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ABSTRACT

This document consists of the first three years worth (9 issues) of the serial, NCRMSE Research Review: The Teaching and Learning of Mathematics. Each issue reviews the research of the National Center for Research in Mathematical Sciences Education (NCRMSE). Major articles appearing during this period (1991-1994) are as follows: "Perspectives on Assessment"; "A Study of Reform of Mathematics Education"; "Cognitively Guided Instruction"; "Reflective Practitioners Reform School Mathematics"; "The Teachers and Algebra Project"; "New Book Considers Domain of Rational Numbers"; "School-Welfare Reform and Teaching/Learning of Mathematics"; "What Gets Graded Is What Gets Valued"; "Geometry in the Primary Grades"; "Integrating Statistics into the School Curriculum"; and "Gender and Mathematics Education Research." (ASK)

NCRMSE Research Review: The Teaching and Learning of Mathematics

Volumes 1-3
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NCRMSE RESEARCH REVIEW

The Teaching and Learning of Mathematics

Volume 1, Number 1 October 1991

In This Issue:

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NCRMSE Begins Five-Year Program

You are reading the first issue of the *NCRMSE Research Review: The Teaching and Learning of Mathematics*. Designed as a newsletter, it will inform the mathematics education research community, educational policy and decisionmakers, and educators about the research and related activities undertaken by the the National Center for Research in Mathematical Sciences Education. The *Research Review* will highlight specific NCRMSE programs, provide summaries of the knowledge developed by the programs, and bring recent national and international developments in mathematics education to the attention of readers.

All NCRMSE research programs emphasize building unified paradigms of study. Research on classroom instruction, student learning, curriculum, and assessment, according to NCRMSE Associate Director Thomas P. Carpenter, often have been conducted in ways that isolate the disciplines, creating separate and distinct streams of inquiry. The NCRMSE plan of research insures that teaching, learning, assessment, and curricular reform are approached from an integrated perspective.

Seven working groups carry out the NCRMSE activities: Thomas P. Carpenter and Elizabeth Fennema, University of WisconsinMadison, direct a group on the Learning / Teaching of Whole Numbers; James Kaput of Southeastern Massachusetts University directs the group on the Learning / Teaching of Algebra and Quantitative Analysis; Walter Secada of the University of WisconsinMadison directs the group on the Implementation of Reform; Judith Sowder of San Diego State University directs the group on the Learning / Teaching of Quantities; Richard Lehrer of the University of WisconsinMadison directs the group on the Learning / Teaching of Geometry; and Thomas A. Romberg of the University of WisconsinMadison directs the group on Models of Authentic Assessment. A Statistics group will begin its work next year.

Each working group involves a small group of productive researchers who share the same field of study, set priorities for research, recruit and train students, communicate with one another, and monitor the rapidly changing structure in a field.

Two of the working groups–Implementation of Reform and Models of Authentic Assessment–cut across or interrelate with the five content working groups. Each content-focused working group has developed preliminary research goals.

While the two cross-cutting working groups have designed preliminary research goals, additional tasks will develop though their relationships with the content-specific working groups. The Implementation of Reform Working group will examine how educational reform becomes integrated into classroom practice. It is undertaking specific studies of efforts to alter curriculum and practice in mathematics classes. These studies will identify the kinds of experiences, resources, and support systems teachers need if they are to carry out the reforms called for in the NCTM *Standards*.

The Models of Authentic Assessment Working Group is identifying a variety of models of assessment practices that are aligned, or in agreement, with the reform goals set out by the NCTM *Standards*. It will develop procedures for rating the validity, reliability, and utility of the models. The procedures will be used to judge models designed for program evaluation by states and schools or for instructional decisions by teachers. After potentially useful procedures are identified, the group will construct or adapt the assessment models, examine them for sensitivity to cultural and linguistic diversity, prepare aggregation and reporting procedures, and demonstrate the viable features of the assessment procedures to educators.

