [subjective knowers], [procedural knowers] retain some trust in authority. The presence of fairly benign authorities may be critical to the development of the voice of reason" (p. 90). I believe that one of her tutors, Kamala, Prema, some other CEPM colleagues, and I were for Thushara such "benign authorities." But since it was very hard to hear Thushara's voice of reason, I believe Belenky et al.'s (1986) continuum holds true, and that Thushara was discovering the limits of received knowing:

> The women ... bring this era to an end with critiques of their tendency to subordinate their own perceptions and judgements to those of others, of their selflessness, of their voicelessness. (p. 51)

Thus at the end of my study, I believe Thushara's voice reflected the "emergence of subjective knowing" where "there is still the conviction that there are right answers" (p. 54). Belenky et al. (1986) found that "almost half [of the 135 women interviewed] were predominantly subjectivist in their thinking" (p. 55). Thushara's experience may thus be quite similar to a significant number of her female colleagues. In its contribution to the aims of my study, Thushara's narrative thus helped me realize the need to encourage teachers' voices, to support their efforts to change and adapt, and to help them learn when to question authority and when to trust their own reasoning.

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TEACHING MATHEMATICS IN THE SPIRIT OF EDUCATION REFORM: PERSPECTIVES FROM THE UNITED STATES AND NAMIBIA

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The outcome of an ongoing, multi-year initiative to enhance mathematics teaching and learning, this transnational study examines the experiences of elementary teachers in the United States and Namibia learning to teach in the spirit of national reform efforts. Qualitative methods were employed to gather and analyze the data. The analysis suggests that several factors played a role in learning to teach mathematics as envisioned within education reform documents: mentoring, reflection, technology tools, reform-based curriculum materials, and collaboration within a mutually supportive environment. The teachers' conceptualizations of an effective mathematics teacher, their conceptual knowledge of elementary mathematics content, and the nature of the learning environment facilitated or impeded this process. Because this study provides some insight into the complexities of implementing mathematics teaching practices as envisaged by education reform agendas, it has the potential to inform the structure and content of professional development workshops and elementary mathematics methods courses.

Introduction

After a protracted armed struggle for liberation from the colonial apartheid government of South Africa, Namibia became an independent nation in 1990. Although the apartheid forces were defeated, the legacies of the illegal and repressive regime, including an educational system shaped by divisive and dehumanising imperialist policies, remained. As they abolished the discriminatory and inequitable Bantu educational system and developed a new praxis, the Namibian government mandated reforms based upon the goals of access, equity, quality, and democracy (Ministry of Education and Culture [MEC], 1993). The transformed educational system is modelled along democratic, empowering, and reconstructive lines. As such, it rejects the authoritarian, teacher-centre Bantu system, emphasises student-centred learning, uses English rather than Afrikaans as the medium of instruction, and takes into account students' prior experiences and their intellectual, emotional, social, physical, aesthetic, moral and spiritual development. The redesigned school curriculum is structured upon a constructivist view of knowledge, learning competencies in content areas, and developing a reflective attitude and creative, analytical and critical thinking (National Institute for Educational Development [NIED], 1998).

In the United States, mathematics education reforms provide vision and direction for school mathematics (Conference Board of the Mathematical Sciences [CBMS], 2001; Mathematical Sciences Education Board [MSEB] & National Research Council [NRC], 1990; National Council of Teachers of Mathematics [NCTM], 2000). A prominent reform-oriented view advocates teaching mathematics in such a way that all students develop "mathematical power" (NCTM, 1989, 2000). Mathematical power includes both ability (to conjecture, reason logically, solve nonroutine problems, and communicate about mathematics) and attitude (self-confidence and a disposition to question, explore, and engage in significant mathematical problems). This goal requires that teachers use knowledge of how students construct conceptualizations of mathematical concepts to select and use materials, develop multiple teaching strategies and instructional activities, design and utilize authentic assessment strategies, and create learning environments in which all students learn mathematics meaningfully. Implementation of these reforms regarding mathematics curriculum and instructional practice requires that teachers and students engage in problem solving, reasoning, communicating, and making mathematical connections.

Although they arose from very different circumstances, in calling for high quality mathematics instruction for all learners, the reform movements of both Namibia and the United States make clear a commitment to equity. In addition to supporting democratic reforms in teaching and teacher education, the emphasis on a constructivist approach to mathematics teaching and learning provides a vision of mathematics education that differs markedly from present practice in many U.S. and Namibian classrooms. To further the implementation of the education reform agendas of both nations, teachers must develop a new perspective on mathematics teaching and learning. This includes possessing a deep understanding of the mathematics they teach and using instructional strategies that promote the active engagement of learners and develop both conceptual and procedural knowledge. The study reported here is part of a larger study designed to explore the process of learning to teach in the spirit of education reform among a small group of elementary teachers in the United States and Namibia. The inquiry was guided by the question, "What strategies are useful in preparing teachers for mathematics teaching as envisioned within education reform documents?"

Theoretical Framework

Elementary teachers' weak mathematical knowledge (Ball, 1990, 1991; Thompson, 1992) and relatively narrow conceptions of mathematics pedagogy (Brown, Cooney, & Jones, 1990; Lampert & Ball, 1998) have been well documented. In an effort to determine whether certain strategies are useful in preparing elementary teachers for reform-based teaching, this study was guided by the intersection of research, theory, and practice in the areas of reflective practice and technology tools.

Reflective Practice

A goal of many teacher education programs is to develop the teacher into a reflective practitioner, "who perceives every experience as an opportunity for growth, change, and development of understanding" (Hutchinson & Allen, 1997, p. 226). The *Principles and Standards for School Mathematics* (NCTM, 2000) and the *Subject Area Curriculum for Educational Theory and Practice* (MEC, 1992) highlight the importance of teachers engaging in reflective practice. *As* defined by Dewey (1933), reflective practice may be viewed as a dynamic process that requires an action outcome. The process of facilitating critical reflection draws upon the notion that teachers are expected to become more self-consciously reflective about their actions (Schon, 1983). However, as Risko, Roskos & Vukelich (1999) insist, "prospective teachers need explicit guidance in reflection so as to advance their natural tendencies beyond mindless ritual towards a critical stance on the pedagogic understandings and actions" (p. 7). During the course of teacher preparation programs, myths and stereotypes regarding diverse student populations may be woven and reinforced time and time again (Flores, Tefft Cousin, Diaz, 1991). Without structured reflection, preservice teachers' misconceptions may be reinforced rather than challenged (O'Grady, 1998). While teacher behavior may be influenced by prior educational experiences, experience is educative only with time for reflection (Richardson, 1990). As teachers become reflective in their teaching experiences and classroom practices, they develop a critical perspective on education and reconceptualize their image of teaching and learning (Howard, 1995). In an attempt to address the gap in the research on reflection in mathematics teaching (Ball, 1996), this study examines elementary teachers engaging in reflective practice on reform-based mathematics teaching.

Technology Tools

Research on the appropriate use of technology reveals that technology tools can enhance mathematical knowledge and conceptual understandings (Grouws & Cebulla, 1999; Wattenberg & Zia, 2000). Because they enable users to visualize and experience mathematics, engage in real-world problem solving, and generate representations of their own learning, technology tools are important resources for teaching and learning mathematics (International Society for Technology in Education, 2000). According to the National Council of Teachers of Mathematics' *Technology Principle*: "Mathematics Instructional programs should use technology to help all students understand mathematics and should prepare them to use mathematics in an increasingly technological world." (NCTM, 2000, p. 40). This suggests that teachers should be provided with adequate preparation and support to implement technology tools in their classrooms. Teachers should be provided with ongoing mentoring and should have time and support to familiarize themselves with software and content to incorporate technology into their lesson plans (President's Committee of Advisors on Science and Technology [PCAST], 1997). Technology tools to enhance teaching and learning should be an integral component of teacher education programs (Knapp & Glenn, 1996). Teachers should be provided with ongoing mentoring, time, and support to learn and to incorporate technology into their lesson plans (PCAST, 1997). An integral component of professional development should be to model for teachers the ways in which technology tools can be utilized to enhance mathematics teaching and learning (Barron & Goldman, 1994). The current study draws inspiration from these earlier studies and seeks to draw together the findings from these works to inform practice and to form a new perspective from which to view elementary teachers learning to teach in the spirit of education reforms.

Methods and Data Sources

The methodological underpinning of this study is derived largely from orientations to research that draw attention to the importance of detailed qualitative fieldwork and the observation and analysis of participants in contexts (Goetz & LeCompte, 1984); in this case, educational settings in the United States and Namibia.

Participants, Site, and Context

Participants included four elementary teachers in the United States and four in Namibia who had voluntarily enrolled in professional development workshops designed and facilitated by the researcher. Two of the Namibian teachers were in their second year of teaching and had completed the Basic Education Teachers' Diploma (BETD). Introduced in 1993, the three-year post grade twelve BETD was designed to support the effective teaching of the post-independence primary school curriculum. Two other Namibian participants were veteran teachers with five or more years of teaching experience and secondary school educations supplemented by post-independence in-service training courses designed to enhance the education of under-qualified in-service teachers. The U.S. participants included two novice teachers with degrees in elementary education. Each was in their second year of teaching in a public elementary school. Two other U.S. teachers who participated in the study were veteran teachers with five or more years of teaching experience.

In an attempt to make teacher preparation congruent with reform documents and reflect a vision of mathematics teaching and learning as described in education reform documents, the workshops are designed to model a learning environment based on constructivist mathematics teaching and learning and attention to issues of access and equity. A fundamental goal of these workshops was to prepare elementary teachers to teach mathematics in the spirit of education reforms. The teachers met in the workshops for one week, then returned to their respective schools. The workshops were designed to challenge the teachers' views about teaching mathematics and to assist them in aligning their teaching with educational reform efforts. During the workshops, the researcher provided a mentoring component and modeled inquiry-based, student-centered, collaborative learning. The participants explored the ways in which reflective practice (journals of reflections), technology tools (*KidPix, TesselMania, Geometer's Sketchpad, and Tabletop Jr.*) and reform-based curriculum materials (*Investigations, Everyday Math,* and *Math Trailblazers*) could enhance mathematics teaching and learning.

Data Collection and Analysis

Data collection and analysis took place simultaneously over a two-year period, was grounded in a symbolic interactionist framework (Blumer, 1969), and guided by the constant comparative method of data analysis (Glaser & Strauss, 1967). Consistent with the qualitative research methodology of the study, data were triangulated via multiple sources of evidence, including: (1) participant observations, audiotaping, and field notes of the workshops and the teachers' mathematics lessons in their own classes; (2) three semi-structured, open-ended interviews (Spradley, 1979) with each teacher before, during, and after the workshops; and, (3) collection of the teachers' journals of reflections, lesson plans, and samples of student work. Major themes were developed using thematic analytic strategies (Spradley, 1979). Findings were shared, discussed, and compared by the researcher and the teachers. In order to provide a measure of external validity (Goetz & LeCompte, 1994), the researcher reviewed transcripts and analyses with the participants and allowed them to react to analyses and clarify and elaborate on their responses. This process also enabled the teachers to reflect upon what transpired in their classes when they attempted to align their mathematics teaching with education reforms.

Results and Conclusions

Three major themes emerged from the data analysis. While informative, these present a picture that is general in nature. In order to provide some insights into the totality of an individual teacher's experience, a profile, in the form of an in-depth case study, was compiled for each teacher. These profiles examine the themes in the context of the teachers' own experiences. Segments of the interviews, reflective journal entries, and field notes of observations illustrate that the teachers' conceptualizations of an effective mathematics teacher, their conceptual knowledge of elementary mathematics content, and the nature of the learning environment influenced the types of teaching that emerged among the teachers. Perhaps because reform agendas in the United States and Namibia promote similar views of education, significant differences between the teachers in the United States and the teachers in Namibia did not emerge.

Conceptualization of an Effective Mathematics Teacher

The findings from the present study suggest that one challenge to facilitating teaching in the spirit of education reforms is the teachers' resiliency toward (Lampert & Ball, 1998) and their tendency to rely upon traditional methods of teaching. This appears to be related to an image of an effective mathematics teacher held by the teachers. As they engaged in inquiry-based, student-centered teaching, there was evidence of change with regard to the teachers' conceptualizations of an effective mathematics teacher. Excerpts from the case studies of Amy and Erna suggest a move away from a conception of an effective mathematics teacher as a transmitter to one of a facilitator.

The Case of Amy

At the time of the study, Amy was a first-grade teacher with two years of teaching experience in a U.S. public school. During her initial interview she described an effective mathematics teacher as "someone who can get through all the material the students need to know and give them the skills they will need." At the onset of the workshop, Amy declared, "an important part of teaching math is showing the children the best way to solve problems." Although she adhered to these definitions throughout the workshop, observations of her post-workshop teaching revealed that she began to supplement her textbook lessons with several of the student-centered explorations from the *Investigations* curriculum. After successfully teaching several lessons from the *Investigations* curriculum unit on measuring, *Bigger, Taller, Heavier, Smaller,* Amy reflected, "I do see how I *was* more of a guide here. The lessons were so open that the children had to discover the math themselves. My part in it was different than just telling them to do it this way."

The Case of Erna

A fourth grade novice teacher in a primary school in Namibia, Erna was exposed to reform-based curriculum theory and practice during her Basic Education Teachers' Diploma coursework. Nevertheless, during her initial interview, she identified an effective mathematics teacher as "a person of authority." Follow-up questions suggest that Erna's conceptualization of an effective mathematics teacher was rooted in her prior educational experiences in authoritarian, teacher-centered classrooms. Observations of workshop sessions demonstrated Erna's engagement and participation in collaborative learning groups and discussions of the role of the teacher as envisaged by reform documents. During her final interview, she expressed one of the most widely noted insights by the teachers: the recognition that "what sense is it hurrying through the content if the learners are not understanding the concepts?" Referring to a discussion that took place during one of the workshop sessions, Erna observed, "It is important for me to remember that I am not the only one with the answers."

Conceptual Knowledge of Elementary Mathematics Content

It was during the interviews that the teachers' weak conceptual understandings were first revealed. Post-workshop classroom observations confirmed a gap in the teachers' mathematical knowledge as they attempted to think mathematically in a new setting (Crawford, 1992). The data suggest that the teachers' attempts to articulate mathematically sound explanations for the concepts underlying the definitions and make connections between the concepts were undercut by their lack of sufficient content knowledge. Selections from the case studies of Selma and Amber reveal that the teachers' weak conceptual knowledge hindered their ability to engage in reform-based mathematics teaching.

The Case of Selma

At the time of this study, Selma was a third grade teacher in her second year of teaching in a Namibian public school. She had completed the three-year Basic Education Teachers' Diploma program and was eager to learn how to implement reform-based teaching practices in her own classroom. In her second interview, Selma acknowledged her weak mathematical content knowledge. For example, after working with the curriculum materials and the technology tools, she declared, "Honestly, before this workshop, I didn't even know what a trapezoid was or that there were meanings behind the numbers and connections to real things." During her final interview, Selma indicated that she found the collaboration and mentoring especially helpful in developing her mathematical understandings, "We were never left alone to struggle through the problems. There was always someone there to show us another way of looking at the problem. " After she returned to her classroom, Selma began to let the students work collaboratively on several of the lessons she explored during the workshop. In a post-workshop journal entry, she reflected, "Teaching this way is more interesting. In fact, I am learning along with the class."

The Case of Katie

Although she had five years of teaching experience in a fourth grade classroom, Katie admitted in a reflective journal entry, "You know, I really don't like math. I know this stems from my own bad experiences. But, I am determined to teach math better than I was taught it." Throughout the workshop, Katie embraced the notion of teaching mathematics for understanding, including the use of technology tools to support the development of conceptual knowledge. After returning to the classroom, Katie struggled with her own weak conceptual understandings. During one observation, she began a conceptually-focused lesson on building understanding of multiplication and division, became confused by a student's question, and quickly reverted to a procedural explanation. Katie reflected on this lesson in her journal, "Today I had an awakening. I was trying to explain a problem and then I realized that I didn't understand it myself so I couldn't explain the WHYS. This experience has taught me to make sure that I understand what I am going teach."

Nature of the Learning Environment

The workshops provided the teachers with opportunities to experience as students reform-based teaching strategies prior to their utilizing them in their own classrooms (Borasi, 1990). However, the data revealed a conflict between reform-based teaching and the learning environments to which the teachers returned. It appears that, once back in their own learning environments, the lack of professional support, including collaboration and mentoring, prompted several teachers to consider a return to traditional methods. Segments from the case studies of Mary and Amber illustrate how the nature of the learning environment influenced the teachers' abilities to implement reform-based teaching.

The Case of Mary

At the time of this study, Mary was an experienced third grade teacher in a Namibian primary school. She had supplemented her secondary school education with several training courses. Mary was an active workshop participant who eagerly embraced the reform-based curriculum materials and acquired competency with the technology tools. During her final interview, Mary expressed a belief that, despite the traditional culture of her school, participating in the workshop and returning to her classroom with a variety of reform-based lessons and instructional strategies would enable her to teach mathematics in the spirit of educational reform documents. However, a post-workshop reflective journal entry suggests that the nature of the learning environment to which Mary returned might have hindered her ability to implement reform-based teaching practices, "I am striving to make my classroom as modeled in the workshop. I did see a new culture. Yet, I am the only one here doing these things?"

The Case of Amber

When this study took place, Amber was in her second year as a teacher in a multi-grade, fifth/sixth grade classroom in a U.S. public school. Throughout the study, she enthusiastically embraced the notion of increased conceptual experiences for mathematics learners. During her second interview, Amber mentioned several ideas that she would implement in her own classroom, "Because I think students learn best through hands-on activities, I plan to bring manipulatives and collaborative group work into all my math lessons." While Amber remained enthusiastic about inquiry-based lessons, the reality of implementing this approach in her classroom of thirty students was overpowering. The culture of the school had made the students comfortable with teacher-centered, procedurally-focused lessons with an emphasis on drill and practice. During her final interview, Amber said, "I am really excited about all this but the kids aren't used to it, you know? I can see that it takes a lot to make it all work, the computers, writing about math, and so on."

The present study suggests areas for continued research that will foster an understanding of how teachers construct an image of mathematics teaching as portrayed in reform documents. This study revealed that mentoring, reflection, technology tools, reform-based curriculum materials, and collaboration within a mutually supportive environment played a role in facilitating mathematics teaching as envisioned within education reform documents. A related study, underway for the past year, is examining the impact of video episodes of reform-based teaching on the learning-to-teach process among preservice elementary teachers enrolled in a field-based methods course (Dunn, 2002). One challenge is to provide for sustained and adequate mentorship opportunities that assist teachers in developing competencies in reform-based mathematics teaching. The implications highlighted by the present study, including preparing teachers to incorporate teaching strategies that address educational reforms support efforts to implement reform-based practices in teacher preparation and have the potential to inform the structure and content of inservice and preservice programs. The next critical step will be to develop strategies to help all teachers engage in and sustain reform-based mathematics teaching.

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WHAT DO ELEMENTARY TEACHERS LEARN FROM REFORM MATHEMATICS TEXTBOOKS?

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This study was undertaken to better understand what happens when teachers implement a reform curriculum designed specifically to communicate mathematics content and pedagogy to teachers. More specifically, the study focuses on what aspects of the curriculum materials teachers consider as they decide what they will enact in the classroom. Information was gathered to determine the extent to which teachers were understanding the main mathematical content of their first unit on number, the kinds of thinking that students were intended to develop, and how useful the supporting materials embedded in the curriculum were in helping them enact the curriculum. The findings in this study about what teachers learn from this supporting information should be of interest to researchers, curriculum developers, and professional development providers as they consider ways to support teachers' implementation of reform curricula.

Research in the last two decades on teachers' use of mathematics textbooks necessarily utilized traditional textbooks as the teachers' main resource. Findings from these studies indicate that teachers often make major alterations to the textbook lessons resulting in an "enacted" curriculum that looks very different from the intended curriculum (Ball, 1988; Grant & Kline, 2002; Remillard, 2000). In addition, the teachers struggled to create a coherent and effective "enacted" curriculum. Therefore, suggestions were made to include in current reform textbooks more information to help teachers learn and to support their creation of an "enacted" curriculum.

In considering what types of information might be useful to teachers, Davenport and Sassi (1995) made available a collection of approximately 600 resources (articles, videotapes, curriculum materials, etc.) to a group of elementary teachers as they worked to think differently about mathematics teaching and learning. The teachers were allowed to choose which resources they wanted to explore and submit reflective writings on the usefulness of these resources. The majority of teachers chose resources that used a narrative structure to provide concrete images of classroom discussions, student thinking, and common misconceptions. Clearly narratives, as an external resource, had a powerful impact on teacher learning. However, it remains to be seen what impact this type of information could have on teacher learning if imbedded within curriculum materials and how this type of information might effect the enacted curriculum teachers create. This study investigates this issue by considering the following question: what about reform curriculum materials do teachers consider and how does this material impact their thinking?

Methodology

Context

This study takes place in the context of a three year funded project designed to support more than 325 teachers and administrators in six local school districts (urban, suburban, rural, public, and private) as they work to improve mathematics teaching and learning. These districts all adopted one of the National Science Foundation-funded reform curricula, *Investigations in Number, Data and Space* (henceforth called *Investigations*), which focuses on reasoning and problem solving where students are encouraged to make sense of the mathematics they are learning and to use procedures that they understand, rather than those they may have memorized but may not fully understand. One of the goals of this curriculum is to communicate mathematics content and pedagogy to teachers by including information, often in narrative form, describing the importance of particular content, describing various strategies students may use and why they work, discussing connections among topics, and providing sample conversations a teacher may have with a student or group of students on a particular mathematical idea. The curriculum is structured into separate modules focusing on particular mathematical topics. There are 6 - 11 modules at each grade level with at least two at the lower elementary level and at least three modules at the upper elementary level focusing on number.

Data

All 329 K-5 teachers in the 6 districts in the project were sent a survey asking them to analyze one module at their grade level. One hundred and twenty-three teachers completed and returned the surveys, representing 5 of the 6 districts in the project in grades K, 1, 3 and 5, and all 6 districts in grades 2 & 4. These teachers represented a range of years teaching in general as well as teaching *Investigations*, although the majority of teachers were less experienced with the curriculum. Sixty-one percent of the teachers who returned surveys had only 1-2 years experience using *Investigations*, while the others had 3 or more years experience, and often more than 5. The first module on number at each grade level was chosen for the analysis, because they involved a topic every teacher would certainly deem important and there was a guarantee that every teacher would actually teach the modules since they appeared in the beginning of the school year.

The intent of the survey was to gather information on what the teachers were considering in each particular module. One section of the survey focused on what teachers thought of the information provided to them in the curriculum materials, referred to as Teacher Notes (TNs) and Dialogue Boxes (DBs). The TNs provide information typically on the mathematics content and on the ways in which children think about the content. The DBs provide examples of classroom discussions around the content. Each survey included a list of all the TNs and DBs in the unit, the number of which ranged from 13 in grade 3 to 30 in grade 1, and asked that teachers rate each item as to its usefulness, on a scale of 1 to 4 if they had read the item; otherwise, they were to indicate that the item was "never read." The teachers were then asked to choose, and provide a rationale for their choice, the TN/DB that they found most helpful and the one they found least helpful. A second section of the survey asked the teachers to reflect on the unit as a whole and describe the mathematics content the students learn, the specific strategies or ways of thinking about number that were developed in the unit, and their overall impressions of the unit.

Approximately a year later, a short follow-up survey was administered asking teachers to assess the helpfulness of a variety of different factors in their efforts to implement the *Investigations* curriculum. They had to rank, from 1 to 4, the helpfulness of: reading the information provided in the curriculum materials; attending various forms of professional development; observing their students during implementation; and talking with other teachers.

Analysis

TNs and DBs

The teachers' ranks for TNs and DBs from 1 to 4 were segregated into ranks of 1 and 2 (generally not useful) and ranks of 3 and 4 (generally useful). The averages for teachers overall as well as the averages by grade level were analyzed for any patterns that would indicate the types of TNs or DBs teachers were choosing as useful or not. Explanations for the teachers' choices of most or least helpful TN or DB were coded into 6 general categories—mathematical content understanding, pedagogical understanding, understanding student thinking, management, time, and general comments. Mathematical content understanding referred to explanations that expressed an enhanced understanding of the teacher's own mathematics. Pedagogical understanding referred to explanations that expressed a better grasp of how to teach the material at hand. Understanding student thinking was used when teachers explained that the TN or DB helped them interpret different types of student responses and what those indicate about the students' understanding. Management and time were used when teachers explained that a TN or DB helped them with management, such as organizing materials, assigning partners, etc., or when they helped them structure the time spent on any given activity in the lesson. Finally, explanations were coded as general when they were not specific to a particular TN or DB. For example, some teachers simply stated that they use all of the TNs and DBs or that they liked this feature in the curriculum in general.

Within these categories, responses were also coded as to whether the teachers had a more critical or negative impression of the TNs or DBs or whether they simply disagreed with the information given in them. For example, a response that was coded as negative for pedagogical understanding was "This shows students thinking aloud, but it doesn't give any help for students who are stuck." Another example of a response that was coded as negative was "I was still confused about what I was supposed to get the students to understand." Finally, these explanations were analyzed by looking for trends among the teachers overall as well as by looking for trends among less experienced teachers (1-2 years experience with *Investigations*) and more experienced teachers (3 or more years experience).

Description of Mathematical Emphasis of Unit and Student Strategies

The teachers' descriptions of the mathematical intent of the unit and the strategies students learned after completing the unit were also coded. The teachers' descriptions of the mathematics were coded to represent the extent to which they captured the mathematical emphasis of the unit. A 4 represented a detailed description of content or a well-developed characterization of big ideas; a 3 represented a description of some big ideas but may have other less developed descriptions included, or a good list; a 2 represented lists with little description, often including less important ideas; a 1 represented no identification or description of big ideas; and a 0 was used for no answer or negative comments.

These descriptions were also analyzed by looking for trends among the teachers overall as well as by looking for trends among less experienced teachers (1-2 years experience with *Investigations*) and more experienced teachers (3 or more years experience).

The descriptions of student strategies were coded for their level of persuasiveness. A response was coded as "persuasive" if it contained a detailed description of the idea or used names of strategies that were well understood and "not persuasive" if it seemed to focus narrowly on more insignificant mathematical ideas or strategies or missed important ideas. Some examples of descriptions that were "not persuasive" were "It utilizes children's many ways of looking at the world and making sense of it" and "I think the emphasis on the idea of more than one way to solve/record results from activities an important concept for children to know." The first example does not describe a strategy per se and the second example speaks about the teacher's objectives rather than strategies.

Results and Discussion

What Materials Were Considered?

In general, the teachers read the majority of TNs and DBs in their unit and rated them as being useful in implementing *Investigations*. See Table 1. The largest proportion of unread TNs and DBs occurred in kindergarten. This may be due to the fact that these units were published last so that many of the teachers had not been able to read all of the support materials in that particular unit. The teachers were also very willing to rate some TNs and DBs as not useful (those rated 1 & 2). There was no pattern in the ratings they made, except that many rated TNs dealing with management issues, such as storing manipulatives, etc., as not useful (or perhaps not necessary).

This overall impression of the TNs and DBs being useful is further supported by the teachers' reasons for their choice of most helpful TN or DB. As Table 2 illustrates, the majority of teachers' reasons dealt mainly with the substantive issues of understanding how children think and orchestrating classroom discourse. In particular, the teachers appreciated seeing sample student work and commented on the usefulness of

Table 1. Usefulness of TNs and DBs

Teacher Notes (TNs)						
	K	**First**	**Second**	**Third**	**Fourth**	**Fifth**
Total # of TNs	12	21	11	7	10	13
TNs rated 3 & 4	77%	71%	72%	67%	69%	68%
TNs rated 1 & 2	23%	29%	28%	33%	31%	32%
Unread of TNs	24%	6%	10%	1%	3%	6%
Dialogue Boxes (DBs)						
Total # of DBs	7	9	5	6	4	6
DBs rated 3 & 4	69%	80%	70%	61%	69%	67%
DBs rated 1 & 2	31%	20%	30%	39%	31%	33%

Table 2. Reasons for Choosing TN/DB as "Most Helpful"

	K	**1**	**2**	**3**	**4**	**5**
helped understand the mathematics content better	14%	6%	13%	16%	42%	30%
helped understand how children think or what they might do	43%	25%	58%	42%	19%	20%
helped make pedagogical decisions related to classroom discourse	36%	56%	23%	37%	31%	40%
helped with general management	7%	14%	6%	5%	8%	10%

having examples of how to handle incorrect answers. Although many teachers felt that the supporting materials also helped them understand mathematics content better, this was more pronounced in the upper grades.

Finally, there did not seem to be any preference for TNs versus DBs. Overall, TNs were chosen as most or least helpful about 70% of the time. This is in keeping with the general proportion of TNs to DBs in each of the units in that typically 67% of the supporting materials in each unit are TNs. This was the case at all grade levels for most helpful, and all but one grade level for least helpful. Grade 3 teachers chose DBs as being least helpful at a higher rate than TNs. In this case the results are understandable as one DB was from an excursion lesson (which is skipped by most teachers) and the other was the first DB in the unit, which is purposefully simplistic.

These results were also analyzed according to years of experience with the curriculum. One might assume that more experienced teachers would make different choices than less experienced teachers in identifying supporting materials that were most helpful or least helpful. For example, more experienced teachers who have had more time to understand the content and student thinking might rely more on the DBs to help them facilitate richer discussions. However, there was no difference in the choices made by less experienced versus more experienced teachers. It could be the case that the more experienced teachers identified what had impacted them the most in all their years of teaching rather than what impacted them in their most recent year of teaching. This would basically remove the effect that experience might have on the teachers' choices.

How Did the Materials Impact the Teachers?

Describing Mathematical Emphases of the Units

In general, the majority of teachers (69%) were able to identify and describe the important mathematical emphases of the units at their grade level. Their descriptions referred to such topics as recognizing and using landmark numbers to navigate the number system, developing number sense, and understanding the meaning of the operations. Some teachers, particularly at the primary level, had difficulty distinguishing between actual content goals versus such process goals as problem solving, communicating, and representing. Although they were asked to describe the mathematics content they thought students would learn, close to 28% of the K-2 teachers' responses dealt with process goals. Grade two teachers spoke more about processes than any other grade. In addition, kindergarten teachers often discussed the notion of "representing quantities with pictures, numerals or words." While this is an important process goal in the curriculum, it can be problematic if teachers are focusing on this without thinking about the mathematics content children are using and exploring by doing the representations. This may suggest that it is necessary to make a stronger distinction between content and process goals in curriculum materials.

The results of analyzing teachers' descriptions of the mathematical intent of the units according to their levels of experience with the curriculum are shown in Table 3. Overall the more experienced teachers were better able to describe the mathematical emphases of their units. This suggests that there is an effect of experience on understanding of content, or at least the ability to describe that content well. As Table 3 shows, this result holds true for all grade levels except kindergarten. In light of the fact that there is only one kindergarten teacher with 3 or more years experience, this exception is not significant. Any additional analysis of grade level variations, like the fact that more experienced second and fifth grade teachers seemed to show particular strength in describing the mathematical emphases in their units, does not seem reasonable given the relatively small numbers of teachers at any particular grade level.

Finally, there were four teachers (two more experienced, two less experienced) who commented negatively about the mathematics content in some of the units. Their comments focused on two issues: some believed that the content of the first number module was mainly review; others felt that there was not enough practice with memorizing basic facts. Both issues speak to a major philosophical difference between reform-based curricula and traditional curricula – the reform-based curricula are purposefully designed to provide students with more time developing both conceptual and procedural ideas before encouraging students to work towards fluency.

Table 3. Quality of Description of Mathematical Emphases of Units

Teachers with 1-2 Years Experience with Investigations							
	K	1st	2nd	3rd	4th	5th	K-5
Ranked 3 or 4	77%	50%	64%	53%	77%	50%	64%
Ranked 1 or 2	11%	50%	28%	47%	23%	25%	33%
Negative	0%	0%	7%	0%	0%	25%	3%
Total # of Teachers	8	12	14	15	13	4	66
Teachers with 3 or More Years Experience with Investigations							
Ranked 3 or 4	0%	60%	92%	67%	77%	100%	78%
Ranked 1 or 2	100%	40%	8%	33%	0%	0%	17%

So, for example, in *Investigations* students are expected to develop fluency in single digit addition by the end of second grade only after spending two years working on addition. They memorize some facts and developing efficient retrieval strategies for those facts they do not have memorized throughout this time. This is contrasted with the practice of drilling addition facts in first and second grade before students have had time to develop their own strategies. These concerns are fairly typical for teachers first encountering a reform-based curriculum. The fact that there were not more negative comments by the less experienced teachers is most likely due to two factors: teachers with these negative opinions about the units may have been less likely to fill out the survey, and all teachers sent this survey were part of a long-term professional development project which addressed these concerns.

Description of Student Strategies

When asked to describe specific strategies or ways of thinking about number that their students developed during their unit, the teachers were generally able to identify important strategies. Overall, 82% of the teachers who responded to the question were able to provide persuasive descriptions of these strategies. For example, a persuasive response related to using relationships among problems was "using landmark factor pairs, like 2 x 25 and 2 x 50, and building on those to pairs like 40 x 25, 40 x 250, 20 x 50 and 20 x 500." Those responses that were not persuasive often focused on content issues rather than student strategies.

Other Factors Impacting Teachers Learning

One may argue that it is difficult to tease out which factors are really impacting teachers' learning as they implement a new curriculum and specify the impact of the curriculum materials themselves. For example, there is no question that teachers also learn from the professional development in which they participate as well as from working with their students as they experience the activities in the curriculum. In a follow-up survey on which teachers were asked to rank (on a scale of 1-4) the impact of some of these factors on their implementation efforts, reading the curriculum materials (average rank 3.3) was second only to talking to other teachers (average rank 3.5). This suggests the teachers certainly value the supporting curriculum materials and find them to be a critical component in their efforts to implement the curriculum. Having said this, it is important to point out that the professional development experienced by these teachers did make deliberate use of these supporting materials on occasion, and thus may have influenced the value teachers placed on them.

Conclusion

The supporting materials in the *Investigations* curriculum clearly had a significant impact on the teachers, demonstrating that it is possible to produce curriculum materials that help teachers learn at the same time as supporting them in teaching. The teachers' choices of most helpful TNs and DBs, and their rationales for those choices, focused on the issues the curriculum developers intended – the mathematics content, ways of thinking students might display, or pedagogical support – rather than less substantive issues of general management and time. In addition, the fact that more experienced teachers were better able to describe the mathematics content focus of their units suggests that teachers continue to learn and deepen their understanding the more they use the curriculum materials. However, many teachers' propensity to blend content and process goals suggests that more discussion is required in the curriculum materials themselves to help teachers distinguish among these ideas. Finally, it would be useful to conduct a similar study with teachers who were not part of a long-term professional development project to better assess the impact of professional development on their interpretation of the curriculum in general, and the supporting materials in particular.

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QUESTIONS THAT ENGAGE STUDENTS IN MATHEMATICAL THINKING

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Teacher questions are an important part of a student-inquiry classroom. This research examines two different student-centered settings to determine the teacher questions that engaged students in mathematical thinking. It reports on questions asked in both a research setting and a high school classroom. Discoursive and retracing questions are defined as asking a student to contribute to an ongoing discourse and consider an old idea, respectively. These questions started strands of student engagement in mathematical thinking. Confirmation, justification, and clarification questions were also asked by both teachers and kept students engaged in mathematical thinking.

Recent reform efforts in mathematics education promote communication as an essential part of the mathematics classroom. Communication makes ideas objects of reflection, discussion, and refinement as part of the process of organizing, consolidating and giving meaning to these ideas (NCTM, 2000). If students are expected to explain their ideas, a question arises about what influence, if any, a teacher's response to those explanations has on student thinking. While a range of pedagogical responses is possible, a key way to facilitate communication and discover student thinking is teacher questions.

If mathematics teachers teach in an environment filled with conversation and where student ideas are valued, then teachers must know what questions engage students in mathematical thinking. While there have been many studies published on teacher questioning, most of them focus on the teacher. Some of this general education literature examines the effectiveness of teaching (Gall & Rhody, 1987) and categorizes teacher questioning (Cotton, 1989; Cunningham, 1987) according to student achievement. Other literature looks at how questioning strategies can involve more students in the learning process by provoking thoughtful responses (Wigle, 1999; Mewborn & Huberty, 1999). While bodies of mathematics education literature examine teacher questioning of students (Maher & Davis, 1990; Maher, Davis & Alston, 1992; Maher & Martino, 1992; Martino & Maher, 1999; Martino & Maher, 1994; Vacc, 1993) and students' learning and understanding of mathematics (Cobb, Wood & Yackel, 1992; Davis & Maher, 1990; Yackel & Cobb, 1996), there has not been an integration of these two research areas. Influenced by the call for communication in student-centered classrooms, the purpose of questioning is to help students explore their ideas during the communication process. Therefore, teachers need guidelines for questions that engage students in mathematical thinking. During a student and teacher conversation, teachers cannot use prepared questions or prescribed strategies. Teacher questions need to be based on the responses received from the student so the teacher can continue the conversation in order to engage the student in mathematical thinking.

The purpose of this study is to address this imbalance in the literature and inform mathematics education practice by classifying and describing teacher questions that engage students in mathematical thinking within a student-centered setting. The following research questions frame the study: 1) What kinds of questions do mathematics teachers in two different settings ask?; 2) What questions engage students in mathematical thinking?

Theoretical Framework

The call for communication is influenced by research in constructivism and discourse. These two research areas lead to the promotion of classrooms where students take an active role in learning and gain mathematical knowledge through social interaction and experience (Noddings, 1990; von Glasersfeld, 1990). As a result, teacher questions are an important aspect of student engagement in mathematical thinking. Since this study focuses on teacher questions, literature in this area supports the framework for the study.

Teacher Questioning and Questions

Questions are a valuable teaching tool and by far the most used technique of teaching (Clegg, 1987). Questions serve many purposes, such as initiating discussion and reviewing material, but the purpose of the question determines the kind of question asked by the teacher (Cunningham, 1987). Research, from general education classes, on questions provides many classification schemes, which label questions according to a particular cognitive level of student thinking.

Woolfolk (1998) suggests categorizing questions into divergent questions, which have many possible answers, or convergent questions, which have one right answer. Cotton (1989) found the majority of researchers conducted similar dualistic comparisons about questioning and Cunningham (1987) provides a more extensive list of questions for teachers to ask based upon the cognitive level of student responses.

Another view of questions is to categorize them in a hierarchy. The most widely used hierarchy is Bloom's taxonomy, where questions are labeled from simple to complex cognitive objectives (Woolfolk, 1998). Wolf (1987) suggests a different hierarchy, which focuses solely on what he considers challenging questions, from observations in his classroom.

A third view of questions is their role in effective teaching. This more current research shows that communication and questioning are part of a larger equation for effective teaching (Glenn, 2001). Effective teachers tend to ask more process questions, asking for explanations, though the majority of questions were product questions, asking for a single response (Reynolds & Muijs, 1999).

Questioning in Mathematics

Research regarding questioning in mathematics classrooms also focuses on classification schemes. Hiebert and Wearne (1993) identify four types of questions: recall, describe strategy, generate problem, examine underlying features. Vacc (1993) cites three categories of questions that occur in the classroom; factual, reasoning, and open based on a study by Barnes (as cited in Vacc, 1993) on questioning in classroom instruction. Vacc (1993) concludes that teachers asking factual questions will find out the specific facts their students know, but teachers who ask questions in the open category gain information about their students' cognitions. While the conclusions are helpful, there are no specific questioning guidelines offered or connections to student mathematical thinking.

Additional research examines questioning strategies and how questions can get students to communicate mathematical ideas. Mewborn and Huberty (1999) advocate a question-listen-question strategy in order to encourage discourse. The study reports how teachers, using questions from the NCTM Standards, improved discourse, but does not provide a classification for the questions in the study.

An examination of teacher questioning, by Martino and Maher (1999), explains how the timing of questions can determine a student's understanding of a mathematical idea after students construct their own ideas. In order to build upon student ideas, the research proposes asking students questions that lead them to justifying their ideas. Dann, Pantozzi and Steencken (1995) also examine teacher and student discourse and recommend that teacher's ask questions, which promote student interaction to help extend their ideas and justify their conclusions. Research in this area supports the idea that questions can encourage students to talk about mathematics, but this research does not classify questions, which promote mathematical thinking.

Research Design

This case study examines two settings in order to provide insight into questions teachers ask and what questions engage students in mathematical thinking.

Participants and Settings

The first setting for data collection is an urban high school classroom. Thirty-five honors students, comprised of sophomores and juniors, work on calculus problems at individual desks in rows. The teacher being observed is a 30-year veteran of the school system and holds a doctorate in education.

The second setting is a component of a longitudinal study on the development of proof making in students[1]. Eighteen high school students, entering their fourth year, work in groups on an open-ended precalculus mathematics problem in a library of a high school. The students are seated in groups and five teacher/researchers interact with the students. Six students, sitting at the same table (two males and four females), and one teacher/researcher, an experienced professor of mathematics and mathematics education at the university level, have been selected for this study.

Data Collection and Analysis

Eighty-minute videotape observations of the high school classroom sessions and two-hour videotape sessions of the two-week institute, both from a consecutive three-day period, comprise the data for this study. Field notes from both settings account for events beyond the view of the camera and summarize the class activity.

The analysis of the data for this study follows the model provided by Powell, Francisco & Maher (2001). Videotape were digitized onto CDs and each CD was summarized. Repeated viewing of the data allowed for the identification of critical events. A critical event, for this study, is defined as a teacher-student interaction where student mathematical thinking could be followed. The codes emerging from the data are: T(r): Teacher asks a student to consider an old idea; T(d): Teacher asks a student to contribute to the ongoing discourse; T(c): Teacher asks the student to clarify their statements or ideas; T(j): Teacher asks the student to justify their statements or ideas; T(con): Teacher confirms the student and teacher both agree on what has been done or said; and T(f): Teacher follows the student's idea or suggestion. It is possible for one question to receive multiple codes.

Results

The episodes selected demonstrate strands of student engagement in mathematical thinking. The strands presented here show teacher questions, with codes, that were common in both settings.

Summer Institute

The task the students are working on during the Institute is called Placenticeras, developed by Bob Speiser. The first part of the task is to draw a ray from the center of the shell in any direction. Then using polar coordinates as a way to describe the spiral of the shell, the students are to make a table of r as a function of theta. The question for the task is what could be said about r as a function of theta.

Episode 1

During the morning session, the students make measurements from a picture of the shell and create a table. They talk with each other about how to enter the points from their table into the TI-89 calculators in order to graph a function. Alice interacts with the group throughout the morning session about the where their measurements come from and what the number representing an average means. The students tell her their measurements go from the origin out to the shell, a radius, and the average is the distance traveled along the shell per 90 degrees. The students develop the equation r equals .039337 times theta, where .039337 is a result of their work on determining an average. In this day's afternoon session, Alice returns to the group and questions Victor.

Alice T(r) If we're saying, you told me the radius, which is he distance out to an point, is equal to that number that you came up with which was .039337

Victor mm, hmm.

Alice T(con) Is that right?

Victor I guess that, that's right for what I said, not right, like correct.

Alice Oh, no. I'm, I'm just saying.

Victor Okay.

Alice T(con) That's what you got and that's what you, that's what you based your spiral on.

Victor mm, hmm.

Alice Times theta.

Victor Right.

Alice T(c) And theta for this number was in degrees or in radians?

Victor Degrees.

Alice T(j) Okay, and so then shouldn't you be able to plug in a number of degrees and get one of

your points on the spiral?

Victor Say that again now.

Alice T(r, c) If this is an equation that says r is equal to .039337 times theta, what happens if you plug in a number of degrees? And multiply it by .039337. What should you get?

Victor A number of radius I guess. The number of the, the radius, right?

Alice T(con) The, the length of a radius? Is that what you should get?

Victor Uh huh.

Alice T(con) But now, isn't that what these things are?[points to values on their spiral]

Victor Right.

Alice T(r, j) Well shouldn't she be able to check by going backwards or not?

Victor shows some frustration with Alice, but agrees to check out if this idea works.

Alice T(c) What's the degree to get this guy out here[points to the last value on their spiral]?

Victor	Uh, 6.55 radians. Which is about, 6.5 um, 6.5 times 180. 1107.
Alice T(con)	Which is almost your total, wasn't it?
Victor	.0, hmm, .0 what, three.
Alice	9337.
Victor	No, you don't get the radius. You don't get the radius. Oh dang.
Alice	No, I just.
Victor	You don't get the radius, you get the length.
Alice T(con,c)	What do you get? do you get the length?
Angela	Is it length from like, length like this, that's what you get?
Victor	Yeah.
Alice T(c)	So is that what you are getting?
Victor	That's what I mean. [Referring to the term length rather than radius.]

Alice asks the students to check two more values with their equation.

Alice T(c)	So, so what, what is the thing over here? What is this thing over here? I mean what is it, what's the name of it?
Victor	I don't know. um, (inaudible) of the spiral.
Angela	Some kind of arc length type deal. I don't know.
Alice	T(con) Well arc, arc, whatever. Arc length, Isn't that what you call that?
Victor	Arc something.

Episode 2

During the last day of working on this task, the students are talking about what they are going to present to the rest of the groups at the summer institute. Alice asks Robert about the two equations the group developed to model the growth of shell. Robert explains how the equations came from the scatter plot the group created, which was made from their table. He also relates the x and y values represented on their scatter plot to the values of r and theta measured on the picture of the shell. After finishing his explanation, Alice asks Angela about what Robert said.

Alice T(d)	Angela does that make sense to you at all? or not?
Angela	Sort of. I like get lost with all this stuff. I hate this.
Alice T(f,c)	What, what are the, what is the sort of question that throws you?

Angela	Like, I get like little bits and pieces of what he's explaining, but I don't really get all of it.
Sherly	What don't you understand?
Angela	I don't know.
Alice T(c)	I think I hear (inaudible), she doesn't even know what she's asking. Um what, what are you asking...in this?
Angela	I don't have like a specific question. I just don't like understand the whole, like everything you just explained. Like why that, the whole thing, like what you were saying like why it's like the spiral unraveled or something like that. Like I don't, you explained the points and just trying to follow and just didn't.
Robert	It's kind of hard to explain.

The students ask each other if they understand what Robert explained.

Alice T(d)	Could you, could you try Michelle, to explain. Cause every time one of you explains it, it helps me a little, little more. This is really just as foggy for me, Angela, as it is for you. I am even further away than you, from this stuff, because I don't understand the calculator either. So Michelle could you try it again.
Michelle	Okay, um. Ooh. Alright if you took, let me draw a piece, the spiral, and you picked like certain points, whatever ones they were. Right? Robert?
Robert	Yeah.
Michelle	Okay (laugh), just checking. And like um, you're doing the ninety degree intervals, which is like the pi/2, you know what I mean, and like you graph them...Okay, then you're saying that r would be the radians?

Michelle, with the help of Robert, continues to discuss with Alice the meaning of the values the group measured from the picture of the shell. Alice leaves the discussion after asking the group to show how they used a rubber band to get the lengths in their table.

High School Classroom[2]

The task the students are working during these observations is to find the area under a curve over a given interval. The teacher provides the students with a function and then calls students to the board to work on a specific part of the task. For episode one, which occurs on the second day of observation, the function given is $f(x) = 2x^3 + x^2$ over the interval [1,3].

Episode 1

The students take the derivative of $f(x) = 2x^3 + x^2$ to get $f(x) = 6x^2 + 2x$. Using the properties of its first and second derivatives, the class draws a graph for f(x). The students find the area under the curve using four rectangles. The teacher asks the students to express the area using sigma notation and then assigns various students in the class a number of rectangles to use to calculate the area with a TI-89 calculator. After the class sees the area is approaching 60 as the number of rectangles increases, the teacher asks a student to write the area using limit notation. The student, Cedric, writes $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}$.

DrG		Question Keri.
Keri		Shouldn't it be 2 over *n* cause that is how you did everything else.
Cedric		(inaudible) [calls on a student to answer]
Student		It should be. That first expression should be *n* over 2, which that would be. So that top number which is *n* is four. So the number in the denominator is four.
Keri		But that.
Student		What?
DrG	T(r, f)	Keri, let us hear your argument again please.
Keri		Okay, it should be 2 over *n*. Because like to get one half, you do two over four. Cause it was like you have four triangles and the area two. Two. So it was two over four, which is one half and you had two over ten which is one fifth. And so on. So wouldn't it be two over *n*.
DrG	T(con,d)	Any disagreement? Everyone understand that. What'd she say Chuck?
Chuck		She said that since those four triangles and those were split into two parts. I mean four rectangles. It was two over *n*, which was reduced to one half and that's how we got one half.
DrG	T(con)	Is that what you said Keri?
Keri		Uh, not really.
DrG	T(r,c)	Try it again, Keri. And then we'll back it up and try and have him run it through again.
Keri		Okay, I don't know how. I'm sure how to explain it. Yeah, You had

okay. You had like two, one to three, three minus one is two. So, you have like two spots and you're doing it for four rectangles. So you did two over four is equal to one half and then you did two over ten is one fifth and so on. So to get it for n, you get two over n. I think that's what I said.

DrG T(d,con) So, now you try and say it Chuck.

Chuck Yeah, fine. Uh, Since we're going from one to three, the top part is two. And since there's four rectangles, the bottom part is four.

DrG It's the length of the interval divided by the number of rectangles. [points to the expression written on the board]

The student finishes the expression for the area and writes "= 60". The class uses the calculator to graph the derivative function and find the area under the curve for the interval [1,3]. The teacher draws the students' attention to the integral notation on the calculator and introduces paper and pencil integral language using formal, graphical and $f(x)$ notation. The teacher also explains how to use the TI-89 to calculate the integral of their given derivative function. After this explanation, a new problem to find the area under the curve $f(x) = 2x^2 + 3x$ over [0,1] is introduced, and a student is called to the board to calculate the parent function. After calculating the parent function by reversing the power rule for derivatives and finding the area using this method, the teacher asks another student to set up the limit equation, which they will determine for homework.

Marc So just, uh, write the equation. [Student writes $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{n}$]

DrG T(r,j) Now, why one over n? Awhile ago it was two over n.

Marc Because the uh, zero to one, not one to three.

DrG Oh, okay.

Conclusions

The two teachers asked many questions during their interactions with students. However, two questions started student engagement in mathematical thinking, while other questions kept students engaged in mathematical thinking. Retracing and discoursive questions, coded T(r) and T(d) respectively, started the strands of student thinking in the episodes selected. The additional questions of clarification, T(c), justification, T(j), and confirmation, T(con), allowed the teachers to keep the student(s) engaged in mathematical thinking.

Both teachers used these questions to engage students in mathematical thinking, but each teacher asked questions for various reasons and purposes. Alice would ask a question after listening to the students' conversation. Dr. G would ask questions with a goal in mind for the students. His initial questions did not always come from something the students said.

Each teacher utilized retracing questions for different purposes. In episode one of the summer institute observations, Alice questions students about what their measurements mean so they could revisit their own ideas. She continually returns to this idea throughout the day to get the students to clarify their thinking. Finally, in the afternoon session, Victor realizes it is not the radius he is talking about, but rather the arc length of the spiral. Retracing questions allowed Victor to revisit his mathematical thinking about this topic until he clarified the meaning of his measurements.

In the high school setting, the teacher asks retracing questions to emphasize an earlier idea to the entire class. Keri indicates the width of interval should be two over n and justifies her reasoning. The teacher returns to this idea when a different student is writing the value of the width of the interval for a new problem to emphasize to the class what Keri stated earlier.

Both teachers ask discoursive questions to check for other students' understanding of one student's idea. In episode two of the summer institute, Alice asks Angela if she understands what Robert just explained. Since Angela states she does not understand, Alice asks Michelle to see if she understands and can explain Robert's ideas about the group's work. Michelle becomes engaged in mathematical thinking and, with the help of Robert, restates the group's ideas. In the high school classroom, the teacher mimics Alice's objective. He has another student explain what Keri said about the interval width for the rectangles in order to check for understanding of other class members.

Retracing and discoursive were the two main types of questions the teachers in both settings used to engage students in mathematical thinking. The transcripts presented also show both teachers using many clarification, justification, and confirmation questions to extend student mathematical thinking. Even though both teachers used these types of questions, Alice asked more clarification and confirmation questions, while Dr. G asked all three kinds of questions. However, the episodes presented only show a very small sample of the overall student-teacher interactions from the two settings over a short observation period. Further quantitative analysis of the entire transcript may provide further insight into the quantity of types of follow-up questions asked by each teacher.

Studies about questioning in the mathematics classroom call upon teachers to ask initial questions that provoke thoughtful responses from students, but to follow initial questions with others that help students clarify, justify and extend their thinking (Dann, Pantozzi & Steencken, 1995; Mewborn & Huberty, 1999). This research shows that teachers can use discoursive and retracing questions as initial questions to engage students in mathematical thinking. Teachers can follow these initial questions with clarification, justification, and confirmation questions to extend and continue student mathematical thinking.

Notes

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[2]Names of students in this setting have been changed.

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A LOOK AT THE "LONG AND SKINNY": EXAMINING IDEAS ABOUT AREA AND PERIMETER IN THE CONTEXT OF STUDENT PRESENTATIONS

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This paper presents one aspect of a qualitative study that examined one middle school teacher's use of questioning strategies to help students construct, revise, refine and extend their understanding of the relationship between area and perimeter. The teacher was involved in a professional development project where she and several colleagues worked with university researchers to consider ways in which to improve their practice. In this case, they were focusing on the use of questioning techniques to improve the discourse in their classrooms. This paper details one lesson dealing with the relationship between area and perimeter and the questioning strategies used in the context of student presentations. Results indicate that deliberate and careful use of questioning strategies within the context of student presentations clearly aided students in understanding the relationship between perimeter and area. Many misconceptions were identified and discussed through whole class discussion. An additional benefit of reflecting on discourse was that the teachers were able to build new ideas about the ways in which students develop mathematical understanding.

Introduction and Theoretical Framework

Prompting students to talk about mathematics is an important goal of mathematics education (NCTM 1989, 2000; Cobb, Boufi, McClain, and Whiteneck, 1997). One aspect of this involves providing students with opportunities to discuss their ideas with each other and their peers, defend and justify their thinking, and reflect upon the mathematical thinking of others. Facilitating this type of discourse is no easy task (Ball 1991; NCTM 1991). This study, which is part of a larger project, will examine how one teacher helped to facilitate substantive mathematical discourse in the context of student presentations.

Our analysis will be done in the context of a "models" and "modeling" perspective. In this research both teachers and students were involved in modeling situations where they could continually identify, revise, refine, test and extend their current models of particular situations involving the teaching and learning of mathematics. Briefly, a model can be considered to be a way to describe, explain, construct or manipulate a complex series of experiences. Models help us to organize relevant information and consider meaningful patterns that can be used to interpret or reinterpret hypotheses about given situations or events, generate explanations of how information is related, and make decisions about how and when to use selected cues and information. Models tend to develop in stages and over time (c.f. Schorr and Koellener-Clark, in press), and early conceptualizations may be quite different from later iterations.

Our hypothesis was that student presentations, if used effectively, could potentially help students to move beyond their initial or intermediate conceptualizations about the mathematical ideas involved. As students reflect on their own thinking—both as they prepare their presentations, and react to the questions that are precipitated by their peers and the teacher, they have the opportunity to revise, refine, and extend their ways of thinking about the mathematics, which in this case involved the relationship between area and perimeter. As they do this, their early conceptualizations become increasingly refined. To this end, we analyzed their discussions by identifying models of the relationship between area and perimeter,and revisions and refinements of these models. An additional hypothesis was that by focusing on discourse as one part of a teacher development project, participating teachers would also refine, revise, test and share important aspects of their evolving models for teaching and learning mathematics.

Background

In order to help teachers make the transition from teaching mathematics in more traditional ways that emphasize the execution of procedures and rote memorization, mathematics education researchers met regularly with teachers -both in the context of workshops and in the context of their own classrooms. During these meetings, opportunities were provided for teachers to: deepen their own understanding of the mathematics they were expected to teach, develop insight into the ways in which students build these ideas; and, consider the pedagogical implications of teaching mathematics in new ways. Teachers were encouraged to provide opportunities for the students to develop logical and persuasive predictions, explanations, and justifications; analyze or critique their own solution, or the solution of others; and, consider the ways in which they could refine, revise, monitor, and assess their own work. Teachers were also encouraged to consider the mathematical discourse that was elicited, and how they might begin to use new methods of questioning to encourage students to, for example, defend and justify their solutions, pose interesting hypotheses, and share strategies.

The student activities used in this research were considered to be "modeling problems". That is, these problem activities had been deliberately designed to provide students with the opportunity to actively explore the mathematics (in this case, concepts involving area and perimeter) as they made conjectures, tested their ideas, and continually refined and shared their understanding of these ideas. (A more complete description of the design characteristics can be found in Schorr and Koellner-Clark, in press). The particular activities used in this research will be further described below.

Method

This research took place in a suburban district in the southeast, over the course of a year, as part of a teaching experiment involving middle school mathematics. In this research, we will focus on one seventh- grade teacher. Twenty-four lessons were documented and we will share the12th lesson of this sequence. The class consisted of 28 students, 15 girls and 13 boys. All sessions were videotaped and audio taped.

Researcher notes were also carefully collected and transcriptions and detailed narratives of the data were recorded.

A coding scheme was designed to flag elements for the study and then categories were identified similar to qualitative analysis described by Glaser & Strauss, (1967). After this, interpretative and observational procedures were employed, based upon our theoretical framework.

Results

As noted above, the type of professional development that was employed throughout the study was designed to help the teachers make conjectures about their own classroom practice, test their ideas, and continually refine these ideas based upon classroom activity. The particular aspect of the project, that is, the topic of this paper, related to discourse in the context of student presentations. The activity, which is the focus of the excerpts below, was the third of a series involving the relationship between area and perimeter. The excerpts and analysis provided below will be used to show how this teacher facilitated substantive discourse through the use of questions and interventions during group presentations. The problem activity presented here had several components. First, the students were asked to trace their footprint on a piece of centimeter graph paper. Next, they were to cut a piece of yarn that had the same perimeter as the footprint; and then use that yarn to create a new rectangle. Finally, they were asked to determine if the area of the newly created rectangle would be equal to the area of the original footprint. Students were also asked to explain whether or not finding the perimeter of the footprint was an accurate way to find the area of the footprint. They were also asked to write a letter to their parents explaining the relationship between area and perimeter using the footprint example to lend support to their findings. When discussing her reason for choosing this particular activity, the teacher stated that her goal was to help students determine the relationship between the perimeter and area of a region, especially when perimeter is held constant. In order to encourage students to talk about their ideas as they were working on the problem, she arranged them into groups of three or four. She felt that by using small groups, students would be more likely to share their thinking with each other.

Throughout the lesson, the teacher used various types of questioning techniques including reiterating what the student(s) said, and requesting explanations. When the students had finished working on the problem, the teacher asked the students in each group to justify their solutions using a presentation style format. Each group had to present their ideas, and then respond to questions from the teacher and their peers. The teacher noted that she chose this format because justification of solutions often drives students to examine the validity of their own ideas by testing and refining previously held notions. This, she felt, could potentially cause the students to extend, reject, or revise their previously held notions.

The following excerpt, in which Andrew shares his work with the class, will illustrate this process, as implemented by the teacher.

Andrew: You can see that I estimated the area of my foot to be 158 cm and the area of the rectangle was exactly 178. I think these estimates are fairly accurate because it was difficult to get a perfect estimate of my foot. I underestimated because I didn't count all of the partial squares. So I think this method is pretty accurate-as accurate as estimating goes..[he goes on to read his letter that is aligned with this thinking]

Robin: I think Andrew's method was accurate-I was doing it a different way but when he showed me why his way worked I could see it. When I did it like him...my estimates [of the rectangle and the footprint] were close too.

Teacher: How did you think about the problem before you used Andrew's method?

Jenny: Well, our feet are smaller, [referring to herself and Robin] so we weren't sure about the shape of the rectangle...my rectangle was more square but then this other way made ours more long and skinny like our foot-doing it Andrew's way made our estimates close.

Teacher: Okay let's have the next group go and try and work off of Andrew, Robin, and Jenny's presentation. Make comparisons between your strategies. You might want to refer back to Jenny's comment about how she made her rectangle more square but then decided to make it more "long and skinny" if you can.

In this excerpt, the teacher attempted to capitalize on the differences between Jenny and Robin's strategy, and the strategy that Andrew used, by calling attention to Jenny's previous method [her "rectangle was more square"-making the area much larger than that of her foot]. She was hoping that by doing this, the students might reflect upon their own strategy and in the process, be able to revise and refine the model they had for thinking about the connection between area and perimeter. In other words, the teacher was attempting to set up a situation where students could make connections and/or revisions to the models they had created.

After careful reflection, she made the decision to select a second group whose solution could potentially push the conversation forward.

James: Well... I think that the perimeter could be the same and the area rectangle could be long and skinny and then the areas would be different because the longer the rectangle the smaller the area. That would be the opposite of the pentomino activity we did last week because then the area was always 5 but we found different perimeters.

Teacher: Sylvia and Renee [James' other group members] were your ideas similar or how did you look at this problem?

In this excerpt, one student, James, said that the area could be the same and the perimeter different. He was able to extend his previous model for thinking about area and perimeter to include the relationship he noted here. However, he also made a new conjecture when he noticed that the longer the rectangle was, the smaller the area became, even if the perimeter stayed constant.

In order to include the other group members and provide them with an opportunity to add to the discussion, the teacher asked for their ideas.

Sylvia: See each of us did our rectangle different and so we can justify our solution or letter by showing the different areas of the rectangles and none of them are the same as our estimated feet.

Teacher: Jenny, how do their findings relate to what your group found?

As the students considered their justification, the teacher requested that the previous group clarify what appeared to be conflicting solutions and provide valid mathematical evidence for them.

Jenny: Well, Sylvia's rectangle looked like mine before I redid it to look more like the shape of the foot. So they were saying the area of that rectangle was not the same as the area of her foot.

Teacher: What might that tell you about the relationship between area and perimeter?

Robin: Well...I guess I could take the string and make rectangles that had different areas.

Andrew: I still would have thought that since we didn't change the length of the string that the area would have been the same...but I just made two different rectangles [one close to a square and the other with two long sides and two short sides] and found different areas using that piece of string...

Jenny: So I guess the area can be different and the perimeter can be the same or vice versa.

It appears that Robin, Jenny, and Andrew considered the other group's solution and revised their own thinking regarding the relationship between area and perimeter. Robin was the first to verbally consider a different solution but it appeared she became even more convinced as Andrew and Jenny tried to model the new interpretation using the string and graph paper. Andrew actually tried to make different rectangles and found different areas using his piece of string. This suggested that he was verifying or justifying his new model of the relationship between area and perimeter. Jenny clarified the relationship verbally and then related this idea back to a previously held model.

Conclusions

Results indicate that this teacher was able to use questioning in the context of student presentations as one way to help students build new models for considering mathematical ideas involving area and perimeter. In this case, the teacher was able to establish a classroom atmosphere in which students could openly talk about their mathematical thinking, and be receptive to the thinking of their peers. This type of discourse and refection should not to be taken for granted, for as Schorr and Bulgar (2002) point out, few of the teachers involved in a large-scale study of classroom practice provided similar opportunities for students. Second, by carefully selecting the order of the presentations, she was able to help students consider discrepancies in their own and their peers thinking. Her careful selection was key in helping students to reconsider their own ways of thinking about the mathematics.

As illustrated in the first excerpt, not all models or notions about area and perimeter remained constant throughout the course of this lesson. In fact, many students directly attributed changes in their own way of thinking to the comments, questions, or ideas of their peers. For example, Robin explained that she changed her method based on Andrew's comments. Her new model for thinking about the relationship between area and perimeter was based on the notion that the rectangle should be the same shape as your foot-long and skinny. More specifically, when perimeter is held constant then the area of the rectangle and the area of their foot should be the same or similar, due to the estimates involved. In discussions between and amongst the teachers, they reported that this type of solution was typical of many groups that they had observed. The teachers felt that it reflected many students' first model of the solution to this particular problem. As this teacher noted, their first attempts "did not take into account all of the information in the problem" and were often "unstable ideas in regards to the concepts of perimeter and area."

In addition to discussions involving the mathematical ideas, the teachers also used the workshop settings to compare the individual characteristics of the group members. Not surprisingly, the development of student ideas varied in all of their classrooms. For example, this particular teacher shared that, as indicated in the first excerpt, Andrew was confident in his first idea and easily persuaded the other group members to use his method. She felt that Andrew "dominated" the discussion in this group and initially was not open to revising his ideas based on Robin and Jenny's ideas. Andrew's "method" provided a visual justification for Jenny and Robin. This finding, albeit not new (Cohen & Goodlad, 1994), clearly impacted the discourse and how students interpreted the relationship between area and perimeter. However, when the teacher became aware of these differences and was able to use them as a way to leverage discussion of the mathematical concept at hand, many students were able to move beyond their initial ways of thinking.

It is worth noting that the teachers became increasingly aware that deliberate and careful facilitation of student presentations provided at least one way to help students (re)consider the ways in which they thought about the relationship between perimeter and area. Many misconceptions were identified and discussed, and the teachers noted that this didn't happen because they "told" the students what to do, and how to do it. The teachers were able to see firsthand, that as students prepared their presentations, and considered their own, and their peers thinking they were able to move beyond initial or early conceptualizations. Equally important, by focusing on this particular aspect of classroom practice, teachers were able to reveal, test, and refine their overall approach to teaching and learning. They became better able to recognize that students can go through many different "modeling" cycles as they solve a problem activity. And, they were able to consider the role that questioning, both their own, and the students' played in this regard.

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THEORY AND PRACTICE OF MATHEMATICS EDUCATION REFORM: INCLUDING THE TEACHERS' VOICE[1]

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One hundred and eighty five high school mathematics teachers completed a survey that elicited their level of familiarity with reform-based curriculum and instruction. Teachers were also asked to comment on the value they attached to current guides for instructional redesign as well as the findings of research in mathematics education. Although a majority of the teachers were familiar with and supportive of Principles and Standards for School Mathematics, they were unsure of whether they applied to their own settings. Nearly all teachers found implementing teaching roles difficult to adopt and implement. Only three teachers attached professional value to current reports of research findings on learning and teaching mathematics.

Introduction

What do mathematics teachers think about the recommendations of the reform for teaching and learning? What value they attach to research in mathematics education? How do they characterize their needs and their expectations of research in helping them meet those needs?

These questions address the very nature of what goes on in mathematics classrooms, the perceived images of reform and research by those that serve as both a target and an audience for their agendas, and have implications for mathematics education research community and those responsible for professional development of teachers. The purpose of the current research was to gain insight into the phenomenon of teacher thinking by collecting data relative to the above questions. The study had three specific goals. The first goal was to document teachers' views on the current recommendations for reform in curriculum and instruction. The second goal was to elicit teachers' assessment of the usefulness of current research in mathematics education in advancing their skills in implementing reform-based practice. The third goal was to identify those elements, from the teachers' perspectives, that facilitate instructional change.

Background

National calls for reform in mathematics education propose a vision of teaching that is student centered and inquiry based. The reform documents encourage teachers to focus on helping students develop a conceptual understanding of mathematical ideas. It is recommended that in doing so, teachers must rely on various innovative instructional approaches such as using group discourse, technology, and open-ended exploratory activities (NCTM, 1991; NRC, 1999). These recommendations are grounded in the current convictions shared within the mathematics education research community about how mathematics is best learned and taught. These convictions are endorsed by a large number of teacher educators across the country and presently guide the design and content of various professional development opportunities teachers are offered.

Despite this widespread enthusiasm of researchers and teacher education practitioners for implementing reform-minded teaching, innovative instruction remains a novelty in school settings (Ball & Cohen, 1998). There is evidence that many teachers are unaffected by the reform (Ball & Cohen, 1996) and continue to teach in a traditional manner (Ball & Cohen, 1998). Researchers have identified several teacher attributes that contribute to this dilemma. Four persistent themes include: teachers' lack of familiarity (and comfort) with a gradual development of the subject matter which prevents them maintaining instructional coherence (Manouchehri, 1998; Manouchehri & Goodman, 1998, 2000); teachers' inability to present mathematics as a chain of interpenetrating concepts rather than as isolated skills due to their narrow understanding of the subject (Kennedy, 1998; Fireston, 1998); teachers' reluctance to conform to new methods of teaching due to a mismatch between their own beliefs about the nature of mathematics and the philosophies that guide the reform recommendations and teachers' reliance on locally driven theories about practice rather than current research on learning and teaching (Tobin, 2001).

In reviewing the content of this body of work on teachers a common denominator prevails. Namely, the results of these studies represent the researchers' (research team's) perceptions of what ought to happen in classrooms, what teachers do not or can not do and the value they attach to the teachers' choice of pedagogy or student learning. In discussing implications, the researchers highlight what may be done to assist teachers to develop the skills they need along with issues that they consider as worthy of further investigation. Although we learn a significant amount from these studies, they also evidence the absence of teachers' voice in deepening our understanding of the complexities associated with instructional change and in shaping the direction of research within the discipline (Goodson, 1996). This omission of teachers' perspectives and views is problematic (Clark, 1996). This is an issue that needs to be addressed if efforts to reform mathematics education are to be successful in real school settings.

Research Goals and Methodology

The overarching goal of the current study was to better understand ways in which mathematics teachers interpret and assess recommendations for instructional redesign, messages of research, and their own role in the broad context of educational reform. In this work I focused on high school mathematics teachers. This was for several reasons. A majority of past research on teaching and teachers has focused on elementary teachers. There is very little knowledge within the mathematics education research community about the professional perspectives of high school mathematics teachers, or their needs.

Data Collection Instrument

A survey was designed and mailed to 400 high school mathematics teachers across the state of Michigan[2]. In addition to obtaining biographic information on teachers, the survey elicited information on: Factors that serve as catalysts for change in the teachers' knowledge for teaching, elements that teachers find most problematic when implementing reformed-based curriculum and instruction; dimensions of teachers' instruction most influenced by current research on mathematics learning and teaching.

Questions parallel to items on Ohio State Teaching Confidence Scale (Hoy, 1998) were developed and include in the survey. These questions asked teachers to rate their confidence with innovative instructional roles and techniques. Teachers were asked to rank their level of proficiency in implementing aspects of reform minded practice (i.e., use of technology, teaching for conceptual understanding, using problem solving in instruction, using inquiry based methods, sustaining classroom discourse, and mathematics teaching as connection making). They were also asked to identify areas in which they felt they needed professional development.

Lastly, teachers stated the extent in which they were familiar with current research in mathematics education, whether they found such research reports beneficial to their work, and to identify areas that they felt must gain research attention.

Data Analysis

I used factor analysis to determine relationships among various teacher variables (number of years of teaching, age, gender, post-graduate training, school district, types of professional development activities in which they were engaged within the last 5 years, courses they taught) and their level of knowledge about, and support of, reform based curriculum and instruction as well as research in mathematics education. In doing so, we used a principal-axis factor analysis using Kaser's criterion of Eigenvalues greater than 1 (Borg & Groenen, 1997) in combination with Cattell's Scree Test to determine the number of factors that needed to be considered in the analysis (Kim & Mueller, 1978). In addition, I looked for common patterns of comments among teachers' responses to open-ended questions. Descriptive statistics were obtained to highlight common themes.

Results

One hundred and eighty five surveys were returned. The sample consisted of 110 male and 75 female teachers. The mean teaching experience of the participants was 21 years. Thirty-two teachers had less than 10 years of teaching experience. Seventy-seven teachers taught in rural, 66 in urban and 42 in suburban school districts. One hundred and sixty one participants were white. Those remaining were of African American and Asian heritage. All teachers taught 9-12 grade mathematics. Thirty-nine participants had teaching responsibilities that included upper division mathematics courses (Pre-calculus, calculus, and statistics). Others' teaching load ranged from courses in fundamentals of mathematics, mathematical modeling, geometry, Algebra I and II. Forty-two teachers had participated in professional development activities that were content specific within the last five years. Topics of these professional development included technology in instruction, using new textbooks, and authentic assessment techniques. Nearly a half of the participants ($n = 90$) had completed at least one graduate course at the university within the last three years. Sixty-nine teachers had completed a master's degree in curriculum and instruction within the last ten years. Approximately 70% of the participants were familiar with NCTM's Principles and Standards for School Mathematics. Only 8% of the participants stated familiarity with Professional Standards for Teaching Mathematics.

There was a positive correlation between the level of mathematics preparation of the participants and their degree of support of the NCTM's guides for practice. Those teachers with a more sophisticated mathematics background (more coursework in mathematics education) were more supportive of, and confident in their ability to implement innovative curriculum and instruction. Female teachers tended to be more supportive of recommendations of reform for learning and teaching mathematics. In addition, teachers in suburban school districts were more familiar with the standards and found them more applicable to their work. Over 95% of the teachers in urban and rural districts expressed that they found the current guides for instructional change impractical for their settings.

Age, number of hours of involvement in professional development activities, and years of teaching experience did not serve as significant factors on participants' level of support of the reform based instruction. Lastly, only three teachers in the entire sample found merit in current research findings for advancing their professional efforts.

A summary of the teachers' responses to questions that assessed their level of comfort with innovative teaching practices, professional needs, and usefulness of research findings in mathematics education is reported below.

Assessment of Reformed-Minded Practices

Table 1 summarizes the participants' assessment of their own comfort with implementing the type of teaching roles currently advocated by various reform documents. As the data indicate, a majority of the teachers (approximately 70%) found teaching for conceptual understanding, connection making, supporting classroom discourse, technology enhanced instruction, and inquiry based mathematics either difficult or very difficult to sustain. The same percentage of teachers found connecting children's thinking to a mathematical structure as a difficult task to implement as well. Nearly 83% of the participants found accommodating all children's needs also a critical issue in instruction. It is also interesting to note that nearly seventeen percentage of the teachers were unsure of what "teaching for conceptual understanding" and "supporting classroom discourse" entailed.

Table1. Teachers' Reports of Their Level of Comfort With Reform-based Instructional Behaviors

Teaching Actions	Very Easy	Easy	Difficult	Very Difficult	Not Sure What it Means	No Response
Teaching for problem solving	27	59	72	17	0	10
Teaching for conceptual understanding	8	11	17	116	31	2
Teaching for connection making	2	25	83	56	19	0
Supporting classroom discourse	7	11	114	18	35	0
Inquiry based mathematics instruction	9	31	88	38	9	10
Technology-enhanced explorations	17	37	116	15	0	0
Teaching reasoning	46	34	67	36	2	0
Utilizing multiple representations	11	35	87	24	25	0
Accommodating all students' needs	13	10	132	22	0	8

Table 2 summarizes the teachers' assessment of their ability to foster the type of student behaviors closely tied to reform-based instructional goals. Seventy percentage of the participants expressed it was difficult for them to help students develop the skills to pose mathematical questions in class, learn to apply mathematics, communicate mathematics effectively, listen to peers' arguments and debate accuracy of the ideas. In addition, 73% of the teachers found it either difficult or very difficult to help students appreciate and enjoy mathematics.

Assessment of Their Past Professional Development Opportunities Offered

In response to the question which elicited teachers' assessment of their past experiences with various professional development opportunities they were offered, an overwhelming majority of the teachers (*n* = 143) rated their experiences as ineffective. One hundred and nine of these teachers claimed the content of the sessions they had attended was either too difficult, or too trivial. Fifty-seven teachers from this

Table2. Teachers' Reports Of Their Level Of Comfort On Fostering Reform-based Learner Behaviors

Learner Actions	Very Easy	Easy	Difficult	Very Difficult	Not Sure What It Means	No Response
Problem solve	23	42	93	8	17	2
Develop conceptual understanding	6	45	106	7	21	0
Use technology to pose and solve problems	13	36	118	17	1	0
Pose mathematical questions in class	2	21	87	58	11	6
Express their thinking both verbally and in writing	38	59	44	23	16	5
Make mathematical conjectures	12	57	87	23	2	4
Approach problems in different ways	59	52	40	34	0	0
Work through ambiguous and challenging problems	23	33	89	21	14	5

category expressed the content of their workshops either too narrow to have practical merit for long term instructional planning, or too general to have mathematical merit. Twenty-one teachers rated those workshops that focused on the use of technology in instruction as extremely helpful.

Areas in Need of Professional Development

Teachers were asked to identify areas in which they felt professional development opportunities were of special need to them. Topics most frequently sited by teachers included: Writing lessons that utilize applications of mathematics (n = 117), using calculators in lessons (n = 174), using computers in lessons (n = 112), implementing discovery learning activities (n = 153), ways to maintain productive discussions about mathematics among students (n = 162), ways to involve/engage all students in mathematic learning (n = 177), dealing with diverse abilities and background students bring to class (n = 151), convincing students that mathematics is important and useful (n = 120), implementing open-ended exploratory activities (n = 76), establishing interest in mathematics and mathematics learning among students (n = 171), helping students take charge of their learning (n = 84). A moderate number of teachers expressed a need for assistance in finding meaningful activities to use in my instruction (n = 53), organizing and monitoring cooperative group activities (n = 75), using students' life experiences in my instruction (68), using assessment techniques other than standard tests (n = 49), A small number of teachers identified a need for further training in how to facilitate learning rather than telling students what to do (n = 13), do long term instructional planning (n = 4), explaining "why" mathematical algorithms work the way they do (n = 21), deciding which mathematical conventions are important for students to know (n = 16), connecting mathematics to other subject areas (n = 36), and making connections among various mathematical topics (n = 11). Nearly all teachers from urban schools also identified the need to learn more about how to deal with students for whom English is a second language, and help reduce absenteeism. In addition, of particular interest to over 65% of the teachers in rural schools was learning about how to work with limited resources in classroom.

Views on Research in Mathematics Education

Three of the survey questions solicited teachers' views on the agenda and findings of research in mathematics education. First question asked teachers to rank their familiarity with findings of current research in mathematics education from very high (4) to very little (0). Second question asked teachers to rate the findings of current research along a continuum from very useful (4) to not useful at all (0). They were also asked to state the reasons for their rating. The third question asked the participants to identify areas in which they felt additional research was needed.

The mean score for the teachers' ratings on the first question was 3.1 with a standard deviation of 0.4. However, the mean score for the teachers' ratings of the second question was 0.2, with a standard deviation of 0.8. A majority of the teachers felt the current research did not address the level of mathematics they taught (n = 117). One hundred and eleven teachers stated they found research reports difficult to understand. One hundred and nine teachers expressed that they felt the body of research failed to provide concrete guides for practice (109). Ninety-two participants claimed those research studied they had reviewed did not focus on answering questions that related to their particular settings. Over 95% of this population taught in rural and urban school districts. Seventy-six teachers claimed the research reports they read did not make sense to them. Fifty-four teachers objected found the research settings artificial, thus, their findings not applicable to their own particular situation. Lastly, twenty-five teachers made statements that indicated a lack of trust for those conducting research in mathematics education (i.e. University people like to tell us what to do).

In order to benefit from research, 141 expressed a need to learn more about how to motivate all students in learning. One hundred and nineteen teachers found it crucial to study and learn about methods to bridge informal explorations to formal mathematics. Other categories identified by teachers included: Long term student outcomes of teaching concepts ($n = 51$), the influence of block scheduling on learning ($n = 37$), the impact of technology on mathematics learning ($n = 46$). Fourteen teachers stated that they did not know how to respond to the question. Seven teachers did not respond to this question item.

Discussion

The findings of the study have implications for both research and practice in mathematics education. On the one hand, the results speak to the need for a substantial redesign of professional development opportunities planned for high school mathematics teachers. On the other hand, they point at the chasm between theory and practice in mathematics education.

The participants in the current study found the content of the current visions for practice as manifested in the NCTM documents difficult to understand and implement. They felt they needed specific guides for long-term implementation of the reform minded practices in addition to having access to sustained and relevant professional development opportunities. The teachers' assessment of their own past educational experiences highlighted the ineffectiveness of professional development activities that fail to take into account the various background knowledge and experiences they posses. For a large number of teachers issues surrounding motivating student learning and meeting individual needs of a diverse student population were of paramount complexity. Certainly, as the teachers' reports evidenced there were differences in what teachers identified as critical to their work according to whether they taught in urban, suburban, or rural school districts. Differences among intellectual and professional needs of a diverse teaching population certainly calls for planning differentiated educational opportunities for them. This is much in line with current guides for teachers' own classroom practice.

The data also indicate that the participants in this study found little merit in current reports of research findings in mathematics education. A majority of the participants found this body of work of little help and value when organizing and implementing instruction and instructional change. For a large number of them both the context and content of reported research on learning and teaching mathematics was foreign- too abstract to have practical merit. Although some of the participants objected to the form in which the findings were reported, their major dissatisfaction rested upon what they perceived as the narrow focus and range of these studies and their inability to shed light on how to resolve dilemmas in their settings. This pragmatic expectation of research by practitioners is neither new nor unreasonable. Certainly, good practice is grounded in sound theory. In turn, the test of a good theory is whether it can guide practice. In order for practitioners to systemically rely on research findings when designing instruction they must first find them meaningful and relevant. They must also believe the conditions under which studies were completed matched the circumstances under which they work. Areas of research identified by teachers provide promising grounds for conceptualizing ways in which professional needs of teachers may be met and possibly link the domains of research and practice.

Notes

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[2]For a detailed discussion of the survey see Manouchehri, A. (2002). Mathematics teaching and learning: Views' survey. http://www.cst.cmich.edu/users/manoula

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LEARNING TRAJECTORIES AS TOOLS FOR SUPPORTING TEACHER CHANGE: A CASE FROM STATISTICAL DATA ANALYSIS

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This paper provides an analysis of a Teacher Development Experiment (cf. Simon, 2000) designed to support teachers' understandings of statistical data analysis. The experiment addresses the following research question: Can the results from research conducted in a middle-grades mathematics classroom be used to guide teachers' learning? In both cases, activities from an instructional sequence designed to support the development of ways to reason statistically about data were the basis of engagement. Analyses of the episodes in this paper document that the learning trajectory that emerged from the teachers' activity did, in many significant ways, parallel that of the students.

Purpose

The purpose of this paper is to provide an analysis of the development of one group of teachers' understandings of statistical data analysis. The analysis builds on the literature on students' understandings by taking prior research as a basis for conjectures about means of supporting teachers' development. In particular, the work reported in this paper is part of a larger project focused on supporting the development of professional teaching communities.[1] The project builds on prior work in classrooms by taking analyses of two classroom teaching experiments as its starting points. The first teaching experiment was conducted in the fall semester of 1997 with a group of 29 seventh graders. The focus of the experiment was on ways of supporting students' understanding of statistical data analysis. As part of this work, the research team[2] developed an instructional sequence and accompanying computer-based tools for analysis. The intent of the instructional sequence was to support middle-school students' development of sophisticated ways of reasoning statistically about univariate data. The overarching goal was that they come to reason about data in terms of distributions. Inherent in this understanding is a focus on multiplicative ways of structuring data. The second teaching experiment was conducted the following year with a subset of the students and focused on covariation. A second instructional sequence and computer-based tools were developed for this phase of the project. Analyses of these teaching experiments therefore provided a basis for collaborations with the teachers by offering a tested and refined conjecture about a learning route for students and the means of supporting its development (cf. Cobb, 1999; Cobb, McClain, & Gravemeijer, in press; McClain & Cobb, 2001a; McClain, Cobb, & Gravemeijer, 2000).

The intent of the teacher collaboration was to build from the mathematical practices that emerged in the course of the classroom teaching experiment. Fundamental to this effort was support of the development of the teachers' content knowledge. The hypothesis was that the same general learning trajectory (i.e., a parallel progression of conceptual development) could serve as a basis for guiding the mathematical development of the teachers. This trajectory served as a conjecture about the learning route of the teachers and the means of supporting their development. During the collaboration, the conjecture was continually being tested and refined in the course of interactions with the teachers. The trajectory therefore offered a conjectured route through the mathematical terrain. This conjecture included not only taking the classroom mathematical practices as a basis for the learning route of the teachers, but also taking the accompanying *means of support* as tools for supporting the emergence of the mathematical practices. These tools included the choice of tasks, the use of computer-based tools for analysis, the use of the teachers' inscriptions and solutions, and the norms for argumentation.

In the following sections, I begin by outlining the instructional sequence. I then describe the methodology used in the analysis followed by a description of the data corpus. Against this background, I provide an analysis of episodes from the work sessions intended to document the teachers' developing understandings of statistical data analysis. I conclude by returning to the conjectured learning trajectory as a means of supporting teachers' mathematical development.

Instructional Sequence

In developing the instructional sequence for the seventh-grade classroom teaching experiment, our goal was to develop a coherent sequence that would tie together the separate, loosely related topics that typically characterize American middle-school statistics curricula. The notion that emerged as central from our synthesis of the literature was that of distribution. In the case of univariate data sets, for example, this enabled us to treat measures of center, spreadout-ness, skewness, and relative frequency as characteristics of the way the data are distributed. In addition, it allowed us to view various conventional graphs such as histograms and box-and-whiskers plots as different ways of structuring distributions. Our instructional goal was therefore to support the development of a single, multi-faceted notion, that of distribution, rather than a collection of topics to be taught as separate components of a curriculum unit. A distinction that we made during this process which later proved to be important is that between reasoning additively and reasoning multiplicatively about data (cf. Harel & Confrey, 1994; Thompson, 1994; Thompson & Saldanha, 2000). Multiplicative reasoning is inherent in the proficient use of a number of conventional inscriptions such as histograms and box-and-whiskers plots.

As we began mapping out the instructional sequence, we were guided by the premise that the integration of computer tools was critical in supporting our mathematical goals. The instructional sequence developed in the course of the seventh-grade teaching experiment in fact involved two computer minitools. In the initial phase of the sequence, which lasted for almost six weeks, the students used the first minitool to explore sets of data. This minitool was explicitly designed for this instructional phase and provided a means for students to manipulate, order, partition, and otherwise organize small sets of data in a relatively routine way. When data were entered into the tool, each individual data value was shown as a bar, the length of which signified the numerical value of the data point (see Figure 1).

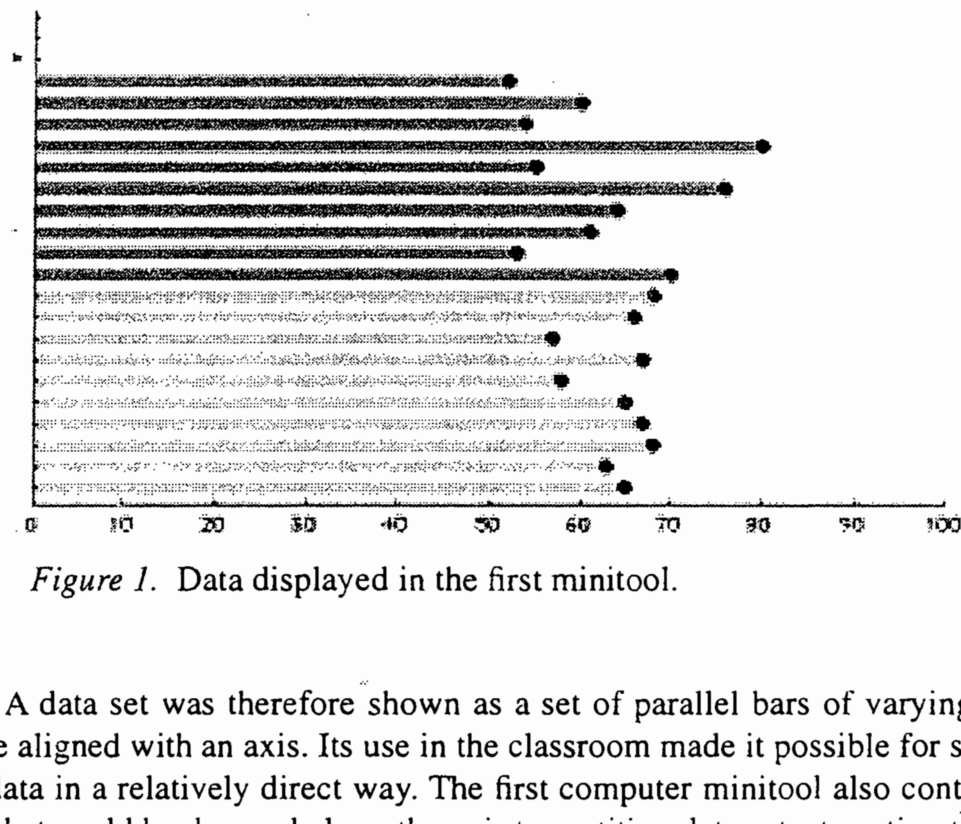

Figure 1. Data displayed in the first minitool.

A data set was therefore shown as a set of parallel bars of varying lengths that were aligned with an axis. Its use in the classroom made it possible for students to act on data in a relatively direct way. The first computer minitool also contained a value bar that could be dragged along the axis to partition data sets, to estimate the mean or to mark the median. In addition, there was a tool that could be used to determine the number of data points within a fixed range. Students' activities with this tool supported the emergence of the first mathematical practice, that of *exploring qualitative characteristics of collections of data points* (Cobb, 1999).

The second computer minitool can be viewed as an immediate successor of the first (see Figure 2). As such, the endpoints of the bars that each signified a single data point in the first minitool were, in effect, collapsed down onto the axis so that a data set was now shown as a collection of dots located on an axis (i.e. an axis plot as shown in Figure 2). The tool offered a range of ways to structure data. The options included: (1) making your own groups, (2) partitioning the data into groups of a fixed size, (3) partitioning the data into equal interval widths, (4) partitioning the data into two equal groups, and (4) partitioning the data into four equal groups. The key point to note is that this tool was designed to fit with students' ways of reasoning while simultane-

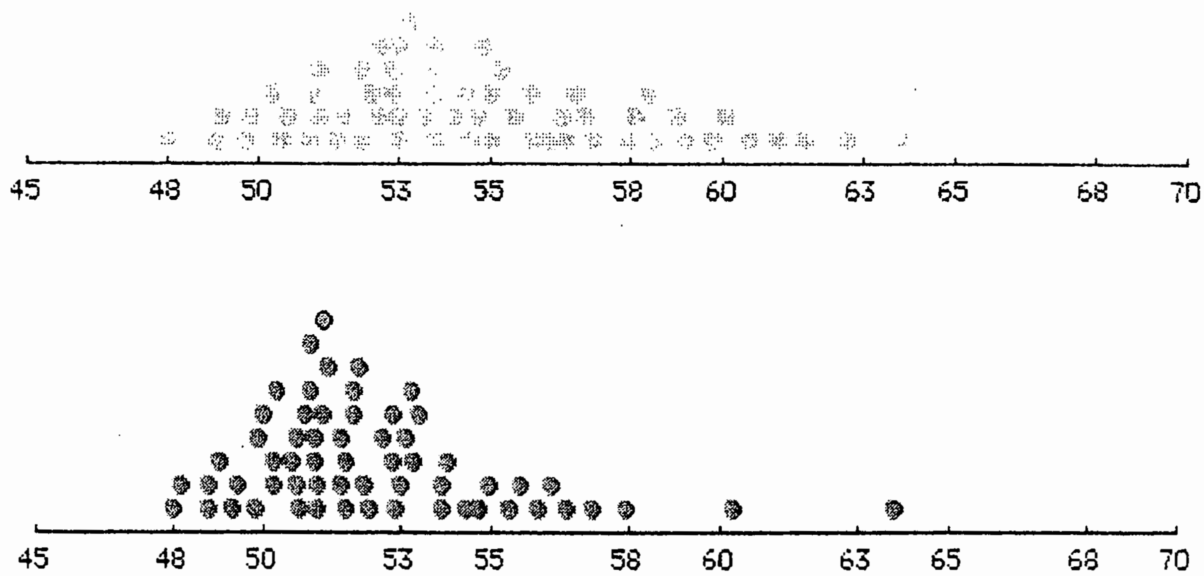

Figure 2. Data inscribed in the second minitool.

ously taking important statistical ideas seriously. The second tool made possible the emergence of the second mathematical practice, that of *exploring qualitative characteristics of distributions* (Cobb, 1999).

As we worked to outline the sequence, we reasoned that students would need to encounter situations in which they had to develop arguments based on the reasons for which the data were generated. In this way, they would need to develop ways to analyze and describe the data in order to substantiate their recommendations. We anticipated that this would best be achieved by developing a sequence of instructional tasks that involved either describing a data set or analyzing two or more data sets in order to make a decision or a judgment. The students typically engaged in these types of tasks in order to make a recommendation to someone about a practical course of action that should be followed. An important aspect of the instructional sequence involved talking through the data creation process with the students. In situations where students did not actually collect the data themselves, we found it very important for them to think about the types of decisions that are made when collecting data in order to answer a question. The students typically made conjectures and offered suggestions about the information that would be needed in order to make a reasoned decision. Against this background, they discussed the steps that they might take to collect the data. These discussions proved critical in grounding the students' data analysis activities in the context of a recommendation that had real consequences.

Methodology

The general methodology falls under the heading of a Teacher Development Experiment (TDE) (cf. Simon, 2000). This methodology is derived from the constructivist teaching experiment (cf. Cobb & Steffe, 1983; Steffe & Thompson, 2000) and whole-class teaching experiments (cf. Cobb, 2000) by acknowledging that a team of "knowledgeable and skillful researchers can study development by engaging in fostering development through a continuous cycle of analysis and intervention" (Simon, 2000, p. 336). The distinction between the TDE and the teaching experiment is that the TDE is concerned not only with the mathematical development of the participants (i.e., the preservice teachers), but also with their professional development. In this way, the TDE can be characterized as a "whole-class teaching experiment in the context of teacher development" (p. 345).

The breath of issues both addressed and acknowledged in the TDE indicates the complexity of teacher professional development. Simon (2000) indicates one source of this complexity when he observes that whereas the focus in a classroom teaching experiment is on students' mathematical development, collaborations with preservice teachers are concerned with both their pedagogical development and their mathematical development. I build from Simon by accounting for both the mathematical and pedagogical issues that emerged over the course of the TDE.

The approach I took when conducting retrospective analyses of the data generated during the TDE involves a method described by Cobb and Whitenack (1996) for analyzing sets of classroom data. This method is an adaptation of Glaser and Strauss' (1967) constant comparative method. The initial orientation for a retrospective analysis is provided by the tentative and eminently revisable conjectures that were developed both prior to and while actually conducting the TDE. The method involves continually testing and revising conjectures while working through the data chronologically. These conjectures might focus on such issues as the evolution of the preservice teachers' understandings of the mathematics or the means by which individual students' learning was supported and organized. This constant comparison of conjectures with data results in the formulation of claims or assertions that span the data set but yet remain empirically grounded in the details of specific episodes.

The particular lens which guided the analysis of the data was a focus on the norms of *what counts as an acceptable mathematical argument and justification* and *what counts as an acceptable pedagogical argument and justification* (cf. McClain & Cobb, 2001b; Yackel & Cobb, 1996). I was interested in documenting changes in the teachers' conceptions of what counts as an acceptable mathematical argument in the course of mathematical investigations of data. It was therefore important for the teachers to participate in exploring the mathematics that underlies the analysis not only to ground their own investigations, but to have them wrestle with the mathematics and how best to justify solutions in the context of the posed analysis.

