DOCUMENT RESUME

ED 466 962 SE 066 540

AUTHOR	Mitchelmore, Michael C.
TITLE	The Role of Abstraction and Generalisation in the Development of Mathematical Knowledge.
PUB DATE	2002-00-00
NOTE	12p.; Paper presented at the East Asia Regional Conference on Mathematics Education (EARCOME) (2nd) and the Southeast Asian Conference on Mathematics Education (SEACME) (9th, Singapore, May 27-31, 2002).
PUB TYPE	Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE	EDRS Price MF01/PC01 Plus Postage.
DESCRIPTORS	Abstract Reasoning; *Cognitive Processes; Concept Formation; Elementary Secondary Education; Foreign Countries; Generalization; Geometry; Mathematics Instruction; Proof (Mathematics); Thinking Skills
IDENTIFIERS	*Angles (Geometry)

ABSTRACT

Although mathematics deals with generalizations relating abstract ideas, very little attention has been given in the mathematics education literature to the role of abstraction and generalization in the development of mathematical knowledge. In this paper, the meanings of "abstraction" and "generalization" are first explored by examining various definitions given by mathematics educators and others. These concepts are illustrated by reference to the author's research on children's understanding of angle concepts. The question of how mathematics teaching would be affected if more explicit attention were given to abstraction and generalization in the classroom is then considered. Finally, some recent experiments in "teaching for abstraction" are presented and discussed. (Contains 29 references.) (Author/MM)

THE ROLE OF ABSTRACTION AND GENERALISATION IN THE DEVELOPMENT OF MATHEMATICAL KNOWLEDGE

Michael C. Mitchelmore
Macquarie University, Sydney, Australia
E-mail: mmitch@ted.educ.mq.edu.au

Although mathematics deals with generalisations relating abstract ideas, very little attention has been given in the mathematics education literature to the role of abstraction and generalisation in the development of mathematical knowledge. In this paper, the meanings of "abstraction" and "generalisation" are first explored by examining various definitions given by mathematics educators and others. These concepts are illustrated by reference to the author's research on children's understanding of angle concepts. The question of how mathematics teaching would be affected if more explicit attention were given to abstraction and generalisation in the classroom is then considered. Finally, some recent experiments in "teaching for abstraction" are presented and discussed.

Children's Knowledge of Angles

Many of the ideas to be presented in this paper derive from research on children's understanding of angles conducted over the past decade by Paul White and myself. So this first section summarises our research on angles. Those who are familiar with our research may confidently skip to the next section.

Australian students (and they are probably not alone in this) find angles difficult to understand. For example, Grade 7 teachers always despair at teaching students to use a protractor to measure angles. Our research suggests that the source of students' difficulties is their inability to link angles in different contexts. Learning to use a protractor is difficult because they have been taught in Grade 4 that "angle is an amount of turning", whereas the angles they are confronted with in Grade 7 textbooks do not turn at all—and neither does the protractor. To learn how to use a protractor to measure angles, they have to link two very different contexts: the dynamic turning context and the static diagram context.

On the other hand, we have found that young children can easily link physically similar angle contexts. For example, almost all Grade 2 children could show how the angles in two turning contexts (a toy ballerina and a regulator knob) are related (Mitchelmore, 1998). A similar success rate was found among Grade 2 children in relating the angles in scissors and road bends (Mitchelmore & White, 1998) - but not the angles of opening in doors and tilting windows (White & Mitchelmore, 2001).

Two questions arose from our earlier research: (1) Among all angle contexts, which do students find most similar and why? (2) How does their recognition of angular similarity change as children grow older? Do distinct clusters of angle concepts develop (e.g., turns and corners) or only one angle concept? To investigate these questions, we presented 192 students in Grades 2, 4, 6, and 8 with examples or models of 9 different angle contexts and asked them to show us how the angles were similar—for example, by using a bent straw to indicate the angles in each context (Mitchelmore & White, 2000a). The variation in the similarity matrices at each grade level was then explored using cluster analysis.

Figure 1 summarises the results. At each grade level, there was one main cluster. In Grade 2, this consisted of junction, tile, and wall; scissors was nearby and joined the main cluster in Grade 4. By Grade 6, the main cluster consisted of junction, tile, wall, scissors, fan, and signpost. Even by Grade 8, the remaining contexts (door, hill, and wheel) had not joined the main cluster. The results suggest that the angle concept develops from a single core concept which gradually expands to include other contexts. They also suggest that the major factor affecting children's recognition of angular similarity is the presence of visible lines and vertices: the angle arms in the scissors, fan, and signpost contexts are rather indefinite; the doors and the hill only have one arm of the angle visible; and a wheel has no visible arm.