Thomas A. Romberg is responsible for the overall direction of the NCRMSE. Joan Daniels Pedro is assistant to the NCRMSE director, and Donald Chambers is director of dissemination. A National Advisory Panel of seven members advises the NCRMSE on the management of its research programs and reviews its work. Members of the Advisory Panel include Merlin Wittrock, chair, University of California-Los Angeles; Robert Davis, Rutgers University; Audrey Jackson, Parkway School District, St. Louis; Harvey Keynes, University of Minnesota; Jeremy Kilpatrick, University of Georgia; Mary Lindquist, Columbus College, Columbus, Georgia; and Edward Silver, University of Pittsburgh.

The NCRMSE is funded by grants from the Office of Educational Research and Improvement, United States Office of Education, Washington, D.C. In addition to publishing the *NCRMSE Research Review* and carrying out its 5-year research program, NCRMSE distributes research reports, publishes monographs relating to mathematics education, and provides informational programs to the mathematics education community.

*New Book Cites Significant Advances in the Study of Teaching and Learning**

"Connecting Mathematical Teaching and Learning," one of the chapters in *Integrating Research on Teaching and Learning Mathematics*, presents initial discussions on the development of a unified paradigm for the study of the teaching of mathematics that incorporates both cognitive and instructional research.

During the last decade, significant advances were made in the study of student learning and problem solving in mathematics, as well as the study of classroom instruction. Mathematics educators have been concerned that these two research efforts have been conducted as separate fields of inquiry. A number of them have agreed that there is an increasing need for an integrated research program that unites the two areas. Each of the book's eight chapters presents the perspective of its author on integrated research programs. The chapters include:

"Research and Cognitively Guided Instruction" by Thomas P. Carpenter and Elizabeth Fennema;
"Diversity, Equity, and Cognitivist Research" by Walter G. Secada;
Research on Learning and Instruction in Mathematics: The Role of Affect" by Douglas B. McLeod;
"Curriculum and Teacher Development: Psychological and Anthropological Perspectives" by Paul Cobb, Erna Yackel, and Terry Wood;
"Connecting Mathematical Teaching and Learning" by Magdalene Lampert;
"Methodologies for Studying Learning to Inform Teaching" by James Hiebert and Diana Wearne;
"Intermediate Teachers' Knowledge of Rational Number Concepts" by Thomas R. Post, Guershon Harel, Merlyn J. Behr, and Richard Lesh; and
"Improving Research in Mathematics Classroom Instruction" by Douglas A. Grouws.

Editors of the book are Elizabeth Fennema and Thomas P. Carpenter, professors in the School of Education at the University of Wisconsin-Madison, and Susan J. Lamon, an assistant professor in the Department of Mathematics, Statistics, and Computer Science at Marquette University. It is part of a series, Reform in Mathematics Education, edited by Judith Sowder of San Diego State University.

Integrating Research on Teaching and Learning Mathematics
Elizabeth Fennema, Thomas P. Carpenter and Susan J. Lamon, Editors
1991, 142 pages, $12.95 paperback, $39.50 hardcover

Available from: State University of New York Press, c/o CUP Services, P.O. Box 6525, Ithaca, NY 14851; phone (607) 277-2211.

***The book was developed at the National Center for Research in Mathematical Sciences Education with funding from the Office of Educational Research and Improvement, U.S. Department of Education (OERI/ED).**

For further information on NCRMSE contact Donald Chambers, Director of Dissemination, NCRMSE, Wisconsin Center for Education Research, School of Education, University of Wisconsin-Madison, 1025 W. Johnson Street, Madison, Wisconsin, 53706, or call (608) 263-0761.

Perspectives on Assessment

National and international attention increasingly is focused on the role of assessment in education. The test scores of American students, particularly the mathematics scores they achieve on standardized achievement tests, once were widely used to compare individuals, schools, and states. In the past, these comparisons–developed with psychometric sophistication–inspired public trust and were used to support varied political and educational agendas.