I was also interested in documenting shifts in the teachers' conceptions of what counts as an acceptable pedagogical argument in the course of discussions of classrooms. It would not only be important that the teachers understand the numerous and varied aspects of practice, but also that they be able to justify in the course of discussion their rationale for a particular stance or judgment.

Data Sources

Data for the study include videotape of monthly and summer work sessions in addition to field notes taken by a research assistant. Copies of the teachers' work and their students' work are also included. In addition, data sources include classroom observations and follow-up interviews to develop accounts of practice (cf. Simon & Tzur, 1999) that can be used to document changes in the teachers' practice. For this reason, sites of data collection include the work sessions and the teachers' classrooms.

Results of Analysis

The reader will recall that the first mathematical practice that emerged in the seventh-grade classroom was that of exploring qualitative characteristics of data points. The initial activities in support of the emergence of this practice involved the teachers analyzing data on the braking distances of 10 each of two makes of cars, a coupe and a sedan. I introduced the task by first talking through the data creation process with the teachers. I then presented the data by giving the teachers paper copies of the data inscribed in the first minitool as shown in Figure 1. I asked the teachers to work at their tables to decide which make of car they thought was safer, based on these data. My decision to use printouts of the data was based on my own experience in working with students on these tasks. I had noticed that when students were asked to make initial conjectures based on analysis of the printouts, their activity on the computer tool seemed more focused. They used the features of the tool to substantiate their preliminary analysis instead of exploring the structures that resulted from the use of the features. In addition, they focused more on features of the data sets such as clusters. I was also curious to see if the tools we had designed offered the teachers the means of analyzing data that fit with their initial, informal ways of analyzing the data.

As the teachers began their analyses, most of them initially calculated the mean of each set of data. They subsequently judged that measure to be inadequate for making the decision and proceeded to find ways to structure the data that supported their efforts at analysis. In this process, they used vertical lines drawn in the data to create cut-points and to capture the range of each set. As an example, one teacher noted that all of the coupes took over 55 feet to stop whereas four of the 10 sedans were able to stop in less than 55 feet. Other teachers focused on the "bunched-up-ness" of the coupes and reasoned that a consistent braking distance was an important feature.

I found these ways of reasoning significant for two reasons. The first was that the ways of structuring that they were creating with the drawn lines paralleled the features that we had designed on the tool. This implied that the tool would be a useful resource in supporting their analysis. The second was that their ways of reasoning about the data were consistent with the methods that we saw emerge in the seventh graders' activity. It therefore appeared that the conjectured learning trajectory could guide the mathematical development of the teachers.

As the teachers discussed the results of their analysis in a whole-group setting, I introduced the first computer minitool as a resource for sharing their ways of structuring the data. I used a projection system to make the data sets visible to the group and, as the teachers explained their analysis, I used the features on the minitool to complement their explanations. As an example, as the teachers talked about the "bunched-upness" of the data sets, I activated the range tool so that they could identify the extremes in each data set. The teachers found this support helpful and were easily able to use the features on the tool to mirror their earlier activity with drawn lines.

In the remainder of the session, the teachers analyzed a second set of data on the longevity of two brands of batteries. Their analyses were again focused on cut points and clusters. These ways of reasoning parallel what we found in analysis of the seventh graders' activity. In particular, Cobb (1999) notes that in the seventh-grade classroom "[t]he characteristics of data sets that emerged as significant in this discussion and in the subsequent classroom sessions in which the first minitool was used included the range and maximum and minimum values, the number of data points above or below a certain value or within a specified interval" (p. 17). The significant difference between the seventh-grade students and the teachers was that the teachers were able to talk about the number of data points above or below a cut point in terms of percentages (or ratios) of the whole. Further, they could reason probabilistically such as arguing that "you have a 30% chance of getting a bad battery" with a Brand A. Their arguments therefore appeared to be multiplicative in nature.

A shift to the second minitool and a focus on the second mathematical practice—exploring qualitative characteristics of distribution—began with the introduction of the speed trap task. After a lengthy discussion of the data creation process, the teachers were shown printouts of data on the speeds of two sets of sixty cars (see Figure 2). The first were recorded on a busy highway on a Friday afternoon. The speeds were recorded on the first sixty cars to pass the data collection point. The second set of data was collected on a subsequent Friday afternoon after a speed trap had been put in place. The goal of the speed trap (e.g., issuing a large number of speeding tickets by ticketing anyone who exceeded the speed limit by even 1 mph) was to slow the traffic on a highway where numerous accidents typically occurred. The task was to determine if the speed trap was effective in slowing traffic.

As the teachers worked on the printouts of the data, most of them created cut points at the speed limit and reasoned about the number of drivers exceeding the speed limit both before and after the speed trap. They used a range of strategies including ratios and percentages. Further, none of the teachers calculated the mean. One teacher focused on the shape of the two data sets and noted that at first it "looked like a Volkswagen Beetle" and then it "flattened out like a large Town Car." I found this particularly significant because it was the first occasion where a teacher found a way to describe the shape of the distribution. I capitalized on her solution and recast it in terms of "hills." My reason for doing so was grounded in our earlier work in the seventh-grade classroom where a similar incident had occurred with this same data set. For the students, the notion of the hill was pivotal in shifting their reasoning from viewing data sets as collections of points to distributions (cf. Cobb, 1999; McClain & Cobb, 2001a; McClain, Cobb, & Gravemeijer, 2000).

The significance of this shift became apparent in the next session when I introduced a task containing data sets with unequal numbers of data points. In the task, two sets of AIDS patients were enrolled in treatment protocols — a traditional treatment program with 186 patients and an experimental treatment program with 46 patients. T-cell counts were reported on all 232 patients (see Figure 3).

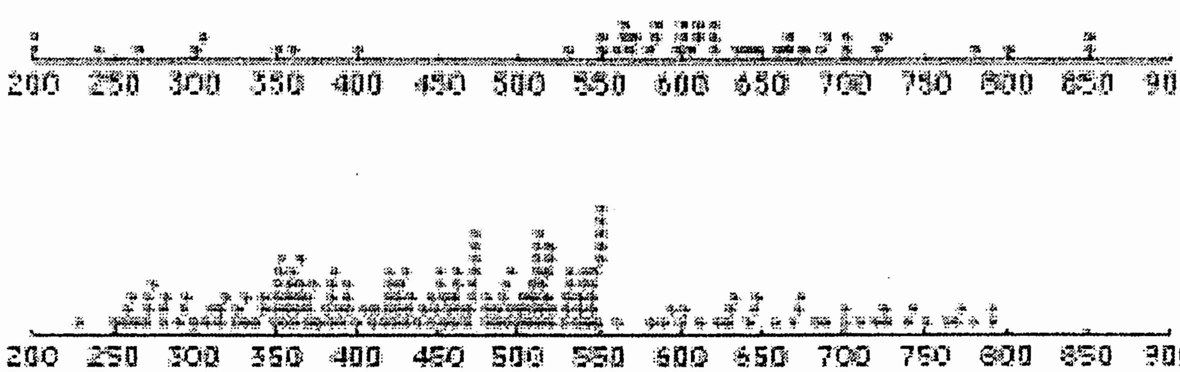

Figure 3. AIDS data displayed in the second minitool.

As the teachers worked on their analysis, they initially noted that the clump, cluster, or hill of the data shifted between the two groups. This was a significant aspect of the distributions and one on which they focused as the continued their analysis using the second minitool. In their work at the computers, they found ways to characterize this shift including creating cut points and reasoning about the percentage of patients in each group with T-cell counts above the cut point. They, like the seventh grade students, noted that that cluster of T-cell counts in the traditional treatment program was below the cut point whereas the cluster of T-cell counts in the experimental was above.

The teachers were also able to use the features of the computer tool to partition the data to support their arguments. One such feature, *four equal groups*, partitions the data into four collections such that each contains 25% of the data. From a design standpoint, we viewed this as a precursor to a box and whiskers plot. The teachers were able to use this feature in conjunction with another that caused the data points to be hidden, leaving only the partition lines. When the teachers structured the data with the *four equal groups* option and the data hidden as shown in Figure 4, they were able to formulate arguments such as, "Seventy-five percent of the experimental treatment is in the range of only 25% of the traditional treatment." "If you use the experimental treatment, you have a 75% chance of getting a T-cell count above 525 and only a 25% chance with the traditional treatment." Arguments such as these indicated their ability to reason multiplicatively about the distributions.

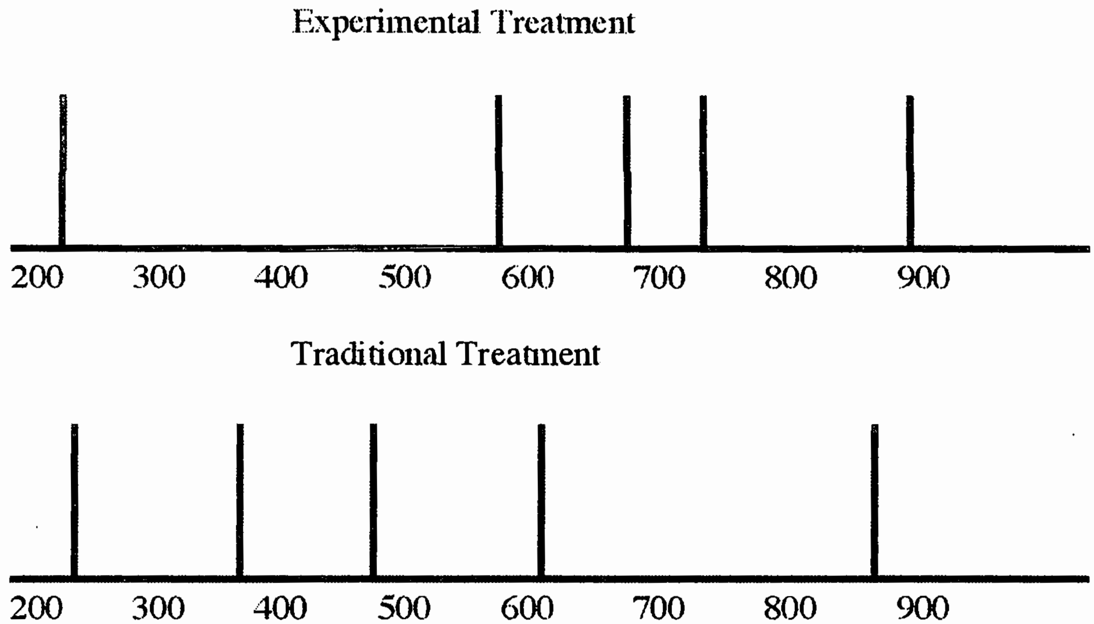

Figure 4. AIDS data structured into four equal groups with data hidden.

Conclusions

As a result of the parallel developments between the seventh-grade students and the teachers, it appears that the underlying learning trajectory that guided the seventh-grade students' development offers the means of supporting teachers' development of understandings of statistical data analysis. However, there were significant differences in the mathematical development of the teachers. In particular, multiplicative reasoning was a goal in the learning trajectory for the seventh grade students. The tasks and tools were designed to support the emergence of problem situations that would problematize direct additive comparisons and create a need for multiplicative ways of reasoning about the data. For the teachers, multiplicative ways of reasoning were a tool that was available from their initial explorations. The goal for the teachers then became to make explicit the importance of task situations where multiplicative ways of reasoning are necessary (e.g., unequal numbers of data values in the two data sets). For this reason, discussions of multiplicative structures among the teachers took the perspective of a *teacher* instead of a *student.* It was necessary to create situations where the teachers had to reason about ways to initiate shifts in their own students' strategies.

A similarity that emerged across the two situations was the use of cut points as an accepted way to make a statistical argument. A choice of cut point was a common form of warrant in creating databased arguments. The similarities between the students' and teachers' choices of cut points are very striking, as is the choice of backing for these warrants. This normative way of reasoning about data can then inform future efforts by feeding forward to the realized learning trajectory in both situations.

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Notes

[1]The work reported in this paper is part of a larger project supported by the Office of Educational Research and Improvement and the National Science Foundation. It is being conducted in collaboration with Paul Cobb.

[2]The research team was composed of Paul Cobb, Kay McClain, Koeno Gravemeijer, Maggie McGatha, Lynn Hodge, Jose Cortina, and Cliff Konold.

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RESPONSIVE DESIGN: CREATING A SCAFFOLDING SYSTEM TO SUPPORT TEACHER PROFESSIONAL DEVELOPMENT

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In this presentation we look at the implementation of instructional design principles to support teachers using the InterMath professional development system. In response to needs identified through pilot testing and interviews, we have explored ways to further support participants in the online system as they become engaged in solving complex, open-ended problems. Our needs analysis identified two specific kinds of scaffolds that needed to be developed: one focused on supporting teachers in applying problem-solving principles, the other focused on supporting geometric constructions as a tool for problem solving. This presentation will explore the issues teachers raised, how we determined the scaffolding needed, and what the forms of the two systems is taking. We will also discuss the rapid prototyping design path we are taking in our development effort.

Introduction

InterMath[1] (http://www.intermath-uga.gatech.edu) is a technology-intensive professional development experience for middle school mathematics teachers. The professional development effort is aimed at furthering teachers' mathematics knowledge by immersing teachers in a technology-rich mathematical investigation environment. The model upon which InterMath is based assumes that teachers who experience learning in a rich, exploration-based environment will more readily transfer these kinds of experiences to their classrooms. The centerpiece of the InterMath experience and website is an extensive set of open-ended mathematical investigations that can be explored using various technologies (e.g., Excel or Geometer's Sketchpad). These investigations are central to the semester-long workshop as teachers are encouraged to explore a particular set of explorations (e.g., triangles or functions), choose problems that intrigue them, work those investigations, and write-up the solutions, along with extension activities.

Unlike the traditional "make and take" professional development experiences, InterMath does not aim to provide teachers with activities they can take back to their classroom and use. Instead, InterMath provides teachers with opportunities develop both their understanding of mathematics and their ideas about what it means to learn and teach mathematics. Interestingly, in our pilot workshops, many teachers had some degree of success using the InterMath materials as "make and take" – they take the investigations back to their classrooms and implement them almost exactly as they experienced those activities in the InterMath workshop.

InterMath is in its fourth year of development and implementation. During the pilot workshop offerings in Year 2, there were two InterMath workshops conducted by two different faculty members. One workshop was offered near Atlanta as the first course in a cohort-based graduate degree program for middle school teachers. This workshop was taught by a mathematics education professor who is also one of the developers of the InterMath program. The other workshop was offered at the University of Georgia campus by a professor of mathematics. The participants in the UGA workshop received staff development credits for their experience, as well as a stipend.

As a part of the pilot workshops, we were able to interact with teachers using the InterMath materials to better understand how they worked. The information collected as part of this effort has allowed us to carefully examine both how InterMath works and how we might improve it. This kind of iterative, responsive design based on participant feedback had formed the basis of our continuing design efforts. This paper explores this design effort.

Instructional Design

There are a number of instructional design models available to use in the development of an instructional environment (e.g., Gustafson & Branch, 1997). They provide a guide for the developer to use in creating a meaningful experience for the learners. While we relied on the general assumptions of instructional design including the need to create a learning environment that is effective, efficient, and motivating (Smith & Ragan, 1999), we developed our own framework of instructional design, borrowing from a number of sources, as we moved forward in the design and development cycle. In this way, we have been able to use approaches that are context-specific – that is, we have been able to develop a model that allows us to respond to the needs of our particular learners. To this end, we have relied on three key instructional design approaches: formative evaluation, needs assessment, and rapid prototyping. In our work thus far, the pilot workshops have provided us with an opportunity to do formative evaluation as we are provided with an opportunity to consider the intervention (the InterMath experience) and determine what revisions are needed (Seels & Glasgow, 1998). We have used needs assessment frameworks in our analysis of the formative evaluation data to identify the critical needs of the learners (Kemp, Morrison, & Ross, 1998). And, we have implemented a rapid prototyping approach in developing the hints system (Tripp & Bichelmeyer, 1990). Every facet of our work has been tightly linked to teacher needs and teacher feedback, thus our decision to view our efforts as responsive design. Further, as will be discussed, most of our design effort has arisen out of learning and motivational principles, therefore, we characterize the approach we are taking to this facet of InterMath development as principled, responsive design.

Formative Evaluation

The purpose of formative evaluation is to determine what revisions are needed in order to optimize learning in a given instructional setting (Seels & Glasgow, 1998).

We had the opportunity to pilot test InterMath two times in 2001 allowing us to collect formative evaluation data. These data were collected during each InterMath session in each workshop through both formal and informal means. InterMath instructors and facilitators observed the teachers each week in the workshop as they worked on a variety of mathematical problems. Fieldnotes and written reflections were completed in each of the classes. In addition, many participants (eight in one class and all four teachers who completed the second pilot) were interviewed at the conclusion of the workshops and all participants were surveyed both at the beginning and the end of the workshop. These data provided information regarding difficulties the workshop participants experienced and how the InterMath experience could be changed to better benefit teachers in future workshops.

An analysis of the data collected provided insight into participant frustrations over the difficulty of many InterMath investigations. Some of the participants noted that they do not have strong backgrounds in mathematics, while others described the mathematical 'rustiness' that has occurred while teaching middle-grades math. These comments gave sufficient cause to consider creating some type of mathematical scaffolding system for the InterMath workshop, as InterMath personnel are not always on hand when the teachers are working on their investigations. Further, our observations showed that the teachers struggled, at times, to use the technology. While the most evident problems were with webpage development software, we also noted many occasions when the teachers were unable to use tools such as Geometer's SketchPad (GSP) without assistance.

Using instructional design principles, we determined that some form of computer-based scaffolding system would be beneficial for teachers participating in InterMath. The goals of this system would be to encourage teachers to select problems that are more difficult than those they might otherwise attempt, thereby assisting them in furthering their mathematical thinking and understanding, and decrease the frustration that can occur when they come to a point where no more progress can be made without some type of assistance.

Needs Assesment

The goal of the needs assessment is to identify particular needs associated with completing a job or task (Kemp, Morrison, & Ross, 1998). In our case, the needs assessment focused on helping us understand what the teachers were unable to do or had to struggle with in order to succeed in the workshop. The first phase of our needs assessment focused on defining some of the critical needs of our participants. It became apparent that the teachers needed more support than could be offered in the workshop and because our team could not be available all the time, we decided that an online system to support teacher learning anytime, anywhere was a critical element of the support system. One clear goal we had was to maintain a balance between providing too little and too much support to the teachers. Our concern was that the hints

– whether for GSP or mathematical problems solving – not only helped the teachers achieve their goals, but also engaged them in thinking about the mathematics they were working with. We realized that by adding a "hint" to the more complicated problems, we may be able to aid teachers in persevering to become more successful problem-solvers. These hints may focus on the mathematics or on the use of technology to support the mathematics.

One of our primary concerns in considering a support system was that it may inadvertently limit the cognitive growth potential for users of InterMath. Based on our initial field test of the hints, this concern was well-founded. Teachers in our test group clicked on the hint before they attempted to solve the problem on their own. The static hint system cannot provide the cooperative problem-solving effort between the learner and a more knowledgeable other that defines scaffolding (Collins, Brown, & Newman, 1989), nor can it anticipate exactly when the learner may require assistance. Additionally, our learners would need to take responsibility for "fading" the scaffolding. It is unclear that learners can effectively self-monitor to determine where they are in their own ZPD (Vygotsky, 1978), however, this is a critical part of the scaffolding fading process (Kao & Lehman, 1997; Oliver & Hannafin, 2000). Further, learning in this environment comes from engagement in the problem-solving process, therefore the learners need to be actively involved in the process in order to benefit. If they click on the hint before struggling on their own, the potential of the learning experience may be significantly weakened.

As we continued working in the design and analysis phase of this undertaking, we began to consider the purpose of the hints. We identified three distinctly different purposes hints might serve: hints as motivators, hints as models of expert thinking, and hints as generalizable strategies. We also discussed why there may be good reason to include no hints at all.

No Hints

The "no hints" argument arose from our beliefs about learning and our previous work with teachers supporting students in computer-based environments (e.g., Hawley & Duffy, 1998). As mentioned previously, we were concerned about the ability of an online scaffolding system to provide the kind of help the learner needed at the time she needed it. After all, the online system cannot sense where the learner is within a problem, what the learner's mathematical knowledge is, or whether the problem is a technical one (e.g., not knowing how to construct a shape or write a formula) or a mathematical one (e.g., not knowing how to approach a problem). It was the "inside help" that we felt the system could not provide (Polya, 1981). That is, the computer is not sensitive or "intelligent" enough to provide the learner with the kinds of questions or suggestions that may have occurred to the student – and those are the questions that move the student to new levels. Further, the static hints system is problematic in that scaffolding should be an interactional approach (Driscoll, 1994). It, ideally, provides for an interaction between two people that can evolve and/or fade over time. The reciprocal teaching research (e.g., Palincsar & Brown, 1984) provides a compelling description of a scaffolding system that exemplifies this approach. Our online system could not be like this. Our design team considered how or if the hint system could serve as a scaffolding device without this kind of teacher-learner interaction.

While compelling, the arguments against the hints system did not provide alternatives for supporting teachers in attempting or completing the problems that offered the most promise for their mathematical development – the "hard" math problems.

Approaches to Hints

Polya's four-step problem-solving process (Polya, 1957) provided a basis for our thinking about hints as generalizable strategies. We determined that our hints would be most productive if focused on the first two parts of the process: (1) Understand the problem and (2) Devise a plan. To this end, we explored the development of a set of hints that provided both the strategy (e.g., "Can you restate the problem?") and a specific hint for the problem of interest (e.g., "Before you answer the given problem, think about the definitions and formulas of circumference and area. On what part of each circle do you really need to focus? How do these parts relate to each other?").

We also explored expert thinking as another foundation for the hint system. Wood, Bruner, and Ross (1976) define the expert's role in scaffolding as directing and maintaining the learner's attention, while also modeling the task and highlighting the critical features of that task. We considered the fundamental differences in expert and novice organization of knowledge (e.g., Bransford, Brown & Cocking, 1999). Experts tend to organize thinking and strategies around core concepts and more readily see where to apply concepts whereas, novice thinkers tend to exhibit signs of more linear and procedural understandings. Modeling has been shown to be an effective means for supporting the development of expert thinking (e.g., Palincsar & Brown, 1984; Schoenfeld, 1991). Our primary concern was if learners do not engage in reflective adoption and adaptation of the modeled example, they learn only a prescriptive task rather than a generalizable approach. With this hint system, supporting the learner's reflection would be one way to move toward the benefits of the modeling system.

Finally, we looked at the motivational aspect of the hints. We recognized the importance of providing indirect facilitation for goal attainment, rather than controlling learners' actions (Ford, 1992). In our case, the hint system would be the determining factor for whether they chose a more challenging problem over a simpler one. We felt that, along with the motivational boost, there would also be a teaching self-efficacy boost from this system (e.g., Bandura, 1989; Pajares, 1996). It was our hope that providing the system would increase learner's feelings of self-efficacy, increasing their likelihood of expanding their mathematical horizons.

Once we had considered all the evidence and the development issues involved with the various hint system formats, we began working toward developing a system that attempted to support mathematical development by relying on questions and examples that are readily transferred to other mathematical situations. To achieve this, the mathematical hints systems uses only questions the learners will be able to use on other problems and in other ways (Polya, 1981). In this way, we aimed to develop tools that, while static, might support the development of transferable knowledge for the learner.

A critical and ongoing effort in our design and development phase has focused on the identification of investigations most in need of hints and what kinds of hints they might need. This process was begun by having a new graduate student whose mathematical background was similar to our target audience's work several of the investigations to identify places where they became difficult as well as ways she was able to overcome those difficulties. This provided a beginning guide for the hints system. However, her activity on the project team and in graduate-level mathematics education courses quickly moved her beyond the mathematical content knowledge of our target audience. We are currently exploring new strategies for determining which investigations require more support.

To better understand what a static, computer-based hints system might look like, we explored existing systems to see how they provided support for learners. These systems included the support system in the Knowledge Integration Environment (WISE or KIE) (Slotta & Linn, 2001), as well as EMILE (Guzdial, 1994) and CSILE (Scardamalia et al., 1992). These tools provided very different approaches to scaffolding learning, ranging from static to very dynamic and including various levels of teacher/knowledgeable other support within the system.

The tools that we reviewed ranged, also, from technologically complex to those that were relatively simple. In the end, the decision to follow a simpler path was tied to two issues: budget and time. The enormous undertaking of developing a dynamic support system was outside the scope of our efforts and was not included in the budget. Further, InterMath is a multifaceted system that includes communications tools, a dictionary, materials for instructors, and, of course, investigations. Because each of these pieces requires support, the need to keep the scaffolding system to a manageable size was considerable. Further, the InterMath team recognized that teachers in need of more specialized support could contact an InterMath team member or could use the communications tools to pose questions to other members of the community. The trade-off with this was the loss of momentum caused by the delayed communication in an asynchronous environment. However, if we paired a just-in-time scaffolding system with these other tools, we felt we could offer support for the learners that would be meaningful.

The most critical decision made as a result of our process was the decision to split the hints system into two distinct pieces. One piece would be the hints as discussed in the analysis section. The other part would be the "Constructionary," which is a system to support teachers in completing mathematical constructions using Geometer's Sketchpad. While making constructions in this program is still a mathematical process, it is the point where mathematical concepts intertwine with technology. The support that teachers struggling with the technology needed was not the same kind of support they needed to conceptually understand the problem.

The Constructionary borrows heavily from the Lego™ approach to instruction – teachers are provided with pictures that show the construction being put together. They are provided with minimal text as well because our rapid prototyping process indicated that the images alone were too confusing. The goal of the Constructionary is to provide teachers with the support they need, while forcing them to remain engaged in the mathematics of the construction. In the development of the Constructionary, we have relied on the principles of rapid prototyping (Tripp & Bichelmeyer, 1990). As we develop constructions to be included in the Constructionary, we show people who are like our target audience the hint. Then, we ask them to decipher what they think that hint is trying to show. The data gathered are used to immediately alter the hints to optimize their effectiveness.

For the content-specific hints system, we chose to use the scaffolding system in WISE (Slotta & Linn, 2001) as our key model. It is a hints system that provides hints on command. A single process step may have more than one hint, but only one appears at a time. The users, in this case middle school students, must click to request a second hint. Further, the hints point the users back to information or processes they have already worked with. As a scaffolding system, the WISE approach aims to develop student inquiry skills.

Rapid Prototyping

The instructional design and development process is ongoing. Our design process has run parallel to our evaluation process and has taken a variety of forms. The analysis and design phases have thus far employed three instructional design strategies: 1) having a person similar to our target audience work through investigations to identify potential problems and discuss how those complications affect the successful completion of the problem; 2) reviewing existing computer-based scaffolding systems to explore the characteristics and possibilities of effective systems; and 3) working with subject matter experts on our team to determine how to translate face-to-face questioning strategies to static online strategies. Further, as we move into the development phase we are using a fourth key strategy – rapid prototyping.

Rapid prototyping is an approach to instructional design in which the product is created in a minimal form in sequential iterative cycles so that it can be evaluated by target users. In our case, we have developed three separate rounds of paper-based Constructionary entries to determine how much text is necessary to convey our meaning while still maintaining cognitive engagement on the part of the user. Further, we have conducted one mock-up version of two of the hint types to determine teacher reaction to them. This approach allows a responsive design system to emerge that constantly relies on feedback and input from the end-users. Further, rapid prototyping allows developers the opportunity to evaluate ideas before considerable time and money have been spent in their creation. As we move forward with our ideas, we will begin to move away from rapid prototyping and will turn again to formative evaluation simply because we now need to do some full-scale development which does not lend itself to the iterative rapid development processes.

Moving Ahead

Both the Constructionary and the content hint system have gone through multiple rapid prototyping iterations. Even as early in the process as the pilot courses, we were able to bring in different kinds of hints and test them out with members of our target audience. This provided valuable feedback as we were able to put the hints system to work, watch the users interact with it, and talk to them about their reactions to it. Interestingly, in the first test of the hints strategies, the participants claimed that the hints were not useful, despite having relied on them to complete their problems.

We are currently involved with the development stages of both the content hint system and the Constructionary. There are still obstacles to overcome, including making the final decisions about which approach to the content hints system to adopt. There are very practical considerations that must be weighed in this decision. For example, we must choose a hints system that members of our team can create without the need for extensive work on any one problem. This has been a critical factor in our exploration of hints approaches.

Another critical obstacle is determining which investigations need hints and what kinds of problems teachers might have with them. This is problematic because all of our team members have either extremely high mathematical ability, or mathematical ability that is too limited. To address this problem, we have considered a number of approaches including conducting more teacher observations during our next workshop offering and using our best guesses to find problematic investigations.

Conclusion

The purpose of this paper was to explore the issues and approaches being used to develop a dual-purpose hint system in response to user needs in the InterMath workshop. Thus far, we have included users in every aspect of the development. We watched them struggle and spoke to them about difficulties to identify the original need, we have worked with them to refine our scaffolding ideas, and we will continue to work with users as we develop and implement the system. Our position is that this is a powerful approach to instructional design as it allows instructional tools to be developed that support learners in the process of learning.

Note

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CHANGES IN MATHEMATICS TEACHING: A CAREER PERSPECTIVE

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This report addresses issues of teacher change over time. It presents factors that, from the teachers' point of view, were important in the establishment of their ways of teaching mathematics to young children. The project involved 5 elementary school teachers with ages between 45 and 60, all with over 20 years of teaching experiences. It focused on the stories the teachers constructed about their careers as they reflected upon their professional trajectories. Analyzing data from one of the participating teachers, Sharon (pseudonym), teaching appears as job that is impregnated with changes. Sharon presents the development of her mathematics teaching as a continuous search process that has gone through many phases and incorporated many recommendations for change. In Sharon's stories, educational trends in general, and new recommendations for school mathematics in particular, have promoted changes in her mathematics teaching. Her school environment, her colleagues, and the professional development activities in which she participated appear as main change agents in Sharon's professional story.

The publication of the *Curriculum and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 1989) in the late 1980s strengthened mathematics educators concern with teacher change. The constructivist perspective that underlined the document implied a vision of mathematics teaching that greatly differed from the widespread teaching model in which teachers hold and transmit knowledge to students. For many teachers, aligning their instruction with the vision espoused by the council represented a new and demanding challenge. In this context, studies of mathematics teachers' change thrived.

Four positions on the process of change in mathematics teaching developed in the late 1980s: a Piagetian stand, an approach rooted in cognitive science, a socioconstructivist perspective, and a view based on teachers' knowledge of mathematics itself (Nelson, 1997, p. 4). Despite their differences, the four theoretical positions pointed to the fact that transforming mathematics teaching was not a simple process. Rather, for most teachers, it required ongoing effort and support. Recent studies have shown that despite a history of resistance to change, teachers can successfully incorporate NCTM's suggestions into their mathematics teaching practices (e.g., Ferrini-Mundy, Graham, Johnson, & Mills, 1998). Thus, mathematics education researchers have come to perceive change in school mathematics as a difficult but feasible process.

This paper discusses change from the teachers' perspective. In particular, it addresses the ways in which teachers portray change in their professional stories. How do teachers' perceive their own professional changes? What are factors they consider as catalysts for change? Analyzing data from Sharon (pseudonym), one learns that teaching is a dynamic job, impregnated with changes. Sharon presents the development of her mathematics teaching as a continuous search process that has gone through many phases and incorporated many recommendations for change. In Sharon's stories, educational trends in general, and new recommendations for school mathematics in particular, have indirectly promoted changes in her mathematics teaching. Her school environment, her colleagues, and professional development opportunities appear as main change agents in Sharon's professional story.

Framework

An epistemology of professional practice is needed for the comprehension of elementary school mathematics teaching. Tardif (2000, p.13) defined this epistemology as the study of the set of knowledge used by professionals in their everyday working space to accomplish all their tasks. This epistemology is charged with revealing teachers' content-knowledge, attitudes, and know-how, as well as the ways in which these different factors are combined and used in teachers' performance of their professional duties. From this perspective, teachers' knowledge is personal, situated, and time dependent. It relies on teachers' life histories and develops throughout their careers. Tardif and Raymond (2000) proposed that to understand teaching, one should investigate the history of development of teachers' personal knowledge base for teaching. It is necessary to understand teachers over time, asking them about the history of their knowledge and practice.

Within this perspective, this project studies teacher change over time. It looks for changes teachers embrace as they experience their lives in the classroom. The study is based on teachers' own explanations for their mathematics teaching and for their accomplishments over the years. In particular, it explores factors that, from the teachers' point of view, are important in the establishment of their ways of teaching mathematics to young children. It searches for factors that, over the span of the teachers' careers, served as catalysts for change.

The study of teachers requires a methodological approach that allows for the representation of action; an approach that can cope with the many ambiguities and dilemmas that emerge from experiences in the classroom. This project uses narratives to study teachers. A narrative is a "symbolic presentation of a sequence of events connected by subject matter and related by time" (Scholes, 1981, p. 205). Narratives are "the closest we can come to experience" (Clandinin & Connely, 1994, p. 415). In education, narrative inquiry presents teachers' stories while searching for patterns of understanding. Stories are the phenomena under study and narratives offer a way to studying them (p. 416).

A story is made of organized information. Humans seek to provide a sense of coherence for their experiences and arrange their life episodes into a meaningful story (McAdams, 1993). They bring together different parts of their selves to compose a convincing story that illustrates essential truths about them. This patterned integration of "remembered path, perceived present, and anticipated future" (p. 12) allows individuals to construct the stories that make them unique. These stories are more about meaning than they are about facts. "In the subjective and embellished telling of the past, the past is constructed" (p. 28). People's self-stories are particularly important in mid-life. At this point, many face the challenge of recreating their identities to enhance their "sense of unity and purpose in life" (p. 202). "In mid-life we endeavor to put many pieces of our life story together into a more integrative and generative whole" (p. 220).

Looking at teachers between the ages of 40 and 50-55 years of age, Sikes, Measors, and Woods (1985) reported that this is a time when teachers come to terms with where they are in life, assessing what they have and have not achieved. Despite the frailty of the assessment moment, many teachers during this stage recognize their knowledge and experience. They feel respected and proficient (p. 229). Within teachers' professional cycles (Huberman, 1993), those with 4 to 6 years of teaching typically experience a feeling of stabilization and of consolidation of their pedagogical repertoire. Following that, between 7 and 25 years of experience, teachers may either attempt to diversify their practices, enlarging their professional repertoire and searching for new challenges, or they may go through a reassessment phase in which they question their path. Teachers in the diversification path can be highly motivated and engaged. Those who experience reassessment face a period of self-doubt and a sense of routine. After 25 years of teaching experience, many teachers achieve serenity, a feeling of greater confidence and acceptance of who they are.

Teachers' professional personal stories and their organization of experiences over many years in the classroom are the kernel of this study. The project involved 5 elementary school teachers with ages between 45 and 60, all with over 20 years of teaching experiences. It focused on the stories the teachers constructed about their careers, as they reflected upon, structured, and organized their own trajectories. Working with very experienced teachers offered the possibility of looking back to life in the classroom in order to pinpoint important events and influences that marked the teachers careers. It allowed teachers to talk about different changes they made over time.

Situating teachers' personal stories within school contexts, one needs to consider that, at schools, teachers live in two fundamentally different spaces: one where they are with students behind the classroom door, and another where they are with their peers (Clandinin & Connelly, 1996). Outside the classroom is a space "filled with knowledge funneled into the school system for the purpose of altering teachers' and children's classroom lives" (p. 25). Clandinin and Connelly call sacred the stories that are part of this theory-driven, out of the classroom space shared by practitioners, policy makers, and theoreticians. They contrast sacred stories with teachers' secret lived stories, which tell about real classrooms, and teachers' cover stories, which portray teachers in a way that is acceptable within the school or the school system.

In this study, NCTM's recommendations are among the sacred stories told to teachers in the last decade. The question is whether these recommendations become part of teachers' stories: cover stories, at least, lived stories desirously. Because this study takes a career perspective, it deals mainly with the stories teachers tell about themselves. Teachers have lived many different stories in their professional progression. This study presents the meaning they make of the stories they lived in many different contexts. Within teachers' organization of their many stories, this reports concentrates on the sacred stories that teachers incorporate to their own, personal professional stories.

The Project

The 5 elementary school teachers involved in this project worked in a K-5 suburban school near a major, southern metropolitan area. During spring 2001, in three individual, open-ended interviews, each teacher worked with a timeline of her career, a logic-spider, and a lifeline. In the first meeting, teachers were presented with a timeline developed from an inventory of professional experiences they had completed. They were asked to comment on different jobs, different schools, and different moments in their careers. In the second conversation, teachers were presented with a "spider" that had knowledge for teaching mathematics as the central theme, and different topics for discussion in the "legs." Teachers choose what they wanted to discuss, jumping from topic to topic. During this meeting teachers also shared lesson plans and mathematics activities they had collected and used in their classrooms. In the last meeting, teachers constructed their own lifeline (Connelly & Clandinin, 1994). They included what they saw as fundamental issues, turning points, and important people in their careers. The different instruments offered teachers opportunities to tell stories from different schools and classrooms, as well as a story of their professional path. All interviews were conducted at the school and most of them took place in the teachers' classroom — during planning time, when students were in other classrooms, or after school hours. They lasted between 45 and 60 minutes. In a final group interview, teachers discussed issues that were repeatedly mentioned in individual interviews. They shared experiences, talked about their school, and discussed what they considered most important in becoming the mathematics teachers they are.

Interviews were audiotaped and transcribed. Teachers received copies of the transcripts of their own individual interviews as well as of the group interview. They were offered the opportunity to comment on the interviews, pointing to modifications or clarifications they would see fit (although they did not make comments). Data were analyzed through constant comparison method using initial descriptive categories such as early teaching, later years, approaches to mathematics instruction, and catalysts for change. Small episodes in the teachers' stories were considered as the unit of analysis, and all information was examined within its context. Although change was part of the initial research interest, it became prominent during the data analysis process because it was such a fundamental part of all teachers' career stories.

Sharon's perspectives were selected for presentation in this report because she began teaching during the Back-to-Basics period and has been teaching through the Standards Movement. Sharon's professional story parallels the story of NCTM's most recent call for changes. Sharon is also currently enrolled in a Specialist Degree program in mathematics education. Therefore, she has been particularly reflective about her mathematics teaching.

Sharon

Sharon graduated with a Bachelors of Arts in 1975 and got a Master's in Education in 1981. After her first year of teaching, she got married, moved, and did not teach for 2 years. Since the 1978-1979 school year, Sharon has been teaching third grade. She defines herself as a resource person. "My strength is I am a good resource person, I have places to get lots of ideas and I am willing to sit down and work at finding them, if I can find the time to sit down and do that."

In 1988 Sharon began teaching at the school where she now works. She defines her school as a place where teachers have a lot of support. Teachers from the same grade level have common planning time and quarterly plan-it-take-it meetings. They have money to get the resources they need, opportunities to visit other schools that are implementing different ideas, and plenty of chances to participate in professional development activities. All five teachers in the study considered their school as a place of high expectations for students and for teachers. They also considered it a supportive working environment. When asked to make her professional lifeline, Sharon began her line at her current school. "Interesting," she said, "because I probably wouldn't go back longer than being here."

One of Sharon's early mathematics stories is her recollection that she could not memorize her facts in fourth grade. "In our family room there was a coat closet and on either side of the coat closet [my mom] put two nails. And every night we would pull out this chalkboard that hung on those two nails, and we would do multiplication facts till I was crazy." Later, Sharon successfully completed her Algebra and Geometry high school courses. But Trigonometry and Advanced Algebra were a challenge. "I was so lost, I mean, absolutely lost. I had no idea what was going on and I was, you know, in the college prep courses. My friends basically got me through the course and taught me what I learned; which was very little." In college, Sharon took as little mathematics as she could.

When Sharon began teaching, teachers were doing individualized instruction, all very structured. "That was the swing of the pendulum. And everybody had their little folder and you were working on this and that, and everybody was in a different place." Sharon's main teaching resource at that time was her textbook. "When I came in the first time, I had nothing basically to come in and start with, other than the textbook that they handed me."

In 1980 Sharon began her Master's program and later joined a study group. This group discussed different ideas for elementary school teaching.

> So then I started doing centers. So that was the last big wave of centers, and then we are back to it again, but, uhm... So I started doing some stuff, but it wasn't as much hands on, but more theme oriented math, I guess. You know, where you tied the math into what you were studying. ... [We did] fun math worksheets even at that time. I mean, it gradually progressed, but you know, at that point in time it was like we would do all things that tied into March, or wind, or lions, you know. So all your math tied into some theme or whatever you were doing in that center. Like there would be a little pattern, or a puzzle at the end that answered the riddle about lions or something. Lions and lambs, it all ended up having some cutesy little puzzle at the end.

In moving from individualized instruction to working more in small groups, Sharon began moving away from the textbook and toward using more "activities." "Very rarely do we just use the book," Sharon says when describing her lessons now. She explains she uses her textbook mainly to organize which lessons she needs to group together. Once she begins to work on a specific unit, she uses the topics of the lessons listed in the textbook to order her teaching. However, for some topics, she uses substitute activities that she considers more meaningful. Sharon does not teach all the lessons that are in the book. Also, during the year, she does not follow the order in which chapters appear in the book. "I move them around," Sharon says.

As an example of activities that she uses, Sharon tells about "racing to one hundred."

> My kids have probably played racing to one hundred to learn place value for fifteen, twenty years, you know, where you roll the dice and you add the cubes, and then you do the trading to the tens and you trade until you get to one hundred. We have done that for twenty years, probably, but it still works. ... You know, they understand why you have to do the trading so, if it works, I keep it.

Talking about "keepers" (good activities) Sharon says she doesn't have anything left from when she first started teaching, or from the "cute worksheets" she had.

> I kind of dumped off all of the worksheets that were outdated, from when I was in my center, do this worksheet that ties into the month or whatever. So no, I probably don't have stuff that I keep from the very beginning. That racing to one hundred is probably the thing that I have used the longest out of all the games and things that I have used.

Sharon explains that her definition of what constitutes good mathematics teaching has changed over the years. "I used to think that good math teaching was being able to explain it in a way that the students understood," Sharon says. Although good explanations are still important for Sharon, she now thinks they are not enough. "Different children learn in different ways," Sharon states. So Sharon is aware that some students will not get the ideas she is trying to present just from listening to her. They have to try it for themselves. Sharon particularly notices that children who have a hard time paying attention to her explanations are very good in solving problems and "discovering" mathematics. Also, when children are solving problems or "discovering" mathematics, Sharon has a better idea about whether or not students are on task, understanding what they are doing.

> Sometimes you think they are involved when you are doing problems together or, you know, if we are working something on the overhead and talking about why this works and why this doesn't work. But I am basically working the problems and they are just telling me what to do. You think they are involved and they are not. ... So then, with them trying to discover, it is much more obvious if they are not doing anything, that they are not with you, so you can see.

Sharon attributes the initial changes she experienced in her mathematics teaching to colleagues and their collective search for new ideas. "The group of teachers who were teaching third, some of those older people retired, and some other people came in and together we said, this is boring, lets try something different. And it has gradually progressed." Still from that time in the early to mid 1980s, Sharon tells she remembers when her school group became a more social group, and the teachers began to plan together. "So, at some point, like about half way through that 10 or 11 years I was at that school, it changed. It became more of a, what are you teaching, what are you doing, that kind of thing."

Sharon enjoyed the interaction and the collective work with her colleagues—a practice she has cultivated for the past 20 years. Sharing with colleagues and working in a grade-level group is an ongoing theme in Sharon's story. At her current school, she says:

> Probably this hall more than any other hall in the school plans together. I mean, everybody knows we plan together. We get together once a week and sit down and do lesson plans and we know what we are teaching, everybody brings what they have. ... In fact, we only turn in 1 set of lesson plan for our hall [grade level].

Some colleagues have particularly influenced Sharon's mathematics teaching. One of the teachers Sharon met at her previous school was particularly important in her development as a mathematics teacher. They were friends at that school and they came together to the school where Sharon is now when it first opened. For 8 years they taught together at third grade in this school, before Sharon's friend moved to another state.

> [She] was one of those people, when you came up on something and you just thought, they are not getting this, and I have taught it every way I know how, and you know. Do you have an idea? ... She was into math, that was her thing, what she really loved. Every now and then, when we were teaching something, she would say, now, this would be a good time to use that activity or whatever. She'd remind me to pull something out or just talk about what she had done that day. ... Some of her stuff went too far for me. ... I was more middle-of-the road and she was far out here. But it was far enough that it pulled me that way anyway.

Sharon's friend used to teach mathematics professional development courses and Sharon has taken many of those courses. In general, Sharon values the opportunities she's had to participate in staff development activities. "I guess staff development changed it more than anything," Sharon says. At her school, there are a lot of opportunities to attend these courses.

> We have staff development after school. We have staff development during school. We have special staff development days. We do staff development in the summer, I mean. And then you can go away to a staff development course. So, I mean, it is just constant, you know, and every time you go, you bring back a couple of things, you know. You always go and say, oh, I am going to do that, just completely, that was just wonderful. But then you come back and it is not realistic. So you come back and do a couple of things, and that becomes a constant. And the next one you go to, you add a couple more things that turn out to be something that you really keep and that becomes a constant.

Sharon believes staff development opportunities are more accessible now, at her current school, than they were in the past, in previous schools. On the one hand, she attributes these opportunities to the school system in which she teaches. "This is a much more progressive system," Sharon explains, "and we have always had mathematics consultants." On the other hand, Sharon thinks that, in general, there are more resources and opportunities for teachers nowadays. "There is a ton of stuff out there," Sharon says, referring to books, lesson plans on the web, materials one can purchase in teaching stores, and staff development events that show new materials or ideas and how to implement them.

For Sharon, the staff development classes she took provide the connection between her practice and educational research. Sharon claims she does not like to read research. She reads teachers magazines, but for the most part they are practitioners magazines that do not deal directly with research. However, Sharon says the mathematics consultant in her county and the people in charge of offering staff development courses should be in tune with "what is going on out there." Whereas educational researchers are looking for new ideas on how to better teach, mathematics consultants' job is to translate the new ideas into what relates to classroom teaching. Sharon explains that, as teachers, "we are not necessarily looking at the research. But we are looking at somebody else who has worked at the research." Through this mediated path, Sharon believes educational research reaches her classroom. She tries out the new proposed ideas and if they teach what they are supposed to, Sharon keeps them.

Discussion

Sharon's story is a story of search. Search for new ideas. Search for resources, for colleagues, for opportunities. Sharon is always looking for what can help her students learn, what can make them understand, what can make her teaching more meaningful to herself and the children in her classroom. As she searches, Sharon incorporates new activities and new approaches to her teaching in general, and to her mathematics teaching in particular. She learns from different sources and uses all she can to implement what she believes is good mathematics teaching. As Sharon looks back at her professional paths, she sees many changes.

Sacred mathematics education stories are incorporated into Sharon's teaching stories. She talks about the swing of the pendulum and about the different educational trends she has lived. As education swings, Sharon learns new ideas, incorporates new suggestions, and develops her teaching. Some of the trends become part of her own search process and she adds those ideas to her own story. In mathematics, Sharon talks about moving from teaching the book, to cute worksheets, to using classroom activities. She mentions having only the textbook as resource to having a plethora of classroom supplies. She says she went from individualized instruction to working more in groups. As her career comes into scrutiny, Sharon organizes it into different times when she used different teaching strategies and resources. Currently, her story incorporates some of NCTM's recommendations for school mathematics such as more hands-on activities and letting students work on problems by themselves to learn mathematics.

Staff development plays an important role in Sharon stories. From her perspective, research results and recommendations for change in mathematics teaching make their way into her classroom mediated by those who offer staff development. Staff development professionals present Sharon with new ideas, which she brings to her classroom. These ideas are incorporated in small doses but, over time, the incremental changes have transformed Sharon's teaching. Sharon sees the past 13 years as especially important in terms of her professional growth and her participation in staff development classes. These 13 years coincide with the release of NCTM's standards. They represent a time of increased calls for teachers to become more professional. They saw a strong push for teachers' ongoing development. Thus, Sharon's stories incorporate and present many of the features that have characterized mathematics teacher education in the past decade or so.

Another important venue for promoting changes in Sharon's teaching over the years has been sharing experiences with her colleagues. Opportunities for professional exchange among peers are part of Sharon's life. She values these opportunities as a way to bring about changes in her classroom. Sharing among colleagues works for Sharon as a continuous professional development environment in which she improves her teaching.

Trying to understand Sharon's teaching by understanding the trajectory of her mathematics practice and examining her teaching from a career perspective, it is quite striking to notice the many changes she perceives in her mathematics teaching. Sharon's career in no way resembles the image of teaching as a job in which changes seldom occur. It also does not present change as a difficult, hard process. Quite the contrary, Sharon's ongoing quest for new ideas, together with changes in working milieu and new educational trends, have led her onto a path of many revisions and alterations of her mathematics teaching practice. Sharon understands change in a way that is in tune with her everyday attempt to improve teaching. From her point of view, change is a natural part of teachers' everyday experiences; it is an obvious and necessary component of good mathematics teaching.

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THE IMPACT OF TEACHERS' CONTENT KNOWLEDGE AND ATTITUDES ON INSTRUCTIONAL BELIEFS AND PRACTICES

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This study investigated 407 in-service elementary teachers' level of mathematical content knowledge, attitude toward mathematics, beliefs about effective instruction, use of reform-orientated instruction and modeled the relationship among these variables. Upper elementary teachers (grades 3-5) were found to have greater content knowledge and more positive attitudes toward mathematics than primary teachers (grades K-2). There was no difference in teachers' beliefs about effective instruction, but primary level teachers were found to use reform-oriented instruction more frequently than upper elementary teachers. Overall, it was found that teachers with higher content knowledge were less likely to believe in the effectiveness of reform-oriented instruction and less likely to use such instruction in their classrooms. However, teachers with more positive attitudes towards mathematics were more likely to believe in the effectiveness of reform-oriented instruction and use it in their classroom. Findings from this study have implications for the goals and objectives of elementary mathematics methods courses.

Mathematical content knowledge (Ball, 1990a, 1990b, 1991) and positive attitudes toward mathematics (Quinn, 1997; Richardson, 1996) are both important components of being an effective teacher. Also important are teachers' beliefs about mathematics and mathematics instruction (Grossman, Wilson, & Shulman, 1989) which often impact instructional practices (e.g., Pajares, 1992; Richardson, 1996; Thompson, 1984). In order to better prepare teachers to use instructional practices that follow from the ideals of reform-oriented mathematics teaching and learning (National Council of Teachers of Mathematics [NCTM], 1989, 2000; Simon, 1995) it is important to understand how teachers' content knowledge and attitudes impact their beliefs, and ultimately impact their instructional practices. Thus, the purpose of this study is to: (1) investigate elementary teachers' level of mathematical content knowledge, attitude toward mathematics, beliefs about effective instruction, and use of reform-orientated instruction, and (2) model the relationship among these variables.

In particular, this study investigates the following questions: (1) Does the level of teachers' mathematical content knowledge and attitude toward mathematics differ for primary (K-2) versus upper elementary (3-5) teachers? (2) Do beliefs about effective teaching strategies differ for primary and upper elementary teachers? (3) Does the frequency of usage of reform-oriented instruction differ for primary and upper elementary teachers? (4) Are teacher beliefs about effective teaching strategies related to use of reform-oriented instruction in the classroom? (5) How does the level of teachers' content knowledge and attitude toward mathematics impact their beliefs about effective instruction and their use of reform-oriented instruction? Information gained from investigating these questions can provide useful information for teacher educators to better develop methods courses that address the many different components of effective teaching.

Methods

Sample

This study involved 407 K-5 in-service teachers from two school districts. These teachers were part of a professional development project focusing on the implementation of NCTM-based mathematics curricula.

Measures

At the beginning of the professional development project teachers' content knowledge, attitudes, beliefs, instructional practices, and other background characteristics were surveyed. Teachers' mathematical content knowledge was measured using a 44-item mathematics test (reliability of test: KR-20 = .88). Items were related to content that would most likely be found in grades K-8. The test was made up of 32 multiple-choice items and 4 open-ended items (multiple-part questions constituting the remaining 12 items). Teachers' attitude toward mathematics was measured using 14 items rated on a 6-point Likert scale (1=*strongly disagree* to 6=*strongly agree*; reliability of the scale based on Cronbach's a = .88). Teacher's were also asked to rate on a 4-point Likert scale (1=*not important* to 4=*very important*) their beliefs about the importance of 14 activity/techniques/strategies for effective mathematics instruction in the grades they teach. Through exploratory factor analysis, a subset of 10 of these items was determined to represent a single factor. Further inspection of the items found them to be consistent with the reform efforts outlined by the NCTM (1989; 2000). These items were used to create a measure of teachers' beliefs about effective instruction (Cronbach's a = .80). Teachers were also asked to rate the frequency of occurrence of 29 instructional activities in their mathematics lessons on a 5-point Likert scale (1=*never*, 2=*rarely*, 3=*sometimes*, 4=*often*, 5=*all or almost all mathematics lessons*). Again, through exploratory factor analysis, seventeen of these items were determined to form a single factor. Inspection of the items found that they aligned with instructional activities advocated by the NCTM (1989; 2000). These items were used to create a measure of teachers' use of reform-oriented instruction (Cronbach's a = .87). Measures of background characteristics of teachers were also included such as years of teaching experience (categorized into 7 ranges: 0-2 years; 3-5 years; 6-10 years; 11-15 years; 16-20 years; 21-25 years; 26 or more years), number of mathematics courses taken, and highest degree attained (Bachelor's vs. Graduate).

Procedures

Teachers were categorized as either primary (grades K-2) or upper elementary (grades 3-5) and compared using descriptive statistics related to their level of content knowledge, attitude toward mathematics, beliefs about effective instruction, and use of reform instruction. Based on the full sample of teachers, a series of path models was used to investigate and compare the relationship between teachers' level of mathematics content knowledge and attitudes toward mathematics and their belief about effective instruction and use of reform-oriented instruction while controlling for number of mathematics courses taken, years experience, and degree level.

Results

Means and standard deviations of all variables are reported in Table 1 for all 407 teachers as well as by grade level. An investigation of means found that upper elementary teachers have significantly higher levels of mathematical content knowledge and a more positive attitude toward mathematics than primary teachers. There was no difference between primary and upper elementary teachers beliefs' about effective instruction, but primary teachers reported more frequent use of reform-oriented instructional activities than upper elementary teachers. Considering teacher background character-

Table 1. Means and Standard Deviations for Variables

	All Teachers (N=407)	K-2 (N=259)	3-5 (N=148)
Content Knowledge (KR-20 = .88)	27.02 (8.18)	25.68 (7.89)***	29.37 (8.18)
Attitude (α=.88)	4.67 (0.77)	4.57 (0.75)**	4.83 (0.80)
Effective Instruction (α=.80)	3.32 (0.42)	3.34 (0.41)	3.30 (0.44)
Reform Instruction (α=.87)	3.17 (0.53)	3.27 (0.52)***	2.98 (0.51)
Years Experience	3.70 (1.96)	3.64 (1.97)	3.79 (1.96)
Math Courses	3.43 (1.45)	3.35 (1.39)	3.57 (1.54)
Highest Degree	0.39 (0.49)	0.35 (0.48)*	0.46 (0.50)

Note: *p<.05; ** p <.01; *** p <.001; Asterisks in the K-2 column represent significant differences between K-2 and 3-5 teachers.

istics, a higher percentage of upper elementary teachers reported having a graduate degree than primary teachers. There was no difference in years teaching experience or mathematics courses taken across the two groups.

Intercorrelations between variables are reported in Table 2 for all teachers. Overall, teachers' beliefs in effective instruction was found to have a moderate positive relationship with teachers' reported use of reform-oriented methods ($r = .42$). That is, teachers who believe that reform-oriented instruction is effective tend to use more reform-oriented instructional techniques. However, the magnitude of this relationship suggests that beliefs and actions do not always match.

In order to investigate the relationship between teachers' content knowledge, attitude, instructional beliefs, and instructional practices a series of path models was estimated. Initially, a saturated model, including all possible paths between the variables was estimated. Subsequent inspection of the path coefficients revealed that some effects were negligible suggesting that a more parsimonious model existed. After deleting these paths a final model was estimated. This model, with significant standardized path coefficients, is presented in Figure 1. The chi-square value for the model was not significant suggesting a good fit with the data, $c^2(9, N = 407) = 6.43$, $p = .696$. Other fit indices further support a good fit of the model with the data. The goodness-of-fit index (GFI) was .996; adjusted goodness-of-fit index (AGFI) = .986; comparative fit index (CFI) = 1.00. These indices can take on a value from 0 to 1 with values closer to 1 showing a better fit and values greater than .90 usually indicating a relatively good fit (Hoyle, 1995). The standardized root mean square residual (SRMR) was .02. This index represents the average residual between the observed and hypothesized correlation matrix. The root mean square error of approximation (RMSEA) was .00. This index takes into account the complexity of the model, that is, the number of

Table 2. Intercorrelations Between Variables for All Teachers (N = 407)

	1	2	3	4	5	6	7
1. Content Knowledge	--						
2. Attitude	.32	--					
3. Effective Instruction	-.11	.13	--				
4. Reform Instruction	-.18	.10	.42	--			
5. Years Experience	-.21	.05	.03	.03	--		
6. Math Courses	.05	.16	.02	.07	.13	--	
7. Highest Degree	.24	.06	-.06	-.07	.11	.02	--

parameters being estimated. These two indices can range from 0 to 1 with values less than .05 representing good fit (see Byrne, 2001). Theoretically, it is possible for the direction of the path between content knowledge and attitude to be in either direction (see Figure 1). However, based on an inspection of the residual covariance matrix, representing the difference between the estimated and observed covariances, it was decided that the model presented in Figure 1, with knowledge predicting attitude, better fit the data.

Results from the path analysis found a significant negative direct effect of teachers' content knowledge on their beliefs about effective instruction (b = -.17, $p < .001$). However, teachers' attitude toward mathematics was found to have a significant positive direct effect on beliefs about effective instruction (b = .19, $p < .001$). Similar results were found for the occurrence of reform-oriented instruction in the classroom. Teachers' content knowledge was found to have a significant negative direct effect on instructional practices in the classroom (b = -.18, $p < .001$) and teachers' attitude toward mathematics was found to have a positive direct effect on teachers' instructional practices in the classroom (b = .10, $p < .05$). Teachers' content knowledge was found to have a positive direct effect on teachers' attitude toward mathematics (b = .34, $p < .001$). Teachers' beliefs about effective instruction was also found to have a significant positive direct effect on instructional practices (b = .39, $p < .001$).

Indirect effects show a relationship between two variables that is mediated by one or more other variables. For example, there was a significant positive indirect effect of content knowledge on teachers' beliefs that was mediated by teacher attitude (b =

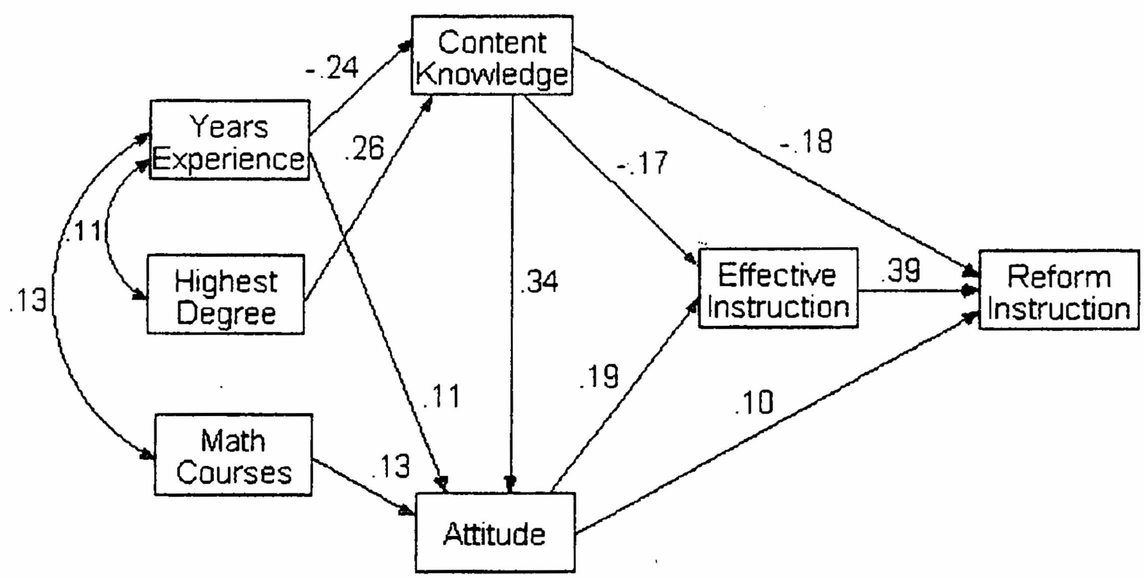

Figure 1. Final path model. All path coefficients are standardized and statistically significant at $p < .05$. Fit indices for the model: $\chi^2(9, N = 407) = 6.43$, $p = .696$, GFI=.996, AGFI=.986, CFI=1.00, RMSEA=.00, SRMR=.02.

.06, $p < .05$, calculated by multiplying intermediary effects, i.e., .34 x .19 = .06). This indirect effect is represented by the path that leads from content knowledge through attitude to effective instruction (see Figure 1). However, the total effect of content knowledge on beliefs, found by summing the direct effects and indirect effects, was still negative (b = -.11, $p < .05$). There was also a significant indirect effect of attitude on instruction which was mediated by beliefs (b = .07, $p < .01$) resulting in an overall greater total effect (b = .18, $p < .01$).

Considering teacher background characteristics, there was a significant negative direct effect of years experience on content knowledge (b = -.24, $p < .001$) and years experience was found to have a positive direct effect on attitude toward mathematics (b = .11, $p < .05$). There was a positive direct effect of degree earned on content knowledge (b = .26, $p < .001$). The number of mathematics courses taken was found to have a positive direct effect on attitude toward mathematics (b = .13, $p < .01$). Teachers' background characteristics were not found to have any direct effects on either beliefs about effective instruction or instructional practices. However, the impact of these variables was mediated indirectly by both teachers' content knowledge and attitude. For example, mathematics courses taken was found to have a positive indirect effect on both instructional beliefs (b = .02, $p < .01$) and practices (b = .02, $p < .01$). Years of teaching experience was also found to have a positive indirect effect on both instructional beliefs (b = .05, $p < .01$) and practices (b = .06, $p < .01$). However, the indirect effect of degree earned was negative for both instructional beliefs (b = -.03, $p < .05$) and practices (b = -.05, $p < .01$). Because there were no direct effects for these variables the indirect effects represent the total effects.

Conclusions

Results from this study suggest that primary teachers (K-2) and upper elementary (3-5) teachers differ in their level of mathematical content knowledge and attitude toward mathematics. Upper elementary teachers were found to have higher levels of content knowledge and a more positive attitude toward mathematics than primary-level teachers. However, primary teachers reported using reform-oriented methods of instruction more frequently than upper elementary teachers. Beliefs in the use of effective instruction were not found to differ for primary and upper elementary teachers. Overall, teachers' beliefs in what constitutes effective mathematics instruction was found to be positively related to more frequent use of reform-oriented instruction. However, further investigation of this relationship suggests that teachers' actions do not always align with their beliefs as many teachers who professed the importance of reform-oriented instruction did not report frequent use of such instruction in their classroom (cf. Cooney, 1985).

After controlling for teachers' years experience, mathematics courses taken, and highest degree earned, teachers' content knowledge was found to have a negative relationship with teachers' beliefs and use of reform-oriented instruction. Teachers' attitude toward mathematics was found to have a positive relationship with teachers' beliefs about effective instruction and ultimate use of reform-oriented instruction. Perhaps teachers who have higher content knowledge feel that since they were successful as a result of traditional instruction that such methods are adequate for their students. In other words, they tend to teach as they were taught. Although teachers' content knowledge is important in helping students to learn and understand mathematics, findings from this study suggest that higher levels of content knowledge alone may not transfer into instructional practices that help promote students' mathematical power beyond basic content knowledge. Teachers who have a more positive attitude toward mathematics may be more able to transfer positive beliefs about mathematics by incorporating what Bishop (1991, 2000) referred to as a mathematically enculturating curriculum that not only promotes students' content knowledge but also instills in them the ideas of mathematics as inquiry-based, as a way of thinking, and as an important part of everyday life. Thus, the importance of teacher education programs goes beyond the development of content knowledge and pedagogical content knowledge to also help teachers develop a positive attitude toward mathematics (e.g., Quinn, 1997) which based on this study may be more likely to transfer into positive instructional beliefs and practices.

The design of this study and the variables used limit the causal links that can be made between the predictor variables and teachers' instructional beliefs and practices. It is quite possible that there are other confounding variables that were not included in this study that may further explain the relationships that were found. For example, perhaps teachers are influenced by the beliefs and policies of their school or school principal. In other words, teachers who are in schools in which reform-oriented instruction is not supported may be less likely to use it even if they believe that it is more effective. Also, consistent with the findings of this study, past research suggests that upper elementary teachers are less willing to use reform-oriented strategies with their students than teachers of students in younger grades (e.g., see Hatfield, 1994 and Gilbert and Bush, 1988, for research on teachers' use of manipulatives). However, this may be due more to the fact that they teach the upper grades than that they have a higher level of content knowledge. In order to take this possibility into account it would be necessary to investigate the relationship between content knowledge and instructional practices across different grades levels. Future models of the relationship between teachers' content knowledge, attitudes toward mathematics and their instructional beliefs and practices would be strengthened by including variables to control for school administrative climate and grade level of teachers. However, even with these limitations, the findings from this study offer interesting insights into the relationship between these variables and provides teacher educators with additional evidence of the importance of enhancing teachers' attitudes toward mathematics as a possible way of encouraging the use of more reform-oriented instruction in the classroom.

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COMMUNITIES OF PRACTICE: THE POSSIBILITIES AND LIMITATIONS FOR BEGINNING URBAN MATH TEACHERS

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This paper draws on sociocultural theory to explore both the possibilities and limitations of deliberately establishing a community of practice for first year elementary math teachers. The paper describes the findings of a study of one such group of teachers who graduated together from a teacher education program that specifically aimed to prepare teachers for urban settings. The goal in establishing the group was to maintain the learning trajectory begun in a reform oriented math methods course in order to help these teachers resist simply being enculturated into existing practices. The focus of this paper is the ways in which limiting participation to first year teachers influenced the outcomes.

Introduction

The beginning of the most recent wave of math reform was marked in 1989 by the National Council of Teachers of Mathematics' publication of *Curriculum and Evaluation Standards for School Mathematics*. Inspired by a concern about students' inability to operate in an information society, this document suggested a reevaluation of the way we think about mathematics curriculum. Although it has been more than ten years since these documents calling for reform were published, the norms in most mathematics classrooms remain unchanged. School cultures are powerfully resistant to any kind of reform. Because teachers and their practices are at the heart of establishing these norms, teacher change is critical to achieve reform. Indeed, the persistence of traditional practices in America's classrooms is frequently attributed to many teachers' resistance to change. Accomplishing significant changes places demands on teachers to change the way they think about mathematics, teaching, and learning. Teacher change, however, is not a straightforward process. Teachers do not approach the teaching of mathematics with a clean slate. They are products of the very system they are expected to reform (Ball, 1996). They bring to their teaching previous experiences, both positive and negative. Most importantly, many of these teachers were not taught math in the way reformers are asking them to teach it now. Teachers come armed with their own beliefs in place about mathematics and the teaching of math. These beliefs are often in conflict with those espoused by math reformers, and they are usually difficult to change (Ball, 1990; Ebby, 2000).

Theoretical Framework

Given this connection between teacher change and the establishment of new classroom norms, approaching the reform of mathematics education for students must begin with an understanding of how teachers themselves learn about mathematics, teaching, and learning. This study draws on sociocultural theory to begin to develop such an understanding. According to theorists such as Lave and Wenger (1991) learning occurs through increasing participation in the work practices of a group of people who are guided by a shared sense of value and meaning. Newcomers begin participation on the periphery, and over time 'move toward full participation in the sociocultural practices of the community" (p. 29). In the view of the authors, community of practice is defined as a group with shared understandings about what they are doing and why, as well as what it means for the community. Here, learning is both defined as and measured by increasing participation in the community's practices, both in terms of breadth and depth of activity (Lave & Wenger, 1991). The relationship between the community and learning is not a unidirectional one. Lave and Wenger (1991, p. 35) write, "learning is not merely situated in practice – as if it were some independently reifiable process that just happened to be located somewhere; learning is an integral part of the generative social practice in the lived-in world." Thus, the relationship between learning and participation is a dialectical one.

The sociocultural perspective on learning also sheds light on challenges to accomplishing the goals of the mathematics education reform. As communities of practice, schools are likely to prevent changes in practices by enculturating participants, including students, teachers and administrators into traditional ways of thinking about school (Putnam & Borko, 1997). If teachers are to change their thinking and practices, new professional communities of practice need to be created that emphasize active learning, inquiry, and critical reflection. Research on the reproductive nature of school cultures raises a number of questions about how such professional communities can be fostered within or outside of a school. This paper explores whether a community established outside of school settings can interrupt or counter the effects of traditional communities within schools. Stein and Brown (1997, p. 156) claim, "Teacher change is not viewed as happening one teacher at a time, but rather through a process that takes advantage of the synergy, support, and motivation supplied when a "critical mass" of teachers undertakes reform for all students in a given school." Can the same level of synergy and support develop amongst teachers from differing school, even when they do not represent a critical mass of teachers from one school?

Methods

Data collection for this study focused on the work of an inquiry group for first year teachers. The group met once every three to four weeks, and focused on math related topics of the participants' choosing. This group served as a source for learning about the experiences of first year urban teachers, and the ways in which those experiences were shaped by participation in a community of practice outside of school. The four participants were volunteers who graduated from the same urban-focused certification program in a large northeastern city. Data collection focused on the ways in which each teacher negotiated conflicts between the various communities in which she participated. Sources of data included the videotaped meetings, individual interviews, classroom observations, reflective journals, and documents that shed light on the norms and practices of the each teacher's school setting.

Results and Discussion

The composition of the group created an interesting learning dynamic. Because the participants were all beginning teachers, the discourse was often focused on sharing mutual frustrations. In spite of the researcher's efforts to maintain an emphasis on learning about math education, the discourse most often evolved into expressions of frustration with working conditions. Advice centered on dealing with parents, bureaucracy, discipline, time management and exhaustion. As one participant said in a mid-year interview, "in some ways it's like the blind leading the blind." However, in an interview at the end of the year this same participant revised her earlier view, indicating that she felt "less threatened" sharing her needs honestly with this group. She suggested that new teachers were less likely to judge her than her experienced colleagues, and that she was more comfortable asking for help.

These findings suggest that the needs of beginning teachers with respect to support in their first year may be very different from the interests of professional developers. While those working towards math reform have a vested interest in seeing new teachers continue to focus on developing their practices in the direction of reform, these teachers may have a greater need for support in simply "surviving" the first year of teaching (Huberman, 1994). In interviews before the school year began, participants indicated that they hoped to get "moral support" from the group. Further, at the conclusion of the study all four cited "support" as the primary benefit of participation. However, in an informal conversation with the participants, each indicated that they had plans to spend a significant amount of time over the summer developing a math curriculum that emphasized what they most valued. The characteristics they mentioned included problem solving, discussion, collaboration, and journaling. This suggests that willingness and an ability to focus on math education may be more present once teachers have overcome some of the other concerns confronting them as novices. This also insinuates that participation in a community that shared the values of math reformers can indeed support teachers in their efforts to resist enculturation into practices that conflict with their own ideals.

The findings also underline the importance of what Lave and Wenger (1991) call "oldtimers." Though the researcher's initial intent was to act only as an observer of the development of this community of beginning teachers, she found herself increasingly called upon throughout the study to offer advice, share resources, and provide insights into the experiences of the participants. While limiting membership to novices had advantages that should not be disregarded, a need did exist for the "wisdom of experience."

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A CALL FOR TEACHER LEARNING GOALS IN PROFESSIONAL DEVELOPMENT

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The purpose of this review is to draw lessons from the research on learning environments for students to aid in thinking about learning environments for teachers. The classroom is a learning environment for students just as professional development activities are learning environments for teachers. Research on effective teaching is guided by the learning goals for *students*, and how *these* goals can best be achieved. It follows that research on effective professional development should be guided by the learning goals for *teachers*, and how *these* goals can best be achieved.

The National Council of Teachers of Mathematics (NCTM) has established detailed academic standards to guide K-12 learning goals for students. However, it is unlikely that many K-12 teachers have learned to adjust their teaching with these new learning goals and Standards as the product. Teachers are expected to teach more content in a deep and meaningful manner without sufficient support (Lampert & Ball, 1999). To compound the situation once teachers are in the field, the U.S. education system has no proven mechanism to systematically improve teaching in our classrooms (Stigler & Hiebert, 1999). Research in the area of professional development for mathematics teachers must become more rigorous and attend to the specific features that benefit teachers' development. Yet, the research will not become more systematic until learning goals for teachers and teacher development programs are more clearly defined.

Theoretical Framework

Learning Environments

Students and teachers as learners present different cognitive profiles, but their learning environments are similar in a number of ways. A *deep understanding* of the mathematical content is a goal for both students and teachers as learners. Both learning environments should also be built upon the *prior knowledge* the learner brings to the task or learning situation. *Collaboration* and group tasks have become significant components of students' classroom learning as well as teachers' professional development. Finally, *feedback* and verbal assessment are components of both student and teacher learning environments that might guide the learners' mathematical trajectories toward the learning goals. These similarities suggest that research on student learning and classroom teaching might inform research on teacher learning.

Three Studies with Increasingly Specific Learning Goals

It is helpful to review mathematics educational research in the *students'* learning environment. The reviews of the three studies that follow show different levels of specificity with which classroom learning environments have been investigated. These three studies provide empirically supported guidelines for specific classroom components: (a) the use of classroom time, (b) behaviors of effective teachers, and (c) the order of mathematical task presentation. What is of interest is that the more specific the learning goals of each study and the more specific the components of the learning environment investigated, the more specific the claims that are made about the nature of an effective learning environment (See Table 1).

Table 1. Comparison Of Three Studies

Study	Learning Goals	Mechanism Studied
Zahn (1966)	General - Standardized Assessment	General - Class Time
Good, Grouws, & Ebmeier (1983)	General - Standardized Assessment	Specific - Teachers' Behaviors
Wearne & Hiebert (1988)	Specific - Assessment Self-Designed	Specific -Task Presentation

Zahn (1966) examined the impact of separating class time into various combinations of "developmental activities" and "practice work" (e.g., 67% developmental, 33% practice) with student learning measured by performance on a three-part posttest. Zahn's data support developmental activities, but the *nature* of the developmental activities that support learning are not revealed and cannot be reconstructed from the data. The Good, Grouws, & Ebmeier (1983) studies looked for relationships between the teacher's use of classroom time and student achievement. Because Good et al. (1983) described the classroom components investigated more explicitly and in more detail than Zahn (1966), they drew more specific conclusions about the features of the classroom practice that affected students' learning. However, because they measured students' learning with a general test, they were not able to make claims about the *nature* of learning influenced by the features. In the final study reviewed, Wearne & Hiebert (1988) designed a program to develop students' conceptual understandings of decimals. Specific assessment tasks were designed to align with the learning goals of the experimental teaching program. Unlike the previously mentioned studies, these researchers designed an assessment closely aligned with the explicit learning goals to evaluate the effects of the experimental mechanism.

Lessons to be learned from these studies follow. (1) When students' *learning goals* are clear and specific, and the feature of the learning environment is described in detail, it is easier for investigators to analyze the specific features of the learning environment that contribute to the achievement of the goals. The more specific the learning goals and features investigated, the more specific the research claims about how to achieve the desired learning goals. (2) The *research design* employed to investigate the effectiveness of a learning environment for students or for teachers *must* include explicit goals for the existence of the learning environment. For professional development, how can researchers investigate a program's progress if it is not designed around clear learning goals?

Recently, the field in mathematics education has expanded to include an influx of research on teacher development, teacher change, and professional education. Professional development is a growing area of research in mathematics education yet, the field as a whole remains unfocused. Empirical data have not systematically accumulated, so it is difficult to confirm which features of professional development programs are *most* critical for teachers' ongoing learning. Why does professional development research lag behind its counterpart? One obvious reason is the natural developmental lag between studying students' learning, then teaching, and then learning to teach. However, another reason emerges through examining much of the work on professional development. The literature has not yet provided a set of well-defined learning goals for teachers in professional development in the same way that it has provided learning goals for K-12 students of mathematics.

Learning Goals for Teachers

The spirit of the learning goals for teachers suggested in this paper, parallel the learning goals or NCTM Standards for students, with a focus on depth of understanding rather than discrete behaviors. The literature provides some empirical support and a good deal of conjecture about the characteristics of an effective teacher. Why do these characteristics rarely serve as a basis to develop cognitively based learning goals for teachers?

There are a number of possible explanations why teachers' learning goals have not been made explicit. (1) The current measures of a successful professional program are usually not based directly on teachers' learning (CTGV, 1997). (2) Teaching, for the most part, has remained a hidden profession with teachers working behind closed classroom doors in isolation making it difficult to assess teachers' desired support, their learning needs, and their performance as learners (Shifter & Fosnot, 1993). (3) Professional development has traditionally treated teachers as technicians and not as learners and practical researchers (Richardson & Placier, 2001). (4) The social norms and expectations in a professional learning environment are frequently not supportive of teacher inquiry, learning, and collaboration (Clark & Florio-Ruane, 2001). These issues will probably continue to perpetuate the non-specification of learning goals for teachers until teaching is viewed as a profession that requires continuous learning, careful self and peer evaluation, and perpetual goal setting.

Summary and Implications

There is widespread agreement in the mathematics education community about the goals for students' learning. There is also agreement about the characteristics of an effective learning environment for students and the characteristics of an effective teacher, but learning goals for teachers remain ambiguous. Until teachers' learning goals are made explicit, empirical research on the methods to support teacher development will not accumulate.

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VOICES OF NEW MATHEMATICS TEACHERS: IMPLICATIONS FOR STAFF DEVELOPMENT

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With teacher shortages nationwide it is critical that new motivated teachers are encouraged and supported in their profession. New teachers with less than seven years of experience participating in a middle school mathematics reform initiative were interviewed about their reactions to their professional development support. Overall these teachers were disappointed with interactions in their departments and with their mentors and found support in their interactions with other new teachers across districts. They also expressed a desire for the opportunity to observe their peers.

Huberman (1995) describes new teachers as in a "survival and discovery" phase of their career. As evidenced by teachers' voices reported in this paper, the word "survival" describes not only new teachers' feelings about their teaching and classroom environment but also characterizes their view of professional relationships and support within their school environments.

Middle school mathematics teachers participating in a Local Systemic Change Initiative funded by the National Science Foundation with less than seven years of teaching experience were interviewed to explore the challenges new teachers encounter. This paper focuses on the new teachers' report about their experiences in professional relationships and development activities. The teachers' voices organize this paper into four areas: (1) mentoring, (2) collaboration, (3) content knowledge, and (4) classroom observations. A pseudonym is used for each teacher and all interviews were conducted in 2001.

Mentoring

All the new teachers interviewed for this study had been assigned a mentor who also taught mathematics. Eileen is in her second year of teaching, just finished undergraduate studies, and is in her early twenties. She reported, "my mentor at my school is, she's older, she's extremely negative. Very hard to talk to. . . (it is) frustrating not to have someone to be able to talk to". Eileen thought barriers in communication might be due to her positive outlook. "I don't know if they think I'm too optimistic. I'm not sure . . . They're so negative. . . I know the kids can do it. I've seen them do it . . . .but (my mentor) is just the queen of depression". Mark is also a new mathematics teacher, but has moved to teaching from another career and is in his late thirties. He too communicated a discomfort interacting with his experienced peers. "Conversations are usually dominated by the ones that have been there the longest . . . I thought somebody would ask me my opinion. . . Very many times I felt left out . . . I didn't want (them) to feel like I was questioning the way certain issues or policies were being implemented".

Although Eileen and Mark recalled negative experiences with their mentors and more experienced teachers, they reported benefits from collaborating with other inexperienced teachers, detailed in the next section.

Collaboration

Two new teachers interviewed for this study met other new teachers at cross-district professional development meetings. They built relationships within their peer group across districts to fill the void of in-school collaboration. Eileen enthusiastically shared the following about her young peer in another district. "We bounce ideas off of each other a lot. . . The fact that she's my own age, she's doing the same thing. We're both new, both new teachers. We have so much in common that I go crazy if we don't talk to each other". Lisa, in her second year also established her own support system. She works closely with a new teacher in a different building. "I talk to her more about what's going on in the classroom than anybody else".

None of the inexperienced teachers interviewed reported positive collaborative experiences in their departments. Patsy recalled,

> "we sit around and talk about the (standardized test) scores. Okay, this is what we need to work on. But we never really address how we want to do it. Nothing is ever really hashed out. Your know, it's brought up, and then it's kind of put away . . . I don't think we've ever had a discussion about how to teach a math concept ever . . .it's more like paperwork and numbers".

Eileen added that her department meetings were "more like, housekeeping issues. We need this, we need that". David has been teaching for six years. He spoke specifically about other teachers' attitudes at department meetings. "There are some teachers (that) . . . are just biding their time until they retire. Others are just . . . want to just get it done". Any meaningful conversations about teaching and learning, he noted have been with peers.

The typical teacher isolation that dominates school culture (Ball & Cohen, 1999) does not support new teachers' desire for communication. Lisa forcefully expressed her feelings of isolation and frustration with the other department teachers' apathy and lack of collaboration. "The same way it's got to be told to the undergrads that it's not acceptable for you not to like math, it's got to be said to teachers that it's not acceptable for you to not be willing to talk about what's going on in your classroom". Each of the new teachers interviewed, communicated a need for meaningful collaboration to share ideas about teaching and learning which was not currently supported in their departments.

Content Knowledge

In addition to their craving for meaningful collaboration, the new teachers interviewed claimed they are equipped with a stronger mathematics content knowledge than their more experienced peers. During professional development activities when grouped with teachers with weaker content backgrounds those interviewed were disappointed. Eileen shared, "I know I could have gotten a lot more out of it, if it had just been math people, like, just math nerds, ready to talk about math and how to help the kids. There was a lot more time spent . . . catching up the non-math people". Patsy concurred reporting, "We were far ahead of the group, as far as reasoning, figuring out the children's reasoning, I mean, we were sitting there, bored to death".

Most of the new teachers interviewed reported (1) disappointment with the mathematics content reviewed during professional development activities and (2) surprise at their more experienced peers' lack of mathematical understanding. They added that they would have preferred to focus on students' learning, methods of presentation, and the new curriculum rather than the content itself.

Classroom Observations

The new teachers looking for suggestions and ideas from other teachers reported that classroom observations would help them learn about effective teaching methods and might encourage in-school collaboration. Mark suggested that in addition to the administrator observations which he described as only a "snapshot in time," he would like "for somebody to come in who's a peer. You know. 'Oh, I think we do that or we can do this.' Somebody who's not in that pressure cooker". Sally, has been teaching for five years and also shared a desire to observe other teachers. "I'd like there to be some sort of program where we can go and see other people doing what they do. . . I don't even have the opportunity, really, to see other people in my own building, much less at another building. That, I think would be so helpful". David, in his sixth year of teaching suggested that observations by teachers across disciplines might also be helpful. "They'd come into the lesson with the perspective of more on line with the students than of somebody with a math background it might be more in line with what the students might be thinking".

In a separate study (Cwikla, 2002) Joanne, a very experienced teacher, who served as an assistant principal, a principal, and then chose to return to the classroom described the powerful impact of observing other teachers in her capacity as an administrator. "I was not a good math teacher and I learned how to be a good teacher by all the observations. That's what made me better. Was watching so many *successes.* Watching so many *failures*." Joanne had the advantage of serving as an administrator as well as a classroom teacher and supported the voices of the inexperienced teachers interviewed for this study who *all* reported a hope to see other teachers in action and how they operate in their classrooms.

Implications

Staff developers might better support incoming teachers by using these new teachers' views to help structure professional learning environments specifically for new teachers. One possibility suggested in the teachers' interviews is the development of peer groups of new teachers across schools and districts, providing a caucus for new teachers. Staff developers might also take advantage of these new teachers to aid in helping and teaching those with a weaker content knowledge. Finally observing other teachers through video presentations or classroom visits is an activity this group of new teachers thought might be helpful for improving their teaching. New teachers should be encouraged to practice the research supported reform methods learned at university and feel they are encouraged to perpetuate educational reform.

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A TEACHER-RESEARCHER FOR THE 21ST CENTURY

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One of the central problems in front of the educational profession is the integration of educational research and teaching practice into a reliable, trustworthy and dynamically developing body of knowledge, which can address successfully the present challenges brought by the requirements of the reform in mathematics education. The urgent calls to address this problem come from several different components of the educational enterprise: a mathematics teacher from NYCity, Mark Saul asserts that till now the education research has had very little to offer the classroom teacher flatly (Saul, 1995), while math education researcher Jeppe Scott (2000) observes that the "Present development in mathematics education provide the teacher with little assistance as far as recommendations for teaching methodology is concerned and leaves him/her in an ironic situation of classroom autonomy". Finally Martin Simon , warns that "although constructivism has the potential to inform changes in mathematics teaching, it offers no particular vision as to how mathematics should be taught" (Simon, 1993). The teaching practice/research integration contains two related components:

(1) from the "pure" research point of view, it involves the change of the site of the research from laboratory to classroom (Brown, 1992).

(2) from the teacher point of view, it involves the integration of her or his intuitive professional knowledge of teaching with the educational knowledge base in the service of bettering students' learning.

Whereas the first component is being dealt at present by the emerging "Design Research" movement (Collins, 1999; Lesh & Kelly 2002), here we will formulate the basic issues confronting the realization of the second component, the new integrated profile of the teacher-researcher. In fact, King and Lonnquist (1992) inform that action research (teaching research, classroom research) has been unable till now, to make a clear contact between the particularity of classroom research and the general educational knowledge base of the profession. For Noffke (1994), one of the present challenges ahead of Action Research is to clarify precisely if and how, action research can contribute to the traditional knowledge base of the profession. The absence of clear-cut, unambiguous answers to this fundamental question has seriously limited the role of teaching - research as a viable research tool.

In Mathematics Education, this limitation can be now eliminated both conceptually and practically by the introduction of the notion of constructive or cognitive theory of learning into professional discourse. Since the time such theories, which apply across wide range of studied phenomena (e.g., process-object duality theories (Asiala, 1996; Sfard, 1994; Tall, 2000)), Anderson's cognitive psychology, (Anderson, 1995) or Vygotsky theory (Vygotsky, 1986), appeared in the context of mathematics education, the bridge between the particular classroom situation and educational base knowledge can be much easier established. *As soon as a teacher in the classroom can coordinate her or his teaching-research methodology with one of the theories proven useful and general in the profession, at that moment he or she leaves the particularity of the classroom situation and can view the same classroom as one of the many instances where the particular theory is known to work.* At this moment, the whole conceptual apparatus and methodology of the particular theory can be applied to bear upon the particular classroom teaching-research situation. Thus the general character and trustworthiness of the classroom teaching-research becomes grounded in the generality and trustworthiness of a particular theory of learning with which the classroom situation was coordinated. What's more, the very process of coordination between a theory and practice, one of the modes of reflective abstraction, becomes the central skill required from the teacher-researcher.

The Professional Development of Teacher-Researcher, therefore, must be focused on the acquisition of that skill. Taking into account the essential qualities of Action Research such as improvement of classroom practice in the context of instruction/analysis cycles (King & Lonnquist, 1993), this professional development has to be organized through cyclical methodology around common teaching-research problems and difficulties present in the classrooms. On the other hand, in order to provide a possibility of theory-practice coordination, it has to be a project, a teaching experiment, which is designed on a basis of a general theory of learning (and teaching, possibly), it needs to be designed to provide converging as well as generative results. That means that the research questions of the teaching experiment need to be formulated so as provide an opportunity for both types of results to afford both directions of theory - teaching practice coordination. Finally, in order to provide a sufficiently large sample of students, sufficient number of different classrooms and their teachers need to participate.

Thus the Professional Development of the Teacher-Researcher takes the form of a large-scale cyclical teaching experiment designed on the basis of a general cognitive or constructivist theory to address and to investigate some class of common students' difficulties in the participating classrooms. The goal of such a Professional Development is to transform the participating teachers into highly qualified teacher-researchers, that is individuals

1. whose classrooms are scientific laboratories, the overriding priority of which is to understand students' mathematical development in order to utilize it for the betterment of the particular teaching and learning process;
2. who as teachers can have the full intellectual access to the newest theoretical and practical advances in the educational field, know how to apply, utilize

and assess them in the classroom with the purpose of improving the level of students understanding and mastery of the subject;

3. who as a researchers have a direct view of, and the contact with, the raw material of the process of learning and development in the classroom, act as a researcher in the context of the daily work of and use that process to derive new hypotheses and general theories.

Conclusions

We see that one simple methodological condition of coordination between research and teaching on the part of classroom teacher fully determines the design and the variables of appropriate large scale teaching experiments. A large-scale experiment in the context o teaching-research methodology means an experiment performed in several different classrooms which serve as different sites of the experiment and the conduct of teaching-research in each conforms to the particularization of the general research questions. It can also be shown that derived here structure of the large scale experiment is a careful composition of the principles of Action Research with the principles of Teaching Experiment of Vygotsky. From Action Research methodology which relies on the individual improvement of classroom teaching by the individual teacher, we employed the cyclic nature of the classroom experimentation and the choice of the focus on the common students' challenges in the participating classrooms. From the Vygotskian Teaching Experiment we took the notion of a large-scale classroom experiment - macro-scheme, according to (Meshchinskaya, 1967), which, in that methodology was performed "to study mental changes under the effect of instruction".

The comparison of the model with other relevant methodologies attempting to bridge the gap between research and teaching practice will be discussed during the presentation together with several examples of such a design.

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THE CHALLENGES IN CONFRONTING TEACHER CHANGE

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The purpose of this study was to document factors that influenced a secondary mathematics teacher to teach in a manner supportive of the mathematics reform movement. The research was designed as a case study with the aim to understand one teacher's classroom practices and factors that contributed to her success or failure in implementing reform based practice in classes she taught. Findings illustrate block scheduling was an instrumental factor to her success in creating an inquiry based instructional practice. Questions concerning the influence of the textbook on one's teaching practices were raised.

The *Professional Standards for Teaching Mathematics* (NCTM, 1991) proposes that teachers must be more proficient in: selecting mathematical tasks to engage students' interests and intellect, orchestrating classroom discourse in ways that promote the growth of mathematical ideas, helping students seek connections to previous and developing knowledge, and guiding individual, small group, and whole-class work (p. 1). The *Principles and Standards for School Mathematics* (NCTM, 2000) supports these teacher roles. Although these expectations are sensible, how teachers gain expertise in these areas is not always well defined. Moreover, these visions are in complete contrast to what is common practice among a large amount of the teaching population (Manouchehri & Goodman, 2000). Therefore, the current dilemma in the mathematics teacher education community is conceptualizing ways in which teachers may be motivated to comply with these new ways of teaching. In particular, there is a need to understand how teacher change occurs and factors that might motivate Standards-based instructional practices. This study was designed to address these issues.

Background and Methodology

Clarke (1997) provided a basis of the framework used to guide this study and is tied to other valuable research in the area of teacher change and practices (Cobb, Wood, & Yackel, 1990; Fennema, Carpenter, & Franke, 1992; Stephens & Romberg, 1985). His framework possessed two key categories for the role of the classroom teacher—what the teacher does and her related beliefs about the teaching and learning of mathematics. Both of these areas were critical in trying to understand the teacher I studied.

The initial study began with a sample of five secondary mathematics teachers, who were students in a graduate-level course where a component of the course was focused on the mathematics reform movement and how teachers viewed the movement "fitting into" their own classroom practices. These teachers taught in four different counties in the Appalachia region of the United States and content ranged from general mathematics to geometry. After initial interviews and visits to their respective schools, it was concluded that only one teacher had truly embraced mathematics reform teaching in her classroom instruction. This teacher, Ruth, is the case that is presented.

Case studies offer a method of investigating complex areas, composed of numerous parts, necessary in understanding the whole phenomenon being studied (Merriam, 1998). The case of Ruth deals with a teacher in a rural area where the community at large does not believe education beyond high school is necessary. The high school she teaches at has approximately 350 students and operates on a block schedule. Her geometry classes became the focus of this research since she was using a somewhat nontraditional textbook, *Discovering Geometry* (Serra, 1997), as well as nontraditional teaching methods.

Data, collected over a period of one and a half years, included in-depth, open-ended interviews, observations and videotapes of classroom instruction, and written documents such as class handouts, laboratories, homework, and tests. State test scores, used to measure student learning, were also reviewed at the conclusion of the school year. Analysis of data was an ongoing process throughout the course of the study. Data was coded to search for recurring themes, which in turn were used to support or make sense of the factors involved in teacher change.

Analysis and Findings

Analysis of data was a constant process—searching for patterns with which it could be grouped together. At times there was anywhere from 10-20 themes that were coded and reorganized. These categories were then used to illicit additional information from Ruth either by interview or observation of class instruction. Upon completion of the data collection process, six major categories emerged: knowledge in three specific areas—content, curriculum, and students, making mathematics relevant, confidence level, and comfort level. These themes were presented to Ruth for her review and reaction to determine if there were any additional themes that she believed were lacking from these findings. In one of the final interviews, she indicated an additional factor—her principal's support—being important. While she viewed this as an important factor, it had become a component of her "comfort level" grouping rather than a "stand-alone."

Using Clarke's work (1997) as a framework for this research was valuable in that the findings were in two specific areas—"what the teacher does" (her actions) and "beliefs about teaching and learning." Both areas were at the forefront in trying to make sense of what influenced Ruth's practices to change and what factors influenced her to teach in a reform-based manner. Her actions provided evidence that she was willing to take risks in providing her students with nontraditional types of experiences and that she also was willing to put in the time it took to accomplish this. She wanted the mathematics to be relevant for her students – whether they would work in factories, the logging business or teach school – and was willing to take whatever measures were necessary to do this. She also believed that she would not have changed her practices if she had not had the opportunity to collaborate with other teachers in her graduate work. Since she was in such a small school district, her professional development came out of her graduate studies. She had been a traditional teacher—lecture and lots of examples—but came to realize that not all students would learn the mathematics in this way and she wanted them all to be successful.

Discussion and Conclusions

Although the reform movement's perspectives on teaching and learning encouraged Ruth, she was not always clear about how to make change occur on a regular basis. She expressed concern that "I might not be able to do the things I do with my students if the school wasn't on block." She had a difficult time finding literature that would help her in her transition and she voiced this on a number of occasions. While there exists research in teacher change, much of it relates to elementary and middle levels. By and large, high school teachers still have not "bought into" the reform movement practices. Thus, it is important that further research be conducted in this area.

Questions related to the textbook used for classroom instruction were also raised. Ruth believed the textbook helped her think about geometry differently than how she had learned it and it gave her some flexibility in organizing inquiry-based activities for her classes. This raises the recurring question on how textbooks drive what occurs in the classroom.