EARCOME 2002 Proceedings
Volume 1, Plenary & Regular Lectures, 157-167

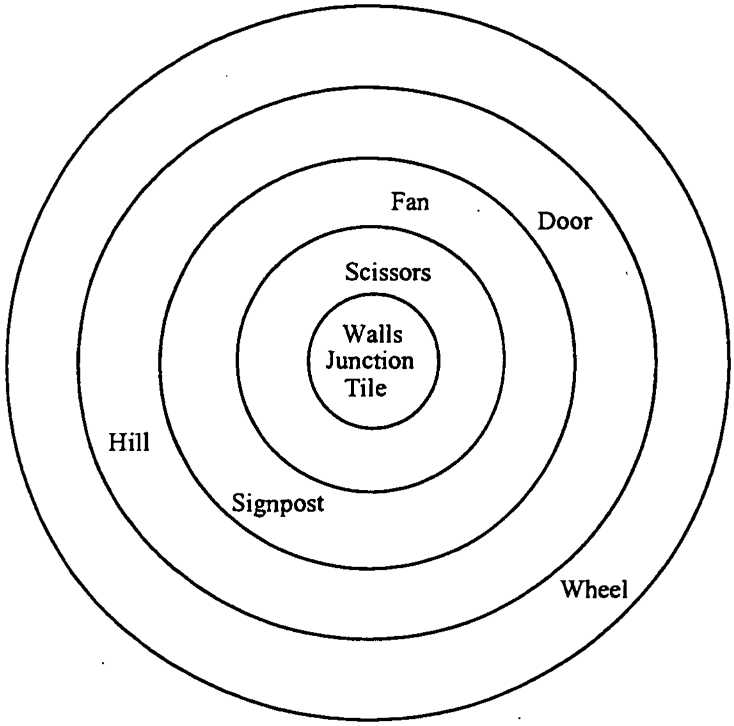

Figure 1 Development of children's angle concepts.

The theoretical basis for the design and interpretation of the above investigation was abstraction theory. The next section describes this theory.

Abstraction

There are many definitions of abstraction—going back at least to Aristotle, who defined it as the omission of qualities from concrete experience (Damerow, 1996, p. 72). Davidov (1972/1990, p. 13) defined it as the process of "separating a quality common to a number of objects/situations from other qualities", and Sierpinska (1994, p. 61) defined it as "the act of detaching certain features from an object". Paul White and I prefer the definition given by Skemp (1986):

> *Abstracting* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept*.

It is most important to notice that all these definitions emphasise abstraction as a *process*. The direction is *from* a set of contexts *to* an abstract concept, and not the other way around.

Common terminology (e.g., "counters are an embodiment of the whole numbers" or "displacements are a model of the integers") suggests that concepts exist prior to the identification of contexts in which they are to be found, a possibility which only Platonists could defend. It is preferable to say "counters are a whole number context" or "displacements are an example of integers". One can also say that a concept *encapsulates* a similarity (Dubinsky, 1991).

Piaget and others have distinguished between *empirical* and *reflective* abstraction. (See Dubinsky, 1991, for a succinct summary of Piaget's writings on abstraction.) Empirical abstraction (*abstraction à partir de l'objet*) is based on superficial similarities and is the type of abstraction involved in everyday concept formation. For example, the concept of "dog" is an encapsulation of the similarity between certain animals in a child's experience.

Our research shows convincingly that empirical abstraction occurs early in the process of angle concept development. Well before Grade 2, students have formed concepts related to many individual angle contexts by empirical abstraction (Mitchelmore, 1997). For example, the concept of "tile" has been formed by recognising physical similarities between all the tiles children have experienced—but few children would be able to say exactly what a tile is. The contexts in the core of Figure 1 (walls, junction, and tile) are also superficially similar in that they contain two clear linear components meeting at a point; in this case, the similarity is easily defined in words. Empirical abstraction undoubtedly occurs in the early stages of the development of many other spatial concepts, such as triangle, symmetry, and circle.

Reflective abstraction (*abstraction à partir de l'action*) is, according to Piaget, based on reflection on one's actions. For example, reflection on the fact that when you put one object with two objects you always obtain three objects leads to recognition of an invariance (later expressed as $1 + 2 = 3$). The objects of the invariance become concepts (the numbers 1, 2, and 3) and the invariant action becomes a relation between these concepts (addition). In reflective abstraction, concepts and relations are frequently abstracted together.