Today educational scholars in the United States and abroad are rethinking the educational value of tests made up predominately of multiple-choice items or of items not related to the instructional programs of schools. These scholars agree on the need to provide students, teachers, and parents with information about student performance in relation to a set of established standards. They also agree on the need to provide administrators and policy makers with information about how well the educational system is performing. Their professional attention is directed to developing alternate types of assessment items, tasks, response formats, and scoring rubrics. They are convinced that alternate assessment strategies also will require new psychometric approaches.

alternate assessment strategies also will require new psychometric approaches

During the last year, national and international educational conferences have featured agendas with a primary focus on educational assessment:

- President Bush's education strategy is described in a publication, *America 2000*, released in March, 1991. The strategy calls for the development of national examinations, "American Achievement Tests," that will be used to assess student performance and to indicate the effectiveness of educational systems. The concern generated by this requirement led to the creation of a National Testing Council in July, 1991. The council consists of 32 members, 22 of them to be appointed by U.S. Secretary of Education Lamar Alexander.

- An International Commission on Mathematics Instruction (ICMI) held a conference on Assessment in Mathematics Education and Its Effects April 11-16, 1991 in Calonge, Spain. This conference provided a forum for discussion of the assessment tasks and strategies used by many countries, particularly Japan, Russia, and Australia.

- The Mathematical Sciences Educational Board (MSEB) held a National Summit on Mathematics Assessment in Washington, D.C., April 23-14, 1991. It was a political meeting designed to acquaint policymakers with assessment issues and had the support of such prestigious associations as the National Academy of Sciences, the National Academy of Engineering, and the Institute of Medicine.

- The National Center for Education Statistics (NCES) held an International Review of Preliminary Recommendations for the Third International Mathematics and Science Study (TIMSS) in May, 1991. The meeting was planned to enable the discussion of frameworks from which the questionnaires and items for the TIMSS would be developed. Experts from many countries participated. The results of the meeting may provide models of appropriate means of gathering data that will indicate how well educational systems are performing.

- The National Assessment of Educational Progress (NAEP) has been making state-by-state comparisions of levels of performance in mathematics and other instructional areas. The items they use have been viewed by many as an appropriate way of gathering data to indicate the quality of educational systems' performance. Those responsible for the revision of the test are weighing the questions of how well it presently is aligned with the NCTM *Standards* and how well it addresses the concerns regarding the limitations of multiple choice items.

- The New Standards Project intends to create a national examination system, as compared with a single examination. It will begin by involving a large number of people in establishing consensus on a framework. The framework will then be used to develop an examination and standards for grading. Developed by the Learning Research and Development Center and the National Center on Education and the Economy, the project will produce a "first draft" of a prototypic Grade 4 mathematics assessment framework that will have been reviewed by the Mathematical Sciences Educational Board's Study Group on Mathematics Assessment and by the MSEB Board. By the end of October 1991, the project will complete a "refined draft" of prototypic Grade 4 mathematics assessment materials and the needed ancillary materials such as scoring rubrics and teachers' guides.

national and international educational conferences have featured agendas with a primary focus on educational assessment

This first issue of *Research Review* reports on some of the NCRMSE research on assessment. That research supports the need for some new thinking about assessment–particularly the kind exemplified by the New Standards Project. It also carries a related article by Susanne Lajoie that defines authentic assessment, develops a framework for its identification, and provides several examples of assessment strategies that hold promise as authentic forms for assessing mathematics learning.

Anecdotal evidence has long suggested that the tests required of their students by schools, districts, or states affect the instructional practices of teachers. Teachers, according to this evidence, "teach to the test." Several years ago the Mathematical Sciences Education Board held a national conference on The Impact of Testing on Mathematics Education. If tests influence what mathematics is taught and how it is taught, said conference participants, testing programs can be used to stimulate instructional change or they can become impediments to change. Three members of that group, Thomas A. Romberg, Jeremy Kilpatrick, and Tej Pandey, proposed a set of studies that would provide a research base and extend what is known about this issue. Several of the proposed studies on the influence of mandated testing on school mathematics instruction were recently completed by researchers at The National Center for Research in Mathematical Sciences Education. One of the studies of testing obtained information on the experiences and the perceptions of a nationwide sample of 1,200 8th-grade mathematics teachers. The second study of testing examined the alignment of six of the most widely used 8th-grade mathematics tests with the NCTM *Curriculum and Evaluation Standards for School Mathematics*. The findings of the two studies are reported here.