Another area that needs additional attention is how teacher change occurs in rural schools. Ruth is one of many teachers who are in isolated areas with little to no support for teacher development. A large part of this is due to funding—or lack of funding. Since the populations served are smaller in number, a large percentage of education dollars must be spent on updating or renovating schools. Little money is available for the professional development of teachers, updating laboratories, and purchasing technology equipment. Thus, the research community cannot overlook the difficulties teachers in rural schools must struggle with in order to make change happen.

If we, as a mathematics education community, expect for teachers to change their practices, then we need to be leaders of reform in teacher education programs. This is especially important in mathematics content courses pre- and inservice teachers take in their programs. We must continue to make progress in defining what mathematics reform looks like and how one gets there. Future research must provide society at large with such evidence.

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INTERMATH: TECHNOLOGY-ENHANCED, LEARNER-CENTERED PROFESSIONAL DEVELOPMENT[1]

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In this oral presentation, we discuss findings from the implementation of InterMath (http://www.intermath-uga.gatech.edu), an NSF-funded, web-based initiative geared toward improving mathematics teaching and learning in the middle grades. We examine two workshops to determine what goals were and were not met by participants in this professional development environment. Furthermore, we focus on the experiences of the learners, offer insights from across both cases, and provide suggestions to improve the experience for later learners.

InterMath has been developed based on the recommendations of leaders in professional development and mathematics who have clearly defined a working plan for improving mathematics teachers' content knowledge and pedagogical content knowledge—two factors critical to student learning. InterMath responds to calls for professional development that occurred over an extended period of time, emphasizes teacher thinking and development of reflective dispositions, and pushes teachers to learn more in their content areas (e.g., Ball, 1994; Hawley & Valli, 1999; Krajcik, Blumenfeld, Marx, & Soloway, 1994). As one facet of a larger effort, InterMath addresses the vision of teacher learning laid out by Kilpatrick, Swafford, and Findell (2001): "Teachers' professional development should be high quality, sustained, and systematically designed and deployed to help all students develop mathematical proficiency" (p. 12). This statement is a call for professional development experiences that depart from the "make and take" model that has pervaded teacher professional development. InterMath provides one format for meeting the growing need for constructivist, learner-centered environments (Hawley & Valli, 1999; McCombs & Whisler, 1997; National Staff Development Council [NSDC], 2001).

The InterMath experience includes a 15-week technology-rich workshop, a website with mathematical investigations and tools to support learning, and an ongoing support system designed to provide continued support for previous participants. The InterMath workshop itself supports its participants in developing content knowledge and technology integration by allowing the participants to use a variety of technological tools (e.g., Geometer's Sketchpad, Excel, and NuCalc) to explore open-ended investigations. In implementation, InterMath allows participants to choose their own paths–they select which problem(s) they want to work; the approach they want to use to solve the problem; and ultimately, the depth of their learning experience.

For this study, we collected data in two InterMath workshops. The larger of the two workshops included approximately 26-28 participants (this number varied due to attrition and late enrollment). These participants were full-time middle school teachers enrolled in a district-sponsored graduate degree program in mathematics education. The workshop served as their introductory course for a three-year degree program (separate from the InterMath workshop). The workshop instructor was a mathematics education professor and one of the InterMath developers. Two graduate assistants served as participant observers during the workshop, and the project manager visited the class three times during the semester to act as an outside observer and to conduct interviews with participants.

The second workshop was held at the University of Georgia (UGA). The class began with seven participants from nearby middle and high schools; however, only four completed the workshop. Two of these participants taught eighth grade mathematics at a rural middle school. The other two participants came from a private school--one was a sixth grade mathematics teacher; the other was the school's technology support person who also had a mathematics education background. A professor from the mathematics department at UGA led the workshop. There were three graduate assistants who were responsible for supporting the teachers and taking field notes. Again, the project manager conducted interviews with this group.

The classes both met once per week for three hours. During the first portion of each class, the instructor demonstrated explorations and answered participant questions. The participants then shifted to the computers to work on investigations and write-ups of investigations for the remainder of the class. For their final product, participants were expected to complete a web-based portfolio comprised of their write-ups.

The data considered for this report include observations from three meetings of each class (an early, a middle, and an end session), interviews with some of the participants, and the online materials produced by those participants. We included tape-recorded interviews with eight participants in the first workshop, all four participants in the second workshop, and both instructors. There were also informal interviews conducted half-way through the course in the first workshop. Participant portfolios of the people interviewed were also analyzed.

Observation and interview data were analyzed using coding techniques (similar to those described by Merriam, 1988) to identify key themes. Each subset of data was analyzed by at least two people on our research team. Once the themes were identified and agreed upon by both researchers, cross-case analysis was used to determine which themes were characteristic of both workshops. The findings and recommendations are based on those emergent ideas that crossed the cases.

Participant portfolios were analyzed by two researchers for each class. The analysis consisted of using a rubric that considered the quality of mathematical thinking and the demonstrated technology use. By using this approach, we were able to watch for changes over time and for trends among learners both within and between the two workshops. Our findings show that support and interaction became intertwined. We noted that there were two kinds of interactions: affective (aimed at providing positive feedback or other motivation) and intellectual (those that provided the information learners needed in order to make progress on their investigations).

There were two main barriers across the two cases: technology and goals. Participants' inexperience with the software and hardware significantly impacted their learning experiences. Moreover, the participants' goals and the workshop designers' goals were not always in alignment, leading the two groups to focus on different aspects of the learning experience.

Another major finding in the cross-case analysis was disturbing. There was a trend among some participants to implement the InterMath problems into their classrooms by structuring their students' learning experiences exactly as their workshop experience had been structured. In short, it seemed that the participants borrowed InterMath rather than adopted it. It may be argued that this is the first step of adoption; however, it was problematic because middle school students are not necessarily ready to have a learning environment quite as open as the InterMath workshop had been. Perhaps worse than directly implementing InterMath, however, was the overwhelming number of participants who were afraid to provide their students with any learner-centered instruction. This fear led them to avoid implementing any InterMath ideas at all.

In conclusion, we have developed suggestions for the improvement of the InterMath workshop experience. First, it is vital to the success of the workshop that we simplify the technology for developing the online portfolio because when the participants were struggling with technology, they did not engage in content knowledge development. Further, we learned that simply telling the participants about the intentions of the workshop was not enough. The structuring of the workshop needs to challenge the participants to push themselves. The participants need to have the opportunity to develop their own goals in order to establish individual ownership of the workshop itself. Finally, participants need to feel that they can work with the instructor(s) to steer this professional development program in the direction that maximizes their success.

From our experience in these two workshops, it seems reasonable that a first step toward more meaningful adoption by the participants would be to attack the problem head on: to discuss ways to implement InterMath with the participants, to look at ways the investigations might be modified to more appropriately meet the needs of middle school students, and to discuss the value and learning that might come from using the investigations. A second step might be to actually model a classroom approach to using the investigations with middle school students.

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CRITICAL EXPERIENCES FOR ELEMENTARY MATHEMATICS TEACHERS

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We define as critical those experiences of epiphany that cause teachers to reflect on their knowledge and beliefs and see mathematics and mathematics teaching in a new light. When such moments of epiphany occur mathematics education artifacts – such as curriculum documents, classroom experiences, ideas from professional development workshops, journal articles, and so forth – can be thought of as inkblots where the image appears to shift and something new is seen, something that was not apparent before. As one teacher in our study commented, "I feel like [this experience] has cleaned my spectacles." Similar findings are reported by McGowen and Davis (2001) where one teacher noted that course experiences "opened [her] eyes to a new outlook on mathematics" (p. 444).

Some research indicates that for significant change to occur in teachers' beliefs and practice, teachers need the opportunity to engage in practical inquiry and reflection about mathematics and mathematics teaching (Borko, Mayfield, Marion, Flexer and Cumbo, 1997; McGowen & Davis, 2001; Stipek, Givvin, Salmon and MacGyvers, 2001). One of the "practical inquiry" experiences of the 20 teachers in our study was to mentally solve arithmetic problems such as 16 x 24 and 156 + 78 + 9. In an asynchronous discussion forum, teachers shared the mental solution methods they used, and they reflected on the pedagogical implications in light of two articles they read on children inventing personal algorithms for arithmetic operations (Burns, 1994; Kamii, Lewis & Livingston, 1993). Teachers also shared their personal views and stories of mathematics. Transcripts of online discussions were analyzed using a comparative case study approach to determine common and disparate characteristics of teachers' perceptions of mathematics and mathematics teaching.

In the beginning of the online course, teachers shared their views of mathematics by telling stories of personal experiences, which we categorized into three types of stories about mathematics. One type of story involved an aesthetic connection to childhood and family – a view of mathematics that is both social and playful. This was manifested in statements, such as "I LOVE math. I always have. I, like many of you, had many problem-solving car trips. I still get excited when I see a license plate that I can make ten with (using any means)." Note the aesthetic nature of the teacher's statement – the verbal expression of delight and the use of capital letters to convey her emphasis. Also note the aesthetic qualities of the open-ended mathematical problem of "making ten" using "any means" which contrasts sharply with the more traditional approach that insists that one finds *the answer*, to, say, 5 + 5. The person "making ten" has the opportunity to use her imagination and to find personal, creative ways of looking at mathematically combined digits on license plates. Sinclair (2001) suggests that aesthetically rich mathematical experiences enable us to "wonder, to notice, to imagine alternatives, to appreciate contingencies and to experience pleasure and pride" (p. 26).

A second type of personal story of mathematics was characterized by a persistent negative aesthetic connection to traditional school experiences and feelings of incompetence and failure: "I grew up with very negative feelings towards math. I felt that there was a time pressure to answer quickly and I panicked when confronted with this. I feared the thought of failure or looking stupid when others could answer so quickly." In contrast to positive stories about mathematics that were in part family-based, the negative stories shared by teachers were school-based. This is not to say that all home experiences of math are positive and school experiences are negative, rather, that in our study mathematical activity in the home was remembered fondly and its pleasurable affects continued to be felt in adult life. A third type of personal mathematics story that emerged was characterized by a passive attitude towards the subject. There wasn't a feeling of failure but neither was there a feeling of excitement or enthusiasm. One teacher commented, "I did not hate math in school but I didn't love it either. I went through the motions." In our study we noted that teachers across these three groups appeared to have been positively affected by the critical experiences offered in the online course. Teachers in the last two groups expressed optimism about a renewed interest in mathematics.

We suggest that the experience of mentally solving 16 x 24 or 156 + 78 + 9 is aesthetically rich in that the mental processes involved do not demand rule-based procedures. How people solve 16 x 24 depends greatly on how they personally interpret the problem. For example, some people may multiply 16 and 25 and then subtract the extra 16. Others may deconstruct the problem as 10 x 24 + 6 x 24. Many other solutions processes are possible – even ones that use algebraic structures like (20 - 4)(20 + 4). Given such problems, people are eager to share their solutions, they express interest and sometimes surprise in the solutions of others, and are motivated to try to come up with different solution processes. Open-ended inquiry, interest, surprise, and motivation are characteristics of an aesthetic approach.

Teachers noticed that their mental solution processes for 16 x 24 or 156 + 78 + 9 differed from the standard algorithms for addition and multiplication, and that several different solution processes emerged: "When I added 156 + 78 + 9 I started with 156 + 9 = 165. Then I added 165 + 80 = 245 - 2 = 243. This is different than when I did it with paper and pencil because I [...] solved my problem by starting with the bigger numbers first (left to right, not right to left!)." Such experiences appear to have helped teachers develop a deeper understanding of what constitutes mathematical activity and mathematical understanding in the context of addition and multiplication. "To me, the implications are that doing arithmetic mentally requires real understanding. The traditional way (on paper, doing the "ones" first) is more of a procedure to be memorized that requires little understanding." As was the case in the study by McGowen & Davis (2001), teachers in our study made important connections between their experiences and ideas in the articles they read. "I do agree with Kamii and Burns' points of view. [...] I think that by having the student discover a successful method they will be more likely to internalize and understand the concept. In coming up with their own methods they are doing the thinking the way their mind works. We can see [in our discussion] that everyone processes things differently." The combined experiences of doing mathematics and reading the scholarly papers on children inventing algorithms resulted in some change in how teachers interpreted curriculum documents.

> After seeing how different people calculate, I better understand this last overall expectation in [the curriculum document]. Hmph! It is not until I do something myself, do I more fully understand the language and what the curriculum is really driving at. Thus students need to explore different ways of doing calculations, talking about it, communicating their ideas in a variety of ways. [...] I am reading the document with new vision.

Discussion

Contrary to the usual alignment of "critical" with "rigorously intellectual," participants in our study revealed that critical engagement occurs within an aesthetic context. This was true for teachers with diverse mathematical backgrounds and attitudes that ranged between positive and negative. The interplay among the mathematical experiences, journal readings, and online reflections and discussions created a critical tension that helped teachers understand mathematics and mathematics teaching in a new light. The experience of struggling to solve simple arithmetic problems mentally helped teachers realize that these problems may be solved in many different ways. The journal articles placed their experiences in the broader context of mathematics pedagogy. The opportunity to share, compare, and discuss ideas online also benefited the teachers. No one experience can create comprehensive or permanent changes in teachers' perceptions of mathematics and mathematics teaching. Neither does one experience significantly affect teachers' classroom practice; teaching is also greatly affected by accepted teaching practices in the wider school community (Buzeika, 1999; Ensor, 1998) and by conflicting priorities (Skott, 1999). However, critical experiences are important starting points for change in teachers' perceptions and classroom practice.

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THE STRUGGLES OF A COMMUNITY OF MATHEMATICS TEACHERS: DEVELOPING A COMMUNITY OF PRACTICE

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The purpose of this study was to understand how participants in a professional development program initiated, developed, and supported a learning community focused on students' understanding. The analysis focuses on teachers' activity as they participated in a community of practice and how that participation promoted (or restricted) their learning and therefore theirs students' learning. The components that characterize a community of practice are 1) mutual engagement, 2) a joint enterprise, and 3) a shared repertoire. The study took place in a bilingual (English-Spanish) high school in a large district in the Midwest, with a high percentage of Latino/a students, and situated in an urban setting. The study shows how the community was transformed by the practices and resources available to the members.

Introduction

Mathematics teachers' participation in communities of practice has been identified as a promising strategy for professional development. Studies focused on school organization have found strong relationships among teachers' participation in professional communities, innovative practices, and student learning (Lee, Smith, & Croninger, 1997; Louis, Marks & Kruse, 1996). Mathematics teachers who participate in communities of practice are more willing to change their practices and their learning becomes generative (Franke, Carpenter, Levi, & Fennema, 1998). While studies have found that teacher' participation in communities of practice is one significant element for supporting teacher change, we do not know how those communities are constructed and supported. In many studies, the communities were already organized or, in some cases, the communities emerged from the reform process in ways that are not clear. Regardless, communities are important in supporting both mathematics teachers' and their students' learning.

The purpose of the present study was to understand how participants in a professional development program initiate, develop, and support a learning community focused on students' understanding. The analysis focuses on teachers' activities as they participate in a community of practice (Wenger, 1998) and how that participation promotes (or restricts) their learning and therefore their students' learning. The three dimensions characterizing a community of practice are 1) mutual engagement, 2) negotiation of a joint enterprise, and 3) development of a shared repertoire. This study seeks to describe how a specific design for professional development serves the different requirements for learning defined on a framework in which communities of practice and participation in those communities are the fundamental unit of analysis.

Such a conceptual framework emphasizes the negotiation of meaning and the building of identities, which take place during the development of practice (Wenger, 1998).

Methodology

This study focused on the community formed by teachers and researchers. There were two opportunities in which researchers and teachers were engaged in common activities. One was the general meetings and the other was the conversations after class—reflection meetings. The data correspond to two year's texts produced through transcripts of teacher-researcher conversations during the general meetings and the reflection meetings. The analysis began reading through all data as a complete corpus to get a sense of what happened over time. Codes for analyzing participant characteristics were used in this first stage. The result of this first reading was the description of the history of the community and the identification of themes that were later used on the second stage to fracture the data.

The second stage, explanation, was focused on the three questions related with the mechanisms for community formation: a) What are the possibilities for mutual engagement? b) How is alignment secured? and c) What material supports imagination? To answer these three questions, the coding system was organized to reflect characteristics of the community. Based on the framework, three aspects were defined as fundamental to support the formation and development of the community and to provide facilities for each one of the components.

Change and Development

Every community of practice evolves in its own way. Looking at the facilities for engagement, alignment, and imagination available each year it is possible to see how they changed. There are at least four aspects that deserve attention—the structure of the meetings, the content of the meetings, the discourse of the community, and the forms of engagement. During the first year, there was no clear structure for the meetings. While the researchers wanted to be open to teachers' suggestions, teachers' concerns directed most of the meetings. The amount of time spent in discussing scheduling for following meetings was very large during the first year but it was not an issue during the second year. From the very beginning of the second year, the meetings frequency was set and most scheduling conflicts were discussed during the reflection meetings or via e-mail.

During the first year, the teachers were in charge of setting the agenda for some of the meetings and the researchers were often uninformed about the topics or questions that would be addressed that day. On other occasions, the researchers had prepared some activities for the teachers and during the meeting the lack of preparation or other teacher's concerns changed the way the meeting was run. In contrast, during the second year, the researchers prepared all the meetings, having goals they expected to meet and discussing with the teachers in advanced the activities they would like to be involved.

The second difference deals with the content of the general meetings. Looking at the facilities for engagement, especially the aspects related with continuity, there is a clear difference in the relation among the topics covered during the meetings. During the first year, each meeting could be described as 'addressing one topic'; during the second year the continuity of activities and contents across meetings was more evident.

These differences in activities and content were translated to differences in discourse of the community, the third difference. During the first year, most concerns were related to issues external to the community—school restrictions, proficiency exam, and curriculum. During the second year, the conversations were directly related with teacher's practice. Even though, during the first year, the teachers shared stories from their classrooms and the community had conversations about specific mathematical content areas, such as algebra; it was only during the second year that the group discussed specific examples of students' thinking, criteria for selecting worthwhile tasks, and the role of the teacher in classrooms that promote understanding.

Finally, the forms of engagement changed drastically from one year to the next. During the first year, all the teachers participated in the discussions during the meetings, some more involved than others, but the attempts to make changes in practice took place outside of those discussions. During the second year, there was a conscious and deliberate attempt to involve the teachers in making changes in their practice. It was not forced, of course, but it was the ultimate goal. All the activities were directed to give teachers tools to understand student thinking, to have clear goals, to select appropriate tasks, and to take new roles in the classroom. And because of these intentional actions, the teachers had more opportunities to participate in meaningful activities, to work with others, to produce artifacts, and to be really engaged in the process of community building.

Implications

The study contributes to research in professional development in two ways, (1) understanding how a community of practice emerges and (2) validating the relevance of the content on professional development programs. The framework used helped to describe the different stages of the community and the way the group matured over time. Looking at the facilities available for each component during the process of community formation has given a detail description of what activities the participants were engaged in, what goals they negotiated, and what tools they produced during these activities for reaching the goals. The description brings to the front the importance of the content in the development of the community.

The second contribution to the research literature is the relevance of the content in professional development programs. The changes observed from the first year to the second year, show how important is the content of the conversation in a professional development program. Only when student understanding was the real focus of conversations and the community had tools to talk about it, did the teachers engage in more meaningful activities. The focus of the discussion was always highly related with the mathematical content, specifically with student development of algebraic reasoning.

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INTERWEAVING AN ONGOING PROFESSIONAL DEVELOPMENT PROGRAM WITH EVERYDAY CLASSROOMS

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This report is based on a professional development program project funded by the state of Ohio and the Regional Professional Development Center. The program was designed to provide middle school mathematics teachers with both content and pedagogical content knowledge. Eight graduate credits were offered from January 2001 through December 2001, and a total of twenty-five middle school mathematics teachers in grades 5-8 participated. Overall, it was observed that the project was conducted in a manner consistent with the proposed goals, objectives and stated outcomes.

Research Questions

The purpose of this study was to find strategies for enhancing professional learning, as well as developing effective professional development models. The focused research questions were (1) What are teachers' expectations from the professional development program? (2) How do experienced teachers adopt/adapt new content/ pedagogical knowledge in their classroom?, and (3) What are effective systematic aids to support teachers' professional development, including their content and/or pedagogical knowledge?

Theoretical Frameworks

Professional development is a critical ingredient of mathematics education reform. Effective professional development experiences are designed to help teachers build new understandings of teaching and learning via the trying out of strategies that help students learn in new ways. Loucks-Horsley, Hewson, Love, & Stiles (1998, p. 36) listed the principles that are addressed in effective professional development experiences: such experiences are driven by a well-defined image of effective classroom learning and teaching; provide opportunities for teachers to build their knowledge and skills; use or model with teachers the strategies teachers will use with their students; build a learning community; support teachers to serve in leadership roles; provide links to other parts of the education system; and are continuously assessing themselves and making improvement to ensure a positive impact on teacher effectiveness, student learning, leadership, and the school community.

Jones, Swafford, and Thornton (1992) discussed the major concerns that evolve around the need for professional development programs. For example, programs should actively promote "individually guided" teacher activities; generate conditions for significant follow-through and feedback on new teaching practices; provide opportunities for teacher input and involvement in establishing and developing the professional development program; support an inquiry approach for addressing teachers' pedagogical problems; and generate a knowledge base for effective teacher decision-making. Nowadays, these concerns are still critical to the success of programs. In addition, more issues to be considered for the success of professional development programs are ensuring equity, building professional culture, developing leadership, building capacity for professional learning, generating public support, supporting the effective use of standards and frameworks through professional development, finding time for professional development, and evaluating professional development.

Professional development does not occur as an isolated strategy. Every program uses a variety of strategies in combination with others. According to the National Staff Development Council (Sparks & Loucks-Horsey, 1990), five different models of effective staff development for teachers were: training, individually-guided staff development, observation/assessment, involvement in development/improvement process, and inquiry. These can be used singularly or in combination (Brown & Smith, 1997). These strategies covered the professional development models adopted by several different institutions or organizations, as well as by this project.

Research Methods

This study was conducted from January through December 2001, during which an introductory workshop in the winter quarter and three consecutive workshops (spring, summer, and autumn quarters) were provided. Twenty-five (6 male and 19 female) middle school mathematics teachers in grades 5-8 participated and earned 8 graduate credit hours. The participants experienced four workshop sessions throughout the year, including theoretical trends in mathematics education, hands-on activities, analyses of student work, using technology, the use of curriculum materials, understanding NCTM Standards and the Ohio model, collaborative work, issues related to assessment, and discussions followed by activities. Reflections on practice such as reflections on audio and video taped self-taught lessons were required. WebCT (Web-based communication tool) was used to communicate among teachers and instructors. Poster presentations and presentations in a mini-conference were conducted to share what they learned with other teachers. In addition to the in-class program, project instructors and a consultant visited participants' classrooms. These site visits provided a better understanding of the participants' needs, as well as the project impact on teachers' content knowledge and pedagogical techniques.

The main data sources of the study were observations, interviews, and written documents such as concept maps, journal entries, teacher reflections on practices, and pre- and post-project survey questionnaires, and commentary notes from site visits by the project consultant.

Findings

(1) What are teachers' expectations from a professional development program?

Overall, participants' expectations from the professional development program were to learn: new ways to effectively teach math; how to get my students more actively involved in learning mathematics; and strategies to show students why things work. They were more interested in actual practices that can be taken to the classroom rather than theories. Teachers were also eager to make new professional connections through the program.

(2) How do experienced teachers adopt/adapt new content/pedagogical knowledge in their classroom?

Teachers' reflections in their journals show what activities/contents influence their learning to teach, as well as what factors act as incentives/hindrances to enriching their professional development. Teachers appreciated visual representations, links to literature, cooperative learning, hands-on and fully engaged learning, talking about things, understanding "Why," opportunities to brainstorm with other teachers, alternatives to traditional texts, connecting to the way children think, and so on.

(3) What are effective systematic aids to support teachers' professional development, including their content and/or pedagogical knowledge?

Each activity of the workshops was developed to accomplish the project goals and anticipated outcomes. The following are few of systematic aids used for the workshops: encouraging teachers to learn mathematics in an active participatory manner; solving problems and justifying their solutions to enhance mathematics content knowledge; using technology; using videotapes to improve instructional practices; clarifying beliefs about teaching and learning mathematics; assessing student understanding of mathematics concepts and procedures; monitoring student performance in mathematics tests; conducting inservice workshops/seminars for mathematics teachers; and so on.

Discussions

The paradigm shift in professional development suggests a change in emphasis from transmission of knowledge to experimental learning; from reliance on existing research findings to examining one's own teaching practice; from individual-focused to collaborative learning; and from mimicking best practice to problem-focused learning (Loucks-Horsley, 1995; Sparks, 1994). What teacher educators and teachers should keep in mind to enrich professional development programs are that professional learning must be relevant to student learning. Also teachers should stop receiving one-shot workshops and become active decision makers in the process of designing and choosing professional development opportunities. Professional development initiatives in mathematics should have an appropriate level of challenge and support, provide activities demonstrating new ways to teach, use a team approach, provide time for reflection, and evaluate the effectiveness and impact of the activities. In addition; follow-up to the professional development program should be provided – such as opportunities for practice in the classroom; the professional development program designer's challenge is to assemble a combination of learning activities that best meet the specific goals and context. In summary, professional development should be viewed as a critical component of reform. The goals of a professional development program must be linked to the goals of students' learning and assessment, preservice teacher education, and school leadership.

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CURRICULUM MATTERS: THE IMPACT OF A PROFESSIONAL DEVELOPMENT PROJECT ON TEACHERS' UNDERSTANDING OF CURRICULUM AND CONTENT

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If we want teachers to apply research to their practice, then we should demonstrate how the application impacts the very issues that concern them daily in the classroom. Currently Standards-based curricula and teaching held accountable by state testing programs are at the center of education concerns and reform. Gandel & Vranek (2001) report, "When they are well devised and implemented, academic standards and tests, and the accountability provisions tied to them, can change the nature of teaching and learning. They can lead to a richer, more challenging curriculum . . . When they are poorly devised and implemented, [they] can become a distraction and a source of frustration in schools." They also conclude that standards matched by equally rich and rigorous tests that become more challenging at each successive benchmarking grade and intensive professional development make a difference between high and low quality programs.

Since January 2000, the Professional Development and Curriculum Alignment Project (PDCA) has used the state criterion-referenced test as an opportunity for research and reform in teacher knowledge and practice. The project, which involves teachers in grades 3-8 at Site 1 and grades K-8 at Site 2, investigates the effectiveness of a professional development model based on coaching (Joyce & Showers, 1988; Weathersby & Harkreader, 2000) and alignment of the intended (written), the taught, and the tested curricula (Glatthorn, 1999; Hayes Jacobs, 1997).

Motivated by a concern over low state test scores, administrators from each site initiated the contact with university mathematics educators. The criterion-referenced tests benchmark student achievement at grades 4, 6, and 8. These tests are rich in that they assess computation in problem contexts and heavily weigh open response items in the raw score. The two sites were not looking for quick fixes. Teaching to test items and adopting a "test-guaranteed" textbook program would inadequately address their problems. However, the university educators proposed that the released test items could be used to help teachers begin to understand the complexities of curriculum and to wean them away from textbook reliance.

PDCA has two components, summer institutes and site visits. The goals of the summer institute are to:

- To develop teacher knowledge by focusing on a major mathematical idea that can be vertically articulated through the K-8 curriculum and that connects to all curriculum strands (Number, Property, and Operations; Geometry and Spatial

Sense; Measurement; Patterns, Functions, and Algebra; Data, Probability, and Statistics)

- To involve teachers in a curriculum development process that is informed by Benchmark test data and moves teachers from a static, traditional model based on covering a textbook to a dynamic model based on student learning.

Each site visit consists of a planning session, a teaching session (either model-teaching by a mathematics educator or team teaching), and a reflection session. The goals of these visits are:

- To implement the curriculum written in summer institutes and integrate multiple Standards-based resources into plans and lessons
- To model inquiry-based teaching that not only develops higher order thinking skills in students but also assesses students' conceptual understanding
- To develop short written assessments of daily lessons that reveal student understanding
- To reflect on all assessment data (from daily written assessments to state tests) so that teachers will continually monitor and adjust the written curriculum.

To document the changes in teacher practice, we have collected teacher-made curriculum documents, lesson plans, student assessments, and observational and interview data from the classroom visits.

The way in which teachers combine content and curriculum is emerging as a most interesting aspect of the study. In the initial activities of the curriculum alignment process (Summer 2000), teachers could only list topics. They had trouble describing exactly what they taught about a topic. While some teachers reported that they felt responsible for covering too many objectives, others revealed that they repeated major teaching objectives that should be only revisited at their grade. Within this week, the teachers did begin to articulate conceptually based objectives for each grade. The discussions between different grade level teachers were invaluable. For example, an eighth grade teacher explained to a sixth grade teacher how she wanted her to informally introduce algebraic equations. A group of fifth grade teachers at Site 2 decided to find material that would emphasize the meaning of fractions and equivalence. Previously, they had emphasized rules and covered procedures for all four operations. The sixth grade group would use this foundation and extend their teaching to the operations.

While we have observed modest success in the curriculum writing process, we have found problems in the classroom. At Site 2, the third grade teachers resisted the written curriculum and returned to the former textbook. A dramatic fall in norm-referenced test scores for their students amidst gains at the other grade levels convinced them to implement the plans and resources they had opposed. They have seen differences in student understanding on a daily basis and have had gains in the Spring 2002 scores. At Site 1, the third and fourth grade teachers are no longer in the project. Their resistance is in part due to adherence to a textbook series. However, during the two years in PDCA, they did identify weak areas and found Standards-based resources for them.

The conflict between using the written curriculum and relying on the textbook also affects grades 5-8. In Summer 2001, the focus of the institute was proportional reasoning. In its report, *The Mathematics Education of Teachers*, the Conference Board of the Mathematical Sciences, recognizes proportional reasoning as a major set of ideas that permeates the middle school curriculum. Yet, teachers fail to recognize how this big idea connects the curriculum strands and instead focus on rules and procedures for fractions. While we documented gains in teacher knowledge at the institute and assisted with curriculum plans that we thought would circumvent typical problems, we did observe the same lessons on fractions at grades 7-8 as those being taught at grades 5-6. I watched an eighth grade teacher at Site 1 who relied heavily on the textbook and re-taught operations with fractions out of context (bare-number skill problems). Not only did her students not do well on an end of unit test, but it also seems she wasted many days re-teaching procedures when she might have had a better result with interesting problems in which students had to make sense of the numbers and operations. In a reflection session, this teacher reported that she would like to do something different. We decided that first we would investigate the fraction topic in grades 5-8 with the following questions for teachers in each grade:

1. What knowledge about fractions do your students bring to your class?
2. What weaknesses and misunderstandings do they bring?
3. What are your major teaching objectives for rational numbers?
4. Ideally what should your major teaching objectives be?

To summarize the responses, we found that teachers reported that student understanding does not change much from understanding basic fractions as parts of a whole. The fifth grade teacher realized she needed to do more with fraction concepts because she was "concentrating on decimals and place value." The seventh and eighth grade teachers were re-teaching the sixth grade objectives. The sixth grade teacher wanted to integrate fractions into lessons from all five curriculum strands, and the seventh and eighth grade teachers wanted to teach fractions in problem solving lessons.

After teachers reviewed these responses, model-teaching sessions were planned. The most effective work occurred with the sixth grade teacher who requested lessons from *Bits and Pieces II* in the Connected Mathematics Project, which is only supplementary material in this school. By being able to watch student interactions with a university educator in several lessons, she saw that a concept could be launched in the context of a rich problem that would become a reference point for work done with fractions throughout the year. She also began to see tasks and lessons that were proportional in nature so that fractions could become a part of many lessons. Assessments were planned that would track student progress with fraction concepts.

In the short oral session we will present teacher responses to the fraction questionnaire as well as resulting revisions (Summer 2002) in the written curriculum and the sixth grade teacher's analysis of the fraction assessments and implications for her teaching practice. While two and a half years of work in PDCA has not produced the "perfect curriculum," it has produced groups of teachers who can identify problems, openly look for solutions, and make data driven decisions.

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INTERMATH - ENHANCING CONCEPTUAL DEVELOPMENT OF MIDDLE SCHOOL TEACHERS WITH TECHNOLOGY

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InterMath is an NSF funded initiative aimed at improving the conceptual understanding of middle school mathematics teachers. Teachers in InterMath participate in a workshop course that promotes mathematical exploration and problem solving using technology. InterMath content includes a series of mathematical investigations that require deep examination of middle grades mathematics concepts. Teachers create a web-based portfolio that includes essays about their investigations. In addition, teachers make connections of their learning to their classroom by designing technology-enhanced activities for their students.

InterMath (http://www.intermath-uga.gatech.edu) is an NSF funded, Web-based initiative. The InterMath environment engages teachers with technology-enhanced investigations in an effort to deepen their understanding of mathematical concepts related to the middle school curriculum.

InterMath seeks to encourage changes in teaching and learning based on the use of technology and the vision set forth in NCTM's standards. InterMath embraces the notion of mathematics as problem solving, reasoning, and communicating so that students are empowered to confidently "explore, conjecture, and reason logically [about the world around them]" (NCTM, 1989, p.5). This change in learning philosophy reflects a need for mathematics that is based in an information-rich and technology-based society. Learning goals should incorporate values that reflect mathematics for life, mathematics as a part of cultural heritage, mathematics for the workplace, and mathematics for the scientific and technical community (NCTM, 2000).

InterMath participants include primarily middle school mathematics teachers and workshop facilitators in the state of Georgia. Any middle school teacher or administrator in the state can participate in the InterMath workshops. Priority in participation selection is given to teachers that have introductory technology skills and those who can attend with peers from their school. Participating teachers can receive university credit or staff development credit for their involvement in the semester long workshop. The classes meet in three hour time periods, once a week, for a semester. During the class, the InterMath facilitator often models thinking, problem solving, and technology use of an investigation that is found on the InterMath website. Teachers are then provided considerable time to conduct their own mathematical investigations, often in teams. During this time, teachers are given an option to closely examine an investigation of their choice based on a particular concept discussed that day, such as fractions or graphing. Teachers use technology tools, such as spreadsheets, function graphers, and dynamic geometry, to support their thinking to develop and verify conjectures.

After using technology to test and verify their conjectures, teachers are encouraged to examine *why* they arrive at a particular finding, usually through proof or additional representation. Further, teachers are encouraged to extend their learning by posing related extensions to the activity so they can broaden their findings and make connections between concepts or to consider how they might be able to teach this concept in their middle school classrooms. Finally, teachers write up and display their findings in a web-based portfolio that can be accessed at the InterMath website.

The poster presentation will demonstrate InterMath investigations, describe the teachers' professional growth through InterMath workshops, and provide opportunities for others to use and contribute to InterMath resources for their own instructional purposes. The investigations will be shown from the presenters' laptop computer, giving viewers an opportunity to experience the dynamic environment that the technology-enhanced activities afford. InterMath teachers' web pages that reflect on these investigations will be shown to illustrate their problem solving methods, conceptual understanding, and lesson plans throughout the InterMath experience.

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ON BECOMING A REFORM-ORIENTED MATHEMATICS PROFESSOR BY REFLECTION

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The purpose of this study was to describe the process of change that a senior mathematics professor with thirty nine years of teaching experience had in his upper level course in secondary mathematics education under the reform influence of a faculty development program and to explain the role of *reflection* in the change process. The researcher had an opportunity to observe the professor's teaching practice for a semester and served as a support provider for him to sustain the change process during, before, and after the class through in-depth conversations based upon the observation of his classes in the fall semester. The method of teacher *reflection* was utilized for the teacher who wanted to bring change (Jalongo, 1992) in the on-going conversations between the professor and the researcher. Both the teacher and the researcher understood teacher *reflection* in the way as John Dewey (1933) described; *reflection* as "behavior which involves active, persistent, and careful consideration of any belief or practice in light of the grounds that support it and the further consequences to which it leads" (Grant and Zeichner, 1984, p. 4). This method of *reflection* addressed the professor's personal experiences as a teacher and their influence on shaping his teaching practice in a systemic way. The change in his beliefs about the nature of mathematics teaching was reshaped in conjunction with his understanding of learning represented by his students that he obtained through the *reflection* on his teaching. Even though the *reflection* process was a time intensive process, the process served as an encouragement and structure for the change to happen in his teaching practice. In this study, he was a professional who can make reasonable decisions, and the process of *reflection* served for him. Consistently, this study made a close connection between practice and research by influencing each other. Understanding his beliefs about the nature of mathematics teaching and many other different issues related to teaching practice helped the connection be established. For the connection between research and practice, the researcher tried to communicate with the teacher much insight about mathematics teaching practice that the reflective professor tried to implement in his mathematics course. The learning from the researcher's presence in the classroom where the practice of teaching was vividly experienced provided the professor with on-going feedback and this acted as a vital source for betterment of his teaching in practice. As the result of it, the professor continuously reshaped his teaching practice in the reform-oriented way. In addition, the learning through the communications was used as an importance vehicle for providing a continuing support for the professor's sustaining of the change. This connection between practice and research was significant. I, as the researcher, learned the importance of the environment and support for a university faculty member to learn to be reflective about his teaching and about his students' learning. The professor, as the practitioner, became thoughtful through the reflective process so that he developed his own professional thinking and continued as a life-long learner with those many years of teaching experience. As he always put it, everything was developing in the process.

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K-3 TEACHERS' APPROACHES TO MATHEMATICAL ARGUMENTS

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Developing and evaluating mathematical arguments is one of the goals that students should reach while they are learning mathematics (National Council of Teachers of Mathematics, 2000). To develop these abilities, students need to have opportunities to engage in classroom activities that allow them to justify their own ideas and analyze others' ideas. They should be able to communicate and provide reasons for their ideas. As such, teachers need to provide their students with appropriate opportunities for making and criticizing mathematical arguments from primary grades. In addition, teachers themselves should develop their abilities of making and evaluating mathematical arguments.

In this presentation, we share elementary teachers' approaches to mathematical arguments while they were engaged in K-3 teacher professional development project activities during the 2000-2001 school year. During the project, 34 teachers had opportunities to define mathematical terms such as even numbers, rational numbers, what constitutes a straight line on a plane and on a sphere.

During the project, many different types of data were collected. For this report, however, we used transcriptions of class videotapes, field notes made of classroom observations, teachers' class notes (containing reflections and solutions to tasks), and telephone interviews conducted with eleven of the teachers. Our analysis not only clarifies how individual teachers reasoned about particular mathematical ideas, but also how their ideas where developed during discussions.

To analyze the data, we first condensed the data by coding, categorized emerging themes, and compared them. Finally, we looked for common themes. These steps, however, were not linear. We had to continue to revisit the original codes and earlier categories to see if there was a counter case and to look for the main themes related to the teachers' conceptions of mathematical arguments.

From data analysis, several common themes emerged among the teachers' approaches to mathematical arguments. First of all, for the teachers to be convinced, mathematical arguments needed to be something tangible or visible. They tended to reason using their everyday knowledge when they made and evaluated mathematical arguments. In such instances, they often discarded their previous claims, even when these claims had been made with full confidence. Second, not surprisingly, these teachers' knowledge of mathematics restricted their ways of developing and evaluating mathematical arguments. Their understanding of rational numbers and of mathematical terms, for instance, sometimes constrained how and when they developed mathematical arguments. As we will illustrate during the presentation, some teachers were not sure that $2/3 = .\overline{6}$ because they thought that $.\overline{6}$ did not fully represent $2/3$. In addition, they were most comfortable with relying on other external sources to confirm their claims. In such cases, the teachers asked the instructors to just confirm the "rightness" or "incorrectness" of their ideas. That is, their ideas, and more generally the evolving collective arguments did not provide sufficient convincing power.

On the other hand, by engaging in these activities, the teachers had opportunities to experience what constitutes a mathematical argument as well as to develop ways to evaluate claims and provide reasonable explanations. They did not just take mathematical arguments as given. They wanted to be convinced of why the arguments worked or not. Even though some formal proofs offered by the course instructors made the teachers frustrated and confused, the proofs certainly challenged them to look at mathematical arguments in a different way. They also realized the importance of using precise language when they described mathematical objects and situations. While defining terms and evaluating others' claims, they experienced that imprecise terms caused confusion and vagueness.

In conclusion, elementary teachers should have opportunities to engage in activities that allow them to explore and develop mathematical arguments and proofs. Such experiences can help teachers conceptualize or reconceptualize how they might guide classroom discussions that facilitate mathematical arguments. In so doing, their students may develop ways to participate in the argumentative process and ultimately, make "proofs" about certain claims.

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THE CHANGE OF TEACHERS' CONCERNS WHILE PARTICIPATING IN A SYSTEMIC PROFESSIONAL DEVELOPMENT PROJECT

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This study investigated the extent to which elementary teachers involved in a systemic change project changed their practice during the first year of implementing reform curriculum using a concerns-based scale. Professional development focused on meeting teachers' immediate needs while engaging them in activities that would help them modify their practice. Researchers rated teachers on the personal, managerial, and collaborative concerns that arose during the implementation process. Results showed that teachers' concerns changed significantly over the first year. An ANOVA test indicated the concerns of teachers at different grade levels did not differ significantly. By addressing the concerns that arose during the implementation of reform curriculum, teachers at all grade levels were able to reflect on their practice and experiment with new modes of teaching.

The *Principles and Standards for School Mathematics* (NCTM, 2000) describe teaching and learning mathematics as a process in which teachers engage students in mathematical explorations and discussions. There is growing recognition that without sustained professional development, teachers do not implement new content, pedagogy, or assessment practices in their classrooms . The Primary Mathematics Education Project (PRIME) is a collaborative change initiative between Illinois State University and a large midwestern school district which established a sustained development program for the district. This study investigated the extent to which elementary teachers involved in PRIME changed their practice during the first year of implementation to reflect a reform vision of mathematics instruction using a concerns-based scale s and whether the grade levels taught influenced the adaptation of reform practice.

Previous research suggested professional development sensitive to the concerns of teachers encouraged generative growth and formed the theoretical framework for this study (Brosnan, 1994; Confrey, 1990; Hewson & Thorley, 1989; van de Berg, 1999). Professional development was designed to support collaborative learning communities where teachers discussed new teaching strategies. It was hypothesized that teachers would begin the innovation with concerns centered on how the innovation affected them personally and over the first year their concerns would change from a need for knowledge to concerns about the impact of the innovation on students.

Professional Development

PRIME teachers followed daily routines typical of mathematics classes throughout the United States. The instructional goal for students was to solve problems quickly and accurately. Teachers assumed student understanding would evolve with repeated practice. PRIME's overriding goal was to help teachers develop a reflective practice as they implemented *Investigations in Number, Data, and Space* (TERC, 1997) to improve K-5 math instruction. To meet this goal, PRIME extended teachers' mathematical content and pedagogical knowledge through summer institutes and school-year seminars, which linked theory with practice. Through reflection, Teachers critically analyzed the process of teaching and learning. A major focus of the project was to improve the questioning skills so teachers would (1) attend to student thinking; (2) use student thinking as a basis for instructional decisions; and (3) engage students in discourse. These efforts were reinforced with on-site visits by PRIME mentors throughout the school year.

Method

Each semester, teachers participating in the PRIME Project scheduled three classroom observations with PRIME mentors. During each half-hour classroom observation, the classroom teacher and their mentor discussed the lesson's task and the use of questions to develop students' mathematical thinking. To appraise the change of teachers' practice during PRIME, the concerns-based adoption model (CBAM) (van de Berg, 1999) was utilized to document the concerns teachers held toward innovation during implementation. The CBAM describes three clusters of concerns with 7 stages: self concerns (stage 0-little concern, 1-need for information, 2-consequences for students); task concerns (stage3-management); and other concerns (stage 4-collaboration, 5-revision, 6-exploration of benefits). The mentors rated each teacher in October 2000, April 2001, and October 2001.

Results

A paired sample t test was conducted to evaluate whether teachers' concerns changed during the first school year of implementation. Using .05 level of significance, the results indicated the mean change (.60) during the first school year was statistically significant, $t\,(100) = 9.67, p = .001$. The mean change (.12) during the summer was not statistically significant, $t\,(100) = 1.01, p = .31$. During the first year of implementation, the mean change (.70) was statistically significant, $t\,(100) = 5.50, p = .001$.

A one-way analysis of variance was conducted to evaluate the relationship between teachers' grade levels and their implementation of reform practice. The independent variable was the grade level taught by the teacher and the dependant variable was the CBAM rating. The ANOVA test showed there was not a statistically significant difference between groups F (4,96) = 1.09, p = .37 for October 2000 or October 2001, $F\,(4, 96) = .41$, $p = .80$ and the relationship was moderately weak, $0^2 = .29$. While the ANOVA test indicated grade levels did not differ from each other for the time periods, review of the mean level of concerns for teachers at different grade level suggested trends of concerns for the grade levels might differ. A paired sample t test was conducted to determine whether grade levels concerns changed in similar patterns. K - 3 grade level teachers showed a significant change of concerns over the school year and fourth grade teachers showed significant change of concerns over the school year, during the summer, and during the first year of implementation.

Discussion

This study investigated the extent to which teachers changed their concerns during the first year of reform curriculum implementation. Initially teachers mean CBAM score was .66, indicating a limited awareness of reform teaching practice. A year later the mean score was 1.37 suggesting that teachers shifted their concerns from an awareness to interest in gaining more information and its impact on developing student thinking. Not surprisingly, the teachers showed the most growth during the school year when they could put into practice the ideas discussed during the seminars. Each semester, teachers were assigned a reform lesson to use with students for discussion at the school year seminars. These assignments and meeting with PRIME mentors supported teachers as they considered the consequences of the intervention on their personal teaching practice and student learning.

Based on the CBAM, ratings, teachers in the PRIME Project were struggling to make sense of how reform teaching affected them personally and its consequence on students. Webb, Heck, and Tate (1996), and Confrey (1990) described the process of teacher change to be slow, as teachers first must understand the nature of the change. As teachers reflected on the plausibility of the endorsed changes, they reassessed its usefulness and decided the degree to which they would adopt the changes. This study supports a description of the change process as a rocky road, to be traveled slowly and carefully negotiated. Hewson and Thorley and Pfaff (2000) found that critical reflection, dialogue, and collaboration between teachers was critical for change to become systemic. With continuous opportunities for critical reflection during the school year, teachers' concerns changed. Without these supports during the summer, teachers reverted to previous concerns. By addressing the concerns that arose during the implementation, teachers reflected on their practice and experimented with new modes of teaching.

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THE ROLE OF LISTENING IN ONE TEACHER'S ENVISIONING AND IMPLEMENTATION OF AN EMERGENT MATHEMATICS CURRICULUM

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Envisioning alternatives for education is a difficult and complex task often met with challenge from outside forces and influences and sometimes even hampered by our own inability to imagine that something else is possible, an inability to see beyond the current and traditional ideas about education. Dewey, a century ago, identified this as our inability to see beyond the oscillation between the extremes of "forcing the child from without, or leaving him entirely alone" (1902, p. 195). The purpose of this study was to examine the pedagogic practices of one middle school teacher who has "seen" an alternative. He has envisioned and enacted an emergent mathematics curriculum in his classroom with and for his students. As his story of transformation unfolded the important role of listening for this teacher became apparent. Listening to his students became an inherent and integral part of the process of his evolution as he initially tried to transition from traditional pedagogic practices to a problem centered learning approach to teaching mathematics. The results of this study reveal the importance of listening as this teacher developed as a constructivist teacher and now as he participates with his students in an emergent mathematics curriculum. Hermeneutical listening, as described by Davis (1997) occurs when the teacher becomes "a participant in the exploration of … mathematics" (p. 369). This teacher participates, listening from a hermeneutical perspective, in a flow of conversation with his student that is not predetermined or controlled by lesson plans or teacher perceived outcomes. This conversation is driven by his, and often the students', presentation of worthwhile mathematical tasks, problem centered inquiry, and hermeneutical listening. This poster will present a portion of data collected for my dissertation research which was a larger study focusing on two teachers and their enactment of an emergent mathematics curriculum. The role of listening as described above was one aspect of the middle school teacher's case that emerged. The larger study was designed as a qualitative case study employing a narrative inquiry approach and a constant comparative method for data analysis throughout the study. The data collected for this study included classroom observations and field notes, video-taping of class sessions, extensive audio-taped interviews with the participant teacher following each class session, and a variety of classroom documents (i.e., problem solving tasks, assessment, and class syllabus). Data was collected over the period of six months during the school year.

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Volume 4: Research Reports, Short Oral Reports, Poster Presentations

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History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implementation thereof.

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Preface

It is with great pleasure that we at the University of Georgia host the 24th annual meeting of PME-NA as we also hosted the 4th annual PME-NA meeting exactly 20 years ago in 1982. A quick glance at the differences between then and now shows how the field has grown and changed: The proceedings that year were about 250 pages long, and there were 34 papers presented in 7 topic areas. This year the proceedings will be nearly 2000 pages long with over 200 presentations in 15 topic areas. Despite the growth of the field and the organization, the hallmarks of collegiality and open intellectual exchange remain.

The theme of this year's conference is *Linking Research and Practice*. The theme is intended to highlight the interplay between the ways that research is used in practice and the ways that research grows out of practice. The invited plenary speakers were asked to address the theme in their areas of expertise by challenging the audience to think critically about the research we do–the questions we ask, the methods we use, the contexts in which we do research, the people with whom we do research, how we communicate the results of our research, etc. We hope that those in attendance as well as those who will read these papers in years to come will be stimulated to think deeply about our roles as researchers and consumers of research.

We received over 250 proposals for sessions at PME-NA and are grateful for the work of the many reviewers who helped shape the program. We undertook only structural editing (format and references) on the final papers so as to leave intact the integrity of the authors' work.

We wish to express our appreciation to the many people at the University of Georgia who have made these volumes and this conference a possibility, including, but not limited to, Patricia S. Wilson, Margaret Caufield, Elizabeth Platt, Salli Park, Bernice Peters, Teresa Banker, Nancy Williams, Joseph Allen, Brian Wynne, and all of the faculty and graduate students in the Department of Mathematics Education.

The Editors

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Rational Numbers

A LENGTH MODEL OF FRACTIONS PUTS MULTIPLICATION OF FRACTIONS IN THE LEARNING ZONE OF FIFTH GRADERS

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We developed and implemented a curricular unit concerning operations on fractions. The unit used drawings of lengths fractured into equal parts for all operations. Students initially had 2 interpretations for multiplying by a unit fraction (i.e., $1/n$), one that generalized and one that did not. The predominant method, dividing by equal shares, did not generalize when the denominator of the unit fraction did not divide into the whole number evenly. The second interpretation, which did generalize to a unit fraction or a non-unit fraction times any whole number, was taking the unit fraction of each "1" (or single unit) in the whole number and then finding the sum of these products. The class and teacher then extended this second method to multiplying a fraction times a fraction, relating via written and oral explanations the length drawings to a general written algorithm of multiplying the top numbers and multiplying the bottom numbers. Students outperformed samples of U.S. students using traditional textbooks and Japanese and Chinese fifth graders.

Purposes, Perspectives, and Theoretical Framework

Multiplication of fractions is on the Grade 5 curriculum in many states, but tests indicate that most U.S. students using traditional textbooks cannot solve such problems (Stigler, Lee, & Stephenson, 1990). Most East Asian students also do not solve such problems accurately (Stigler, Lee, & Stephenson, 1990). The question then arises whether this topic is developmentally inappropriate or whether the usual approaches to such teaching are not effective and fifth graders could learn with an effective approach.

A common visual representation of fractions used in instruction is a circular (e.g., pie, pizza) model of fractions (for discussion see, for example, Behr, Harel, Post, & Lesh, 1992; Clements & Del Campo, 1988; Lamon, 1999; Moss & Case, 1999). Pies and pizzas are within the experiences of most students, but both the real-world objects and the circular drawings of fractions frequently violate the central characteristic of fractions: fractions must have equal parts (equal shares). It is not easy to divide a circular model into equal parts except for 2, 4, and perhaps 8 parts. Examples of errors in circular drawings made by students at the beginning of the present study are given in Figure 1.

We instead propose a length model as a generalizable model that is easier to partition into an arbitrary number of equal parts and that will support all of the concepts and operations on fractions. Length models are also good for decimals and

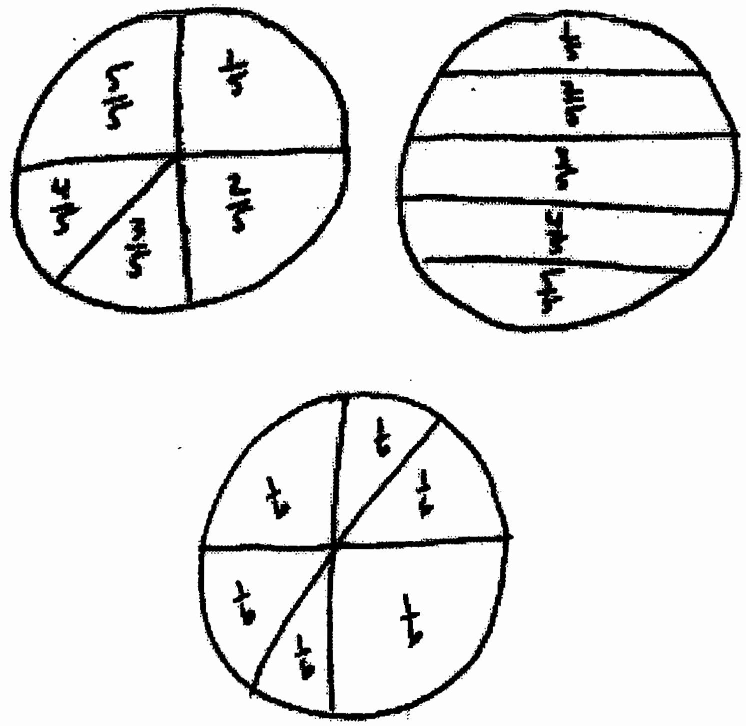

Figure 1. Unequal fractions in student circular models.

for metric measurement, so they would also enable fractions to be related to decimals and metric lengths. In this paper we focus on multiplication of fractions, and we examine whether a length model will pull multiplication of fractions into the learning zone of fifth graders.

Our theoretical framework uses both a Piagetian constructivist model of learning and a Vygotiskiian socio-cultural model of teaching. From our Piagetian perspective, we assume that students are continually interpreting their classroom experiences using their own conceptual structures as well as continually adapting their conceptual structures to their on-going classroom experiences. From our Vygotskiian perspective, we assume that a major goal of school mathematics teaching is to assist learners in coming to understand and use cultural mathematics tools. One means of assistance is drawings, which are semiotic tools that can support sense-making both individually and in the classroom discourse about mathematical thinking.

Our research question is whether our length models fit the learning zones of fifth graders well enough that length model drawings can be used to develop general methods of multiplying with drawings, which can be related to general numerical methods for multiplying fractions. This study is part of a larger project 1) examining the utility of the length model for helping students build understanding of all the concepts, situations, and operations involving fractions, 2) seeking to identify students' general learning paths when using a length model, and 3) understanding how this approach to fraction understanding relates to concepts in other multiplicative domains (e.g., functions and ratio and proportion).

Methods and Data Sources

Participants were 25 fifth-grade students in a classroom in a mid-western multi-racial small city with a considerable number of immigrants, a large minority of the students on free lunch, and a substantial number of students with highly-educated parents. This diversity was chosen to test the accessibility of our approach to a broad range of students.

The teacher, Ms. H., was an experienced teacher of reform mathematics who was recommended by a district-wise administrator for the quality of the mathematical discourse in her classroom. Working with such a teacher ensured that instruction would involve sense-making by all students and that alternative student methods using the length model drawings linked to numerical methods would be discussed. The students used the *Everyday Mathematics* curriculum as their regular mathematics program and were used to discussing their methods.

The present fraction unit was developed by the authors as an alternative approach to the one suggested in the *Everyday Mathematics* curriculum. In total, the fractions unit involved approximately an hour a day of class-time for a total of 15 days. Four days were spent on the multiplication of fractions. These days are described below in the results and conclusion.

The authors co-developed the initial version of the multiplication of fractions approach. The second author and Ms. H. met regularly to discuss the ideas in the approach, to address any concerns of either of the authors or of Ms. H., to make any necessary changes to adapt the unit to district goals and classroom idiosyncrasies, and to assess and adapt to on-going student progress. The second author, who is also an experienced elementary teacher, went to each class, videotaped each class, and contributed to the instruction and discussion when appropriate. The students understood her to be a co-teacher of the unit and someone with whom they could talk, of whom they could ask questions, and from whom they could seek help.

Data for analysis of the classroom instruction were videotapes, notes taken during and after class, and copies of the overheads made by Ms. H. and students during class discussions. Data for student learning were interviews of target students, students' work, and a test given at the end of the unit. This test was comprised of numerical items comparable to and word problems identical to those given by Stigler, Lee, and Stevenson (1990) to U.S. fifth graders using traditional textbooks and to Japanese and Taiwanese fifth graders. The test included items on most fraction concepts and was given 5 days after the multiplication teaching was ended and following one day in which all of the concepts were reviewed.

Results

On the first day, students' methods were elicited for simple multiplication problems in which a unit fraction would divide a whole number evenly (e.g., 1/3 x 6). Students had two different interpretations of such fraction times a whole number problems (see Figure 2). Some students used an equal sharing notion of multiplying by a unit fraction. They found "one-third of (the whole group of) 6" and so divided 6 into 3 equal groups of 2. Others found "six one-thirds" or "six one-thirds of one" and then accumulated these thirds numerically or on a length drawing to get six thirds, which totaled 2 (thus 6 was formed by two groups of 3 thirds).

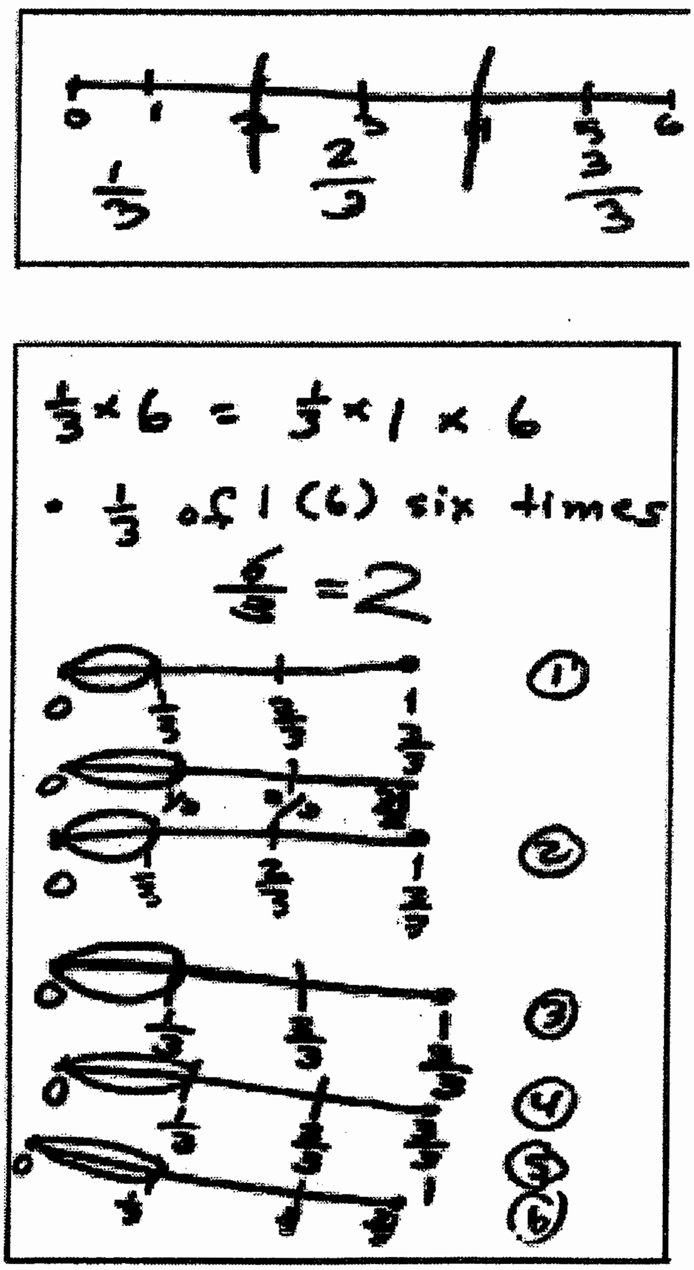

Figure 2. Student Length Models showing 2 interpretations of the Meaning of 1/3 x 6.

Instruction then moved to more difficult problems in which the fraction did not divide the whole number evenly. Students were unable to extend their own methods to these more difficult kinds of problems because most of their initial methods depended on the idea of a fraction as the equal sharing of a whole number. Students could not proceed if the fraction did not divide the whole number evenly. Using the length model, the piece-by-piece multiplying method that had arisen on Day 1 for 1/3 x 6 (see bottom of Figure 2) was extended first to a fraction times a whole number (Day 2) and then to a general fraction times a fraction case on Day 3. Each of these extensions was made in a whole-class co-constructing session led by the teacher with considerable input from students. Students then chose their own "fraction times a fraction" problem, drew a length model for their problem (see an example in Figure 3), and wrote an explanation of their step-by-step thinking.

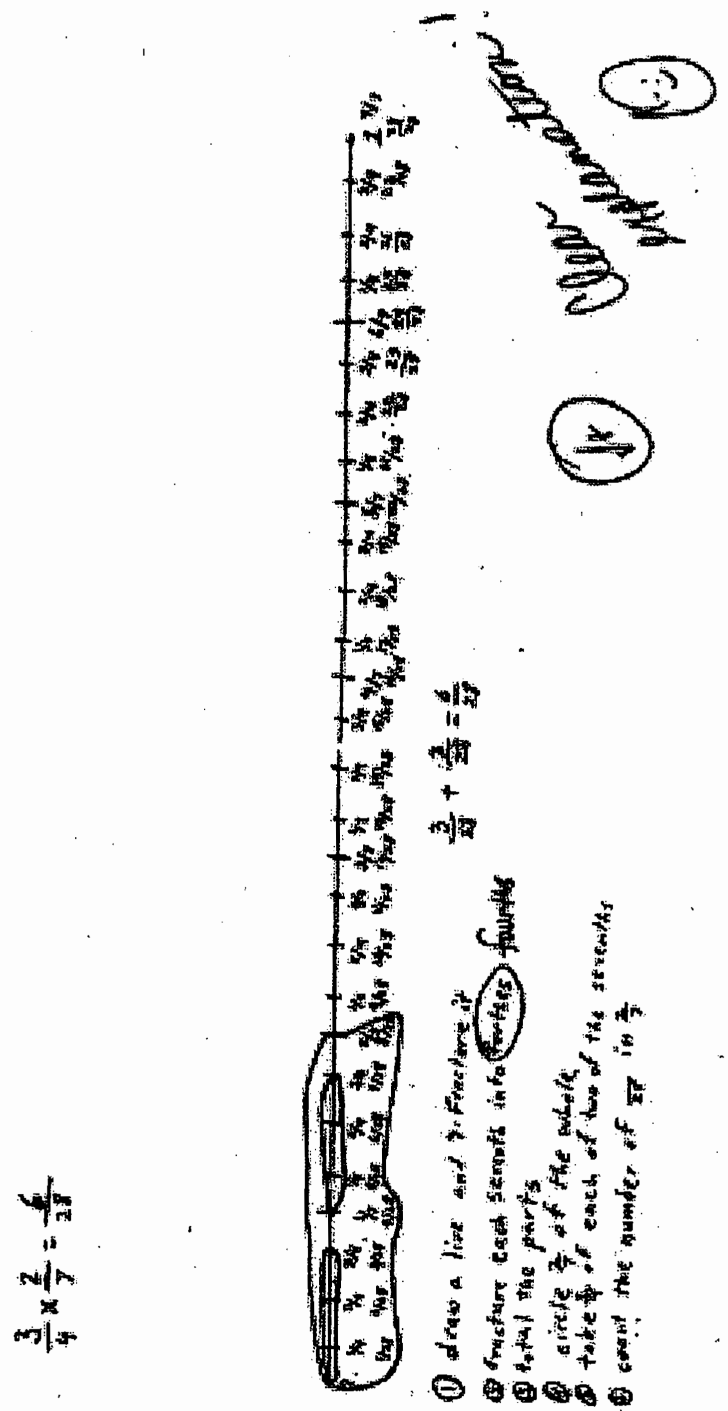

Figure 3. A student drawing and explanation of 3/4 x 2/7.

An example of a student explanation at the overhead projector of multiplying a fraction times a fractions is as follows:

JT: Here's my fraction, four-fifth times two thirds (writes 4/5 x 2/3 on a transparency). So I'm going to draw a line five segments long...(draws a horizontal line five inches long and places hash marks at each inch along the line). So now that I've labeled my fifths I'm gonna divided them each into thirds (places three evenly-spaced hash marks between the five-fracturing hash marks he drew previously). So I divided each fifth into, I divided it into thirds. [So now JT has done 1/5 x 1/3, the bottom part of the algorithm.] Now I have to circle four-fifths. That's the number of parts I have. Four-fifths is my fraction, so I circled that (retracing the large circle he made around four-fifths of the original line). Um...so now I have, I have to circle every two thirds (pause) because I have two-thirds times four-fifths. I circled the four-fifths, now I have to circle the two-thirds (pause). And you don't circle past the circled part (referring to the large circle around four-fifths). That's your extra. You don't need that. If you count every third, one, two, three (pointing to hash marks on his line) every, um, three thirds.

Ms. H: And you have five sets of three thirds? [seeking to get explicit naming and labelling of the common fracture: 1/5 x 1/3 = 1/15]

JT: Ya.

Ms. H: Which is a total of how many spaces?

JT: Fifteen.

Ms. H: Fifteen for your whole, right? For your whole unit.

JT: So, I totaled my whole and that was fifteen (writing "15" below his line). So now I already circled it [the two-thirds] so I have one, two, three, four, with two in each (pointing to circled sets of 2/3). Four times two equals eight (writing the equation vertically) and you have to multiply that because you labeled it two-thirds. You circled two-thirds for each one. You have four sets of two-thirds. [explaining the meaning for the top part of the multiplication algorithm] With two in each and that gives you a total of eight (writes 8 above the 15 he wrote earlier).

The fourth day focused on doing more examples and discussing why the algorithm of multiplying top numbers and multiplying bottom numbers worked. These explanations were related to the fracturing experiences students had when making the drawings, for example, "For the bottom, you fractured one fraction by the other fraction, so you get a fraction that is their product; for the top, you are taking the top number of groups of the size of the other top number, so again you multiply."

Over the 4 days in which students drew length models of fraction multiplication problems, almost all students were able to make such drawings correctly. Some required help along the way with various steps, and not all students could explain all of the steps as clearly as did the student in Figure 3 or at the overhead. However, all students were using the length models with most of the steps drawn correctly and linked to accurate numerical labeling.

The students did very well on the multiplication items given at the end of the unit (see Table 1). More than 80% of our length-model students solved the numerical fraction computation problems correctly, compared to means of 20%, 21%, and 14% in Japan, Taiwan, and the U.S., respectively. Our length-model students also did comparatively well on the word problems, with 63% correct answers compared to 54%, 49%, and 20%, respectively.

Comments by students during whole-class discussion and individually indicated that particularly powerful parts of the length model seemed to be its generality (students could choose their own fraction times a fraction problem and draw it with the length model), its affordance of seeing and writing multiplication as repeated addition (see the 3/28 + 3/28 in Figure 3), and the visual ease of the connections of the partitionings in the length model to the multiplications in the numerical algorithm.

Table 1. Percentage Correct on Multiplication Problems by Students in Japan, Taiwain, and the United States

	Background of Students			
Item	Japan	Taiwan	U.S. Traditional	U.S. Length Model
Whole Number X Fraction Fraction X Whole Number	14	8	2	85
Fraction X Fraction	25	33	26	88
Dad cut a cake into 16 pieces. George ate one fourth of them. How many pieces were left?	65	63	30	56
A stamp collecting club has 24 members. Five-sixths of the members collect only foreign stamps. How many members collect only foreign stamps?	43	35	9	69

Conclusions

Multiplying fractions by fractions is within the learning zone of fifth graders if a length model and a piece-by-piece partitioning method is used in a sense-making classroom environment in which drawings are referents for discussions and explanations of student thinking and the multiplying tops and bottoms algorithm is based on experience with these drawings. The piece-by-piece multiplying method works because one can gather the unit fractions into any mixed number (e.g., 1/3 x 7 instead of 1/3 x 6 is just one more 1/3, i.e., 2 1/3). The length drawings for a unit fraction times a whole number (e.g., 1/3 x 6) are the same as those for a unit fraction times a unit fraction (e.g., 1/3 x 1/6) except that the labelling is different (labelling 1 through 6 instead of 1/6 through 6/6). This similarity simplifies the transition from multiplication by a fraction x whole numbers to fraction x fractions.

A recurring issue throughout the unit is why does "of" mean "times"? Students had no trouble understanding "one-third of 6" as dividing 6 into 3 equal parts, but this experience then made them think that they were dividing, not multiplying. In fact, multiplying by a unit fraction is dividing by that whole number, but the operation *within fractions* is multiplication. Either multiplication grouping or comparing language can provide a basis for understanding why "of" means times for fractions as well as for whole numbers. Across different countries, people interpret 4 x 2 in two ways: as "4 sets of 2" or as "4 taken 2 times." Using both of these meanings relates "of" and "times" within the English language. So 4/5 x 2/3 can mean "4/5 of 2/3" or "4/5 taken 2/3 times." Within multiplicative comparing situations, English shifts from the "times as many" to " fractional parts of" language, again providing a relationship between these two. For example, we say, "Joe has 3 times as many as Mary has" but "Mary has 1/3 of Joe's amount."

Finally, we wish to highlight our view that 4 days is not sufficient for mastery of fraction multiplication. These days were part of a larger coherent approach to fractions using length drawings. Follow-up work would also be necessary to maintain the understanding built during this unit. We see three phases in building understandings in such a complex domain. First, the domain is approached using intuitive easy numbers in situations where objects or drawings can help students develop meanings. Second, these meanings are connected to generalizable numerical methods through discussion and linking to drawings. Third, a longer period follows of remembering and explaining in which occasional practice with numeric methods by students is accompanied by explaining why the method works. This phase is required to keep the meanings connected to the general numeric meanings. The first phase must be done with a view to the second and third phases. We have seen in this study how the use of easy intuitive numbers that divided easily led students to methods of multiplying by a unit fraction that did not generalize to numbers that were not evenly divisible. Curriculum development must keep all of these phases in mind from the beginning if students are not to be led to develop methods that will not generalize.

Note

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CONCEPTS OF RATIO AND PROPORTION IN BASIC LEVEL STUDENTS: CASE STUDY

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This research report shows two of the three cases which were used to conclude and evaluate the advances achieved within a broad qualitative field work. The study was conducted among eleven-year-old students in a public primary school of Mexico City. The main issue in this research is the enrichment of the conceptual contents in both, ratios and proportions, related with the strategies employed in problem solving. Said cases are analyzed starting from the recognition of the identifiable profiles in both subjects through the interpretation of the results obtained in a teaching program carried out with the group of students they belonged and with two questionnaires applied before and after the instruction process. The follow up of these cases was done in three interviews of didactic nature. These interviews made up the prime methodological study instruments, which are part of a doctoral study to be concluded soon.

Theoretical Background

For Piaget (1978) the notion of proportion always begins in a qualitative and logical fashion before its quantitative structure. In a similar way, Kieren (1988) highlights the transition form concrete to abstract, pointing out that the intuitive part is not totally left out. The aforementioned allowed to carry out the teaching proposal and the case studies here shown because it brought about a cognitive, global and evolutive point of view.

Karplus (1983) has been one of the first researchers to categorize the subjects´ answers in problem solving, not as indicators of a general development phase, but as illustrative of a proportion comprehension level linked to teaching. This author also establishes that the additive reasoning does not lie in the beginning of a non-variant sequence of proportion construction, but that education strongly influences it and represents the student´s effort to take on a task the best possible way there is instead of doing it systematically.

Hart (1988) describes different types of arithmetic problems through the recognition of several hierarchical levels of ratio and proportion comprehension. For this purpose, she resorts to children´s methods of problem solving as the predominant errors and the respective mathematics contents. Hart was able to detect that most students found it easy to solve mathematics problems which dealt with proportion. Even so, there is evidence which points out that younger and high school students who are not very successful have a sense of what "looks right" or what "seems to be a distortion". For Hart, some generalization levels such as ratio solving and the methods to create equivalence relations are brought about when the multiplication strategies are used systematically. The didactic program and the case study presented here makes emphasis on it.

The other researches taken into account such as Freudenthal (1983), Streefland (1984, 1991) and Coll (1990) had the main purpose of combining the didactic aspects and the mathematics reflection about the elemental school teaching of ratio and proportion. From this base the instructional program was designed. The mentioned program was applied in the doctoral study partially exposed here.

In this way, the students were encouraged to work with the idea that the multiplication operator was only a support resource for the generation of "internal and external ratios" (according to Freudenthal, 1983) in regards to the proportional variation problem solving due to the fact that, in order to obtain the missing quantities in a table, several pupils made multiplications or divisions using a "scale factor" or "factor-function", according to what Vergnaud (1991) stated.

Research Problem

Using the developmental, cognitive, mathematical and didactic approaches which were fully explained in the previous section, a constructivist focus is adopted. Thus focus grants a privilege to the highly active role exerted by the sixth grade students when constructing the meanings, notions and concepts of ratio and proportion in the field of arithmetic problem solving using different representation registers, specially drawings, tables and numbers.

Method

The study of the two cases shown in this work is part of the doctoral investigation near its formal conclusion. Through the follow up of some cases, the field work came to an end, with the purpose of conducting a deep assessment of the teaching within a school group (Ruiz, 2000, 2001; Ruiz & Valdemoros, 2001a, 2001b).

Methodological Instruments

The following instruments were employed for the study: an exploratory questionnaire, the constructivist-didactic teaching program, a final questionnaire and three interviews of "didactic-nature" (according to the bases given by Valdemoros, 1998).

Questionnaire

The "initial questionnaire" was later introduced as a "final questionnaire". What changed between the both of them was the aim of its application; the first showcased a clear explanatory nature and the second was more focused on evaluation. The resolution space between one and the other was of seven months and was subject to a qualitative analysis of the results in both applications.

It was made up by 13 tasks. In some problems, justifications supported on the qualitative interpretation were required, since it was interesting to see the way in which thought is constructed without the use of quantities related to ratio and proportion. For another problems, it was necessary to do a drawing against the grid with the sole purpose of achieving a transition towards quantification. The remaining tasks were explicitly quantified because some values were needed and some others were provided. For some of these tasks, a table was used as a way of representation for the recognition of internal or external ratios.

Teaching program

This proposal dovetailed with current scholar of Mexico City. It was made up by eight teaching models used in different sessions, taking into account the definition related to the "teaching model" given by Figueras, Filloy and Valdemoros (1987). The teaching proposal started form the qualitative side, so that the student would be able to find sense and meaning to the ratio and to the proportion, gradually making the transition from qualitative to quantitative. All these without setting aside the qualitative level of thought at the end of said process. The transition was obtained by the application of several teaching models in different instruction sessions.

In the quantitative field, the student established relations between magnitudes, worked with natural numbers and they also used fractional expressions, making an elemental incursion into the field of rational numbers. The student was able to define the ratio as the relation between two magnitudes and proportion as the equivalence relation between ratios. These coincides with the definitions provided by Hart (1988).

Didactic-nature interviews

Three girls from the sixth grade teaching study group were interviewed. They were picked based on their performance when taking questionnaire and during the teaching program. The interviews showed a natural and semi-structured character. Each one of the participants was subject to tasks of common design, similar to those included in the teaching proposal, in such a way that once finished, the results from all three cases were able to be compared.

The Snow White and the Seven Dwarves Model was used for the first interview (with newer tasks than in previous programs). Said model allowed to work with the issues regarding the student´s qualitative and quantitative thought in the field of proportion. They intended to see up to what point the students were able to keep consistency in both thoughts, as well as verifying the fluency they displayed in the transition form one representation register to another: the drawing, the table and the numerical, checking their conceptual and semantic skills.

The main purpose for the second interview was to double check if they used ratios in their problem solving, and if so, weather they chose internal or external and why. Verifying the scope that the children gave to the concepts of ratio and proportion was of utmost importance. This was reflected in the way they interpreted said terms and their notation registers.

The third interview allowed to dig deeper into the way the subjects establish an equivalence relation based on ratios determination. At the same time, the usage of the table was checked for its filling and for the relation between the table and the stated situation. There was an enrichment of the conceptual handling follow up, the basic technical language and the different representation registers added during the previous teaching process.

The didactic nature of the interviews was determined by the succession of two moments differentiated during the process: a) an initial program assessment phase in which the advance of each subject by his or her own means was determined and, b) a subsequent phase of constructivist-didactic nature , when the interviewer tried to encourage the interviewed to overcome the cognitive difficulties shown in his or her previous development, providing a feedback to it , but doing without solution proposals and without obstructing new searches by the student. The interview was a fundamental instrument in the development of this case study.

Subjects

The students involved in the doctoral research were 29 eleven-year-old children, attending sixth grade in primary school. We present two of the three case studies in this work: a) the case of Nuria, who showed characteristics common to a certain sector of the school group, identifiable by the ease with which they used any of the three modes of representation (drawings, table and numbers) when solving the problems, and b) the case of Wendy who showed a poor initial scene, but achieved tremendous breakthrough. This situation was also acknowledged by her classmates.

Result Analysis

We highlight the follow up of the representation modes adopted by both girls, the ease and difficulties related to the usage of internal and external ratios as well as the establishment of proportions. The contents that Nuria and Wendy gave to the concepts of ratio and proportion, the argument with which they supported the respective processes of problem solving.

The Case of Nuria

Before the teaching program Nuria did not know the concept of ratio, since it was not included in the school´s program; *after the didactic program*, she was able to give meaning and sense to the terms of ratio and proportion.. In the *questionnaires* and in the teaching program, Nuria used several strategies in order to solve them.

During the interviews and in regards to qualitative issues, she used verbal categories such as, "bigger than..." or "smaller than..." When dealing with tasks related to the transition from qualitative to quantitative, Nuria worked with number extension, once she detected the factors for the corresponding scale (x2), (x3) and for the cases of reduction she divided between two or three.

In those tasks demanding the filling of tables, she was able to do so multiplying by the natural operators which she determined by watching how the values increased in each column. To solve other tasks, in the quantitative field , first she obtained the unitary value to determine what she was asked. The language she used to express herself was somewhat technical sometimes, when using words such as "a half", "double" and "triple". She showed a good usage of concepts and an ease to employ symbols and terms loaded with meaning in different work sessions. This was reflected in the way she solved the tasks through the establishment of relations in both, the qualitative and quantitative fields. All this led her to determine ratios within different contexts, as well as the equivalence relation between them, to finally arrive with success to the solution of the stated situation.

She managed to use the three types of representations: the drawings, the table and the numbers, she could also make the transition form one to another. At no time did she leave out the qualitative, despite working with the quantitative, because for all her explanations she used ratios and proportions through the fractional notation and through the use of the table, but she also showed her intuitive and common sense ideas as a way of corroborating them. She could understand the concept of likeness as two figures which are proportional.

The Case of Wendy

The girl showed great advance *after the teaching sessions*. In the *interviews* it was plain to see that she overcame the idea of "focusing" in only one dimension of the plane figures shown to her (thing which was deeply rooted in her at first), being able to extend or to reduce both linear dimensions of said figures. She was able to give sense to the algorithms she used to solve the proportion tasks. She used ratios and proportions adequately, getting them from the table or from the text of the problems were stated in. She used natural or fractional operators to make the transition form one scale to another or from one value to another within the same scale. In regards to this, she showcased a clear command of halves and thirds but she acknowledged the existence of another fractional operators.

She used the table as a representation register since through it she was able to express ratios as fractions. This shows an important aspect of the great breakthrough achieved by Wendy, because at the beginning of the teaching program she had difficulty when extracting data from a table to give solution to the problems. She also gave signs of being unable to organize data provided in a table and to identify the ratios in it. After the didactic program, not only was Wendy able to compose ratios when interpreting and relating the magnitudes involved in the tables, but she was also able to express them with fluent fractional notations.

Conclusions

The progress achieved by Nuria in the interviews, as the crowning of a whole process, were the integration of the different representation systems, the broad usage of all sorts of multiplication operators, the link of figure measurement with the establishment of proportional ratios, the use of ratios as fractional expressions and the elemental construction of the concepts of ratio and proportion.

For Wendy, the teaching experience in particular, made easier for her the use of proportional variation tables, the full sense granted to the multiplication operators (natural as well as fractional) and the active establishment of the ratio and proportion elemental concepts.

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WHEN "THE SAME" IS THE SAME AS DIFFERENT DIFFERENCES: ALIYA RECONCILES HER PERCEPTUAL JUDGMENT OF PROPORTIONAL EQUIVALENCE WITH HER ADDITIVE COMPUTATION SKILLS

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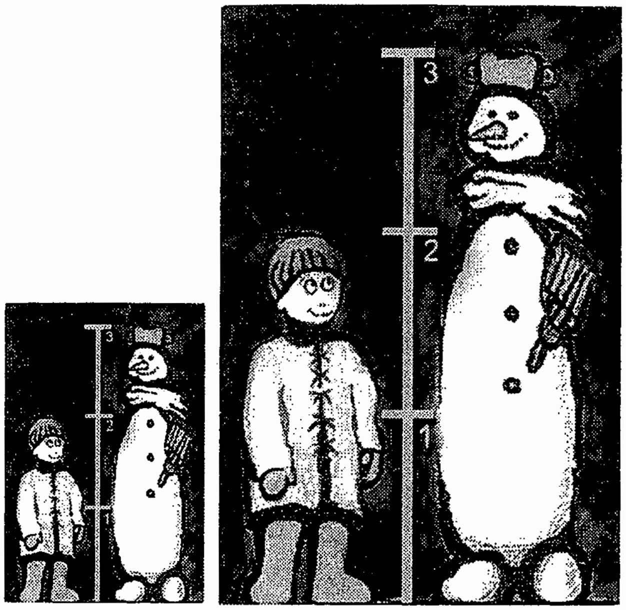
The paper introduces the "eye-trick", an optical illusion, and argues for its viability as a didactic means to mediate between young students' naturalistic perceptual judgments and mathematical descriptions of proportional equivalence classes (e.g., 2:3=4:6=6:9=8:12=...etc.).

Cognitive-psychology (Suzuki & Cavanagh, 1998), biological (Thinus-Blanc, 1988), and developmental (Piaget & Inhelder, 1946) studies all suggest a human capacity to perform *perceptual* judgments of proportional equivalence, e.g., between two geometrically similar rectangles. Such performance appears to rely on what Cobb and Steffe (1998, p. 55; see also Gelman, 1993) call "concepts in action, enactive concepts, rather than [on] abstract concepts embodying a structural relationship between...quantities", as evidenced in students' notoriously low achievement in *numerical* proportion problems (e.g., Kaput & West, 1994). By embracing students' domain-appropriate 'enactive' knowledge, we hope to create "instruction [that] is in harmony with [learners'] schemes" (Cobb & Steffe, 1998, p. 48), and may thus preempt "discontinuities between the child's procedures and the child's concepts" (p. 58; see also Vygotsky, 1978; and Freudenthal, 1981, on mathematization). Specifically in the domain of ratio and proportion, the "eye-trick", a perceptual illusion (see below), may afford students an opportunity for "logico-mathematical structuration that....*goes beyond perception*" (Piaget & Inhelder, 1969, p. 49, my italics).

The motivation of this work is our belief that ratio and proportion is an advantageous conceptual entry to rational numbers (e.g., see Confrey, 1998) because ratios do not require embedded numbers, as fractions do. Fractions are *parts-to-1-whole*, and thus present the perceptual-logical challenge of 'inclusion' (e.g., Singer & Resnick, 1992). The simpler visual physical instantiations of *whole-to-whole* ratios in geometrically similar shapes suggest a simpler approach. This work was done as part of our larger project to utilize the multiplication table as a source for teaching/learning rate, ratio, and proportion as coming from iterated addition (see also Abrahamson, 2002; Cobb & Steffe, 1998; see Abrahamson & Cigan, 2002, for an outline of our curricular unit).

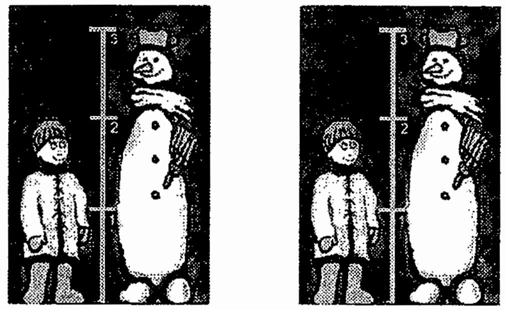
Method

The eye-trick involves two proportionate pictures (e.g., of heights 2cm&3cm and 4cm&6cm, Figure 1a). Children are asked to shut one eye to eliminate their stereo-



Card A Card B

Figure 1a. Cards A and B as seen laid flat on the table.



Card A Card B

Figure 1b. Cards A and B as seen through the eye-trick illusion.

scopic vision. Holding both pictures, they then move the larger picture away from the smaller picture (farther from their eye) until they find a point where the two pictures produce images of the same size (Figure 1b). This proportion is then examined numerically by attending to the embedded ruler in each image (2&3 units in both pictures), and through measurement, using a stretchable rubber ruler. The unstretched units of this ruler correspond to 2 and 3 units of height in Card A and to 4 and 6 units of height in Card B, but the stretched units of the ruler correspond to 2 & 3 units in Card B. The entire set of materials included a total of five cards per ratio set (e.g., 2:3, 4:6, 6:9, 8: 12, 10:15) as well as additional sets of cards (a 3:4 set and a 3:5 set) bearing different images of object pairs.

We have employed the eye-trick tasks both in whole-class design studies (Abrahamson & Fuson, in preparation), and in clinical interviews, of which Aliya's (8.5-year-old) interview was typical. I worked with and video-taped Aliya over three 1-hour periods spanning 15 days.

Results

Aliya (a) *saw* that two cards of different size appeared "the same"; (b) measured these cards with the stretchable ruler and tabulated these data (2&3, 4&6, in Figure 2a); (c) claimed these data were mathematically nonsensical since 3-2=1 but 6-4=2 (the differences are different); (d) sought an alternative numerical pattern to explain what she saw, wondering aloud whether the differences of 1 and 2 units, respectively, could possibly signify a trend of 1, 2, 3, etc., which would predict a difference of 3 in an additional card; (e) explored and verified her hypothesis by using a card in which the relevant dimensions were 6cm:9cm (6 & 9, Figure 2a); (f) compared these data to a case of head-start equal-rate growth (Figure 2b, Bob and Joe were born exactly one year apart); (g) discussed the viability of each table as a mathematical descriptor of some real-world class of situations; (h) practiced using her hands to simulate and differentiate equal-rate and different-rate growths: starting from holding her hands 2 and 3 "units" above the table, respectively, she raised her hands whilst either maintaining a fixed difference between them or by gradually increasing the difference; (i) re-interpreted the proportion table as modeling "unit-splitting", e.g., 3 "becomes" 6 because each 1-unit became 2 smaller units but the visible total remained the same size (Fig. 3, compare to 2* 3 as 3+3 where the total visibly doubles in size); (j) came to accept proportional equivalence as the numerical phenomenon corresponding to the stretch/ shrink or "change unit" classes of real-world situations.

Ratio Change	
Danny	Snowman
2	3
4	6
6	9

Figure 2a. Tabulated measurements.

Head-Start Equal Rate	
Bob	Joe
2	3
4	5
6	7

Figure 2b. Sibling ages.

Conclusions

The eye-trick provides a powerful sensory support for understanding proportional equivalence. This visual support was successful in overriding the well-documented "additive frame", by which 2:3 cannot equal 4:6 because 3 is 1 more than 2 but 6 is 2 more than 4. It enabled Aliya to build an additive-multiplicative frame for proportion situations, initially as additive increasing-difference situations (within the ratio-table rows), and then as a multiplicative interpretation of unit splitting within both the eye-trick pictures and the ratio table (between its rows).

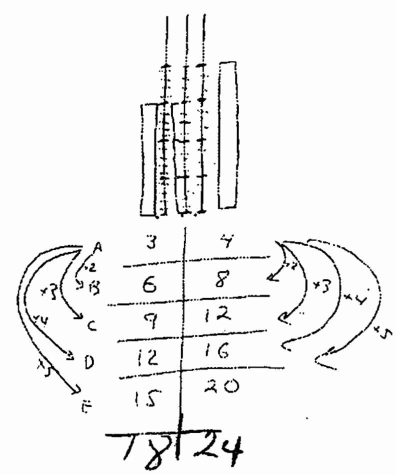


Figure 3. Explaining proportional equivalence as coming from unit-splitting interpreted multiplicatively.

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UNITIZING: THINKING IN PACKS AND PIECES

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The purpose of this report is to examine how fifth grade students in whole class teaching experiment conceptualized and symbolized a problem containing composite units. The analysis presented in this paper is significant because it sheds light on students' unitizing and reunitizing processes as they solve a fair sharing problem containing composite units. The findings from this study indicates the importance of having students symbolize and explain their thinking as they work with composite units through pictorial representations. It forces students to explicitly think about the shifting nature of the unit as they decompose the original unit and reunitize to solve fair sharing problems.

The ability to form reference-units based on the problem context and the capability to flexibly decompose the unit and convert it into different forms while problem solving is important for working with rational numbers. This process is called unitizing (Lamon 1993, 1999). It is a cognitive process for conceptualizing the amount of given commodity or share, before, during and after the sharing process (Lamon, 1996). A unit can signify one continuous object, a set of discrete objects or a unit of measure (Behr, Lesh, Post, & Silver,1993). Once a reference unit has been determined based on the problem context, it can be decomposed into smaller chunks through partitioning and recomposed while problem solving. Students have more difficulty reuniting the decomposed pieces into different size chucks as opposed to being able to decompose the whole into smaller parts, which appears to be a more natural process (Lamon, 1996).

Partitioning involves separating an entity into equal parts. The numerator is the quantity to be divided by a particular number of parts, resulting in the value of one part. In real life situations, continuous units are cut into parts where discrete units are sorted. This is a multi stage operation that involves marking objects, cutting them and clearly indicating one person's share (Lamon, 1996). Lamon (1996) points out that students have to use more sophisticated unitizing processes when working with composite units. She claims that it is important to understand children's models of unitizing and partitioning and how they develop so that instructional decisions can serve to facilitate students' construction of increasingly abstract ideas. To this end, we provide an example of how a fifth grade student in a whole class teaching experiment solved a fair sharing problem containing a composite unit. We point out the importance of having the student symbolize their solutions while representing the problem through pictorial representations. Having students symbolize and explain their thinking forces students to think explicitly about what they are unitizing and reunitizing. The analysis presented in this paper is part of a larger whole class fifth grade teaching experiment on children's development of the quotient construct.

Method

A five-week whole class teaching experiment was conducted in a fifth grade classroom in an urban school district in the Southwestern United States. Nineteen students and the classroom teacher participated in the study. Two cameras video recorded the teaching episodes. One focused on the teacher and the other on a focus group of 4 students. All students written work in math journals were documented along with the whole class inscriptions. Student work and whole class inscriptions were coded and analyzed. The videotapes were transcribed. The classroom discourse was analyzed in relation to the activity of creating and translating inscriptions to make meaning of the quotient construct. The emerging hypothetical learning trajectory of the whole class was documented. A students' solution to a problem from the teaching experiment is used in this report to illustrate the unitizing process involved in solving a fair sharing problem containing a composite unit. The students were asked to solve the problem of 5 children equally sharing 6 packs of gum with 5 pieces in each pack.

Results

Students conceptualized the unit of 6 packs of gum with 5 pieces in each pack as either packs, pieces or both as they solved the fair sharing problem containing a composite unit. Students were asked to symbolize their thinking along with their pictorial representations. Students symbolized their actions of partitioning through pictorial representations as opposed to symbolizing the number sentences. Even though students were able to figure out the correct answer and justify their reasoning, most students did not write number sentences to represent the original problem and solution. The difficulty in doing this was due to the fact that students had to explicitly think about their unitizing and reunitizing processes related to the problem context as represented in relation to their answers. For example, Karen conceptualized the problem as packs and pieces. She drew a 5 x 6 matrix to represent the 30 sticks of gum (see figure 1). Each column represented a pack of gum containing 5 sticks of gum and the symbol inside the matrix represented pieces of gum in each corresponding pack. Therefore, she needed the visual aid of the discrete pieces to keep track of the packs.

She wrote the fraction 5/5 to represent 5 pieces of gum shared by 5 kids. This fraction represented how one pack was split among 5 kids. She treated the pack as 5 discrete pieces. Next, she indicated that 5 (packs)÷5 (children)=1 pack of gum. In doing so, she conceptualized the 5 packs as divided among 5 kids. Each pack was treated as a composite unit. She checked her answer using the inverse operation to check her arithmetic that 5 x 1=5 and she indicated that the resulting 5 represented 5 pieces of gum. Karen had confused herself during the process of reunitzing by shifting her thinking from packs of gum back to pieces of gum. However, later on she caught

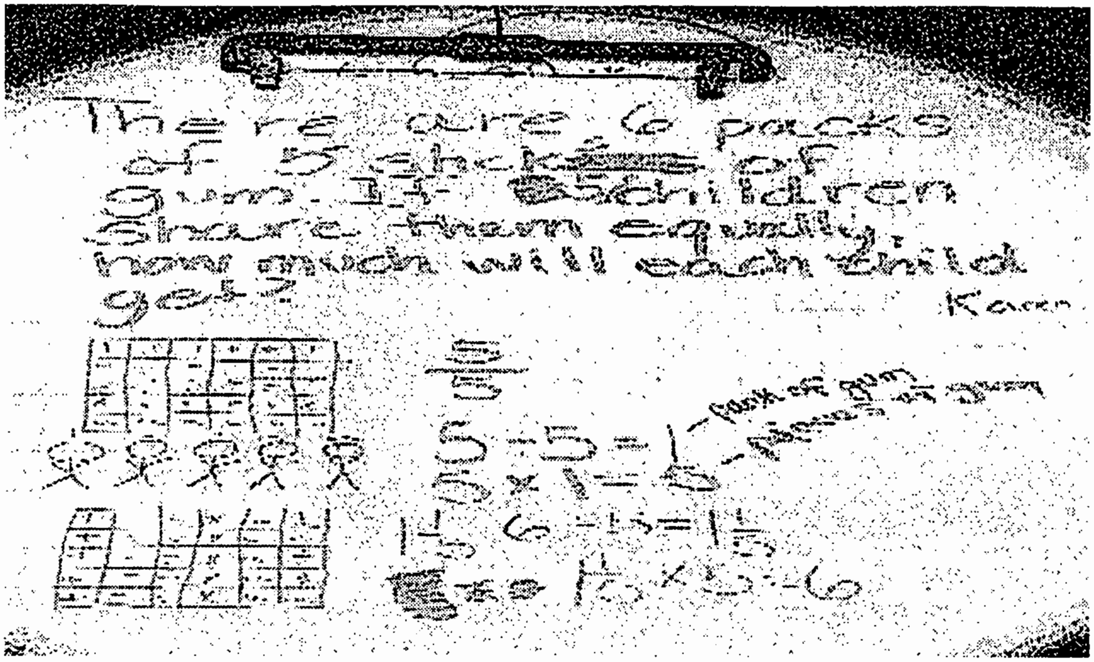

Figure1. Karen's Method.

her error and re-explained her thinking as indicated by the matrix listed at the bottom of figure one. She concluded that each kid would get 1 1/5 packs of gum. She treated each column as representing one pack and each piece as representing on fifth of the pack. She used the Mark-all strategy as defined by Lamon (1996). Karen concluded that each child would receive one column of 5 pieces and 1 piece that represents 1 1V5 of the pack. She was thinking in packs, but used the visual support to simultaneously think about the pieces.