Reflective abstraction occurs late in the development of children's angle concepts. There is no superficial similarity between tiles and wheels, for example. To recognise their angular similarity, a student must firstly focus on one spoke of the wheel and then relate its starting and finishing positions. This process is particularly difficult if the wheel does not have spokes! But even if it does, in order to form the angle either the starting or the finishing position has to be imagined or remembered—whereas the two arms of the angle on a tile are both present at all times. The formation of a general angle concept (i.e., one that includes all possible angle contexts) is thus a particularly analytical, constructive process. One may ask what action, in the Piagetian sense, drives such abstraction. Perhaps it is angle combination and measurement. For example, students may recognise the similarity between four square tiles fitted around a point and a toy ballerina making four quarter-turns, and then begin to compare corners and turns in general. Whatever the action, it is probably more the result of explicit instruction than spontaneous investigation.

Several authors have attempted to identify stages in the abstraction process. Thus Sfard (1991) described abstraction as an interiorisation-condensation-reification cycle, and Dreyfus (1991) spoke of a generalization-synthesis-abstraction cycle. The important point is that abstraction does not consist of similarity recognition alone: The similarity must be abstracted and formed into a mental object in its own right. For example, a child might be able to tell you that "one more than two children is three children", "one more than two cars is three cars", "one more than two elephants is three elephants", and so on for every imaginable object. But we cannot claim abstraction of the whole numbers 1, 2, and 3 until the child can confidently assert that "one more than two is three".

In the popular mind, "abstract" tends to mean not concrete or decontextualised, therefore unreal or even meaningless. In particular, mathematical concepts are often regarded as meaningless symbols that have to be manipulated according to stated rules. Indeed, a professional mathematician must at least act as if this is the case. We call such concepts *abstract-apart*, meaning that they exist apart from any contexts from which they might have been abstracted. Concepts that have been abstracted through the recognition of similarities between contexts we call *abstract-general* (Mitchelmore & White, 1994). Such a concept derives its (initial) meaning from the set of contexts from which it has been abstracted.

Generalisation

As for abstraction, there are many meanings for the term "generalisation". By analysing the literature, Paul White and I have been able to group them into three categories, which we call G1, G2, and G3 (see Figure 2).

G1. A synonym for *abstraction* or *concept.*

G2. An *extension* of an existing concept:
  Empirical extension
  Mathematical extension
  Mathematical invention

G3. A *theorem* relating existing concepts.

Figure 2 Three meanings of "generalisation".

Many writers use generalisation as a synonym for abstraction (G1). For example, Davidov (1972/1990, p. 10) defines it as "finding and singling out [properties] in a whole class of similar objects" and, as we have seen, Dreyfus (1991) considered it the first stage of abstraction. Paul White and I use the same meaning when we talk of "abstract-general" concepts.

There are at least three senses in which generalisation means extension (G2).

- Empirical extension (called *primitive generalisation* by Dienes, 1963, and *expansive generalisation* by Harel & Tall, 1991) occurs when a person finds other contexts to which an existing concept applies. For example, an infant recognises a large, white, shaggy animal as a dog even when she has only had experience of small, brown, sleek dogs. Or an older child accepts a scalene triangle as a triangle even though his textbook only has diagrams of isosceles and equilateral triangles.
- Mathematical extension (called *mathematical generalisation* by Dienes, 1963, and *reconstructive generalisation* by Harel & Tall, 1991) occurs when one class of mathematical objects (e.g., the whole numbers) is embedded in a larger class based on a different similarity (e.g., the integers). In this case, the similarity relating members of the first class has to be viewed in a new light (i.e., reconstructed) in order to also relate members of the larger class.
- Mathematical invention (called *Cartesian abstraction* by Damerow, 1996, and *creative generalisation* by Fischbein, 1996) occurs when a mathematician deliberately changes or omits one or more defining properties of a familiar concept to form a more general concept. The most familiar example of this kind of generalisation is probably the invention of non-euclidean geometry by varying the parallel axiom. Such generalisation can only create abstract-apart concepts.

The term generalisation often refers to a relationship that holds between all members of some set of objects (G3). Typical examples of this meaning of generalisation are "dogs don't live as long as humans" and "$x \quad 2x = 3x$". Similar to Piaget's distinction between empirical and reflective abstraction, Davidov (1972/1990) contrasts empirical and theoretical generalisation (in the sense of G3). An *empirical generalisation* is one based on experience whereas a *theoretical generalisation* can be explained by reference to other knowledge in some sort of theoretical structure. Both "dogs don't live as long as humans" and "$x \quad 2x = 3x$" could be empirical generalisations, but only the second example is likely to become a theoretical generalisation for most school students.

It seems preferable to use generalisation in a sense different to abstraction, so meaning G1 will be ignored. Both G2 and G3 then involve the notion of widening a *domain*—either of a concept (G2) or a relation (G3)—without necessarily leading to a new concept. Wherever a new concept is formed, abstraction is also involved.