Discussion

Students should have the ability to conceptualize a composite unit in a fair sharing problem situation in flexible ways such as packs and pieces in order to successfully solve the problem. The student did not have much difficulty partitioning the unit into smaller pieces, however she did get confused as she reunitized the problem into packs and pieces. This is consistent with Lamon (1996) findings that the students have easier time decomposing the units as opposed to reunitizing. Furthermore, Lamon (1996) had indicated the importance of having perceptual support specially when students use a mark all strategy as illustrated in the example provided in this paper. Having students symbolize their thinking and providing explanations of their thinking of their symbolization and visual representations appear to help students think explicitly about unitization and reunitization processes while problems solving. Furthermore, having students gain initial experiences with discrete and continuous problems prior to tackling problems that involve nested composite units might be helpful.

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A COMPARISON OF CONTENT TREATMENT FOR THE TEACHING OF PERCENT IN SELECTED AMERICAN AND CHINESE MATHEMATICS TEXTBOOKS

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This study examined selected American and Chinese mathematics textbooks in terms of their ways of presenting and organizing content for the teaching and learning of percent. The results provide a glimpse of different metaphors of mathematics learning that have been employed in American and Chinese textbooks. In particular, much more differences than similarities are found between the selected textbooks in what students need to learn and how to facilitate their learning. Further studies are needed in near future to include more textbooks and more content topics.

Efforts to identify contributing factors for cross-national differences in students' mathematics achievement have led to the contention that the curriculum is one of the key factors (Lapointe, Mead, Askew, 1992; McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers et al., 1987). Large-scale international comparisons of students' mathematical achievement have provided broad measures of the differences and similarities in mathematics curricula but not a fine-grained analysis of mathematics textbooks. Mayer, Sims, & Tajika (1995) revealed the pivotal role of textbooks for teaching mathematics in their cross-national study on mathematical content presentation in texts. Other cross-national studies on mathematics textbooks (e.g., Fuson, Stigler, & Bartsch, 1988; Li, 2000; Stigler, Fuson, Ham, Kim, 1986) suggested the importance of understanding textbooks' influence on the teaching and learning of mathematics from different perspectives. In particular, studies on the selection of content topics and content treatment in textbooks (e.g., Li, 1999) show the promise of revealing different approaches that textbooks take for the teaching of mathematics. Along this line of curriculum inquiries on content, this study compared selected American and Chinese textbooks on the teaching of percent.

Methods

One American mathematics textbook series and one Chinese mathematics textbook series were examined to locate the content instruction of percent in elementary school. The Chinese math textbook series (People's Publishing House, 2000) bears the approval of Chinese Ministry of Education and is the most popular one in China. The American textbook series is "Progress in Mathematics" (McDonnell, Burrows, Murphy, LeTourneau, Geschke, & Kelly, 2000). This selected American textbook series is, by no means, a representative one in the United States. The selection of this series, only as the first step, tries to illuminate possible similarities and differences between American and Chinese textbooks.

The two textbook series were examined first to locate the grade level(s) when the topic of percent is introduced. Textbook treatment of percent and other content topics in the textbooks was described. Further comparisons were then given to specify how the content topic of percent was introduced. Textbooks' uses of different representations, worked-out examples and exercise problems were also examined to show their approach for the teaching of percent.

Results

Textbook Placement of Percent Topic

In both American and Chinese textbook series, the content of percent is first being introduced at the fifth grade. The American textbooks take a spiral approach and introduce the topic of percent again at the sixth grade. In contrast, the Chinese textbook takes a sequential approach to introduce the topic of percent only at the fifth grade. Although both American and Chinese textbooks place the topic of percent after the content of fraction, they differ in dealing the topic of percent with ratio and proportion. In particular, Chinese textbook introduces the content of percent before ratio and proportion whereas American textbooks combine the introduction of ratio and proportion with percent as one chapter at both fifth and sixth grade levels.

The Introduction of Percent Concept

The concept of percent is defined as "A fraction with a denominator of 100 can be written as a percent. Percent means the part of each hundred. The symbol for percent is %." in the American fifth-grade textbook, as "A percent is a ratio that compares a number to 100." in its sixth-grade textbook. In contrast, the Chinese textbook introduces the concept of percent based on the concept of fraction. It differs from the American textbook by defining the percent as "the number that represents one number as how many parts of 100 parts of another number. It can also be called percentage."

Specifically, American textbooks present the definition of percent directly and then ask students to work with 10x10 grids. For example, students are asked to shade 32 squares in a 10x10 grid or to figure out what percent of squares is shaded. In contrast, Chinese textbook presents a problem situation first: There are 17 students being evaluated as excellent in 100 fifth graders, and 30 in 200 fourth graders in an elementary school. Which grade has a higher rate of excellent students? To solve this problem, the textbook uses the concept of fraction equivalence, and leads to the definition of percent. Mathematically, the differences in their introduction of percent show that the American textbook defines the concept of percent as a fraction or ratio, whereas Chinese textbook defines the concept of percent based on the concepts of fraction and fraction equivalence.

The Relationships Between Percent and Fraction, Decimal

In American texts, the relationships between percent and fraction and decimal are introduced in the fifth-grade text first, and then repeated in the sixth-grade text. Similar problem contexts and 10x10 grid are used in the texts of both grade levels. The general rules for converting percent and fraction, percent and decimal are also introduced at both grade levels. Similar to the American text, the Chinese text also introduces the conversion between percent and fraction, percent and decimal. However, the general rules of conversion are not given first. Students are prompted to generalize from several examples being given. Moreover, the conversions made in the examples are not explained with 10x10 grid, but based on the definition of percent. For example, 0.123 = 123/1000 = 12.3/100 = 12.3%.

The Application of Percent

The application of percent is included in both American and Chinese textbooks, but shows quite a big difference. The American textbook presents very limited application of percent, but places much more emphasis in introducing problem-solving strategies. In contrast, the Chinese textbook places heavy emphasis on the application of percent. It includes examples of percent application in all three different cases: (1) finding the percent when the whole and part are known, (2) finding the value of whole when the percent and part are known, (3) finding the value of part when the percent and whole are known. The examples given for each case are also varied, if possible, after solution. Practice problems contain many more variations in problem context. Other applications such as "interest", "chen (a Chinese way of using 10%)", and "discount" are also introduced with examples and practice problems.

Discussions

The study shows much differences than similarities between the selected American and Chinese mathematics textbooks on their treatment of percent. Taking together, these differences may suggest different metaphors of mathematics learning that have been employed in American and Chinese textbooks to organize and present mathematics content. Specifically, the selected American textbooks try to decrease the difficulty of math content by using spiral organization of content, direct introduction of content being taught, use of visual representations to facilitate students' learning, and include examples and exercise problems that are not mathematically challenging. In contrast, the selected Chinese textbook tries to maintain the content coherence and challenge by using a sequential organization of content, expecting students' involvement in learning from examples, emphasizing logical reasoning with concepts and problem analysis, and include examples and exercise problems that are mathematically intriguing. The differences, in general, indicate that the selected American and Chinese textbooks were developed with different assumptions about what students should learn and how to facilitate their learning. Because of the limitation of textbook selection, however, further efforts will be needed in near future to include and analyze more textbooks.

Note

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Reasoning and Proof

EXPLORING SOCIOCULTURAL ASPECTS OF UNDERGRADUATE STUDENTS' TRANSITION TO MATHEMATICAL PROOF

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This paper explores how undergraduate students come to understand mathematical proof and how classroom norms of argumentation evolved in support of this. We present here data from a discrete mathematics course to document the change in classroom norms over the course of one semester, as well as data illustrating how students' capacity for proof evolved in this context. In particular, we show how the instructor of the course established an expectation for explanation and justification, and how students' interactions developed in accordance to this through the semester. That is, we trace students' development from the passive acceptance of the instructor's authority to the expectation that students become contributors to the class and that they share common understandings. Finally, we explore linkages between this social aspect of development and the emergence of students' ability to reason deductively.

Perspective

Arguments such as "proof is not a thing separable from mathematics....[but] is an essential component of doing, communicating, and recording mathematics" (Schoenfeld, 1994, p. 76) reinforce the centrality of proof in mathematical thinking. As such, one might expect proof activity to be embedded in curricula at all levels. However, mathematical proof has been disproportionately represented in school mathematics (Wu, 1996), and only a small percentage of students understand it (e.g., Chazan, 1993; Senk, 1985; Usiskin, 1987; Williams, 1980).

In response to this, reform efforts are calling for an increased focus on the learning of mathematical proof, especially at the college level (American Mathematical Association of Two-Year Colleges, 1995; Kaput & Dubinsky, 1993). The fact that students find proofs hard to understand and to write (Balacheff, 1988; Bell, 1976; Chazan, 1993; Porteous, 1986; Senk, 1985; Usiskin, 1987) stems from a variety of sources, including difficulty with the problem solving skills necessary to construct mathematical arguments (Schoenfeld, 1985), difficulty in understanding the mathematical language and symbolism associated with proof (Laborde, 1990; Leron, 1985), and difficulty with the concepts about which a proof is to be made (Tall & Vinner, 1981). In addressing these issues, recent studies have focused on cognitive issues in learning about proof, particularly the development of students' proof schemes as their capacity for justification progresses toward generality (Harel & Sowder, 1998). However, what is still needed is an understanding of the role of the social dimension in how students, particularly in undergraduate mathematics, come to know and do mathematical proof.

We situate an emphasis on the social character of proof within the broader theoretical perspective that development cannot be understood apart form the *social* context in which it occurs (Vygotsky 1934/1978, 1934/1986). Vygotsky maintained that one's development of self-regulatory thinking occurs through a process of internalizing events that originate on the social plane and involves the transformation of public discourse into private speech. This perspective underscores the importance of social factors when accounting for students' mathematical growth.

Our purpose in this study was to explore the role of the social in how undergraduate mathematics students came to know and do mathematical proof. In particular, operating from the premise that learning is a consequence of one's internalization of interactions that originate on the social plane, we considered the following questions:

(1) How do classroom norms evolve in an undergraduate mathematics course focusing on mathematical proof?

(2) How does mathematical proof and the language of proof come to be internalized by students in the context of existing classroom norms?

Method

Participants for the study were a group of 50 undergraduate mathematics students enrolled in a two-semester, first-year discrete mathematics course emphasizing mathematical argumentation and proof. For this study, we focused primarily on the first semester of the course. The course was taught by one of the two investigators, while the other investigator collected data and provided technical assistance. Each class was videotaped and coding focused on the forms of explanation and justification used by students. Additionally, students were presented with a pre-, mid-, and post-assessment, all of which were analyzed to identify the generality, form and competency of students' arguments. That is, we examined transitions in students' written arguments for increasingly generalized and more formalized expression of those arguments, which we took as evidence for shifts in students' capacity for self-regulatory thinking.

Results

While it is difficult to establish a direct link between a succession of classroom conversations and, for example, students ability to construct written arguments for proofs, we do claim that the norms established in this classroom supported the development of argumentation required for constructing mathematical proofs. We take shifts in student responses to one proof problem given on three occasions as part of the evidence for this claim.

A typical class began with a problem introduced by the instructor followed by student group work. Students were encouraged to ask each other questions and help each other clarify concepts and problem requirements. The small group work usually alternated with whole class discussion of students' approaches, thinking and questions. Throughout the course there was a concerted focus on both written and verbal expression of student thinking. The instructor worked to establish the expectation that students explain their reasoning, try to make sense of each other's explanations, and challenge each other's reasoning and justifications. Our analysis shows that there were shifts in how students came to operate within classroom norms. As the following episode illustrates, students were initially hesitant to participate in discourse and to challenge the thinking of their peers or the instructor, and even lacked the language to do so. The class was introduced to combinations and permutations and students were asked to find the number of different combinations of pastries one can purchase from a bakery. The instructor prompted students to collaborate with a partner and to "think about their thinking" and to question each other's approaches and arguments, thus implying an expectation that students would share their reasoning. The interaction, which occurred during the third week of the semester, took place between Isabelle and Josh (pseudonyms):

1. Isabelle: What did you do?
2. Josh: You multiply them all out and you get 10×9×8×.....
3. Isabelle: Oh, OK.

The level of discussion described in this short episode illustrates the discussions that took place among almost all groups. Although students were asked to collaborate in solving the problem, making sure to question each other's thinking, they instead tended to ask each other (or the instructor) for a procedure for solving the problem and accepted each other's solutions without further questioning. In the few cases where a student asked another student for further clarification or explanation for his answer or approach, the response often was "it worked for me!"At this point, it was not the norm for students to challenge each other's thinking. In fact, it seemed that another cultural norm was still at work, namely, that the goal in solving a problem was to get a solution (as quickly as possible) and that, if one would merely *describe* the procedure, then analyzing the process would be unnecessary for understanding. That is, students would be able to simply replicate their own or their peers' procedures in solving other problems. As students came to learn, however, the nature of the content was such that no single procedure worked for a broad class of problems (in the way that, for instance, techniques of differentiation allow students to solve large volumes of similar problems). Thus, it was imperative that they begin to question the process of one's thinking in order to tease apart 'techniques'. This was further supported by the instructor's routine of allowing students to work with partners in lengthy exchanges (for example, it was not uncommon to allow students 10 to 15 minutes to begin to sketch out a proof), or by the instructor removing herself from the conversational space during whole-class discussion, thereby requiring students to build cases whose validity was argued exclusively by students. Moreover, the instructor worked to make her own process of argumentation explicit for students and to focus their thinking on anticipating the steps in an argument while keeping track of how given information could be used.

As the semester progressed, we observed that the *quality* in students' oral and written explanations and their capacity to express their mathematical thinking in increasingly formalized ways changed substantially. Students' arguments gradually shifted from empirical and procedural to deductive and conceptual. Moreover, students grew accustomed to engaging in explanations and justifications and seemed to internalize a more formal language of proof reflecting that used by the instructor during regular instruction. We take the following, which illustrates how students came to argue the simple conjecture that "the sum of an even and odd number is always odd" (Odd/Even Conjecture), as part of the evidence of our claim.

The Odd/Even Conjecture was administered as part of a larger assessment on proofs on the first day of class (before any instruction occurred), 9 weeks later, and at the final class of the semester. We note that it was administered in partner pairs for the mid-test, and while this no doubt contributed to the significant improvement, the individual post-test indicates that students had reached a comparable (slightly better) individual level of performance by the semester's end for this particular problem. Of the proofs assigned throughout the semester, this was one of the simpler problems. To interpret the results, we include here a possible solution to the conjecture.

> *Odd/Even Conjecture: What happens when you add an even and odd number? Prove that your conjecture will always hold.*
>
> *Possible solution: Let x be even and y be odd. They $x = 2m$ and $y = 2n+1$, for integers m and n. Then $x + y = 2m + 2n + 1 = 2(m+n) + 1 = 2k + 1$, where $k=m+n$ is an integer. But $2k+1$ is odd, by definition, so $x + y$ is odd. Thus, the sum of an even number and an odd number is always odd.*

One student (Daniel – pseudonym) wrote the following on his pre-test, a response that was fairly representative of the range of responses:

> *The result will always be odd. $1+2=3$. $x+x+1$ is odd.*

He then tried to solve the equations 2x+1=3 and 2x+1=4. His comments about his 'proof' were

> *[I] can see intuitively that odd + even = odd. But how to prove, I don't know what to say, generically. If you solve after setting 2x+1 = [sic] to an odd number you get a whole number, [sic] to an even number you get a fraction. Hmm... Obviously need some generic method to prove. Oh well.*

At this point, Daniel does not have a formalized way to think about arbitrary even or odd numbers (i.e., 2x + 1 for some integer x), although in the expression 'x+x+1' he does recognize that consecutive even and odd numbers are separated by a unit. Although he records this information, he does not use it to build an argument. The class proved a set of problems similar to the Odd/Even Conjecture, although they solved this particular one only on the pre-, mid-, and post-tests. Daniel's response on the post-test was much different:

> *Let n, m be integers. 2n is even. 2m+1 is odd. (2•any integer is divisible by 2 and hence even. Add 1 and it is odd.) 2n + 2m + 1 = 2(n+m) + 1 = 2p + 1, where p is an integer. 2p + 1 is odd, so even + odd is odd.*

In this response, Daniel identifies arbitrary even and odd integers and uses this definition to symbolically argue that the sum of an even and odd is odd. While this sequence of maneuvers is not complex, he initially did not have the language (even everyday language) to build this type of argument. Following are the class results on the Odd/Even Conjecture.

These results clearly indicate significant gains in students' responses. Only one person (2%) gave either a correct or essentially correct proof on the first attempt, while 92% of the class gave correct (67%) or essentially correct (25%) proofs on the second

Table 1. Summary of Student Responses to the Odd/Even Conjecture

	Pre-Test (individual) (50 responses)	**Mid-Test (paired)** (51 responses)	**Post-Test (individual)** (33 responses)
Correct proofs			
completely correct	2%	67%	67%
almost correct (minor error)	---	25%	27%
Total	2%	92%	94%
Incorrect proofs:			
Used examples as a "proof"	52%	4%	0%
Used illogical reasoning	20%	4%	6%
Looked at a narrow case	20%	4%	0%
No attempt made	14%	0%	0%
Total incorrect proofs	98%	8%	6%

attempt (paired mid-test), and 94% of the class gave either a correct or essentially correct proof on the final attempt. Students whose response was scored 'almost correct' had expressed a more limited argument by using the same variable in the generalized expressions of an arbitrary even and odd number (i.e., 2x and 2x + 1, rather than 2x and 2y + 1). In addition, 52% of students on the first attempt used examples as their method to 'prove' the conjecture, while only 4% of students used this as a strategy on the second attempt, and no students used this as a strategy on the final attempt. Moreover, there was a significant increase in students' level of formalization, particularly, their capacity to express their thinking in increasingly formal ways via symbolic language. Only 16% of respondents on the pre-test used some form of symbolization, whether correctly or incorrectly (otherwise, if students attempted a proof, they used everyday language). Meanwhile, on 94% of the mid-tests and 97% of the post-tests students expressed their proof or proof attempts symbolically in a manner similar to the possible solution given here. We take this as evidence that students were beginning to internalize a more formal or symbolic language of proof used in instruction (that is not to say that they did not at times use symbols in illogical ways).

Students' also shifted in their conception as to what counts as a strong mathematical argument. For example, during the first weeks of class the instructor's request for explanation or proof often resulted in procedural descriptions or the listing of several examples. However, ultimately students attempted to explain the generality of their arguments. We maintain that they came to act within the norm of justifying their thinking in ways that required increasingly sophisticated notions of what constituted acceptable mathematical justification or proof. As an illustration, we include the following episode that occurred as students were trying to prove 2 is irrational.

4. Jared: I set 2 = p/q. Then I…

5. Daniel: What are p and q?

6. Jared: Two integers

7. Daniel: Any integers?

8. Jared: Two integers

9. Daniel: If it's not *any* integers, then it's not true for all cases, and then someone can come up with a case where it fails and your argument is gone.

10. Mike: To me, the important thing to remember is that 2 is written as a specific ratio, not any p/q. We are trying to show it can't be rational….

It seems here that the norm at work is the expectation among students that their peers should explain their reasoning and could freely challenge one's thinking. As Jared started sharing his thoughts, Daniel asked for further explanation regarding the generality of Jared's use of integers. Mike joined the discussion in an attempt to clarify the argument further, which seems to be determining whether 'p' and 'q' represent a fixed but unknown pair of integers, or whether they represent any two arbitrary integers. Mike's language further suggests that the argument is a collective one. He notes that "*we* are trying to show it can't be rational" (emphasis added). It is no longer Daniel's attempt to show that the square root of 2 is irrational. It is an argument embraced by the class (or some subset thereof) as their responsibility.

We maintain that active dialogues such as this were essential for and reflective of an internalization of knowing and doing mathematical proof. Moreover, we argue that as students were increasingly able to create these conversations, it changed the structure of public discourse and allowed for a more advanced form of collective debate so that interactions such as that given by 4-10 became both tool and result of the mediated language of students. In contrast, the protocol given earlier shows no argumentation and suggests a passive interchange between students. We maintain that these structural differences in the forms of interactions among students indicate an evolution in how they thought about mathematical proof. Moreover, this shift did not occur in a vacuum or as a result of watching (passively) a teacher record steps to proofs which students were expected to mimic with similar problems. Instead, it required classroom norms that particularly supported the opportunity for public debate and scrutiny. As such, it seems that discussions such as that between Jared, Daniel, and Mike helped to build a habit of mind whereby students could internalize public argumentation in ways that could facilitate private proof construction.

Constructing a proof is ultimately a process of logical argumentation. We maintain that, for most students, this process is learned in the context of *public* argumentation that makes not only the arguments, but the thinking behind the arguments an explicit part of the conversation. It is in the context of active public debate that students learn how to question, dissect arguments, and build justification as they engage in the act of doing so. In this study, classroom norms were established in which proofs could be socially constructed and in which the components of an argument and one's reflective and anticipatory thinking about a proof were a part of the public discourse. We maintain that these supported students' emergent capacity to construct proofs as well as their internalization of an increasingly formal language for expressing their ideas. More work is needed to understand how diverse kinds of classroom norms, both social and sociomathematical, lead to different types of knowledge and understanding about proof.

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ENABLING STUDENTS TO MAKE CONNECTIONS WHILE PROVING: THE WORK OF A TEACHER CREATING A PUBLIC MEMORY

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The possibility to connect present inquiries to prior knowledge is crucial in students' ability to prove conjectures. How to organize the storing and recall of that prior knowledge is an important piece of the work of teaching. The complexities of such task are particularly revealed in the case of engaging students in making and proving conjectures in geometry. Using records from a classroom intervention in high school geometry, this article examines what are some of those complexities and what is involved in a teacher's management of such complexities.

Problems and tasks can be useful contexts for students to use mathematical reasoning to construct new knowledge (Brousseau, 1997; Lampert, 2001; NCTM, 1991, 2000). As Ball & Bass (2000) show, the possibility for reasoning to play such role in the construction of knowledge hinges on the existence of a base of public knowledge that students can use and language appropriate to formulate and represent claims and arguments. The present article takes that conversation a step forward and argues that students' access to such base of public knowledge as they engage in reasoning and proving depends on how existing knowledge is organized for public use. Furthermore, that the nature of such "public memory" can be problematic for a teacher.

Proof is a prominent tool to articulate connections between mathematical ideas: Proof plays that role not only in the producing new knowledge (Lakatos, 1976) but also in the re-presentation and organization of that knowledge inside larger theories (Rav, 1999). Proof is also prominent in NCTM's Principles and Standards for School Mathematics (*PSSM*) for similar reasons—as the document invites teachers to help students understand the crucial role that reasoning and proof play developing new knowledge from old knowledge and making connections.

But classrooms are complex environments. They are complex not only in regard to the interpersonal dynamics that may develop in the contexts where individuals come to know and use mathematics (Chazan, 2000; Voigt, 1994). Classrooms are complex environments also in regard to the positions and responsibilities that teachers and students have vis-à-vis each other and the subject matter they must teach and learn inside the schooling institution (Brousseau, 1984). There are specific complexities associated with managing the use of existing knowledge in proving new conjectures. These complexities may be especially manifest in classrooms where the subject of studies is organized in a deductive fashion and yet students' opportunities to develop new knowledge are constituted in and through their work on problems. I focus on one particular aspect of this complexity, managing the creation of *labels* for (or public records of) the new ideas that students develop and their later use in constructing public arguments. I want to argue that such work of developing a public memory that students can use in proving may create tensions in the work of teaching.

The argument that I want to present has two parts. First, I argue that labels of shared classroom experience may play an analogous role in developing arguments for new conjectures as postulates and proven theorems play in the mathematical presentation of a new result. In that first part of the argument I suggest that the creation of conditions for students to prove new conjectures require deliberate actions on the part of the teacher. One of those deliberate actions is the sanctioning of labels for chunks of students' previous mathematical experience, labels that students may use to think with. I discuss two examples of how students' public solutions to problems were publicly recorded into labels that they could use later to make and argue for geometric conjectures that were not perceptually evident. Second, I observe that in the examples considered, those public records of experience that were used to prove new, non-evident propositions did not correspond univocally to mathematical objects that students were supposed to learn—they were in fact made up labels, meaningful only to the class that would use them. Thus, I argue that no matter how usable, the nature of those labels may confront a teacher with a problem. Some of those records and labels for mathematical ideas and experiences that students might find useful in making arguments might also be transitory items of knowledge that need not (and perhaps should not) be remembered in the long run. They may compete for attention with other, more general, items of knowledge that students must remember for future use. To keep those records in memory, though helpful for the work that students do in a particular period of their studies as they investigate a particular set of questions, might conspire against the general organization of the subject that they are supposed to study. The making of labels for experience could also make the knowledge to be learned more fragmentary, even idiosyncratic. The two parts of the argument suggest that there is a potential tension for a teacher that concerns the work that he or she might do creating a public memory. I discuss the issues at play in managing that tension, using records from a collaborative classroom teaching experiment in high school geometry—a particularly fertile context where to inquire on this problem.

Theoretical Framework

The Didactical Contract

This paper contributes to the development of theory for understanding how the interactions between teacher and students about their subject of studies in classrooms shape the particular ideas and practices that students have access to. I build on Brousseau's theory of the didactical contract (Brousseau, 1984,1997) which postulates that those interactions respond to a hypothetical contract that establishes global, mutual responsibilities for teacher and student vis-à-vis the subject matter. The teacher is responsible to teach mathematics to the student; the student is responsible to study mathematics with the help of the teacher; the subject at play in their interactions must defensibly correspond to the discipline that they are expected to teach and learn. Teacher and students negotiate the particular ways in which those responsibilities apply for the specific tasks that they do. Those negotiations play a role in shaping the specific ideas and practices at stake in those tasks, and how teacher and students interpret the work done around those tasks in relation to the global responsibilities that they have contracted with each other.

Thus the theory recommends paying attention to the relationship between the mathematical activity constituted in and through the interactions between teacher and students in specific tasks and the roles and responsibilities that teachers and students have vis-à-vis the study of mathematics in a classroom. Structural factors (formal relationships between the positions of student and teacher inside an institution, institutional expectations on the knowledge to be taught and learned) and emergent factors (viable relationships between actual students and teacher engaging in specific, viable conceptions and forms of work) combine to shape a knowledge ecology. Some ideas or practices are more likely than others to exist and endure in a classroom; the theory seeks to give the means to explain why.

As teachers are those who might effect change in the knowledge ecology of a classroom, a crucial, broad problem within that theoretical approach is to understand what is viable for teachers to do within the conditions and constraints afforded by the global didactical contract. A growing body of research on mathematics teaching in classrooms (Ball & Bass, 2000; Chazan, 2000; Cohen, 1990; Lampert, 1985, 2001) has examined the work of teaching mathematics in terms of managing dilemmas, problems, and tensions that are endemic to practice (Ball, 1993). That language is being used here to describe the work of the teacher shaping the knowledge ecology of a classroom. Of particular interest here is the work a teacher does creating and managing the use of labels for students' experiences working on tasks and the relationship between those labels and the production of proofs for new conjectures.

Labels and Public Memory

Students develop awareness and meaning for mathematical ideas through their experience in classroom tasks (Doyle, 1988). "One of the key devices human beings have for learning from experience is the use of a label to refer to elements of experience" (Mason, 1999, p. 187). What Mason calls *labels* include words, phrases, sequences of words, or slogans that may be used to refer to generic "fragments"—the building blocks for descriptions of experience. Yet labels don't just organize the storage of past experiences, they can also be oriented toward triggering the recall of experiences—in particular labels may serve action (know-to act) in context. Thus if "labels are the constituents of theories" (Mason, 1999, p. 190) that can act as "nodes in a web of connections [and] inform our actions" (p. 205) their creation, nature, and function in a classroom are of special interest. Brousseau (1997, p. 246) notes that students' memories are not merely cognitive entities but didactical (or instructional) ones, in that teachers organize them in complex ways, to serve not only the reactivation of what is needed when it is needed but also the forgetting of what is not needed. Thus the creation or endorsement of labels for students' experience is an aspect of the work of the teacher managing what can be known, in particular what can be proved, in the classroom.

The theory of the didactical contract is particularly helpful in turning that interest on the creation of labels for activity into a problem to be studied. In terms of the global didactical contract, the roles and responsibilities between teacher, student, and subject of studies have special entailments in the case of geometry that relate to the creation and management of public labels. Among the high school mathematical subjects, geometry is not only one that has had a longstanding presence in the curriculum but is also one with evident ties to a specific mathematical domain. The objects that teachers must teach and students must learn are highly prescribed by tradition and by the discipline (a discipline whose foundations have not changed much in about a century). The subject tends to be presented and structured in accordance with the axiomatic-deductive method. In fact the teaching of geometry is often predicated on the argument that it involves students in the study of a mathematical system (Moise, 1975). Of all subjects, geometry is a special place where students have the chance to develop mathematical values such as the notion that a general proposition implicitly includes particular propositions as consequences or corollaries, and that those need not be explicitly labeled to be taken as known. And the subject is populated by many technical names for concepts and propositions that students must remember to be able to operate within. All of that to say that insofar as the structural relationships between teacher, student, and subject of studies, it is a plausible hypothesis to say that geometry imposes stringent constraints on teacher and students regarding what should and could be labeled and committed to memory. Namely, there are plenty of definitions and theorems that must be labeled and remembered in accordance with the discipline; others that are dictated by tests; those are the ones that students are accountable to study and learn. There also are substantive disciplinary reasons why some other things, propositions for example, should not be labeled and committed to memory. If those propositions were particular instances of known theorems, they could suggest tasks that teachers could use to gauge students' ability to move from the general to the particular.

Yet the teaching and learning of specific geometric ideas occurs in and through engagement in particular activities. To the extent that an activity may have been chosen to mediate the teaching and learning of an idea, securing students' engagement in that activity is a way for the teacher to fulfill what the didactical contract requires of him or her. But fostering that engagement or, more generally, making those activities viable, requires specific negotiations between teacher and student, negotiations that specify a context for the development of the new idea and thus affect its meaning.

The negotiations undertaken to make an activity viable may include a variety of things (e.g., undertaking a simpler activity first, legitimating certain forms of participation, or changing the stakes). Negotiations can be accomplished interactively (as in bargaining the price of an automobile), or they may result from unilateral moves (as when setting goals for a fundraising campaign upon analysis of donors' likely behavior). The creation of labels for previous experiences is another possible outcome that may proceed from (either kind of) negotiation and be oriented toward making an activity viable. This is particularly interesting to examine when the activity includes the expectation that students come up with a conjecture and provide a proof in a place like the geometry class. Indeed proof activities in geometry customarily require the justification of the steps of an argument by reference to labels (for theorems and concepts). It is therefore quite plausible that students' capacity to develop a proof for a conjecture hinge on having labels available to be used in justifying their arguments and hence, that the teacher's disposition toward endorsing a label responds to the need of making the proving activity viable.

The foregoing discussion thus permits to anticipate that enabling students to connect new knowledge to existing knowledge through proving might require the use of labels that identify experiences in the public memory. Those labels might or might not necessarily belong in the knowledge to be studied. Thus the theoretical approach sketched above permits to anticipate that in managing the constitution of a public memory useful to enable students' proving, a teacher must ponder two kinds of considerations. Considerations of what elements are useful to keep in memory in order to make the work possible need to be balanced with considerations of what elements must be kept in memory in order for the work done to comply with the global responsibilities imposed by the didactical contract. In the sequel I illustrate how these issues played out in the teaching and learning of area of plane figures in a high school geometry class. I report on the work of a particular teacher enabling her students to connect new knowledge through proving by endorsing labels that were not part of the subject of studies.

A Collaborative Classroom Teaching Experiment on Area of Plane Figures

I worked collaboratively with Megan, an experienced high school geometry teacher to design and teach a three-week replacement unit on area of plane figures. The unit (called "Speaking the truth about area without calculating areas") was structured as a series of problems and questions where students would have the opportunity to conjecture and prove key properties of area. It started from acknowledging what students knew about how to calculate the area for a triangle and intended to use that knowledge to develop the formulas for area of other plane figures. Further, the unit was designed so as to give students a chance to use the formulas to make and prove claims about area relationships, modeling geometric problems algebraically, and connecting them to mathematical ideas they had studied before (such as ratio, similarity, and the medians of a triangle). I refer here to two of the problems that students investigated. One was to construct a point inside a triangle so that the three triangles it determines when connected to the vertices are of equal area. The second was to find the ratio between the area of a given quadrilateral and that of its midpoint quadrilateral (the one whose vertices are the midpoints of the sides of the given quadrilateral).

The unit was implemented in three simultaneous classes taken mostly by sophomores. Megan taught most lessons for two of those classes with the help of Norah, a student teacher that collaborated with us in the project; Norah took care of teaching most lessons for the remaining class; I taught the lesson on the midpoint quadrilateral in all three classes. Data collected included conversations about design and planning, videotaped lessons, students' interviews and written work, teachers' journals, and frequent interviews to Megan or to both teachers during the time we implemented the unit as well as later on in the year and during the following summer.

The two sample problems mentioned above (splitting a triangle into three equal-area triangles and finding the area of the midpoint quadrilateral) turned out to be important highlights of the three-week experience. While working collectively in those problems students were able to produce proofs for two propositions about area that are not perceptually evident. The production of those proofs was made possible through ways of referring to existing knowledge that are central to the present discussion on the constitution of public memory. I describe briefly how labels were used in producing those proofs and then discuss how that use brings up issues regarding the work of teaching.

Two Conjectures and Their Proofs

In the second day of the three-week unit, students were presented with the problem of finding a point inside a triangle so that when connected with the vertices, the triangle would be split into three equal area triangles. By the end of the third day all three classes had arrived at the conjecture that such point would be the intersection of the medians;[1] in two of those classes students had come to the board and given a proof of that conjecture. In all three classes students were given a chance to understand the general problem first. But then, as an intermediate step they had been asked to work on a simpler problem, to find a point inside a triangle and connect it to the vertices so that at least two of the three triangles thereby constructed would be of equal area. In all three classes (though not equally fast in all of them) students found that if they considered one of the medians of the triangle, and picked a point on that median, such point connected with the three vertices would create two triangles of equal area. Students were asked to justify their solutions and they did so by noticing, in diagrams like that of Figure 1a, that if AM is the median of ABC and O is on AM, then OM is also the median of BOC. Thus not only BAM and CAM are equal in area but also BOM and COM are equal in area, and hence subtracting BOM from BAM and COM from CAM, yields that two of the triangles determined by O, ABO and ACO should be equal in area.

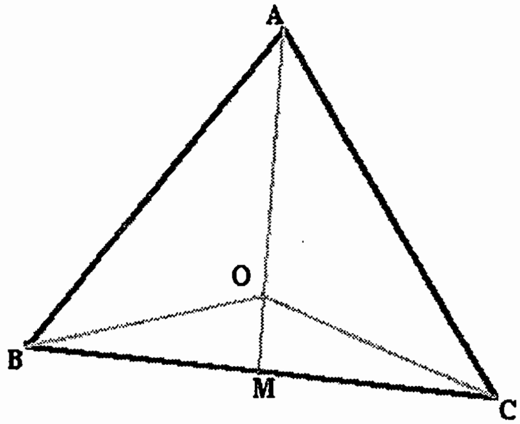

Figure 1a. Using "the median rule" to find two triangles of equal area.

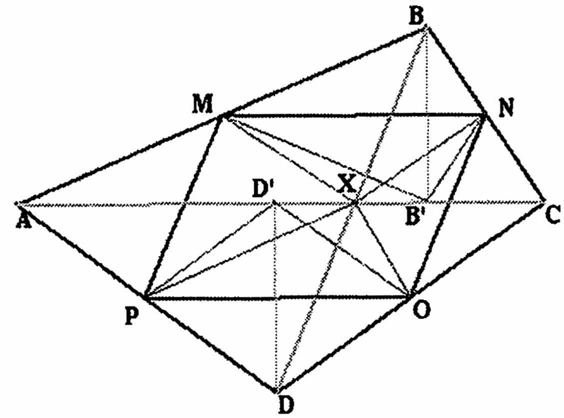

Figure 1b. Using "the sliding triangle" to cover the area of the midpoint quadrilateral.

Two of the classes arrived at that argument by the end of the first day working on the problem, whereas the third class did so during the second day. By then many students also thought the point that would yield three equal-area triangles was the intersection of the medians. On the second day of working on the problem, students in the two morning classes had realized that the argument used to construct a solution to the simpler problem could be used to justify why the intersection of two medians would split the triangle into three equal areas. One of those students, whom I call Connor, provided a proof at the board. When Connor gave his proof that the centroid of a triangle splits the triangle into three equal areas, he used the label "the median rule" to justify four of the steps in his proof. His proof included running twice through an argument like the one sketched above for the solution of the simpler problem, to establish that if O were the centroid, AOB and AOC would be equal in area and also BOC and AOC would be equal in area.

On the last day of work on this three-week unit on area, I took charge of presenting students with the midpoint quadrilateral problem. We gave each student a sheet with a quadrilateral drawn on it (different students had different quadrilaterals) and asked them to find the midpoints of their sides, join them to form a new convex quadrilateral, and estimate the ratio between the areas of the midpoint and the original quadrilaterals. Some students folded in the corners trying to tile the midpoint quadrilateral (see Figure 1b). Many of them suspected that, even though these corners would as a rule not tile the midpoint quadrilateral, their aggregated area was about the same as the area of the midpoint quadrilateral. They also started to notice that everybody in the room had a midpoint quadrilateral that was a parallelogram. We asked them to consider a simpler problem—if the original quadrilateral were a rhombus, what could you say about the relationship between the areas? Many students in the second period class observed that folding in the corners would in that case tile the midpoint quadrilateral without overlaps, and some suspected the fact might have to do with the diagonals of the rhombus being perpendicular (hence one of them parallel to the corner fold). Prompted to think about whether the folding argument could be made to work in the more general case, they first realized that all four vertices were reflected at different points (unlike in the rhombus case where they all would land at the intersection of the diagonals). Yet they also realized that all four vertices would always be reflected on a diagonal and that the challenge really was to make all four points coincide. Two students, whom I call Dan and Saul, collaborated in articulating for the whole class an argument. They said that if after folding vertex B onto B' one would slide B' along the diagonal AC (say, making it coincide with the intersection of the diagonals, X) the area of the triangle should not change. They justified that assertion, and other similar ones they had to make as they proved that the area of the original quadrilateral was double the area of the midpoint quadrilateral, on "the sliding area triangle."

Surprising Accomplishments and the Role of Labels

The descriptions of the development of those two conjectures and their proofs are evidently too synthetic. The point in reporting them here is to note that students in these classes were able to do things that many, themselves and their teachers included, would judge surprising: Finding out and justifying the solutions of two non trivial problems, by drawing deductive connections to what they knew about area. The descriptions above also draw attention to the role played by two labels—"median rule" and "sliding area triangle"—in making those proofs possible. I now discuss what those labels meant and how they were instrumental in helping students construct these arguments.

Partly in response to Megan's doubts that students would be able to figure out the triangle splitting problem, the work planned for the first day of the unit had included asking students to split triangles into two triangles whose areas stood in a given ratio. In the context of splitting a triangle into two whose areas were in a 1:1 ratio, students had been reminded that they already knew the name for the segment they had drawn to do the splitting—they knew it as the *median*. And they had been asked to prove that two such triangles were indeed equal in area—which they did by noticing that if one calculated their areas they would come out equal as both had same height and equal bases. Upon suggestion of one student, Megan had labeled that observation, that if one wants to splits a triangle into two equal areas one draws the median, "the median rule."

The "sliding triangle" label had appeared first in the context of discussing the problems for the unit; in the context of discussing what would students have the opportunity to learn if they had to work out the "division of the land" problem.[2] As the design of the unit took shape, various problems were selected for students to develop awareness that triangles that share a side and whose opposite vertex lie on a parallel to that side necessarily have the same area. In some of those problems students had had to use that observation proactively—to choose a triangle that had the same area as a given one but that satisfied an additional constraint.

From a perspective centered in the textual presentation of mathematics, both items of knowledge referred as "the median rule" and the "sliding triangle" are immediate corollaries of the triangle area formula. They are particular cases, packed inside the formula so to speak; they don't ordinarily occupy a prominent place in the geometry curriculum. Yet, from the perspective of delivering intelligence that might inform action, those two labels are not contained in the triangle area formula—they evoke connections between situations that, I argue, were instrumental in enabling students to produce the two proofs described above. They were available not only as justifications for the arguments that students made, but also as building blocks for finding out how each conjecture could be argued. The label "sliding area triangle," for example, did something to mediate a difficult connection—it triggered in students a way of creating and recognizing classes of equal-area triangles that was relevant in making the rhombus solution work for the general case. Similarly, the label "the median rule," evoked for students a way of getting two triangles of equal-area, this was part of what was called for in the simpler version of the triangle splitting problem. To the extent that proving a proposition involves not just justifying statements but also strategically finding out what statements will make for a cogent argument (Greeno, 1980), these labels contained proactive elements that students could use in building up the proofs they came up with. Yet, to say that proactive labels like these are useful in enabling students to make deductive connections is only half of the story.

Tensions that Endorsing Labels for Experience Present for a Teacher

The two cases observed show that students' investment in proving important propositions was particularly enabled by the availability of labels that they could use to refer to their experiences connecting the triangle area formula with particular situations and problems where it applied. These labels triggered ways of acting that were instrumental for students in thinking about the elements of the arguments they had to construct. To enable students to make those connections as they elaborated their arguments, Megan had had to make room in her class for students to work on tasks that built awareness of those consequences of the triangle area formula. But that was not all. In addition to having students work in those tasks she had to endorse the labels—the median rule, the sliding triangle—as mathematical, as components of the subject of studies. Beyond naming experiences, labels were of particular significance, as Megan herself recognized, in enabling students to develop proofs. Students not only could recognize resemblance between situations, say between the folded corners in the midpoint quadrilateral and the division of the land problem. They also had an official name, sliding triangle theorem, they could plug in as "reason" in a proof, to justify the statement that two triangles would be of equal area. As Megan noted, having labeled those experiences with earlier problems meant for students that "[they were] gonna use it later."

Whereas Megan eventually had no personal objection to endorsing those labels, deciding that building awareness of those ideas was worthwhile spending time on was not immediate. Not only those tasks and problems themselves implied a detour from the tasks and problems usually included in the chapter on area, the insights about area of plane figures that they pointed to were not mainstream ideas of the curriculum either. To the extent that labels such as "sliding triangle" were evidently made up by ourselves in planning the unit, was it legitimate to include experiences geared toward students' learning of something that needed a made-up label to be identified? Was it worthwhile to give a label to something that eventually was merely an application of the triangle area formula? Part of the justification that indeed it was worthwhile came right on the third lesson, as students were able to do something uncommon in geometry classes—to prove that those three seemingly distinct triangles split by the centroid were actually equal in area. But precisely, given that reasons in proofs are supposed to be theorems, definitions, or postulates, wasn't it problematic to let students justify statements on something like "the median rule"? Furthermore, given that there are so many terms, definitions, and theorems that students are asked to remember in a course such as geometry, was it fair to clutter students' memory even more? The impression that students were indeed doing surprising things helped Megan decide that it had been worthwhile to spent time developing awareness for the ideas that those labels pointed to. But the questions posed were still recognized as legitimate. If their impact was not more noticeable, it may be because the chapter on area in the text that Megan regularly uses is structured more around providing practice with formulas than around proving area properties. Similar creation and use of labels in a chapter more structured around postulates and theorems might have been more problematic for her to manage.

The discussion of the use of "the median rule" and "the sliding triangle" in Megan's classes helps bring to the fore a foreseeable tension that a teacher may need to manage in relation to the development of a usable public memory. This tension operates in particular when the development of new knowledge is done in and through students' engagement in problems and tasks and yet the knowledge students' have to acquire is an axiomatized, hierarchical body of knowledge, such as Euclidean geometry. This tension may be observed as a teacher engages students in proving propositions that build on prior knowledge developed through students' work on tasks. The tension is by no means presented here as a recommendation to avoid the use of labels in proving, rather as a construct that may help classroom observers understand what is at play in teachers' management of that use of labels. I state that tension in more general terms.

Students experiences working on problems and tasks may help them become aware of how certain mathematical ideas can inform their actions. For a teacher to endorse labels that trigger that proactive knowledge may be instrumental in enabling students' to construct arguments for new conjectures. Yet to the extent that such labels are more evocative of action than subordinated to deductive structure and generality, building a public memory based on those labels may conflict with expectations about the deductive organization of the subject of studies. Labels that enhance chances to produce an argument may also discourage search for the minimal conditions needed for those conjectures to be true, hence conspire against the development of mathematical theories. There is thus an opposition between the short-term availability of knowledge-to act and long-term organization of general knowledge. To the extent that a teacher is responsible for students' mastering a deductively organized subject and for their successful participation in daily tasks, managing that opposition may be difficult. Successful management may require engaging students in mediating tasks where they confront various ways of organizing the public memory.

Notes

[1]Intermediate problems were posed that aimed at raising the issue of whether that solution was unique. In Herbst (2002) I have reported on some tensions that teachers had to cope in presenting those problems. Students never questioned that medians intersect at a point—they had studied (though not proved) that in the chapter on triangles some months before.

[2]In that problem (seen in the videos of Japanese classrooms collected by TIMSS; see Stigler & Hiebert, 1999) students are told that the fence between two neighbors is a crooked line (composed of two segments) and asked to find a single segment that will divide the lands without altering the area.

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MAPPING THE CONCEPTUAL TERRAIN OF MIDDLE SCHOOL STUDENTS' COMPETENCIES IN JUSTIFYING AND PROVING

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This paper presents preliminary results from the first year of a multi-year research project[1] exploring the development of middle school students' competencies in justifying and proving and the conditions and pedagogy necessary to promote that development. In particular, in this paper we present and discuss results from a written assessment completed by approximately 350 6^{th} through 8^{th} grade students. The assessment focused on students' abilities to both evaluate and generate mathematical arguments, their understanding of various aspects of deduction, and their understanding of the nature of proof. Consistent with previous results, students demonstrated an overwhelming reliance on the use examples as a means of demonstrating and/or verifying the truth of a statement. The results also suggest that there are differences among students (by grade level) with regard to their understanding of various aspects of the deductive process.

Many consider proof to be central to the discipline of mathematics and the practice of mathematicians. Yet surprisingly, the role of proof in school mathematics has traditionally been peripheral at best, usually limited to the domain of high school geometry. According to Wu (1996), however, the scarcity of proof in school mathematics is a misrepresentation of the nature of proof in mathematics: "Even as anomalies in education go, this is certainly more anomalous than others inasmuch as it presents a totally falsified picture of mathematics itself" (p. 228). Similarly, Schoenfeld (1994) claimed that "proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels" (p. 76). Sowder and Harel (1998) also argued against limiting students' experiences with proof to geometry, but more from an educational rather than mathematical perspective: "It seems clear that to delay exposure to reason-giving until the secondary-school geometry course and to expect at that point an instant appreciation for the more sophisticated mathematical justifications is an unreasonable expectation" (p. 674).

Reflecting an awareness of such criticism, as well as embracing the important role of proof in mathematical practice, recent reform efforts are calling for substantial changes in school mathematics with respect to proof. In contrast to the status of proof in the previous national standards document (National Council of Teachers of

Mathematics [NCTM], 1989), its position has been significantly elevated in the most recent document (NCTM, 2000). In fact, not only has proof been upgraded to an actual standard in this latter document, but it has also received a much more prominent role throughout the *entire* school mathematics curriculum and is expected to be a part of the mathematics education of *all* students. More specifically, the *Principles and Standards for School Mathematics* (NCTM, 2000) recommends that the mathematics education of pre-kindergarten through grade 12 students enable all students "to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (p. 56).

These recommendations, however, pose serious challenges for school mathematics students given that many students have, traditionally, found the study of proof difficult. In fact, research has painted a bleak picture of students' competencies in justifying and proving (e.g., Bell, 1976; Chazan, 1993; Healy & Hoyles, 2000; Porteous, 1990; Senk, 1985; Usiskin, 1987). Although such research has made significant contributions to our understanding of students' competencies in justifying and proving, such research also has its limitations. As a collective, the studies do not provide a coherent picture of students' competencies over typical grade spans (e.g., middle school, high school), since each study was framed differently, studied different student populations in different school contexts, and relied on different instrumentation and methodology. Thus the goal of this session is to present and discuss results concerning the range of middle grade (6^{th} - 8^{th}) students' understandings of justification and proof—in short, to provide a map of the conceptual terrain of middle school students' competencies in justifying and proving.

Theoretical Perspective

Researchers have hypothesized that the development of students' competencies in justifying and proving might follow a developmental progression; that is, students' understandings of mathematical justification are "likely to proceed from inductive toward deductive and toward greater generality" (Simon & Blume, 1996, p. 9). Waring (2000), building upon the work of other researchers (e.g., Balacheff, 1991; Bell, 1976; Fischbein, 1982; van Dormolen, 1977), proposed six levels of proof concept development:

Level 0: Students are ignorant of the need for, or existence of, proof.

Level 1: Students are aware of the notion of proof, but consider checking a few cases as sufficient (i.e., Balacheff's naive empiricism).

Level 2: Students are aware that checking a few cases is not sufficient, but are satisfied that either i) checking extreme cases (i.e., Balacheff's crucial experiment) or random cases is proof, or ii) use of a generic example forms a proof for a class of objects (i.e., Balacheff's generic example).

Level 3: Students are aware of the need for a general argument, but are unable to produce such arguments themselves; however, they are likely to be able to understand the generation of such an argument (for example, by a classmate). This also includes the ability to follow a short chain of deductive reasoning.

Level 4: Students are aware of the need for a general argument, are able to understand the generation of such an argument, and are able to produce such arguments themselves in a limited number of (familiar) contexts.

Level 5: Students are aware of the need for a general argument, are able to understand the generation of such an argument (including more formal arguments), and are able to produce such arguments themselves in a variety of contexts (both familiar and unfamiliar).

This framework provided a lens for interpreting our assessment data; however, our analysis of the assessment data also resulted in further delineation of the framework. As proposed, Waring's (2000) framework focuses primarily on students' judgments of proof and approaches to proving; our revision includes further elaboration of Level 3. In particular, we have extended Level 3 to include students' understandings of various concepts (e.g., definitions, necessary and sufficient conditions) viewed as prerequisite to being able to understand and produce deductive arguments.

Methods

Approximately 350 middle school (6^{th} - 8^{th} grade) students participated in this study. These students responded to a written assessment designed to measure their competencies in justifying and proving. In all, students responded to fifteen different items during a 45 minute class period; using matrix sampling, each individual student responded to eight of the fifteen items. The assessment items required students to evaluate various arguments (cf. Healy & Hoyles, 2000), generate their own arguments (e.g., Show that the sum of any two consecutive whole numbers is always odd.), demonstrate understanding of implication rules and deduction (e.g., Given the statement: If two even numbers are multiplied, then their product is even. Decide if the following statement is true or false and explain why: 286 is even, so it is the product of two even numbers.), apply definitions (e.g., Given a definition of a quadrilateral, determine whether a particular figure is an example of a quadrilateral), and discuss the nature of proof (e.g., What does it mean to prove something in mathematics?). In several cases, the assessment items were adapted from prior research studies. Data analyses were informed primarily by the theoretical perspective described above.

Results

Due to limitations of page length, results are presented from a subset of items (full results will be presented during the conference session). Representative excerpts from students' written responses are provided to illustrate particular findings.

Producing Mathematical Arguments

Consistent with previous research, students demonstrated an overwhelming reliance on the use examples as a means of demonstrating and/or verifying the truth of a statement (Level 1/Level 2). Further, students did not indicate in their responses any recognition of the limitation of empirical evidence as a means for establishing the truth of a statement. As an example, 70% of the student responses to the two assessment items that required them to construct an argument justifying the truth of a given statement were based on the use of examples. The following are the two aforementioned assessment items: (1) *The sum of two consecutive numbers is always an odd number. For example, 5+6=11 and 8+9=17. Show that the sum of any two consecutive numbers is always an odd number*; and (2) *Show that when you add any two even numbers, your answer is always even. Provide an explanation that would convince a classmate that the answer is always even.*

One the one hand, the majority of these students appear to believe that providing one or more unsystematically selected examples is sufficient to prove a statement (Level 1). The following is a representative justification to Item 2: "I would take 5 different [pairs of] even numbers and then answer them [i.e., find the sum] and if she/he does not believe me, I will keep on doing it." In this case, the student seems to suggest that any five examples provide sufficient proof that the statement is always true and, if this justification fails to convince a classmate, then the selection of more examples should suffice to provide the necessary evidence to change the classmate's mind. On the other hand, a smaller percentage of these students strategically selected different types of numbers (e.g., very large/small numbers) in their efforts to produce a convincing argument (Level 2). For example, one student responded to Item 2 by selecting five pairs of even numbers: three pairs of small even numbers, one pair with one small and one large even number, and one pair of two large even numbers (including one over 100 million). In this case, the student employs a variety of numbers and combinations to show that the statement is true for different classes of numbers, including "very large" numbers. Responses of this nature suggest a certain level of sensitivity to the need to provide evidence that the statement is true, regardless of the size of the number.

A minority of students attempted to produce more general arguments for the two assessment items, with varying degrees of success. As an example, one student responding to Item 1 wrote: "Consecutive numbers are *always* one odd and one even number no matter whether I started with one even or one odd. Sums of an even number and an odd number are always odd." In this case, the student uses a feature of consecutive integers and then proceeds to utilize a mathematical declaration ("Sums of an even number and an odd number are always odd.") of his own in presenting an argument that the initial proposition is true. Although the student has attempted to present a general argument, acceptance of his argument as proof is dependent upon the acceptance of his mathematical declaration as true (to an extent, his argument is somewhat circular).

Students' Application of Definitions

On items in which students had to apply a mathematical definition, younger students had a tendency to use information external to that provided in the definition and, as a consequence, the use of such information often interfered with their interpretation of and ability to correctly respond to the assessment item. In other words, there seems to be a tendency, especially strong among younger students, to bring intuitive and familiar notions of a concept to bear in applying a definition; this tendency, however, appears to abate as students progress further in middle school. As an example, one assessment item required students to consider an unconventional, but technically correct, interpretation of the following definition of a quadrilateral: *A quadrilateral is what you get if you take four points A, B, C, and D and join them with four straight lines.* The "unconventional, but technically correct, interpretation" of the definition consisted of a figure with two of its four sides formed by connecting non-consecutive vertices (produced by a hypothetical middle school student named Carla). The students were then asked if this figure was a quadrilateral according to the stated definition.

In Grade 6, about two-thirds of the responses referred to a more generic definition of quadrilateral, whereas in Grade 8, about two-thirds of the responses used the specific definition provided to decide if the figure was indeed a quadrilateral. In the former case, the following is a typical 6^{th} grade student response, one that appeals to a more generic definition: "No, it is not a quadrilateral because the sides cross." It seems clear that the student is referring to his or her own definition of a quadrilateral rather than to the specific definition provided with the assessment item. In the latter case, the following is a typical 8^{th} grade student response: "According to the definition, Carla has drawn a quadrilateral. In the definition, it says you find any 4 points and connect them with straight lines, not in what order you have to connect them."

Understanding Conditional Statements

The data suggest that students are far more likely to correctly interpret the conditional nature of a mathematical statement if they could readily visualize a counter-example. For example, students responded to an item consisting of two conditional statements of the form "If (A and B), then C." More specifically, students were asked to state whether each of the following statements were true or false and to explain their choice: (a) *A quadrilateral that has four equal sides* must *be a square*, and (b) *A quadrilateral that has four right angles* must *be a square*. In (a), it was apparent that, at all grade levels, the students' lack of knowledge (or recall) regarding the existence of rhombi interfered with their ability to generate a counter-example and, thus, their ability to correctly respond to the question. Interestingly, this lack of knowledge (or recall) was also evident in (b), however, more so with the 6^{th} grade students than the 8^{th} grade students (statistically significant, $p < 0.05$), that is, more 6^{th} grade students responded incorrectly than did 8^{th} grade students. Apparently, the 8^{th} grade students had a greater familiarity with the properties of rectangles than did the 6^{th} grade students—the 6^{th} grade students may not have viewed a square as a special case of rectangle—and as a consequence, the 8^{th} grade students were better able to generate a counter-example. This result may be a curricular effect in that 6^{th} grade students may not have yet learned to classify quadrilaterals.

Concluding Remarks

Previous research has suggested that many teachers have inadequate conceptions of proof (e.g., Harel & Sowder, 1998; Jones, 1997; Knuth, in press; Martin & Harel, 1989) and that they have limited views regarding the nature and role of proof in school mathematics (Knuth, 2002). Consequently, engaging teachers in discussions focused on the details of students' competencies in justifying and proving may provide a basis for enhancing both teachers' own understandings of proof and their perspectives regarding proof in school mathematics. In addition, such detail on student reasoning may also provide a basis for continued growth and development of teachers' understandings of their students' reasoning and, consequently, their abilities to support the development of their students' mathematical reasoning (cf. Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, 1996).

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AN INVESTIGATION OF CLASSROOM FACTORS THAT INFLUENCE PROOF CONSTRUCTION ABILITY

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This paper on classroom factors influencing students' proof construction ability reports findings from the data collected in the first two years of a three-year National Science Foundation-funded project[1]. Four different classrooms, two from each participating school, were involved in the project. Data sources included videotaped classroom episodes, interviews with the participating teachers and with focus students from each class, as well as students' responses to items on the *Proof Construction Assessment* instrument. Student ability to construct proof was interpreted in the context of the classroom microculture (Cobb & Yackel, 1996). The results show that students performed poorly on items that required them to write a formal proof with no support. They also had difficulty on items that required students to make a single deduction from a given piece of information.

Objectives

Although proof and reasoning are seen as fundamental components of learning mathematics, research shows that many students continue to struggle with geometric proofs (Chazan, 1993; Harel & Sowder, 1998; Hoyles, 1997). In order to relate students' understanding of geometric proof to pedagogical methods and other classroom experiences, our three-year project investigates two components of student understanding of proof, namely, students' beliefs about what constitutes a proof and students' proof-construction ability.

In this paper we focus on students' proof-construction ability. We summarize findings from our second year of data collection that connect student ability to construct proof in geometry to classroom factors that may influence that ability. More specifically, we address two objectives:

1. To characterize the psychological aspects of students' evolving proof-construction ability in proof-based geometry classes in order to update and expand existing research in this area;
2. To link students' geometric proof-construction ability to aspects of the classroom microculture as well as to teachers' pedagogical choices.

Perspectives and Theoretical Framework

Existing research documents students' poor performance on proof items and identifies common, fundamental misunderstandings about the nature of proof and generalization in a number of mathematical content areas (Chazan, 1993; Hart, 1994; Martin & Harel, 1989; Senk, 1985). In particular, Senk's seminal study has set the benchmark for high school students' performance on geometry proofs. Other researchers (Balacheff, 1991; Harel & Sowder, 1998) have proposed frameworks that describe increasingly sophisticated strategies used by students to construct proofs. At the least sophisticated level, students appeal to external authorities for mathematical justification. At the next stage, students base their justifications on empirical evidence. Finally, students are able to use more abstract and mathematically appropriate techniques when proving statements. We have used the existing research to help us identify student misunderstandings and to characterize their strategies for constructing proofs.

The theoretical lens through which we view the classroom activities and students' mathematical development as participants in the community of the classroom is associated with the emergent perspective as described by Cobb and Yackel (1996). This perspective is useful in that it attempts to describe individual learning in the social context of the classroom. Thus student understanding of proofs (their beliefs about proofs and their ability to construct proofs) is seen as constructed on both a social level and a psychological level. In other words, the students' interactions with the teacher and peers lead to the development of taken-as-shared knowledge, or an understanding of social and sociomathematical norms. These interactions also influence individual students' developing understanding of proof. Our research focuses on individual student performance as well as the classroom microculture and teachers' pedagogical choices. By classroom microculture, we refer to social and sociomathematical norms, as well as classroom mathematical practices as defined by Cobb and Yackel (1996). We define pedagogical choices to include the choice of mathematical tasks, the ways the teacher allocates time for activities, the instructional strategies (direct instruction, cooperative learning, investigations), and the teacher's expectations about student ability that may be reflected in choices.

Methods of Inquiry and Data Sources

During the two years of the project, we captured the nature of proof instruction, as well as classroom interactions and student activities, through daily observations, video recordings, and written observer field notes. This allowed us to characterize the four participating teachers and their proof-based geometry classes. An initial pair of teachers participated for two years (Mrs. A and Mrs. B). A second pair of teachers participated for only the second year (Mrs. C and Mr. D). The multiple sources of data provided information about the context for the development of proof-construction ability in order to interpret this information and connect it to the classroom norms and mathematical practices.

Student proof construction ability was determined using three types of data collected during the two project years. First, we designed and administered a performance assessment instrument to measure students' varying levels of ability to engage in formal logical reasoning. This *Proof Construction Assessment* instrument includes items in which students must construct partial or entire proofs, as well as generate conditional statements and local deductions. In addition to some original items, the instrument include d items modified from Healy and Hoyles (1998), Senk (1985) and from the Third International Mathematics and Science Study (TIMSS) (1995). Second, data was collected during classroom observations, including video recordings, field notes, and student written work. Third, selected focus students participated in clinical interviews with researchers. The video-recorded interviews focused on some aspects of the Proof Construction Assessment and required focus students to create at least one original proof during the session.

Results and Conclusions

Results of the Proof Construction Assessment provide information about individual student ability to construct proofs. Analysis of the videotapes and field notes illuminate aspects of the social context in which individual learning developed. In addition, we discuss how the classroom social context may have influenced the individual understanding constructed by the students.

Proof Construction Assessment Results

Table 1 summarizes results from the Proof Construction Assessment from year two. The table provides a brief description of each item on the assessment instrument as well as measures of student performance for all four classes (n=84). Proof items were scored using detailed rubrics similar to those used in scoring the TIMSS (1995) and the National Assessment of Educational Progress (National Assessment Governing Board, 1994). The items were scored using rubrics of 5, 3, or 2 points, depending on the item. The table shows the average score for all participating students on each item. These average scores are also reported as percentages of available points, to facilitate comparison of student performance among items scored with differing numbers of available points. The last six columns display the percentage of students receiving a particular score on each item.

Item 1 on the Proof Construction Assessment required students to fill in statements or reasons in a two-column proof. The proof was a justification for "supplements of congruent angles are congruent." In general, students in all four classes did relatively well on this item. However, focus students in Mrs. C's and Mr. D's classes expressed frustration with the limitation of a pre-structured proof. Many of these students said that they would prefer to write their own proof.

Students also did relatively well on Items 3a, 3b, and 3c on the Proof Construction Assessment which required translation from an informal conjecture to a formal con-

Table 1. Proof Construction Assessment Results: Year 2

Item	Title of Item (Content Area)	Average Score (%)	% of Students Receiving Score					
			0	1	2	3	4	5
1	Fill-in proof (Supplementary angles)	3.29/5 (66)	0	3	20	37	23	17
2	Analytic proof without hints (Midpt of parallelogram)	2.54/5 (51)	16	20	10	19	21	14
3a	Conditional statement (Similar triangles)	2.18/3 (73)	2	25	25	48		
3b	Stating the *given* (Similar triangles)	1.25/2 (63)	3	68	29			
3c	Stating the *prove* (Similar triangles)	1.92/2 (96)	2	4	94			
4	Synthetic proof without hints (Isosceles, overlapping triangles	2.55/5 (51)	0	43	8	13	23	13
5	Synthetic proof with hints (similar triangles)	3.49/5 (70)	2	14	11	13	24	36
6a	Local deductions (Segment midpoint)	1.71/3 (57)	18	30	15	37		
6b	Local deductions (Intersecting segments)	0.64/3 (21)	70	10	6	14		
6c	Local deductions (Congruent triangles)	1.89/3 (63)	13	25	21	41		
6d	Local deductions (Non-parallel lines)	0.67/3 (22)	71	7	5	17		

ditional statement. The items also required the identification of "given" and "prove" statements. Students in all four classes had the most difficulty specifying all required conditions in the conditional statement. However, they had very little difficulty constructing "given" and "prove" statements that correctly corresponded to their conditional statement or to the original conjecture.

Items 6 a-d, which required students to draw a conclusion from a given set of conditions, proved to be much more difficult for all students than we expected. For these items, students were provided with written statements describing given conditions, none of which were accompanied by diagrams. Average student performance ranged from 21% correct on part B to 63% correct on part C. Because each part only differed by the geometric content, it appeared that the combination of item type and context were influential in determining item difficulty. Students had very little difficulty concluding that a point equidistant from and between two other points was a midpoint, but had much more difficulty drawing any correct conclusion from two intersecting segments. During clinical interviews, many of the focus students noted that they were uneasy with the open nature of the local deduction items. They claimed to be much more comfortable proving a "fact," than making one or two deductions (for which they had to provide reasons) of their own choosing.

Although all four teachers emphasized the importance of drawing a diagram and marking it with given information before beginning a proof, this approach was not found to be useful for students in solving items 6b and 6d. For example, of the 11 focus students (out of 16) who chose to draw a diagram for Item 6b, none of them provided strong conclusions (using all given conditions) with valid reasons and only two of them gave valid but weak conclusions (using only some of the given conditions) with valid reasons. In addition, for Item 6d, out of the 12 focus students who used a diagram only four of their conclusions (strong or weak) and reasons were valid. Some of the incorrect conclusions drawn by students were restatements of the given information or answers left blank. Other incorrect conclusions involved assuming additional information that was not provided (such as assuming that *intersecting* segments *bisect* each other) or carelessly choosing one conclusion from a family of conclusions, without checking for the validity of the conclusion (such as assuming that some relationship between special angles associated with parallel lines would prove that the lines were parallel). Figure 1 shows an incorrect response given by a student who did not draw a diagram. By using a diagram, the student may have been more likely to check the validity of her conclusion or reason. Another common incorrect response is shown in Figure 2. In this case, the student came to the conclusion that P was the midpoint of the line segments XY and ZW. Two of the participating teachers conjectured that the students may have assumed that the segments bisected each other, because point P was said to be between the endpoints of *segments* rather than between two points on given *lines*. Although we are not able to determine the exact cause of the misconception, it is clear that students did not attend to the precise meaning of the symbols and words that were used to specify the given conditions.

6c. Given: $\triangle LMN$ and $\triangle PQR$. $\angle L \cong \angle P$. $\overline{LM} \cong \overline{PQ}$. $\angle N \cong \angle R$.

Conclusion: Triangle LMN is congruent to triangle PQR

Reason: Angle-side-angle postulate

Figure 1. Sample student response to item 6c.

6b. Given: $\overline{XY}$ intersects $\overline{ZW}$ at point P. Point P is between X and Y.
Point P is between Z and W.
Conclusion: XP=PY and ZP=PW
Reason: P is a Midpoint of XW and ZW

Figure 2. Sample student response to item 6b.

Item 5, which required students to write a multi-step synthetic proof based on given information, a diagram, and a collection of hints (essentially an outline of the proof), was less difficult for students than we expected. Although students in Mrs. C's and Mr. D's classes said that they ignored the hints (preferring, again, to do things their way), the average score for all four classes was about 70% correct. The item required students to prove that two triangles were similar that were embedded in a diagram with two pairs of parallel lines. Some of the incorrect responses included unnecessary steps, which may have taken students off-track. These students did not appear to have a sense of the direction of the proof, despite the outline provided with the hints. Some students wrote little more than the given information.

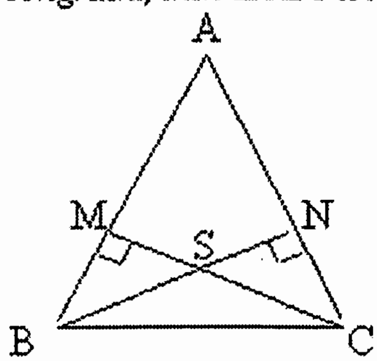
The full proofs, without hints, (Items 2 and 4) were very difficult for most students. Very few (14% and 13%, respectively) students gave fully correct responses. An additional 21% and 23%, respectively, wrote proofs that were satisfactory, containing roughly 4/5 of a correct argument. Correct responses tended to be mostly two-column proofs, but paragraph proofs were not uncommon, particularly from students of Mrs. C or Mr. D, who often used paragraph proofs in class.

For Item 2, the analytic proof, correct proofs either relied on the midpoint formula or properties of a parallelogram. Incomplete or incorrect proofs lacked justification for using the midpoint formula or lacked enough information to make a formal argument. This was particularly true in Mrs. A's and Mrs. B's classes in which students rarely encountered analytically represented figures associated with a proof. When figures were presented on a coordinate axis, it was in the context of the application of a theorem. These problems only required students to perform computations, without providing justification. During clinical interviews, students in Mrs. A's and Mrs. B's classes said they were unsure of what was required beyond the computations.

For Item 4, students were asked to show that a triangle was isosceles if the altitudes from the two base vertices were congruent (see Figure 3). Several students attempted to prove the two smaller triangles MSB and NSC congruent, although there was insufficient information to do so. Students either left out steps in their proofs, provided erroneous reasons for statements, or left reasons blank (perhaps, hoping for partial credit).

One of our focus students, Kevin who earned one out of a possible five points on Item 4, gave the following proof (copied from original script).

4. Consider the conditional statement and the accompanying diagram.
"If two altitudes, $\overline{BN}$ and $\overline{CM}$, in ΔABC intersect at point S and are congruent, then ΔABC is isosceles."



Write a proof of the statement.
Give geometric reasons for the statements in your proof.

Figure 3. Item 4 from Proof Construction Assessment.

Statement

1. BN ≅ CM (Given)
2. ∠NBC ≅ ∠MCB (Opp. ∠'s of ≅ opp. seg. are ≅)
3. triangleNBC ≅ triangleMBC (ASA)
4. ∠NBC + ∠MSB = ∠MCB + ∠NSC (Angle add.)
5. ∠ABC = ∠ACB (Substitution)
6. ∠ABC ≅ ∠ACB (If ∠'s are =, then they are also ≅)
7. AC ≅ AB (Opp. seg. of ≅ ∠'s are ≅)
8. triangle ABC is isos. (Def. of isos. triangle)

It can be noted that Kevin has a beginning and an end in his proof, but he writes in between does not make much sense. For example, he makes a hasty conclusion, with insufficient reasons, that triangle NBC and triangle MCB are congruent. It shows that Kevin focused on many irrelevant details and left out essential ones.

In the next section, we provide examples of teachers' pedagogical choices and describe aspects of the classroom microculture. We use this data to further explore possible connections between the social environment of the classroom and students' ability to construct proof.

Analysis of Teachers' Pedagogical Choices and Classroom Microculture

As noted earlier, we conjectured that the classroom teachers' pedagogical choices would have an impact on the students as they began to construct an understanding of geometric proofs. It might be more accurate to say that the teachers' pedagogical choices influenced the classroom microculture, including students' expectations for acceptable and valid proofs, hence influencing their understanding of how to construct valid proofs.

In some ways, the four participating teachers were alike. For instance, all four teachers followed the chosen textbook quite closely for structuring daily lessons and for assigning homework. In addition, the teachers and students used very little technology (one to two days a semester for Mrs. A and Mrs. B, four to five days a semester for Mrs. C and Mr. D) and very few hands-on investigations to help students explore geometric ideas that they studied in class. All teachers allowed the students to work with partners or groups on occasion to discuss proofs or other related problems. However, one major pedagogical difference between the classes was apparent. Mrs. A and Mrs. B chose to use geometric proofs as applications of the theorems and concepts they studied in class. In other words, Mrs. A and Mrs. B would introduce a new concept or theorem, demonstrate the concept, then show how to use the concept in a proof. In fact, this often led to students learning a particular kind of proof for a particular concept. In contrast, Mrs. C and Mr. D were more likely to use proofs to introduce new concepts or theorems to students. As a result, proofs assigned by Mrs. C and Mr. D were often the basis of the next day's lesson, whereas proofs assigned by Mrs. A and Mrs. B were rote applications of proof-writing procedures with limited student autonomy in problem solving and proof writing. This may explain the poorer performance of students in Mrs. A and Mrs. B's classes on the Proof Construction Assessment. In particular, the students in these two classes had more difficulty than other students in writing unsupported proofs (Items 2 and 4). Only with the guidance of the interviewers were the focus students from Mrs. A's and Mrs. B's classes able to make progress on these items.

A sociomathematical norm that appeared to be accepted in at least three of the four classrooms was the expectation that all mathematical problems can be solved in a relatively short period of time. Teachers contributed to the perception that mathematical problems can be solved quickly by providing examples that were always provable and usually in a few steps. As a result, students developed very little perseverance, in terms of reasoning ability, and gave up quite quickly on challenging tasks. Students rarely spent very long on a particular proof. This may be the reason why some of the students' responses to the proof items without hints (items 2 and 4) on the Proof Construction Assessment tended to be brief.

To determine what the students' deemed as valid mathematical proofs, the focus students from each class were given three different proofs of the same statement that were constructed by other students. The focus students were asked to examine and grade the proofs. Most of the focus students claimed that they were able to follow the reasoning provided although many were not able to detect mistakes in the logic of the proof. Three separate norms were revealed as students from the various classes expressed the reasoning for grading the proofs as they did. For instance, three of the four focus students from Mrs. C's class commented on the overall creativity of the proofs. They were willing to assign a higher grade to a proof they saw as creative or one that was built on an idea that they did not come up with themselves. Similarly, the focus students from Mr. D's class commented on the elegance or lack of elegance in the proofs, which is something Mr. D stressed with his students during the school year. The focus students from Mrs. A's and Mrs. B's classes appeared to be more concerned with the level of detail in the proofs. These students were less likely to consider the overall structure of the proof. Again, this seems to be in concert with the choices Mrs. A and Mrs. B made to spend class time critiquing the details of students' proofs, rather than the choice of Mrs. C and Mr. D to also reflect on the overall proof as a way of making sense of the geometric concepts involved.

A classroom practice that appeared to have an influence on students' ability to construct proofs was the taken-as-shared perception that drawing and marking a diagram was a necessary prelude to constructing a proof. All four teachers frequently reiterated the importance of this step in class by marking diagrams when they wrote out formal proofs and when they "talked through" a proof without recording statements or reasons. Diagram marking gave students an opportunity to make some progress on just about every proof. The practice of redrawing complicated diagrams, also emphasized by the teacher, was not very well followed by the students on the Proof Construction Assessment. The students who used this strategy were generally successful in the problem with overlapping triangles (Item 4, as shown in Figure 3 above).

Interviewer: Do you usually draw your own diagram or do you use the diagram given?

Aaron: Well, like if we are trying to prove overlapping triangles then I'll usually separate the triangles out, ... And then I'll also look at the one given ... so I could see the reflexive property, because if you break it apart you can't really see it. So I draw them out and I also use the diagram given.

Interviewer: So you mark things [on the diagram] as you are going through?

Aaron: Mrs. A usually tries to preach to us ... that it would probably help us to mark stuff down to like ... mark the givens and mark the ... things we see down, and write it out before we go through and start the proofs. I've caught on to that and it really helps.

A second mathematical practice that was noticed in both Mrs. A's and Mrs. B's classes was that students seem to know that they should identify the *given* and *prove* for the geometry proofs presented in their homework or other class discussions. This practice helped the students to articulate the logical beginning and the end of their written proofs. Some students who often failed to go beyond the beginning and end perceived this practice negatively. This was noted during clinical interviews when a couple of the focus students indicated that they believed their proof was valid because they knew where to start and end and they had some statements and reasons in between (which may not have been logically connected). This may be tied to the "follow-the-pattern" proof writing that was emphasized by Mrs. A and Mrs. B.

Two of the teachers (Mrs. A and Mrs. B) openly encouraged students to memorize definitions, theorems, and postulates. This became a taken-as-shared method of learning these important facts of geometry. Even so, the students in these classes often claimed to not remember the substance of the definitions, theorems and postulates and demonstrated this by incorrectly recalling the statement of the theorems when asked. For the Proof Construction Assessment, the students were provided with a list containing the definitions, theorems, and postulates as stated in their textbooks. Surprisingly, most students claimed they did not use this as a reference. The dialogue below occurred during an interview with one of the focus students from Mrs. A's class.

Susan: Cause like when she told us about Angle-Angle and all the ones that you can get with right angles, like Hypotenuse-leg, I never understood it. And I never remembered any of it...

Interviewer: Did you look at the sheet that was given that had the theorems and postulates listed?

Susan: It wasn't helpful, because I was kind of like Ok, so what does that say?

Interviewer: So as you were reading them off the sheet, that was difficult for you?

Susan: Yeah, since I didn't grasp the concept the first time when she taught it to us, it didn't really even matter that it was on the paper.

Although the other two teachers (Mrs. C and Mr. D) did not explicitly state the need for memorization of definitions and theorems, they expected the students to begin to learn these through frequent use. Students were encouraged to write out theorems in their proofs, rather than to write a title for the theorem, such as the Angle-Angle Similarity Theorem as referenced in the dialogue above. At this point in our research, it is not clear how familiarity with and understanding of theorems and definitions influence student proof writing ability. It is becoming apparent, however, that the teachers' pedagogical choices greatly influence the students' views of what constitutes a valid proof, as shown in the discussion of the interview questions related to students grading proofs done by peers. The students' growing sense of what constitutes a valid proof appears to have played a role in the students' proof construction.

As the analysis of our rich data continues, we hope to identify more clues related to the teachers' choices and aspects of the classroom microculture that influence student understanding of proof. At this point, our findings support those of Senk (1985) who identified aspects of proof construction that were difficult for students. It was most difficult for students to write formal proofs of statements with no hints given. We are beginning to notice, however, that teachers' expectations play a key role in student proof construction ability. In particular, it appears that if teachers focus on the overall structure and need for proofs in understanding the underlying mathematical concepts, students will also develop a better sense of the need for proof. On the other hand, if teachers expect students to learn to do proofs in a more mechanistic way, the students are likely to see proofs as just another exercise or application and will not develop a more complete understanding of proofs and how to construct proofs. These findings are supported by Battista and Clements (1995) who suggest that the teaching of formal proof should follow from helping students make sense of mathematical ideas. In other words, proofs should be seen as a way to establish the validity of ones ideas. Further analysis of our data is needed to determine how the teachers' expectations influenced student ability to construct proofs.

One surprising result was the difficulty students had with the local deductions problems. Although it is not part of our current plan to investigate this further, these open-ended deduction items may be a key step in creating formal proofs. Another avenue of analysis will be to investigate the students' proof construction ability in terms of the proof schemes described by Harel and Sowder (1998). These proof schemes are defined as what the student believes to be a valid way of ascertaining truth for herself or himself as well as persuading others of the truth of a situation or observation. For those students in Mrs. A's and Mrs. B's classes, it is unlikely that they hold very strong proof schemes, since from interviews with focus students, they appear to view proofs as exercises and not necessarily a means of ascertaining truth.

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CONSTRUCTING MATHEMATICAL ARGUMENT IN CONTEXT: AN ANALYSIS OF PROCESSES OF CONTEXTUALIZATION IN INTERACTION

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In this paper, I will report on an analysis of a group of undergraduate calculus students working on the construction of a mathematical argument. The analysis is centrally concerned with how participants use the *context of ongoing interaction* as a resource for their mathematical meaning-making. I borrow the sociolinguistic notion of *frame* (Goffman, 1974; Tannen & Wallat, 1993) to highlight the construction and use of context in the participants' activity. The analysis demonstrates that, in the collaborative development of a mathematical argument, (1) participants interactionally construct frames of interpretation through sequences of claims-assessments-justifications in talk (Goodwin & Goodwin, 1992), and (2) the coordinated use of talk, gesture, and representation serves to re-frame mathematical work, supporting the generation of new understandings of the mathematical activity. I will conclude with implications of analyses in the sociolinguistic tradition (and this analysis in particular) for understanding and improving the teaching of mathematics.

Introduction

Mathematical discourse in classrooms has increasingly become a focus of reform and research in mathematics education. Mathematics classrooms can be seen as communities in which the standards, norms, and practices of the communities of practicing mathematicians are developed and supported. As such, the fostering of mathematical discourse—the disciplinary discourse of the larger mathematical community—is a central focus of some classroom reform efforts in mathematics classrooms (NCTM, 2000). The processes of mathematical justification and argumentation are among the aspects of mathematical discourse that have received attention in reform efforts and the research literature (Harel & Sowder, 1998; Lampert, Rittenhouse, & Crumbaugh, 1996; Schoenfeld, 1985). Effective mathematics instruction should "de-emphasize the transmission of factual or otherwise incontestable information from teachers to students and should encourage the active involvement of students in discussing ideas, making convincing arguments, reflecting on and clarifying their thinking" (Forman, Larreamendy-Joerns, Stein, & Brown, 1998, p. 528).

However, despite an increasing rich body of research addressing mathematical justification and argumentation in classroom interaction (Forman et al., 1998; Hall & Rubin, 1998; Lampert, Rittenhouse, & Crumbaugh, 1996; O'Connor & Michaels, 1996) the nature of *in situ* mathematical justification from the students' perspectives is still not well understood—what and how students decide what is convincing, what is necessary to an argument, and what counts as valid in mathematical argumentation.

As with any social interaction, these issues are negotiated in ongoing interaction by the participants within their particular settings (social, cultural, historical, and technical). Understanding students' mathematical activity, in particular in reform environments that are organized to foster aspects of mathematical discourse, involves the examination of how students interact and collaborate to construct situated meanings of the mathematics. As such, the *context of ongoing activity* as a resource for participant interpretation should be a primary focus of an analysis of mathematical meaning-making (Cobb, 2000). In this analysis I will borrow the sociolinguistic construct of *frame* to highlight the construction and use of context in ongoing mathematical activity. How participants collaboratively construct frames for understanding their work and how those frames shift in ongoing activity are the central questions in this investigation of students' joint construction and evaluation of mathematical argument.

Theoretical and Analytical Orientation

This paper is centrally concerned with how *context* shapes and is shaped by ongoing interaction. Although the notion of context holds a wide variety of meanings in different research perspectives, for my purposes, context is the set of relevant psychological, cultural, environmental, and technical resources that participants bring to a situation to attempt to accomplish particular goals or make meaning of an interaction (adapted from Van Oers, 1998, and Goodwin & Duranti, 1992). This is not to suggest that all the participants in a particular situation necessarily hold the same meaning of the interaction or are deploying the same set of contextual resources. An individual makes particular situated meaning of her activity supported by that context in which that meaning is functioning. Neither is context static or objectively determined by the characteristics of the situation. Contexts change what is considered relevant to an activity and what can be drawn upon to support that activity changes. And, finally, these changes do not occur independently of the participants and their activity; participants "dynamically reshape the context that provides organization for their actions within the interaction itself" (Goodwin & Duranti, 1992, p. 5).

How is this *process of contextualization* accomplished? How do interactants collaboratively construct contexts that support the situated meaning-making of their activity? How do they use context to make meaning? One approach to this question, rooted in Activity Theory (Engestrom & Miettinen, 1999; Van Oers, 1998), emphasizes *activity*—the interaction of participants and their purposive use of resources to accomplish goals—as the context for particular actions that provides resources for appropriate interpretation of those actions. Further, as the activity system shifts (as a consequence of the completion of an objective, in response to changes in the environment, in order to accommodate changes in the interaction order, etc.) the context of the interaction shifts. Thus, the set of relevant resources with which interactants make meaning of particular actions is reshaped via the previous actions of the interactants themselves—actions provide a frame for interpreting subsequent actions (Van Oers, 1998, p. 480). In discursive activity, talk, gesture, and purposive tool use can be seen as forms of action. Interactants shape context and are shaped by context through their talk, use of gesture, and use of cultural artifacts, which serve to accomplish goals in ongoing activity (Hall & Greeno, 1997; Saxe, 1991).

In this analysis, I will be using the sociolinguistic notion of *frame* to describe the construction and use of context in ongoing interaction. In the most general sense, Goffman (1974) has defined frames as "schemata for interpretation" or "the organizational and interactional principles by which situations are defined and sustained as experience" (p. 21). In other words, a "definition of what is going on in interaction, without which no utterance (or movement or gesture) could be interpreted" (Tannen & Wallat, 1993, p. 59). Frake (1977/1997) warns, however, that frames should be seen as accomplishments of *interaction* through which people "propose, defend, and negotiate interpretations of what is happening" (p. 37). Similarly, Tannen and Wallat (1993) argues that since interactants communicate messages about "what is happening" by their behavior, "frames emerge in and are constituted by verbal and nonverbal interaction" (p. 60). Thus, frames are resources for 'seeing' or making meaning of ongoing interaction that are interactionally constructed via the actions—both verbal and nonverbal—of the participants. Thus, the process of *contextualizing* can be seen partly as a problem of framing.

The notion of frame has recently been used to examine discourse in mathematics classrooms, and argumentation in particular. In their analysis of how a teacher frames collective argumentation in a mathematics classroom, Forman et al. (1998) use the concept of frame as a way to understand the development of the shared expectations among community members about mathematical argument. They focus on how the students bring different frames to the discussion, and how the teacher re-frames the activity through revoicing moves. I will be using the notion of frame in a related but somewhat extended fashion—not only as a way of talking about how the participants interpret the kind of activity they are engaged in (e.g., collaborative problem-solving, collective argumentation), but also how the participants are interpreting, or 'seeing', the objects of the activity—the mathematics.

Data/Participants/Setting

The study was conducted in a discussion section of an introductory calculus course where the sections were organized as "workshops" (Fullilove & Treisman, 1990). In these sections, students worked in small groups at blackboards during problem solving sessions while the graduate student teaching assistants (TAs) were expected to actively support student interaction and to emphasize the need for mathematical explanation and justification. With the TA of our focal section, we selected one group of three students and several problems assigned during the section for close analysis. We chose problems that explicitly required proof or justification and that contrasted analytic/geometric content in their design. The subjects of the study are three undergraduate students—two female (Ariel and Amy) and one male (Josh)—and a male graduate student TA (TK). The data was collected in a university classroom in which this section was regularly held. There were a total of about 24 students in the section and they worked at the blackboards in groups of 3 or 4. Data consist of (1) field note observations of three weeks of section (six sections total) that occurred about a third of the way into the semester, (2) primary video and audio recordings of the group during the two days when the groups were working on the focus problems, and (3) video-elicited interviews subsequently conducted separately with the group of students and the TA. Selected video/audio segments from the group's problem solving session in their section were reviewed and served as the basis for a common set of semi-structured questions for these interviews.

Orientation to the Episode

For this paper, I will focus my analysis on one particular episode during the two days of problem solving that were videotaped. On the day of this episode, the discussion section is working on problems from the problem sheet "Monotonicity and Concavity." At this point in the course, they had covered the concept of derivative and some applications of derivative (especially curve sketching). The episode consists of the students' work on part (b) of the third problem they worked on that day, a three part problem about the quadratic polynomial: (a) Show that a quadratic polynomial $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, always has one critical point and no points of inflection. When is f concave up? When is f concave down? (b) How can you tell if a quadratic polynomial has two roots? One root? No roots? (c) Suppose that f has two real roots, r and s. Show that $f'(r) + f'(s) = 0$. Also show that the critical point of f is midway between the two roots.

Part (b) is intended to be a derivation of the discriminant test for the number of roots of a quadratic polynomial, starting from the relationship between the critical value and the concavity. One possible way to approach the problem is on a case-by-case basis, considering the conditions on the critical value and concavity for each of the three cases: two roots, one root, and no roots. This approach requires a geometric understanding of the relationship between the concavity of the parabola, the location of its critical point, and the x-intercepts of the parabola.

Analysis of Episode[1]

Phase One: Making Sense of the Task (lines 1-57)[2], [3]

This episode begins as the students turn their attention to part (b) of the problem. Amy reads the problem aloud—"how can you tell if a quadratic polynomial has two roots, one root, or no roots." Josh and Amy each suggest possible methods of approaching the problem: factoring, using the quadratic formula, and using the discriminant. There is some discussion (and a misidentification) of the discriminant, but Josh and

Ariel soon agree that these methods are not appropriate for this task—they should not have to "remember back to algebra" to solve this problem, and they should be using "stuff that we know how to do right now," i.e., calculus. Josh then considers whether one can tell anything about the roots of a quadratic polynomial, which is what the task is asking about, from the derivative, which is what they are currently studying; Ariel and Josh tentatively dismiss this notion. Ariel then suggests graphing "it," and Josh responds with a short laugh and reiterates that "there are a lot of ways but it seems like they want us to use calculus and not (.) um algebra." After some moments of silence, Josh suggests that they "imagine a quadratic." After a brief pause, he produces an explanation, in talk and gesture, of when a quadratic has no roots—"since it only has one maximum, it only has a maximum or a minimum (.) right? then if the maximum is greater than zero then...it has no roots." This statement is immediately questioned by Ariel ("well...it depends"). Finally, Josh produces another explanation, again in talk and gesture, that describes when an "open up" parabola has no roots ("it's greater than zero"), one root ("if it's equal to zero"), and two roots ("if it's less than zero").

Constructing shared frames for seeing: How talk and gesture shape context.

For most of this first phase of the episode, the students can be seen to be negotiating a way of seeing this task—a way of seeing that is relevant in and appropriate for their particular situation as calculus students solving problems in this section. In other words, they have framed their task as one involving the mathematics of calculus. How is this work of framing achieved? How is the negotiation of a shared frame of understanding accomplished in interaction? This discussion of phase one highlights the participants' use of particular structures in talk to interactionally construct context; specifically, how sequences of *claims/suggestions*, *assessments*, *justifications,* and *repairs* construct frames for shared understanding of activity. Before beginning the discussion, I will illustrate how I used these constructs in the analysis. *Claims* are assertions of fact: "no you shouldn't you can't tell anything about the roots from the derivative..." or "well, thats one of those places that you can't use the: the second derivative test..." *Suggestions* are statements that suggest the possible utility of some idea or procedure: "you can (...) isn't there that discriminant thing..." or "graph it and?" Throughout the episode, claims and suggestions were used to introduce ideas, concepts, or methods into the context of the ongoing interaction.

Following Goodwin and Goodwin (1992), *assessments* are seen to be statements or other types of action that (implicitly or explicitly) evaluate the claim/suggestion, usually affirming or rejecting it with some justification. In the simplest case (without justification), a claim is affirmed or rejected and the activity progresses:

Josh: (4) no you shouldn't you can't tell anything about the roots from the derivative (.) can you? (3) the roots of the for of the form of the // function] itself?

Ariel: //no::::]

In another example, Ariel's suggestion is implicitly rejected by Josh and this assessment is justified on the grounds that her suggestion is inappropriate:

Ariel: graph it and?

Josh: ((LF)) (2) there are lots of ways but it seems like they want us to use **calculus** and not (.) um algebra

A turn sequence of claim/suggestion-assessment-justification can sometimes include repairs which then serve as claims that are also evaluated. In the following example, Josh gives a verbal and gestural explanation of the conditions under which a quadratic polynomial has no roots, a claim of fact. Ariel challenges his claim ("it depends"), implying that his statement did not sufficiently address the conditions necessary for his conclusion. He then repairs his initial claim, again employing both talk and gesture, producing a statement that is more complete, coherent and consistent in structure (lines 41-57):

41 Josh: =if the (.) it's [since it only has one maximum]
42 *[{traces a concave up parabola in space}],*
43 it only has a [maximum or a minimum (.)] right?
44 *[{points above and below the traced parabola}]*
45 then if the [maximum is greater than zero]
46 *[{points to a 'maximum' then points down to 'zero'}]*
47 Josh: //then] right then //then] [it has no roots]
48 Ariel: //oh] //well] *[{points to a point above 'zero'}]*
49 Ariel: //it depends]
50 Josh: //if it's]
[if it's] whatever [if it's open up] right?
51 *[{traces cc up parabola}]* *[{traces cc up parabola}]*
52 and [it's greater than zero then it has no roots]
53 *[{points to a point above 'zero' in space}]*
54 [if it's equal zero then it has one root]
55 *[{points to 'zero'}]*
56 [if it's less than zero then it //has:] two roots.]
57 *[{points below 'zero'}]*

By identifying the occurrences of these kinds of turn sequences and their relations to each other in the ongoing interaction, we can see how shared frames are interactionally constructed. As claims are made, evaluated, and repaired, the frame of activity shifts or is refined, thereby shaping the kinds of claims that can subsequently be made and how they are evaluated. In this way, the participants negotiate in ongoing interaction the space of possible contexts and content, mathematical or otherwise, in and with which they do their joint work.

As mentioned above, this phase begins with Josh suggesting two possible approaches to investigating the roots of a quadratic polynomial, which he is then immediately skeptical of—"by factoring generally by using the quadratic formula (.) (but??) ((LF))." Overlapping Josh's expression of doubt, Amy begins to suggest "the discriminant," though she self-repairs twice, softening her statements with each repair, until she finally asks "what's the discriminant." Josh then implicitly evaluates and rejects these suggestions on the basis that they should not need to appeal to content from previous mathematical experience: "we should be able to do it without having to like (.) remember back to..." At this point in the interaction, we see that Josh is beginning to frame their task as one located in the present, without having to reach back to concepts like the discriminant or methods like factoring or using the quadratic formula. Ariel immediately agrees and Josh elaborates: "it should it should it should show us this without having to like (..) remember back to algebra." Overlapping with Josh's utterance, Ariel then asserts that "it should be stuff that we know how to do right now." Josh's next utterance and shift in gaze—"so:: *{looking at the board}*"—indicates that he has resolved for himself a way of understanding what realm of mathematics is relevant for their task and is moving on. This extended and overlapping turn sequence shows that Josh and Ariel are interactionally constructing a frame for understanding their activity. By limiting the space of possible approaches to those involving the current content, which at this point in the course involves derivatives, they have established a frame within which subsequent claims regarding the specific mathematics of the task are generated and evaluated. However, they have not yet have a way of *understanding* the mathematics of the problem; they must further refine their shared frame and use it as a resource for seeing how one can tell the number of roots of a quadratic polynomial.