Generalisation (G2) plays an important role in the development of the angle concept. We have already seen how similarity recognition expands outwards from a core (see Figure 1). The expansion from contexts where the two arms of the angle are easily recognisable (walls, junction, tile) to include contexts where the two arms are somewhat disguised (scissor, fan, signpost) is a perfect example of empirical extension. Expansion of the angle concept to include contexts where only one or no arms are physically present (hill, door, wheel) is best thought of as mathematical extension: The initial concept of an angle has to be reconstructed in order to encompass the wider variety of contexts.

Generalisation (G3) occurs alongside the development of the angle concept. Within each physical angle concept, children make empirical generalisations (G3) early on. For example, Grade 2 students know a great deal about the effect of slope on speed of descent (Mitchelmore, 1997). Theoretical generalisations (e.g., two right angles make a straight line) cannot be made until a general angle concept is formed, and probably start to emerge at the same time as the general concept.

Teaching Generalisations (G3)

Generalisations, in the sense of G3, are at the core of school mathematics—numerical generalisation in algebra, spatial generalisation in geometry and measurement, and logical generalisations everywhere. A great deal has been written in mathematical methods books on how to teach such generalisations. I have identified three categories of methods of teaching generalisations (Mitchelmore, 1999).

In the *ABC method*—Abstract Before Concrete—generalisations are taught as abstract relations which have to be learnt before they can be used in any concrete application. Many textbooks and curriculum guides reflect this order of teaching (Mitchelmore & White, 1995). In theory, "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). In practice, the ABC teaching method can only lead to *abstract-apart* knowledge, which students cannot apply to problem situations and quickly forget once examinations are over (White & Mitchelmore, 1996). Furthermore, as Dreyfus (1991, pp. 28) writes, such students "have been taught the products of the activity of scores of mathematicians in their final form, but they have not gained insight into the processes that have led mathematicians to create these products."

By contrast, *exploratory methods* proceed from the concrete to the abstract. For example, students may discover the relation $x \quad 2x = 3x$ by finding that $1 + 2 \quad 1 = 3 \quad 1$, $2 + 2 \quad 2 = 3 \quad 2$, $3 + 2 \quad 3 = 3 \quad 3$, and so on, whatever number you use for x. Or students discover that the angles of a triangle add up to 180 by drawing a variety of triangles and measuring their angles. These are clearly examples of empirical generalisation. Students perceive mathematics taught this way as more "relevant", but the results are still mysterious. No answers are provided for such natural questions as "*Why* does $x \quad 2x = 3x$?" and "Are you *sure* the angles in a triangle always add up to 180 degrees? Exactly?" If these questions are not answered (or even asked), there is a constant danger of making false inductions. Moreover, students still perceive mathematics as a series of disconnected facts to be committed to memory (Mitchelmore, 2000).

A third method of teaching generalisations is through problem solving. An example I often use is: *Can you tessellate the plane using a scalene triangle?* All sorts of generalisations concerning congruence, angles, and parallels arise and become connected—in particular, theorems about angles on a straight line; the angle sum of a triangle; and corresponding, alternate, and co-interior angles formed by a line intersecting several parallel lines. In my experience, a period or two wrestling with this problem produces a far deeper understanding of these concepts and their relation to each other than teaching them by the ABC method or through empirical investigation. Although this problem-solving method essentially consists of theoretical generalisation, a large element of empirical generalisation is used to generate conjectures and even, in some cases, axioms. Davidov (1972/1990) cites many experiments carried out in the days of Soviet Russia which, he claims, show that an emphasis on theoretical generalisation provides a superior method of teaching mathematics.

Teaching Abstraction and Generalisation (G2)

Generalisations (G3) are relations between abstract concepts, and in order for a generalisation to be meaningful, applicable, and memorable, the concepts involved must be abstract-general and not abstract-apart. It is strange, then, that teaching abstraction and the related generalisation (G2) is almost completely neglected in the mathematics education literature. For example, a well-known book on mathematical knowledge (Hiebert, 1986) does not mention abstraction at all, and an oft-cited reference on mathematical understanding (Hiebert & Carpenter, 1992) only mentions it in passing. However, there are some recent signs of interest in the works of Noss and Hoyles (1996) and Hershkowitz, Schwarz, & Dreyfus (2001).

The only explicit discussions of *teaching* for abstraction (and generalisation G2) are to be found in the works of Dienes in the 1960s, and more recently in the constructivist Dutch *Realistic Mathematics Project.*

Dienes and Multiple Embodiment

Dienes (1963) attempted to teach young children abstract structures by leading them to identify similarities between isomorphic structures. For example, place value concepts were abstracted from communalities between his Multibase Arithmetic Blocks and tree diagrams. His use of abstraction is characterised by one of his *multiple embodiment* principles: "To abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties" (p. 158). (Notice the word "embodiment".)

Dienes was not happy with the outcomes of his experiments. "We assumed ... that abstraction would arise from a multiple embodiment of the concepts to be abstracted. By this I mean that situations physically equivalent to the concept-structure to be learned would, if handled according to specific instructions leading towards the structure, result in abstracting the common structure from all the physical situations. ... But as we observed children going through the 'abstraction exercises', it soon became clear that the picture was far more complex than we had assumed" (1963, p. 68). It appears that children were not used to looking for deep (structural) similarities, and were frequently distracted by superficial characteristics. A major criticism of his experiments is that the various materials he used were not familiar objects in children's experience but already represented abstractions—abstractions made by the researcher and not by the children. This view is borne out by findings that children often have difficulty relating Dienes' blocks to arithmetical procedures (Boulton-Lewis, 1992).

Constructivism

Many elements of constructivist teaching could be expected to promote abstraction. Writers frequently emphasise discussion of existing knowledge or experience, small-group cooperative learning, and the admission of contrasting methods and the reconciliation of conflicting solutions. All of these would be expected to promote reflection which could lead to the recognition of communalities and hence either the abstraction of new concepts or the generalisation of existing ones.

However, the Dutch *Realistic Mathematics Education* movement appears to be the only constructivist curriculum project that explicitly teaches for abstraction and generalisation (G2)—although their publications do not use these terms. According to Treffers (1991, p. 26), their approach to teaching a topic consists of three stages:

1. Develop rules of operation in several specific, familiar, everyday contexts
2. Demonstrate that the same structure is present in several such contexts
3. Formulate, symbolise and study the common structure

Treffers (1991, p. 32) calls the first step *horizontal mathematising*: "The modeling of problem situations [so] that these can be approached with mathematical means." The second step consists of the recognition of structural similarities and the third step the construction of a new mental object. Treffers calls these steps *vertical mathematising*, which is "directed at the perceived building and expansion of knowledge within the subject system, the world of symbols." We can recognise the three steps as together constituting abstraction and generalisation (G2).

Teaching for Abstraction: Principles and Examples

Paul White and I believe that it would be valuable to design mathematics teaching in such a way as to explicitly promote the abstraction of crucial mathematics concepts (Mitchelmore & White, 2000b). Based on the theory outlined above, study of *Realistic Mathematics Education*, and various informal experiments we have conducted, we have formulated three principles of what we call Teaching for Abstraction: familiarity, similarity, and reification.

The familiarity principle. Students should become familiar with several examples of the concept (i.e., several contexts from which the concept will be abstracted) before making any attempt to abstract the concept itself. These examples—which may be objects, operations, or ideas—should be discussed using the natural language peculiar to each context, not the mathematical language related to the concept to be abstracted. All the examples should be familiar to children's experience, and not include abstract models "embodying" the concept. However, the teacher should anticipate the abstraction to be made later (e.g., by drawing children's attention to crucial characteristics that may not be obvious).

The similarity principle. The teacher teaches the concept by leading students to identify the similarities underlying familiar examples of that concept. The similarities may be superficial or structural. Whichever it is, the teacher directs the students' attention to the critical attributes which define these similarities and which are encapsulated in the concept to be abstracted. The teacher then introduces the mathematical language associated with the concept and uses this vocabulary to show how the concept relates to the similarities on which it is based. Abstract physical models that embody the concept may be introduced at this stage if they help children recognise the similarity between different contexts.

The reification principle. As students explore the concept in more detail, it becomes increasingly a mental object in its own right, detached from any specific context. Almost any use of the concept is likely to assist its reification, providing the relation between the abstract concept and familiar examples of the concept is maintained. Some possibilities include applying the concept in practice, investigating how to operate on the abstract concept, working with special cases, and looking for generalisations (G3) relating the concept to other concepts already learnt.

Teaching for Abstraction, as defined by the above three principles, is clearly constructivist. Unlike a radical constructivist method, however, it has no problem with the fact that much of the content of school mathematics is pre-determined. Instead of merely hoping that abstract mathematical ideas will develop as a result of cooperative learning, reflection on experience, and so on, it makes a more deliberate attempt to foster the abstraction of crucial mathematical concepts.

To show that Teaching for Abstraction could constitute a value method of teaching mathematics, two examples of its realisation in practice will now be presented.

Teaching Angles for Abstraction

In Mitchelmore and White (2000b), we used the three principles of Teaching for Abstraction to outline a hypothetical sequence for the teaching of angles in primary school. Last year, we had the opportunity to test our proposals in practice through an experiment we carried out for the *Count Me Into Space* project of the New South Wales Department of Education and Training (Mitchelmore & White, 2001).