Josh next begins to consider a hypothetical ("if it has") which Ariel picks up ("if it has two roots then::"). Josh then responds with "no you shouldn't you can't tell anything about the roots from the derivative (.) can you? (3) the roots of the for of the form of the function itself?" It is significant that this is the first time that the word "derivative" is mentioned in this episode. It is a reasonable conjecture that Josh is making a self-assessment of his own hypothesized consideration of the roots of a quadratic polynomial in relation to the derivative, answering the unstated question "can you tell anything about the roots of a quadratic from its derivative." His negative self-assessment highlights the salience of the frame (he is now considering how concepts in "calculus" not "algebra" relate to the number of roots) as well as his current framing of the task—as one in which derivatives are important players. Ariel then agrees with this statement and Josh restates his claim tentatively: "I don't think you **can**" At this stage, Ariel suggests that they "graph it?" to which Josh responds "((LF)) there are lots of ways but it seems like they want us to use **calculus** and not (.) um algebra." Josh is clearly assessing Ariel's suggestion as inappropriate (and perhaps even laughable). Though it is not obvious what Ariel meant by "graph it," it seems clear that Josh interprets this from the "calculus" frame and rejects her suggestion, reiterating that it is the current "calculus" material that they should use rather than "algebra."

However, her suggestion to "graph it" has introduced another element into their frame for understanding the task—the graphical representation of the quadratic polynomial. As he stated in the interview after viewing the videotape, "it's possible that Ariel cued me with hmm saying "You should graph it," or something like that, 'cause then I started thinking about the shape of the quadratic." In the next utterance (line 37), Josh suggests "well let's imagine (2) a quadratic," an invitation to the other participants to join him in visualizing a quadratic function, to imagine the graph of the function. And he does, in fact, through this new frame, start to see "how it should be conceptually." His statement in lines 41-47 (see above), his first try at an explanation, draws upon a geometric understanding of the situation, the language of derivatives ("minimum," "maximum"), and gestures illustrating and clarifying his talk.

With this performance, he appears to have not only produced an explanation of how you can tell if a quadratic polynomial has no roots which locates the quadratic in the realm of "calculus," but also created a shared frame for Amy and Ariel by animating the parabola in a plane in shared gestural space. They can *see* how he sees it because he has literally drawn his way of seeing for them. It is plausible that Ariel's overlapping response of "oh" (line 48) indicates that she is also seeing a way of understanding how you can tell if a quadratic has no roots (Schiffrin, 1994). This interpretation is corroborated by her subsequent challenge: "well...it depends" (lines 48-49). Josh's performance in lines 41-47 is not unproblematic; what he is saying—that if the parabola has a maximum above the x-axis then it has no roots—is the not the case for the concave up parabola he is gesturing. A concave *down* parabola has no roots if its maximum is above the x-axis, and a concave up parabola has no maximum at all. By saying "it depends," Ariel shows that she understands that his explanation depends on the concavity of the parabola, and that what he is *saying* is only true for a concave down parabola. She is sharing his frame for understanding the problem, but is pushing for further refinement of the explanation. Finally, he repairs his explanation, producing an argument that is more coherent, structurally consistent, and complete (lines 50-57 above).

This first phase ends with a fairly clear verbal and gestural expression of Josh and Ariel's current frame for interpreting this problem. It involves concepts in calculus that they have recently studied (extrema (maxima/minima) and concavity), is based upon a geometrical understanding of quadratic polynomials and derivatives, and is organized in a logical case-by-case structure. As will be seen, this way of seeing the problem is carried by the participants and used to accomplish further objectives in their activity. It is not merely an explanation, a historical artifact in their ongoing interaction, but a way of seeing that they come to use flexibly as a contextual resource.

Phase Two: Writing up the Argument (lines 58-122)

This second phase begins after Josh's final explanation as the focus of the activity shifts to the blackboard. Ariel begins writing down symbolically the first of the cases that Josh has just explained. She writes "if a>0" (concave up) and the condition that the critical point is "greater than zero"—meaning that it is above the x-axis—which she represents as "-b/2a >0." At Josh's suggestion, she has incorrectly taken "-b/2a" to mean the y-value of the critical point; however, none of the students dispute this and they continue to jointly write up the conditions for each of the three cases in the original task: no roots, one root and two roots. At several points there is some confusion, in particular about the case of "one root" and what happens if a=0. At these points, Josh gives an explanation that satisfies Ariel and they continue constructing their table of cases.

Coordinating frames for seeing the mathematics to produce a solution.

Throughout the episode, the participants coordinate their talk with gesture and representational use in order to not only elaborate and illustrate their speech but also to organize and support their understanding. In other words, coordinated talk, gesture and representations provide frames for interpreting their ongoing activity. At the start of this phase, an uncontested verbal and gestural explanation of one set of conditions under which one can tell whether a quadratic has two roots, one root or no roots has been produced. As stated earlier, the work that the students now begin is the transformation of this understanding into a complete, coherent, clear and durable representational artifact. This activity begins with Ariel announcing "so::", picking up chalk, and moving to the board. In doing so, she is implicitly affirming the existing argument and re-framing their activity to be one in which they will make durable and public what had up to this point been verbal and gestural. In the several minutes that follow, Ariel and Josh jointly produce a chart of the different cases of quadratic behavior. At the start most of this work is unproblematic—Ariel writes the condition and conclusion, narrating her actions: "if a is greater than zero which means it's (.) concave **up** {writes 'if a>0' on the board}." She is also sometimes directed by Josh and follows his suggestions without challenge. During these kinds of exchanges, their talk refers exclusively to the symbols being written and to features of the inscription itself. They are sharing a frame for seeing their activity, writing up an argument, and in what manner that argument should be represented, symbolically in a case-based logical structure.

As they continue, however, there are points of contention at which the nature of their interaction changes. At these points, their talk is no longer *in* details of the production of the table, but *about* the content of the table and how it is coordinated with other ways of seeing how the cases are organized. For example, Ariel and Josh run into difficulties when she proposes "a=0" as a case. Josh responds by reasoning that if a=0, you will have a case where the second derivative test cannot be used. Rather than draw upon their geometric frame for seeing the cases, he uses a symbolically derived result from their work on part (a) of the problem, the second derivative test, to justify why they should not use a=0. And although Ariel affirms this reasoning, it is clear in lines 88-107 that she has not yet come to a stable way of seeing the cases symbolically:

88 Ariel: [if negative b over two a is equal to zero] (.) then [one root]

89 *[{writes -b/2a = 0 underneath a>0}]* *[{writes '1 root'}]*

90 Josh: across the board (.) for [that one also]

91 *[{points to a<0}]*

92 Ariel: no because=

93 Josh: =if it equals zero=

94 Ariel: =a isn't gonna exis

95 Josh: [if it's if it's concave up and it equals zero]

96 *[{makes cc up parabola with rt hand, then points to vertex of his parabola}]*

97 Ariel: right *{draws dividing lines in grid}*

98 Josh: then it's then it has one root.

99 [if it's concave **down** and the critical point equals zero (.) then it still has one root]

100 *[{makes cc down parabola with rt hand, then points to vertex of his parabola}]*

101 Ariel: right ok

In this strip, there is again confusion about the relation between "a" and the "no roots" case, in particular whether or not "–b/2a=0 and a<0" is a valid case of "no roots." Rather than appeal to the second derivative test again, Josh re-frames the case in terms of their shared geometric understanding. He again provides referents for his verbal statements using gestural animations of a parabola, its vertex, and relation to "zero." By re-framing the case in such a way, he is coordinating the geometric/gestural understanding they shared at the end of phase 1 with their current, somewhat unstable, way of seeing the problem in the symbolic/inscriptional arena. In his coordinated performance, he not only displays his own competence with respect to moving in and between the multiple 'ways of seeing' this situation, but uses these multiple frames effectively enough to get Ariel to see the logic of the table more clearly and coherently. Thus, we see the refinement of Josh's verbal/gestural/symbolic coordination.

It is an important to note that during their work, the mathematical object "-b/2a" takes on a meaning that is mathematically incorrect—that of the value of the critical point rather than the x-coordinate of the critical point. This object comes to represent something to them, via the coordination of gesture, talk and inscription, whose meaning is shared by the participants, yet is problematic in the view of normative mathematical meaning.

In this analysis we see that when the production of a particular inscription is unproblematic focus of the talk is on the inscription itself, often narrating what is being written. However, when there is disagreement about how to proceed or the accuracy of something written, the participants draw upon gestural and representational resources to help them re-frame their own or others' ways of seeing. In this way, the coordination of talk, gesture and representation can be seen to support the progress of the ongoing activity.

Conclusions

By examining the processes of framing that these individuals interactively engage in during their mathematical work, this analysis highlights the contextual resources that individuals bring to bear in a particular activity and how participants re-construct, re-negotiate—in short, re-frame—their interactions as their activity evolves. As we have seen, participants' frames may shift when the objectives of activity are met and new objectives arise. As the focus of their activity shifts from making sense of the task to showing their argument (first part to second part), their geometric/gestural frame of reference shifts to a symbolic frame of reference, which are, of course, not unrelated ways of seeing, but ways are complementary and rely on different contextual resources. These frame shifts impact not only the mathematical (psychological, representational) resources that individuals draw upon, but also the social, cultural and historical resources that individuals find relevant in doing their mathematical work. In addition to frame shifts that arise out of shifts in the objectives of activity, participants' use of assessment moves in talk contributes to the collaborative construction of context through the ongoing negotiation of frames for interpreting further activity. As claims are made, evaluated, and repaired, the frame of activity shifts or is refined, thereby shaping the kinds of claims that can subsequently be made and how they are evaluated. In this way, the participants negotiate in ongoing interaction the space of possible contexts and content, mathematical or otherwise, in and with which they do their joint work. Additionally, their coordinated use of talk, gesture, and representation serves to re-frame their work in alternate 'representational worlds', supporting the generation of new understandings of the mathematics.

On a general level, this analysis underscores the collaborative and contextual nature of this kind of mathematical work. In these discussion sections, students and instructors are expected to engage in conceptually challenging mathematics and interact with each other about that mathematics so that the participants' thinking processes are made visible and mathematical justification is a norm of their problem-solving activity. These students make relevant aspects of their thinking visible to others as they work to frame and re-frame their ongoing activity, and through their assessment sequences in talk their justifications for mathematical claims are brought into conversation. Thus, to differing degrees, these students can be seen as skilled practitioners of this classroom culture who deploy various representational, language, artifactual, cultural/historical resources to engage in those practices. Although an analysis of the students-TA interaction is missing from this paper,[4] I believe that analyses of this kind can inform the design and implementation of classroom environments like these. As instructors are increasingly being expected to skillfully support and manage student-student and student-instructor conversations in ways that are mathematically rich, personally meaningful, and relevant, it is important that we have a nuanced understanding of how students engage in mathematical activity as they struggle to make sense of the mathematics and their identities as students of mathematics. With this understanding, instructors may be able to more effectively capitalize on students' ways of talking and seeing as they support and manage interactions around mathematical content.

Notes

[1]In this following analysis, the episode will be presented in two phases. For each phase, I will first give a rough sketch of the events, then provide a fine-grained analysis of the construction and use of frames in each phase.

[2]Line numbers refer to the full transcript (not included for length reasons).

[3]Transcript conventions: , (.) short pause; (#) pause # seconds; . falling tone; ? rising tone; :: extended syllable; -- self-interrupt; = latching; no pause between utterances; **bold** emphasis/stress; //] overlapping talk; [ ] overlapping talk and gesture; ( ) text in parentheses is unclear; ((LF)) laughing; *{italics}* gestures

[4]This is included in a complementary analysis of the data corpus.

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Research Methods

ASSESSING STUDENTS' CONCEPTIONS OF REFORM MATHEMATICS

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Introduction and Perspective

As the use of NSF-sponsored, reform-oriented mathematics curricula has become more prevalent across the US, an increasing number of researchers are attempting to study the "impact" of reform. In particular, mathematics educators are interested in determining whether reforms are having the desired effects on students, particularly with respect to the learning of mathematical content and the improvement of attitudes about mathematics. In this effort, researchers have used a variety of methods, and have looked at a variety of variables, in order to assess the impact of reform. In many cases, such research assesses reform by looking closely at students' scores on tests or their strategies for solving certain kinds of problems. For example, Riordan & Noyce (2001) assessed reform's impact by comparing students' scores on standardized achievement tests. Other researchers have used structured interviews, classroom observations, and more interpretive or ethnographic methods to assess the impact of reform (e.g., Boaler, 1997). Both of these methodologies are useful in assessing the impact that reform mathematics curricula are having on students. An alternative evaluation of the impact of reform that has not been as widely used is through the use of survey instruments. Surveys have been widely and reliably used to assess students' motivation (Pintrich, Smith, Garcia, & McKeachie, 1993), beliefs and attitudes (Kenney & Silver, 1997), and interest (Köller, Baumert, & Schnabel, 2001). We propose to add to this literature by using a survey to study the impact of reform on students' conceptions of mathematics.

Background

Few survey measures have been developed for the purpose of studying secondary students' conceptions of mathematics. One such measure is the Conceptions of Mathematics Inventory [CMI] (Grouws, 1994). The CMI is intended to assess students' beliefs or conceptions about mathematics. Its 56 questions asked students whether they agreed or disagreed with certain statements about what it means to do, learn, and think about mathematics. The survey questions fall into seven categories (the composition of mathematical knowledge, the structure of mathematical knowledge, the status of mathematical knowledge, doing mathematics, validating ideas in mathematics, learning mathematics, and the usefulness of mathematics), each of which assesses a different aspect of students' beliefs toward math. Students' responses for each question were between 1 and 6, with "1" expressing strong agreement and "6" expressing strong disagreement. A student who mostly agrees with all questions would seem to have an attitude that the survey designers felt was consistent with the aims of recent reform documents. Such a student would view mathematics as being composed of a useful, coherent, and dynamic system of concepts and ideas, where learning is accomplished by sense-making and authority is found through logical thought. A student who mostly disagrees with statements on the CMI would find mathematics an irrelevant, unchanging collection of isolated facts and procedures, handed down from a book or teacher, that must be memorized; this is a view that the CMI designers consider more typical of traditional math curricula.

Grouws and colleagues (Grouws, Howald, & Colangelo, 1996) gave the CMI originally to 163 9th, 10th, and 11th graders in traditional mathematics classes. Their intent was to determine whether students in advanced mathematics classes had different conceptions of the discipline than students in "regular" track. Although the questions on the CMI are intended to indicate whether students' conceptions are consistent with the goals of reform, the CMI has never been administered to a large group of students in reform mathematics classes to "validate" its effectiveness. In other words, although it may seem obvious that students with extensive experience in reform mathematics would have more reform-oriented conceptions than those with extensive experience in traditional mathematics, we made an effort to empirically explore whether students in reform and traditional settings would respond differently on a validated survey instrument.

In this paper, we describe our attempt to validate the CMI. In addition to providing evidence of the effectiveness of the instrument, such a validation also allows the CMI to be used as a measure of the impact of reform, in that it establishes "baseline" values for reform and traditional students' conceptions.

Method

In conversations with the authors of the CMI (Grouws, personal communication, October 5, 2001), we became convinced that all of the 163 original respondents to the CMI (Grouws et al., 1996) came from traditional backgrounds in mathematics. In other words, although no data were collected about the schools, courses, or instruction experienced by students in Grouws and colleagues' original sample, we are confident that no student in this sample had any recent experience with NSF-funded reform-oriented curricula, and thus experienced something of a default approach to teaching mathematics that was more traditional than reform-oriented.

For the present study, and as a contrast, we sought to assess a sample of students who had as "pure" of a reform experience in mathematics as we could find. As part of our work in a number of teacher professional development and curriculum writing projects in Michigan, we developed relationships with several schools with exemplary enactments of reform curricula, particularly at the middle school level. Through these connections, we recruited 134 9th grade students from a high school in Michigan to complete the CMI, early in their 9th grade year. All students completed at least three years of reform-oriented instruction (in 6th, 7th, and 8th grades) in a middle school whose curriculum (Connected Mathematics Project, Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997) and instruction we were quite familiar with and that we are confident was an extremely well-enacted version of reform. Students were administered the CMI in their regular mathematics classes by their teacher.

Results

To summarize our main results, we found that students from a reform background responded differently to the items on the CMI than did students from a traditional background. In particular, students' responses in the reform setting were more aligned with reform-oriented ideas on the scales of the CMI than traditional students' responses. Table 1 shows students' mean scores on the six scales of the CMI, for both groups of students. Recall that all items used a six-point scale, and that the lower the number, the more reform-oriented the response. The differences on each scale are statistically significant, $p < .01$.

Table 1. Participants' mean scores (standard deviations) on CMI scales

	Compo-sition	Structure	Status	Doing	Validity	Learning	Useful-ness
"Pure" traditional	3.90 (0.9)	3.69 (0.8)	3.69 (0.9)	3.89 (0.9)	3.96 (0.9)	3.73 (0.8)	3.50 (0.8)
"Pure" reform	3.34 (0.5)	2.76 (0.7)	2.75 (0.7)	2.65 (0.6)	3.01 (0.6)	3.01 (0.5)	2.20 (1.1)

Discussion

Our results provide evidence that the CMI is a useful and valid instrument for assessing the impact of reform. We found that after experiencing several years of exemplary instruction in a reform curricula, students do develop conceptions of mathematics that are aligned with NCTM reform documents.

Assessing the impact of reform is a complex endeavor, requiring the investigation of many aspects of students' experiences and using multiple methods. We believe that assessing students' beliefs about mathematics is vital to this effort.

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CHANGING THE WAY WE STUDY STUDENTS' MATHEMATICAL MEANING MAKING AROUND A GRAPHING CALCULATOR USING INNOVATIVE VIDEO DEVICES

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This short oral report intends to examine the innovative use of video devices as a research method for better understanding the nature and process of students constructing mathematical meanings as they use graphing calculators. A TI-Presenter was used along with a video camera to record students as they solved college algebra problems around a TI-83 graphing calculator. The TI-Presenter is a tool which allows graphing calculators to output the calculator screen to video media such as a TV monitor or VCR. In this way, it is possible to view what students enter into the calculator and to see the screen as students see it.

Current reforms in mathematics instruction set forth the expectation that students are to explore, discover, make conjectures, and negotiate mathematical meanings with others by sharing and communicating about mathematical objects and representations. Such learning activities are particularly important in advanced mathematics because secondary students encounter complex mathematical ideas such as functions that often require multiple representations such as tables and graphs. Furthermore, advanced mathematical topics often involve students using technological tools such as a graphing calculator which allows them to switch back and forth between the multiple representations of functions (e.g., Dunham & Dick, 1994; Lauten, Graham, & Ferrini-Mundy, 1994; Wilson & Krapfl, 1994).

Earlier studies which examined students using graphing calculators experienced difficulties in really understanding what students thought or how they came to understand complex mathematical ideas since it was not easy to see students' calculator screens or keystrokes (e.g., Doerr & Zangor, 2000; Wilson & Krapfl, 1994). The calculator screens and keystrokes could be thought of as windows into students' thinking. The TI-Presenter, a tool originally intended for classroom instructional purposes, solves the methodological problems experienced in previous studies.

With the aid of the TI-Presenter and digital video editing software, it was possible to combine and integrate multiple video data sources (i.e. the video data from small group interactions around a graphing calculator and the video data from the graphing calculator screen) into a "picture in a picture" format. This made it possible to view simultaneously multiple data sources while analyzing and interpreting data.

This was a critical methodological step for it allowed the researcher to record students' calculator screens and keystrokes as they used the graphing calculator. The significance of this methodological innovation is that it fills in a serious gap in understanding how students make sense of ideas such as end behavior or asymptotes and how they use graphs and other representations to make this sense. The use of the multiple video data sources provides a much fuller and accurate account of students' developing mathematical understandings.

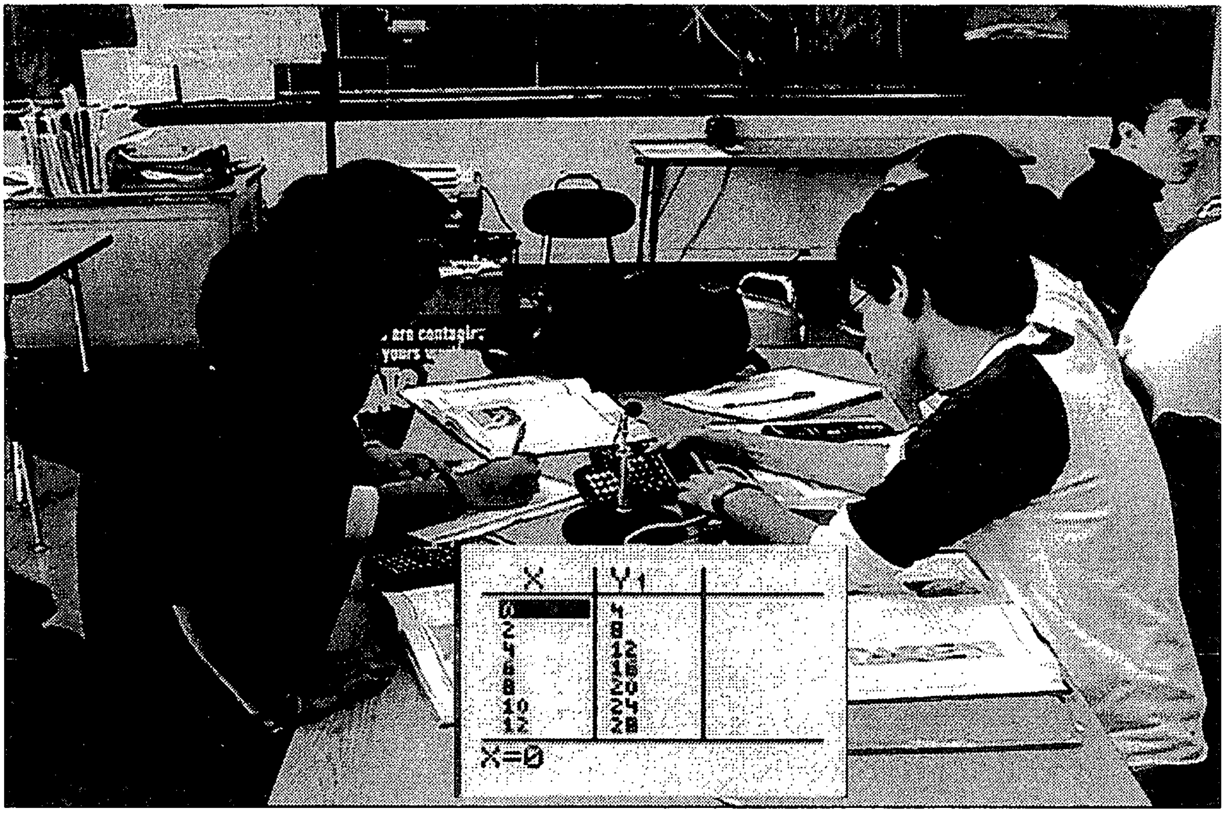

Figure 1

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Socio-Cultural Issues

THE NEXT GENERATION OF EMERGING SCHOLAR PROGRAMS:IMPACT, REPLICATION, AND ADAPTATION

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This study examines the impact of Emerging Scholars Programs (ESP) on student achievement, faculty beliefs, and institutional structure in mathematics-based major programs. Based on a case study, this investigation identifies key program elements that may be critical for successful replication and adaptation of ESP programs in mathematics at other institutions.

Theoretical Framework

In the early 1980's, Uri Treisman demonstrated the effectiveness of using highly structured workshop groups in the learning of calculus for African-American college freshmen (Fullilove & Treisman, 1990; Treisman, 1985). Drawing on the constructivist view of learning and the impact of social function on mathematics learning and self-efficacy, Treisman documented the effects of workshop participation on African-American students' achievement in first-year calculus. Bonsangue (1993, 1994) and Bonsangue and Drew (1995) replicated this result for a group of primarily Hispanic students majoring in science, mathematics, engineering, or technology (SMET), as well as documenting the positive effects of the program in the students' subsequent courses in their respective majors. In the ten years subsequent to Treisman's initial report, the workshop model, or Emerging Scholars Program, has been adapted for numerous professional and medical schools, universities, two-year and four-year colleges, and secondary schools across the country (Drew, 1996). Reports of the effectiveness and sustainability of such programs have varied considerably, based on the institution's expectations for success, structure for funding, and participation of students, key faculty, and institutional leaders. These reports have also documented the impact of such programs on faculty beliefs relative to student achievement, as well as fiscal benefits related to timely completion of degree requirements (Bonsangue, 1994, 1999).

Methods

This study examined an on-going ESP program at the University of Houston and its impact on six participating institutions within the UH system which adapted such programs at for their own students. These institutions include both four-year comprehensive universities and two-year colleges. As part of a National Science Foundation grant awarded to UH, the authors served as external evaluators to the Houston Louis Stokes Alliance for Minority Participation (H-LSAMP) project. This study was based on data collected as part of that external evaluation.

The HSLAMP alliance includes The University of Houston, Texas Southern University, Southwest Texas State University, The University of Houston - Downtown, The University of Houston - Victoria, The Houston Community College System, The San Jacinto College District, Rice University and the Houston Independent School District. The first seven institutions are funded directly by NSF; the latter two institutions are project participants but do not receive federal funding.

Both qualitative and quantitative data were collected over a 1.5 year period from November, 2000, to May, 2002. Quantitative data included student achievement in SMET courses, institutional financial support, and faculty participation. Qualitative data included interviews with students, faculty, and site administrators. Results were based on comparison of student achievement, including graduation and transfer rates, with baseline data for each institution, together with both documented and perceived impact by faculty, students, and institutional leaders.

Results

Initial results showed positive effects on student achievement and institutional adaptability for participating institutions. Interviews with participating students showed extremely enthusiastic support for the continuance and development of such programs. Interviews with faculty and site leaders showed enthusiasm mixed with concern relative to faculty rewards and fiscal priorities. Identifying specific areas of success and topics of concerns may help to clarify the critical elements of an ESP program to be successfully adapted and developed, not only at the institutional level, but at the system-wide and state-wide levels as well.

Increase in SMET Degrees

Based on discussions with AMP participants and leaders, it was evident that there was a strong consortium commitment to meeting the AMP goal of doubling the number of underrepresented minority graduates in science, mathematics, engineering and technology (SMET) majors over the next five years. Each campus had made considerable progress towards defining its specific goals and adapting its programs to identify and support AMP scholars who are planning careers in SMET disciplines. Moreover, in the first year of operation, the number of bachelor's degrees given to minority students in SMET fields increased dramatically from 458 to 640, a jump of nearly 40%.

Importance of Outreach Activities

One top administrator mentioned that many students define their first semester as a time to have fun—after all, for many this is their first time away from home—and, consequently, receive a low GPA which, then, undermines their future opportunities in college. Several of the STEM students with whom we met emphasized the importance of reaching out to high school freshmen and sophomores to let people know about the options for support, mentoring, and achievement in college. Based on our conversations with students, faculty members, and staff at the LSAMP institutions, we can suggest several mechanisms that might facilitate more effective recruiting:

- Contacting high school students early in their career.
- Reaching out to home schooled students. Home schooling is a growing movement and these students represent an increasing segment within the undergraduate population.
- Communicating with high school counselors. The high school counselor, the person who advises students about college options, is a key person in the STEM pipeline. Counselors vary in quality and background, as do deans of instruction and head counselors. Perhaps special meetings could be held each year with the counselors, or head counselors, from all feeder high schools in the area, not just those high schools in the Houston Independent School District. Also consider personal phone calls and separate meetings with each of the counselors.

Strength of Diversity

The SEP program at the University of Houston seems to be an enormous success. We met dozens of students who participate in the program, and interviewed seven of them at length. More to the point, we were able to visit the rooms in which the SEP program is housed and to observe the student culture created there. We saw a diverse group of students representing many ethnicities and different points in the college continuum, from freshmen to senior. It is clear that students are drawn to the SEP in part because of the community atmosphere that is created; that is, they are drawn in part because this is a fun place to hang out with friends. However, while the term is relaxed and friendly, virtually all students realize they are there for serious study. The result is that a student who seeks out the community and friendship of others in the SEP will find that the norm is to study and likely will find himself or herself working hard in STEM courses. As in all successful workshop programs, mentoring, working together on homework problems, and solving extra homework problems to excel, all are emphasized. We heard comments that the SEP "brings a lot of intelligent people together." At the University of Houston, there are about 140 scholars, of which 50-75 are in their first or second year. One student said the SEP is "more than friends, a family."

The University of Houston, under the leadership of Dr. Sylvia Foster, has created the largest, and arguably the most successful, SEP program in the country. The mentoring and community building activities at the University of Houston can be held up as a model for the rest of American higher education.

Most participants are students of color. Many are the first in their family to attend college. But their self-confidence, their achievements in STEM courses, and their aspirations are impressive. At the University of Houston, we talked with students majoring in STEM disciplines who have serious aspirations toward graduate study, toward becoming teachers, a future attorney, a future physician, and a future astronaut (a graduate student at Rice also indicated aspirations to become an astronaut. Given the proximity of the Johnson Space Center, and the achievements of these young people in scientific disciplines, these aspirations do not seem unrealistic).

Role of Expectation

The concept of aptitude often has been misused as an excuse not to educate many young people in mathematics and science. This is because many adults who guide and control the fate of young people in elementary and secondary school, e.g., teachers, counselors (and sometimes parents), often hold the erroneous belief that some groups of students lack the aptitude to master these subjects. These groups include girls and young women, students of color, and students from poverty. The data to show that, while the assumptions these adults hold about lower aptitude are invalid, these negative adult expectations often translate into organizational barriers that prevent students from taking certain subjects, and psychological barriers when the students incorporate those negative judgments into their self-images. Programs like the Uri Triesman (1985) workshops attempt to reverse the pattern: to expect students to excel and to create a mentoring and community support system that guides them toward achievement. At the University of Houston, we saw a wonderful example of how this type of program can succeed. We were told that the SEP students are welcomed by instructors, that the instructors know which students are in the SEP program (often from the t-shirts that they wear proudly), and that the instructors know that "we are smarter than the other students." This comment was made by the same student who acknowledged, in response to a question, that the SEP program is open to any student who wants to participate, and who meets some minimal standards. This is a classic example of the concept of aptitude being employed in a positive domino effect. Students join a community of scholars; they work hard to complete their assignments and to excel on tests; when they do so with the aid of the mentoring available through the community of scholars, they, and others, begin to assume that they are smarter than other students.

Just as high school students need to be told that they can be college material, that they can succeed in college, so, too, do undergraduates need to be told that they have what it takes to succeed in graduate school. We also discussed the need for successful STEM undergraduate and graduate students to balance the twin goals of successfully developing their careers while giving back to their community.

In the exit interviews that Southwest Texas State University conducts with LSAMP scholars who are graduating, they ask about the greatest benefit they received from the program. The most frequent responses center about leadership skills and communication and presentation skills

Implications and Recommendations

Based on our visit and the documentation of the program, several specific suggestions emerged. These are relevant not only for the program under study, but as a national model for other programs as well.

1. The H-LSAMP is important, not only for its effects on student achievement, but for it effects on the institutions involved. The presence of the community colleges in the consortium makes both of these even more significant. Strong communication and effective cooperation between participating two-year and four-year institutions will play a major role in the success of this AMP. While discussions between neighboring institutions have begun, all H-LSAMP institutions are encouraged to work together to track the progress and success of AMP scholars that transfer from one institution to another. Typically, academic departments or schools are not directly involved with this process. However, direct involvement by AMP leaders will not only help facilitate a successful transition for its students, but will help define and measure the success of specific goals, such as doubling the number of SMET underrepresented minority students who transfer to a four-year institution.

2. Each institution should articulate in its strategic plan what activities it proposes in the areas of recruitment and retention, if it has not yet done so. Different institutions should, and will, emphasize each of these two activities to varying degrees. For example, in one of the larger institutions with substantial minority SMET enrollment in the junior and senior year, a greater increase in degrees could be linked to aggressive retention efforts, for a smaller investment, perhaps than recruitment efforts. At the other extreme, recruitment will be the first priority for institutions that currently enroll few minority SMET students.

3. Several of the institutions participating in the Houston LSAMP have successfully integrated the LSAMP student support program with their existing and ongoing programs, for example the SEP program at the University of Houston. This is an effective implementation procedure and we applaud the skill with which it has been carried out. However, it does make assessment and evaluation difficult, as it is sometimes a challenge to sort out the effects on a given individual, or group of individuals, of specific components of a highly effective group support program. For example, one of the most articulate students with whom we met at the University of Houston spoke about the contributions of the SEP program and made observations that, we're sure, apply also to the LSAMP program—despite the fact that she is not an LSAMP scholar.

4. Under the reasoning that the most successful students at four-year institutions will be those who attend full-time, LSAMP administration has required that awards be given only to full-time students. But the community colleges are a rich source of potential high achievers in STEM disciplines. Many of these students work full-time and have family responsibilities and other financial obligations. At best, some of them

can attend college only on a part-time basis. Relaxing the full-time requirement at community colleges so that these highly-motivated potential future scientists, mathematicians and engineers can receive some funding could have a dramatic positive impact on the success of this LSAMP.

Summary

The development of ESP-type programs has perhaps been the single greatest factor underlying the mathematics reform movement in calculus at the university level (National Research Council, 1990; Selvin, 1992). Indeed, in 1992, Dr. Treisman was given a MacArthur "genius" award for his pioneering work in this area. The current study may be important because it examined and identified critical elements for initiating and sustaining ESP-type programs at the institutional and systemic levels. This study also documented the interactive effects of ESP programs on student and faculty beliefs and expectations.

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LISTENING AS A VITAL CHARACTERISTIC OF SYNERGISTIC ARGUMENTATION FOR ENHANCED MATHEMATICAL LEARNING IN A PROBLEM-CENTERED ENVIRONMENT

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From research observations of activities during a second-grade mathematics problem-centered learning environment classroom, synergistic argumentation emerged as a class norm for discussing the students' mathematics. In this paper we analyze the contrast between two students who were participants in this class. One students Brett, depicts the use of argumentation successfully while the other, Miriam, depicts the use of argumentation ineffectively. An important aspect of Brett's argumentation was identified as hermeneutic listening. Brett's engagement enhanced the learning environment whereas Miriam's stance was counter-productive, at least for her.

Purpose

Our research over the last several years has investigated a second grade mathematics classroom where the teacher has enacted a problem-centered learning environment (Wheatley, 1991; Wheatley & Reynolds, 1999). During the last three years we have focused in particular on the quality of discourse and argumentation that occurs during the whole class sharing time in each lesson. The two-fold purpose of the study has been to describe the mathematics whole-class sharing session and to analyze its function for effective learning occurring through conflict and disagreement in the open explanation of solutions and strategies.

In the second year of this investigation two students in particular provided us with contrasting pictures of discourse and argumentation during their involvement in this class. One depicts the use of argumentation successfully, while the other depicts the use of argumentation ineffectively. While our overall goal was to analyze the whole-class interaction patterns of a variety of students, focused in particular on the conflict of ideas and the ways of solving problems, in this paper we will describe and analyze episodes that involve these two particular students. By presenting their contrasting stances as they engaged in the sharing of ideas we propose that a deeper understanding of the nature of discourse and argumentation from the young student's perspective is provided.

Theoretical Framework

In recent years there have been increasing numbers of investigations into classroom environments in which children talk openly about mathematics and explain their mathematical solutions in order for teachers to facilitate students' mathematical understandings. In this environment the teacher acts as facilitator and organizes instruction so that students are actively engaged in the doing and talking interactively about mathematics. The National Council of Teachers of Mathematics (NCTM, 1991, 2000) calls for students to construct meaning, learn different ways to think about ideas, reflect on and clarify their own thinking, and develop arguments.

The problem-centered learning environment classroom encourages students to collaborate and work together in developing mathematical communication based on openly shared mathematical activities (Wheatley, 1991). One characteristic of the problem-centered classroom is student small-group collaboration, where students work in pairs on problematic tasks for which solution procedures have not been provided or demonstrated by the teacher. Collaboration provides students opportunities to talk, to listen, and to negotiate not only the social norms (rules of behavior) but the socio-mathematical norms as well. The socio-mathematical norms are different from the social norms in that they are specific to students' mathematical activity (Yackel & Cobb, 1996), whereas social norms are rules of behavior developed and accepted by the students and teacher and can apply to any subject. In pairs mathematical meanings are under continuous negotiation as children work collaboratively to complete an assigned task. Mathematical meanings that arise between them are formed as part of their interactions within their small groups and then with the wider community in the whole-class sharing time.

Another characteristic of the problem-centered learning environment is the whole-class sharing time. The teacher and students negotiate mathematical meanings as they present their solutions and strategies and talk about their mathematics in this wider community (Cobb, Yackel, & Wood, 1992). The negotiation of socio-mathematical norms defines what kind of talk is valued in the mathematics classroom (Cobb, 1998; Kazemi, 1998). These norms are not rules that students must fulfill, but they deal with the process of making mathematical contributions toward learning.

Mevarech (1999) argues that collaboration and whole-class sharing allow children to learn through opportunities to explain, justify, and listen to one another's ideas. Further, he contends that explanations are the best means for elaborating meaning and making connections. These conditions help students to construct rich networks of meaning. As students share their explanations they must actively communicate with each other and the teacher. In order for that communication to be successful, they must negotiate meanings, not just recite facts. As they negotiate they adjust their interactions by presenting rationales for their strategies. Working together gives rise to learning opportunities as students express their thoughts while attempting to understand each other's ideas.

In a culture that demands students' understanding, teaching is more than merely informing or demonstrating for pupils. Teachers must enable pupils to create meaning through their own thinking and reasoning (Wood, 1999). Teachers and students facilitate mathematical discourse by questioning and challenging each other to defend answers. In this way students have opportunities to talk about what they hear and understand and learn from one another's interactivity, and are not limited to just taking turns in a discussion (Kazemi, 1998). Their discussions relate to the mathematics at hand, making comments to gain understanding and/or clarification.

Setting for this Research

This research was conducted in a second grade classroom in a low to middle income suburban public school that was also a site for the district's emotionally handicapped program. The students were from various backgrounds and had varying abilities and disabilities. The teacher had enacted a problem-centered learning environment for a number of years (Wheatley, 1991; Wheatley & Reynolds, 1999). The research reported here is part of a three-year study into the discourse and argumentation process students engaged in during the whole class portion of their mathematics class. The data analyzed here is taken from the second year of this study.

Methodoogy and Data Sources

To frame this research, we used Cobb and Steffe's (1983) teaching experiment methodology where the researcher is a facilitator participant observer, working with the teacher and students, and listening to their interactions and ideas (Steffe & Thompson, 2000). We observed the second-grade mathematics lesson at least one day each week throughout the school year, monitoring and questioning students where appropriate to clarify their thinking. Extensive field notes were taken during each observation period and supplemented by video recordings. Observations of students in their small group problem solving, as well as interviews with teacher and students, provided a variety of explanations and interpretations during the whole-class events. The individuals' and groups' mathematical activity are interdependent. Therefore, students' participation in the small groups and the whole-class discussion gave insights into their mathematical development, and supplemented and helped when interpreting the whole-class discussion in greater depth. Particular attention was given to the mathematical activity during the whole-class discussions in which discourse was prevalent in students' construction of mathematics.

A constant comparative method guided the analysis of socio-mathematical interchanges that might lead to individual mathematical constructions (Strauss & Corbin, 1998). After each session, we examined the whole-class discussion data, which was first separated into specific event sections as frames in which to focus subsequent classroom observations, interactions, and interviews. Independently, we coded and categorized each of the data sets within each event, looking for regularities and patterns in the discursive process of students across the sets. Using the constant comparative method and more than one analyst's coding furthers the study's trustworthiness or fidelity by considering the classroom in its natural state without prior presupposition (Schwandt, 2000).

Analysis of Data

It should be noted that when the majority of the students entered the second-grade they were not used to presenting their solutions to the class in the ways they were now encouraged to do in this second-grade mathematics classroom. Although they had been encouraged to share their solutions and strategies previously their sharing experiences during their first-grade year had been more like those attributed to Ms. Andrew's class in Kazemi (1998) or as McClain & Cobb (2001) reported. They were used to taking turns to present their solution strategies but had not been encouraged to critically listen to and challenge their own and other students' ideas. Much of the focus of the first few weeks in second grade, particularly during whole-class discussion, was on the negotiation of social norms including listening, making sense of others' explanations, asking questions of clarification, and challenging ideas that were presented by students and the teacher (Geoghegan, Reynolds, Lillard, 1997).

As the students began actively explaining, justifying, and questioning, synergistic argumentation (Cassel, 2002) emerged as the norm for the mathematics discussion, and this argumentation focused on mathematical constructions. There are several characteristics of synergistic argumentation such as interactive and open communication, reflection, listening, meaning making, and adjustment of intentions. These are all interwoven in the argumentation process of this problem-centered learning environment. During the argumentation process students are involved in learning what others expect of them in terms of participation as well as learning mathematics. Ideas are laid out in the open for others to make sense of. In this type of discussion emphasis is on the students' ways of knowing.

The emergence of synergistic argumentation brought about opportunities to observe changes in how students participated in the whole-class discussions. Whereas at first it was simply turn-taking, as teacher and students negotiated what was expected and that sense making was the focus, students began to listen and ask thought provoking questions. They were willing to actively participate without the teacher having to call on them. They knew sense making was valued and expected. They became uninhibited about explaining, justifying, or questioning. The teacher's role as an interactive facilitator, discussion stimulator, and encourager helped to develop a high level of open interactive argumentation among students that became the norm for this second-grade class. However there was one notable exception that became a focus for our analysis. As we observed students explaining and justifying their solutions, two students' actions emerged portraying a contrast between interactive arguing and turn-taking. These two students, Brett and Miriam, were both competent in mathematics and at times demonstrated sophisticated mathematical thinking and solutions to tasks presented.

We observed a number of students, particularly Brett, who became more reflective and critical of their own ideas in response to others' questions and challenges of ideas presented by others during the whole-class sharing time. However, each time we analyzed the episodes in which Miriam presented her solutions, we noticed a somewhat rigid stance as she adhered to her solutions in response to other students' questions and challenges. This occurred at times where her solution was "correct" as well as at times where her solution was "incorrect." For example on one particular occasion was when there was disagreement about the "correct" solution for the following task:

> There are 8 red flowers and 4 yellow flowers. How many more red flowers are there than yellow flowers?

Some students thought the answer would be 4 and some thought it would be 12. The class gave explanations and justifications as they discussed both of these solutions. Brett and Miriam agreed with a few other students that the answer would be 4. Miriam at one point in the discussion reread the task orally and stated, "It is take away." Another student said, "No, it says *more* so it is not take away." The teacher at this point asked Miriam, "Can you prove to them that it is take away?" Miriam's *partner* drew 8 circles in a column with a column of 4 circles next to it. A line was drawn under the first 4 circles in both columns leaving 4 circles below the line, thus indicating that 4 and 4 matched and there was a set of 4 circles without a match (see Figure 1).

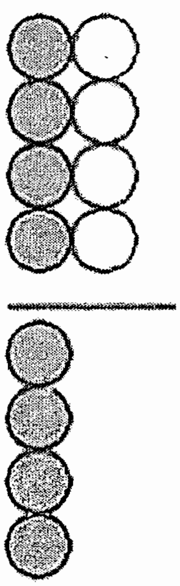

Figure 1

She explained that therefore, there were 4 more red flowers than yellow flowers. Miriam did not participate in the explanation of the solution either by commenting or by indicating involvement through her body language; she simply ignored what her partner was doing and saying. As the argument about whether the problem was "add" or "take away" continued several times Miriam simply restated, "It means to take away." Brett on the other hand offered three different explanations of why the answer was 4. For example he drew the following on the board (Figure 2):

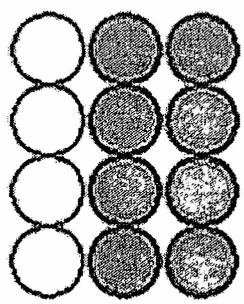

Figure 2

Note that his drawing is not the same as Miriam's partner had presented earlier; instead it is provides a different justification in support of subtraction. In pointing to the 2 rows of red flowers (shaded circles in his drawing) he says: "8 of these....how much more red than yellow? So it means 4 more red than yellow (as he points to the extra row of red)." Later, he tries again in a slightly different way, referring to each row in his drawing (from left to right) as he says: "This is one row, this is one row, and this row is left" as he tries to explain that one row of yellow flowers matches one row of red flowers and that there is still one row of red flowers left.

Brett took into consideration what others said and then re-explained trying to help other students understand why the task was to subtract. Although she was one of the few students who believed at this stage that the task was one of subtraction, Miriam did not try to help Brett explain nor did she try to re-explain the columns of circles from either her partner's or Brett's drawings. This was one occasion early in the year where our subsequent analysis of the dialogue and the videotape of the discussion showed that all of the class was engaged and interactively participating in the argument, trying to make sense of each others' ideas and proposing counter arguments in an attempt to resolve this conflict, that is all except Miriam.

As a result of our noticing Miriam's rigid stance, the teacher initiated a variety of strategies to try to challenge Miriam to become more reflective and critical of ideas, but by the end of the year we saw very little change in her stance. This is clearly evident during one particular episode when students were given the following task:

From research Mrs. Stolt's class discovered that there are three basic fingerprint patterns: loop, whorl, and arch. After the 18 students recorded their thumbprints, they discovered there were two 2 more loops than whorl thumbprints. There were 5 less arches than whorl thumbprints. How many loops, whorls, and arches?

In this particular episode several students willingly volunteered to share their ideas, including Brett and Miriam. Miriam explained her solution and Brett and a few other students questioned her solution. Brett and Miriam became intensely "locked" into a stalemate while discussing each other's solutions during this particular episode. Brett's solution was correct but Miriam's solution was incorrect. She was just stubbornly arguing in favor of her own solution and was not listening to Brett as he tried to explain his strategy in arriving at his answer. What Brett was saying was going past her, she was not listening.

Brett: (Wrote on the board 9, 7, 2.) 9 + 7 = 16, 9 and 7 are like our class then add 2 more. I knew because on paper it said 3 less...and 5 more than whorl is 2.

Miriam: (Drawing tally marks on the board.) ...there are 18...6 arches because 2 more loops than arches so 6 here and 8 here then whorl added 4 more.

Both solutions equal 18. It is not clear at this stage whether they had made sense of the task because they have given numbers, but with no explanations.

Two other students disagreed with Miriam's solution. She ignored them. From her perspective, she has demonstrated her solution method and sees no need to respond to either one. From a "turn-taking" perspective all she needs to do is state her solution without listening to others. The two students put their solution on the board without an explanation.

Miriam: Can you prove it up here?

Brett: (Reads part of the problem over again.) Because it says 2 more loops than whorls, not arches.

Miriam: (Has her back to Brett the whole time he is talking, and she is doodling on the white board.) more loops than whorls...

Teacher: Did you hear what he said?

Active listening involves body language as well as verbal reply. The teacher notices Miriam's stance. The teacher and students have negotiated that sense making is important. The teacher looks at Brett indicating for him to continue. In this way the teacher is reinforcing that the students are to make sense of the mathematics.

Miriam: No!

Brett: (Waits for her to turn around. When she does he then rereads the problem while looking at her.) 2 more loops than whorls...

Miriam: What!!

Brett: On the paper it says there are 2 more loops than whorls.

Brett rereads the task to bring into focus the question asked. By doing this, he gives Miriam, and the rest of the class, a chance to hear and reflect on what is being said.

Miriam: There is!! On my paper there is 2 more arches than whorls. There is because 2 more equals 6...so there is!!

Brett: Loops than whorls?

Again Brett tries to use another approach to help Miriam see the conflict. This time he repeats two significant words in the question instead of the whole question, thus highlighting the specific part Miriam needs to look at. He is again bringing the focus back on the task, hoping she will realize she has arches, not loops.

Miriam: Same thing, that's the same thing! (a bit irritated)

Brett: 5 more.

Miriam: 2 more that's it, that's the same!

Brett: 5 less arches than whorls and that's 2 more. (Keeps trying to refer to the paper with it in his hand. He was trying to repeat part of the problem.)

Brett now uses Miriam's words to argue from her point of view. He is trying to help her understand her reasoning in comparison to what the task asks. Miriam becomes defensive and does not acknowledge his solution. She did not ask him how he got 5 thus, the argumentation process for sense-making was hindered.

Miriam: That's what I have!!! (She emphatically repeats several times, "That's what I have," but doesn't read the paper or asks him any questions.) That's the same!!

As other students listen to the argument, they try to follow along and make sense of the discussion. Several students try to aid in the sense making process. As each student understands what is being discussed, s/he interjects with important ideas. Miriam dismisses their ideas and tells them they can't prove it with the paper.

In a problem-centered classroom where argumentation has been negotiated as a norm, students listen eagerly to each other while they explain and justify their solutions. They know that each will have opportunities to comment and/or question what the others have done or discovered. In the literature two terms, taken-to-be-shared (Voigt, 1994) and taken-as-shared (Yackel &Cobb, 1996) have been used to describe the collective sense making process. In this particular second grade classroom, a difference between the two terms has emerged. In this classroom the ideas are laid out for everyone to try to make sense of. In this way the ideas are meant to-be-shared with all learners. This does not necessarily mean that the ideas will be shared. If a student is not listening or does not understand, how can the ideas be shared? For example in this episode, Brett and Miriam are discussing their strategies. His are laid out to-be-shared. But, at no time did Miriam share his ideas. She did not attempt to make sense of them and they had no common ground from which to communicate because she was not listening. Thus, the argumentation process was hindered.

In this situation Brett was an active listener using Miriam's words to restate the question trying to promote sense making for all involved in the discussion. He even repeated a significant part of the question, "loops than whorls?" in response to her explanation. He used *her* words to argue from her point of view. He tried to understand her reasoning in comparison to what the task asked. Miriam became defensive and did not acknowledge his solution or questions. Each time Miriam answered or responded to questions, her arms were crossed (closed minded) and she simply stated that she had the answer.

Following is a brief contrast of the two, active listening and not listening (See Table 1).

Table 1

Brett	Miriam
looks at person talking	does not have eye contact
rereads task	ignores written problem
questions answer	gives definitive answer
open minded	closed minded
positive body language	negative body language
uses other person's words	repeats her own words
mutual adjusting	no adjusting

Discussion

This contrast between Brett and Miriam provided data supporting what happens when one person talks past another person verses when one person is truly engaged in discourse and arguing/talking to one another. During the thumbprint episode, Miriam could not appreciate the differences in the interpretation of the solutions, thereby causing an impasse. In argumentation, listening emerges as a critical factor. Substantive mathematical learning occurs when students experience conflict with their previous ways of knowing (Wood, 1996). When students are not listening actively they cannot make sense of their flawed thinking nor can they make sense of other students' thinking. Miriam had her whorls and arches confused in her solution. A couple of students tried to explain, asked her questions, and reread the task in hopes that she would see her error. She did not look back at the original task to show a relationship between her solution and what the task asked. Nor does she acknowledge the other students' solutions or questions. Her stubborn response of, "I have that as my answer," inhibited her from making sense of the situation, thus not changing her answer. Learning breaks down when interactive listening is not achieved.

Students in this whole-class argumentation process attempted to accommodate each other in listening for understanding, to properly orient each other's mathematical constructions and ideas for meaningful learning, and to resolve conflicting solutions. Davis (1997) states that hermeneutical listening is where the mode of attending has been negotiated to reflect the participatory manner of interacting with other learners. Listening becomes the attentiveness to the historical and contextual situations of the students' actions and interactions. The students who listen hermeneutically, as illustrated with Brett here, are willing to interrogate and to be interrogated, reflect, and adjust their thinking. Thus, students who listen hermeneutically further mathematical discussion and constructions. Listening is vitally important in argumentation because it aids the teacher and students in their understanding of each other's solutions, explanations, justifications, and questions. As they become aware of each other's perspectives listening enables them to ask appropriate and constructive questions that help to deepen their own understanding and each other's mathematical understandings.

Many times during the mathematics lesson conflicts of interpretation arise naturally as students give various answers, and in this particular class the teacher expects the students to resolve them. The students' own personal constructive activities are valued. The students in this class are also expected to make sense of ideas presented by other students. Thus, listening emerges as a key factor within this problem-centered classroom's mathematical argumentation process. The students, as listeners, are responsible for making sure they understand other students' explanations and justifications. In this way students are responsible for making sense of the mathematics as well as being active listeners. Active listening involves reflective thinking as students attempt to understand others' explanations and justifications.

In this whole-class argumentation students are attempting to accommodate each other in listening for understanding, to properly orient each other's mathematical constructions and ideas for meaningful learning, and to resolve conflicting solutions. We chose to label this open, mutually interactive and continually adjusting mathematical discussion "synergistic argumentation." As we observed this synergistic argumentation in action, and its positive effects on learning, we noted that an important aspect of its functioning is hermeneutic listening. If a student or students fail to listen, as with Miriam in this instance, it is easy for interactive mathematical communication to break down. This situation becomes evident when students talk past one another, if one is not listening to another, or if they are failing to make sense of other students' perspectives. Constructive interactions occur when both the sharer and receiver are actively involved in communicating and listening, which in turns help students develop mathematical understandings. Synergistic argumentation is more than talking about differences and misunderstandings, and it is more than seeing mathematics from various perspectives. It is the total interaction when students present their ideas and solutions, defend them in the face of questions, and question other students' ideas. In order for students to be able to defend and question they must be interactively listening or listening hermeneutically (Davis, 1997). Listening with the intent to understand creates a context in which the students are reflecting on their own methods while attempting to understand the other methods. Thus, it is up to the listening community of learners to ask questions, to agree or disagree or to make challenges, thus soliciting other arguments. As students listen interactively they become aware of interpretations from other students and continually try to adjust their thoughts and questions until understanding is reached among students. Hermeneutic listening provides opportunities for students to construct meaningful mathematics.

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MATHEMATICS FOR PARENTS: FACILITATING PARENTS' AND CHILDREN'S UNDERSTANDING IN MATHEMATICS

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The study examines ethnically and linguistically diverse parents' learning in an eight-week Math for Parents course on fractions, decimals and percentages. Analysis of classroom observations, interviews, focus groups, written feedback, videotapes, and task-based/clinical interviews provided rich data yielding findings about both parents' affective and cognitive learning as well as their understandings of how to support their children's learning of mathematics. The research points to an appreciation among the parents for the importance of learning mathematics with understanding. It also raises some questions concerning the goals of mathematical learning experiences for parents.

Context

Our research takes place in the context of a large parental involvement project in K-12 mathematics. MAPPS[1] (Math and Parent Partnerships in the Southwest) is a four-year long project that focuses on parental involvement in mathematics. It is now in place at four sites: in Tucson since 1999 and at the other three sites since 2001. The implementation at the different sites varies somewhat according to local needs, but overall we share some common goals. One such goal is to develop leadership teams (parents and teachers/administrators) that will help in the mathematics education outreach effort throughout the districts involved. We have three main types of activities related to our goal: a) Leadership development sessions for the members of the Leadership Teams; b) Mathematics Awareness Workshops (MAWS) ranging over key topics in K-12 mathematics and open to all the parents and children in a given district; c) Math for Parents (MFP) courses in which parents in the Leadership Teams and other parents have an opportunity to explore mathematical topics in more depth and to learn about reform mathematics and its implications for their children's education.

In this paper we consider our work in Tucson where we have three leadership teams in place and have offered a total of five Math For Parents courses. All the Math Awareness Workshops are currently facilitated by members of the Leadership Teams. In particular here we look at the experiences of an ethnically and linguistically diverse group of parents (all women) in one of the Math for Parents eight-week courses that was being taught for the first time. We discuss the affective and cognitive impact of the course that centered on developing an understanding of concepts related to fractions, decimals, and percentages. We also examine parents' reflections on their roles in their children's learning of mathematics (K-12) as they participated in this course. Research on the effect of experiences such as these MFP courses on the participants' understanding of mathematics and on how they bring these experiences home to their children is particularly relevant for those of us concerned with equity issues.

Theoretical Framework

Mathematics educators often stress the importance of involving parents as we embark on reforming school mathematics. These calls are predicated on the assumptions that parents have great potential to influence children's mathematics learning (Ford, Follmer, & Litz, 1998; Epstein, 1994; Henderson & Berla, 1994; Kliman & Mokros, 2001) and that parental support is necessary for successful implementation of reform mathematics programs (Kliman & Mokros, 2001). Researchers (Lehrer & Shumow, 1997; Peressini, 1998) have documented, however, the frequent mismatches between the aspirations of reformers in mathematics education and those of parents and other community members in the communities where reforms are being implemented. Peressini (1998) has observed "in both the larger arena of general educational reform and the subset of school mathematics reform, these calls for parental and community involvement have been at an abstract level and have not been closely examined" (p. 557).

We argue that in order to move beyond the abstract level in working with parents in mathematics, educators and researchers must begin by interrogating deficit models of parenting (Henry, 1996; Vincent, 1996) and question prevailing assumptions about the necessary skills base for parents' work with children in mathematics (Merttens, 1993). Our work reflects an awareness that, as Weissglass and Becerra (n.d.) write, "often classes or programs for parents are one-way transmissions of information and materials from school to parents. Rarely do parents, particularly those from groups underrepresented in mathematics, have an opportunity for their beliefs, ideas and concerns to be heard. ... All parents need a safe place to share and explore their early experiences with schooling, their thoughts about their children's learning, and their attitudes toward mathematics" (p. 2). Beyond discussions about learning, schooling, and mathematics attitudes, we have suggested in previous work that parents also need opportunities for meaningful learning of challenging mathematics content (Civil, 2001).

In addition to research literature on parent involvement, research on adult education, especially that grounded in critical pedagogy (Benn, 1997; Frankenstein & Powell, 1994) has been valuable in broadening the view of potentials for parent involvement in mathematics education. One of the key premises in this research is to view parents as intellectual resources. These researchers stress that there are different forms of mathematics and push us to reflect on what we count as mathematical knowledge, while suggesting pedagogical approaches that seem to be quite powerful when working with adults who have often been marginalized.

Method

We followed a phenomenological methodology (Van Manen, 1990) that relies heavily on participants' contributions to the experience. The lived experience of each parent is considered significant and thus we try to capture it in our analysis and writing. Our sources of data include observations and field notes, interviews and focus groups, evaluation protocols, and videotapes of all the MFP sessions. These multiple sources of data allow for triangulation. The course we studied was being taught for the first time by an instructor who had written the course curriculum. The area of mathematics education for parents is so new that research and understanding of how it works and what approaches are most effective are still in very nascent stages. Necessarily, then, our study incorporates goals and approaches found in formative evaluation research as we consider questions of what parents learned and how they learned it. To examine the impact of the course at the cognitive level, we report on two task-based / clinical interviews, one with two mothers (conducted in Spanish) and the other with one mother (conducted in English). The data analysis for the overall research in MAPPS follows Glaser and Strauss (1967) constant comparative method. The different pieces of data are looked at and codified and as conjectures emerge, we go back to the data for affirmation or rejection.

The Course

Most sessions of the MFP course on Fractions, Decimals and Percents started with a brief activity in which the parents were asked to make connections between the mathematical concept under discussion and their everyday experiences. The rest of each two-hour class involved activities to help parents develop or revisit concepts. About four of the sessions focused on fraction concepts (mostly around the idea of what is a fraction; very little was done on arithmetic with fractions), two sessions centered on decimals, one on percents, and the last one was a wrap-up session in which the ideas about fractions, decimals, and percents were connected. The content goals were primarily on concept development. As a college educated participant explained,

> It's not like they give you a book telling you that we're going to learn decimals, start at page one and do a whole bunch of problems about decimals. It's not really about learning how many fractions you can reduce, it's just about starting from the very basic level and developing the concept… it doesn't matter how much math you've learned or how many courses you took in college. Everyone starts at the same point and then we work and develop it together. It's not about just doing the math but understanding why it is the way it is.

Each session included a look at one aspect of the NCTM Principles and Standards as well as a handout with ideas to try at home with their children. In an exit interview, the instructor described her goals for the course as threefold: a) that the parents learn mathematics; b) that they increase their interaction with their children about mathematics and about school; c) that they have fun. She reiterated these goals often during the sessions and the participants picked up on this,

> One the biggest things [the instructor] mentioned, one of her goals was to have fun and see fun in using and doing math. I have really enjoyed it; to me that's really strange because I don't like math. I started with an attitude that I don't like math and this has been a really fun class.

Parents worked in groups, used a variety of hands on materials, and communicated the results of their investigations to the whole class. The activities were largely based on the instructor's collection of activities that she has written over the years and has used in her work with teachers and children. The 23 people who attended the course were women from a working class, largely Hispanic community. The course was conducted bilingually. The parents' mathematical autobiographies as well as comments in the interviews and in informal conversations conveyed an array of largely negative experiences in their prior learning of mathematics. Issues of lack of confidence, of not being good at mathematics, and of feeling alienated were quite common among our participants.

The course pushed parents to revisit "elementary concepts" and in doing so, allowed us to discuss some of the typical difficulties as documented in the literature on rational number, such as the different meanings for fractions, the concept of unit, and the influence of informal knowledge (Lamon, 1999; Mack, 1993). For example, towards the beginning of the course, in an activity with the tangram pieces, some parents quickly said "each piece is 1/7." Others disagreed and pointed out that the pieces were different sizes. This led to a discussion on the meaning of "1/7" in this context. As the course went on, the participants engaged in contrasting arithmetic/algorithmic approaches with conceptual/manipulative-based approaches to the different tasks proposed. In the next sections we present some of the findings in three areas: affective impact; cognitive impact; and issues related to the parents' interactions with their children about mathematics.

Affective Component

Parents were asked, in small group interviews that were taped and transcribed, to reflect on their learning experiences in the Math For Parents course. A theme that was woven throughout their responses was their recognition that their understanding of mathematics concepts was expanding as a result of their work in the MFP course and the importance that they gave to learning with understanding. A bilingual college educated mother of a preschooler and a first grader, stated:

> I always liked math… and…. the formulas… I mean the teacher would give us a formula and I just accepted it, I didn't question I said, okay, this is it, and I applied it. But with these classes we start from the very beginning to see

> how they developed... How and why a fraction is a fraction, why a decimal is a decimal. So from the very basic [concepts] ... *and then we have to develop our own formula, that's the biggest thing...*yeah we develop it, and then so when you understand, okay, you can say you know why this is. Because instead of just accepting it, you know it is... Before they would just give us a formula... but now that you understand it you can use it and you'll know how to use it. Developing the concept from the very beginning, starting at the very elementary level and then developing it, that's what I like.

This excerpt is significant in terms of one of our overarching goals for the project which is related to raising parents' awareness about what reform in mathematics education may look like and about what teaching for understanding means. This mother is reflecting on how powerful it is for her to "develop our own formula, ... you know why this is." She also points out how in her previous learning experience, they would just give her the formula and her comments indicate that she does not view this as being particularly powerful. We wonder, will experiencing for herself this kind of teaching that emphasizes meaning make her more likely to expect this for her children's mathematics instruction? Will it make her more likely to focus on helping her child develop meaningful understanding of mathematics concepts?_

Parents cited the real-world problem solving activities in their MFP course as contributing to their own understanding of mathematics concepts and they also reported using the real world applications in helping their children make sense of the mathematics they were learning in school:

> I liked that she [the instructor] explained what her goals were going to be for the class and then tying the material that we were working with, to real life. Fractions is not just something in a math book it's everywhere.

> Some of the projects that we did, you know like going into the newspaper or magazine and finding the percentages, finding decimals, ... So..., in a way that's relevance in the real world where ... you see all these numbers all these fractions, all these decimals, all these percentages... and... you take them in.

> You have a lot of relevance in the outside world because you can start to see what you are doing and then apply it. You see it elsewhere, then working with your child, you know now you are in the very basic math level. Then you can also help your children see it somewhere else, that's just not in your math books or math worksheet ... Math can be found everywhere...

Parents also commented on feeling more confident, but showed an awareness that they had just started scratching the surface with this course. In fact, several participants expressed a need for more classes on the course topics:

> I feel more confident to teach now.
>
> When I started my confidence was way down. I was ashamed of even being here. But now I even enjoy it, I come because I'm open-minded. I feel better now. I'm not going to say that I learned everything, but I learned a lot.
>
> We were getting into it, and then we changed to something else. We need more time; I hate to say this but maybe more classes.

Cognitive Component

The lessons on fractions focused on concept building. For example, in one activity the participants had to rename each pattern block in terms of what a given one was worth (e.g., if the hexagon is 1/2, then what are the other blocks worth?). In another activity, the participants explored equivalent fractions using color tiles. A few months after the course was over, we conducted two interviews with tasks very similar to the ones the participants had worked on in the course. One interview was with Georgia, who was in her first year of MAPPS. She has three children aged 5, 7, and 13. She is English-speaking and understands Spanish, is a homemaker and completed twelfth grade in Tucson where she studied basic mathematics. She explained that she had enjoyed mathematics in school until the eighth grade when she became ill, missed school and fell behind. When she failed geometry the next year, she found that she just struggled in mathematics and was placed in a basic mathematics course just so she could pass and graduate. The other interview was with two Spanish-speaking mothers, Monica and Elena. They were both schooled in Mexico, where they completed high school. They are both homemakers. Monica's experience with learning mathematics was rather negative while Elena's was overall positive. Monica has two daughters ages 7 and 9, and Elena has a boy age 7 and a daughter age 5.

The interviews revealed a tendency among the three mothers to look for connections between the questions posed to them and everyday life experiences. For example, when asked to give the percent represented by 3/4, Georgia looked at the collection of 4 tiles and said, "Let's pretend this is dollar, so each tile represents 25%, so 3/4 is 75% or 75 cents." Elena and Monica were at a loss in many of the school-like tasks. For example, when asked to make a rectangle that was 1/2 blue, 1/4 green, and 1/4 red, they reached for 2 blue tiles, 4 green tiles, and 4 red tiles. Also, throughout the interview it became clear that they were not understanding the questions: they understood the language (the interview was in Spanish), but not the mathematical language. For example, they had a particularly hard time with questions aimed at assessing their concept of equivalent fractions such as in a task where they had said that blue was 6/8 and they were then asked, "can you give me another fraction to describe the blue part?" They did not seem to understand what the question was asking.

Georgia also seemed somewhat uncomfortable with expressing herself in a mathematical context, though she was much more successful at solving the interview tasks.

For example, in the task of making a rectangle that is 1/4 green, 1/3 yellow, 1/6 red and the rest blue, her first approach was to take 3 blue, 1 red, 1 yellow, and 1 green. She wanted 6 tiles "because that is the largest number" (and that had worked in a prior task). But she quickly realized that this was not going to work and revisited her work. It took her two more attempts before she came up with a collection that worked. But then she said, "It's not the biggest number on the bottom, but the number that is times all bottom numbers that give me the number." So, she seemed to realize a connection to the concept of common denominator. Georgia was quite successful at thinking through the tasks but showed little confidence in her answers. She often turned to the interviewer for validation.

We wonder to what extent these parents would be able to help their children with the typical school-type fraction tasks. But we also wonder how much of their difficulties were related to language, at two levels: the language of mathematics and in the case of the Spanish speaking parents, the fact that the course was taught in English, with Spanish translation. Our classroom observations have raised the question of whether the translations of the English instruction provided effective and equitable learning opportunities for the parents who spoke Spanish. This task-based/clinical interview, in which Spanish-speakers struggled with understanding and expressing their understanding of fraction concepts explored in the course, confirmed our suspicion that the bilingual teaching and learning environment of the math for parents courses needs to be critically examined.

Interactions with Children

This MFP course was modeled after content taught to children in grades K-6. Parents engaged with the material as adult learners, but one aspect of our research is how what parents learn makes its way into the home. The topics of this course were particularly relevant in that they span over all grade levels and are often perceived as difficult to learn. At this point, what we have is self-reported data in which the parents comment on how these experiences are impacting (or may impact) their interactions with their children. For example, several participants commented that they could now help their children with their homework or at school:

> It has helped me to be able to help my children with fractions, in their homework.
>
> Today I worked with one of my daughters, and I learned more in this year with MAPPS. So I'm happy because I can help my daughter with the work. I can understand more if I practice more and more.
>
> This has helped me a lot. Now that I go to my child's classroom, it's easier for me to teach them when I sit with them at their table.

Another important aspect that our research shows is the effect that these experiences have on the nature of the interactions between the mothers and the children. It is not only whether they help them with their homework, but it has more to do with how the children view their parents as a result of their attending classes (Brew, 2001):

> And then taking the course helps you reinforce the idea that you're your child's first teacher. We were their first influence, so it empowers us. We are also their teacher and we can help them. They go to school and there's a teacher, and they do their homework and learn. But at home we're their teacher also, then we can help them. They look at us differently.
>
> I have learned ways to interact with my children and also it's fun to have them showing the games to their friends and visitors.

Although these courses are for parents only, some of them occasionally bring their children. Several mothers have commented on the fact that they enjoy sharing their learning with their children, "it's very helpful for the kids to come in and see what they're teaching."

Parents' comments indicate they feel they are gaining both knowledge and confidence regarding mathematical concepts such as fractions that translates directly into confidence in working with their children to understand similar concepts.

Conclusion

Parents' experiences with the teaching and learning of mathematics for understanding are the focus of our study. Others (Lehrer & Shumow, 1997; Peressini, 1998) have shown that the theoretical assertion of the importance of understanding in learning mathematics can be a confusing and controversial one for parents. Our study illustrates an approach that appears to support parents in bridging mathematics pedagogical theory into their own practice of mathematics learning and their own work with their children in mathematics. The parents in the MAPPS program have enthusiastically embraced the idea of the importance of understanding in mathematics; no controversy on this point has been evident.

The study reported here of parents' experiences in a Math for Parents course adds to the small body of literature examining the impact of intervention programs with parents in mathematics (Kliman & Mokros, 2001; Lehrer & Shumow, 1997; Merttens, 1993; Morse, 2001). Our findings suggest that giving parents opportunities to actively construct their own understanding of mathematics concepts provides a critical foundation for their work with their own children. Furthermore, as parents themselves learn mathematics with an emphasis on understanding rather than rote memorization, they become quite vocal about the importance of understanding for their children's mathematics education. As one mother very eloquently said, "I don't want them [teachers] to teach to the test. You have to be versatile in many things. If you don't understand, what's the point?" A key aspect of mathematics reform is the goal to ensure that students understand the mathematics they are learning (rather than memorizing procedures). Yet, this important aspect is beginning to be left behind as the testing pressure increases. In our study, parents reported working with their children to develop mathematical understanding. They talked about incorporating strategies that they had learned in their MFP courses in exploring their children's understanding of concepts in their schoolwork.

This MFP course provided opportunities for parents to engage as adult learners of concepts related to fractions, decimals, and percents. It also provided links to what their children may be doing in school. In a sense there seemed to be several goals for the course, as the instructor captures very insightfully in her reflection,

> When teaching children, the goal is mastery as demonstrated on tests and other measures. When working with teachers, the goal is that they understand the material well enough to explain it to students. I felt the goal for the parents was fuzzy. Were they learning fractions, decimals, and percents for their own personal knowledge? Should I have "tested" and "retaught" to make sure they were mastering it? Were they there to get a "glimpse into a classroom of today" and see what learning is like for their children? Were they there to learn enough to be able to assist their children with homework and studying? I know that all of these are goals of MAPPS, but I wonder which one should be the primary one for a MFP course. If the goal is content mastery, then the course probably needs more than eight sessions.

Our findings corroborate that, indeed, in order for parents to be able to help their children with school tasks in these topics, one course like the one we just described is probably not enough for many of the participants. The nature of the interactions about mathematics that these parents developed with their children may be even more relevant and crucial than their actually being able to help them with specific content. Our future plans are for more case studies of parents and their children utilizing both open-ended interviews and task-based clinical interview methods. In addition, a number of questions to guide future research have arisen. How do parents' reported growth in confidence and embracing of reform mathematics pedagogy such as a focus on understanding translate into actual support for their children's understanding in mathematics? How can bilingual instruction for parents in mathematics best be structured to support full participation and learning for Spanish-speaking parents? And finally, what are we learning about appropriate goals and realistic and wide-ranging outcomes for mathematics education programs for parents?

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CULTURAL INFLUENCES ON TEACHING MATHEMATICS: CASE STUDIES OF INTERNATIONAL TEACHING ASSISTANTS

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In recent years, teaching assistants (TAs) have assumed a greater role in the teaching of university mathematics courses throughout the United States. As a result, the adequacy of their preparation to teach such courses has come into question. To address this issue, a mathematics pedagogy course was developed and taught and several research studies examining TAs' mathematical beliefs and classroom practices were conducted. Results of these studies indicated that while the TAs appeared to adopt new beliefs about the teaching and learning of mathematics, they perceived several obstacles in trying to employ more reform-minded classroom practices. Further, the background and experiences of international teaching assistants (ITAs) appeared to present unique challenges in their development as teachers. This study, involving four ITAs, was conducted to learn more about the role of culture in the beliefs and teaching practices of ITAs. Data were collected through interviews and observations and analyzed using qualitative methods. Findings, in the form of case studies, revealed aspects of the ITAs' culture that may have impacted their mathematical beliefs and classroom practices.

Introduction

Beginning in 1997 a mathematics pedagogy course was developed and co-taught by education and mathematics faculty at the University of Connecticut to mathematics teaching assistants (TAs). Over the past five years, several research projects have been conducted examining the beliefs and instructional practices of the TAs who participated in the course. Results of these studies indicated that while the TAs appeared to adopt new beliefs about the teaching and learning of mathematics, they perceived several constraints in trying to employ more reform-minded classroom practices. In particular, culture (i.e., individual, departmental, and institutional) served as a lens to examine the barriers the TAs faced in trying to change their instruction. The background and experiences of international teaching assistants (ITAs) have presented unique challenges in their development as teachers. In order to understand this unique group of TAs, a qualitative study of four ITAs was conducted to examine how their past experiences and participation in the day-to-day routines of the mathematics department shaped their beliefs of teaching and learning as well as their instruction.

Background of the Study

Research has indicated that culture plays a significant role in shaping cognition, which in turn influences behavior. Culture refers to the totality of socially transmitted values, beliefs, aesthetic standards, linguistic expression, patterns of thinking, and behavioral norms and styles of expression. ITAs, who are often expected to maintain the norms of U.S. classrooms, exhibit behavior and communication styles that are significantly influenced by their strong cultural orientation (Jenkins, 1997). Further, ITAs often possess beliefs about student-teacher relationships and educational norms that conflict with the educational system in the United States (U.S.), leaving them unprepared to address the different learning styles of U.S. students (Jenkins, 1997; Travers, 1989). Travers (1989) has indicated that ITAs must participate in preparatory courses or seminars to discuss differences in culture and philosophies of instruction if the needs of U.S. undergraduate students are to be met. In order to design such programs and support structures, more research needs to be conducted that examines the nature of ITAs' belief systems and the role of culture in shaping these beliefs.

Research (Olaniran, 1996) on ITAs has identified five variables that influence their teaching performance—including *language, age, academic classification, cultural similarity* and *friendship communication network patterns*. First, due to language concerns, ITAs often resort to teaching techniques that involve lecture-only, memorized presentations. Second, given that different cultures view age and its associated roles and responsibilities differently, ITAs from high power distant countries (i.e., where power and authority are influenced by age) may experience difficulty in U.S. classrooms. Third, teaching assistantships place greater responsibilities, and stress, on ITAs who have to interact with U.S. students (e.g., responding to students' questions or explaining grades received). Fourth, the degree to which an ITA's native culture resembles the host culture may contribute to an ITA's instructional effectiveness. Finally, the lack of host nationals (i.e., in this case, Americans) in ITAs' friendship networks (i.e., circle of friends) may result in teaching behavior characterized by lack of self-disclosure, failure to invite class participation, and cocooning tendencies.

Other research on ITAs has focused on variables affecting intercultural communication, and in particular, the notion of a collectivist versus an individualistic culture (Jenkins, 1997). Collectivist cultures embrace a "strong in-group orientation" centered in relationships with others. In this type of culture, teachers control the classroom, are viewed as the dispenser of knowledge, and don't expect student interaction or participation. In contrast, individualistic cultures emphasize the development of students' individuality of expression and teachers are seen as facilitators rather than fonts of knowledge (Jenkins, 1997). For example, in the Chinese culture, which is collectivist in orientation, "interactions in academia are characterized by a status differential between teacher and student, with higher status accorded to the teacher. Status is maintained by politeness strategies that maintain "positive or negative face" (Jenkins, 1997, p.6). Silence, avoidance, and tentativeness are just a few examples of politeness strategies that may be the preferred and necessary communicative style of a person from a collectivist culture, yet inappropriate and often unnerving for U.S. undergraduate students (Jenkins, 1997). Other research on ITAs has found the transition from a highly selective educational tracking system to an open-admission university is very difficult and ITAs are often unable to identify appropriate levels of instruction, maintain realistic expectations of students, and remain patient when students lack understanding (Damarin & West, 1979).

Beyond these variables specific to ITAs, researchers have long understood that an individual's beliefs about mathematics and the teaching and learning of mathematics play a significant role in shaping characteristic patterns of instructional behavior (Thompson, 1989). Some researchers (Green, 1971) have used the notion of a *belief system* as a metaphor, to describe the organizational structure of beliefs acquired by an individual. In this view, belief systems are dynamic and subject to change (Thompson, 1992) and may help explain certain behavior with respect to teaching and learning. Green (1971) has identified three dimensions of belief systems-a quasi-logical structure, the psychological strength between beliefs, and the clustering nature of beliefs. The quasi-logical structure of a belief system permits beliefs to be "primary" (e.g., a belief that is used as a reason for other beliefs) or "derivative" (e.g., a belief derived from some other belief). The notion of which beliefs are most important, and thereby more resistant to change, has to do with the strength in which these beliefs are held. Psychologically "central", or "core", beliefs are held with greatest conviction and are least susceptible to change while "peripherally" held beliefs are more likely to be altered or changed. Finally, the third dimension of the belief system indicates that beliefs are held in "clusters" and generally in isolation from other clusters (Green, 1971). According to Thompson (1992), "This clustering prevents cross-fertilization among clusters of beliefs or confrontations between them, and makes it possible to hold conflicting sets of beliefs. This clustering property may help explain some of the inconsistencies among the beliefs professed by teachers, documented in several studies" (p. 130). In summary, by virtue of participating in the day-to-day routines of family and school life in their native countries, ITAs have been enculturated into the profession of teaching. Further, ITAs' beliefs about teaching and learning are continually shaped through classroom interactions and participation both as a student and teaching assistant in the mathematics department. The purpose of this study was to build on previous research regarding the factors influencing ITAs teaching behavior. In particular, the research question that guided this study was, "What is the role of culture in ITAs' beliefs about the teaching and learning of mathematics?"

Methods and Procedures

This research project was conducted during the 2001-2002 academic year. Four ITAs, who were participating in the mathematics pedagogy course, were purposively selected to participate in this study. There were two males and two females all of whom were born and raised in different countries (i.e., Kenya, China, Kazakhstan, and Belgium). All participants were teaching two freshmen level mathematics courses below the calculus and were teaching for the first time in the United States. All subjects participated in four 45-minute semi-structured interviews, which were conducted during the Fall and Spring semesters. The purpose of these interviews was to learn more about their

(1) early home and school experiences,

(2) beliefs about mathematics and the teaching and learning mathematics,

(3) experiences teaching U.S. students,

(4) perceptions of the culture of the mathematics department, and

(5) network of people with whom they discussed issues surrounding teaching and learning.

In addition, each subject was videotaped teaching a lesson during the Fall semester and participated in a stimulated recall session. All interviews were audiotaped, transcribed, and coded and these data were organized in a matrix. This information, along with the researchers' field notes, was used to develop four case studies and to identify themes across participants regarding the role of culture in their beliefs on the teaching and learning of mathematics.

Results

Blidi

Blidi* (all names are pseudonyms) was one of 12 children raised by his parents in a small village in Kenya. Both of his parents earned high school diplomas and valued education. Blidi indicated that age and gender had an influence on one's standing in his community. For example, individuals were expected to show respect for their elders and that if an equally qualified man and woman were to apply for a job, the job would be given to the man. Blidi attended a typical elementary school where the average class size was 50 students. As he entered the middle grades he began preparing for the high stakes national exam to gain admittance to high school. To do this, he remained at school until 6 p.m. and often continued his studies at home for several more hours each night. Blidi attended a prestigious private high school and continued to study and attend classes for approximately 15 hours a day. He described his high school mathematics classes as emphasizing practical applications and places in which students felt comfortable asking and answering questions. He added that if a student didn't understand a topic a committee of students would meet to help him learn the course material. Blidi worked diligently in high school to prepare for the national exam to enter university in Kenya since approximately 50,000 students apply for only 8000 slots at the four main universities. Blidi came to the U.S. to pursue a M.S. in Actuarial Science. At the time of the study, Blidi had been in the United States for three years, and had developed a large circle of U.S. friends whom he met in the classes he was enrolled in at the university.

Blidi indicated that mathematics is a way of understanding the world and should be taught in a more practical and less theoretical way. He commented that he learns mathematics through reading the text and doing examples, but he also regularly visits his professors during office hours to discuss questions he has about the material. Blidi viewed his teaching as an opportunity to educate his students rather than simply as preparation to pass a common mathematics exam.

The classroom observation of Blidi revealed a model of teaching in which instruction was student centered, incorporated both large-and small-group work, and included graphing calculator applications. Acting as a facilitator, Blidi fostered a sense of community in his class by calling on students by name, encouraging student-student dialogue, and circulating around the room throughout the lesson. Further, approximately 90% of his students were in attendance with all students arriving on time and no one leaving early.

Blidi attributed his teaching style to his personality and to the fact that he "understands" U.S. students—that is, they want to participate in class and understand why they need to learn mathematics. He felt very comfortable teaching in English and viewed language, not as an impediment to his teaching but as a tool to engage students in the learning process. For example, Blidi used his misconceptions of the English language to infuse humor in the class, which he believed helped him establish a rapport with his students. He indicated that he routinely reflected on his teaching, reviewed his grade sheet weekly to identify students who may need extra help, and called these students over the weekend to suggest meeting to review the course material.

Throughout the year, Blidi struggled with finding an appropriate way to pace his teaching to meet the demands of the course syllabus and still ensure that students understood the material. He also felt that at times he struggled with maintaining "control" of the class. Blidi appeared to take his teaching responsibility very seriously often planning extensively for lessons that included a mixture of lecture and group work. During the Spring semester, he revised his instruction by trying to encourage more student participation in class. Finally, he indicated that members of the mathematics department, and the chairman in particular, were very supportive of him and helped him improve his teaching.

Ling

Ling was raised by her parents in a large city in China. Her parents were both college educated and worked in research institutes in China. Ling indicated that in her culture, age had an influence on one's standing in the community and that individuals demonstrated respect for their elders by not arguing with them. Ling attended a public elementary school and scored well enough on an exam to gain entrance to a boarding school at age 11 where she studied until age 17. According to Ling, the school day was highly structured (e.g., students were scheduled to be either studying, in class, or in the cafeteria between the hours of 6 a.m. and 10:30 p.m.) and the courses prepared students for the high stakes national exam needed to enter university. Ling indicated that her teachers pushed students to work very hard by assigning many homework problems, testing frequently, and announcing the names of students who scored poorly on quizzes and exams. The general style of instruction was that teachers lectured, students took notes, and there were rare exchanges between teachers and students. Ling did well on the exams and was one of only 15 students in her graduating class of 100 that went on to university. Again, once at university Ling prepared for an exam to enter graduate school and was one out of 6, in a class of 25, which was admitted to graduate school. Ling came to the United States to pursue a M.S. in Actuarial Science. At the time of this study, she had been in the United States for two years. Ling indicated that since in the U.S. she had not made any friends from the U.S. and that she spent most of her time with fellow Chinese graduate students.

Ling indicated the purpose of teaching mathematics is to help students understand the material and to prepare students for the future. Her image of good teaching included being well-prepared, fostering eye contact with students, being organized, writing notes neatly on the board, and providing motivation to learn the material. Ling indicated that she learned mathematics by practicing and solving many problems and studying the textbook.

The classroom observation of Ling revealed a transmission model of teaching, which she described as "a typical lesson". In particular, her instruction was teacher directed and based on a pattern involving the teaching of mathematical theory followed by examples. Her lesson mirrored the textbook presentation of the topic and did not include applications of technology. Ling used the blackboard as the sole means of dispensing information, solicited little to no participation from the students, and with notes in hand, followed a scripted lesson, teaching from the front of the room. Finally, on average, approximately 50% of the students were in attendance with students arriving late and leaving early.

In reflecting on her teaching, Ling believed she had to follow the course syllabus closely since it was well planned by the coordinator who "knew how much time to spend on each topic". In addition, she closely followed the text because she believed her students would understand it better than her. Ling indicated that she modeled her instruction, which is predominantly lecture-style, after teachers in her graduate mathematics courses. She stated that she took her teaching responsibilities very seriously but sometimes was unable to plan sufficiently for class, especially when she was studying for her actuarial exams. Ling believed that the biggest challenge in teaching U.S. students was her difficulty with the English language. In comparing U.S. and Chinese students, Ling indicated that U.S. students "hate" mathematics and do not spend enough time studying while Chinese students do many practice problems and consult reference books when they don't understand a particular topic. She also believed that in China it is the students' responsibility to learn while in the U.S. more responsibility is placed on the teacher. To improve her teaching Ling consulted her fellow Chinese TAs, her course coordinator, and her husband. For example, when she was wondering how to deal with a student who she perceived was asking "tricky questions that were not relevant to the lesson" her husband advised her to begin class each semester with some very hard problems to show students how smart she was so that students wouldn't question her ability in class.

During the second semester Ling indicated that her teaching was much improved—she felt more comfortable speaking English and was assigned to teach a higher-level course with brighter and harder working students. However, she indicated that attendance was still a problem and could only encourage the same three or four students to answer questions in class. Finally, she indicated that members of the mathematics department were very supportive in helping to improve her teaching.

Olga

Olga was born and raised in the second largest city in Kazakhstan. Her family consisted of her mother, father, and two older sisters. Both parents were college educated and very supportive of their children's education. Olga indicated that age and gender influenced one's standing in her community—that is, individuals would not challenge elders even when they were wrong and generally, men were more respected and had more opportunities than women. Olga attended the best schools in her city from elementary school through high school. She commented that all schools used the same curriculum and class schedules typically consisted of 7 classes a day, each class lasting 45 minutes. Technology (i.e., computers and software) tended to be outdated and rarely used during class. Generally, she spent 2 to 3 hours a night studying and the material was easy to master. Olga commented that teachers were very respected and did not establish a close relationship with students. Mathematics instruction in high school followed a transmission model in which teachers lectured and students listened, with some time reserved for students to ask questions. To enter university, students needed to score well on several exams. Olga stated that professors at university lectured with little student interaction. Student-teacher relationships at university were much more formal than in the U.S. and professors did not hold office hours. Olga indicated that, in her country, "We never blamed the teachers...we think it is our [the students] fault. It doesn't matter if he is a bad teacher it is our mentality that if you are smart, you will do well, it doesn't matter how your teachers are. In our country we think students are responsible for learning." Olga came to the United States to pursue a M.S. in Actuarial Science and at the time of this study, she had been in the United States for two years. She indicated that while living in the U.S. she had not made new friends from the U.S. and that she spent most of her time with students from the former Soviet Union.

Olga stated that mathematics involves the laws of nature and the purpose of teaching mathematics was to help students understand basic problems. She felt that acquiring mathematical knowledge provides individuals with more opportunities in life. Olga commented that great teachers are kind and make people feel comfortable in class. Further, such teachers get to know their students, respect what they say in class, are flexible, and actively involve them in class discussions. She believed that students learn mathematics by passing the common exams and in one-on-one situations during office hours. To accommodate students' different learning styles she excuses students from class who may already know the material and tells students who are having difficulty to meet her during office hours. In her own learning of mathematics, Olga did homework, attended class, and did what she was told to do in class. It was only at the university that she began to learn how to think about solving problems.

During her classroom observation, Olga relied heavily on her notes and delivered the material in a way that closely mirrored the textbook presentation of the topic. She spent most of the class time writing notes on the board, rarely facing the students. During the period, several students arrived late and left early. Of the students who stayed in class, most sat passively and copied notes. Olga asked a few questions during the period but had a difficult time getting students to answer questions.

Generally, Olga was not satisfied with her teaching performance throughout the Fall semester. She believed that she had little autonomy in her planning and teaching since she had to follow a course syllabus and prepare her students for the common exams. She indicated that she planned between 45 minutes and two hours for each class and that in her planning she thought about examples and how she could explain the material in English to the class. Olga commented that she felt more comfortable using material from the textbook to ensure that it was familiar to her students even though she believed she could explain if more effectively in a different way. To determine whether students understood the material in class, Olga tended to read students' facial expressions. In particular, she looked to one student—when he was happy she believed she was doing well in class. She reflected on her teaching immediately after each class and discussed her teaching with her office mates, husband, and course coordinator. She believed that teaching was the most important aspect of her academic responsibilities since it is embarrassing to her if she didn't do well and because it affected her students and their grades. She indicated that in the future she would like to make fewer mistakes in class, be more confident in her teaching, and be stricter with her students.

Throughout the Fall semester, Olga struggled to balance the demands of her graduate coursework and teaching responsibilities. However, during the Spring semester Olga commented that she felt more confident in her teaching and better understood the expectations of the courses she was teaching. Further, she indicated that while teaching she was more patient with her students, her communication skills had improved, and she relied less on the textbook and more on "her own words" to explain material in class.

Karl

Karl, an only child, was raised by his parents in a large city in Belgium. His mother attended school until age 11 and his father until age 14. Both parents valued education and expected Karl to earn good grades in school. Karl indicated that the role of age and gender in his country is similar to that in the U.S. His parents paid tuition to send him to a Catholic elementary school and high school since they believed that these schools provided a better education than public education in his city. Mathematics instruction in the schools he attended at this time mainly involved pencil and paper, with little or no integration of technology. Karl indicated that teachers in his country were "information givers" and were the central authority figures in the classroom. Generally, his teachers taught from the front of the class while students remained silent. Although students could raise hands to ask questions, they remained generally passive. Karl mentioned that high school teaching was a well-respected profession but at the university level research is valued over teaching. According to Karl, the university system in his country was "harsh" since in four years students earn what would be equated with a Master's degree in the U.S. At university, Karl studied approximately 30 hours per week. He came to the United States to pursue a Ph.D. in mathematics and at the time of this study had been in the United States for less than one year. Karl indicated that since in the U.S. he had met people from all different countries and considered himself to be friendly with some people from the U.S.

According to Karl, mathematics is the "queen of the sciences". He indicated that he decided to study mathematics because of the aesthetic beauty of the discipline. Further, he commented that the purpose of teaching mathematics was to create the next generation of mathematicians and provide students with a sense of its utility. In his view, good teachers are clear, patient, flexible in dealing with different answers, able to say when an answer is correct or incorrect, and able to say in a polite way that an answer is wrong. Further, good teachers are precise, meticulous, and orderly and are able to show the relevance of the topics they are teaching. With respect to his own learning, Karl indicated that he rarely understood lectures and learned by doing exercises alone at home. Further, he said that he learned mathematics by looking at definitions and considering examples and counterexamples. He indicated that he believed students learned mathematics by viewing a solution or by reading the textbook rather than trying to solve problems. He also commented that it was unimportant to concern himself with the different learning styles of students in his classes but it was necessary to recognize when a student was doing poorly.

Throughout his classroom observation, Karl's instruction modeled a transmission method of teaching. Generally, he taught from the front of the room and students spent the period copying notes in their notebooks. His teaching mirrored a pattern involving mathematical theory followed by concrete examples. He rarely asked for student input—two students asked questions during the period, he addressed them and then returned to lecturing from the board. Many students arrived late to class and several students left early.

Several times during the Fall semester Karl indicated he was "quite happy" with his teaching performance given that it was his first semester teaching. Upon reflecting on his videotaped lesson, he commented that he felt his English was very clear and was satisfied that he could "read his own writing". He also believed that he handled students who came in late or were not paying attention appropriately by ignoring them since "learning is the student's responsibility". Karl indicated that the model of teaching in his graduate courses is modeled after the Bourbaki-style of teaching and learning (i.e., definition, theorem, proof) and that while he preferred this approach, and tended to teach in this manner, it was not effective with his students. According to Karl, he spent approximately 3 hours planning for every lesson and used the book as a guide. While planning, he spent considerable time thinking about how he could present the material in the book in a better way, how to best say something, which topics should receive emphasis, and how to make connections between chapters or other material in the book. He said that he used humor often in class since he believed it created a personal bond and a rapport with students and "grabbed the attention of someone falling asleep". Throughout the Fall semester Karl noted differences between students from his native country and students from the U.S. He explained that students in Belgium universities are more disciplined than students in the U.S. He felt that U.S. students spend as much time working (in jobs) as studying and noted that they speak more in class, are sometimes rude, and are not as well prepared for college level mathematics. In reflecting on his teaching, he indicated that he would be "stricter" with students in subsequent classes.

According to Karl, teaching during the Spring semester was easier than the Fall since he was assigned to teach the same course and as a result had less work to prepare. He commented that teaching during the Spring semester was "less work for the same results". He said that he made some minor changes to his notes, including different examples, and began each class on time. Unlike the other ITAs Karl did not believe there was support in the department to improve his teaching and often felt there was contradictory information given to TAs on how to teach and handle student problems in class.

Discussion

While it is clear there existed cultural differences among all the participants, the data revealed a number of common threads in their backgrounds and experiences. For example, all of the participants had to pass high stakes examinations to continue their education in their native countries and were exposed to models of instruction that involved lecture with little to no student participation. Throughout their schooling they came to view and learn mathematics as an isolated activity—that is, they solved problems independently and read textbooks to understand difficult topics. At the time of the study, all of the participants were in their first semester of teaching in the U.S. and took their teaching responsibilities seriously. They were expected to follow demanding course syllabi and prepare their students for common exams. In addition, they relied heavily on the course text in planning and teaching the material. All of the participants indicated that models of teaching in graduate school in the U.S. were overwhelming lecture with little student participation. Finally, all participants indicated that teachers in their countries were very respected and they recognized marked differences between U.S. students and students in their respective countries. For example, they indicated that in their native country learning was placed squarely on the shoulders of the students while in the U.S. teachers are expected to shoulder a large part of the responsibility in helping students learn mathematics.

Classroom observations of the participants indicated that Ling, Olga, and Karl employed a transmission model of teaching with little student interaction while Blidi appeared to create a classroom environment where students were actively engaged in learning mathematics. Looking across the cases, what might explain these differences? Differences in the participants' teaching styles may be attributed to cultural variables identified by Olanirin (1996), their academic mathematical preparation, their views about teaching, models of teaching they have been exposed to as students and their enculturation into the mathematics department as researcher and teacher.

The classroom observation of Blidi revealed a model of teaching aligned with a constructivist perspective on teaching. His instruction was student centered and fostered a sense of community by calling on students by name, encouraging student participation, and circulating around the room throughout the lesson. His circle of friends included U.S. students, which may help explain his willingness to create a classroom environment in which students felt comfortable asking and answering questions and participating in class (Olaniran, 1996). Further, just as a strong sense of community permeated his own learning throughout his high school years, he fostered a "community of learners" among students in his own classes. For example, he reviewed his grade sheet weekly to identify students who needed extra help, and called these students over the weekend to suggest meeting to review the course material. Blidi attributed his teaching style to his personality and to the fact that he believed that U.S. students want to participate in class and understand why they need to learn mathematics. He felt very comfortable teaching in English and viewed language not as an impediment to teaching but as a tool to engage and connect with students in class. For example, he used his misconceptions of the English language to establish a rapport with his students.