As stated earlier, our research had suggested that even Year 2 children could learn to identify 2-line angles (i.e., angles in contexts with two visible arms) but that generalisation to angles with one or no visible arms would be difficult. We designed a 10-lesson unit for Year 3 intended to teach 2-line angles and then see how well children could generalise to a few angle contexts which only one arm was visible.

The first three lessons focussed on corners, including the corners of pattern blocks, corners in the room, and measurement of the size of a corner using a primitive "angle tester" (a paper protractor consisting of six lines through a point intersecting at 30). Students investigated how scissors moved in

Lesson 4, and in Lesson 5 they investigated other scissors-like objects. In all lessons, students matched angles in different contexts by superimposing one angle on the other. It was hoped they would be able to recognise that all the examples of angles involved (a) two lines, (b) a point where the lines meet, and (c) an amount of opening between the lines.

Lessons 6-8 each introduced one 1-line angle context: the hour hand of a clock, a door, and a sloping ruler. In each lesson, students firstly studied how the object moved and the significance of this movement. (For example, the hour hand of a clock moves from 2 o'clock to 5 o'clock in 3 hours.) They then investigated how to describe the size of such movements (e.g., by matching the 3-hour movement to a corner of a square pattern block), thereby linking back to the angles developed in earlier lessons. The learning activities were designed to help children identify the second, missing line of the angle in each context. Lesson 9 was an attempt to highlight the similarities between all the angle contexts studied in the unit, and Lesson 10 was an open-ended, creative activity designed to generalise the angle concept to other curriculum areas besides mathematics.

It may be seen that this unit incorporates the three principles of Teaching for Abstraction. Firstly, each context (pattern blocks, angles in the room, scissors, clocks, doors, and slopes) is investigated in its own right in order to increase students' *familiarity* with it—especially as regards possibly hidden features common to all angles. Secondly, the *similarity* between each new context and previous angle contexts is continuously stressed. Thirdly, *reification* occurs as students measure and compare angles in different contexts and edge towards generalisations (G3).

12 teachers from 5 schools taught this angles unit. They reported that students particularly liked the hands-on nature of the activities, which was apparently also new to the teachers. They found that the "angle tester" and the use of superimposition were particularly useful in linking the angles between different contexts. All students made substantial gains on a set of similarity recognition tasks given (by the teachers) before and after teaching the unit. It appeared that, by the end of the unit, almost all the students had mastered the idea of a 2-line angle. By contrast, only just over a half of the students had generalised this idea to 1-line angles. Some technical problems had been experienced with the door model used, and students had experienced some difficulty drawing lines at an appropriate orientation.

It was concluded that the method of Teaching for Abstraction was promising, but that more time was needed for students to study the 1-line contexts before they could be expected to recognise their similarity to 2-line angles. We have since revised the unit into two 8-lesson units: one mainly on 2-line angles for Year 3 and one mainly on 1-line angles for Year 4.

Teaching Decimals for Abstraction

At the end of 2001, we were able to test the generality of our principals for Teaching for Abstraction by applying them to the teaching of decimals. This exploratory investigation took place in a single classroom containing a mixture of Year 3 and Year 4 students.

The teacher had closely followed the official syllabus (New South Wales Department of Education, 1989). She had taught her Year 3 children to represent 2-place decimals on a 10 10 grid, in particular by using Dienes "longs" and "shorts" placed on a "flat" (see Figure 3). The Year 4 students had been taught to identify the first decimal place as representing tenths and the second as representing hundredths. She told us that students had considerable difficulties with the subject, and asked us to design a 6-lesson revision unit.

We asked ourselves, how could children learn 2-place decimals by abstraction from familiar experience? Working with the teacher, we identified three contexts: (1) money—dollars and cents; (2) measurement—metres and centimetres; and (3) chocolate bars. (This third context was one we invented to make children's experience with the hundreds square and Dienes blocks more "relevant". Children were simply asked to imagine that the "flat" represented a chocolate bar divided into 100 pieces.) Looking back on it, these contexts now seem obvious. In fact, they are all there in the official syllabus—but in three different strands, with little guidance given the teacher on how to link them. Our aim was to help children abstract the common idea of a 2-place decimal from these three contexts.

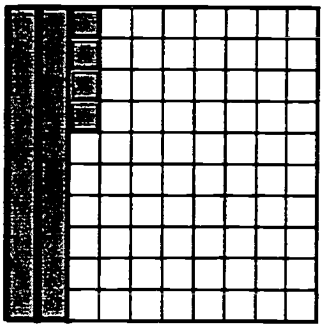

Figure 3 Representation of 0.24 on a 10 x 10 grid using Dienes blocks.