In contrast, classroom observations of Ling and Olga revealed classroom teaching that would be characterized as teacher directed, with little or no student participation. They each followed a script and students passively took notes from the board. Their instructional behavior can be explained in part by some of the variables identified by Olaniran (1996) that influence ITAs teaching performance—*language, age,* and *friendship communication network patterns*. Their concern to communicate effectively in English influenced their teaching practice—each resorted to a "lecture-only, and memorized" classroom presentation. Further, both Ling and Olga hailed from high distance power countries where power and authority are influenced by age and in addition, both have maintained close ties with students from their native countries and had no U.S. students in their circle of friends. In addition, in collectivist cultures (e.g., China) a status differential exists between teacher and student and is maintained through "politeness strategies" such as avoidance and silence, resulting in classrooms dominated by teachers with little or no expectation of student participation. Together, these factors provide some justification for classroom instruction in which the teacher is the central authority figure in class (as with Ling and Olga) with little or no student participation (Jenkins, 1997; Olaniran, 1996). Finally, the models of teaching mathematics that both Ling and Olga have been exposed to as students and in graduate school provided a powerful image of a traditional model of teaching and further evidence in explaining their teaching behavior.

Similar to Ling and Olga, Karl exhibited a transmission model of teaching during his classroom observation—he lectured and students quietly took notes from the board. Currently, Karl is studying for his Ph.D. in mathematics and expects to acquire a position as a research mathematician at a university in the future. According to Schoenfeld (1989), "there is a cultural component to learning to think mathematically: Becoming a mathematician involves a process of enculturation, in which initiates become members of, and accept the values of, a particular community" (p. 87). By virtue of being a research institution mathematics faculty at the University of Connecticut direct their energies toward research and mentor graduate students into the process of becoming researchers. Further, anecdotal evidence indicated that most instruction within the mathematics department tends to model a traditional method of teaching—that is, faculty lecture while students take notes, with little opportunity for classroom discourse. Therefore, participation in the day-to-day routines of the mathematics department provides a cultural milieu in which teaching assistants acquire a perspective on mathematics and on the teaching and learning of mathematics closely aligned with a transmission model of teaching. It appeared that Karl has acquired such a perspective on teaching based on his exposure to models of teaching as a mathematics student and his day-to-day interactions with faculty at the university.

Final Remarks

Research has indicated that culture plays a significant role in shaping cognition, which in turn influences behavior and ultimately one's teaching performance. In this study, the cultural experiences (e.g., one's schooling, models of teaching, language, members of the mathematics department, views about teaching, etc.) of the four ITAs have shaped and continue to shape their beliefs about teaching and learning mathematics and as well as their classroom practice. However, there appeared to be varying levels of consistency between the participants' acquired view of teaching and learning mathematics and their instruction. For example, Blidi appeared to act on his core belief that community plays a central role in educating students and therefore strove to create a sense of community in his classroom. In contrast, while Ling believed the purpose of teaching was to help students understand the material she seemed to act on her core belief that her job was to cover the material and closely follow the textbook and the course coordinator. This is not surprising since individuals can hold conflicting beliefs while acting on their core beliefs (Green, 1971). Further research on the role and impact that core beliefs may have on teaching performance among ITAs needs to be conducted. Finally, the cultural variables outlined by Olaniran (1996) with respect to ITAs and teaching performance seemed to have an impact on their classroom instruction. These variables need further attention. An examination of these variables, with an eye toward improving mathematics pedagogy courses, may lead to an improvement in the teaching performance of ITAs within the mathematics department.

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SOCIOCULTURAL PERSPECTIVES ON MENTORING MATHEMATICS STUDENT TEACHERS

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This case study of mentoring in a middle school reform-based mathematics classroom examines learning to teach from a participatory perspective on development. The participants' orientations toward mentoring were considered to capture previous experiences and understandings of the mentor and mentee. In addition, an analysis of the forms of participation and the apprenticeship of the mentee (Rogoff, 1995, 1997) were considered to understand how learning to teach took place on multiple planes of analysis. The study showed that learning to teach was grounded in the mentor and mentee's professional communities. Their guiding orientations of mentoring and forms of participation were influenced by these communities. An implication of the study is that we need to construct mentoring education to include a strong reform-based mathematics content in which normative practices of reflection and collaboration on teaching and learning are second nature to instructional practice. And, mentee education must be coordinated with mentoring education to capitalize on field and teacher education opportunities.

Research Problem

Student teaching is considered by preservice teachers to be one of their most important experiences during teacher education. Undeniably, mentor teachers' actions, beliefs and knowledge play a large role in influencing the nature of this experience for student teachers (McIntyre, Byrd & Foxx, 1996). Research on mentoring documents the importance of mentors' beliefs and provides descriptive accounts of mentor and mentee interactions (Stanulis & Russell, 1998; Hawkey, 1997). However, there are other areas of research such as the context in which student teaching takes place that informs the experience but has typically been ignored by mentoring research (Warren-Little, 1990; Zeichner, 1986). Acknowledging the abundance of research that does not capture both the individual and the environment, Barbara Rogoff (1995, 1997) comments on learning and development in general, explaining that traditional research in these areas have either focused on the individual or the environment. This disconnect effectively limits the researcher's perspective by separating the individual from the environment. This separation of the individual and the environment and the nature of the research on mentoring points to a need for research to coordinate the influence of beliefs, the descriptions of interactions and the impact of the context in a way that acknowledges the interdependence of the these influences. This research coordinates these contributing factors to development posing the following research questions. What is the nature of apprenticeship in mentoring activity? What is the nature of guided participation in mentoring activity? And, what is the nature of individuals' influence on mentoring activity?

Conceptual Framework

This study analyzed mentoring as a sociocultural practice in which learning to teach took place through participation (Rogoff, 1995, 1997). Rogoff's interpretations of sociocultural theory provided a means to examine mentoring as a system of cultural practices, social interactions and individual understanding. More specifically, this system was (1) individuals' understandings or orientations brought to and developed by participation in mentoring, (2) the mentor's and mentee's participation with one another – the forms of participation constructed, and (3) the apprenticeship of student teachers within institutions or pertinent cultural contexts impacting mentoring. Participants' orientations to mentoring were frameworks, developed from past experiences and understandings of learning to teach and mentoring, that guided mentors' and mentees' participation. Rogoff (1995) suggests that development, or the taking on of more responsible roles in activity, takes place through changes in forms of participation. For example, a student teacher learns to teach by learning to taking on more responsibility for classroom teaching and student learning through changes in forms of participation. Apprenticeship in learning to teach is the way that a new member is brought into full membership through the systems of normative practices associated with mentoring learning to teach. To analyze these levels of influence on mentoring and record development I examined individuals' orientations to mentoring, the forms of participation constructed by the mentor and mentee, and how the normative practices of the professional communities to which participants belong were imported into mentoring practice.

Methods

This study was undertaken within a university reform-based teacher education program. The secondary school selected for the study used reform-based curriculum (Connected Mathematics Project) as the site for learning to teach. The data collected were participant observations of the student teacher, cooperating teacher and other faculty engaged in mentoring practices inside and outside the classroom. All conversations were audio taped and transcribed, including formal initial and final interviews and informal interviews after each cycle of observation. During the informal interviews participants clarified their intentions and understanding of participation. In addition, they provided feedback on my interpretations of events. Participant quotes used in this text are followed by the data source in parentheses. Artifacts from lessons, meetings and emails were also collected. Data were analyzed using external codes based on the conceptual framework for the study and internal codes generated out of the data. Coding checks were conducted by an external researcher to verify consistency and thoroughness of coding. Analytic memos (Strauss, 1987) were constructed for each external code to identify the internal codes or patterns in the data. Vignettes were also constructed to capture the events within the case.

Findings - Early in the Semester

This paper reports on the data from mentoring a middle school mathematics preservice teacher to illustrate how the coordination of the three planes – cultural, social and individual -- of the conceptual framework provided greater understanding of learning to teach. Ms. Lane, the cooperating teacher, and Ms. Sekwiter, the student teacher, were paired in student teaching because Ms. Sekwiter suggested that she wanted a reform based middle school placement in which she could continue to develop her ideas of teaching in a problem based classroom. Ms. Sekwiter had previous field placements in reform based classrooms and suggested that she was very comfortable with problem-based curricula and student inquiry. Those previous placement were pivotal to Ms. Sekwiter's understanding of learning to teach. She said,

> [my teacher] just wanted me to get in there and get going, she just really wanted to see what I could do, what I thought... And it went really well and then she always helped me modify so that it would work even better the second time around (NS, I#1, p. 6).

Ms. Sekwiter's understandings of learning to teach developed from this experience. She thought that she was to try her ideas in student teaching and her mentor would tell her how to modify the "little things of teaching" that she needed to improve. Trying ideas was her orienting framework implying that Ms. Sekwiter was only interested in learning through experience. She comments,

> I need the chance to just do everything and try to handle everything all at once. ... how am I going to schedule the things that I need to do and is my perfect theory of what I would like to do going to be able to work ... I think the most important part for me was just I know what I want but can it really work (NS, I#1, p. 8).

As a result of focusing on trying her own ideas, Ms. Sekwiter resisted reflecting on ideas or incorporating Ms. Lane's suggestions into her teaching.

Ms. Lane's orientation to mentoring – reflective practice -- differed dramatically from Ms. Sekwiter's trying ideas. Ms. Lane commented, "... I want to know what they are thinking about, why they would do [something in instruction], tell them that it [is important] to have a reason to do it and they don't just do it without thinking about it" (PL, I#1, p. 33). Reflection was integral to Ms. Lane's ideas of teaching and learning to teach. Ms. Lane attempted to collaborate with Ms. Sekwiter by planning together, asking questions, and co-teaching. Ms. Lane reflected on the nature of these activities early in the semester saying,

> ... with Nan I don't see her asking as many questions back. So if she comes with an idea and I ask, what about this, what about that, she is already convinced [that her idea is fine], she doesn't modify...[T]hat is what I have seen

> so far. So it doesn't have any influence if you ask a question back. I want her to develop so that she understands that teaching, like for me, teaching is not done yet and it is never done, and [for Nan] it is like you want to student teach and then you are done and you know how to teach....(PL, I#1, p. 28).

Ms. Lane thought teaching and learning to teach was a process of developing ideas, questioning the ideas, trying ideas and modifying these ideas through reflecting.

The following is a vignette of Ms. Lane and Ms. Sekwiter's interactions early in student teaching. The vignette highlights Ms. Lane's collaborative efforts and Ms. Sekwiter's responses.

Vignette

Ms. Lane shows Ms. Sekwiter a pattern problem that she wants students to work with in class. She asks Ms. Sekwiter what they might want students to understand from the problem. Ms. Sekwiter answers quickly, "if they can show a general rule then you know they got it." Ms. Lane asks if there are other ways that students might show that they understand? Ms. Sekwiter appears confused and frustrated that Ms. Lane is not accepting her idea that if the students have a rule then they understand. Ms. Lane discusses the idea that they should allow for other ways for students to show what they know, suggesting that they try the problem in class and see if the students have different solutions. Ms. Lane teaches the lesson and students work in groups while Ms. Sekwiter helps a few groups.

The next day Ms. Lane asks Ms. Sekwiter about what she saw in class the day before and how they might use another pattern problem with the students. Ms. Sekwiter responds that she saw a lot of students not getting an algebraic rule and she guided them in their group to see the rule. Ms. Lane asked if there were other ways that students were seeing the pattern, like did she notice a group of students who described the rule recursively. Ms. Sekwiter sat listening passively. Ms. Lane suggested a way for Ms. Sekwiter to explore alternative solutions in class that day. Ms. Sekwiter never confirmed Ms. Lane's suggestion and added that maybe they could try modeling an algebraic general solution for a pattern to the whole class. Ms. Lane explores what Ms. Sekwiter thinks students will learn from this. She asks if Ms. Sekwiter thinks that they have time during the lesson to do all of this and allow the students to work. Ms. Sekwiter says, "I guess not." Ms. Sekwiter again looks frustrated that Ms. Lane is not acknowledging her idea for the lesson. Ms. Lane asks Ms. Sekwiter to introduce the problem and they both will work with groups during class. Ms. Sekwiter presents the problem without much discussion and begins to work with groups using her idea of showing them a model.

The following day, Ms. Lane has copies of students' work that use different methods to find the pattern. Ms. Lane starts planning by asking questions about the work. Ms. Sekwiter looks frustrated with Ms. Lane and walks away from the table changing the subject asking about entering grades. Ms. Lane asks Ms. Sekwiter what sense she made out of the students' solutions yesterday. Ms. Sekwiter says from across the room that she thought that most of them got it but some were still not coming up with a rule. Ms. Lane looks frustrated and suggests that they try another problem and that Ms. Sekwiter listen to two groups with different solutions discuss how they saw the patterns. Ms. Lane taught the lesson and Ms. Sekwiter worked again with students. Class ended before Ms. Sekwiter asked two groups to compare solutions.

The final day of working with patterns, Ms. Lane and Ms. Sekwiter start planning for another problem that was more complex than the days before, the triangle problem. Ms. Lane asks, "How are we going to do this?" They each look at the pattern. Ms. Lane asks, "What should they have on their papers that shows that they understand the pattern?" What would we want to hear them discussing? They each worked the math in the problem. Ms. Lane creates different ways that students could find a rule for the figures. She is talking to herself about the various methods students could use and solutions. Ms. Sekwiter has found a solution to the problem and turns to organizing warm ups and worksheets that need to be copied for the day. Ms. Lane has her solution notes in front of Ms. Sekwiter working on them. Ms. Sekwiter walks away to go make a stack of copies near the door and find the attendance book

Ms. Lane looks up at Ms. Sekwiter and asks her, "What would be the a way to support students getting into the problem?" Ms. Sekwiter sits back down and says,

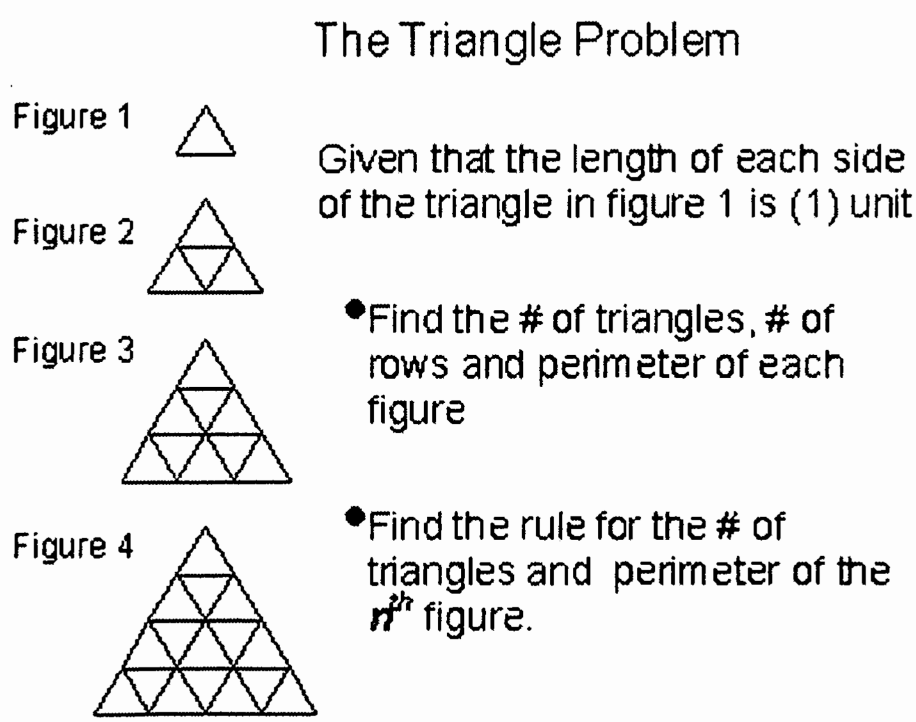

Figure 1. The Triangle Problem.

"Well we could draw a chart and then have them follow us filling in the chart to find the patterns for each piece. And we could find one rule for them." Ms. Lane replies, What do you think some of them will understand about the numbers in the chart? Ms. Sekwiter says she is not sure, but, "I just think that if we show them one of the rules then they can follow what we have done and find the other rules." Ms. Sekwiter continues to elaborate on her idea that they should show the students a chart to organize their work. Ms. Sekwiter says that she thought about this all the way home yesterday, that a chart was what the students needed to see patterns more clearly. After three days of asking Ms. Sekwiter about making sense of students' solutions and Ms. Sekwiter suggesting ideas for teaching the lesson, Ms. Lane has reached a dead end with posing questions to foster reflection and assessment of students' thinking – Ms. Sekwiter is unwilling to modify her idea. Ms. Lane suggests to Ms. Sekwiter that she teach the pattern problem based on her idea of the chart.

Ms. Sekwiter teaches the lesson and Ms. Lane observes from her desk. Some students have trouble creating the chart and don't understand where Ms. Sekwiter is finding the numbers for each of the columns.

They ask questions and Ms. Sekwiter suggests they aren't paying attention. Ms. Sekwiter is working out each step of the chart. Some students have solved the problem

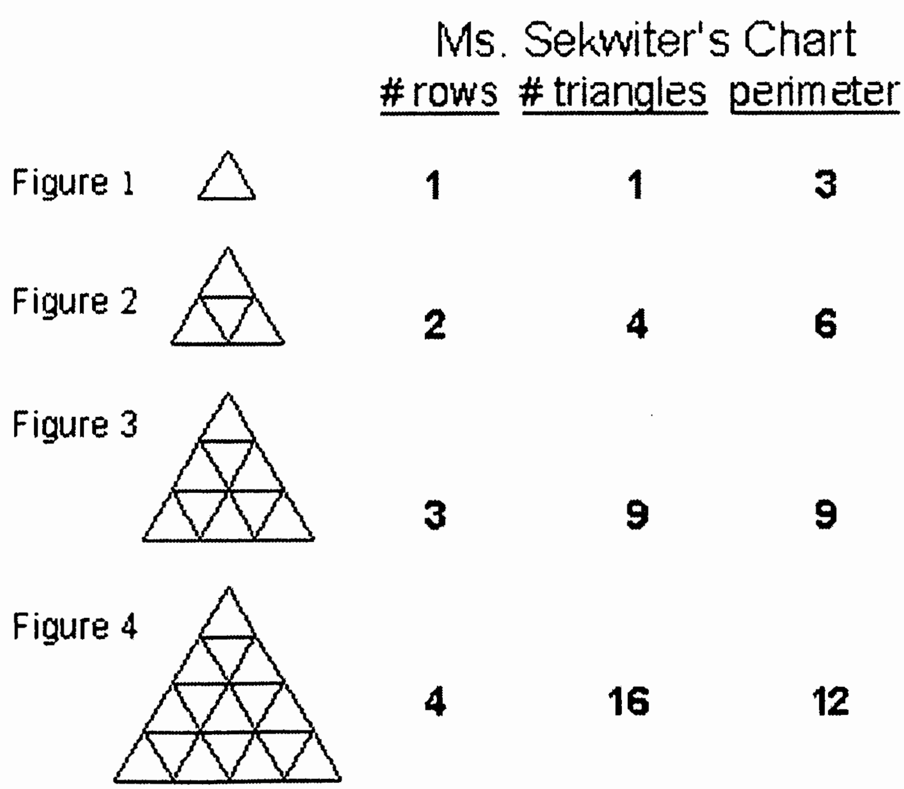

Figure 2. Ms. Sekwiter's chart.

and are discussing their different solutions while Ms. Sekwiter is working at the front of the class with the chart. Class ends before Ms. Sekwiter models finding a rule.

Ms. Lane asks Ms. Sekwiter, "How do you think it went?" Ms. Sekwiter responds, "Well it didn't go too badly. Those that needed the chart at least saw how to organize their work." She adds, the class had quite a few students who had been absent so those students lacked experience with patterns. Ms. Lane asked Ms. Sekwiter what one student was thinking when he asked "look the numbers are growing from one step to the next." Ms. Sekwiter said that she saw that he was basing the pattern on the previous number but she wanted to get to general rule. She added that the student that brought that up was also not paying attention during most of her discussion. Ms. Lane asked Ms. Sekwiter about running out of time and how she might modify the lesson. Ms. Lane suggested that she didn't realize she had run out of time until there was only three minutes left. Adding, that the clock in the room was difficult to see since it was at the front of the class. Ms. Sekwiter suggested that the students in the afternoon class were not as easily distracted so it would probably go better the next time.

Ms. Sekwiter had difficulty making sense of lesson and knowing how she could modified the lesson to address the issues. Ms. Lane said that she would teach part of the afternoon lesson and that Ms. Sekwiter could watch how she worked with the pattern problem.

Patterns of Interaction

Early in the semester Ms. Lane worked with Ms. Sekwiter on developing, questioning, and modifying ideas through reflection. Ms. Lane was attempting to build Ms. Sekwiter's content and pedagogical content knowledge through refection on experience. Refection focused on mathematical ideas, students' conceptions, and formative assessment of student discourse and written work. This reflective process was difficult for both Ms. Lane and Ms. Sekwiter. Ms. Lane had created a form of participation in mentoring in which she attempted to engage in meaningful conversation, yet Ms. Sekwiter was focused on trying ideas. Mentoring early in the semester developed into a pattern of miscommunication, in which both Ms. Sekwiter and Ms. Lane communicated their ideas of how to participate in mentoring without the other listening.

The participants' orientations to mentoring learning to teach framed how they participated in mentoring events and the resulting miscommunication. Ms. Sekwiter's trying ideas orientation guided her to pursue ideas and gain experience teaching. Her orientation did not include deep reflection on ideas. This orientation was built on Ms. Sekwiter's previous field experiences from which she came to understand learning to teach as gaining experiences and a mentor's job was to tell her how to improve on her teaching strategies. Contributing to Ms. Sekwiter's guiding orientation was her understanding of the work of teaching. For Ms. Sekwiter teaching was about collecting interesting and fun activities that engaged students in doing math. However, she was not clear on what the important mathematical ideas were and how she would make the math explicit in her fun activities. Ms. Sekwiter gathered these activities from teacher friends, searching the internet and teacher resource workbooks. Most prominent in her discussions of these resources was Ms. Sekwiter's associations with teachers that would share activities with her to try during student teaching. Ms. Sekwiter was creating her bag of tricks by gathering activities from other teachers.

Ms. Lane believed the collaboration was at the heart of mentoring learning to teach early in the experience. Ms. Lane's reflective orientation to mentoring framed how she understood the job of learning to teach and her role in that process. In-depth conversations about teaching, learning and mathematics were essential to developing as a teacher. Ms. Lane was a long-standing member and leader in reform-based professional development. She suggested that she based her understanding of learning to teach on these experiences, importing the purpose of mentoring learning to teach from this professional community.

As the semester progressed Ms. Lane became more frustrated with Ms. Sekwiter's lack of reflection and willingness to question and modify her ideas. Ms. Lane came to understand that Ms. Sekwiter was not able to see all the complexity of teaching while they were collaboratively teaching. She commented that Ms. Sekwiter was "not open to learning" anything more from sharing the responsibility for teaching. As a result, Ms. Lane shifted how she and Ms. Sekwiter participated in mentoring. Ms. Lane suggested that Ms. Sekwiter begin her solo teaching time in which Ms. Sekwiter would be responsible for all the teaching.

Findings - During Solo Teaching

Even though Ms. Sekwiter expressed that she saw her placement as "fitting right in with [her] previous placements" her orientation of trying her ideas, conflicted with Ms. Lane's orientation of reflection and discussion that framed her participation in mentoring. The differing orientations precluded Ms. Sekwiter from engaging in reflective discussion with Ms. Lane. Ms. Lane's process for collaboratively teaching was not understood by Ms. Sekwiter as an appropriate means of engaging in mentoring. Ms. Lane struggled to understand Ms. Sekwiter's ways of engaging in mentoring. About a month into the semester Ms. Lane suggested that she would have to "figure out" how to work with Ms. Sekwiter. Prior to soloing, her process for figuring this out included trying a variety of methods to engage in reflection, each garnering only limited success.

Soloing proved to be a vehicle for Ms. Sekwiter and Ms. Lane to begin to discuss the complexities of teaching reform-based mathematics. After a number of unsuccessful teaching episodes in which Ms. Sekwiter was baffled by students' behavior, Ms. Sekwiter began to reflect, ask questions and examine her practice with the guidance of Ms. Lane. The change in participation – soloing-- facilitated by Ms. Lane, was a catalyst for Ms. Sekwiter's development in learning to teach. In addition, Ms. Sekwiter reviewed the text and surmised that she did not know how to facilitate the mathematical ideas suggested in the teacher's notes. She came to Ms. Lane with questions, asking for help. This was a shift in their ways of participating in mentoring. Ms. Sekwiter began to develop new ways of communicating about teaching. She began to look deeper and reflect on her instruction.

Analysis

The forms of participation that constituted mentoring practices early in the semester did not provide for Ms. Sekwiter to try her ideas to the full extent that she believed was appropriate. These forms of participation were constructed by Ms. Lane, with the intention of incorporating Ms. Sekwiter's ideas, but these opportunities were less than successful. Ms. Lane thought that Ms. Sekwiter's lack of reflection constrained their collaboration and Ms. Sekwiter's learning to teach. Ms. Sekwiter thought that Ms. Lane's need for reflective discussion on ideas was not part of her job as a mentee and simply was Ms. Lane controlling her access to trying her ideas. The participants' guiding orientations framed how they participated in mentoring. The forms of participation did not facilitate development in learning to teach until Ms. Lane shifted the nature of the work. Consequently, Ms. Lane backed away from her attempts to make Ms. Sekwiter engage in reflection *prior to* taking full responsibility for teaching and suggested that she start her solo teaching time. This shift in participation was a marker of development in the case.

Development in learning to teach in this study was examined on three planes of analysis. On the individual plane, participants' orientations were the unit of analysis which uncovered how participation in mentoring was guided by the mentor and mentee's experiences and understandings of mentoring leaning to teach. These guiding frameworks informed how participants engaged in mentoring interactions. On the second plane of analysis, forms of participation constituting mentoring were examined for shifts in participation that marked development of the participants. On the third plane of analysis, the nature of the apprenticeship was examined to understand how mentoring was a sociocultural practice. Apprenticeship, "focuses on the *system* of interpersonal involvements and arrangements in which people engage in culturally organized activity." (Rogoff, 1995, p. 143) The emphasis on this plane is not the actions of the participants, but the ways these actions are bound by normative practices of the larger institutional or cultural context.

Wertsch, Minick and Arns (1984) suggest that to understand individuals' actions when engaged in activity we must look to the sociocultural context in which the activity is situated. Participants import ways of acting from these familiar settings to guide participation in other cultural activity. In this mentoring case the other cultural activities that related to mentoring were the professional communities in which Ms. Lane was a member and that Ms. Sekwiter had associated with prior to and outside of student teaching. These were different communities that had different normative practices.

Ms. Lane imported the reflection, discussion and content she engaged in while attending and leading reform-oriented mathematics professional development. Ms. Lane participated in two long-term professional development projects during the year of the study, devoted to mentoring and reform-based teaching. These opportunities were formative to her mentoring of Ms. Sekwiter. She suggested that her purpose for mentoring was to facilitate a student teacher's entrance into the profession by adopting the normative practices of the reform-based community. For example, Ms. Lane's discussion of students' conceptions, mathematical big ideas and formative assessment were topics in her professional development experiences as well as embedded in her instructional practices. Her experience of reflecting on these topics in professional development permeated what she understood as important to learn during student teaching. Furthermore, Ms. Lane was being mentored during this semester of observations by an experienced reform-based teacher that she respected. Ms. Lane capitalized on this experience by reflecting and incorporating into her mentoring of Ms. Sekwiter what and how she discussed teaching and learning with her mentor. These professional development opportunities were Ms. Lane's more familiar activity settings that informed her purpose for mentoring and her interactions.

Ms. Sekwiter's imported the ways of interacting that she had developed in her early field experiences and what she understood to be important to teaching gained through conversations with teacher associates. Each of these contexts provided Ms. Sekwiter with evidence that teaching was about gaining experience and trying different ideas. She commented that she did not witness her previous mentor needing to reflect on their experiences nor engage in a process of discussion and modification. Ms. Sekwiter's orientation to mentoring and her participation were guided by the normative ways of thinking and acting that she witnessed in previous teaching experiences and associations with teachers.

Implications

Ms. Lane's purpose for participation in mentoring, the forms of participation constructed and her orientation to mentoring, was commendable and embodied many of the suggested best practices of mentoring (Peterson & Williams, 1998). The content of her mentoring focused on learning to teach reform-based mathematics using notions of professional skills such as reflection that are central in the NCTM Principles and Standards. However, when examining the outcomes of the mentoring interactions Ms. Lane's orientation and forms of participation were less than fruitful. Ms. Sekwiter's focus on trying her ideas was typical of student teachers, but it did not facilitate reflection on her practice (Borko & Mayfield, 1995; Elliott & Calderhead, 1993). What guided these the participants' orientations to mentoring and the ways of participating were the participants' membership in professional community and previous experience with the profession. The differences observed between the mentor and mentee were not merely individuals' different perceptions, rather they were the participants' familiar activity settings that served to orient their mentoring activity (Wertsch, Minnick, & Arns, 1984).

The data in this paper illustrated how the cultural practices of professional communities impacted the social interactions and individual orientations that inform mentoring activity. This research coordinated the individual, social and cultural influences on learning to teach by examining development in sociocultural activity. One implication of the work suggests that mentors and mentees need support to develop cohesive orientations to mentoring. In addition, mentor education, long reported as important for developing strong mentors, proved to be important to the content of mentoring (Feiman-Nemser, Parker & Zeichner, 1993). However, the nature of mentor education has been broadened by this study to include the professional development focused on reform-based mathematics. Furthermore, mentee education must also be included in the call for support of mentoring activity. There is a need to leverage the learning opportunities in student teaching and previous field experiences coordinated with the theoretical perspectives represented in teacher education to develop sound mentee education.

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WOMEN, MATHEMATICS, AND SOCIAL RESPONSIBILITY: THE ROLE OF WISDOM IN CHOOSING A COLLEGE MAJOR

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This study addressed the question of what factors impact mathematically talented college women's choice of academic major. Twelve participants engaged in a 12-week on-line bulletin board discussion group, and the data were collected and analyzed using the qualitative method of constant comparative analysis. This study found that four central factors influenced choice of major among these participants: environment, behavior, talent, and value. Value had the highest relative influence, as these women were deeply concerned about the positive social impact of their future careers. Many participants had conflicts with the perceived values of the mathematics departments at their colleges and universities, and those conflicts were often cited as the central reason (and occasionally the only reason) that a participant was not majoring in mathematics. This paper also discusses how university mathematics departments can use these findings to address the problem of female underrepresentation in undergraduate mathematics programs.

Purpose and Background

The purpose of this study was to examine how mathematically talented college women come to decisions concerning their academic careers. This study focused primarily on the decision of college major. While a large body of research already exists on this topic, previous research studies concerning mathematically talented girls and their academic plans have focused almost exclusively on the differences between mathematically talented girls and boys (Eccles, 1985, 1989). Because previous research has indicated that mathematically talented boys have been more persistent and more successful than mathematically talented girls, the often unstated question being asked in many of these studies addresses the ways in which girls can be changed to be more like boys (Benbow & Lubinski, 1993; Cramer & Oshima, 1992; Heller & Ziegler, 1996). This approach does not necessarily shed any light on the potentially unique issues that talented women in mathematics find to be important in their decision of a college major. Therefore the results cannot necessarily be used to determine how college and university mathematics departments might attract talented women to their programs. A greater understanding of how mathematically talented women make the decisions that they do requires a study of the women themselves rather than a comparison of them with men.

Methodology and Data Sources

The participants in this study were 12 mathematically talented college women from various colleges and universities throughout the United States. I gained access to the participants through the use of a network selection process called *snowball sampling* (Patton, 1990, p. 176). A small initial group of potential participants forwarded a proposal to other mathematically talented college women (at their own university and others) who they thought might be interested in participating. Each new potential participant also passed on the names of a few interested individuals, which resulted in a network of 24 potential participants. Using biographical questionnaires, I selected 12 women who as a group represented a wide range of colleges, ages, and majors and individually best met the essential criterion of exceptional mathematical experience, looking particularly for a balance between traditional measures of mathematical ability and interest in mathematical activities. All participants also met the criteria of having a declared major, although only 4 of the 12 were majoring in mathematics.

The participants engaged in a 12-week on-line bulletin board discussion group, moderated by the researcher, in which they discussed the variety of issues that had influenced their choice of whether or not to major in mathematics in college. In order to keep the group size and reading load for the participants manageable, I assigned the participants to two separate bulletin boards (6 members each) operating simultaneously. This decision also allowed for the possibility that topics and lines of conversation might arise in the two groups independently, adding strength to the finding that such an issue was essential and important to the participants' choices. Assignment to the two groups was not random but purposeful (Patton, 1990). I wanted to have two groups that were as diverse as possible in age, college affiliation, and major. Because of the nature of the network selection process, several participants knew each other well and might possibly have recognized their friends despite the use of pseudonyms. Therefore, some participants were placed in separate groups to protect their anonymity. Each participant was able to access only the discussion board for her assigned group. In addition to all electronically collected data from the two discussion groups, the women also participated in an individual interview, either in person or over the telephone, at the conclusion of the on-line portion of the study.

The process of data analysis in this study followed the traditional format applied in grounded theory research: constant comparative analysis (Corbin & Strauss, 1990; Creswell, 1998; Glaser, 1992; Glaser & Strauss, 1967). An essential component of the constant comparative analysis method is that data collection and data analysis are interrelated, often simultaneous, events. This method of data collection and data analysis allows for categories and related questions to emerge from the data, at which point these questions are taken back to the participants for further discussion. These categories and questions arise primarily during the initial open coding process. Categories are developed, discussed, and explored in an iterative fashion (gather data from participants, analyze the data, return to the participants for more data, etc.) until the categories are "saturated." One way in which this level of analysis was employed was through summaries posted to the bulletin board by the researcher. Participants could respond to and amend these summaries throughout the 12-week bulletin board discussion period. This method of ongoing responsive data analysis also helped guide the formation of questions for the closing interview.

According to the techniques of grounded theory, additional levels of coding and theorizing take place after the initial open coding process. I used axial coding to take the initial categories from the open coding phase and assemble them in a logical fashion. The purpose of this stage of analysis was to identify a central phenomenon and to explore causal conditions, strategies, and intervening conditions that affected it. I was initially influenced by previous models used to describe this phenomenon, such as Eccles (1985, 1989) model of choice, but I soon found that previous models were not adequately describing the conditions and factors that affected the participants' choices. In particular, these models did not account for the fact that while many factors affected the participants' choices, not all of these factors carried the same weight in the decision process. Simply put, some factors were more important than others. For example, the code *positive social impact in career* (which related to value) had a much stronger quantitative and qualitative grounding than the code *grades* (which related to expectancy), and the organization of the codes into categories needed to reflect this finding.

After axial coding, I used selective coding to integrate the categories that emerged from the axial coding into a "story line" that would outline a conditional proposition or theory. I assembled four core categories and their subcategories into a model that described the relative importance of a variety of factors that influenced the participants' choices about their majors. Some of the initial codes were combined and others abandoned, but this decision was made based on the need to include only the factors that the participants claimed influenced their choice of majors. The four central categories and their subcategories therefore represent the result of the collapsing, combining, and elimination of codes throughout the analysis process. Although the labeling of the central factors was my own (as a result of the analysis) and not a direct phrasing by the participants, the data supported the use of the terms.

Findings

On the first day of online data collection, I posted a question to both groups concerning the participants' beliefs about their mathematical talents. I had the idea that their beliefs about their mathematical talents would have had a large impact on whether they chose to major in mathematics or not major in mathematics. I was quite surprised when, on the same day, one of the participants, Elizabeth, posted the following entry:

> *I think in the overall scheme of things, talent is not the most important thing. Talent is secondary to passion.*

The other members of the discussion group strongly echoed and supported her sentiment. Surprisingly, a similar sentiment arose in the other discussion group, independent of Elizabeth's comment. It was almost as if the participants were saying to me, here's what's really important to us. They continued to explore this idea of passion throughout the entire 12-week online discussion, gradually elaborating on what they meant by *passion*. Passion was not simply enjoyment of a subject (although that was certainly a part of it), but it was more the sense that what they were doing had purpose and value. Any theoretical representation of the factors that influenced the participants' choices of majors would have to focus on this sense of purpose and value that was at the core of their decision processes.

It was soon obvious, however, that value alone was not sufficient to describe why these women had made their choices. Note that, in the above quote, the participant claimed that talent was secondary to passion and not that talent was irrelevant. Talent was necessary in order to be successful, but that success was considered less valid if it did not match a participant's sense of values. The online discussion was peppered throughout with other factors that influenced the participants' choices of majors, but each of these factors was discussed in a way that showed how they culminated in a need for value and worth. This sense of relative and culminating importance led to the development of the model of academic choice.

Regardless of major, the women in this study made choices about their major based on four domains: environment, behavior, talent, and value. These domains are listed in order of relative importance, from least important (environment) to most important (value). Although the relative importance of these domains does not necessarily reflect the order in which the participants addressed the issues present in the domains, the culmination of the four domains can be demonstrated by the following sequence of questions:

- How comfortable am I in the environment of my major department?
- What kind of behavior is expected in the department in order to be successful?
- Do I have the necessary talent to be successful in the department?
- Do I believe those talents and behaviors are valuable for me to cultivate?

This sequence of questions, and the relative importance of the domains that they represent, illustrates the model of academic choice arising from the data. This model does not assume that the student will necessarily reach a total agreement between the four domains, as new information and experiences will cause the student to constantly reevaluate and reexamine her choices. It is unlikely, therefore, that a student would ever reach a point in her decision process when she experiences no internal conflict whatsoever with the academic choices that she has made, and the data from the participants in this study supported the perspective of academic choice as a dynamic set of experiences rather than a single discrete event. The relative importance, however, of these domains does reflect the likelihood that a conflict would cause a participant to change her major. When a conflict was experienced at any level, the participant resolved this conflict by either reaching a compromise within the self about her current major or by changing her major outright. A conflict in the environment domain often resulted in a internal compromise which did not cause the participant to leave her department, whereas a conflict in the value domain often resulted in the more dramatic decision to change majors. It was therefore essential to the participants that they experience little if any conflict with the values of their major departments, as a conflict in this domain most likely resulted in their leaving the department.

The findings for the general model of choice are consistent across participants when applied to the particular decision of whether to major in mathematics or not. The participants cited few if any conflicts with the environment of the mathematics departments at their colleges and universities, and this domain had relatively low importance in their decision of whether or not to major in mathematics. On the other hand, many participants had conflicts with the perceived values of the mathematics departments at their colleges and universities, and those conflicts were often cited as the central reason (and occasionally the only reason) that a participant was not majoring in mathematics. Therefore, the domain of values held a relatively high level of importance in the participants' choices of whether or not to major in mathematics. Also, all participants who were majoring in mathematics claimed that they found their majors satisfying in the value domain, confirming the general finding that satisfaction in the value domain was essential to a participant's overall satisfaction with her choice of major.

Given that all of the women in this study had considered (albeit a very brief consideration for some) mathematics as an option for a major, they were particularly qualified to speak about how their experiences with mathematics and mathematics departments and classes affected their choice of major. It is particularly interesting that the large majority of the participants expressed little conflict with their mathematics departments in the first three domains of the model. In general, they found the environment to be hospitable, the activities and behaviors to be purposeful and enjoyable, and their talents to be more than sufficient to be successful. As noted above, the largest source of conflict for the participants lay in the domain of values, where they began to question if the environment, behaviors, and talent that they possessed fit in to their larger system of what they believe is important about their role in the world. These women expressed great concern that they use their talents not just for their own enjoyment, but also for the larger good of society as a whole, as the following quotations demonstrate:

Jane: *I would like for my job to involve cool people and be intellectually challenging and interesting, but ideally I'd like to think it was worthwhile too. . . . What I learn is never as important to me as personal relationships and helping people in some way. What I study is just the means for attaining that goal.*

Elizabeth: *I had to write an essay for college about why I wanted to study math. I didn't [want to study it]. It was a game...a game I was good at, but merely a game...for me at least. It was just puzzles, and oh, this is fun. But then to say that I'm going to make this my life? I don't believe that I want to spend the rest of my life doing something that's not going to help people.*

Esther: *I don't think I could feel good about succeeding in my field if I couldn't see it in some sort of wider context. I've said this a lot, but some concrete result, which has a tangible impact on the people around me, has become important. And hopefully I'll make a positive impact in my career. It's all about people, right?*

Hadley: *Whether or not people acknowledge it, I think everything and everyone has some sort of social impact, and so for me, it makes more sense to strive to have a positive one.*

To this end, it was important to the participants' sense of values that their majors, and subsequent career choices, encompass the goal of social responsibility.

This finding aligns with theories of talent as the application of knowledge and can be used to discuss how a person might make use of his or her mathematical talent. These characteristics are sometimes referred to as *evaluative skills* (Bloom, 1985), and they provide a link between the development of talent and the appropriate use of that talent. Sternberg (2000) refers to this perspective as *wisdom*, which he defines as "the application of tacit knowledge as mediated by values toward the goal of achieving a common good" (p. 253). This goal is achieved by balancing multiple interests (intrapersonal, interpersonal, and extrapersonal) among responses to environmental contexts. The perspective of talent as knowledge application does not eliminate the need to discuss the development of mathematical talent in terms of practical, analytical, and creative talent, but it places a goal of the common good and social responsibility as central to the individual's need and desire to develop his or her mathematical talent in these ways. In short, wisdom involves not only intelligence, talent and creativity, but also concern for others, judgement, intuition about right and truth, and the sense that knowledge is not value-free (Csikszentmihalyi & Rathunde, 1990; Sternberg, 1990, 2001). Unfortunately, the large majority of participants did not find the idea of a career in mathematics to satisfy their need for social responsibility, and therefore they chose other majors and careers in which they felt that they could better utilize these components of wisdom.

Conclusions

The results of this study draw attention to the importance that values have played in mathematically talented women's academic decisions. These findings demonstrate a level of concern deeper that the one addressed by the common practice of attempting to draw talented women into mathematics departments by making the environment more female-friendly (Hanson, 1996; Henrion, 1997). A congenial, supportive environment certainly played a role in these women's decisions. However, the findings indicate that research on women in mathematics that focuses almost exclusively on the environment may be misplacing it's efforts. Rather, this research points toward a practice of drawing women to college and university mathematics departments by addressing all domains of the choice model, with particular emphasis on whether the program addresses the individual's sense of values.

How might undergraduate mathematics departments use the findings of this study indicating that values are the most important factor in talented women's decision whether or not to major in mathematics? Much of the decision rests with the impressions that students form of the values of a mathematics department in the first years of undergraduate study. If a student does not have the impression within the first two years of college that a major will be satisfactory in the value domain, then she is unlikely to make that major her final choice. Considering that many of the participants viewed the first year of college mathematics as consisting of "weed-out" survey courses, it is not surprising that very few saw an undergraduate major in mathematics as an attractive option. They were already concerned about the relevance and social implications of their academic work, and their early mathematics courses provided few indications that mathematics met these concerns. Instead, these courses, as well as many subsequent courses, focused almost entirely on proofs rather than relevant applications.

These findings are similar to those of a recent study of female undergraduate computer science majors at Carnegie Mellon University (Margolis & Fisher, 2002). The researchers found that female students were drawn to computer science because of what they perceived as the positive social implications of computing innovations, yet their experiences in their early computer science courses focused almost exclusively on programming. As a result of these findings, Carnegie Mellon made serious changes in the structure and curriculum of their undergraduate program in computer science. One of the changes put in place was the institution of an "immigration course" (IC) for students entering the undergraduate program. Previously offered only to incoming doctoral students, this course was designed to provide students with a broader, contextualized view of the field than they tended to receive in their early programming-oriented courses. As a result of the new course, freshman and sophomore computer science majors who had participated in IC had a much clearer view of the field and its applications than students who had not had the opportunity to participate in IC. As a result of this change as well as many others (including student recruitment efforts and increased focus on quality teaching), the proportion of women entering Carnegie Mellon's undergraduate computer science program rose from 7 percent in 1995 to 42 percent in 2000. This dramatic change in the proportion of women in a department speaks to the power of early efforts to place undergraduate coursework within a broader context.

The findings of this present study support the idea that undergraduate mathematics departments could also benefit from instituting such an "immigration" opportunity for their students that focuses on the social implication and value of mathematical work. Although it is true that undergraduate mathematics departments have to some degree always addressed the issue of the value of mathematical study, much of their emphasis is on the aesthetic value of the subject—the inherent beauty of mathematics and the pursuit of knowledge for knowledge's sake. This aesthetic value was certainly important to the women in this study, as they were unlikely to pursue a field that they did not find enjoyable on a personal, intellectual level. However, equally if not more important to these women was the value of their work in context. How was their work going to affect the work of others? If the culture in their major departments was not able to provide an answer to this question, then they were less likely to remain in that department. This view challenges undergraduate mathematics departments to place early emphasis on not only the intellectual value of mathematical study but its social value as well.

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DEVELOPING MATHEMATICAL IDENTITY: A LOOK AT A TWELFTH GRADE MATHEMATICS CLASSROOM AS A COMMUNITY OF PRACTICE

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Using a socio-cultural lens, this paper employs a community of practice framework (Lave & Wenger, 1991) to examine the mathematical identity formation of a group of twelfth Grade 12 in an urban Discrete Mathematics classroom. Within this community, the individuals re-negotiated their mathematical identities as they participated in the practices of the classroom. In this classroom, it appears that (1) the majority of the students developed their own mathematical identities in relation to the valorized practices (de Abreu, 2002) of this particular classroom, and thus articulated such positive development when they engaged in such practices, (2) the practices of this classroom did not occur in isolation from other social and institutional practices, and (3) students positioned themselves and others with respect to valorized and sometimes contradictory practices.

While being in this urban Discrete Mathematics classroom, I became aware that this classroom was one where the teacher and regularly attending students co-created a community in which three interrelated practices were valorized: working collaboratively in groups, students' and teacher's sharing of knowledge, and clear explanations of mathematical processes. The majority of students engaged fully in these practices, and as a result, articulated changes in what I call their mathematical identities. Although there were certainly limitations to what the teacher and students were able to accomplish, in this paper I seek to understand the processes that contributed to these students' positive development of mathematical selves. However, there were also contradictory practices within this classroom that sometimes made negotiation of one's participation difficult. I also seek to understand the choices students made in their participation, and how these choices affected their positioning in the classroom.

Methods and Data Sources

This paper presents an ethnographic study of a semester long Discrete Mathematics classroom in an urban high school. Within this high school of 1800 students, almost 99% were African American and 86% were from low-income families. This 12th grade Discrete Mathematics class was in the Business Small Learning Community (SLC)[1]of the high school.

This class was taught by Ms. Lorraine Smith[2], an African American veteran math teacher of 27 years. She had taught in this particular school for 12 years, and was responsible for much of the secondary mathematics curricular reforms in this urban district. In this class, there were 40 registered students, but on the days I observed, an average of 25 students were present. Of these 25 students, there were on average 15 females and 10 males. All students identified themselves as African American, with the exception of two females who came to the United States in the fall of 2001 from Sierra Leone. The desks in the classroom were arranged such that four desks were pushed together to make a table. Students chose their own seats around a table with three other students. The majority of the students grouped themselves according to sex.

At the time of this study, there was no district-wide curriculum for "Discrete Mathematics." Thus, Ms. Smith composed the curriculum on a daily basis. She defined Discrete Mathematics in this context as "problems that take days to solve" and found problems from the World Wide Web and other math books. Typically, the mathematical projects took three or four days to complete. For example, the class began the year by examining patterns that could be solved with combinatorics; students were urged to solve these problems without using formulae, but instead by conceptually mapping the problems, finding rules, and using these rules to generalize to solve subsequent problems.

My data included detailed field notes from over 50 hours of classroom observation, informal conversations with the teacher and students, 30 student surveys, a 50-minute teacher interview, and 21 student interviews, each averaging 8 minutes. Analyses for this paper focus primarily on classroom observations along with information gathered from the written student surveys and follow-up interviews.

In analyzing my classroom observations, written surveys, and interview data, I concentrated on both what appeared as consistencies and inconsistencies among students' articulations of selves and classroom practices, the teacher's verbal accounts, and my observations of practices within the classroom. I also examined the role of assessment tools of the classroom in relation to the students', the teacher's, and my perceptions of their emerging mathematical identities.

Theoretical Framework

My analyses are guided by sociocultural perspectives on learning, practice, and identity. I begin with Lave and Wenger's (1991) notion of a community of practice. Communities of practice are defined as:

> a set of relations among persons, activity, and world, over time and in relation with other tangential and overlapping communities of practice. A community of practice is an intrinsic condition for the existence of knowledge, not least because it provides the interpretive support necessary for making sense of its heritage. (Lave & Wenger, 1991, p. 98)

If we consider the mathematics classroom as a community of practice, and specifically this discrete mathematics class, then the students and teacher are constantly negotiating and mediating the learning that takes place through their interactions. "Students do not just learn methods and processes in mathematics classrooms, they learn to *be* mathematics learners and their learning of content knowledge cannot be separated from their interactional engagement in the classroom, as the two mutually constitute one another at the time of learning" (Boaler, 2000, p. 380, italics in original).

Lave and Wenger (1991) define learning as an aspect of changing participation in evolving communities of practice. As Boaler and Greeno (2000) note, "according to this view, *participation in social practices* [italics added] is what learning mathematics is" (p. 172). Wenger (1998), building on Lave and Wenger's (1991) theory of communities of practice, writes of identities as developing in *social* practice. "An identity … is a layering of events of participation and reification by which our experience and its social interpretation inform each other" (p. 151). He uses the concept of trajectories of participation to explain his conception of identity. "As we go through a succession of forms of participation, our identities form trajectories, both within and across communities of practice" (p. 154). Wenger contends that identity is a relational process that is both individual and collective and incorporates the histories and practices of both the self and others, and that it is temporal.

It follows that students are indeed developing identities in the social practices of their mathematics classrooms, and in conjunction with Lave (1996) and Wenger (1998), we need to examine this identity development in order to examine the learning that is taking place in mathematics classrooms. I am interested in the mathematical identities that are constructed and re-constructed in relation to the valorized mathematical practices of the classroom (de Abreu, 2002). I say this keeping in mind that the mathematical content knowledge we take with us is an integral part of our "mathematical identity."

I make the distinction between mathematical identity and identity as I only observed these students' formation and development of their identities in the figured world (Boaler & Greeno, 2000; Holland, Lachiotte, Skinner & Cain, 1998) of this specific mathematics classroom. Holland et al. (1998) define a figured world as "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (p. 52). One's identity in the figured world of classroom mathematics depends on past participation (history) in mathematics classrooms, or communities of practice. It also depends on a number of other "identities": larger structural identities such as "gender, race, ethnicity, or any other durable structural feature of the society" (Holland et al., 1998, p. 7) or identities based on other less structural figured worlds. In similar communities of practice, such as mathematics classrooms that value similar practices, students may construct more durable mathematical identities. But, depending on past participation or participation in other communities, this may not prove viable for all students.

Research Findings

I present the results of this research in four sections. First, I examine the valorized practices of this particular community of practice. Second, I show the relationship between students' perceptions of what constitutes a "good" mathematics student, their definitions of selves as learners of mathematics, and the valorized practices of the classroom. Third, I examine the contradictions inherent in some of the valorized practices, and fourth, I document how some students negotiated their mathematical identities with respect to the valorized practices of the classroom. Throughout these findings, I present the students' emerging mathematical identities as a relation between how they positioned themselves, how they were positioned by others, and their participation in the valorized practices in this community.

Valorized Practices

From my analyses, I concluded that, within this classroom, three classroom practices were valorized; that is, the teacher and the students engaged in these practices successfully and articulated this engagement well. The three interrelated practices were: (1) working collaboratively in groups, (2) students' and teacher's sharing of knowledge, and (3) clear explanations of mathematical processes. Ms. Smith explicitly endorsed the practice of collaborative learning in mathematics in both an interview and while teaching. As noted in the methods section, Ms. Smith chose to group her students into tables of four. Students were expected to work on all assignments in groups, including formal assessments. Her belief in collaborative learning was closely tied with her belief in inquiry. As a teacher, Ms. Smith articulated her job as to help the students construct their own mathematical understandings through doing. It was rare for Ms. Smith to conduct class didactically; and when she did, I never saw her do so for more than fifteen minutes. She encouraged questions from students, but more often than not, would either re-direct such questions back to the collective, or would suggest that a student ask another student his/her particular question. Ms. Smith's consistent inquiry endorsed the second valorized practice of students and teacher sharing their knowledge with others. Students discussed their assignments both in and across groups. Certain students, as will be addressed in subsequent sections of this paper, moved from group to group either as "information gatherers" or as resources for the other groups. The third valorized practice, that of providing clear explanations, was an effect of both the practice of collaborative learning and the teacher's ability to provide cogent explanations of mathematical processes. Students questioned one another further if they did not find another student's explanation valid or clear. Certain students were praised for their exceptional explanations by their peers and by the teacher.

Important to note, though, is that whereas the practices identified above were both agreed upon by the teacher and the students, there were spaces within the classroom that did not fully support such practices. Although students were expected to complete all assignments, including formal assessments, within their groups, Ms. Smith consistently graded the students on the basis of individual work. At the end of each period, students individually handed in their work to Ms. Smith. She then awarded "power stamps" to each person, depending on the amount and quality of the individually written work. At the end of the week, students turned in their stamps from the week and received a grade for their class work. Thus, stamps, on the basis of individual written work, were the valued currency of this community. In addition, Ms. Smith posted an updated ranking of the students each week along with all of their grades. Students regularly consulted the rankings to check their grades and to find out what assignments they were missing.

The Interrelation of the Students' Perceived Ideal Learner of Mathematics and Perception of Self

Students appeared to develop a situated perspective of what an ideal learner of mathematics was with respect to the valorized practices of this classroom. Further, regularly attending students re-constructed their own mathematical identities with respect to these notions of the practices of a "good" math student. In this section, I document the interconnection between the perceived ideal learner of mathematics in this classroom and students' perceptions of their mathematical selves.

On a written survey, I asked students to give characteristics of a "good" math student. Students tended to agree that successful learners of mathematics were those students who paid attention in class, who helped others or asked for help when necessary, and were who willing to work at building mathematical understanding. There was relatively little in these responses that conjured up typical images of successful students of mathematics (numbers, counting, speed, etc.). In fact, only two students made any such reference in their definition of a "good" math student. Students, for the most part, defined a "good" math student on the basis of what was rewarded in this particular community of practice. These characteristics clearly reflected the valorized practices earlier discussed, and demonstrated the students' valorization of these practices.

Another frequently cited characteristic of a successful student of mathematics was "coming on time." While in some schools, arriving to class before class begins may be a norm, in this school it was not. First period began at 7:45 am. Yet, there were usually 7 or 8 students present at the official beginning of class. About 10 more arrived by 8:00. After 8:00, both Ms. Smith and the school[3] as a whole began marking students as officially late. Students were given one power stamp if they arrived on time and began working on their "pre-class" problems. Coming "on-time" to class was a desired practice, one which included a valued external reward (one power stamp) and was reinforced at both the classroom and institutional levels. With this information, it makes sense that students perceived a "good" math student in this particular community as a student who arrived on-time for class.

The absence of "typical" mathematical discourse in these definitions of "good" math students is significant in understanding this community. As I mentioned before, only two students made direct reference to typical mathematical descriptors. Both were students who did not attend class regularly[4] and were failing the class. One wrote, "[A good math student is] a number cruncher or someone who has no problem being able to break out the ruler or the tape measure and go to work" (Mark, Survey, 12/12/01). Important to note is that Mark transferred to this school in November 2001. The second student wrote, "[A good math student is a] fast thinker, good with numbers, me" (Tommy, Survey 12/12/01).

Their perceptions of "good" math students are vastly different from the majority of the class. In the case of both of them, it could be argued that because neither one regularly attended school, they were "less-engaged peripheral participants" (Lave & Wenger, 1991), and due to their limited engagement, did not value the same practices as the other students. Lave and Wenger (1991) write,

> Peripherality suggests that there are multiple, varied, more- or less-engaged and -inclusive ways of being located in the fields of participation defined by a community. Peripheral participation is about being located in the social world. *Changing* locations and perspectives are part of actors' learning trajectories, developing identities, and forms of membership. (pp. 35-36, italics in original)

I argue that Mark's and Tommy's location in the social world of this particular community, as members who are not regularly present, contribute to their different perceptions of "good" math students.

I now share students' perceptions of their mathematical selves. I argue that one's developing mathematical identity is in part relative to what is taken to be the valued perception of a "good" or "successful" math student in this community. In an attempt to have students articulate how they perceive themselves as learners of mathematics, I gave them a written survey about their practices in this particular community. After receiving written responses from 30 students, I was able to interview 21. At the time of these interviews, I had visited the class 20 times and was able to ask them how their classroom practices related to their perceptions of self.

An overwhelming majority of students claimed that they were "good" in mathematics. Only 2 of 30 students said that they were not good in math. Of the 21 students I interviewed, 1 said she was not good at math and was not improving, 18 said that they felt they were currently becoming better at mathematics, and 2 claimed that they have always felt "good" at mathematics, and felt the same in this particular classroom. Because student surveys and interviews were both measures of students' perceptions of their participation within this particular community of practice, I recognized the necessity to triangulate this data with my own classroom observations, the teacher's observations, and forms of assessment. Thus, I offer multiple perspectives in the following discussion on students' perceptions of their mathematical selves.

The fact that the majority of students identified themselves as becoming better at mathematics was very encouraging. My own observations agreed; I noted throughout my formal observations that most students appeared to be gaining proficiency in not only using their mathematical content knowledge, but also in their ability to work effectively with others. I was curious to find out why it was that students in this class consistently articulated a positive movement in their trajectories of participation in this particular community.

Elijah consistently earned a B in this math class. He completed all his assignments, was on time for class everyday, and regularly participated in whole class discussions. On his written survey, he claimed he generally did not think he was good in mathematics, but he did feel he was successful in this particular mathematics community. I interviewed Elijah the following day. I was curious about his response in relation to his relative success. I asked him if he had always felt unsuccessful in mathematics.

Elijah: Not always. It's just like, when I started, probably eighth grade on up I stopped doing math well and I stopped going to math classes. By the time when I come back in, I don't know what's going on. I'm lost, and everybody ahead of me and stuff. So I felt like I wasn't that good in math. I can do this stuff in here. Because the way Ms. Smith teaches.... Cause it's like, we in a group, it's four of us. There's always somebody in the group that knows it, like say there's something on the board, and like three people don't know it, somebody in the group gonna know it, and he's gonna help everybody in the group. That's how it is. ... It seems like nobody in the group's smarter than anybody else. Everybody has trouble with something, and everybody just helps each other when you're doing this stuff.

K: What specifically about Ms. Smith's teaching has made you feel this way?

E: (3.0)[5] She just, (2.0), the way she just teach. She's like, you could have, you could have, I mean she can crack a joke while we're doing our work, it's like fun in there, and when we having fun, you just do your work. It's not like she's talking you to death. She, she like energetic, and we can do our work in there and stuff, and we having fun. We're doing it. ... It don't even feel like work. Don't even feel like work. (Interview, 12/14/01)

Throughout this passage, Elijah attributed his evolving mathematical identity to practices within the community, and his relations with both his group members and Ms. Smith. It would be important to find out why Elijah stopped doing well in math after eighth grade in a subsequent interview, but for now, it seems crucial to note that first, his self-perception changed within this particular community, and second, that he ascribed this change to relationships in the collective. He noted Ms. Smith's ability to "crack a joke" and her energy; he also noted that within his group, there was a sense of shared power and ability. Each person was valued within the group as an equal, and each person was willing to help the other. My observations concur with those of

Elijah. I frequently noted in my field notes the successful practices of Elijah's group with respect to sharing equally the responsibility to complete assigned projects, and their proficiency at sharing their knowledge with one another. With these interactions acting as an integral part of the learning of mathematics in this community, Elijah was able to alter his perception of self, and in particular, his mathematical identity.

Eboni was consistently either the Number 1 or Number 2 student in the class. Ms. Smith said the following about Eboni: "Eboni is at the top of her class. She pretty much understands what's going on. You know she's like Nicole [another top student in the class]. If she don't understand it they're not going to rest themselves. ... She likes being at the top" (Interview, 10/25/01). In a written survey, Eboni wrote, "I like math a lot. [It] happens to be one of my favorite subjects. On the weekly chart this week I am number one and last week I was two. I think of myself as a good math student." In an interview, I asked Eboni if she had always thought of herself as a "good" math student and she answered no. She appeared to equate success in mathematics with grades, and said that in the past, she had received B's and C's, but now was earning A's.

K: Okay. So since you've been in this class, you've been getting an A the whole time. Why do you think you're doing so well?

E: Because, because she [Ms. Smith] is so good. When she give you problems, she don't make you make it through the problems by yourself. She'll help you, and she'll explain it on the board. She do a good job of explaining stuff. I like how she breaks stuff down. (Interview, 12/13/01)

Just like Elijah, Eboni credited her teacher for at least some of her improvement in math. She characterized Ms. Smith as helping students, explaining mathematics well and "breaking concepts down." Many other students cited these same characteristics as part of the reason why they were improving in mathematics. My own observations indicated that Eboni regularly participated in whole group discussions and often helped other students when they were having problems. Several students cited Eboni as being a top mathematics student. When I asked them why, they said that Eboni understood mathematics, but more importantly, was willing to share her knowledge with others, and could explain clearly. Eboni valued collaborative practices, and because she successfully enacted such practices, perceived of herself as a successful and an improving student of mathematics. In addition, Ms. Smith and the other students positioned Eboni as a successful student for those same reasons.

In fact, 20 of the interviewed students cited Ms. Smith's teaching style (quality of explanations, willingness to help) and working in groups as reasons for either their enjoyment of mathematics or their changing perception of their mathematical ability. Students recognized that they were changing as learners of mathematics and tended to cite more socially constructed practices as promoting such change. What these students articulated was that one's perceived identity in mathematics was not fixed. These students were in their senior year of high school, and almost all expressed changes in their mathematical identity. They were articulating and naming changes in their trajectories of participation (Wenger, 1998). In addition, the articulated changes were made in respect to the valorized practices of collaborative learning, which included the sharing of knowledge and the production of clear explanations. Because the students were able to envision their own participation with respect to some of the valorized practices of the classroom, I argue that these practices were valorized by both teacher and students. Within this particular figured world, students positioned themselves and sometimes others with respect to such practices. However, as noted before, not all the practices of this classroom were coherent, and consequently, students' trajectories of participation were more complex than perhaps shown thus far.

Contradictory Practices

As noted before, not all practices in this classroom supported the practice of collaborative learning. Students were assessed and ranked individually, and thus although students worked collaboratively, their collaborative effort was not necessarily rewarded in regards to grades. This disconnect between the collaborative learning practices and the assessment practices appeared to cause some tension within the classroom. This tension was a result of not only the teacher's assessment practices, but of larger institutional practices. Ultimately, Ms. Smith had to award individual grades to students; she chose to do so on a daily basis. Thus, students had to negotiate this tension and depending on their positioning within this classroom, took different participatory paths. Hence, different mathematical identities emerged as students participated differentially.

First, I offer the case of Syltee, one of the students who transferred to the school from Sierra Leone in September of 2001. In a written survey, Syltee offered that a good math student was someone who shared her knowledge with others, gave clear explanations willingly, and was "not selfish." Syltee expressed to me privately that some students in this class, especially in her group, did not want to share their knowledge with her.

> I used to be good [in mathematics], but when I came here, you know, everybody, they don't share their knowledge, so I just stick on what the teacher told me and when I get home I practice. I think the way I could learn more is the when I mix with my fellow classmates. And when they give me their own knowledge, share the knowledge that they have. When I came here, you know I'm new, when I came in especially this school, most of the students in this math class never share their ideas. They are just selfish. (Syltee, Interview, 12/13/01)

Syltee's perception of a good math student was in part determined by what was valorized in this classroom, and presumably other practices in her life history. Because

Ms. Smith emphasized the value of collaborative work in this classroom, most students expected that their peers would share their knowledge. Syltee's dissatisfaction with some of her peers was relative to the valorization of this particular practice in this classroom. Because she found she could not rely on others to work well with her, she often chose to work alone, and in completing assignments, to use only what she was offered by Ms. Smith. This caused difficulties for Syltee. As mentioned before, Ms. Smith rarely gave explicit directions on how to solve a problem. Rather, she expected students to work out solutions to problems with one another. The case of Syltee helps to identify that not all students fully engaged in collaborative practices. Without peer support, Syltee adopted more individualized practices within the classroom. She experienced a contradiction between what she perceived as valued practices of the community and the enacted practices, and was not satisfied with this inconsistency.

Second, I recall the case of Eboni. While she valued Ms. Smith's clear explanations, and was noted for sharing her knowledge willingly, she identified her self as an improved and successful learner of mathematics in part because she was regularly located as Number 1 or 2 on the wall chart. Thus, it appeared that practices of collaborative learning coupled with individual assessment both contributed to her positive identification of mathematical self.

Negotiation of the Valorized Practices

Students negotiated the valorized practices of collaborative learning and individual assessment, as well as their participation in past, mathematical and otherwise, communities. While the majority of students accepted the collaborative practices of this classroom, and adopted their own participation to meet that of the perceived ideal readily, others expressed more difficulty in negotiating their participation. I turn to the case of Nicole to examine how one student negotiated her position in this classroom. Nicole, mentioned briefly in a previous section, was a top student. She felt that she had always done well in mathematics, but that this year was better for her because she felt she understood more of what she was doing. However, Nicole complained that her group-mates did not always contribute as much as she did. She expressed discomfort with her positioning as a top student in the classroom, and more specifically within her group.

> When I work in groups, like most of the time, I'm doing all the work and everybody else copies. Cause they don't like understand it themselves. Instead of trying to take the time to understand it themselves, they just copy my answers. So sometimes I'd rather work by myself. (Interview, 12/13/01)

Nicole was recognized by the members in her group as a quick math student. In class, I observed numerous times that students from her group and other groups asked Nicole for help. She usually agreed to help, but once she began to offer help, was sometimes perceived as not wanting to share her knowledge. It appeared that Nicole only offered clear and slow explanations when she perceived that the student in need of help was also identified as one of the more successful students in the classroom, such as Eboni. With other less successful students, she spoke quickly and sometimes would take their paper and write the answer with little explanation. Syltee, mentioned earlier, complained that some members of her group did not want to share knowledge; Nicole was one of the members of her group. In part, I attribute Nicole's selectivity in sharing her knowledge to the contradictory practices of the classroom. Nicole's position as top student in the class was reinforced weekly as she consistently was ranked as either Number 1 or number 2 (just as with Eboni). In order to maintain such a position, Nicole chose to share her knowledge with other students who potentially were also beneficial to her, and whose quickness in understanding would not detract from Nicole's own work. Thus, as she participated in exchanges with others, she solidified for some her knowledge of mathematics whereas, for others, was perceived as not wanting to share her knowledge. Nicole's negotiation of this contradiction was tricky; she managed to maintain her position as top student but was not identified by others as successful in enacting the practice of collaborative work.

Discussion and Conclusion

It appears that a majority of the students in this class constructed an "ideal" math student in relation to the valorized practices of this particular community of practice. This implies that our perceptions of ideals change as we change location within our figured worlds. What may have been perceived as traits of a "good" math student in one classroom may not hold true for another classroom. This also suggests that one's (mathematical) identity is temporal (Wenger, 1998). In addition, the experience of the majority of these students suggests that when students and teachers respect and engage in the valorized practices of the classroom, they experience positive movement in their trajectories of participation. For high school seniors who may have completed their formal mathematical training, it is possible that their positive experiences in this classroom will benefit them outside the classroom as well. It would indeed be informative to follow these same students into the next stages of their lives, tracking their mathematical participation in other communities. This study has shown that individuals develop mathematical identities with relation to their participation in valorized practices of the communities in which they reside.

However, this study has also shown that individuals' negotiation of classroom practices is not a simple process. As with the case of Nicole, it was difficult for her to negotiate contradictory practices successfully. Eboni was more successful at this negotiation in that others noted that she exemplified mastery of the collaborative practices of the classroom, and was a top student. Meanwhile, Nicole sacrificed the way she was identified by some students in order to remain individually a top student.

Wenger (1998) writes, "[F]ocusing on identity brings to the fore the issues of non-participation as well as participation, and of exclusion as well as inclusion. Our identity involves our ability and our inability to shape the meanings that define our communities and our forms of belonging" (p. 145). This paper argues that as students come to share meanings of what are valorized practices in the classroom, they shape their practices in relation to this valorized ideal. As noted in an earlier section, only 25 of 40 students attended this class on a given day. The students who were more peripheral participants did not experience the same development in their mathematical identities, although they certainly did develop. Non-participation, as Wenger notes, is just as important as participation is in temporal identities. It is indeed difficult and challenging to document the trajectories of students who do not regularly attend class, yet their non-participation is also critical to their own mathematical identities as well as the mathematical identities of the more central participants.

Analyzing students' developing mathematical identities can be quite informative, as the process of identity is a relational one, and requires researchers to document students' trajectories of participation. As Lave and Wenger (1991) note, changes in one's participation indicate that one has learned. As a student's participation changes, her identity is also developing. As Wenger (1998) argues, learning and identity are interrelated. When we take identity to be a unit of analysis, we are able to make the social practices in which one engages the focal point of our studies. Studying social practices in mathematical communities lends insight into the relationship between valorized practices and success as learners of mathematics. Such analysis also allows us to examine closely the relationships actors develop with one another, and how students and teachers position each other and their selves with respect to the practices of the classroom.

Notes

[1]A small learning community (SLC) is a school within a school. They are implemented in this district to promote more intimate settings in large high schools.

[2]All names used in this paper are pseudonyms. All participants in this study have chosen their own pseudonyms.

[3]The school has an explicit lateness policy for first period. If a student arrives after 8:00 am, s/he is sent to a "Late Room" and is released to go to class at 8:30 am.

[4]For the purpose of this paper, I define "regularly" as a student who is present 2/3 of the time.

[5](3.0) represents that it took three seconds for Elijah to respond.

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SOCIAL CONSTRUCTIVISM IN PRACTICE: CASE STUDY OF AN ELEMENTARY SCHOOL'S MATHEMATIC PROGRAM

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This research report investigates the following questions: 1) Can social constructivist theory be implemented to raise achievement levels of African American students? 2) What impact does social constructivist theory have on the structure and culture of school? This research report tells the story of a K-4 elementary school that struggled to reform mathematics education (1990-2001) by implementing social constructivist theory in classrooms. After eight years (1990-1998) of focusing on restructuring and reculturing mathematics classrooms, fourth grade students' scores on state mathematics tests dramatically improved for a three year period: 1999-2001. The test results attracted local and statewide attention because African American and white students achieved at about the same high level. The school enrolls 525 students: 60% African American, 33% white and 7% other racial/ethnic groups.

A Bit of History

In 1990, the school principals and teachers implemented social constructivist theory in most classrooms. State mandated testing began in 1995. Educators at this school then faced the political pressures of state tests. The publication of the test results disturbed the climate of this learning community. In 1998 only 69% of fourth grade students passed the mathematics test, which was below the average score of other elementary schools in the district. Because of the test results, some parents, teachers, school board members and central office administrators questioned the credibility of the reform. School principals and teachers searched for ways to blend social constructivist practices with preparation for the state mathematics test. In addition to closing the achievement gap between white and African American students on the state test, the reform confronted the challenge of teaching high-level mathematics to all students so that students and educators could meet the goals of a postmodern society. The potential of social constructivist learning theory became evident when parents actively supported this framework for mathematics learning even when test scores were low. Parent support, confidence and enthusiasm increased when the 1999 mathematics scores showed a dramatic improvement (69% to 90%). Mathematics scores continued to improve over the next two years and the achievement gap continued to narrow.

Theoretical and Philosophical Assumptions

The theory and philosophy of this study were influenced by social constructivist epistemology (Cobb & Yackel, 1996). Social constructivist theory posits that learning and knowing is built via active and interactive activities in a classroom. This theory values time for discourse among members of the learning community and time for building or drawing models of mathematical situations. In addition, the theory recognizes prior and present experiences, relevancy of context, and the value of multiple perspectives. Social constructivist theory assumes a teaching-learning environment beyond the rote and routine learning of basic mathematics skills, not in place of learning basic skills. Social constructivist teaching practices emerged as an important feature for students' understanding of mathematics beyond the limited memorization of basic facts and mechanical procedures. In this sense, classroom practices focus on: dialogue, prior knowledge, mathematical modeling, multiple solutions, students' preconceptions, problem-solving/writing and the importance of context for building understanding.

Methods and Study Design

The study was grounded in the constructivist inquiry of Lincoln and Guba (1985, 1994). Participant-observation, long interviews and review of public documents were used to collect information. Interview data included the transcription of audiotapes from interviews with teachers, principals, assistant superintendents, community officials, parents, and students. Field notes were used when the primary researcher observed and participated in the teaching-learning process. Most data were collected over a three-year period (1999-2001). Additional data sources were triangulated and negotiated among the researchers for trustworthiness of data analysis. Based on multiple data sources, several themes emerged and categories were developed.

Lessons Learned

> Seeing how you ask probing questions, and how you get students to talk about what they are seeing. That's probably my favorite part. It's really not only teaching me about math, but also teaching me how I can be a better teacher. (First year teacher, involved with extended time programs)

Major mathematics reform at the elementary level probably requires structural and cultural changes. Implementing social constructivist practices in elementary mathematics classrooms may call for changing conventional teaching-learning-assessment practices.

> I taught, years ago, in a very traditional classroom. And, I taught exactly like my advising teacher taught. She introduced me to a college professor who was doing math differently. I had been a very apprehensive math student but she was teaching this [math] and showing it through games, manipulatives

> and stuff. It was fascinating to me. I lapped up everything I could. I'd tell the kids that I didn't understand this [mathematics] when I was your age. I didn't have a clue what mathematics was. It's kind of neat, look! (Fourth grade teacher at the K-4 school)

Mathematical illiteracy in America is documented. Most parents and teachers did not learn mathematics in a social constructivist classroom where instruction focuses on the development of mathematical concepts. Many teachers and principals may need to relearn mathematics. Some parents and concerned citizens may also need to view mathematics education differently. Most school schedules (day and year) do not provide enough time for the effective implementation of social constructivist practices. Classrooms are often not equipped with resource materials that support active, interactive instruction. Some researchers believe that computers and calculators are also necessary in a social constructivist classroom. Therefore, schools may need more technology to support changes in mathematics education.

Structural Changes

> I think it's a shift in the way that you think about yourself as a teacher. I think the biggest change is getting people to want to devote more time and more energy to improving mathematics education. (Principal, K-4 school)

> It's a very focused curriculum. Focused on big ideas and on the skills necessary to take those big ideas and apply them to real world situations. The problems that we give the kids are highly contextual. So if you were asking me what makes this different from what you might see in other classrooms, I would think that in other classrooms mathematics is driven by textbooks. There are no books here. (Assistant Principal, 1995-2001)

Teaching all students to understand and apply key mathematical ideas within a social constructivist framework necessitates extending instructional time. Some students need extended time to learn and understand mathematics. Mathematics classroom instruction at this K-4 elementary school was extended from 45 minutes to 90 minutes for all students.

> If you are going to apply it [social constructivism], if you're going to build and draw models, role play it, and have time to correct and discuss misunderstandings, change your mind and rebuild things, and come back and revisit things, and try to put it all together, that takes time. (Principal)

Students who were still underachieving at the third and fourth grade level were invited to participate in an extended school day, week, and year. Morning and afternoon tutoring was provided (75 min. before school for 4 days each week, 45 min. after school for 3 days each week). Tuition-free summer school was offered for 6 weeks, 3.5

hours per day. Saturday School for fourth graders was held from early September until mid-March for 3.5 hours each Saturday. Students who participated in extended time initiatives had about 250-300 more hours of instruction prior to taking the state mathematics test. About 90% of participating students in extended time programs were African American whose parents drove them to school early each Saturday morning and early to school 4 days a week.

Most instruction during these extended sessions was conducted within a social constructivist framework. Computers and mathematics software programs were used extensively to support classroom instruction. Social constructivism in practice demanded more instructional time because it went beyond basic calculation skills toward an emphasis on understanding how calculation skills are used in real life contexts. The Kumon Mathematics Program which is more representative of behavioristic instructional practices, was used in after school tutoring sessions and offered free of charge to students. The researchers were uncertain that all students could effectively learn basic facts within a social constructivist framework. Also, the researchers were aware that there was little time to teach both basic skills and conceptual understanding in order to improve student achievement on state tests. The Kumon program was done after school and away from regular classroom time so that teachers could focus on learning mathematics within a problem-solving and problem-finding environment.

Cultural Changes

> It's the leadership of the building, number one. There is a real commitment to the fact that all students can learn. There is not a student at this building that can't learn. I see an emphasis on the extra time that the students need to learn as a big priority at this school. They are doing whatever it takes here to get it done. A lot of parents almost come to expect it. I also see a focus on curriculum a lot more intense here than at some of the other buildings [district schools] in terms of the most up-to-date practices and what the most up-to-date research says about how we're teaching. Some of those practices are put into play here at this building. (Assistant principal, 2001-2002)

> Mathematics is hard and sometimes it can be fun but it is hard. I want to learn mathematics but also I want to have fun while I do mathematics. I have to think a lot while I am problem solving at [name of the school]. We have to explain our reasoning. We have to draw pictures of money when we are solving problems. We do a lot of discussions while we are solving fraction problems. I like fractions and I like graphing. It is fun to put stuff on the chart and color it in to measure. There is not one way to answer a question, there is always another way you can do it. In problem solving you have to use your mind, you have to go way back when you explain, you have to use all your knowledge (Fourth Grade Student).

Using a learning theory that moved away and beyond behaviorism created disequilibrium for mathematics education. Mathematical processes and mathematical ideas started to be viewed as less definite and more probabilistic than most parents and teachers had learned when they were young. Most teachers learned more about mathematics and the application of mathematical skills and concepts. Some teachers began to view mathematics as a creative process. They taught students to create different patterns, to understand probability, and to design different strategies for solving mathematical problems. Many parents learned to appreciate, enjoy, and view mathematics as a creative art. Many parents also respected their child's use of discourse, illustration, and concrete models to build understanding about important mathematical concepts.

> Perhaps the most interesting lesson learned from this reform process was the importance of "student voice" in mathematical discourse. Discourse and "student voice" provided opportunities for student self-reflection and opportunities for teachers to understand what students did not know and what to do next. This recursive relationship between "student voice" and classroom discourse was pivotal for reforming instruction and curriculum. Guided by "student voice" and classroom dialogue, teachers realized that textbook instruction and standardized courses of study might be contextually limited. It seemed that the value placed on "student voice" and student discourse provided a more open, respectful and democratic environment for the mathematics classroom. "You don't know what they're thinking unless you ask them. I only get it [instructional decisions] from the voices of students. They greatly impact the instructional path that I take." (Assistant Principal, 1995-2001)

Politics and Reform

The political environment that surrounded mathematics reform at this elementary school was challenging and somewhat explosive. Parents were informed regularly about the purpose and goals of changing mathematics education. Most parents were supportive and encouraging. District administrators were somewhat skeptical and cautious. Politics came center stage when the state began "proficiency" tests in mathematics for fourth grade students. The scores were used to compare and rank elementary schools within the district and throughout the state. The predominately multiple choice and short answer tests were not compatible with this school's constructivist approach to teaching and assessment.

Early in the reform the fourth graders were not scoring well compared to other schools. However, when the principal and assistant principal designed early morning, Saturday, after school, and summer school classes for about one fourth of the fourth grade students, the state mathematics proficiency scores soared to the highest in the district and the state. The educators learned to maintain constructivist teaching practices within extra teaching and tutoring sessions and the students learned much and scored very high.

Fourth grade students achieved the following mathematics scores over the last three years (1999-2001). In 1999, mathematics proficiency scores were 86.1% passage rate. In 2000, the result was even better—90.9%. In 2001, the students' passage rate was 95.5% with 58% of the class (90 students) scored "advanced proficient." The three-year average was 90.8%.

Social Constructivism and African American Students

> I'll tell you what keeps me going, is I can't stand the statistics [the achievement gap between white and African American students]. I look at my classroom list and I see more and more problems and quite frankly, I think, Jeez, these are the same children who are going to be taking care of me when I'm older. And if I can't change this trend, what is going to happen to these kids? (Fourth grade teacher)

State mathematics test results in 2001 revealed a 100% passage rate for white students and a 92% passage rate for African American students. The difference between African American scores and white scores was much smaller than differences between these two racial groups' scores at local, state and national levels. One might conjecture that social constructivist teaching practices within an extended instructional program can effectively raise African American achievement in mathematics.

Most African American students eagerly participated in extended time programs. The study evinces that students enjoyed learning and responded well to social constructivist teaching practices that encouraged dialogue, risk-taking, problem-solving, concrete modeling and the use of technology. Parents were surprised by their students' progress. In 2001, 40% of African American students who were considered "at-risk" in third grade scored "advanced proficient" in mathematics on state tests. The study suggests social constructivist practices can be effective in raising mathematics achievement of African American students.

> The problem of African American underachievement in mathematics is filled with complexity, frustration and anger. American public education has been unable to effectively raise African American achievement in mathematics. Failure to educate African American students in mathematics limits their access to jobs in a technological society. This failure has economic and social justice implications. The study may offer some ideas how to raise mathematics achievement among African American students. "The agenda is primarily social...I think that the Emmy, the Oscar, the Grammy will go to the person who can combine the social agenda with the academic agenda." (Principal)

Uncertain Future

> He's been hiring people who are wonderful teachers and wonderful people. But, he's really, I guess he let's them know that a lot is expected of them. So when he is looking for a teacher, he's looking for someone that's going to come to school and not only follow their contract, but also do a lot more. I've noticed the teachers that he's hired in the last couple years, those are the ones who are here on Saturdays. If he's able to hire people that are totally dedicated like that, and continue to hire teacher like this as people retire the program will continue. (Fourth grade teacher)

Maintaining and sustaining reform efforts at this school remain a challenge. Many elements within the restructuring and reculturing process are expensive, time consuming and dependent on dedicated, intelligent teachers and administrators. The changes represent a dramatic shift away from conventional mathematics education at the K-4 level. Maintaining these reforms within a changing and controlling bureaucratic institution may be difficult. For example, some experienced staff members retire. Other experienced staff members leave for family obligations and career advancements. It is not easy to acquire and educate new administrators about mathematics, social constructivist learning theory, and social constructivist instruction. This type of instructional reform requires an ongoing passion toward adult learning, creative instructional design, and more instructional time for some students.

In addition to the uncertainty that comes with personnel changes, this type of reform may be threatened by district decisions to purchase textbooks and instructional guides that package content, predetermined instruction and give teachers a "crutch" that allows them to avoid active and creative instruction. The reforms at the school were achieved without standard textbooks because teachers and administrators trusted the research about textbook-driven mathematics instruction. Some research suggested textbook instruction hinders creative, active-interactive instruction.

Overall, the reforms represent a major commitment to teacher-administrator learning, to teacher as designer of instruction, to teacher as intellect, to teacher as instructional leader and to the education of all children even though it must happen outside the conventional school schedule. This research implies that the reform will not sustain itself without money, time, and the on-going commitment of teachers and administrators.

Final Remarks

Creating learning opportunities for all children to make sense of important mathematical concepts may require restructuring and reculturing schools. Fitting social constructivist theory into traditional school schedules and school resources may produce frustration and disappointment until new teaching-learning environments and new time schedules are designed. Findings suggest that social constructivist theory helps teachers and learners move beyond the limited mathematical information gained through practices designed from behaviorist theory. However, reaching all students with mathematical understanding may also require more money, more time, more resources, and a radical philosophical shift for mathematics teachers and educational leaders.

> And there are so many unknowns. The structure is not big enough for this reform. It doesn't fit. I've used comparisons like a box. The current instructional box isn't big enough to hold this change. It's not just 'thinking out-of-the-box' either. That might be easier. It's that you don't have a box big enough to fit what you need. The instructional frame isn't big enough to fit the art of it, the beauty of it. You need a bigger frame. I've looked for appropriate metaphors or similes that would fit all of this and explain it to people clearly. It's just not so simple to explain this whole effort. (Principal)

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MATHEMATICS CLASSROOM DISCOURSE AND THE EDUCATIONAL EXPERIENCE OF DIVERSE STUDENTS

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This study examined how four third-grade teachers applied their understandings of their professional development experiences to their classroom discourse and whether the discourse enhanced the educational experiences of their diverse student populations. The results indicate that although each teacher changed some aspect of her classroom discourse, some teachers were more equitable in their distribution of questions across student gender, race, and achievement.

Theoretical Framework

School systems across the US are engaging in educational reform that demands higher educational standards and results. Mathematics education has received a large share of the attention because of the growing need for schools to produce more quantitatively literate citizens. No longer can schools allow large number of students, especially African American and Hispanic students, to leave school underprepared for the technological society in which they will live and work. Yet the disparities in mathematics achievement among students are well documented (Strutchens & Silver, 2000; Tate, 1997). For example, data from the 1996 NAEP (Strutchens & Silver, 2000) found that the average proficiency of African American and Hispanic students at all grade levels was considerably lower than that of White students. These differences were especially substantial on tasks that called for extended responses and complex problem solving. Although African American and Hispanic students have made achievement gains in recent years, these gains have been only on low-level, basic mathematics skills. As Secada (1992) notes, basic skill proficiency is not enough for "true knowledge and mastery of mathematics" (p. 630). Instead, all students "need to learn a new set of mathematics basics that enable them to compute fluently and to solve problems creatively and resourcefully" (National Council of Teachers of Mathematics [NCTM], 2000a, p. 1)

The poor academic performance of African American and Hispanic students in mathematics is attributable, in part, to their educational experiences in mathematics classrooms (Campbell & Langrall, 1993; Oakes, 1990; Secada, 1992). According to NCTM (2000b), "students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are all shaped by the teaching they encounter in school" (p. 17). Researchers who have examined the educational experiences of African American and Hispanic students report that these students are disproportionately placed in low-tracked mathematics classes that are largely taught by direct instruction, rely heavily on worksheets, and cover less of the curriculum (Oakes, 1990; Secada, 1992). In these classrooms, teachers spend more time directing students on repetitive tasks, remedial work, and conformity to rules than they spend on developing students' mathematical competence and autonomous thinking. However, research supports the view that students do not learn mathematics effectively when passively listening to teacher directions.

Developing children's mathematical thinking through classroom discourse has attracted considerable attention in recent years (Davis, 1997; Elliott & Kenney, 1996; Kazemi, 1998; Lo & Wheatley, 1994; Martino & Maher, 1999; National Council of Teachers of Mathematics, 1991; Pirie, 1996). Research on classroom discourse often cites the NCTM (1991) recommendations that mathematics teachers initiate and orchestrate discourse by posing questions that elicit, engage, and challenge students' thinking; by listening carefully to students' ideas; and by asking students to clarify and justify their ideas orally and in writing. Classroom discourse centered on mathematical reasoning and sense making allow teachers to stimulate students' thinking and to reflect on students' understanding. Teachers can promotes students' growth of mathematical knowledge by asking more open-ended questions aimed at problem solving and conceptual understanding (Martino & Maher, 1999). More importantly, productive classroom discourse requires that teachers engage *all* students in discourse by monitoring their participation in discussions and deciding when and how to encourage each student to participate. By actively listening to students' ideas and suggestions, teachers demonstrate the value they place on the students' contributions (Davis, 1997; Pirie, 1996).

Engaging all students in classroom discourse has direct implications for equity issues in mathematics education (White, 2000). For example, the nature of teachers' questions, whom she/he selects to respond to these questions and whose answers are accepted and valued can impact student achievement and participation. An equitable learning environment is one where every student is afforded an opportunity to learn mathematics, participate in mathematics lessons and discussions, and have their ideas valued and respected. Thus, if communication is essential to the learning of mathematics, then researchers need to examine the nature and type of communication occurring in classrooms of diverse student populations. As Hart and Allexsaht-Snider (1996) specifically suggest, we need more research of teacher development programs that focus explicitly on teachers of diverse students and the sociocultural contexts of mathematics learning in their school settings.

Method

This research examined how teachers applied their understandings of their professional development experiences to their classroom discourse and whether the discourse enhanced the educational experiences of their diverse student populations. Central questions for this research were:

1. What changes occurred in the nature of teachers' classroom discourse before and during participation in Project IMPACT?
2. What changes occurred in the questioning patterns between teachers and their students across students' gender and race before and during participation in Project IMPACT?

Participants

Four third-grade teachers and their students in a large, urban school district located just outside of Washington, DC participated in this study. The teachers and students were part of a longitudinal research study entitled Project IMPACT (Increasing the Mathematical Power of All Children and Teachers) to design, implement, and evaluate a model for mathematics instruction in schools serving children of diverse ethnic and socioeconomic backgrounds. All of the teachers were White females. Three teachers, Ms. Atkins, Ms. Davis, and Ms. Tyler, were first-year teachers. Ms. Smith was the most experienced teacher in the study. She had taught a total of eighteen years at the primary level, six of those years in third grade. A total of 172 third-grade students were included in this study: 79 students before Project IMPACT and 93 students during Project IMPACT. The students represented various ethnic groups and were classified into the following categories: Asian, Black, Hispanic, and White. This racial categorization was based on the school system's policy for classifying students. Any student who was not considered Asian, Black, or Hispanic, as defined above, was classified as White. In this study, only two non-white students were placed into this category. In both cases, the students were from East-Indian descent. Tables 1 and 2 show the racial and gender breakdown of the students in each of the four classrooms, before and during Project IMPACT.

Project IMPACT

As participants in Project IMPACT, teachers attended a 22-day summer enhancement inservice where they reviewed mathematics reform documents, examined research on classroom discourse and questioning techniques, mathematics equity, and teaching for understanding to culturally diverse populations. Ten mornings in the summer, teachers taught small groups of third grade students to practice the ideas expressed in the inservice. Teachers were also supported during the school year following the summer inservice with an on-site mathematics specialist. For more information on Project IMPACT, see Campbell and White (1997).

Data Sources and Analysis

Classroom observations, supplemented by my field notes, provided the first source of data for this study. Prior to their participation in Project IMPACT, I observed each teacher teaching mathematics on four separate occasions. This provided me with background information on the teachers' initial classroom discourse and their distribution of questioning patterns. To examine the changes in these patterns, I observed the teachers teaching mathematics on eight separate occasions from January to June of the academic year following the Project IMPACT summer inservice. After the last classroom observations, I individually interviewed each teacher to gather data about her perceptions of the classroom discourse, questioning patterns during mathematics instruction, and whether her views were consistent with her actual classroom practices. These semi-structured interviews provided the second source of data for the study. The journals that teachers wrote during the summer inservice also provided a third source of data to document changes in the teachers' beliefs about teaching mathematics for understanding, mathematics education reform, and educational equity issues.

A separate set of analyses was conducted for each teacher using methods of analytic induction (Bogdan & Biklen, 1992). I chose a qualitative perspective because it afforded me the opportunity to describe the teachers' classroom discourse and questioning patterns in a naturalistic setting while attending to both the content and context of the discourse (Carlsen, 1991). Transcripts of classroom observations and field notes were first analyzed by examining the teachers' question and response patterns and categorizing the patterns into themes based on the nature and focus of the discourse and the cognitive difficulty of the questions. Questions were considered either low-level or high-level. Once the themes were identified and assigned to units of data, these themes were analyzed to identify the students that were involved in the interactions based on categories across students' gender and race (Irvine, 1985; Simpson & Erickson, 1983). This analysis helped answer the second research question, regarding the discourse patterns between the teachers and their students across students' gender and race.

Results

Changes in Classroom Questioning Patterns

The teachers in this study entered the Project from various points of instruction. Three teachers were traditional in their approaches to teaching mathematics before their involvement in Project IMPACT. They began their lessons with a review of a topic, followed by assigning a worksheet of problems for the students to solve. These teachers, Ms. Atkins, Ms. Davis, and Ms. Smith, directed students on the mathematical skills, concepts, and procedures they should use to solve mathematical problems. Ms. Davis wanted her students to know that there was more than one way to solve mathematics problems and often demonstrated various solution strategies for the students to use as they solved problems. Ms. Tyler taught mathematics from a hands-on approach where students worked on problems in various contexts. Although she included several open-ended tasks, she expected students to solve problems in a prescribed manner and emphasized the procedures students were to follow. In all the teachers' classrooms, students worked with manipulative materials; however, manipulatives were either used to explore ideas as in Ms. Smith's class, or to solve problems in a specific manner.

The most common question and response pattern for all four teachers, prior to their involvement in Project IMPACT, was defined as Pattern 1. In this pattern, the teacher asks a question, the student responds with the correct answer, upon which the teacher acknowledges the response and moves the lesson along. For these Pattern 1 interactions, the teacher accepts the student's correct answers without question. She does not probe the student to explain how he/she arrived at the correct answer. Instead, she assumes the answer implies correct thinking. The number of teachers' questions coded as Pattern 1 before Project IMPACT were: Ms. Atkins, 78%; Ms. Davis, 58%; Ms. Smith, 58%, and Ms. Tyler, 62%. Thus, a substantial number of all the teachers' questions prior to their involvement in Project IMPACT focused on the correct answer.

The students observed before Project IMPACT mainly asked questions. The majority of their questions were posed to clarify how they should solve problems and to verify whether they were solving problems correctly. Most teachers answered students' questions directly. They assumed the role of mathematics authority as they gave specific instructions on how problems should be solved.

During Project IMPACT, teachers changed the way they approached mathematics instruction. Students were encouraged to work collectively to solve problems and manipulative materials were regularly available and used by the students to solve problems. Mathematics problems were generally in the form of word problems created by the teacher, by the students, or adapted from a textbook. Worksheets of drill and practice exercises were not evident. Rather, problem solving provided the context for classroom lessons.

There was a major shift in focus of the teachers' questions during their participation in Project IMPACT. Most teachers shifted from a focus on the correct answer to the various ways mathematics problems can be solved. They included several new questioning patterns as they taught mathematics and expected students to explain how they arrived at their answers. For example, most teachers posed more Pattern 2 questions in their mathematics lessons. Pattern 2 questions were open-ended and allowed for multiple answers and solution paths. For these questions, teachers selected several students to answer the same question in different ways and often followed up by asking students to explain their answers. In particular, 93% of Ms. Atkins' responses to her Pattern 2 questions were to ask the students to explain their thinking. Ms. Davis, Ms. Smith, and Ms. Tyler also asked students to explain their answers for 47%, 23%, and 68% of their responses, respectively.

Teachers also wanted the class to judge the correctness of the answers offered by their classmates. This pattern was referred to as Pattern 4. In this pattern, teachers asked the class whether and why they agreed or disagreed with a particular answer and/or solution strategy. Ms. Tyler was the only teacher that asked a Pattern 4 question before her involvement in Project IMPACT. However, during the project, all four teachers included Pattern 4 questions in their mathematics lessons. Furthermore, teachers asked Pattern 4 questions for both correct and incorrect responses.

During the project, student talk was not limited to asking questions. While students continued to ask questions for clarification purposes, they were also observed sharing their opinions about the answers and explanations offered by others. Most teachers encouraged student expression with the exception of Ms. Smith. In her class, students were often observed off-task and engaged in non-mathematical discussions. Teachers' responses to student-generated questions also changed during Project IMPACT. Teachers often asked students questions to help them figure out the answers on their own. They wanted the students to be in control of their learning and asked questions to lead them to the correct answer. This was particularly noticed in Ms. Tyler's classroom.

Cognitive Level of Questioning Patterns

The majority of the questions posed by the teachers, prior to their involvement in Project IMPACT, were coded cognitively as low-level. Low-level questions were those requiring students to recall mathematical facts and procedures. In contrast, high-level questions required students to analyze and interpret mathematical concepts based on their prior knowledge and experiences. Teachers asked high-level questions because they wanted students to reflect on and make connections between previously learned mathematical ideas and newly presented problems. During the project, there was a substantial increase in the number of high-level questions posed by all four teachers.

Two teachers, Ms. Atkins and Ms. Smith, posed high-level questions as they taught mathematics before Project IMPACT. Two percent of Ms. Atkins' questions were coded as high-level as were six percent of Ms. Smith's questions. For these teachers, the number of high-level questions posed during the project increased to 26% and 40%, respectively. All of Ms. Davis' and Ms. Tyler's questions were coded as low-level as they taught mathematics the year before their involvement in Project IMPACT. During the project however, the percentage of high-level questions posed by Ms. Davis and Ms. Tyler were 23% and 32%, respectively. Therefore, all four teachers increased the number of high-level questions they posed during their participation in Project IMPACT.

Changes in Distribution of Questions

Before Project IMPACT the majority of Ms. Atkins' questions were posed to Black- and Asian-male students. Black-females were posed a considerable amount of questions, but less than their male counterparts. Furthermore, Hispanic- and Asian-female students were rarely asked to respond to any questions. In contrast, Hispanic- and Black-females were asked a large share of Ms. Atkins' low-level questions, while Asian- and White-females were asked the majority of Ms. Atkins high-level questions. Another interesting aspect of Ms. Atkins' distribution of questions is that in both years she would select only boys in some lessons, while in other lessons she selected only girls. In both years, Ms. Atkins selected students that could answer her questions correctly.

Table 1. Racial Distribution of Students Across Classrooms: Before Project IMPACT

	Sex	Asian	Black	Hispanic	White	Total
Ms. Atkins	Male	2	4	2	0	8
	Female	3	7	1	0	11
Ms. Davis	Male	2	1	1	3	7
	Female	3	2	4	0	9
Ms. Smith	Male	3	3	5	1	12
	Female	1	4	5	0	10
Ms. Tyler	Male	0	5	3	3	11
	Female	0	7	1	3	11

Table 2. Racial Distribution of Students Across Classrooms: During Project IMPACT

	Sex	Asian	Black	Hispanic	White	Total
Ms. Atkins	Male	0	2	5	1	8
	Female	4	2	3	3	12
Ms. Davis	Male	3	4	5	0	12
	Female	1	4	3	2	10
Ms. Smith	Male	4	4	6	1	15
	Female	3	1	4	1	9
Ms. Tyler	Male	0	5	0	9	14
	Female	0	8	0	5	13

Prior to her involvement in Project IMPACT, Ms. Smith selected a White-male student more often than any other student to answer her questions and "rescue" others that gave incorrect answers. This student was selected regardless of whether he volunteered. Ms. Smith's distribution of questions did not dramatically change as a result of her participation in Project IMPACT. There were several students who were rarely asked to respond during both school years. During Project IMPACT however, there were five students selected to answer the majority of Ms. Smith's questions across patterns and cognitive levels. These students included a White male, Black male, Hispanic male, Black female and White female. Moreover, they were often selected to help when other students answered incorrectly. Students in other race/sex categories, especially Asian males, were rarely selected to respond to Ms. Smith questions. In Ms.

Smith's classroom, there were students that either always participated in classroom discussions or rarely participated. Instead, these students were often asked to listen and watch, and when they were asked questions, it was usually to clarify that they were on task.

Ms. Davis selected females to respond to her questions more than the males in her class prior to her involvement in Project IMPACT. With the exception of Pattern 2, where she selected an even share of males to females, Ms. Davis tended to ask more questions of girls than boys. More specifically, Black-females were selected more often across questioning patterns, while Hispanic students and White-males where rarely asked to respond. During Project IMPACT, Ms. Davis selected students of all genders and races to respond to both her low- and high-level questions. White-females were asked the majority of her questions across patterns however, all students were selected in one pattern or another a moderate number of times.

Ms. Tyler selected Black- and White-females to respond to most of her questions prior to her involvement in Project IMPACT. Prior to her involvement in Project IMPACT, Ms. Tyler selected a fair number of students from both genders to respond to her questions. Black- and White-female students were selected more often across questioning patterns, while the one Hispanic-female was rarely asked questions. Males of all races were posed a moderate number of questions across patterns, with the exception of questions that required the students to explain their answers. In this case, one White male was selected the most. During Project IMPACT, Ms. Tyler selected students of all gender and racial backgrounds to respond to both low- and high-level questions. For low-level questions she distributed her questions fairly across gender and race, with the exception of Patterns 1A and 2. In these patterns, she selected more Black males and White females, respectively. For high-level questions, there were differences in Ms. Tyler's distributions across race. While students from all race/sex categories were selected to respond to high-level questions, White students of both genders were selected the most. Moreover, the high-level questions that were posed to Black students were usually answered by Black males.

Influence of Project IMPACT

All four teachers believed their questioning patterns changed as a result of their participation in Project IMPACT. During the summer inservice, teachers often expressed feelings of frustration in their journal entries. They were frustrated with their questioning patterns resulting in feelings of inadequacy. However, by the end of the summer inservice teachers believed they had made progress.

Most teachers, with the exception of Ms. Smith, began to realize how their views and beliefs about teaching are directly connected to students' learning. Teachers often made reference to classroom exchanges and how their questioning may or may not have been a factor. Ms. Smith rarely wrote about the students she taught during the summer. Her main focus was on her questioning with little attention to curriculum matters.

The end-of-year interviews revealed teachers were cognizant of the changes in their instructional practices. The teachers were very candid in their interview responses and cited several instances where change was identified. For example, most teachers recognized that they rarely asked students to explain their answers to mathematics problems before entering the project. In reflection, they believed they changed in both the focus and type of questions they posed.

Discussion

The purpose of this investigation was to examine teacher change as exhibited by four third-grade teachers who were participants in Project IMPACT. The project provided teachers with research on current mathematics reform documents and its implications for mathematics instruction, as well as issues relating to equity. Thus, the study examined how teachers used this information to provide meaningful mathematical experiences for all their students.

The teachers in this study entered the Project from various points of instruction. All four teachers in this study were influenced by their involvement in Project IMPACT and changed their teaching practices. The degree of that change varied but was evident in all four classrooms. Overall, teachers encouraged students to find alternative ways to approach problems and to explain their answers. They used questioning rather than telling to focus students' attention on mathematical concepts. These questions focused more on students' solution strategies and less on correct answers. They also included more cognitively challenging questions into their instruction. More importantly, teachers used questioning to create a classroom environment where thinking and sense making were the norm.

Most teachers encouraged students to exchange points of view. In Ms. Smith's and Ms. Atkins' classes, these exchanges were often initiated by the students. That is, students exchanged their opinions regarding other students' answers or solution strategies without solicitation from the teachers. In contrast, Ms. Tyler was often observed encouraging students to speak to one another as the class listened, especially when they agreed or disagreed. She stepped out of the picture in the sense that she asked another student to share what they thought about their peer's comment. Ms. Davis wanted the class to come to a consensus when they answered questions. She encouraged students to share why they agreed or disagreed with an answer, but she did not select specific students to respond when their answers were debated. Ms. Davis made a special effort to insure that Limited English Proficient students were matched with another student that could translate problems and explanations. However, she still expected all students to do their own work. Ms. Tyler and Ms. Davis seem to be negotiating social norms of discourse in ways described by Lo & Wheatley (1994).

The study also examined issues of equity in relation to the types and distributions of teachers' questions. All students were exposed to the same mathematics curriculum. That is, no form of ability grouping was evident in any of the teachers' classrooms.

However, there were several behavioral problems during Ms. Smith's mathematics lessons. As a result, students were often expelled from the class and moved to another teacher's classroom. These students' exposures to mathematics lessons were limited.

The findings from the study as discussed above suggest the educational experiences of students in some teachers' classrooms varied. Ms. Davis and Ms. Tyler were the most equitable in their distributions of questions and interactions with students. They posed questions to all students across questioning patterns and cognitive levels. They also acknowledged that some students needed more high-level questions than others based on their academic ability. Therefore, students that were higher in academic ability were not selected as much for low-level questions. As Ms. Tyler explained in her interview, lower-achieving students tend to shy away from answering questions if the higher students propose answers they may not have thought of or understood. These teachers wanted to insure that all students shared in classroom discussions, and therefore based their decisions accordingly.

Ms. Atkins selected students based on academic ability. In her classroom, students with the greatest mathematical content knowledge (i.e., those who scored highest on the Project IMPACT mid-year assessment) were selected more often to answer high-level questions. Lower achieving students were selected for more low-level questions and were rarely asked high-level questions. For Ms. Atkins, neither race nor gender seemed to be the deciding factor in her selections. Instead, she posed questions to students she believed could handle the information and respond accordingly. Ms. Atkins' journal entries provide some insight into her selections. In several of her entries, she expressed her concern about teaching mathematics. For example, she wrote: "Many times I find teaching math difficult because there are concepts I don't understand. How do I teach what I don't understand?" Ms. Atkins' selections may therefore be explained by her lack of understanding of mathematical concepts. What is more important, however, is that she wanted to insure that higher-order thinking was included in her lessons. Thus, she selected students that she was confident would give her the correct answer.

Ms. Smith posed the majority of her questions to students with greater mathematical understanding, both before and during Project IMPACT. Before Project IMPACT, one White-male student was disproportionately selected to answer her questions; during Project IMPACT there were five students she selected. The educational experiences with regard to Ms. Smith's questions were clearly different and inequitable for the students in her classes. For Asian-male students in particular, their exclusion from class questions may explain their limited gain on Project IMPACT assessments. Ms. Smith's distribution of questions may also explain the overwhelming numbers of behavior problems in the room.

It is not clear whether Ms. Smith's mathematical knowledge was a factor in her selection of students. She rarely made reference to her mathematical ability in either

her journal writings or interview responses. One explanation for her selections may be her beliefs about which students can and cannot do mathematics. Ms. Smith was the most experienced of all the teachers in this study. On the basis of her teaching experience, she may feel that some students, especially low-level or non-English speaking students, cannot do mathematics. This may explain why she rarely interacted with these students. However, alternative explanations may hold.

For both Ms. Atkins and Ms. Smith, their selections resulted in the differential treatment of students. Unlike Ms. Davis and Ms. Tyler, these teachers did not insure that all students received both low- and high-level questions. Rather, students that were deemed academically ready were posed questions across cognitive levels, while low-level students were mostly asked low-level questions. Thus, the educational experiences of the students in their classrooms varied.

Shaw and Jakubowski (1991) write "the effectiveness of mathematics education reform is only as good as the commitment and willingness of teachers to change" (p. 20). For the teachers in this study there seemed to be a genuine commitment for change. Their commitments developed over time and were a result of several factors incorporated in Project IMPACT. The students also played a role in the teachers' commitment to change. The students in all four classrooms often voiced their opinions about mathematical ideas and solutions. Most students participated in the Project in the previous year and were accustomed to sharing their thoughts. However, the teachers' willingness to accept the students' comments can not be ignored. Recall that most of the teachers were very traditional in their approach to teaching mathematics before Project IMPACT. They told students how to solve problems and rarely asked them to explain their answers. By changing the focus of their questions and acknowledging students' various thinking skills, these teachers fostered mathematical classrooms conducive for learning.

Conclusion

As we begin the 21st century, the challenge to mathematics education researchers is to address the causes for the disparities in mathematics education and to describe possible solution paths within the context of educational reform (Secada, 1992). This study provided a glimpse of the educational opportunities, experiences, and outcomes of students from various ethnic and gender backgrounds in classrooms of teachers attempting classroom reform. The study also described how classroom discourse can provide a context for examining equity issues in mathematics education. The results from this investigation suggest that teachers can provide meaningful and equitable mathematics instruction when given the knowledge of research, instructional support, and when they have a genuine commitment for change. Future research is warranted to examine how teachers' classroom practices evolve over time and the interplay among students' gender, race, and achievement.

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BUILDING A NEW INTERVENTION MODEL

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Latino families are growing in numbers as Latino families, currently the largest minority group in the Southwest, have been visibly transforming culture and society. These growing families are seeking ways to improve their status in the United States and look to the educational process as a way of achieving this goal. By including them in partnerships with schools, we can help them reach these goals. This paper outlines a research model for greater family involvement and family participation among Latino families in the area of mathematics. Children's Math Worlds Family Connection component (CMWFC) is part of a reform mathematics curriculum, Children's Math Worlds, was developed for first through third grades, at Northwestern University. The CMWFC component is in the final stages and is being field tested in Illinois, California and in Arizona. These finding indicate that the project holds promise of improving mathematics teaching and learning at the elementary level among Latino students.

Involving parents in their children's learning has been recognized as necessary and important by scholars who contributed to several reports on the nation's schools. The recommendations reflect the consistent findings in social research that children have an added advantage in school when their parents encourage and support schooling. This intervention model represents changes taken by classroom teachers, along with the families to improve the mathematics knowledge of the children. Data collected in a period of four years in the form of students' mathematics test scores, family and teacher surveys and journal entries, along with interviews resulted in the New Intervention Model.

Context-Embedded Mathematics Instruction

The lack of English proficiency is the major reason for language minority students' school failure. A considerable amount of research from both Europe and North America suggests that minority students frequently develop fluent surface or conversational skills in the school language but their academic skills continue to lag behind grade norms. This surface fluency may mask significant gaps in the development of academic aspects of English. Academic growth will be fostered by context-embedded instruction that validates students' background experiences by encouraging them to express, share, and amplify these experiences.

Children come to school with different degrees of exposure to decontextualized language. When students have had little exposure to such uses of language prior to school, and instruction in school assumes that these uses have been developed already, the resulting "mismatch" can cause considerable confusion for children, as well as facility to help children directly learn to use academic style language. Children's Math

Worlds Curriculum exposes children to contextualized language by using their stories and their experiences as the bases from which mathematical concepts are taught. Mathematical growth is developed by starting where the student is academically and embedding concepts that will extend their language usage through their active participation.

Meeting Particular Needs of Latino Parents

School reform programs were developed without sufficient attention toward how to attain the academic goals required for success among Latino and other linguistic and ethnic minority students. In the CMWFC component, parents were asked what they needed in order to help their children in mathematics. Surveys in both Spanish and English were sent home with children and follow-up calls were made to parents who had not responded to the survey. Parents wanted to learn to help their children at home but did not feel they had enough mathematics knowledge to help their first, second, and third grade children. The Building a New Intervention Model outlines adaptations made by teachers to expand the support base of their students. These adaptations allowed all student to have access to higher learning in mathematics and to be come successful in this learning.

School Miscalculations and Teacher Knowledge

The educative process of all homes are important ingredients for society in the United States. Because of the diversity in U.S. families, we do not find one single home curriculum, but many variations. Too often, educators underestimate the power of the curriculum in the home. When this happens, teachers miscalculate and misjudge the learning that children have acquired outside of school.

Research indicates that no matter what the ethnic or socioeconomic makeup of the family, common parental styles and activities support children's growth toward a productive lifestyle. Unfortunately, in some families situations, parental styles and habits mitigate against children's natural pursuit of knowledge and positive development. All families have an organization structure that defines family members and their roles. Even in homeless families, parents have some kind of organizational structure to provide physical and emotional support (Takaki, 1996). Whatever children's environment, they learn who they are, how to use the space, and what kind of a world they live in. And no matter what family structure exists, home learning will be a consequence of family interaction, use of time and space, routines of the day, sharing of interests and skills, family rituals and traditions, and family outreach to others.

Parents are children's first and probably most influential teachers. Common features of parenting imply a curriculum; some families have specific goals for their children's development, while others' intentions are vague and ill defined. This New Intervention Model hopefully will give new direction to the intervention practices that fail to take the family needs into consideration before they are implemented in schools.

The current knowledge base of mathematics teaching arises from previous studies of practitioners who lack constructivist orientations and rely heavily on traditional images and methods of teaching and learning. Therefore, a need for pedagogical models that support constructivist learning goals for children exists. Documenting the instructional practices of teachers who truly allow students to construct their own mathematical meanings seems to be one productive way to generate fruitful descriptions of successful reform mathematics teaching.

A major aspect of such reform classrooms is establishing a new classroom discourse style that focuses on and facilitates children's description and discussion for solution methods and mathematical thinking rather than concentrating on eliciting brief answers to simple teacher questions. The research literature at present has few detailed models of this more complex discourse.

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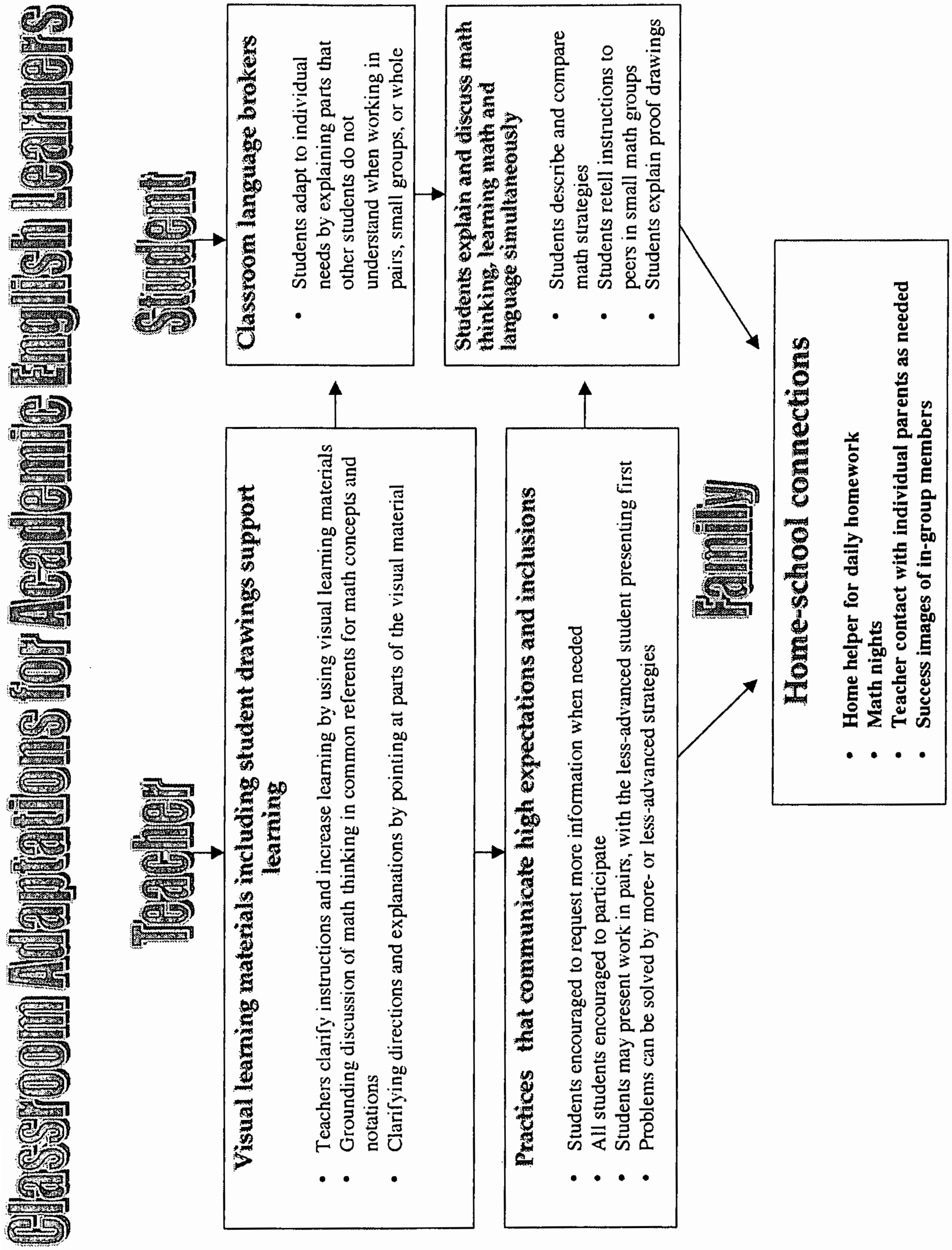
Classroom Adaptations for Academic English Learners
Teacher
Student
Visual learning materials including student drawings support learning
• Teachers clarify instructions and increase learning by using visual learning materials
• Grounding discussion of math thinking in common referents for math concepts and notations
• Clarifying directions and explanations by pointing at parts of the visual material
Classroom language brokers
• Students adapt to individual needs by explaining parts that other students do not understand when working in pairs, small groups, or whole
Practices that communicate high expectations and inclusions
• Students encouraged to request more information when needed
• All students encouraged to participate
• Students may present work in pairs, with the less-advanced student presenting first
• Problems can be solved by more- or less-advanced strategies
Students explain and discuss math thinking, learning math and language simultaneously
• Students describe and compare math strategies
• Students retell instructions to peers in small math groups
• Students explain proof drawings
Family
Home-school connections
• Home helper for daily homework
• Math nights
• Teacher contact with individual parents as needed
• Success images of in-group members

Figure 1

MATHEMATICS IN CANADIAN CLASSROOMS AND SOCIOECONOMIC MATHEMATICS ACHIEVEMENT GAP

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This paper explores three main issues: the extent to which students background characteristics affect their mathematics achievement, the extent to which differences in classrooms affect students' mathematics achievement, and the characteristics of mathematics classrooms where students irrespective of their backgrounds succeed in learning mathematics. These issues are consistent of a vision of mathematics education where all students irrespective of their background characteristics are expected to succeed in learning mathematics. The issues are explored employing the Third International Mathematics and Science Study data for grades 7 and 8 students in Canada. Analysis of the data using multilevel models demonstrates a relatively low mathematics achievement of students from low SES families, especially in Proportionality, Measurement, and Fractions indicating that, a student's success or failure in mathematics learning is domain-specific and is also related to the backgrounds of students. However, the socioeconomic mathematics achievement gap vary significantly among classrooms, and there is some evidence that the gap decreases with increasing classroom mathematics achievement levels. This suggests that mathematics achievement is equitably distributed in classrooms with high achievement levels. Thus, there are successful classrooms in Canada, but the most successful classrooms tend to be those where students from disadvantaged socioeconomic backgrounds excel in mathematics. Another important finding from the analysis is that a more equitable distribution of achievement within mathematics classrooms where teachers avoid practices which involve small grouping, where mathematics teachers are specialized, and in schools where pupil-teacher-ratio is low. This finding calls for a deeper examination of how the effectiveness of instructional practices may vary depending of the background characteristics of students.

SOME REFLECTIONS ON THE DISCIPLINARY CULTURE OF MATHEMATICS AND ITS IMPLICATIONS FOR INSTRUCTION

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The basic beliefs about a discipline, both by its practitioners and its students, form an important part of the context in which education takes place. In this paper, I describe some basic contradictions among common beliefs about mathematics, how mathematicians do their work, and mathematics instruction, and I reflect on the ways these contradictions affect mathematics instruction.

A disciplinary culture, as it is lived at the core of the activity of the discipline, affects the education of the graduate students at that core. Those graduate students go on to become the faculty at many colleges and universities throughout the country (Becher, 1989). Those faculty are then responsible for educating the vast number of students who will become teachers, who then educate pre-college students. Through this "domino" effect, aspects of the disciplinary culture have a profound effect on education at all levels. By looking at the work and attitudes of mathematicians and the experiences of the doctoral students they educate, we can begin to unravel some of the effects of the disciplinary culture of mathematics on earlier levels of mathematics education. This paper is the beginning of the author's attempt to characterize some important aspects of the disciplinary culture of mathematics and their implications on mathematics instruction. As such, I present a series of loosely-coupled reflections on this issue.

The disciplinary culture of mathematics is comprised, in part, of a set of fundamental beliefs and practices that guide mathematical activity. On the surface, these beliefs and practices exist as a series of tensions between some sometimes contradictory ideas. For example, while mathematicians have strong aesthetic and emotional descriptions and reactions to their work (Herzig, 2002), mathematics is often believed to be "objective" and impersonal. These apparent contradictions actually characterize an important feature of the disciplinary culture of mathematics: mathematics itself lies in an n-dimensional space, spanned by a basis whose elements define continua between the personal and impersonal, between collaborative and individual work, between solvable and insolvable problems, between the sciences and the humanities. The remainder of this essay explores four of these basis elements, these continua of activity.

First, mathematicians do a lot of their work collaboratively and cite many advantages to collaborative work (Burton, 1999; Herzig, 2002), contradicting the common myth of the "Mathematical Marlboro Man" (Henrion, 1997). However, when Herzig

(2002) asked mathematicians about the nature of these collaborations, they described partnerships in which most of the work was done independently and individually by the collaborators, and put together periodically; some had even written papers with people they had never met. Indeed, mathematicians have been characterized as being very independent (Fennema & Peterson, 1985).

The work of mathematics students at many levels is expected to be individual, and is often competitive. In contrast, recent reform efforts in K-12 mathematics education have emphasized the importance of students working collaboratively to solve problems. What are the ideal roles of individual and collaborative work in mathematics education? Are mathematicians independent because that is a style that is best suited to the nature of mathematics, or, since the social structure is one that requires independence (Herzig, 2002), is it only the independent learners who succeed in mathematics?

Second, the first few years of graduate study in most U.S. mathematics departments consists of students taking courses in preparation for the qualifying exams, which are important filters along the path to the Ph.D. Students who cannot solve the problems posed on a qualifying exam are judged not to be capable of conducting mathematics research. However, as one mathematician explained,

> Doing mathematics isn't exactly like answering qualifying questions. . . . When I do math I try to keep in my head a lot of questions, most of which I can't solve and then my success is measured by the ones I've solved. It's measured by the papers I've written, and not the ones I've failed to write.

Much of traditional mathematics instruction is built around deriving specific solutions to particular problems, rather than on exploring a host of problems to see where they lead. How and what would students learn about mathematics, if mathematics instruction was built around students' explorations of a host of mathematical problems, even if they are never "solved"? Recent reform efforts at the K-12 level have attempted to move students closer to this exploration.

Third, mathematicians have strong emotional and aesthetic reactions to their work; for example, when they describe the euphoria that comes form solving a problem (Burton, 1999), or when they describe their appreciation for the beauty of mathematics (Herzig, 2002) using words like "beauty," "pleasure," "delightful," and "pretty." Although many of Herzig's (2002) mathematicians devoted substantial time and effort to their teaching, most of them described their goals for teaching to be to cleanly communicate the main ideas of the subject, rather than to share their vision or to lead students to an appreciation of the ideas. They primarily lecture, with a focus on teaching as telling students what they need to know. Not surprisingly, the students described courses in which little motivation was provided for mathematical ideas, connections among ideas were not explored or even provided, and the mathematical "big picture" was missing.

Little if any mathematics instruction includes consideration of mathematics' aesthetic properties. What would be the impact of mathematics instruction that focused on leading students to appreciate the aesthetic, emotional, personal side of mathematics, rather than just passing on the facts of mathematics? Is this important?

Fourth, while mathematics is often classified as one of the sciences, it also bears many similarities to the humanities. Mathematical *knowledge* is a lot like that of the physical and natural sciences (Becher, 1989), which is reflected in the view of many mathematicians that mathematics is an aspect of the real world, which they work to discover (Maddy, 1990; Steen, 1999). However, the nature of mathematical *work* is more like that of the humanities. In the sciences, students often have research experiences as undergraduates, and graduate students generally begin research early in graduate school, which is rarely the case in humanities (Golde, 1996; Tinto, 1993) or in mathematics, where students often don't begin research until they have completed their graduate coursework (National Research Council, 1992). Research in mathematics and the humanities is more likely to be individual and isolated (National Research Council, 1992), compared with a high degree of collaboration in the sciences (Becher, 1989; California Postsecondary Education Commission, 1990; Golde, 1996; Nerad & Cerny, 1993).

How would mathematics instruction look if the focus was on mathematics as one of the liberal arts, if we taught mathematics as one of the great works of human thought?

Mathematics is each of these things, at all points along these continua, in different contexts and at different times. In each of these ways, engaging in mathematics is a very varied activity,. I will argue that the educational experiences of many students represent a much narrower range of activity than those that characterize the work of mathematicians. Although I am not trying to model the work of students directly after the work of mathematicians, I will argue that the ways in which we narrow mathematical activity for students limits their abilities to truly understand, appreciate, and use mathematics.

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STUDENT REASONING DURING IMPLEMENTATION OF TWO GENERALIZING TASKS: ASPECTS OF GENERALIZING AND THE ROLE OF REPRESENTATIONS

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This paper analyzes two lessons where students worked on generalizing tasks. Despite the similarities of the tasks and format of the classes, these two lessons proceeded quite differently. One lesson progressed smoothly and students produced and justified a generalized formula. In the second lesson, students encountered difficulties and the teacher had to heavily scaffold students' engagement with the task. Three distinct aspects of generalizing tasks (particular, formal, and contextual generalization) are discussed and the role of representations in mediating the interrelationships among these aspects of generalizing is considered. An understanding of these interrelationships has important implications for classroom instruction.

Recent work on task analysis and implementation has heightened our awareness of the role of tasks in structuring classroom interactions, and more specifically, how the demands of a task can shift during task implementation (Stein, Smith, Henningsen, & Silver, 2000). The analysis presented here contributes to this area of research by considering the evolution of student reasoning on generalizing tasks during two lessons. I consider how students' engagement with particular representations mediated their progress on the task as they moved between generating examples and generalizing.

The data for this paper are classroom videos collected as part of a yearlong study of a ninth grade Math A (pre-algebra) class. The videos were taken during a unit on patterns. Each lesson focused on a generalizing problem:

Lesson 1: How many diagonals are there in a polygon with *n* sides?

Lesson 2: How many ways can you arrange *n* different flavors of ice cream on a cone?

The goal of each task was to find and justify a general formula for the *n*th case. Despite similarities in the tasks and the instructional format, these two lessons proceeded quite differently. In the diagonals lesson, students' made steady progress. In the ice cream cone problem, students had more difficulty and the teacher had to heavily scaffold students' engagement with the task to keep a high level of cognitive demand. (See Staples, 2002 for further discussion.)

A Closer Look at 'What It Takes' to Generalize

To understand the differences between these two lessons, it is useful to distinguish among different kinds of generalizing opportunities typically afforded by generalization tasks (Driscoll, 1999). I refer to these as *particular, formal,* and *contextual* generalizing[1]. A *particular generalization* is a pattern or rule by which any particular example could be generated (e.g. the next value in a sequence is two more than the last value). A *formal generalization* is a general rule expressed symbolically encompassing all cases at once (e.g., $y = 2(x-1)$). A *contextual generalization* is a general rule grounded in the context or problem situation (e.g., since all but one person in the park has a bike, the number of bike wheels is twice the number of people minus two). Contextual generalizations are often used to justify formal generalizations, and have been a focus of reform-oriented teaching. Although these aspects of generalizing do not have to be addressed in any particular order, and can be engaged simultaneously, often generalizing tasks are structured so students progress from the particular to formal to contextual. The assumption is that the thinking and reasoning students do to produce, for example, a particular generalization is a useful step on the way to a formal or contextual generalization.

The Lessons

In the diagonals lesson students, under the guidance of the teacher, worked with the particular examples of a square, hexagon, and an octagon, finding the number of diagonals of each. The emphasis was on developing a systematic process of drawing and counting the diagonals. Remarkably, while working on the third example, several students articulated a formal generalization and then offered a contextual generalization, explaining how the formalization was connected to the geometry of the *n*-gons. Thus in this lesson there was a tight connection among the three different kinds of generalizing activities. Students' work with the three examples, and more specifically, their process of developing an increasingly systematic method for drawing diagonals, afforded students the opportunity to engage the three aspects of generalizing concurrently.

The ice cream scoops lesson unfolded quite differently. Students produced particular, formal and contextual generalizations as three distinct phases of the lesson. Using a table, students recognized and articulated a particular generalization: moving down the right column, values are multiplied by 2, 3, 4, etc.

# of scoops	# of ways to arrange
1	1
2	2
3	6
4	24
5	120

After producing this particular generalization, students were unable to make headway towards a formal or contextual generalization. Their engagement with the tabular representation, while amenable to a particular generalization, did not provide students with resources for the subsequent generalizing activities. The teacher had to actively scaffold their engagement with the problem, verbally directing their attention to features of the pattern (e.g. 24=4x3x2 not only 4x6) and then employing both a visual and situational representation to help students complete the task. Thus during lesson, the different kinds of generalizing activities in this lesson were not closely linked or mutually supportive. One can imagine alternative approaches such that these generalizing activities would more closely align. Interestingly, the visual representation was not an adequate resource in this task implementation, perhaps because, unlike the diagonals lesson, attention was not given to its systematical production, and so did not provide students with the same kinds of experiences with the structure of the problem.

Summary and Implications

This analysis highlights the importance of attending to the different kinds of generalizing (particular, formal, and contextual) required by a task, as well as the role of particular representations (tabular, visual, situational) in mediating the connections among them. A close connection among the three kinds of generalizing is assumed, but this relationship requires further consideration. The interrelationship among these depends upon task implementation and the affordances of the representation used. An understanding of the interrelationships and representations has implications for classroom instruction. In designing activities and planning instruction, teachers make choices about representations and whether to direct students to preferentially consider one kind of generalizing over others. These choices can significantly alter the progression of a lesson, the learning opportunities afforded students by a particular task, and the work the teacher must to support students in completing the task.

Note

[1]Friedlander and Hershkowitz (1997) use the terms *working* and *explicit* generalizing for *particular* and *formal* generalizing.

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BUILDING A FRAMEWORK TO ANALYZE STUDENTS' SUPPORT NETWORKS FOR THEIR MATHEMATICS EDUCATION

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This paper proposes a framework for analyzing the support networks that students develop when interacting within the embedded contexts of the mathematics classroom, the school, and the community. Instead of looking at the networks within each of these settings, we look at the *type* of support provided to the students. We focus on academic support in mathematics from a broader point of view, including experiences inside and outside the mathematics classroom that may have a great impact on a student's attitude towards mathematics and his/her understanding of mathematics. The framework also proposes to look at all sites where students are exposed to mathematics instruction through the lens of a constructivist approach to learning.

Defining the Framework

The ideas proposed here have emerged from a larger qualitative research (the author's dissertation study). Some of the questions guiding the study are: what experiences in- and out-side the school help English-Language-Learners (ELL) make sense of mathematics when they take English-taught mathematics classes? What circumstances seem to hinder these students' advancement in mathematics? (Note that although these questions were originally designed to explore ELL students' experiences, the results included in the current paper are not limited to ELL students. Specific results related to ELL issues are not presented in this paper.) The original study took place in an innercity high school, within a poor community in New York State. Minority students compose approximately the 50% of the school population--African-American 36%, Latino 9%, and Vietnamese 4%. The English as a Second Language (ESL) program—a combination of ESL classes and ESL push-in strategies—serves 10% of the school population. To better understand students' experiences, the data collection included observations in two algebra classes, and a series of interviews with the algebra teachers, eight students from those algebra classes, a Latino vice-principal, an ESL teacher who pushed into one of the algebra classes, a special education assistant who pushed into the other algebra class, and a Latino ESL teacher assistant. The interviewed students were ELL and non-ELL, male and female students. Latino students were from Puerto Rican (4) and Mexican (2) backgrounds.

Although most students received the core of their mathematics instruction in their algebra classrooms, data from the study revealed that each student had a variety of experiences that he/she perceived to have a significant impact on his/her motivation towards school and higher education, and on his/her actual opportunities for learning mathematics meaningfully.

Listening to the student voices allowed for the creation of categories of relationships based on the *nature of the support* provided to the students. A relationship may provide (#1) *Affective*, (#2) *Academic*, (#3) *English*, and/or (#4) *Social and Cultural support.* Instead of looking at students' relationships within a classroom, school, or community settings, we look *across* these contexts to uncover the relationships that impact students' learning of mathematics. The emerging framework proposes the definition of a *student's support network* as the structure of all students' interpersonal relationships, the student's participation in social and cultural organizations, the student's utilization of resources, and any other relationship that provides the student with some type of support for his/her education. As relationships change, develop, or end, support networks are to be considered dynamic structures.

This framework emerged from the data and was used, later on, to reflect back on the data about students' experiences. Following are some examples: (a) we expect *academic support* from the algebra teachers, but the data showed that the drill-and- practice approach to teaching algebra generated students' frustration, for lack of understanding. The teachers did not provide true academic support but actually hindered the students' mathematical thinking; (b) the ESL teacher greatly helped ELL students to learn English and adjust to the American culture (#3 and #4). ELL students also consulted her for affective support (#1) and for tutoring in mathematics. As she taugh her students in a step-by-step, memorization fashion, she did not succeed in providing proper academic support in mathematics; (c) A sport coach became an affective support (#1) for a Puerto Rican student who played in his team, as well as a support for learning English through the reading and discussing sport magazines.

The proposed framework could be used by practitioners and researchers in other situations; for example, to understand and explore support networks of students with special needs, defining categories of support that might be crucial in these students' inclusion and success in their academic career.

MATHEMATICS AND THE MEME

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In *The Selfish Gene,* Richard Dawkins (1976) introduced a gene-centric view of the history of life. Genes are self-replicating units whose essence (the biological instructions they encode) persists indefinitely (possibly with minor transmission errors); thus in Dawkins's theory genes, rather than individuals or species are seen as the basic units of life and evolution. A key point of Dawkins's thesis is this: any immortal, self-replicating, possibly erring entity in an environment of limited resources will be subject to the laws of Darwinian evolution. It is the inevitability of Darwinian processes that leads in Dawkins's final chapter to an enticing application of the theory, one with important implications for mathematics educators as we think of knowledge as culturally situated and generated.

Dawkins's new application is the *meme.* A gene is a unit of biological information; a meme is a unit of cultural information. It might be a catchy tune or remarriage taboo. It is a thought that replicates by passing from one mind to another. Other criteria are satisfied and the principles of Darwinian evolution govern how memes spread (for a thorough introduction to the study of the spread of memes, see Lynch, 1996).

PMENA XXII spotlighted socio-cultural issues in mathematics education. Several researchers speak of knowledge as culturally held and generated. Attention to the theory of memes provides a mechanism for this process and a point of view leading us to ask new kinds of questions about mathematics learning.

A Mechanism

Knowledge is culturally held and generated. What does that mean? How does it occur? One answer is theories of enculturation (for recent applications see, Gutmann,2000; Tireney & Rhoades 1994). Here the agent is the cultural group, initiating newcomers to traditions and knowledge. Memetics suggests a different point of view: meme-centric. Memes, cultural ideas are independent actors working to perpetuate themselves by infecting new minds. If one adopts this point of view, then one must concurrently adopt a radically different stance to understanding learning. Rather than minds acquiring or constructing knowledge or cultural competencies, knowledge and cultural competencies acquire and construct minds. The task of the teacher is not to help students learn, but to help students become more receptive vessels for memes.

New Questions

Several have attempted to answer "what is mathematics?". Most have focused on what its tools are, what its modes of thought are, and what its applications are. If instead we ask "what is the mathematical meme complex?" "What are the cultural units that combine to create a being called mathematics that as a whole perpetuates its component memes?" then we ask a new breed of question. For example, we might observe that western mathematics has often been thought of as the province of the elite. We should now ask why it aids the mathematical meme complex, aids in an evolutionary sense, a self-perpetuation sense, to be restricted to only a few, elite minds. We should begin to ask not only "what is mathematics?", but "what is the environment that has fostered the evolution of the mathematical meme complex?".

The purpose of the poster is to introduce these ideas and generate discussion. Memetics as a science has developed and changed with many of Dawkins's ideas being alternatively lauded or abused in different camps. My own thoughts on this subject are only beginning to gel, but my sense is that a meme-centric view of mathematics learning can provide a language to describe learning in the classroom, one which focuses on the development of knowledge through social interactions within a community of learners. As mathematics educators take up a cultural lens wit increasing frequency, new mechanisms need new languages to describe them and introduce new kinds of questions.

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EXPOSITORY WRITING AND COMMUNICATION IN GRAPHING LINES

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In response to the proposal of *Principles and Standards* (National Council of Teachers of Mathematics, 2000) that communication be an integral part of the mathematics classroom, we have developed a conceptual framework from which to understand students' expository communication activities. Little attention has been paid to understanding the way communication processes, interaction practices, and functional exposition elements are integrated in expository communication. This general framework is an effort toward understanding the nature of communication in the mathematics classroom. This study initiates the understanding of communication with this framework and explores its utility for identifying the explanatory communication practices of undergraduate remedial mathematics students' expository writing. We explore how each of these communication activities may be realized and enacted in particular explanations of mathematical concepts and processes. We also examine the extent to which these communication activities are related to others' assessments of student writing.

Expository writing in the mathematics classroom engages students in writing as an active part of the learning process and classroom instruction. This writing allows students to catch mistakes in addition to remembering and understanding problems better (Cai, Jakabcsin & Lane, 1996; Drake & Amspaugh, 1994). The natural process of writing requires students to reflect upon their thoughts before solidifying them into words. This process makes their thoughts objects of further thought. Powell and Lopez (1989) describe learning as a dynamic process where students move between experiences and reflections. Somewhere between the experiences and the reflections, students go through critical reflections. Bell and Bell (1985) first discovered that there was a link between writing and the mathematical thinking behind problem solving.

In order to develop our framework, we incorporate classic views on the role of logic in teaching (Ennis, 1969), the current work on expository writing and communication, including the studies mentioned above (Schoenfeld, 1998), technical writing (Wieringa, Moore, & Barnes, 1992), as well as work in communication and discourse analysis that has been focused on the analysis of explanations and explanatory discourse (e.g., Antaki, 1981; Garfinkel, 1981). Together with this literature, research in mathematics education, and our data analysis, we suggest that the following categories of communication activities are present in expository writing in mathematics instruction: orienteering, articulating relevant concepts, legitimizing concepts, stating connections between the concepts and their features, guiding procedural activity, solidifying conceptual understanding, and facilitating linguistic control.

Orienteering is structuring the interaction or guiding the reader's attention and displaying awareness of the informative activity. Articulating relevant concepts with interpretive explanations is providing the reader with a kernel. Legitimizing concepts through reason-giving explanations is using models to display the procedure or concept and provide concrete interpretation with exemplars or displays. Stating connections between the concepts and their features is reasoning and providing coherence with exemplars in procedures. Guiding procedural activity with descriptive explanation is explaining the relevant process or procedure. Solidifying conceptual understanding is making discursive connections to help solidify new knowledge and create elaborative connections. Finally, facilitating linguistic control is the use of mathematical terminology and specific language.

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EQUITY IN THE MATHEMATICS CLASSROOM: A STUDY OF THE CONNECTION BETWEEN PRE-COLLEGE EXPERIENCES AND MATHEMATICS ATTITUDES OF UNDERGRADUATE AFRICAN AMERICAN STUDENTS

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This study examined how pre-college experiences with equity in the mathematics classroom connect to mathematics attitudes in undergraduate African American students. Specifically, from the perspective of selected African American undergraduate students:

(a) To what extent did the mathematics curriculum use contexts that were inclusive of a wide range of student experiences and perspectives?

(b) To what extent did mathematics instruction incorporate and value diverse student experiences, perspectives, and learning styles?

(c) To what extent did the classroom culture promote positive teacher-student and student-student interaction and relationships?

(d) How are mathematics classroom experiences connected to one's feelings toward mathematics?

(e) How are mathematics classroom experiences connected to one's feelings about oneself as a learner of mathematics?

In the study attitudes was defined as feelings about mathematics and feelings about oneself as a learner of mathematics. Equity was defined as the idea that all students, regardless of their race, ethnicity, gender, social class, or other cultural characteristics should have equal educational experiences that support equal achievement. Equity in the mathematics classroom was examined from three aspects: curriculum, instruction, and classroom culture.

The study found that pre-college experiences with equity are integrally connected to the mathematics attitudes of African American undergraduate students in that the kind of mathematics experiences they had either encouraged or diminished positive attitudes toward mathematics and toward themselves as learners of mathematics. How those experiences affected attitudes were influenced by teacher characteristics (the type of math teachers the student encountered), student resiliency (personality and other individual traits), and student ideology (how the student viewed himself and the world).

GROWTH OF MATHEMATICAL UNDERSTANDING IN A BILINGUAL CONTEXT: A CASE STUDY OF TONGAN SECONDARY STUDENTS

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The National Council of Teachers of Matheamtics in the *Curriculum and Evaluation Standards for School Mathematics* (1989) addressed the issues concerning bilingual students' learning of mathematics by announcing that "students whose primary language is not the language of instruction have unique needs" (p. 142). This lack of proficiency in the language of instruction often leads these bilingual students to switch languages (Fasi, 1999) partly because learning mathematics *in* and *through* a second language presents a double challenge – learning of mathematics and the need to understand the language of instruction (Adler, 1998). My experience as a mathematics educator with a bilingual background has allowed me to study the relationship between bilingual (Tongan) students' "language-switching" and their growing mathematical understanding. My research interests centered on "What is the relationship between Tongan bilingual students' *language-switching* and their *growth of mathematical understanding*?"

Theoretical Framework

The Pirie-Kieren dynamical theory for the growth of mathematical understanding offers a *language* for explaining, and way of observing, the dynamical growth of mathematical understanding. Pirie and Kieren (1991) explicitly state that growth of understanding occurs through a continuing movement back and forth between the layers or modes of understanding to re-member and to re-construct new understanding, recognizing the inter-dependence of all the participants in an environment. Standard distinctions within systemic functional discourse analysis, based on Halliday's work (1978), are employed to understand the data better in making the link between *theory/practice* and *action/reflection* (general or specific) as an essential step in enriching one's analysis of video data particularly in this context about how each layer of the Pirie-Kieren Model has a basis in the discourse. Its importance for understanding the role of language (either English or Tongan) in teaching and learning is crucial.

Method and Tasks

Grades 8 and 9 Tongan students were asked to solve mathematical tasks together in pairs or threes. Video was employed to capture both the verbal and non-verbal evidences of these students' growing understanding of fractions and patterns and relations. The videotapes form the main data of the study, although they are supplemented with audiotapes, field notes, written work and any appropriate record of the students' work together with *video-stimulated recall* to clarify and to elaborate what was observed. Bilingual students' growth of mathematical understanding will be analysed and mapped using the Pirie-Kieren diagrammatical model in terms of their uses of images (mostly at the *image*-layers) and the relationship between their *language-switching* and the Pirie-Kieren's notion of *folding back*.

Findings and Conclusion

In doing the assigned tasks, the Tongan students appeared to work mostly at the *image*-layers, either working with their constructed images at the *image having* and *property noticing layers* or constructing new images at the *image making layer*. In addition, the concept of *reflection* resembles that of Pirie-Kieren's notion of *folding back*, and this similarity requires further investigation. Through the framework of the Pirie-Kieren Theory, the systemic functional discourse analysis has offered a thorough understanding of the comprehensive meanings of the bilingual students' language use. Halliday (1978) suggests that since languages "differ in their meanings, and in their structure and vocabulary, they may also differ in their paths towards mathematics, and in the ways in which mathematical concepts can most effectively be taught" (p. 204). As a result, bilingual students might pay attention to different characteristic of a particular language and further research could be done to explore if ESL students create new pathways towards understanding mathematics.

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PROJECT VOICE: LISTENING TO AFRICAN-AMERICAN STUDENTS' VOICES ABOUT THEIR MATHEMATICAL EXPERIENCES

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This poster session highlights the findings and implications of the research project, *Project Voice: Listening to African-American Students' Voices About Their Mathematical Experiences* (Research Advisory Committee Grant No. 2-67968, The University of Alabama). The project used the metaphor of *voice* to investigate the mathematical experiences of four (two males and two females) African-American students. The primary goal of the project was to give voice to African-American students who had been successful with school mathematics to determine what factors led to their success. The data suggest that teachers and particular teaching practices are paramount in fostering successful African-American mathematics students.

"Voice refers to the discourse that is created when people define their own issues in their own ways, from their own perspectives, using their own terms—in a word, speak for themselves" (Secada, 1995, p. 156). Using the *voice* metaphor as the basis of this investigation, this project examined the mathematical experiences of four African-American students to determine what factors led to their succeeding in mathematics. The project particularly investigated the African-American mathematics students' perceptions of and responses to their mathematics education and sought to unravel whether experiences in mathematics classrooms shape African-American students' viewpoints about their mathematical experiences and inevitably impact their succeeding in mathematics.

The research project employed a phenomenological research strategy. Phenomenology is the study of lived experiences and essences that allow researchers to discover that, which makes an experience unique and meaningful for an individual. Further, phenomenology is aimed at interpretive understanding and describes subjective experience from the viewpoint of the individual (Tesch, 1987). The study particularly examined the phenomenon of being a successful African-American mathematics student from the viewpoint of the student.

Data were collected in the form of an initial survey, an autobiography, and interviews, including a third and final interview that consisted of a member-check by the participants. The initial survey was a questionnaire that consisted of open-ended questions. The participants were asked to describe the schools they attended from elementary school throughout college. They were asked to provide descriptions in their own words of these schools, the mathematics courses they took, their mathematics teachers, and the mathematics classrooms in which they participated. For the autobiography, the participants were asked to write a story of their lives as successful

African-American mathematics students in the schools and mathematics classrooms they had described in the initial survey. They were given no instructions on how to write their stories but were asked to describe their interactions with mathematics teachers, peers, and other persons who were important in the telling of their stories. The participants were interviewed three times: once, after the completion of the initial survey, secondly, after the completion of the autobiography, and a third time toward the end of the study.

The results of the study suggest a need to prepare teachers to become *culturally responsive teachers* (Malloy, 1997). "Culturally responsive mathematics pedagogy is predicated on the teacher's interpreting, understanding, and recognizing the students' culture and integrating it into the learning process [and] the teacher's allowing students to construct mathematical knowledge on the basis of their experiences " (Malloy, 1997, p. 28). The results also suggest a need to prepare teachers to become what the participants called *caring educators* who listen to African-American students' voices and endorse their preferred ways of learning mathematics. Finally, the data suggest a need to reconstruct mathematics teacher education programs that raise the critical social consciousness of preservice teachers and help them reflect on the nature of structural oppression as it affects African-American mathematics students. Such reflection has the propensity to help teachers embellish teaching practices that embody equity and counter oppression, which ultimately fosters successful African-American mathematics students.

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STUDENTS' EXPERIENCES WITH COOPERATIVE LEARNING

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Johnson & Johnson (1991) classify the structure of general classroom instruction in three main forms: competitive, cooperative, and individualistic. They claim that "for the past fifty years competitive and individualistic efforts have dominated classrooms" and that "cooperative learning has been relatively ignored and underutilized by teachers even though it is by far the most important and powerful *way* to structure learning situations" (p. 26). For the past twelve years, as a teacher of both secondary mathematics students and pre-service teachers, I have incorporated cooperative learning into my lesson structures and have found myself continually challenged to answer the question: What is meant by this "way" called cooperative learning as it relates to mathematical learning?

Cooperative learning has been widely researched from a variety of perspectives. Researchers have shown that cooperative learning can result in positive effects on students' achievement, motivation, self-esteem, and race-relations (Rottier, 1991; Bellanca & Fogarty, 1991; Kluge, 1990). With a focus more on process than outcome or effect, researchers have investigated students' interactions (Webb, 1991), questions (Walter, 2001), and cognitive processes (Appleton, 1997). An aspect of cooperative learning that has received limited focus in previous research is students' experiences with cooperative learning from their own individual perspectives.

Believing there is a need to provide students' a voice in this research, I conducted an interpretative study in which I observed and interviewed three female, pre-service secondary mathematics teachers. The goal of the study was to investigate the ways in which the students were both simultaneously constructed and constructed by cooperative learning experiences as they engaged in mathematical problem solving. Emphasis during analysis focused on how are these various constructions are related to the students' mathematical activity.

Data was collected as the participants worked cooperatively, on three separate occasions, to solve different mathematical problems. Each problem solving session (PSS) was immediately followed by a group interview in which the participants discussed their perspectives on the PSS. The following day each participant returned for an individual, 90-minute interview where she and I viewed the tape together—pausing often to discuss specific instances pointed to by both myself and the participant. In light of the employed methodology coupled and my belief that we do research with and not on people, the participants were, in a strong sense, co-researchers. They participated in data analysis (by analyzing videotape as they viewed it) and helped formulate discussions and interview protocols (by using the group interviews to identify themes in their own conversations).

St. Pierre and Pillow (2000) claim that "the work and challenge of postmodern theory is to continually question our "taken-for-granted structures of intelligibility." The participants in this study feel that they had the opportunity to question and deconstruct some traditions of cooperative learning. They also feel that they were able to reconstruct what cooperative learning is for them as learners and what it will be for them as teachers. One of the participants will join me in this poster session as we view and discuss 30-second video clips that highlight salient aspects of the research findings with the conference attendees. This will be a unique opportunity for attendees to interact with one of the study participants and to hear about her interpretations of and experiences with cooperative learning.

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FOSTERING CLASSROOM DISCOURSE THAT HELPS STUDENTS LEARN AXIOMATIC GEOMETRY

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As we move into the new century, efforts to enhance school mathematics teaching and learning continue. Within the last decade, many researchers have focused on studying pre-service/in-service teachers' content and pedagogical knowledge. The results of such research made us aware of various difficulties and issues that affect instruction and students' learning. Researchers focusing on classroom discourse point out that teacher's content and pedagogical knowledge influence the richness of class discussion, teacher's instructional style, and student learning (Fennema & Franke, 1992; Knuth & Peressini, 2001). Furthermore, Marks (1990) emphasizes the importance of links of teachers' content and pedagogical knowledge.

In this presentation the author will describe a specific activity identified by the participants of the study as the most valuable for creating mathematical discourse in a college level geometry course. The study represents an attempt to apply and extend the results of research done at the K-12 level. Sixteen students, mathematics and mathematics education majors, enrolled in the 'Euclidean and Non-Euclidean Geometry' course at the large Southeastern American University participated in the study. The course was structured and designed with the intention to put into practice recommendation stated in the National Council of Teachers of Mathematics (1991) -- using various pedagogical methods to engage students actively in the process of teaching and learning of mathematics. In particular, the report will focus on discourse in which students engage as they present in class their solutions to homework assignments. Data collected throughout the semester that will be used for this particular presentation consists of students' journals and video recorded class presentations.

The results of the investigation suggest that dialogical discourse that appeared between student-presenter and his/her classmates was the most powerful and helpful in developing their understandings about those "propositions and axioms that continue to hunt" them. As it could be seen from the videotaped classroom sessions (and confirmed in students journals) students created productive mathematical dialog by questioning and challenging each other to defend their answers. The classroom climate was encouraging for students to 'take risks'. All of them were very involved and responsive. Even when a question appeared trivial, no one ridiculed anybody's thinking. Through active involvement and participation the students developed understanding and acquired knowledge. Detailed analysis of components of such discourse will be presented.

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Teacher Knowledge

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TEACHER IDENTITY AND KNOWLEDGE IN ELEMENTARY MATHEMATICS[1]

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Research shows teacher thinking is influenced by both knowledge and identity. Yet these two constructs are rarely studied in combination. This study documents the knowledge and identity of teachers experienced in mathematics reform, and examines how identity and knowledge resources interact in teachers' thinking about instruction. We interviewed eight teachers involved in either Investigations in Number, Data, and Space or Cognitively Guided Instruction. We found teachers' knowledge of multidigit multiplication and division generally exceeded that of teachers in prior studies, and they held uniformly positive perceptions of their current identities as learners and teachers of mathematics. Furthermore, teachers' high point and/or turning point experiences were closely linked to their conceptions about how to foster understanding in students. Low-point experiences for teachers with early negative stories framed current conceptions about the role of affect in understanding. Two cases illustrating specific links between knowledge and identity themes are presented.

A great deal of attention has been devoted to understanding elementary teachers' knowledge of mathematics and children's mathematical thinking in the context of standards-based instruction (Ball, Lubienski, & Mewborn, 2002). A separate line of work, equally robust, has investigated teacher identity and its influence on teachers' understanding of practice (Carter & Doyle, 1996; Clandinin & Connelly, 2000). With increasing pressure for education reform one unanswered question concerns how knowledge and identities *interact* and *change* over teachers' lives, and what can be done to facilitate change in the directions valued by many in the education community—toward, for example, more student-centered instruction that includes problem solving and constructing mathematical arguments. Although these questions have been pursued separately for knowledge and identity, we believe mathematics education reform can profit from the study of the two constructs in combination.

Framework

We focus on *narratively organized* identity (Bruner, 1990; Drake, under review), or the stories teachers tell about themselves and their experiences, the meanings they give to their experiences, and the narrative coherence they see across their past, present and future professional lives. Teachers use these personal narratives to "define what is recognized as significant in the stream of experience" and to "stipulate how issues and problems can be thought about" (Carter & Doyle, 1996, p. 134). In contrast, teachers' knowledge is usually depicted using *conceptually organized* frameworks representing the aspects of mathematics and children's thinking most salient for instructional decision making (e.g., Franke, Carpenter, Levi, & Fennema, 2001; Ma, 1999).

Bruner (1996) argues that "interpretations of meaning" –such as what to make of a child's unusual invented strategy—"reflect not only the idiosyncratic histories of individuals, but also the [sub]culture's canonical ways of constructing reality" (p. 14). Teachers' understanding thus depends on meanings rooted in their own stories and on those reflecting culturally shared knowledge of mathematics and children's thinking—such as the knowledge teachers may hold as a community by virtue of participating in the same reform agenda. Teacher thinking is influenced by narrative *and* conceptual perspectives, reflecting individual and communal histories.

This framework leads us to expect teachers to use both identity and knowledge resources to make sense of a variety of issues central to their practice. For this paper we focus on an issue that has figured prominently in calls for reform: understanding mathematics. Virtually all teachers involved in reform are committed to helping students understand the mathematics they teach. But beyond this basic commitment, teachers differ in their thinking about what understanding is, its sources, how it develops, its affective components, and so on. These differences are reflected in teachers' instructional practices, and ultimately, we believe, in student outcomes.

Question

Our goals in this paper are to: 1) examine reform-oriented teachers' knowledge and identity, and 2) identify how identity and knowledge resources influence teachers' thinking about reform-oriented instruction, with specific regard to facilitating students' understanding of mathematics. Building on the premise that narrative is a powerful tool for organizing new information, Drake (under review) has established that teachers' stories influence their initial experiences of reform. Our question is, as teachers become experienced at reform teaching, how does their knowledge of mathematics and children's thinking influence their identities as mathematics teachers? And how do teachers' newly recast identities motivate new knowledge? We conjectured that as domain-specific knowledge of mathematics or children's thinking increases, teachers may reconsider or "re-write" their identity stories, and the role of earlier mathematics identities as resources for interpreting practice would shift.

Method and Data Sources

We analyzed the knowledge and identity of eight teachers of grades 3-6 involved in mathematics reform in two low-income urban settings. We chose schools with different reform-oriented approaches to mathematics instruction to achieve a range in teacher knowledge and story type. One school was located in Austin, Texas, and had been implementing *Investigations in Number, Data, and Space* (TERC, 1995-1998) for five years. Over 80% of its students came from low-income families. The school population was classified as 11% African American, 75% Hispanic, 12% White, and 20% Limited English Proficient. The other school was located in Phoenix, Arizona; teachers had been implementing Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Levi, & Empson, 1999) for three-six years. Virtually 100% of its children came from low-income Latino families. All of the teachers from this school worked in bilingual classrooms. Teachers were given pseudonyms beginning with 'I' (for *Investigations)* or 'C' (for CGI).

Instruments

Each teacher participated in 1) a mathematics story interview designed to elicit teachers' mathematics *identities*, and 2) a teaching-scenarios interview designed to elicit *knowledge* of the domain and student thinking in multidigit multiplication and division. The Identity Interview included questions about teachers' high points, low points, and turning points in learning and teaching mathematics, for which teachers were asked to describe the setting, what happened, who was involved, and feelings brought on by the event. The Knowledge Interview included scenarios of students' correct and incorrect invented algorithms for multidigit multiplication and division, questions about the relative roles of invented and standard algorithms in the development of understanding of multidigit operations, and questions about particular students and their mathematics understanding (Table 1).

Coding/Analysis

We began by analyzing the knowledge and identity interviews separately. We then coded across each teacher's two interviews for evidence of their conceptions about sources of understanding (self, community of peers, teacher/knowledgeable other, materials), and mechanisms for developing/acquiring understanding (sample codes include: inventing strategies, interacting with others' thinking, use/application, making connections, exposure to multiple strategies, "seeing"/visualizing, persistence/effort, being told/shown, "it just clicks"). The codes were derived from multiple passes through the data and our knowledge of the literature on mathematics understanding (e.g., Carpenter & Lehrer, 1999). After coding each teacher, we looked for patterns within and across teachers, and generated portraits of each teacher's conceptions about mathematics understanding.

Results and Discussion

We first present findings related to the separate analyses of knowledge and identity interviews. We then consider how different combinations of knowledge elements and identity experiences result in different portraits of teachers' thinking about what it means to understand mathematics, and present two cases illustrating these findings.

Table 1. Selected Items From Knowledge Interview

(Item order) type	Item
(2) Multidigit ÷: Novel invented strategy with misconception	*Lucia was solving 144 ÷ 8. She said, "I know, I can just split it in half. So I will keep dividing by 2. I need to do that 4 times, since 2+2+2+2 is 8. The answer is 9." As she talked, she wrote this* (indicate written strategy). *How would you respond to Lucia?* 144 ÷2 72 ÷2 36 ÷2 18 ÷2 $\boxed{9}$
(3) Multidigit X: Common invented strategy	*You have given your class a problem that involves figuring 24 groups of 32. James is the first child to share his strategy with the class. He figured 24 groups of 32 using this strategy:* 32 +32 = 64 32, 32, 32, 32, +32 = 160 160, 160, 160, +160 = 640 +32 = 672 +32 = 704 +64 answer → 768 *What do you learn about James through this strategy? As James explains his strategy to the class, what questions would you want to ask James or the other children about this strategy? What kind of strategy would you want a child to share next?*
(5) Multidigit X: Common misconception	*You have given your students the problem 27 X 42 to solve. Julia says to you, "Simple. 20 X 40 is 800. 7 X 2 is 14. So it's 814" You know this strategy is incorrect. How do you respond to Julia? When it is time for your students to share and discuss strategies, would you have Julia share her strategy with other students? Why or why not?* (Strategy from Ambrose, Baek, & Carpenter, in press.) Show card with: 20 X 40 = 800 + 7 X 2 = 14 814

Teachers' Knowledge

Every teacher but one, Ms. Ibenez, was able to generate at least two invented strategies (i.e., nonstandard strategies using base-ten concepts and algebraic principles) beyond repeated addition for double-digit multiplication.

All teachers found the novel division strategy with a misconception (Table 1, item 2) difficult to understand, and only one teacher, Ms. Carranza, identified the nature of Lucia's misconception as involving an additive decomposition of the divisor instead of a multiplicative one. There were differences, however, between the *Investigations* and CGI teachers in their hypothesized responses to Lucia. With the exception of Ms. Ives, *Investigations* teachers interpreted the division in quotative terms (the number of groups of 8), instead of partitive terms (the number in 8 groups), on which Lucia's strategy was based. For example, Ms. Ingold said Lucia should be thinking in terms of how many eights are in 144. Two teachers believed operating on the partial quotients (i.e., 72, 36, 18) could not lead to a correct strategy because "this part [half] is still left here, and you're only breaking up half" each time. Three out of four of the *Investigations* teachers reported they would redirect the student to use a strategy discussed in class. Ms. Ives, however, thought the strategy was "clever," and noted that the first two steps of repeated halving created four groups (but did not follow through on this reasoning to explain how eight groups could be created). In contrast, three of the four CGI teachers reported they had seen unusual invented strategies before, and all four recognized their confusion over this one, but wanted to explore it with Lucia rather than redirect her.

All teachers understood and could explain the common invented strategy for multidigit multiplication (Table 1, item 3). They differed in their knowledge of its placement in a developmental progression, and in their interpretations of what it signified about James's understanding. Ms. Immendorf and Ms. Ibenez believed James seemed confused; Ms. Ibenez noted that "sometimes he doubts himself and sometimes he's not sure" because he used smaller "clumps" when he could have used larger ones; she wanted to show him a more efficient strategy. In contrast, Ms. Cabrera and Ms. Cortez believed James's strategy was relatively sophisticated. Both teachers described less advanced strategies, such as repeated addition or using base-ten blocks, and more advanced strategies, such as a partial products approach; and explained how James's strategy could be part of a developmental progression. Ms. Carranza believed that, because of the different-sized chunks, the strategy was complicated to explain; James would have trouble explaining it to the class, and other children would have trouble understanding it.

Teachers differed in their understanding of Julia's strategy for multiplying two-digit numbers using a common misconception (erroneous generalization from two-digit addition where one operates on the tens and ones separately, and combines the results; Table 1, item 5). Three out of four *Investigations* teachers did not think it was possible to multiply the tens and ones separately as viable first steps in a strategy. Ms. Ingold reported, "I would tell her that she's off to a good start, but that she can't break down both numbers, because that changes the meaning of the problem." Three out of four CGI teachers understood the partial nature of Julia's strategy, and reported they would ask her for a second strategy, or have her present her strategy to the class as a way to prompt the four partial products needed for the strategy to work (the fourth teacher was not given the item because she taught lower grades). Ms. Cabrera in particular expressed strong feelings about not showing Julia a correct strategy, because "it shouldn't have to be the way that I do it or the way that I was taught."

In summary, teachers' knowledge of multidigit multiplication generally exceeded that of U.S. teachers in prior studies (NCRTL, 1992; Ma, 1999). All teachers understood multidigit multiplication strategies other than the standard algorithm, and at least half the teachers understood these strategies at a deep level—that is, they could generate novel strategies, place strategies in a developmental progression, and/or make connections between different strategies. Some teachers' understanding of multidigit multiplication appeared limited to a few alternative strategies (such as the clusters approach in *Investigations*), which they did not readily generalize or connect to other versions (such as Julia's misconception). Other differences had to do with their hypothesized pedagogical responses to the students in the scenarios; we return to these themes later.

Teacher Identity

In earlier research, Drake (under review) found teachers' math stories could be categorized based on *early (pre-adulthood) experiences* and *current perceptions*. Teachers' *early experiences* were either 1) negative, with very few recollections of positive events and/or at least one highly specific memory of failure in mathematics, or 2) mixed positive and negative, with positive memories as prevalent or more so than negative ones. In a sample of 20 teachers in their first year of implementing a reform-oriented curriculum, half of the stories were classified as negative, while half were mixed positive and negative (Drake, under review). Similarly, *current perceptions* were categorized as 1) positive about self as a learner of both mathematics content and pedagogy, 2) positive about self as learner only of pedagogy, or 3) negative about self as a learner of content. Almost half of the teachers, 9 out of 20, were in the first category, another 9 out of 20 were in the second category, and 2 out of the 20 were in the third category.

In the current study, all 8 of the teachers were positive about their current identities as learners of mathematics content *and* pedagogy. When discussing key events in their adult experiences as learners and teachers of mathematics, all 8 of these teachers described themselves as learning specific mathematics content as well as learning more generic pedagogical strategies. This consistent focus on mathematics content, in particular, across all the teachers is a stark contrast from earlier research on teachers new to reform, of whom just under half had positive identities as learners of mathematics content. However, this group of teachers experienced with reform was similar to the earlier group of reform novices in that 3 of the 8 teachers (Ms. Immendorf, Ms. Ibenez, and Ms. Carranza) described primarily negative early experiences with mathematics, including at least one instance of failure, while the rest of the teachers described mixed early experiences. For all 8 teachers, their experiences with *Investigations* or CGI figured prominently in their descriptions of their high points and turning points.

Teachers' high-point and/or turning-point experiences were expressed through one of two kinds of story frames. One story frame focused on the excitement of understanding a piece of mathematics for the first time. These experiences usually came in adulthood, after prolonged experience of a different kind of mathematics (as non-sensical, rule-based, failure inducing). The setting usually involved knowledgeable others using concrete materials or multiple representations to express mathematics concepts and strategies. There is a sense of surprise at the possibility of understanding and newfound power to make and see mathematical connections. The second story frame involved teachers' sense of agency in effecting change in students' or colleagues' understanding of mathematics or self-perceptions as learners/teachers of mathematics. There is an emphasis on the power of helping others realize their inherent mathematical potential.

The low points teachers described in their stories generally occurred early, before or during college. While all of these low points depict scenes of struggle, frustration, and often failure, they, like the high points and turning points, were framed in one of two ways. In some stories, teachers' struggles and frustrations were similar to mental blocks; there was no clear reason for it, it just happened. Teachers who framed their low points in this way made blanket statements like, "I just never really understood" to describe entire experiences. Other teachers, however, had developed clear and specific reasons for their low points, often attributing them to the failure of knowledgeable others (teachers, parents, the school system) to provide the kind or level of understanding the teachers sought. These teachers said things like, "I wanted to know why things worked in math the way that they did. And somehow the teachers just never could quite explain it." There is a sense in these recollections that teachers have developed explanations for them in order to avoid repeating these failures with their own students.

Influence of Knowledge and Identity on Teachers' Thinking About Mathematics Understanding

General Findings

Regardless of early mathematics learning experiences, CGI teachers expressed stronger beliefs in children's fundamental ability to create mathematics. Ms. Cabrera's case, described below, illustrates a clear version of this belief. Teachers with the strongest beliefs reported they would almost always refrain from showing struggling students how to solve problems, and help such students rely, instead, on inner resources, such as informal knowledge. *Investigations* teachers, on the other hand, tended towards stronger beliefs that students can make sense of mathematics and that the understanding required for this sense making needs to be provided by teachers—especially for students who may be struggling. Understanding is cultivated from without, rather than within. Teachers from both programs emphasized sense making and connections as critical to learning.

Teachers' high point and/or turning point experiences, however, were closely linked to these conceptions about how to foster understanding in students. All eight teachers provided responses to the student strategy scenarios that reflected their stories of how they had increased their own understanding. *Investigations* teachers were more likely to focus on children's use of conceptually transparent strategies presented by the teacher, whereas CGI teachers tended to emphasize children's invention of strategies based on prior knowledge as ways to develop understanding. A few teachers, such as Ms. Ives and Ms. Carrera, expressed elements of both views. Teachers' responses to James's common invented strategy (described above and in the cases below) highlight these differences.

Finally, early negative experiences with mathematics appeared to exert a pervasive influence on teachers' current conceptions about the role of affect in understanding. The three teachers who reported consistently negative early experiences were more sensitive than the others to potential sources of confusion for students and expressed a stronger conviction in developing confidence in their students as a *prerequisite* to understanding. Other teachers were more likely to view students' explaining, and inventing as routes to mathematical understanding *and* confidence.

Cases

In this section we present cases of one teacher from each program. The purpose is not to compare the two programs, but to examine how different combinations of knowledge and identity resources influence teachers' thinking about what it means to understand mathematics and how these differences may play out in their construction of reform-oriented practices.

Ms. Immendorf

Ms. Immendorf has taught for fewer than five years, and for all of those years has taught *Investigations*. She has participated regularly in district-sponsored workshops and credits her team at school as having the most positive influence on her perspective on mathematics. Although she had some strong negative experiences of failure in mathematics, she says now "I just love math. If I could teach it all day, I would."

Ms. Immendorf believes the teacher is a primary source of understanding for students. Her role is to provide students with alternative strategies, correct mistaken strategies by showing other nonstandard strategies, and find out the origins of students' ideas. She does not, in general, encourage students to invent their own strategies, but if they do, she accepts it. Understanding develops primarily through awareness and use of multiple strategies and making connections, and to a lesser extent, following accepted conventions.

This set of conceptions about the nature of mathematical understanding appears to have been influenced by specific knowledge frameworks and identity experiences. For example, consider James (Table 1, item 3), the fictional student who used a developmentally predictable invented strategy to multiply two two-digit numbers. Ms. Immendorf noted that since James added 32 plus 32, he could have added that sum 12 times (i.e., use 64 X 12 to figure 32 X 24). But without this kind of obvious plan, she worried he was floundering. Next steps for him would include making multiple towers (e.g., 32, 64, 96, 128…) so he could skip count and look for patterns (e.g., counting by twos in the ones place). Instead of building on her insight about what James might have done, she opted for an approach she knew from *Investigations* as a precursor activity to multidigit multiplication. Her knowledge of the development of multiplication was founded on the conceptual organization presented in *Investigations,* rather than a framework for the progressive refinement of student-invented strategies.

Further, Ms. Immendorf brought to bear identity themes involving struggle in her speculation that James may not entirely understand his strategy: "but then the fact that he just keeps switching around and testing different things out without like a pattern, I would worry that maybe he's not quite sure what he's doing with it." This interpretation of the significance of James's strategy appears to rest on two formative ideas: that confidence is required to invent new strategies and children are easily confused. Ms. Immendorf expressed these ideas in her Identity Interview in several key events. She reported as a low point the time she "failed miserably" a college entrance exam for Algebra with no inkling beforehand she would not succeed. This blow to her confidence in herself and the system (she wondered why she had not been adequately prepared) took a great deal of effort to overcome. A high point for her was building the confidence of a boy who, at first, would cower under his desk rather than attempt a mathematics problem. But by the end of the year, "he would try different strategies. Like he wouldn't just take the paper and just totally throw it and get up under his desk."

Ms. Cabrera

Ms. Cabrera has taught for about 20 years, and has spent the last three years in an upper-elementary bilingual classroom. She has been involved in CGI for six years and considers it the biggest "eye-opener" in terms of positive influences on her perspectives on mathematics.

Ms. Cabrera believes understanding originates in the self, and building identities as mathematically competent is a necessary component to understanding. The teacher's role in building understanding is to highlight students' competence, set norms for the discussion of ideas, and encourage students to take risks. Mathematics consists of, among other things, "figuring things out on your own." Understanding is evidenced by inventing strategies, having reasons, making connections, and using multiple strategies.

As with Ms. Immendorf, these conceptions appear to be the result of a combination of knowledge and identity resources. Ms. Cabrera's hypothesized response to James's strategy was influenced in particular by her knowledge of a developmental progression of understanding multidigit multiplication and a strong belief in the validity of child-invented strategies. When presented with the strategy, she reported she had seen strategies like it many times, and noted it was more advanced than strategies involving pictures or base-ten materials. She wanted to question him about why he switched from chunking two 32s per group to chunking five 32s per group, and anticipated it would have something to do with being able to figure 20 32s more efficiently using chunks of 5. Rather than seeing confusion in James's actions, Ms. Cabrera saw an insight he may have had. She would discuss with students the kinds of connections they saw between James's strategy and other more and less sophisticated strategies.

Ms. Cabrera's strong belief in what her students know and can do was expressed in different forms throughout her Identity Interview. One of her earliest memories of mathematics involved a teacher who allowed her to win at the game of Bingo by pretending to choose her number. This act of caring made her feel like she was "successful" and "that somebody was there looking out for me." The theme framing this memory shows up repeatedly in Ms. Cabrera's thinking. With her involvement in CGI, she has also learned to see mathematics as an integral part of who her students are. Thus, knowledge of children's mathematical thinking has facilitated a new kind of caring for her students, for their mathematical identities.

When asked what her high point was, Ms. Cabrera said it involved helping students who do not see themselves as mathematical thinkers to see themselves in that light. She described an incident with such a student, Camila, who had used a strategy to solve 3/5 of 45 in way that "blew me away because I had never even thought of figuring a problem that way." Camila partitioned 45 into 15 groups of 3, and then took 2 from each 3, to get 30. She was having trouble verbalizing this strategy to the class, so Ms. Cabrera—who did not understand the strategy at first herself—questioned Camila until she understood it, and helped Camila explain it to the class. She wanted the class to hear this strategy because she wanted Camila to be successful and be viewed that way by her classmates. The context of this success, in contrast to the earlier experience winning Bingo, involved substantive mathematical thinking. The story frame is the same but the content is different, because knowledge Ms. Cabrera had gained allowed her to anticipate and appreciate children's invented strategies.

Conclusion

All eight teachers in this study were experienced at teaching mathematics in ways that emphasized sense-making, connections, and communication. All felt positively about themselves as teachers and learners, even though uniformly positive identification with mathematics had not always been the case for them. All reported gaining knowledge as a result of implementing innovative mathematics approaches, and cited experiences illustrating the meaning this new knowledge held for their mathematical identities.

We found differences in the kind of knowledge teachers brought to bear to decide how to respond to fictional students, and in how teachers used their stories to interpret the significance of these potential pedagogical moves. Neither identity nor knowledge existed along a continuum that could be labeled "traditional" at one end, and "reformed" at the other. Instead teachers' thinking placed differing emphases on what mathematics educators have argued are basic features of understanding: connections, application, generativity, justification, and identity (Carpenter & Lehrer, 1999; Lesh, Post, & Behr, 1987). However, in the absence of knowledge about the genesis of mathematical understanding in children, teachers tended to draw on identity resources to interpret the significance of unusual strategies; we will explore in future work the possibility that these differences are related to the kinds of opportunities to learn offered in each program.

Extrapolating from these findings, we propose that reflection on identity narratives may serve as a compelling change mechanism for teachers. It is natural to imagine alternative endings to stories or to retell a story with a different emphasis: stories "invite reconstrual of what might have happened" (Bruner, 1990, p. 53). One might ask Ms. Immendorf, for example, what would have happened had the system not failed her? How might the system have been different? This kind of reflection may help teachers distance themselves from their personal stories in ways that help them see other possibilities, and help them articulate salient influences on learning and identity. Equally important is teachers' knowledge of mathematics and children's thinking. Examining identity and knowledge in combination may result in an even more powerful framework for teacher change by identifying explicit relationships between the identity stories teachers tell and the knowledge that helps teachers act on and transform the critical elements of these stories. The way Ms. Cabrera facilitated Camila's presentation of her unusual invented strategy illustrates the kind of mathematical empowerment that is possible under these circumstances.

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VEHICLE TO CONNECT THEORY, RESEARCH, AND PRACTICE: HOW TEACHER THINKING CHANGES IN DISTRICT-LEVEL LESSON STUDY IN JAPAN

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Lesson study is the major form of professional development in Japan and has attracted attention in the United States in recent years. Lesson study currently conducted in Japan varies in terms of length, structure, and scale. District-level mathematics lesson study gathers teachers from different schools who share an interest in teaching the subject, and the discussion can focus more on particular aspects of teaching mathematics or content issues than that of in-school lesson study. The results of the survey indicate that lesson study provides opportunities for better communication among teachers, researchers, and administrators by presenting concrete classroom teaching examples surrounding particular educational ideas and/or issues, thus minimize the gap among theory, research, and practice. Implications to U.S. schools are discussed.

Purposes

In this paper, we begin with describing what lesson study is and how various forms of lesson study differ from one another in their structures. We then report from our survey study how lesson study works to connect theory, research, and teaching practice in schools and how it can affect teacher thinking and learning to improve mathematics education in Japan. Lesson study is the major form of professional development in Japan, and some U.S. teachers have recently initiated it in their own schools (Germain-McCarthy, 2001; Research for Better Schools, 2000; Stepanek, 2001; Weeks, 2001). With the prospect of lesson study being incorporated in U.S. settings, it is important that U.S. practitioners be informed of the ways in which various forms of lesson study influence teacher thinking and practices in Japan so that we can better explore its potential impact on U.S. teachers. The idea of lesson study was introduced only around a small-scale and school-based study so far in the United States, and the effectiveness of lesson study has been discussed from data primarily based on observations. To date, no research has described how various forms of lesson study are organized and conducted in Japan, or what the experience means to Japanese teachers. It is our attempt to increase the international communication by reporting these aspects of lesson study, and thus, contribute to the collaborative effort to define a new model of lesson study in the United States.

Perspectives

What Is Lesson Study?

Lesson study is a form of professional development commonly and widely conducted in Japan (Fernandez, Chokshi, Cannon, & Yoshida, 2001; Lewis, 2000;

Lewis & Tsuchida, 1998; Shimahara, 1999; Stigler & Hiebert, 1999; Yoshida, 1999). In lesson study, teachers work collaboratively to: 1) formulate long-term goals for student learning and development, 2) plan, conduct, and observe a 'research lesson' designed to bring these long-term goals to life as well as to teach a particular academic content, 3) carefully observe student learning, engagement, and behavior during the lesson, and 4) discuss and revise the lesson and the approach to instruction based on these observations. The research lesson is taught in a regular classroom with students, and participants observe as the lesson unfolds in the actual teaching-learning context. Debriefing following the lesson is developed around the student learning data collected during the observation. Through the process, teachers are given opportunities to reflect on their teaching and student learning.

Lesson study embodies many features that researchers have noted effective in changing teacher practice, such as using concrete practical materials to focus on meaningful problems, taking explicit account of the contexts of teaching and the experiences of teachers, and providing on-site support within a collegial network. It also avoids many features noted as shortcomings of typical professional development; e.g., that is short-term, fragmented, and externally administered (Cuban, 1990; Firestone, 1996; Huberman & Gusky, 1994; Kennedy, 1999; Little, 1993; Miller & Lord, 1994; Pennel & Firestone, 1996). Furthermore, researchers have argued that lesson study is the critical system feature that has enabled Japanese elementary teachers to improve classroom instruction in mathematics and science in recent decades (Linn, Lewis, Tsuchida, & Songer, 2000; Lewis & Tsuchida, 1998; Takahashi, 2000; Stigler & Hiebert, 1999; Yoshida, 1999).

Lesson study has become visible in state, national, and international conferences, open houses, high-profile policy reports, and special journal issues in recent years in the United States. Moreover, some school districts in the United States have attempted to use it to change their practice and impact student learning (Coeyman, 2000; Council for Basic Education, 2000; Germain-McCarthy, 2001; Research for Better Schools, 2000; Stepanek, 2001; Weeks, 2001).

Different Types of Lesson Study

The type of lesson study known and tried in the United States so far is small-scaled in-school lesson study. However, lesson study that is currently conducted in Japan varies in terms of length, structure, and scale. Besides the type of lesson study that is conducted within a single school, school districts organize lesson study for groups of teachers who share similar professional interests (e.g., subject matter) or who are at the same professional stage. Thus, district-level lesson study tends to focus on particular issues and topics to connect teachers from different schools. At other times, national-level research organizations conduct lesson study that gathers hundreds of teachers nation-wide.

At district- and national-level lesson study focused on mathematics, the teachers who participate are particularly interested in mathematics. They bring their knowledge of and experience with teaching mathematics to the study, and the level of the discussion is likely to be higher than that of the in-school lesson study. The main purpose of in-school lesson study can be to present opportunities for teachers whose main interest may or may not be mathematics to think about their teaching of the subject in general. At a district-level lesson study, the discussion can focus more on particular aspects of teaching mathematics or content issues. When it comes to national-level lesson study, the level of discussion may become very high. However, because of the large number of participants, they often cannot fit in a single classroom, and the research lesson presented may be considered unrealistic. For that reason, district-level lesson study maintains its unique middle-of-the-range characteristics.

Lesson study also plays an important role in improving curricula, textbooks, and teaching and learning materials in Japan. Most Japanese mathematics textbook publishers employ authors who are also classroom teachers involved in lesson study, and their materials are in some manner examined through the process of lesson study. The same situations can be found in the process of developing teaching and learning materials such as manipulatives and teachers' reference manuals. Japanese textbook publishers often include teachers' ideas and lesson plans that have been examined and discussed at district-level lesson study in particular because they are most likely to exhibit high mathematics content in a realistic manner.

District-level lesson study is often conducted to explore a new educational idea or a goal that is currently being explored, and a number of teachers across different schools come to participate for this type of lesson study. Each time the Course of Study was revised in Japan during the last few decades, lesson study across the country was conducted that particularly focused on exploring and demonstrating the new educational issues. Teachers collaboratively worked to illustrate how new ideas may be used in classrooms through their teaching, make the ambiguous aspects of the ideas come to surface, and set a stage for discussion for better understanding of the ideas in the future. The mechanisms of this type of lesson study will be discussed later.

Methods

Survey questionnaires (Attachment) were sent to selected Japanese elementary school teachers who had played main roles in organizing district-level mathematics lesson study and are considered to be the leaders in the field. They lived in metropolitan Tokyo and its surrounding area and had thorough knowledge of how lesson study is conducted at the district-level. The total of 125 responses were collected.

For the 125 teachers, the median number of years they had taught was 16. They had participated in mathematics lesson study an average of 5 times a year, and taught research lessons in district-level lesson study an average of 10 times. They also reported that they thought approximately 35% of all Japanese teachers had taught a mathematics research lesson at least once in their professional career, thus the average of 10 lessons taught indicated that the teachers who responded to the survey questionnaires were especially experienced in lesson study.

The questionnaires were designed to draw out information from the teachers regarding their experiences with district-level lesson study. They were specifically asked the (1) frequency and duration of district-level lesson study, (2) cooperative mechanisms between schools and districts in organizing a lesson study, (3) advantages district-level lesson study provides for teachers, and (4) problems or factors that should be improved for successful district-level lesson study. Multiple-choice/fill-the-blank questions were used to draw an overall picture, and open-ended questions then followed to gather details in teachers' own words.

After all the survey forms were collected, data were organized and compiled in a matrix to visualize the possible common categories. The data from multiple-choice questions were examined to draw a concrete image of the ways in which district-level lesson study is conducted in Japan. The data from open-ended questions were reviewed multiple times by two researchers to identify underlying threads that connected the responses from different teachers and different questions. The responses were then coded to indicate the teachers' beliefs, knowledge, and practice.

Results and Discussion

How Is District-Level Lesson Study Organized and Conducted?

Teachers responded that 98% of all school districts conduct mathematics lesson study regularly as a part of district-level professional development sessions. For the questions that asked the teachers to identify other activities likely to be included in those professional development sessions, 71% reported lectures by university professors and other authorities, 65% reported sessions for teachers to share each other's research, 40% reported model lessons taught by a known practitioner, and 28% reported workshops. The majority of the teachers (66%) reported that the professional development sessions in those districts were held approximately once or twice every trimester (Japanese school trimester range between 3 and 4 months).

Approximately half of the teachers (48%) responded that district-level professional development sessions are typically held in the afternoon of a regular school day. On those professional development days, schools in the district maintain the regular school day except for the school to be used for the lesson study. After all the students go home, teachers gather at the school where students remain for the extra class period for the research lesson. In other cases, the whole district cancels one school day and devotes it to professional development day (37%). Only the children in the classroom used for the research lesson attend schools for the lesson period.

In most cases, school districts or schools financially support district-level lesson study. Eighty-six percent (86%) of the teachers reported that districts cover the entire expenses (e.g., transportation), and 13% of the teachers reported they do so partially.

For lesson planning and teaching, 64% of the teachers reported that classroom teachers typically lead the district-level lesson study. It is a collaborative process, and 94% of the teachers said lessons are planned by a group of teachers together. They typically prepare lesson plans outside of their instructional time (84%), while on some occasions, schools make extra planning time within the regular school hours (4%) or the teachers do so on their own (6%). When a group of teachers work together to write a lesson plan and prepare for a lesson study, the teachers reported that they meet approximately 3 to 4 times (76%), more than 5 times (14%), or 1 to 2 times (10%) before the actual lesson study day.

Teacher Changes

The teachers felt strongly that lesson study helped them improve their teaching. Almost all (98%) the teachers either agreed or strongly agreed that they had grown professionally by participating in lesson study and observing research lesson. Over ninety percent (91%) of the teachers also agreed or strongly agreed that lesson study is the most effective form of professional development. In comparing different forms of lesson study by ranking, 31% of the teachers reported that lesson study organized by an educational organization is the most effective, followed by district-level lesson study (26%), and lesson study held within a school (21%).

Many teachers described how seeing research lessons completely changed their beliefs about teaching. They described the experiences in their own words (translated from Japanese) as follows:

> When I was young, I used to have the attitude that teachers needed to teach everything, but through lesson study, I came to think that teachers are the learning "assistants" for students and it is important to carefully prepare lessons so that students can investigate and solve problems on their own. (Number of years taught 9, average number of times participated in lesson study per year, 3)

> I used to think the lessons that are aligned closely to textbooks are the best ones for students. After seeing research lessons where teachers tried to make a textbook lesson closer to students, I started to think about and search for good teaching material for my students as I plan mathematics lessons. (Number of years taught 12, average number of times participated in lesson study per year 2)

> When I was young, I thought teaching was to make a point and explain students so that they can understand better. So, to me, back then, it was critical to find the 'technique' to do that effectively. After seeing the investigative open-ended lesson, I have come to think that learning is not what I had thought. (Number of years taught 17, average number of times participated in lesson study per year, 6)

Another teacher noted about his experiences as follows:

> I don't see my teaching as a complete "instructional method" any longer. I now see the methods that may possibly become a part of good instruction. (Number of years taught 13, average number of times participated in lesson study per year 12)

Lesson study helps Japanese teachers experience good teaching practice when the ideas may not appeal to them theoretically. Seeing the successful examples helps teachers understand the benefit of the different and/or unfamiliar practices, and they come to see the good lesson in terms of children's learning across different theories and approaches.

Connecting Theory and Practice

The teachers indicated that they saw lesson study as an important and effective link between educational theories and their classroom practice. Many teachers responded that seeing new ideas demonstrated in research lessons and participating in discussion of the actual practice strengthened their understanding of such ideas. They described the experiences as follows:

> (To US teachers,) I think it is important to actually teach lessons and get feedback from others, and not just reading theories in books or attending lectures. (Number of years taught 1, average number of times participated in lesson study per year 1)

> It is hard to incorporate new instructional ideas and materials in classrooms unless we see how they actually look. In lesson study, we see what goes on in the lesson more objectively, and that helps us understand the important ideas without being overly concerned about other issues in own classrooms. (Number of years taught 22, average number of times participated in lesson study per year 20)

On more practical side, many teachers reported that seeing actual materials used in the research lesson classroom helped them understand the benefit of the materials in the particular lesson to help children's learning.

> For teaching the unit on large numbers for second grade, I saw how one-yen coins (pennies) were used effectively, and the small trays that accompanied the activity. (Number of years taught 1, average number of times participated in lesson study per year 1)

> When I see an effective use of manipulatives, a bulletin board, or a good way the teacher presents an investigative situation using questions, I take the idea to my classroom and use it. (Number of years taught 10, average number of times participated in lesson study per year 5)

> I learned what to prepare for the open-ended investigative problem solving (hint card, etc.). (Number of years taught 11, average number of times participated in lesson study per year 4)

Lesson study provides live examples for teachers to see and experience how educational ideas are played out in actual classrooms. Seeing the actual practice makes it easier for the teachers to think and adapt the ideas. Even for novice teachers, they reported that participating in discussions with more experienced peers using concrete classroom data helped them understand the importance of the new ideas and apply them practically. Lesson study plays a critical role in connecting theory and practice in Japanese education, and may potentially do the same for U.S. schools.

Implication and Conclusion

With the absence of national curriculum and shared vision, U.S. teachers are sometimes left alone to make everyday teaching decisions in their own classrooms. When the larger culture values autonomy and independence, it makes it even harder for them to communicate with one another. This not only makes their work more difficult but also keeps them from learning from one another to improve the learning chances of their students. Well-crafted educational theories may be understood partially and taken by pieces by different teachers and lose their original shape when communication is limited. The Japanese teachers who responded to the survey indicated that this is not a unique situation in the United States, however, and Japanese teachers also struggle to make sense of new ideas and help them come to life in their classrooms. Lesson study provides opportunities for better communication among teachers, researchers, and administrators in Japan by presenting concrete classroom teaching examples surrounding particular educational ideas and/or issues. Discussions focused on the particular classroom examples or data helps people to communicate better and understand one another. Lesson study will potentially help create a place and a reason for all people who work to improve education of our children to gather and discuss their ideas within the actual U.S. classrooms. The commonly seen disconnect between theory, research, and practice may be minimized just by having such a common place for all.

As we continue our collaborative effort to define U.S. lesson study model, future studies should explore the effect of lesson study on teachers and students by going beyond their self-reports and observations. The studies are needed that address the particular aspects of teacher learning and teacher change as a result of their participation in lesson study, and the influence of such changes on students learning in the classroom. Focused case study of a particular teacher, video-tape analysis of the changes in teaching practice, and/or careful examination of students' understanding of mathematics before and after the teacher participates in lesson study may provide us different pictures of its potential effectiveness in U.S. mathematics classrooms.

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Attachment on Following Page

Questions about mathematics lesson study

1. At your district, is mathematics lesson study conducted regularly? Yes / No
2. For the following questions, please take your overall experience with lesson study in consideration (not only of your current school district)
   (a) In general, how often do you think district-level mathematics professional development sessions are held? Once a month / Once or twice a trimester / Once or twice a year / Other
   (b) In general, how often do you think lesson study is conducted as a part of the district-level mathematics professional development? Once a month / Once or twice a trimester / Once or twice a year / Other
   (c) What other activities do you think are part of the district-level mathematics study session? (Please choose as many as you wish) A lecture by a university professor / A model lesson taught by a known practitioner / A session for teachers to share each other's research / Workshops / Other
   (d) How do you think teachers make time to attend district-level mathematics lesson study? The whole district cancels a school day and keeps only the children in the classroom used for the lesson study / The whole district uses the after-school hours and keeps only the children in the classroom that is to be used for the lesson study / The whole district conducts lesson study during the regular school time and teachers assign independent work for students during their absence / Other
   (e) Who is financially responsible for teachers' expenses to attend district-level mathematics lesson study? (e.g., transportation) School district or school covers the whole expense / School district or school covers the partial expenses / Participating teachers pay on their own / Other
   (f) In general, who leads the district-level mathematics lesson study? School district or instructional department of the district / School administrators / Classroom teacher who belongs to the research committee / University professor or researcher (external) / Other
   (g) In general, who makes the lesson plan for a district-level mathematics lesson study? School district or instructional department of the district / University professor or researcher (external) / The teacher who teaches the lesson / A group of teachers including the teacher who teaches the lesson / A group of teachers not including the teacher who teaches the lesson / Other
   (h) How do teachers make time to prepare for the district-level mathematics lesson study? (e.g., writing a lesson plan) For the teacher who teaches the lesson or the others who help the lesson planning, schools make extra planning time for them within the regular school hours / The teachers make time within their instructional time on their own / The teachers plan outside of their instructional time / Other

COURSE CONCEPTUALIZING THAT AIMS TO PSYCHOLOGIZE THE CURRICULUM IN CALCULUS

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A century after Dewey wrote *The Child and the Curriculum*, we still have an inadequate understanding of what it takes to psychologize the study of mathematics. In this report, I present a case study of an Advanced Placement Calculus teacher who aims to psychologize the study of calculus for his students. I highlight the role of course conceptualizing—reasoning about and choosing among course-level alternatives—in his efforts, focusing on choices he makes in three categories of alternatives: describing what calculus is about, organizing the substantive content of the course to support students' learning, and representing knowing in mathematics. The teacher's strategic choices in each of these categories support his efforts to psychologize calculus. Using this case as an illustration, I argue that conceptualizing as a course-level practice can play an important role in psychologizing the curriculum.