To act as a linking aid, we used a simplified version of the Linear Arithmetic Blocks (LAB) developed by the *Teaching and Learning About Decimals* project at Melbourne University (Stacey, Helme, Archer, & Condon, in press). In this model, students represented hundredths by a Dienes "short", tenths by a Dienes "long", and units (when needed) by metre rules. They could lay these objects out end-to-end to represent the size of a number by a length. Alternatively, they could lay them out on an "organiser" consisting of a 1-metre long strip of paper fixed to the students' desk, divided into three columns by a broad blue line and a thin black line. Figure 4 illustrates how children were taught to represent 1.35 (in each of the three contexts) using this organiser. The advantage of this model over the more conventional abacus is that it highlights the relative size of the numbers in each place.

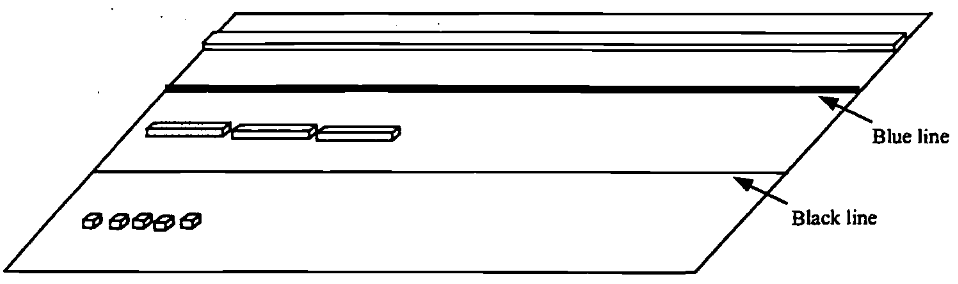

Figure 4 Simplified LAB model showing 1.35

Note that the LAB model was not used as a model for decimals in the traditional sense, where students first learn to use the model and then "apply" it to various contexts. Instead, it was used as an aid to help children understand the structure of each context in a way that would eventually lead to abstraction of the structure common to all the contexts. It corresponds to the use of the "angle tester" in teaching angles by abstraction, as described above.

Following the standard procedure for teaching by abstraction, students were first given activities to familiarise themselves with each context. Within each context, they were shown how to represent numbers using the LAB model. They were then given challenges to add numbers in that context and to multiply by 10, firstly using their intuitive understanding and then using the LAB model. The meaning of each digit was also emphasised. All exercises were embedded in familiar situations such as spending pocket money, comparing heights, and sharing chocolates. After this introduction, students' attention was drawn to the similarities between the three contexts, both in terms of the superficial features (e.g., three digits) and structural characteristics (e.g., the methods for adding and multiplying by 10). A standard place value chart was then introduced as an abbreviation of the LAB model. The revision unit concluded with some abstract exercises which students were expected to complete either by interpreting them in any one of the three contexts or by using their abstract understanding of 2-place decimals.

A target group of 8 Year 3 students was observed in each lesson, and a short posttest was administered to these 8 students and to 8 students in a Year 3 class in the same school. (There was no time to administer a pretest.) At the time of writing, the analysis of these data is incomplete, but the unit appears to have been effective. At the end of the unit, 5 out of the 8 target students successfully calculated 0.34 + 0.7, which the teacher indicated was more than she would have expected of the Year 4 students under normal conditions.

We plan to repeat the experiment on a larger scale at some future date, as well as finding opportunities for applying the principles of Teaching for Abstraction to other mathematical concepts.

Conclusion

Many students have difficulty making sense of abstract mathematical concepts and relations. Even students who do well in mathematics examinations can seldom give everyday examples of concepts or explain the value of generalisation. A major reason for this state of affairs is frequent misunderstanding of the nature of abstraction and generalisation on the part of teachers and educators. Hopefully this paper will have served to clarify the meaning of these two crucial terms and to encourage others to take more account of abstraction and generalisation in their teaching.

References

Boulton-Lewis, G. (1992). The processing loads of young children's and teachers' representations of place value and implications for teaching. *Mathematics Education Research Journal, 4*(1), 1-23.

Damerow, P. (1996). *Abstraction and representation: Essays on the cultural evolution of thinking.* Dordrecht, The Netherlands: Kluwer.

Davidov, V. V. (1990). *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula* (Soviet studies in mathematics education, Vol. 2; J. Kilpatrick, Ed., J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics. (Original work published 1972)

Dienes, Z. P. (1963). *An experimental study of mathematics-learning.* London, England: Hutchinson.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht, The Netherlands: Kluwer.

Dubinsky, E. (1991b). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht, The Netherlands: Kluwer.