In *The Child and the Curriculum*, Dewey (1902/1990) argues that we must move beyond an artificial polarization between child-centered and subject-centered instruction. Rather, he calls for *psychologizing the curriculum*: learning to see important elements of the subject matter in students' current understandings and interests and figuring out how to use the latter to support students' learning. Dewey claims that we must get rid of the prejudicial notion that there is some gap in kind (as distinct from degree) between the child's experience and the various forms of subject-matter that make up the course of study. From the side of the child, it is a question of seeing how his experience already contains within itself elements—facts and truths—of just the same sort as those entering into the formal study; and what is of more importance, of how it contains within itself the attitudes, the motives, and the interests which have operated in developing and organizing the subject-matter to the place which it now occupies. From the side of the studies, it is a question of interpreting them as outgrowths of forces operating in the child's life, and of discovering the steps that intervene between the child's present experience and their richer maturity. ... Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction. It is continuous reconstruction. (p. 189)

He claims that not only is it possible to see a continuity between students and subject matter, but essential, for it is the interaction between the two that lies at the heart of effective education. Moreover, this need not be a forced interaction. Rather, because studies are a human creation, they can be connected to the interests and experiences of all humans. The teacher is responsible for facilitating this interaction:

What concerns him, as teacher, is the ways in which that subject may become a part of experience; what there is in the child's present that is usable with reference to it;

how such elements are to be used; how his own knowledge of the subject matter may assist in interpreting the child's needs and doings, and determine the medium in which the child should be placed in order that his growth may be properly directed. (p. 201)

Dewey goes on to discuss three "evils" that result when "the material is not translated into life-terms, but is directly offered as a substitute for, or an external annex to, the child's present life" (p. 202): the purpose for which symbols are initially created—the attempt on someone's part to symbolize an idea that has genuine meaning for that person—is lost and the symbols become purely formal; the felt need and therefore much of the motivation for ideas is lost; and the organizing function of the discipline—the extent to which the disciplinary structure facilitates sense-making and problem-solving that would otherwise be difficult—is no longer apparent. These concerns are particularly relevant in mathematics, given the importance of symbols and logical foundations within the discipline. Dewey points out that because teachers must always deal with the psychological, when they use a logical approach, they often end up motivating students in inauthentic ways, e.g., by habituating them to routine, penalizing them for poor work, or sugar-coating the material with fun activities.

I value Dewey's insights in identifying the construct of psychologizing the curriculum, highlighting its importance, and elaborating its meaning. Still, having a goal of psychologizing the curriculum doesn't tell one how to go about it. Few detailed images exist of teachers' explicit attempts to psychologize the curriculum, especially in secondary mathematics (see Chazan, 2000, however, for an example). Moreover, as soon as one turns to practice, questions arise about the basis on which one claims that the curriculum has been psychologized. Dewey's discussion focuses on the teacher, e.g., the need for a teacher to interpret students' experience with respect to the subject's "facts and truths" and "the attitudes, the motives, and the interests" central in its development. The role of students is more ambiguous. Students' experiences, attitudes, and interests are clearly central to psychologizing; however, it is unclear whether students must experience efforts to psychologize the curriculum as motivating for such efforts to be seen as effective, especially for students already used to routine, grades as external motivators, or sugar-coating. In addition, Dewey writes about the child (singular), whereas teachers are faced with trying to psychologize the curriculum with a class of students whose experiences, attitudes, and interests often differ in significant ways. And while day-to-day teaching is central to psychologizing because each student's "present" continually shifts, curriculum is more than a collection of individual lessons, and therefore one must also find ways to look across a course.

In this paper, I will focus on the teacher's efforts at the course level, in particular, the role of a practice I call *course conceptualizing*: reasoning about and deciding among alternatives that play out at the course level. I will use examples from a case study of an Advanced Placement calculus teacher, Marty Schnepp,[1] to illustrate a relationship between course conceptualizing and psychologizing, arguing that a desire to psychologize the study of calculus is a central consideration in Marty's reasoning as he chooses among or invents alternatives while conceptualizing and that course conceptualizing can play an important role in efforts to psychologize the curriculum.

Course Conceptualizing as a Construct

Course conceptualizing is a practice of some teachers. It involves reasoning about course-level aspects of a course (e.g., what the course is about, how to organize the course content to support students' learning) rather than unit- or lesson-level aspects, with a recognition that alternatives exist. I restrict my use of this construct to courses intended to have some unity of content, whether by branch of mathematics (e.g., calculus, algebra) or theme (e.g., modeling). Course conceptualizing draws on—and can contribute to the development of—a person's *landscape of alternatives* for a given course: the alternatives that person knows about and the ways in which he or she has made sense of them. Different people's landscapes of alternatives may vary in how they categorize alternatives and think about those categories, which alternatives they know of within a given category, and the level of detail in which a given alternative is known. For example, many people think about the organization of content in terms of sequencing, but Lampert (2001) prefers to think about this in terms of a terrain. Some people know of alternative sequences for a given course, but the alternatives known by one person may be different from those known by another, and a given person might know of one sequence in detail and others more sketchily. Chazan (1999) identifies another category of alternatives, a course's "approach": the fundamental objects of study for the course, the questions one asks about them, and the other important objects of study that arise in the context of the fundamental objects and questions. But, others think about the content of a course in other ways (e.g., scope).

Teachers engage in course conceptualizing for a variety of reasons, for example, to make sense of new curricular materials, to better understand the differences between two alternatives, to figure out an alternative to something unsatisfying at the course level. Although alternatives often come from outside sources (e.g., curricular materials, talking to colleagues), course conceptualizing can involve inventing an alternative of one's own. A common goal of conceptualizing is to develop a *course conceptualization*: a particular constellation of alternatives chosen from the categories one has considered. An important conceptualization is the way in which one has made sense of a course in light of materials one must use; this might or might not be the conceptualization one prefers for that course.

Methods

This investigation is part of a pilot study. I am designing a more comprehensive study of teachers' course conceptualizing and conceptualizations to better elaborate these constructs. My methodology is qualitative. The data consist of two hour-long interviews with Marty about his calculus course, focused on his goals for his students, how he structured the course, and what he valued about these goals and structure; videotapes from a single section of his calculus course, taped daily for one year; audiotapes of presentations Marty made to mathematicians and mathematics educators about his calculus teaching; and copies of Marty's published and unpublished writing. I first searched through the interview transcripts, presentation transcripts, and writing for discussions of choices Marty made about the course and analyzed these to produce a description and interpretation of his calculus course conceptualizing and preferred conceptualization (Sandow, 2002), using my yearlong classroom observation and sample videotapes from different points in the year to better understand Marty's descriptions of his goals and considerations. This analysis was oriented around scope and sequence (Tyler, 1949) and ontological and epistemological content, broadly interpreted (e.g., fundamental objects of study, Chazan, 1999; knowing in mathematics, Lampert, 1990; substantive and syntactic structures, Schwab, 1978), with attention to Marty's ways of considering these (Strauss & Corbin, 1990). I then analyzed Marty's conceptualizing and preferred conceptualization to identify choices that represented efforts to psychologize the curriculum for his students.

Marty's Calculus Course Conceptualizing

An Introduction to Marty

Marty has been teaching for 14 years altogether, 11 of them teaching calculus. For the last 11 years, he has taught at Holt High School (HHS), a Professional Development School (PDS) associated with Michigan State University (MSU). A number of experiences have contributed to Marty's efforts to redesign his calculus course. He has worked as a teacher-researcher for the last five years with the SimCalc project at TERC, piloting the use of "line becomes motion" technology, in which distance-, velocity- and acceleration-versus-time functions can be entered into a computer graphically or symbolically and used to drive the motion of one of two devices connected to the computer: a small wire bicyclist named Eddie who pedals in place and two toy mini-cars that move along parallel straight tracks (Schnepp & Nemirovsky, 2001). Two courses he took while completing his Master of Arts in Curriculum and Teaching at MSU were also particularly influential, one on philosophies of mathematics and their connections to teaching, with Bill Rosenthal, the other on post-modern philosophies of education, with Cleo Cherryholmes. In addition, a broader PDS initiative at HHS supported released time for teachers to co-teach (Chazan, Ben-Chaim, Gormas, Schnepp, Lehman, Bethell, et al., 1998) and contributed to a restructuring of the teachers' workweek to give them a three-hour block of time on Wednesday mornings in which the teachers could work collaboratively or independently on projects or other work supporting their professional development. (Chazan, Lehman, & Bethell, 2002, describes the HHS mathematics department's PDS work and its impact in more detail.) The PDS initiative also created significant opportunities for MSU mathematics education faculty and graduate students to work with members of the HHS mathematics department and carry out research in this context (see, e.g., Chazan, 2000). It was as a PDS research assistant that I first came to know Marty and his calculus teaching.

Developing a Course Conceptualization Aimed at Psychologizing Calculus

Although at the time of this study Marty hadn't read *The Child and the Curriculum*, he talks about his goals in ways that are quite consistent with Dewey's argument. He writes, "I want to hear what my students think, in some detail, and then figure out how to connect what they think to what others have developed" (Chazan & Schnepp, 2002, p. 187). In this section, I present three examples of ways in which Marty's desire to psychologize calculus for his students influences and is supported by his course conceptualizing: describing what calculus is all about, organizing the substantive content to support student learning, and representing knowing in mathematics.

Describing What Calculus Is All About: Choosing the Fundamental Objects of Study for the Course

Marty writes that he wants his students "to appreciate that Calculus is about something" (Chazan & Schnepp, 2002, p. 179). He identifies that "something" as rate, accumulation, and the relationship between them:

> We started out in the fall with looking at examples of something that the kids would have some kind of experience with, where something was changing at a varying rate. And just trying to figure out how much of it had accumulated, 'cause ... they've got lots of experience computing rates and watching rates. ... They're presented with a new idea: what if the rate changes over the course of the time? And that naturally raises questions for them about what a rate is, and whether or not you need a time interval for a rate. And they've been pretty loosely using the term "average rate," and [we explored] what that really means. So we sorted all of that out, and we went into looking at problems where you vary a quantity and you want to know about its rate instead, so going the other direction. ... Then we started looking at the relationship between going back and forth on those. (interview)

Choosing to describe calculus as about rate, accumulation, and the relationship between them makes it possible for Marty to present his students with situations "that the kids would have some kind of experience with" and in which central ideas of calculus are embedded, making it possible for the students to develop solution methods and talk about them before they've been introduced to the formal vocabulary and methods, thereby motivating the latter. To help guarantee that the situations are truly ones that all of the students have experience with, Marty brings physically immediate experiences into the classroom: on the first day of class, he pedals his own bicycle at a varying rate on a stationary trainer in front of the students, and later the class uses the moving TERC devices [(i.e., Eddie and the mini-cars). Although in the quote above, Marty doesn't explicitly contrast his framing of what calculus is all about with alternative choices, he distinguishes between rate/accumulation and differentiation/integration (e.g., "When students discuss integration, we want them to recognize (1) that accumulation always occurs *at a certain rate* and (2) that this rate at any given point *is* the value of the function being integrated," (Schnepp & Nemirovsky, 2001, p. 91)], implying that he sees these as different objects of study.

Marty's choice is relevant to psychologizing in a second way, for as Dewey notes, "the problem of the relation of the child and the curriculum presents itself in this guise: Of what use, educationally speaking, is it to be able to see the end [the subject-matter of the studies] in the beginning [the child's present experience]?" (p. 190). Therefore, teachers must articulate for themselves what the end *is*, which again raises a question of what calculus is all about. Marty identifies a deep understanding of the Fundamental Theorem of Calculus as an important element:

> Although students usually intuitively see adding and subtracting as "undoing" one another, most calculus students do not develop an equivalent sense of why integration "undoes" differentiation and vice versa. ... We want to build the core of calculus education, from day one to the final test, around the development of this insight. (Schnepp & Nemirovsky, 2001, p. 91)

Dewey (1902/1990) goes on to say that this problem is partly one of seeing how a child's "crude beginnings .... may get clarity and gain force" (p. 195), for clarity and force help define studies in that they characterize the elements of studies that make "the net product of past experience ... most available for the future" (p. 199). Some aspects of studies—often referred to as "the big ideas"—clearly provide more insight and power than others. In calculus, the Fundamental Theorem is one of these (hence its name and its role in Leibniz and Newton getting credit for inventing calculus). Thus, although Dewey initially writes about connecting the child's present to the "the facts and truths of studies" as if the facts and truths were all equally worth knowing, studies' big ideas play a special role in psychologizing.

Organizing the substantive content to support student learning.

Describing calculus in terms of rate, accumulation, and their relationship also allows Marty to shift the sequencing of the topics in the course relative to the traditional sequence. Among other things, it allows him to postpone a formal treatment of limits until much later in the course. He notes that starting with limits is problematic because the students are not yet in a position to understand why the limit concept, much less a rigorous limit definition, is needed:

> When I tried starting with limits in calculus, for example, it was completely disconnected for the kids, so that was a drawback. Um, they didn't understand why anybody would be concerned with anything like that. I thought

> that was a pretty legitimate question, so as I started looking back in the history of calculus that, you know, limits evolved out of arguing about how you talk about rate of change at an instant .... And so it seemed to make a lot of sense to me to help make sure the kids understand what kind of problems were being tackled historically that lead them to conversations about limits or ... why different conventions have evolved and just why a given topic exists. (interview)

Instead of starting with limits, the class initially works only with polynomial functions and develops a theory of differentiation and integration for those, using their intuitive understanding of limiting processes. Later in the year, when they work on generalizing their theory to other kinds of functions, more precise notions of limits and continuity are needed and then get developed. Thus, Marty recognizes that there are alternative ways of organizing the course content, and he chooses to organize it such that his treatment of limits is consistent with Dewey's injunction against presenting material as "an external annex to the child's present life" (p. 202) and the concomitant "evil" of losing the felt need and motivation for a concept. Moreover, he recognizes that as they move to more abstract work, there's a risk that students will lose sight of the abstractions ever having been connected to their own experiences. To help the students maintain a connection, the class returns to the moving TERC devices later in the year, reinterpreting the formal theory in terms of their motion.

Marty recognizes that the question of whether particular content is external to students' lives must always be judged in relation to his actual students and the ideas they introduce and are intrigued by. Therefore, although he has a characteristic way of organizing the course content, which he describes in terms of "having a coherent storyline for [the course]" (interview), the organization sometimes plays out differently in different sections, depending on the ideas that students introduce and find intriguing. Marty notes:

> The other major influence on how the course is sequenced is sort of what they choose, or what the students tend to do with the material; you know, that will change different topics, sequences and things, depending on how they start talking about it. You know, if the two-sided [limit] issue had become relevant earlier, back in the fall, we could have done limits then. (interview)

In fact, this happens the following year:

> This year, they got to that point [of working with limits informally], and they just wouldn't let go of it. Um, they didn't know what limit really meant, and so we actually did look at ... limits more formally, where typically that's come second semester. (presentation)

In addition, consistent with Marty's desire to motivate the relationship between integration and differentiation, he pairs several topics in unusual ways: the chain rule for derivatives with u-substitution (representing composition), the product rule and integration by parts (representing products), and implicit differentiation, related rates, and separable differential equations (representing implicit or parameterized relationships).

Representing Knowing in Mathematics: The Nature of Mathematics and Its Practice

In addition to wanting his students to learn about the substantive content of calculus, Marty wants his students to develop certain beliefs about the nature and practice of mathematics, including beliefs about themselves in relation to mathematics. These additional goals are relevant to psychologizing the curriculum, for as Dewey (1902/1990) points out, an important aspect of psychologizing is recognizing how the child's experience "contains within itself the attitudes, the motives, and the interests which have operated in developing and organizing the subject-matter to the place which it now occupies, ... [for this aides in] discovering the steps that intervene between the child's present experience and their [the studies'] richer maturity" (p. 189). These goals are also a part of Marty's course conceptualizing, both in the sense that he recognizes this as an explicit part of the content of his course (even though this is not usually articulated as part of course content) and that other teachers portray a different image of mathematics to their students.

For Marty, one theme of knowing in mathematics is wanting his students to see mathematics as a way of describing the world that then takes on a life of its own, a life in which internal consistency is essential. Because consistency can be difficult to determine oneself, social interaction, especially the comparing of different people's interpretations and understandings of a particular idea, is also an important part of mathematics:

> [I want my students to understand that] people have developed this symbolic system for describing what they seen in the world around them, ... for describing quantities and shape around them. ... And so I want kids to, first of all, think about it as these are just descriptions for things that occur, but then once you create the descriptions, you need to look for internal consistencies and for the conclusions that you can draw using that system as a tool. ... It's checks for consistency, but those checks need to go on from person to person to clarify thinking. ...
>
> What I want kids to think about what mathematics is: ... it's looking at a situation and creating some way of describing it or techniques for dealing with it, and then once you've created those things, let them take on their own life and see what they look like and see how they relate to all the other stuff that you know. (interview)

Consistent with Dewey's exhortation not to address the psychological by sugar-coating the content (i.e., "get[ting] the child to swallow and digest the unpalatable morsel while he is enjoying tasting something quite different," p. 208), Marty says that he doesn't introduce mathematics as describing the world in order to present "real-world applications" or to make the course more fun:

> I don't know if I'd call it a plausibility thing, like it's plausible that someone would actually be interested in doing this or it would be useful to do it. That's where it targets some kids, but mostly ... I want to come up with some kind of problem that has the mathematics we need to study embedded in it, in some way that has some kind of intuitive appeal. I certainly do everything I can to avoid "oh, I do it to make it interesting or make it fun." Because a lot of this stuff may not be interesting or fun, ... and there's no way that you could ever do that for every kid in your room. (interview)

Characterizing mathematics in the manner described above supports psychologizing in several ways. First, it creates an atmosphere in which actual and hypothetical physical experiences are seen as relevant. Students can then use the ideas of accumulation and rate to describe these, helping them make connections between their experiences and the mathematics they are studying. Second, students' differing ideas and interpretations become valuable. And third, it creates a way to arrive at the "richer maturity" of mathematics without starting there. Marty has found that prematurely formalizing language and notation shuts down students' exploration. Instead, as the students themselves introduce ideas about rate and accumulation, he and the students gradually develop language and symbols to capture those ideas. He writes that "frequent classroom discussions focus on revising the meanings of familiar words—speed, velocity, average, distance, and so on—and constructing mathematics related to their technical use" (Schnepp & Nemirovsky, p. 101). For example, a student recognizes that they are using "average rate" in different ways and that they need a definition; Marty has them propose definitions and work on this, revisiting it over time as they make sense of the difference between averaging a discrete set of values and averaging something that varies continuously.

A second theme is that Marty wants his students to develop a sense of their own mathematical authority. The authority of textbooks and teachers poses a significant obstacle to this, especially since his students may have already become habituated to being told "the answer." Thus, Marty intervenes in a dramatic way in his talk and his textbook use. Believing that his students need time to learn to persist and develop greater intellectual independence , he essentially refuses to tell them whether they are right or wrong for the first two months or so of the course:

> In my view, the danger of responding, "yes what you are doing is correct," or "no what you are doing is wrong because ...," is perpetuating students'

> reluctance to make sense of and analyze arguments for themselves. If I pass judgment on ideas and methods—to prevent students from pursuing a dead end approach for example—students will learn to look to me to assess validity of arguments and not learn to do such reasoning independently. (Chazan & Schnepp, 2002, p. 174)

> To get the kind of sense-making I want, there has to be an extended period of time where kids can ... go through their emotional stage of being mad at me because I'm not telling them the answer, [so] they can finally figure out "OK, wow, I *can* make sense of this." (presentation)

In addition, Marty doesn't use a textbook during the first semester, maintaining that his students can only make truly productive use of textbooks once they understand what motivated the development of the ideas in the text and feel that they can critique the text's representation instead of relating to the text with a sole goal of mastering content:

> The textbook tends to be full of stuff that the kids haven't thought about yet, and if it's used as the sole basis for the class, [students are] constantly being thrown into situations where they don't really understand what lead to the kind of stuff the person's going to talk about .... It gets in the way in the sense that it's such a finished polished product in terms of the, the mathematical structure and the presentation of it. ... A high school kid looks at that and says "OK, my job is to try to understand the stuff that, you know, is truth" ..., and it's just such a completely different mind set than what I think I would want people doing. I mean they're not going to think as critically about it if they're not critiquing it or if they don't know exactly what went into creating that. ... There's just something so different about the process, I think, when you're critiquing something versus when you're just trying to understand what it's saying. (interview)

Marty's hope is that by the time they move to the book, the mathematical objects and writing are themselves psychologized to a greater degree, and therefore that using a text is not as disconnected from psychologizing as it might be. Not using a textbook also makes it easier for Marty to move around in the material flexibly in response to ideas that the students introduce.

Revisiting the Construct of Course Conceptualizing

Before discussing the relevance of course conceptualizing to psychologizing, I briefly return to the constructs I introduced above, using this case to illustrate them. Marty has engaged in course conceptualizing in calculus— reasoning about course-level issues with a recognition that alternatives exist—in order to develop a course conceptualization that psychologizes calculus for his students. In his landscape of alternatives, the categories include describing what calculus is about, organizing the substantive content of the course, and representing knowing in mathematics. He makes explicit some of the alternatives that he knows about within a category [(e.g., in the category "organizing the substantive content of the course," he explicitly rejects a common alternative: starting with a treatment of limits; and elsewhere, he writes that with differentiation and integration, "rather than studying one and then the other, students encounter and then reencounter both at once (or in succession) in different contexts, levels of analysis, and representations," (Schnepp & Nemirovsky, 2001, p. 93), contrasting another of his choices with an alternative]. Although he may not explore all of the alternatives he knows about in depth, he recognizes that alternatives exist and that there are choices to be made or invented. He is articulate about his reasons for choosing certain alternatives, and these come together to form his preferred course conceptualization for calculus. This conceptualization is not a simple union of the chosen alternatives; rather, it is a complex coordination in which choices from different categories interact with and often support each other.

The Relevance of Course Conceptualizing to Psychologizing

Much of the work of teaching obviously occurs in day-to-day planning and classroom interactions. And because students' "present," which is always changing, plays such a central role in psychologizing, the day-to-day work of teaching is especially important in efforts to psychologize the curriculum (e.g., in assessing students' current understandings, interests, etc., and using that understanding together with an understanding of the discipline to figure out where to go next). Nonetheless, conceptualizing as a course-level practice has much to contribute to efforts to psychologize the curriculum.

First, when working with students for whom mathematical objects are not yet psychologized, one must look for alternatives to the (mathematically) logical progression (e.g., in calculus, limits are unlikely to be an appropriate place to start the course). Moreover, even if one has a preferred sequence and believes it connects to students' experiences in valuable ways, one must still think about the content in ways that allow one to be responsive to students' thinking along the way; thus, organization might be better thought of as a terrain (Lampert, 2001) in which one chooses the starting point ahead of time and in which the next step is determined jointly by the ideas that students introduce and the one's knowledge of the overall set of understandings one wants students to develop. A second, related, point is the value of looking for a way of framing the substantive content of the course such that the course is about something that can be communicated to students before they've been introduced to formal vocabulary and allows them to engage with problems before they've been introduced to formal solution methods.[2] The big ideas of the discipline likely play an important role in deciding what a course is about. Nonetheless, the fundamental objects of study need not be mathematical objects; rather, they may be objects like rate and accumulation, for which meanings exist in both natural language and the mathematics register. Third, psychologizing the curriculum necessitates attending to the nature and practice of mathematics, and thus to making explicit for oneself the attitudes, motives, and interests that have "operated in developing and organizing the subject matter." Not only would attempting to psychologize the substantive content alone be incomplete in Dewey's eyes, it may actually be difficult to psychologize the substantive content without attending to the attitudes, etc. that lead to its development. Finally, the practice of course conceptualizing—especially the act of considering alternatives (which can involve such things as naming alternatives and seeing both distinctions and connections among different elements)—may set teachers up to listen to their students more flexibly, which puts them in a better position to psychologize on the day-to-day level. In light of its role in psychologizing the curriculum (as well as its potential role in helping teachers understand diverse approaches in this era of curriculum reform), course conceptualizing deserves our attention.

Notes

[1]I use the teacher's actual name with his permission.

[2]Chazan (1999, 2000) reasons similarly in arguing for a functions-based approach to school algebra.

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PRESERVICE ELEMENTARY AND MIDDLE SCHOOL TEACHERS' CONCEPTIONS OF ALGEBRA REVEALED THROUGH THE USE OF EXEMPLARY CURRICULUM MATERIALS

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One of the greatest challenges for mathematics teacher educators committed to reforming and improving mathematics education is to help preservice elementary and middle school teachers develop an appreciation for algebraic reasoning. Preservice teachers' views of algebra are typically derived from their experiences in middle school and high school where they developed a conception of algebra as a body of rules and procedures for manipulating symbols. However, as Moses (1997) observed, the content of algebra is being transformed from a discipline involving the manipulation of symbols to a way of seeing and expressing relationships, "a way of generalizing the kinds of patterns that are part of everyday activities" (p. 246). In elementary and middle school, the crucial issue now appears to be the development of algebraic reasoning, with a focus on relationships, not just the introduction of algebraic concepts (Yackel, 1997). An important goal for mathematics teacher educators is to organize experiences for preservice teachers that will broaden their vision of algebra so that they can effectively promote the algebraic reasoning of elementary and middle school children.

This investigation focused on an algebra course for preservice elementary and middle school teachers who have chosen mathematics as their area of specialization. The goals of the course are (1) to develop preservice teachers' understanding of algebraic concepts, (2) to encourage preservice teachers to focus on children's algebraic reasoning, and (3) to foster more conceptual views about the nature of algebra. The research mission of this investigation was to assess progress toward meeting the goals of the course.

Recommendations from various professional organizations (CBMS, 2001; MAA, 1991; NCTM, 1991) have outlined the algebraic content appropriate for preservice teachers. This content includes investigating patterns, representing problem situations with variables, analyzing functional relationships, and investigating algebraic structure. Although it may be difficult to address all of these topics meaningfully in a one-semester course, we decided to use this collection of topics as the framework for the content of our course (Stump & Bishop, 2001). As we address these topics, we wish to shift the emphasis away from symbolic manipulations and toward a broader and more relationship-oriented view of algebra.

Bednarz, Kieran, and Lee (1996) suggested four conceptual approaches for developing algebraic ideas with children – generalization, problem solving, modeling, and functions – and these four approaches also offer promising alternatives for developing algebraic ideas with preservice teachers. As described by Bednarz, Kieran, and Lee, *generalization* focuses on the construction of formulas that account for general procedures or relationships among quantities. *Problem-solving* emphasizes the forming and solving of equations, using letters as unknowns. *Modeling* involves algebraic representations arising out of real-world situations, and relationships originating in observations or measurements. A *function* approach examines various representations of dependence relationships among real-world quantities, observing how change in one variable produces variation in another variable. Each of these approaches provides a different setting in which the focus is on the development of conceptual knowledge and algebraic reasoning. For our purposes, we have collapsed these four approaches into three: generalization, problem solving, and a combination of functions and modeling. In our view, modeling a physical situation frequently involves identifying the function that explains the situation, and the emphasis on relationships that is inherent in work with functions may help interpret a situation that is being modeled. The three approaches provide a means of organizing algebra content that goes beyond building up layers and layers of procedures.

Although several conceptually-oriented textbooks exist for developing algebraic ideas with children, analogous materials for use with preservice teachers are not easy to find. Algebra textbooks do exist for college students, but they are typically designed to prepare students for subsequent mathematics courses, not to prepare preservice teachers for the mathematics classroom, and they do not necessarily contain the topics in our framework. Lacking a tailor-made textbook, we resolved the issue by selecting a nontraditional college algebra textbook (one that emphasizes a functions-and-modeling approach to algebra) and then supplementing it with exemplary elementary and middle school curriculum materials, including algebra modules from *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998a, b, c, and d) and *Mathematics in Context* (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1998).

This investigation focused on the role of these exemplary curriculum materials in serving the goals of our algebra course for preservice teachers. The research addressed the following questions: (1) What aspects of preservice teachers' knowledge of algebraic concepts are revealed in their use of exemplary curriculum materials designed for middle school students? (2) What aspects of preservice teachers' focus on children's algebraic reasoning are revealed in their use of these materials? and (3) What are preservice teachers' views about the nature of algebra before and after the course?

Method

The participants in this investigation were 30 elementary education majors enrolled in an algebra course for preservice elementary and middle school teachers who had chosen mathematics as their area of specialization. Five of these preservice teachers also indicated plans to complete a middle school mathematics endorsement. All of the preservice teachers had successfully completed two years of high school algebra and 13 had successfully completed a year of high school calculus. All of the preservice teachers were juniors or seniors and had completed the two-course sequence of mathematics required of all elementary education majors. Only two of the preservice teachers had taken a mathematics teaching methods course. For the 26 preservice teachers who had taken the SAT exam, the median mathematics score was 560.

On the first day of class, the preservice teachers wrote their responses to the question, "What is algebra?" They answered the same question again on the last day of class. The class spent the first four weeks investigating patterns and solving problems selected from the NCTM Algebra *Navigations* series (Cuevas, Yeatts, & House, 2001), Driscoll's (1999) *Fostering Algebraic Thinking: A guide for teachers, grades 6-10*, and algebra modules from *Connected Mathematics* (Lappan et al., 1998c) and *Mathematics in Context* (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1998).

In the second week of the course, the preservice teachers read and discussed the article, "A Foundation for Algebraic Reasoning in the Early Grades" (Yackel, 1997). This article describes a classroom discussion in which at least one student moves from a focus on isolated cases to reasoning about the entire range of possibilities. Then each preservice teacher wrote responses to a series of questions.

During this time, the class spent three days exploring *Comparing Quantities*, a sixth-grade algebra module from *Mathematics in Context* (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1998). After working through sections of the module in groups, the preservice teachers completed the Chickens problem (p. 21) individually as an in-class quiz. The following day, the class examined the various strategies used to solve the Chickens problem. Then they received an interview assignment to pose the Chickens problem to a child in grades 4-8 and to ask questions to elicit as much information as possible about the child's mathematical thinking. Each preservice teacher then completed a written analysis of his or her interview with the child.

The class then spent three days exploring various sections of *Say It with Symbols: Algebraic Reasoning*, an algebra module for grade 6 from the *Connected Mathematics* series (Lappan et al., 1998c). Preservice teachers completed Problem 3.3: Finding the Area of a Trapezoid and follow-up questions (pp. 40-41) as a quiz.

In the fifth week, the class began a ten-week exploration of functions and modeling. Throughout this time, the class alternated its focus between selected investigations of linear, exponential, and quadratic functions in *Connected Mathematics* algebra modules and corresponding lessons in the college algebra textbook. They investigated linear relationships in *Thinking with Mathematical Models* (Lappan et al., 1998d), exponential relationships in *Growing, Growing, Growing* (Lappan et al., 1998b), and quadratic relationships in *Frogs, Fleas, and Painted Cubes* (Lappan et al., 1998a). The class spent two or three days with each module and completed a quiz after each one.

The last three weeks of the semester were devoted to matrices and investigations of algebraic structure. This investigation does not focus on that part of the course.

Research Findings

Preservice Teachers' Understanding of Algebraic Concepts

For this investigation, samples of the preservice teachers' work were collected to reflect each of the three conceptual approaches to algebra adapted from Bednarz, Kieran, and Lee (1996): generalization, problem solving, and functions and modeling. The data sources for examining preservice teachers' understanding of algebraic concepts included: (1) solutions to the Area of a Trapezoid problem, (2) solutions to the Chickens problem, and (3) solutions to quizzes focusing on linear, exponential, and quadratic functions.

Generalization

Generalization is described by Bednarz, Kieran, and Lee (1996) as focusing on the construction of formulas that account for general procedures or relationships among quantities. The Area of a Trapezoid problem in *Say it With Symbols* (Lappan et al., 1998c) provides an opportunity to assess preservice teachers' reasoning about relationships among geometric quantities. This problem asks students to interpret four hypothetical children's methods for finding the area of a trapezoid. Table 1 shows preservice teachers' success with the various tasks associated with the problem. Task 4 presented the greatest challenge. Preservice teachers struggled with creating a drawing to match the expression $\frac{1}{2}h(b-a)+ha$ to illustrate finding the area of a trapezoid.

Problem Solving

Bednarz, Kieran, and Lee (1996) define the problem-solving approach as solving problems by forming and solving equations using letters as unknowns. The Chickens problem, from the sixth-grade *Comparing Quantities* (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1998), depicts a situation involving three unknown quantities, the weights of three different chickens. This problem asked students to examine a series of three drawings, each showing a different combination of two of the three chickens in the problem. Given the total weight for each pair of chickens, the problem is to find the weight of each individual chicken. A possible solution strategy is to set up and solve a system of three equations involving three variables. Sixth-graders, though, would be more likely to guess and check or to employ informal reasoning skills to analyze the various relationships among the three unknown quantities. The latter of these two strategies would be considered more algebraic even though the formal procedures of algebra may not be visible. Of the pre-

Table 1. Preservice Teachers' Success With Various Tasks Associated With The Area Of A Trapezoid Problem

	Number of PSTs (n = 30)*	
Tasks	Yes	No
1. Given labeled drawings of trapezoids, explain the meth- ods for finding area.	27	3
2. Write an expression to describe each method.	25	5
3. Show that the expressions are equivalent.	26	4
4. Given an expression, make a drawing and use it to explain the method for finding area.*	6	13
5. Explain why one expression is equivalent to the other three.*	13	6
6. Find the area of a given trapezoid using each of the four expressions.*	17	2

* Only 19 students attempted Tasks 4-6, perhaps because these were offered as bonus questions.

service teachers in this investigation, 8 set up and solved a system of three equations, 13 used informal reasoning skills, and 9 relied on guess and check. All but 2 preservice teachers successfully solved the problem.

Functions and Modeling

The functions and modeling approach focuses on various representations of dependence relationships among real-world quantities (Bednarz, Kieran, & Lee, 1996). In each investigation, the class modeled real-world situations and examined how the change in one variable related to change in a second variable. For each situation, they explored connections among tables, graphs, and equations. The responses to quiz questions taken from *Connected Mathematics* (Lappan et al., 1998a, b, & d) algebra modules provide evidence of some of preservice teachers' strengths and weaknesses regarding functions and modeling. Tables 2, 3, and 4 reveal preservice teachers' success with various tasks associated with linear, exponential, and quadratic functions, respectively.

Linear Functions.

Table 2 shows that more than a third of the preservice teachers had difficulty writing adequate explanations for how to determine a linear relationship from either a table or an equation. Additionally, more than a third failed to write an equation to match the graph of a line.

Exponential Functions

As shown in Table 3, only a third of the preservice teachers adequately described the exponential pattern of change in a given table. When given the equation $y = 2(3^x)$ only three preservice teachers correctly identified 200% as the rate of growth. Most preservice teachers said the rate of growth was 300%, evidently confusing the "rate of growth" with the "growth factor," even though the curriculum module carefully distinguishes between the two concepts.

Table 2. Preservice Teachers' Success With Various Tasks Associated With Linear Functions

	Number of PSTs (n = 30)	
Tasks	Yes	No
1. Recognize linear relationships from tables.	28	2
2. Explain how you can recognize a linear relationship from a table.	18	12
3. Recognize linear relationships from equations.	27	3
4. Explain how you can recognize a linear relationship from an equation.	17	13
5. Recognize linear relationships from graphs.	30	0
6. Given a graph, write an equation.	18	12
7. Explain how you can recognize a linear relationship from a graph.	28	2
8. Find an equation passing through two points on a graph.	26	4
9. Given a point and the slope, find an equation of the line.	26	4
10. Given a situation, make a table.	28	2
11. Given a situation, make a graph.	29	1
12. Write an equation to describe the situation.	26	4

Table 3. Preservice Teachers' Success With Various Tasks Associated With Exponential Functions

	Number of PSTs (n = 30)	
Tasks	Yes	No
1. Given a exponenetial situtation, create a table.	23	7
2. Describe the pattern of change in a table.	10	20
3. Given a exponential situation, write an equation.	26	4
4. Given an exponential equation, fill in a table.	30	0
5. Given two equations, determine in which equation the *y* value increases at a faster rate.	30	0
6. Given an exponential equation, determine the initial value of the function.	28	2
7. Given an exponential equation, determine the rate of growth.	3	27
8. Given tables, distinguish between a constant rate of increase and a percentage rate of increase.	26	4

Quadratic Functions

As Table 4 indicates, all but three preservice teachers were able to draw and label the rectangle represented by a quadratic equation, but a third of the preservice teachers failed to label the area of the rectangle. A more serious difficulty appeared in Tasks 9 and 10, in which preservice teachers struggled to extend a geometric pattern and to write the equation representing the pattern.

Preservice Teachers' Focus on Developing Children's Algebraic Reasoning

The data sources for assessing preservice teachers' focus on the development of children's algebraic reasoning were: (1) their responses to questions about "A Foundation for Early Algebraic Reasoning in the Early Grades" (Yackel, 1997) and (2) their analyses of the Chickens problem interview.

Responses to Yackel Article

The Yackel (1997) article seemed to provide preservice teachers with a language for describing algebraic reasoning. After reading the Yackel article, 16 of the 28 preservice teachers who completed the article response mentioned "patterns" or "relationships" in their answer to the following: "Describe the type of thinking that forms the foundation for algebraic reasoning in the elementary grades. How is this type of

Table 4. Preservice Teachers' Success With Various Tasks Associated With Quadratic Functions

	Number of PSTs (n = 30)	
Tasks	Yes	No
1. Recognize quadratic relationships from equations.	23	7
2. Explain how you can recognize a quadratic relationship from an equation.	28	2
3. Draw and label a rectangle represented by a quadratic equation.	27	3
4. Label the areas of the rectangle represented by a quadratic equation.	19	11
5. Recognize quadratic relationships from tables.	29	1
6. Explain how you can recognize a quadratic relationship from a table.	24	6
7. Write an equation for each table that represents a quadratic relationship.	24	6
8. Given a situation, determine a specific value of the quadratic function.	26	4
9. Extend a quadratic geometric pattern.	18	12
10. Write an equation to describe a quadratic geometric pattern.	15	15

thinking different from numerical reasoning?" Interestingly, though, the remaining 12 preservice teachers used language that suggested they equate algebraic reasoning with the process of problem solving (not to be confused with a problem-solving approach to algebra). In their descriptions of algebraic reasoning, these 12 preservice teachers mentioned such things as the importance of recognizing multiple solutions and having students explain their solution strategies, but did not refer to anything specifically algebraic.

Chickens Problem Interview

Yackel (1997) places emphasis not on the content of algebra, but rather on "the underlying thinking and reasoning of the students"(p. 276). Thus, in order for preservice teachers to develop a coherent view of algebraic reasoning, it is important to provide opportunities for them to examine children's mathematical thinking. Indeed, Fennema and Franke (1992) suggest that knowledge of students' cognitions is more valuable to teachers than knowledge of learning theories. McGowen and Davis (2001) suggest that a way in which preservice teachers can become more effective teachers of early algebra is by seeing children solve problems with which they themselves have struggled. The Chickens problem interview was designed to serve this purpose.

The instructions asked preservice teachers to: (1) describe the student's work with the problem, noting questions asked and strategies attempted, and (2) discuss conclusions about the child's mathematical thinking. Unfortunately, because the instructions did not ask preservice teachers to focus on "algebraic reasoning," it is not appropriate to analyze their descriptions of algebraic reasoning. However, 13 of the 28 preservice teachers who completed this assignment carefully described the child's solution and discussed aspects of the child's algebraic reasoning. Another 9 preservice teachers mentioned algebraic reasoning, but did not provide a specific description. Finally, 6 preservice teachers focused entirely on other issues such as the child's attitude toward mathematics. The instructions also asked preservice teachers to reflect on the interview, describing what they would do differently another time. Eleven preservice teachers responded by saying they would ask more and better questions. Four said they would provide less help, and two said they would provide more help. Four preservice teachers said they would like to interview more children to get a broader perspective.

Preservice Teachers' Views about the Nature of Algebra

The data sources for analyzing preservice teachers' views about the nature of algebra were their initial and terminal responses to "What is algebra?" Preservice teachers' responses to the question, "What is algebra?" were assigned to the following categories: symbolic, problem solving, generalization, and functions. The latter three categories are derived from the three conceptual approaches to algebra adapted from Bednarz, Kieran, and Lee (1996). For coding purposes, *symbolic* responses merely listed various symbols and operations, *problem solving* focused on solving equations, *generalization* mentioned patterns or relationships, and *functions* included the word "functions" or mentioned connections among tables, graphs, and equations. Table 5 shows preservice teachers' responses on the first and last days of class. At the end of the course, none of the students indicated a symbolic view of algebra, instead expressing some combination of generalization, problem solving, and function definitions.

Conclusion

The results of this investigation show that exemplary curriculum materials developed for middle school students present mathematical challenges for some preservice elementary and middle school teachers. Preservice teachers in this investigation were challenged by situations involving generalization, problem solving, and functions. Although most found answers for problems involving linear relationships, the preser-

Table 5. Categories Of Preservice Teachers' Responses To The Question, "What Is Algebra?"

	Definition of Algebra*			
Data Source	Symbols	Generalization	Problem Solving	Functions
Number of PSTs (n = 30)				
Initial Responses	5	5	20	0
Terminal Responses	0	14	16	15

* Some preservice teachers' responses reflected more than one definition of alge

vice teachers were often unable to explain their thinking. They had difficulty answering questions about the rate of change of exponential and quadratic functions. However, they were sometimes bored by materials designed for elementary and middle school students. Preservice teachers often struggle to clearly communicate their understanding of the mathematical relationships embedded in the activities, but they do not always need to explore the same concept from as many angles as are presented in the well-developed materials. We are compelled to use the materials judiciously, selecting critical elements of the activities but avoiding over-repetition.

The interview situation with children proved to be an effective tool for helping preservice teachers begin to focus on children's mathematical thinking, and exemplary curriculum materials offer engaging problems for this purpose. Many of the preservice teachers in this investigation did not provide evidence that they understand the difference between algebraic reasoning and problem-solving skills. Even more discussion may be needed to help preservice teachers strengthen their understanding of what constitutes algebraic reasoning, and more than one of these interview experiences may be necessary to help preservice teachers shift their focus from children's problem-solving behaviors to children's algebraic reasoning.

As they experience exemplary curriculum materials from a learners' point of view, preservice teachers can begin to think about them from a teacher's perspective. It is important to take time to discuss the philosophy that guided the development of reform curricula so that preservice teachers will understand why these materials are so different from the textbooks with which they are more familiar. With guidance, they may come to appreciate the potential of the new curricula for developing children's mathematical thinking.

Before they can successfully promote algebraic thinking in their own classrooms, preservice teachers need to understand algebra as a way of thinking, a way of working with the patterns that occur every day. A comparison of their initial and terminal responses to the question "What is algebra?" suggests that this group of preservice teachers broadened their views about the nature of algebra. Because their terminal definitions of algebra reflect the three conceptual approaches to algebra adapted from Bednarz, Kieran, and Lee (1996), it would seem that these approaches do indeed serve as a meaningful structure for framing the course. Providing an opportunity to explore exemplary elementary and middle school curricula in the context of a college algebra course has helped preservice teachers develop a dynamic vision of algebra they can take with them into the classroom.

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VIEWING A REFORM-BASED MATHEMATICS CURRICULUM THROUGH THE EYES OF TWO VETERAN TEACHERS

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The focus of this research is an investigation of the ways that teachers use reform-based mathematics curriculum materials. Specifically, we explore how two veteran elementary-school teachers make use of a new curriculum that they volunteered to implement for two consecutive years. Based on observations of instruction and interviews with the teachers, we examine the extent to which the teachers relied on information in the written curriculum materials to guide their instruction, and their reasons for doing so. Furthermore, we consider how the teachers' approaches to curriculum use changed over the two-year period. We found that one teacher followed the curriculum quite closely during the first year and more flexibly during the second year. In contrast, the other teacher used the curriculum liberally during the first year, and followed it much more closely during the second year. Discussions of similarities and differences in the teachers' implementation of the curriculum give rise to important implications for curriculum development in the context of reform.

Recently, researchers and mathematics educators have called for curricula that not only support student learning, but also help teachers learn to manage the demands of reform (Ball & Cohen, 1996; Remillard, 1999). In turn, several new mathematics curricula have been developed that strive to be educative for teachers. They include, for example, information on possible student misconceptions, explicit explanations of curricular goals, and discussions of key mathematical concepts. At the same time, however, these curricula look quite different from the traditional materials that teachers have used and as a result may require that teachers develop new approaches to curriculum use.

In this research, we address this issue by attempting to characterize the ways that teachers use reform-based curriculum materials. In particular, we examine the experiences of two veteran elementary-school teachers who implemented a reform-based mathematics curriculum over the course of two years. For each teacher, we illustrate the specific approach that the teacher had towards using the curriculum materials, paying particular attention to how closely the teachers chose to follow the written curriculum and the reasons behind their decisions to do so. In addition, we describe the ways in which the two teachers' approaches to curriculum use changed from year one to year two.

Theoretical Perspective

Two perspectives on the relationship between teachers and curricula help to frame this research. First, prior research emphasizes that curricula do not always dictate instruction (Sosniak & Stodolsky, 1993). Instead, it is teachers who bring these materials to life in the classroom, and the "enacted curriculum" often differs from what was in the "written curriculum" or teacher's guide (Ben-Peretz, 1990). Along these lines, Freeman & Porter (1989) identified several different styles of textbook use among teachers. Particularly relevant for this study was their description of some teachers as "textbook bound," closely following the textbook, while other teachers relied less heavily on the details of the curriculum.

Second, and more recently, researchers have begun to examine teachers' use of curricula in the context of mathematics education reform (Heaton, 2000; Lloyd, 1999; Remillard, 1999, 2000). Some of this work examines the opportunities for learning that occur as teachers use reform-based materials (Davenport, 2000; Remillard, 1999; Sherin, in press). These researchers describe the process through which using a new curriculum helps teachers to increase their knowledge of both content and pedagogy. In addition, their work highlights that this learning can take place as teachers use the new materials with students, and as teachers reflect on these materials outside of the demands of instruction.

Other research in this area explores differences in teachers' approaches to reading, evaluating, and adapting reform-based curriculum materials (Morse, 2000; Remillard, 2000; Sherin & Drake, 2000). Common to all of these approaches is the notion that teachers do in fact develop particular approaches to using new curricula. For example, researchers have described a number of different ways that teachers "read" reform-based materials. Remillard, for instance, (1999) contrasts one teacher who read the curriculum by focusing on the activities for students with a second teacher who read the curriculum in order to get a general idea of the mathematical goals for each lesson. Similarly, Sherin and Drake differentiate between teachers who read for the "big ideas" of a lesson versus teachers who read for lesson details, looking at the precise wording of examples and explanations. Morse also discusses how teachers read reform-based curriculum materials. She emphasizes the need for teachers to learn to read the curriculum in ways that help them understand the intentions of the curriculum designers. These different findings highlight the wide variety that exists in the ways that teachers use new reform-based curriculum materials. Furthermore, examinations of how and why teachers develop these different approaches suggest that curriculum use is influenced by teachers' understandings of the purpose of the curriculum materials and by their goals for instruction.

Research Design

This research is part of a project that investigates and supports teachers as they implement the reform-based mathematics curriculum Children's Math Worlds (CMW). The first- through third-grade CMW curriculum is designed to incorporate the best of both traditional and reform mathematics curricula by attending both to children's natural solution methods and to effective standard procedures. Furthermore, the curriculum builds on the individual experiences, interests, and mathematical knowledge that diverse children bring to their classrooms. Specifically, CMW is based on combining the following three key features: (a) representational and conceptual supports to facilitate individual understanding, (b) developmental sequences of conceptual structures, and (c) real-world contexts for exploring mathematical thinking (Fuson, De La Cruz, Lo Cicero, Smith, Hudson, Ron et al., 2000). The design of CMW creates new challenges for many elementary-school teachers of mathematics because using this curriculum is very different from using a traditional mathematics textbook.

CMW includes a variety of materials for teachers and students. First, there is a teacher's guide with detailed information about each lesson, as well as an overall description of every unit and suggested participant structures for organizing classroom instruction. There is also a book for students with worksheets and activities related to each lesson. In addition, CMW provides an assessment packet and a set of innovative classroom manipulatives to use during instruction.

This study investigated the curriculum use of two veteran elementary-school teachers, Cindy and Tracy, in a large urban public school district where the CMW materials were being field-tested. The study began when Cindy and Tracy implemented the materials for the first time (1999-2000). Both teachers had previously been involved with mathematics education reform and were interested in trying out additional materials that might help their students. At the beginning of the study, the second-grade teacher (Cindy) had been teaching elementary school for 18 years. She was recommended by the acting assistant principal and stated that if mathematics was headed away from traditional instruction, she wanted to be prepared. She was observed monthly during her first year of using CMW and on two consecutive days at the beginning, middle and end of the second year.

The third-grade teacher (Tracy) was in her 20th year of teaching. She had previously implemented a reform-based curriculum for one year, and was described by the school's curriculum coordinator as a teacher who was willing to try to meet the goals of reform. Tracy was observed monthly during the first year (except for two months) and on two consecutive days at the beginning and middle of the second year. Both teachers also participated in post-observation interviews and in monthly meetings with a researcher. During these interviews and meetings, the teachers were invited to reflect on the recent events of their classrooms. In particular, they were asked questions about decisions they had made regarding how they were enacting the curriculum. For instance, if either teacher omitted part of a lesson, the meetings were a place for them to elaborate on their reasons for doing so. In addition, the teachers were encouraged to discuss any questions they had concerning the goals of lessons they had recently taught. All observations, interviews, and meetings were videotaped and transcribed.

Fine-grained analysis was used to characterize how each teacher incorporated the reform-based curriculum into their mathematics instruction (Schoenfeld, Smith,

& Arcarvi,1993). To do this, transcripts of classroom video were analyzed alongside the corresponding CMW lesson to determine the extent to which each teacher either deviated from or followed the lessons as written in the curriculum. As a result, similarities and differences among the written and enacted lessons were noted in several areas including the mathematical ideas that were discussed in class, the order in which activities were implemented, and the vocabulary, explanations, and representations utilized by the teacher. Interview and meeting transcripts were also coded, citing the teachers' reasons for the choices they made as they read and implemented the CMW lessons. Based on this analysis, we characterized each teachers' approach to curriculum use in their first and second years of using the new materials.

Results

Analysis of the data revealed that each of the teachers demonstrated a consistent approach to using the curriculum in the first year of the study. These two approaches, however, were quite different from one another. Furthermore, during the second year of the study, the teachers also demonstrated a consistent approach to using the materials. Yet for each teacher this approach differed from how the teacher had used the curriculum during the previous year. In what follows, we discuss each teacher in turn.

The Case of Cindy

The first year that Cindy implemented CMW, she followed it very strictly. She began with Unit 1, Lesson 1, and from there, implemented each lesson in the order that it was presented in the teacher's guide. Furthermore, Cindy paid close attention to the details of each of these lessons, planning to use all of the suggested activities. She described this approach as wanting to go "straight through everything," and claimed that "I don't think there's anything I don't use." In particular, Cindy carefully read the example problems given in the curriculum materials, and often used them precisely as they were written. She also studied the sample questions provided by the curriculum as potential probes that she could use to encourage students to share and explain their thinking. During instruction, Cindy tended to use these probes exactly as they appeared in the curriculum materials. In fact, during an interview, Cindy noted that these questions were an important part of her instruction, explaining that "the scripted part … it helps me."

The following example is typical of how Cindy used CMW in the first year of the study. On this day, Cindy was implementing a lesson from Unit 1. The lesson began, as did many of the CMW lessons, with two simultaneous activities, "Quick Practice" and "Homework Check." The idea was for student leaders to conduct practice activities with the class, while the teacher checked the students' homework. Cindy's class had done "Quick Practice" on several occasions prior to this lesson, and thus the students were familiar with a range of practice activities. Nevertheless, Cindy asked a pair of students to lead the class in the two practice activities ("Teen Routines" and "Finger

Flash") that were specified in the day's lesson. The students were given explicit directions that these were the activities listed in the curriculum and that is what they would do today.

The next part of the lesson involved having students solve addition word problems. Several example problems were given in the teacher's guide with the objectives stating that students should explore two specific solution strategies: (a) counting on, and (b) counting up. Cindy followed the directions in the teacher's guide precisely and read the story problems verbatim from the teacher's guide. Moreover, as the curriculum suggested, she required her students to illustrate their solutions by using both the 'counting on' and 'counting up' methods of addition. For instance, when solving the problem: *I have 11 gumdrops, and I grouped them by flavor. I put the 3 lemon ones in my mouth and the cherry ones in my pocket. How many cherry gumdrops do I have?*, the first student called upon shared the "counting up" method. Therefore, Cindy next called for a student who used the "counting on" method to explain to the class what she had done.

In addition to the close attention that Cindy gave to each activity in the day's lesson, she also worked more broadly throughout the lesson to satisfy other recommendations that she had read in the teacher's guide and that she viewed as central to implementing CMW. For example, the curriculum encouraged the teacher to allow students to display multiple solution methods. Therefore, in preparing to teach this lesson, Cindy make sure that students would have a way to display how they solved each problem. To do so, she decided to pass out individual chalkboards for her students to use to record their methods. This helped Cindy to easily monitor individual student's solutions and then to choose students with different methods to stand before the class and explain their solutions. Furthermore, after students demonstrated the 'counting on' and 'counting up' methods, Cindy asked students if there were other solution strategies that they could use to solve the problem, and on several occasions, students demonstrated the "make a 10" strategy that had been introduced in a previous lesson. Thus, Cindy worked to meet the specific goals of the day's lesson — to have students use the 'counting on' and 'counting up' strategies — but at the same time she took seriously the suggestion in the lesson to have students "display multiple strategies."

Similarly, Cindy attempted to follow the curriculum's recommendation to promote a "Math Talk" community where students as well as the teacher asked questions of each other and offered explanations for each others' solutions. Thus, as students explained their solutions in front of the class, Cindy facilitated a discussion, encouraging students to talk about their own and others' ideas. For example, at one point during the lesson, Cindy asked a student to elaborate the "make a 10" method that he had used to solve the problem 8 + 9. The student responded, "I took 2 [from the 9] and added it to the 8, and 8 + 2 equals 10. Then I took the 7 [left] and added it to 10 and 10 + 7 is 17." Following this explanation by the student, Cindy turned to another student and said, "David, can you say what Anthony just said? How did he get his answer?"

Also in line with the goals of the curriculum, Cindy tried to use student errors as learning opportunities for the class. Thus, on this day, when one student presented a solution that was incorrect. Cindy used the error as an opening to discuss with the class a variety of misconceptions that students had concerning order of operations. Cindy explained that in previous years, she tended to simply correct students' mistakes and continue with the lesson-in-progress. This year, however, she was working to achieve the goals of CMW one of which involved discussing errors that arose with the class.

As illustrated in this example, throughout the day, Cindy worked with her students so that the lesson unfolded during instruction as it appeared to her in the teacher's guide. She did not deviate from the lesson plan with additional problems for students to solve, and instead kept the central goals of the lesson and of the curriculum at the forefront of her plans. Overall, Cindy's approach to using the curriculum during the first year was to follow it as closely as possible.

In her second year of implementing CMW, Cindy demonstrated a different approach to using the materials. In particular, she used the curriculum much more flexibly than she had in the first year. While she continued to follow the general order of topics as they were presented in the curriculum, she would, at times, skip a lesson entirely, or skip parts of a lesson that she felt were "too repetitive." Despite adapting the materials, the lessons that she taught still conformed to the goals of the curriculum. Cindy explained that she had a sense now of what students were supposed to learn and so she used that knowledge to decide which parts of a lesson to implement.

For example, during a lesson that involved buying items with pennies, nickels, and dimes, Cindy felt comfortable skipping an entire section of the lesson which 'introduced the nickel.' In an interview following that particular lesson she stated that the students were already familiar with the nickel and therefore she did not feel it was necessary to go over that particular section with the class. "They've already used the quarter, so I figured they already knew the nickel part...and I've had them count by fives [already]." Thus, Cindy was aware of her students' understandings and felt that she could still meet the goals of the lesson whether or not she used that particular activity.

At other times, Cindy adapted a lesson not by omitting an activity, but instead by modifying an activity from the way that it was described in the teacher's guide. For instance, the curriculum introduced a manipulative called "penny strips" to help students work with monetary amounts. On one day, Cindy decided to make a set of large penny strips which she laminated and attached to magnets. She then placed the new penny strips on the board in the front of her classroom. This was not a suggestion found in the curriculum, yet it was something that Cindy believed would enhance her presentation of the lesson. Using the large penny strips, Cindy demonstrated on the chalkboard how she expected students to use the smaller penny strips at their seats. Furthermore, she used the over-sized penny strips to work with students on the concepts they were exploring in the day's lesson. She stated afterwards, "...they could see the [money] better, instead of me drawing it, with my [penny] board."

Similarly, in the same lesson, Cindy replaced a "Quick Practice" activity with an activity that was designed to come later in the lesson. The "Quick Practice" activity involved students working in pairs. In an interview following the lesson, Cindy explained that while she regretted not having enough time for students to work in pairs as was suggested in the curriculum, she felt confident that she had met the goals of the lesson through the activities that she had used.

Thus, during the second year of the project, Cindy developed a much more flexible approach to using the CMW materials. Her overarching goal at this point was to help students learn the mathematical ideas of the lessons, and she felt able to do so while adapting the materials somewhat. She still saw herself as implementing CMW, but she had taken more control over the specifics of what that might look like during instruction.

The Case of Tracy

During her first year of implementing the new curriculum, Tracy used CMW as one of several resources that guided her instruction. She explained, "Sometimes I like to just read [the CMW materials] and then have my own ideas. [I] pick...up some of the ideas...I try to make notes in the margin combining my ideas and [the CMW ideas.]" Thus, Tracy would use CMW to learn the main objective for a given lesson, but would then pick and choose pieces of the CMW lesson to implement along with other materials she had used in the past.

For example, during one lesson addressing story problems, Tracy used the initial activity suggested by CMW but then switched to a more familiar method of discussing addition and subtraction story problems. The lesson centered on a Problem Bank of story problems found in the curriculum. Tracy gave the subtraction problem: *Kamal weighs 67 pounds. His older brother Osama weighs 110 pounds. How much more does Osama weigh than Kamal?* The students had a difficult time when they tried to determine how to arrange the numbers in order to perform the necessary computation. Some students recorded the problem correctly as 110 – 67, but then either subtracted incorrectly (the minuend from the subtrahend) or added the numerals. Other students recorded the problem incorrectly as 67 – 110 and could not figure out how to perform the subtraction based on how it was numerically written. Because of the difficulties her students were having, Tracy made the decision to stop the current lesson and to address the issue in a manner she felt would be beneficial to her students. She summoned the students away from their seats to the floor in the center of the room and held a 'mini-lesson' to review how to determine the order of numbers and which arithmetic operation (addition or subtraction) to use when solving certain story problems. It was not uncommon for her to take this approach. When her students did not seem to be grasping the concept, Tracy often stopped the current lesson to review the key mathematical ideas involved.

As a part of the mini-lesson, Tracy rewrote the story problem so that Osama's weight was given first, and then Kamal's weight was given. Tracy then asked her students if anyone knew which number should be written first in a subtraction problem. In response, a student explained that 110 would "go on top" and when Tracy asked, "Why?" the student continued by stating that "the higher number goes on top in a subtraction problem." Tracy further asked how the class could decide if they were supposed to add or subtract the numbers in the problem. Students seemed unsure, and thus Tracy directed the students to the key words 'more than.' In response, a student mentioned that 'more than' means take away so that meant it was a subtraction problem.

During this mini-lesson, Tracy relied on instructional strategies that were not suggested in the CMW materials, but rather, were based on techniques that she had used previously. Specifically, her idea to reorder the numbers in the problem was based on her prior experience that ordering the numbers in the problem in the same way that they will appear in the computation helps students to record the numeric computation accurately. Specifically, she stated, "It's confusing [for the children] when the numbers in the [story] problem are in a different order [than in the numeric computation.]" Furthermore, the key word strategy was also a technique that Tracy had used in the past when helping students learn to solve story problems. Thus, this example shows that Tracy was not tied to using the CMW curriculum. She was willing both to use different kinds of activities with her students, and to help them learn the lesson objectives in ways other than those specifically suggested in the teacher's guide. Tracy was aware of her willingness to adapt the curriculum in these ways. In fact, in an interview she stated, "It's up to me, right? I mean, I could have reworded [the problem] or something like that." In all, Tracy was comfortable supplementing a lesson, in particular when she believed that her students were struggling with the mathematical ideas in the lesson.

Interestingly, during the second year that Tracy used CMW, she followed the curriculum more closely and did not generally look to other resources to teach the CMW topics. When asked how the second year compared to using the curriculum in the first year she stated, "I know it! I kind of know the whole routine...Last year I would bring in other materials and I wasn't familiar with [CMW] enough to know how it would work...Last year I wanted to get away from [CMW] and do it the way I understood it. But now that I understand [the curriculum] better, I'm doing very little adapting." Tracy also stated that having taught the curriculum the previous year helped her to understand her students' responses. She stated, "Compared to last year [I] can hear their mistakes better. I can see their mistakes on the board."

For example, during the second year of the study, Tracy taught a lesson on quadrilaterals whose purpose was for students to explore different four-sided shapes such as squares, rectangles, parallelograms, and rhombi. When Tracy taught this lesson the previous year (during her first year of implementing CMW), she did not use the activities suggested by CMW to help students explore the quadrilaterals given in the lesson. Instead, she focused the lesson on the definitions of the different shapes, and on having students provide a name for each of the shapes given on the students' pages. For example, Tracy was content if her students labeled a figure with four equal sides and four right angles as only a *square*. She did not make the connection for her students that this definition also described a *rhombus* nor did she discuss with the class why such a definition might describe more than one kind of shape. During this lesson, Tracy also omitted the section addressing the heights of the quadrilaterals. This was an additional concept included in the objectives.

In contrast, during the second year, Tracy implemented this lesson quite differently, sticking closely to the activities suggested in the lesson in order to meet the lesson objectives. For example, as outlined in the teacher's guide, Tracy asked students to compare different shapes and to try to define the relationship between a rectangle and a square and between a square and a rhombus. Students responded by making a variety of observations, including the following comparison made by Ravi: "A square has four equal sides and the corners [are] equal, all four corners [are] equal. A square is like a ...rhombus." In addition, Tracy followed the curriculum's recommendation to have students compare the angles in the different shapes, the heights of the different shapes, as well as the relationship between the lengths of the sides of a given shape. As described in the curriculum, Tracy also provided students with toothpicks that they could use to compare the lengths and angles of the different shapes.

Tracy's implementation of this lesson during the two years was quite different. Specifically, during the first year, Tracy taught a lesson on the topic of quadrilaterals, but did not make much use of the suggested CMW activities. During the second year, however, Tracy taught the lesson by attempting to stick closely to the way that it was written in the teacher's guide. This same pattern was evident throughout the second year, with Tracy consistently implementing CMW with more fidelity than she had during the first year. Overall, during the second year of the study, Cindy felt better able to use the CMW activities to try to meet the needs of her students.

Discussion

Each of the teachers demonstrated a particular approach to using the curriculum in the first year of the study. Cindy's approach involved closely adhering to all aspects of the CMW lessons — from using the activities provided to using the language suggested for talking with students. In contrast, Tracy used the CMW curriculum materials as more of a general guide for information about the topic of the lesson that she would teach. She tended to use some parts of the CMW lesson, the students' pages for example, but also regularly incorporated activities that she had used in the past to teach what she believed to be the lesson objectives.

Despite these differences, however, there were important similarities among the reasons for the teachers' approaches to using the curriculum. Specifically, in their first year of using the curriculum, both teachers felt that they did not have a deep understanding of the goals behind the daily lessons and related activities. Yet, this influenced the two teachers' instruction in different ways. Cindy felt that she did not know enough to make appropriate changes in the materials and therefore needed to carefully follow all aspects of the curriculum. In contrast, Tracy decided that because she did not know how a CMW lesson would proceed, it did not make sense for her to implement the lesson as planned. It was for this reason that she used CMW activities along with other materials that she had used in the past.

In the second year of the study, both teachers claimed to have developed a deeper understanding of the purpose of the curriculum and their approaches to curriculum use changed accordingly. Still, as before, there were differences in the particular ways that this greater understanding of the curriculum influenced the teachers' instruction. Specifically, Cindy began to regularly adapt the CMW lessons claiming that she could do so because she now understood "where a lesson was headed." In contrast, Tracy began to use the curriculum with more fidelity explaining that she could now see how the curriculum worked and therefore was willing to follow it more closely. Interestingly, despite the fact that these changes were in opposite directions — Cindy implementing the curriculum with less fidelity and Tracy implementing the curriculum with more fidelity — the teachers' practices had become somewhat similar. They both followed the general outline of the curriculum. Furthermore, while they used most of the activities described for each lesson, they also made changes in light of the needs of their students. This research suggests that while teachers may exhibit broad differences in their initial use of new curriculum materials, by the second year of implementation, their practice can be similarly guided by an understanding of the goals of the materials. Future research is examining the extent to which the teachers' practices during the second year aligned with the goals of reform.

Conclusions and Implications

Generally, when we talk of curricula as being educative for teachers, we think of particular aspects of a lesson that can help a teacher to understand the reasons behind the curriculum designers' intentions or that can help a teacher to predict how students might respond to a specific lesson. Based on this research and on the work of our colleagues (Drake & Sherin, 2002), we suggest that *using* a curriculum can also be educative for teachers. In other words, the process of implementing a new curriculum, can, over time, help teachers to learn new ways of teaching mathematics and to come to new understandings of how their students learn mathematics.

This research has implications for the study of teacher learning, for the design of reform-based curricula, and for teacher education and professional development. First, this study advances our understanding of teacher cognition by offering insights concerning how teachers approach the implementation of reform. In particular, the study reveals that contrasting approaches to curriculum use can occur, even when teachers have similar beliefs about their own ability to implement a new reform-based curriculum. In terms of curriculum development, we suggest that curriculum designers pay attention not only to providing information on specific lessons for teachers, but also to providing broader strategies that teachers can use to implement a new curriculum effectively. For example, are there certain activities that teachers should tend not to omit? And are they particular kinds of adaptations that teachers should consider that would not change the goals of the curriculum? Finally, in terms of professional development, we believe that even veteran teachers can benefit from opportunities to discuss and examine their own approaches to using curriculum, particularly when these materials are reform-based and as such, call for changes in teachers' practices. With appropriate support, perhaps these teachers could have more quickly developed a deep understanding of the curriculum, an understanding that would have allowed them to implement the curriculum with fidelity, while still making appropriate adaptations for their students. In our continued research, we are examining the design of such professional development materials.

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ON THE CHALLENGE OF REFORMING TEACHING: MATHEMATICS TEACHERS AS MEMBERS OF A RHETORICAL COMMUNITY

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The post-apartheid South African government's "outcomes-based" educational policy, *Curriculum 2005*, refers to "all learners" which include adults and out-of-school youth as opposed to "students," in warm and reassuring ways. The policy aims to transcend "rigid divisions" such as theory and practice, knowledge and skills, academic and applied knowledge, and education and training (Department of Education, 1997). The mathematics component of this noble policy is called *Mathematical Literacy, Mathematics and Mathematical Sciences* (MLMMS) (see Government Gazette, 1997). Its lengthy name indicates the seriousness with which policymakers view the mentioned dichotomies. This paper focuses on contact points between rhetoric in MLMMS and ways mathematics teachers talk about their teaching. Such a comparison is important because it shows constraints and opportunities teachers face in espousing and enacting reform teaching practices according to MLMMS.

The participants in the study were three middle school and three high school teachers from very diverse school settings in and around Cape Town in the Western Cape. The latter is a province that has the highest success rate in the high-stakes end-of-the-year 12th grade examination in South Africa. It is also a province that wants to distance itself from the central government. The participants and the site of the study are therefore interesting in terms of seeing how the central government's ambitious policy statements will unfold at the classroom level.

Literature on educational slogans and slogan systems (Komisar & McClellan, 1961; Scheffler, 1964) enables one to identify habitually repeated phrases and words in policy documents like MLMMS and the NCTM (2000) *Standards*. In particular, this literature allows one to recognize rallying cries of different mathematics education traditions such as constructivism (Steffe & Gale, 1991), ethnomathematics (D'Ambrosio, 1985) and critical mathematics (Skovsmose, 1985; Vithal & Skovsmose, 1997) in policy documents. From here the notion of "policymaker rhetoric" is developed. Likewise teachers in schools form a rhetorical community with a recognizable vocabulary. The participants in the study were interviewed using methods adapted from "A study package for examining and tracking changes in teachers' knowledge" (Kennedy, Ball, & McDiarmid, 1993). Mathematics teachers speak in ways that reveal their knowledge and beliefs about the nature of mathematics, their espouse and enacted models of teaching and learning mathematics, the social context of their work, and their use of texts (Ernest, 1987). On a related point Gregg (1989) refers to the school mathematics tradition (SMT) in which mathematics is viewed as an isolated collection of complex rules and procedures, teaching is via direct instruction, learning is achieved through rote memorization of rules, and the textbook is the mathematical authority. From the transcribed interviews it was thus possible to identify "teacher rhetoric" which could be either "inside" or beyond the SMT.

Contact points between the teachers' rhetoric and policy statements come from excerpts in the interview data where I prompted the teachers on rallying cries such as "mathematics as reasoning" and "mathematical connections" which I identified in the "specific outcomes" of MLMMS. The teachers' responses to these rallying cries serve as evidence of how they react as a "rhetorical community" to similar MLMMS policy statements on reasoning and connections.

Results of the comparison show cases of "similar words and similar understandings" and "similar words and different understandings" between the two sets of rhetoric. There is also the case of the effect of the interview process on the teachers' rhetoric. The results point to the challenge of bridging lofty and sometimes jargon-laden policy statements and teachers' rhetoric. In addition, there is the challenge of supporting teachers' rhetoric with enactment in the classroom, implying the "creation of a rich conversation, in and around classrooms, about mathematics, teaching and learning" (Cohen & Ball, 1990, p. 337).

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MAKING MATHEMATICAL CONNECTIONS IN PROGRAMS FOR PROSPECTIVE TEACHERS

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The efforts of the past decade to improve precollege mathematics education have placed increasing demands on undergraduate programs to supply a strong cadre of teachers prepared to sustain these efforts and to help prepare all students to meet the technological demands of the 21st century. Research suggests that the undergraduate experience is critical in the development of prospective mathematics teachers (Fisher & Leitzel, 1997; National Research Council, 1996; National Science Foundation, 1996). The recent report from the RAND Mathematics Study Panel (2002) proposes a direction for further research.

> At the same time, more needs to be learned about the mathematical resources needed for teaching. Measures developed can only be as good as the conceptions of what needs to be measured. We propose systematic analytic and empirical work on what mathematical knowledge is needed in order to teach, and how it is deployed in the course of teaching for what sorts of tasks. (p. 66)

It has been suggested that throughout the undergraduate experience, prospective teachers need the opportunity to explore the connections between the mathematics they are studying and the school mathematics they will be teaching (Fisher & Leitzel, 1997).

This paper is the result of a pilot study designed to explore how prospective teachers of mathematics develop an understanding of undergraduate mathematics. In particular, we examined how prospective teachers' knowledge of mathematics develops in the context of curricula designed to facilitate students' connections between two content areas of undergraduate mathematics. The context for the study was an undergraduate course in geometry required of prospective mathematics teachers. During the course a set of undergraduate curriculum modules were developed and implemented (funding for the project is provided by NSF-DUE CLLI Grant #9981029). The modules provided the students the opportunity to make connections between the mathematical content areas of transformational geometry, linear algebra and abstract algebra.

Ball (1991) states that "although something called 'subject matter knowledge' is widely acknowledged as a central component of what mathematics teachers need to know, little agreement exists about what this means or how to tell whether teachers have it" (p. 82). She suggests that the traditional means of measuring understanding, "course work, grades, and test scores," are not sufficient and that a change in the outcomes of school mathematics will depend in part "on closer and more serious consideration of the mathematics that the teachers need to understand, as well as how, when, and where they can acquire this kind of understanding and how we can assess it" (p. 82). This project provided a context within which to investigate the following question related to Ball's comments: What are the characteristics of a prospective teacher's understanding of undergraduate mathematics in the context of this integrative experience?

Integrated units in transformational geometry and linear algebra provided the context for the investigation of the above research question related to the development of prospective teachers' mathematical understanding. The materials were implemented in a junior level undergraduate geometry course in the fall semester 2002. There were twenty students enrolled in the course. Results reported here are products of journal entry and interview data. Three interviews were conducted during the course of the semester and the interviews focused on the students' beliefs about mathematics, understandings of mathematics, and the connections they were developing between the content areas of mathematics and between precollege and college mathematics.

Interview transcripts were analyzed using open and selective coding techniques (Strauss & Corbin, 1998). Theories of Sfard (1991), Dubinsky (1991), and Tall, Thomas, Davis, Gray, & Simpson (2000) informed the data analysis. In particular, Tall and colleagues (2000) discuss various theories which describe students' transition from viewing mathematical ideas as processes to viewing them as objects. These transitions are sometimes called "encapsulation" (Dubinsky) or "reification" (Sfard).

Tall et al. (2000) also discuss the difference between students' *perceived* geometric objects (drawings, etc.) and *conceived* geometric objects (mind's eye, uniform, etc.). They state "...the construction of perceived geometric objects leads later to conceived geometric objects, which, though, imagined in the mind's eye, and discussed verbally between individuals, are perfect entities that have no real-world equivalent" (p. 236). Student's views of geometric ideas may influence their abilities to reason geometrically and data from this study has shown evidence of two such instances.

Students in this study first saw transformations as processes that map geometric objects onto other geometric objects. Data analysis suggests that students viewed transformations as processes but had difficulty utilizing them in geometric proof. Evidence indicates that students are seeing transformations as acting on perceived objects and that this inhibits their ability to construct geometric proofs using transformations. The curriculum in this study treated transformations as objects that carried a group structure. Data analysis suggests students had difficulty viewing transformations as a group. Evidence suggests that students' views of transformations as processes, not as objects, caused them difficulty in seeing the inherent group structure. There is also evidence that indicates students had not encapsulated, or reified, transformations and therefore had not begun to view them as objects.

This research supports the view that in geometry students' encapsulation of mathematical ideas as objects may be supportive to their mathematical understanding. In the context of this study the following questions arise that form the basis for future data analysis and research: Were students seeing transformations as processes that acted on perceived objects or processes that acted on conceived objects? What role might these views play in a student's ability to construct geometric proofs? In geometry what might enhance student transition from working with perceived to working with conceived objects? What experiences might enhance students' abilities to encapsulate the mathematical idea of transformations?

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TEACHERS' NUMBER SENSE: ITS INFLUENCE ON TEACHING

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Studies have focused on the development of number sense in children, but studies involving adults are scarce, if not, non-existent. Unfortunately, elementary teachers are often among those that do not possess a good sense of number. If reform is to be successful and if all students are to achieve success in mathematics, teachers must possess a good sense of number. How can teachers foster number sense development in students if teachers do not understand what number sense is? Studies addressing how the number sense development of teachers may affect the number sense development of students have not been found. It is vital to research this area to assure competent teachers of elementary school mathematics. This discussion highlights a study involving preservice teachers in an elementary mathematics methods course with a number sense emphasis. The teachers' concept of number sense and that influence on teaching mathematics is investigated within this inquiry.

How do teachers develop number sense in students? Research suggests that activities such as mental computation, estimation, and problem solving discourse (Sowder, 1992) help to foster number sense development. Learning about number and operation is a very complex process (Fuson, 1992). According to the National Council of Teachers of Mathematics' (NCTM) (NCTM, 2000) "...understanding number and operations, developing number sense, and gaining fluency in arithmetic computation form the core of mathematics education for the elementary grades" (p. 32). This must progress into a fully developed sense of number in high school students. The development of number sense does not always occur in this manner. Teachers bring prior experiences and understandings that profoundly shape their learning (Ball, 1996). From these experiences teachers have formed ideas about their role as a mathematics teacher, about who can learn mathematics, and about what it takes to learn and know mathematics. Elementary teachers are the products of the very system they are now trying to reform.

How number sense develops is unclear. Sowder (1992) investigated components of number sense and how they are exhibited: numeration, number magnitude, mental computation, and computational estimation. A richness of student involvement activities provides positive effects versus the negative effects of emphasis on calculations. Shull (1998) investigated the development of number sense in students over a three-year time period in which one essential indicator of number sense appeared to be the ability to generalize. Results from a direct instruction and number sense inquiry (Markovits & Sowder, 1994) indicate that students had reorganized existing knowledge rather than learned new knowledge. Instruction emphasized conceptual knowledge and allowed the students to make connections between prior knowledge and new knowledge. Curriculum and instruction that incorporate a constructivist approach to learning (learner-centered) have a greater potential to offer learning opportunities for developing a sense of number. Students need to develop good number sense to be effective problem solvers. Efforts must be made in the classroom to help students make connections between what they know about numbers and what they know about operations on numbers. Since number sense is not a set of isolated facts, it cannot be assessed just by written paper-and-pencil tests. Sowder (1989) believes that number sense must be assessed in a non-traditional manner. Alternative forms of assessment are warranted.

The Context

A course goal was to develop understanding of mathematical concepts in a non-algorithmic way and to foster the development of numbers sense in teachers. A standards-based, constructivist approach to instruction was implemented. Mental computation exercises were conducted regularly in class. A timed, written number sense test, a mental math test were administered that served as a pretest/posttest assessment along with a free-response questionnaire. Data were collected from three different methods classes, part of a field-based program in southeast Texas. Questions on the 45-item test were developed using the Framework for Examining Basic Numbers Sense (McIntosh, Reys, & Reys, 1992). The majority of teachers' number sense, as measured by the number sense test, seems to be the same. Two or three points in either direction were interpreted to being the same score. A few students seemed to demonstrate a high level of change (40% increase) and others decreased. An item analysis was performed. The results were mixed. The exit assessment (four multi-level, free-response questions) assessed the level of teachers' number sense after receiving instructional strategy sessions. It was assumed that if the teacher displayed the ability to generalize, then this may be an indicator of number sense held. This would reinforce previous findings (Shull, 1998). Data were collected and analyzed for accuracy and completion of response. Responses were rated as high, medium, and low according to displayed level of accuracy and comprehension. Results were mixed and preliminary results are inconclusive. However, further examination, analysis, and re-investigation is warranted.

Discussion

It was questioned if teachers' perception of number sense could affect the development of number sense in students. This discussion reviews an inquiry into the number sense of elementary teachers and the possible relation with teaching elementary mathematics and student development of number sense and mathematical understanding.

Many factors may have been critical in the results obtained. The profileof the teachers reveal that most hold interest in early childhood or language arts (not mathematics) and that many have English as a second language. These factors and the time-constraint factor of the testing should be examined in more detail before proceeding with further reporting and analysis of data. Preliminary results from data collected from the pretest and posttest mental math instrument indicate some growth in number sense ability is possible. Future inquiry and treatment plans include more emphasis on mental math and the development of number sense with qualitative measures such as, "What is number sense?". Research investigating and developing number sense in pre-service teachers is sparse. Instruction on estimation and mental computation provides an avenue for developing number sense in students (Sowder, 1992). Implementing this research into the practice of preservice teacher instruction in methods courses is necessary. Number sense is believed to be the ability to think on one's feet. It is crucial that elementary teachers have a strong, solid sense of number to successfully provide curricula, instruction, and assessment that foster number sense growth in students. From this study, data provide some insight into the deficiency, strength, and growth of number sense and indicate need for further study.

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ELEMENTARY TEACHER STUDENTS' MATHEMATICAL UNDERSTANDING EXPLAINED VIA CONCEPTUAL CHANGE

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In Finland the elementary teachers are responsible for mathematics teaching to children during their first six years of schooling. They teach the basic mathematical basic skills and moreover, during those years they also build a basis for children's future attitudes toward mathematics. Some professors of mathematics consider that these first years of school mathematics, however, give a weak basis for all future mathematics instruction due to the low level of mathematical skills of the elementary teachers themselves (cf. Martio 1998; Pehkonen 1999). In this elementary teacher students' mathematical skills and understanding is discussed.

In learning mathematics, every extension to the number concept demands, not only accepting new concepts, but new ways of thinking as well. This new thinking more or less contradicts the prior fundamental logic of natural numbers (Merenluoto, 2001). Therefore, misconceptions and learning difficulties are possible at every enlargement of number domains. The very fundamental idea of successor, for example, is necessary for learning the notion of natural numbers. It is, however, seriously conflicting with the understanding of the very character of both rational and real numbers (Sowder, 1992). In the domains of rational and real numbers, the principle of successor is not defined but in every interval infinite successive division is possible. This quality makes the sorting of these numbers difficult, because the students' are prone to use their knowledge of whole numbers in the process. Understanding the successive division demands intuition on infinite processes which, however, in students' thinking seems to be dominated by the finite processes of everyday thinking (Merenluoto, 2001).

According to the theories on conceptual change (cf. Carey, 1985; Vosniadou, Ioannides, Dimitrakopoulou, & Papademetriou, 2001) the relationship between learners' prior knowledge and new information to be learned is one of the most crucial factors in determining the quality of learning. Researchers make a distinction between two levels of difficulty in the learning process targeted at conceptual change: a continuous growth and a discontinuous change. In this distinction the easier level of conceptual change is called enrichment kind of learning, suggesting continuous growth or improving the existing knowledge structure. The more difficult conceptual change is needed when the prior knowledge is not incompatible with the new information but needs revision.

Objectives and Methods

The aim of this paper is to analyze the elementary teacher students' mathematical skills and their understanding of school mathematics. Furthermore, we try to explain the results obtained from the viewpoint of theories of conceptual change.

In November 2000, elementary teacher students' basic skills and understanding in school mathematics were measured with a level test in the Department of Teacher Education at the University of Turku (Finland). New teacher students' skills in elementary school mathematics (i.e., in grades 1–6) were decided to check with the level test. The aim of this test was to filter those few students who might possibly still have difficulties in some parts of elementary school mathematics. Most of the first year elementary teacher students ($n = 70$; i.e., 88% of all) participated the level test.

The level test was composed of two different parts: In the first part, there were ten simple mechanical tasks with one correct solution. They are similar to tasks to be found in every elementary mathematics textbook. In the second part, there were six open tasks with several solutions. Such tasks are rare in school textbooks. All tasks in both parts can be solved with elementary school (i.e., grades 1–6) mathematics.

Results

The results from the level test refer to serious problems in elementary teachers' number concept. All ten tasks of the mechanical part of the level test were correctly solved by less than one third of the students (29%). Less than half of the students (48%) received the result needed for passing the level test (at least nine tasks correctly). In the six problem tasks of the level test, the students had remarkably more difficulties. The mean score in the first part was 77% and in the second part 45%, the difference between them was statistically significant ($F(1, 69) = 134.82$; $p < .001$).

When considering the amount of correct answers, almost everybody has the tasks done correctly, when using whole numbers. But in the tasks demanding skills in fractions and decimal numbers, about three fourths of the students (76%) were successful. The most difficult task was to find the decimal presentation for 5/17, where many students divided 17 with 5. The difficulties the subjects had in handling fractions and decimal numbers, refer to serious deficiencies in their number concept.

Problems with rational numbers were especially obvious in the task, where the concept of successor was embedded. For example, the results of task 13 (from the second part of the level test) *"Try to find out which would be the biggest fraction which is still smaller than 3. Give reasons for you answer!"* describe the situation clearly: The task was unanswered by one-fifth of the students (21%), and a clear "one smaller" method of the whole numbers was used by one-fifth (19%). Only one-tenth of the respondents (11%) wrote clearly the correct answer that there is no such number.

Especially in the problem tasks the results refer to a low level of mathematical thinking and to the restrictive nature of thinking based on whole numbers. Task 11 (from the second part of the level test) *"a) Find out numbers in the empty places in such a way that the calculation is valid:* $(\_\_ + 12) : 3 - \_\_ = 5$. *b) Do you find another solution? c) How many solutions do you think that there exit?"* deals with unlimited amount of solutions. Here, about one-fourth (24%) found correctly two alternative solutions and recognized that there are even more, but noted carefully to it with words like "many" or "several". Almost one-third of the students (27%) had an idea of unbounded, uncounted or infinite number of solutions, but they were not able to reason their answer. Only two students (of 70) presented a high-level solution where the infinite number of the solutions was based on the relation between the unknown variables, and where the numbers used were not restricted to positive whole numbers.

Discussion

The results refer to serious problems in the numbers concept, and to a low level of mathematical thinking of almost all the elementary teacher students. The results also suggest that the use of whole numbers and everyday experiences dominates their thinking. Thus it is possible that the difficulties the students had refer to problems of conceptual change in their number concept. Further it is also an indication that the extension of the number domains has not been dealt with thoroughly enough from a theoretical viewpoint, but teachers have rushed into practice with new numbers.

Within the theories of conceptual change, it has been recently emphasized that, the prior knowledge is can be very resistant to teaching, and that the process of a radical change can be very slow, and that the students' are easily prone to 'fall back' to their previous thinking. It is possible that most of the students were not even aware of the need to reconsider their initial thinking on numbers.

In teaching of school mathematics, these problems should be recognized, and the construction of students' number structure needs to be supported. One of the presumptions for the conceptual change is the ability for meta-conceptual awareness of one's thinking. It is suggested that the elementary teacher program could benefit from consideration of the viewpoints of conceptual change.

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TEACHING MATHEMATICS IN NEW WAYS: THE ROLE OF PEDAGOGICAL KNOWLEDGE

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This paper examines the teaching practices of a first-grade teacher who was uncomfortable with mathematics and who identified herself as a "weak" mathematics teacher. In other respects, Ms. Jones was considered an exemplary teacher. In particular, she enjoyed teaching reading and was skilled at engaging her students in authentic and interactive activities related to literacy. Given the discontinuities between her views of mathematics and literacy, this study aimed to examine and compare her practices in these two areas. During this study, Ms. Jones used a reform-oriented curriculum, *Investigations in Numbers, Data, and Space* (TERC, 1998) for the first time and participated in regular mathematics study group meetings. Ms. Jones' feelings about mathematics and literacy were typical of many elementary teachers confronting mathematics education reform (Ball, 1994). Understanding what resources she drew on in navigating this new terrain as well as what challenges she faced can offer insights into what is involved in supporting teachers as they attempt to change their mathematics teaching practices.

Theoretical Framework

Many have argued that teachers' mathematical knowledge (Ball, 1994; Heaton, 1992) and beliefs about mathematics, teaching, and learning (Smith, 1996; Thompson, 1992) significantly influence their practices. Thompson, for example, argues that teachers' beliefs about mathematics and the pedagogy associated with its teaching "play a significant role in shaping the teachers' characteristic patterns of instructional behavior" (p. 130). For example, teachers who view mathematics as a body of rules and procedures, tend to emphasize this conception in their teaching (Remillard, 1999, Thompson, 1992). The depth and nature of a teacher's mathematical knowledge is viewed as integrally connected to her beliefs and practice (Wilson, Shulman, & Richert, 1987).

This depth of knowledge is critical for teachers aiming to respond to current reform efforts in mathematics education that emphasize conceptual understanding, reasoning and problem-solving. Such practices place significant organizational and intellectual demands on any teacher (Ball, 1994). For teachers who lack mathematical knowledge, engaging in such practices can be intimidating. Studies have documented teachers' attempts to use new curriculum materials or experiment with reform-inspired practices that have resulted in mathematically problematic teaching (Cahnmann & Remillard, in press; Heaton, 1992). As these authors argue, helping students learn to think mathematically is challenging work that cannot be prescribed in a teacher's guide.

What does this mean for elementary teachers who teach many subjects? Can pedagogical knowledge developed in one subject area contribute to that in others? Flick (1995) studied an elementary school teacher whose scientific content knowledge was weak, but who readily engaged students in inquiry-based reading instruction. These well developed pedagogical skills served as "assets in negotiating the rapid flow of relatively unstructured information typical of inquiry in elementary classrooms" (p. 1065). In contrast, Spillane (1995) documented a case of an elementary teacher whose ability to facilitate inquiry-oriented discussion amongst her fifth graders in the area of literacy was cut short when it came to mathematics. He argued that "subject-matter is a critical variable in understanding teachers' efforts to reconstruct their teaching" (p. 44). In the current study, we explored when and how a teacher's pedagogical practices might transcend subject areas and serve as resources in the development of others.

Methods/Data Sources

This study is part of a larger study examining teacher learning, classroom practices, and student learning in urban schools. We examined one year in a first grade classroom in an urban elementary school in which all 30 students in the class were African-American and many were from low-income families.

The data were based on about 40 hours of classroom observations of the teacher's mathematics and reading teaching and 4 formal tape-recorded interviews. We also had a number of informal discussions and correspondences with her. The interviews were open-ended with guiding questions that included, but were not limited to, beliefs about teaching and mathematics and pedagogical practices. Analysis began by using field notes to characterize and then compare the teacher's practices in mathematics and literacy. In order to explain the patterns and contrasts, we coded transcripts of the interviews and analyzed them according to emergent and predetermined themes.

Results

Our analyses of the patterns in Ms. Jones' pedagogical practices during reading instruction as well as mathematics revealed several notable practices that seemed to transcend subject matters. These include a) a deep commitment to highly interactive lessons that included significant student participation and interaction, b) an appreciation of student-student interaction during instruction, and c) the inclination to make connections to previous lessons.

One distinction we noted between Ms. Jones' reading and mathematics lessons was in the level of flexibility she brought to the subject. In reading, she frequently veered from her plans to follow the lead of a student or to make connections to other student experiences. In one instance, when a student observed rhyming words, Ms. Jones turned the conversation to an exploration of rhyming words and sounds. At other times, she encouraged students to find relationships between events discussed in science or elsewhere and the content of the reading lesson.

Ms. Jones identified the ability to respond to students spontaneously and to take advantage of their observations and insights as a critical component of her teaching. "I think spontaneous is much better because I think you're teaching to...with the children, rather than just 'ok, . . .this is what comes next'" (Interview, 3-2-02). She was highly conscious of the fact that she was not comfortable making spontaneous changes in her mathematics lessons. As she put it, "I don't feel secure enough about that at all to just be spontaneous" (Interview, 3-2-02). Indeed, during each of the mathematics lessons we observed, Ms. Jones designed a lesson using an activity in the *Investigations* curriculum and then followed her plans rigidly. Her discomfort with responding to students spontaneously was also evident in a pattern we observed in her selection of suggestions from the curriculum; she rarely brought her students back together after an activity to attempt a class discussion—a suggestion that occurs frequently in the curriculum.

It is evident that her strong pedagogical skills in reading, the availability of a reform-oriented curriculum guide, and classroom support were not enough to help Ms. Jones overcome her discomfort related to teaching mathematics. In fact, her discomfort formed a formidable barrier to her efforts to teach mathematics for meaning. Throughout the year, she admitted that mathematics was the first subject to be dropped from her often-packed schedule. We noticed this tendency increased as the year progressed.

In endeavoring to explain the difficulties Ms. Jones had teaching mathematics, we sought to go beyond the claim that she had limited knowledge of mathematics necessary to develop mathematical pedagogical content knowledge (Wilson, et al., 1987). The knowledge gaps that appeared most significant in inhibiting her efforts were related to her understanding of the purposes underlying the activities in the curriculum and their connections to the mathematics she thought her students needed to learn. In several conversations, Ms. Jones stated that she was concerned that her students were not learning basic facts and that she felt unsure how to respond to parents' concerns about the mathematics she was teaching. Interestingly, the conflicts that Ms. Jones felt led her to avoid teaching mathematics, rather then supplement with other material.

Conclusion

It is our view that Ms. Jones' generic pedagogical strategies were critical for helping her experiment with unfamiliar approaches to teaching mathematics. However, they were not enough to help her interpret the ideas in the curriculum and connect them to what she understood about mathematics. While it is evident that Ms. Jones would benefit from a deeper understanding of mathematics, we argue that her needs were specific to her curriculum. Her case illustrates difficulties that arise when teachers use reform-oriented curricula. It also suggests the potential for deepening teachers' mathematical knowledge within the context of exploring new curriculum materials.

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AN EXAMINATION OF THE KNOWLEDGE BASE FOR TEACHING AMONG UNIVERSITY MATHEMATICS FACULTY TEACHING CALCULUS

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This poster session will report on a dissertation research study designed to examine teacher knowledge base (TKB) among university mathematics faculty. Research on TKB at the K-12 level provides a solid foundation for exploring related issues in post-secondary education. This study seeks to uncover the categories and components that constitute a framework for TKB in undergraduate mathematics, identify themes that exist in the TKB of university mathematics faculty, and examine issues of consistency between the TKB of university mathematics faculty and their instructional practice. Preliminary results and implications for future research will be addressed.

Teacher Knowledge Base

Several national mathematical organizations (National Research Council, 1991; National Science Foundation, 1996) have declared a "state of crisis" in undergraduate mathematics education and have recommended that researchers focus more attention on the teaching and learning of undergraduate mathematics. Research (Fennema & Franke, 1992), nearly exclusive to the K-12 level, has shown that one's "knowledge base for teaching" has the most significant influence on his or her instructional practice. What types of knowledge are needed for teaching mathematics and how teachers utilize their knowledge base to inform their instructional reasoning and action are both compelling questions (Wilson, Shulman, & Richert, 1987). Shulman (1987) articulated a framework of TKB that included knowledge of content, context, general pedagogy, curriculum, learners, educational ends and pedagogical content. Considerable research has been conducted in this area, yet existing models of TKB still need to be refined.

This study extends this line of investigation to the undergraduate calculus classroom. A framework of the knowledge base for teaching undergraduate mathematics (beliefs, context specific knowledge, general pedagogical knowledge, personal practical knowledge, the nature of disciplinary mathematical knowledge, and knowledge of research) was developed and piloted. The purpose of this research is to investigate TKB among university mathematics faculty and examine the nature of its consistency with their instructional practice. Themes and patterns that exist related to TKB among university mathematics faculty will be identified and will provide a foundation for future research aimed at reform in the teaching and learning of undergraduate mathematics.

Eight university mathematics faculty from four-year institutions will be interviewed, observed, and video-taped. Interview data will be coded and sorted to identify existing themes, which will be entered into a text matrix for within- and cross-case analyses. Classroom observation field notes will allow for representative segments of the participant's instructional practice to be identified. A representative video lesson segment will be analyzed using a coding scheme adapted from the research (Brenderfur & Frykholm, 2000; Schoenfeld, 1998; Wells, 2002). This model will be used to analyze classroom instruction, including the mathematical discourse evident, at a microscopic level to explain how the knowledge, beliefs, goals and images of the teacher combine to produce his or her actions in the classroom. Preliminary results from this research are forthcoming and will be reported in the poster session.

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