Fischbein, E. (1987). *Intuition in science and mathematics.* Dordrecht, The Netherlands: Riedel.

Harel, G., & Tall, D. O. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics, 11*(1), 38-42.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32*, 195-222.

Hiebert, J. (Ed.). (1986). *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life.* Cambridge: The University Press.

Mitchelmore, M. C. (1997). Children's informal knowledge of physical angle situations. *Learning and Instruction, 7*, 1-19.

Mitchelmore, M. C. (1998). Young students' concepts of turning and angle. *Cognition and Instruction, 16*, 265-284.

Mitchelmore, M. C. (1999). How to teach generalisations in mathematics? In J. M. Truran & K. M. Truran (Eds.), *Making the difference* (Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia, pp. 373-379). Sydney: MERGA.

Mitchelmore, M. C. (2000). Empirical is not mathematical! *Reflections, 25*(2), 13-15.

Mitchelmore, M. C., & White P. (1995). Abstraction in mathematics: Conflict, resolution and application. *Mathematics Education Research Journal, 7,* 50-68.

Mitchelmore, M. C., & White, P. (1998). Development of angle concepts: A framework for research. *Mathematics Education Research Journal, 10*(3), 4-27.

Mitchelmore, M. C., & White, P. (2000a). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics, 41,* 209-238.

Mitchelmore, M. C., & White, P. (2000b). Teaching for abstraction: Reconstructing constructivism. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle, pp. 432-439). Sydney: MERGA.

Mitchelmore, M. C., & White, P. (2001). *Teaching angles by abstraction: A professional development experiment in Year 3.* Unpublished report, New South Wales Department of Education and Training, Sydney.

New South Wales Department of Education (1989). *Mathematics K-6.* Sydney: Author.

Noss, R., & Hoyle, C. (1996). *Windows on mathematical meanings.* Dordrecht, The Netherlands: Kluwer.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1-36.

Sierpinska, A. (1994). *Understanding in mathematics.* London, England: Falmer.

Stacey, K., Helme, S., Archer, S., & Condon, C. (in press). The effect of epistemic fidelity and accessibility on teaching with physical materials: A comparison of two models for teaching decimal numeration. *Educational Studies in Mathematics.*

Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 21-56). Utrecht, The Netherlands: University of Utrecht, Freudenthal Institute.

White, P., & Mitchelmore, M. C. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education, 27,* 79-95.

White, P., & Mitchelmore, M. C. (2001, July). *Teaching angle concepts by abstraction and generalisation: an exploratory study.* Short presentation at the 24th annual conference of the International Group for the Psychology of Mathematics Education, Utrecht.

U.S. Department of Education
Office of Educational Research and Improvement (OERI)
National Library of Education (NLE)
Educational Resources Information Center (ERIC)
Reproduction Release (Specific Document)

I. DOCUMENT IDENTIFICATION:

Title: The role of abstraction and generalisation in the development of mathematical knowledge
Author(s): Michael C. Mitchelmore
Corporate Source: *Mathematics Education for a Knowledge-Based Era* (Proceedings of the Second East Asia Regional Conference on Mathematics Education and the Ninth Southeast Asian Conference on Mathematics Education, Singapore, 27-31 May, 2002), pp. 157-167
Publication Date: 2002

II. REPRODUCTION RELEASE:

If permission is granted to reproduce and disseminate the identified document, please CHECK ONE of the following three options and sign in the indicated space following.

[x] Check here for Level 1 release, permitting reproduction and dissemination in microfiche or other ERIC archival media (e.g. electronic) and paper copy.

I hereby grant to the Educational Resources Information Center (ERIC) nonexclusive permission to reproduce and disseminate this document as indicated above. Reproduction from the ERIC microfiche, or electronic media by persons other than ERIC employees and its system contractors requires permission from the copyright holder. Exception is made for non-profit reproduction by libraries and other service agencies to satisfy information needs of educators in response to discrete inquiries.

Signature: M. C. Mitchelmore

Printed Name/Position/Title: Dr. Michael Mitchelmore
Associate Professor of Education
Organization/Address: Australian Centre for Educational Studies
Macquarie University
NSW 2109, AUSTRALIA
Telephone: (+612) 9850 8655 Fax: (+612) 9850 8674
E-mail Address: mike.mitchelmore@mq.edu.au Date: 4/7/2002

III. DOCUMENT AVAILABILITY INFORMATION (FROM NON-ERIC SOURCE):

Not applicable

IV. REFERRAL OF ERIC TO COPYRIGHT/REPRODUCTION RIGHTS HOLDER:

If the right to grant this reproduction release is held by someone other than the addressee, please provide the appropriate name and address:
Name:
Address: n/a