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Instructional Impact of Testing

The first study on the instructional impact of testing asked mathematics teachers whether mandated standardized mathematics achievement tests were given in their school settings. If tests were given, teachers were asked about their knowledge of the basic content and style of the mandated test taken by their students. It also asked about teachers' efforts to ensure that their students perform well on the tests, about adjustments they make in the curriculum to focus on the knowledge and skills on the tests, and about adjustments in their modes of instruction in response to the style and content of the tests. Teachers were also asked about the effects of testing on the use of problem solving and of calculators during instruction.

The use of mandated tests with 8th-grade mathematics students is widespread. Nearly 87 percent of the responding teachers indicated their students were given mandated mathematics tests. About 68 percent of teachers said their students took a district-level test while 46 percent said their students took a state test and another 18 percent said their students were part of a state assessment program. Only 13 percent of teachers indicated their students were not required to take a mandated test in mathematics.

Test results are used by districts and by teachers in a variety of ways. The results obtained by tests mandated by districts are used, said teacher respondents, by 80 percent of districts. They use them to group students by ability, compare them to national norms, or assign them to special programs.

fewer than 20 percent of teachers say they make no instructional changes based on test results

Most teachers are provided with test data for individual students at an item or objective level. A majority of teachers think their districts are using the test results to stress what is tested or to deliberately affect teaching and the curriculum. When teachers think districts use test data to make decisions about students, teachers are much more likely to consider the style and format of the test as they plan instruction than when this is not the case.

Fewer than 20 percent of teachers say they make no instructional changes based on test results. Those teachers making instructional changes say that as a result of testing they increase their emphasis on some areas and decrease it on others. They spend additional time on basic skills, pencil and paper computation, topics emphasized on the test, and direct whole class instruction. They report a decreased emphasis on extended project work, problem solving, activities involving calculators, topics not emphasized on the test, activities involving computers, and cooperative learning. Finally, those teachers who give students practice test items and set aside time for students to prepare for the test are much more likely than others to reduce their emphasis on activities that involve computers and calculators.

testing does have an impact on instruction

Testing does have an impact on instruction, according to this national survey of 8th grade mathematics teachers. Teachers know what the tests call for and they plan accordingly. These results suggest that if districts adopt or prepare a test that sets standards of achievement they regard as suitable, a test with a style and format likely to promote the kind of mathematics the NCTM *Standards* seek, the use of that test to set standards and make decisions about students can also foster desired instructional changes.

NCTM Standards and Six Tests

Based on the first study of testing, we reported that tests have an impact on the instructional programs of a majority of 8th-grade teachers. A second NCRMSE study used standardized mathematics achievement tests that were identified in the first study. Only six of the tests–those identified as receiving the most widespread district usage nationwide–were examined.

standardized mathematics achievement tests studied do not cover adequately the range of content called for in the NCTM Standards

Our purpose was to determine whether the widely-used tests reflected the recommendations of the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989). The NCTM *Standards* were developed as a means of improving the quality of school mathematics. The Board of Directors of the National Council of Teachers of Mathematics established the Commission on Standards for School Mathematics. The Commission drafted a document that represents the consensus of NCTM's members about the fundamental content that should be included in the school mathematics curriculum and about key issues regarding student and program evaluation.

To complete the analysis of the tests, each item on each of the six tests was categorized by content, process, and level using the NCTM *Standards* for Grades 5-8. The content category included Numbers and Number Relations (nr), Number Systems and Number Theory (ns), Algebra (alg), Probability and Statistics (p&s), Geometry (geo), and Measurement (mea). The process category included Problem Solving (ps), Communication (com), Reasoning (rea), Connections (con), Computation and Estimation (c&e), and patterns and Functions (p&f). And the level category included Conceptual (conc) and Procedural (proc).

There were few differences across the six tests as shown in Table 1. When each item was identified according to content, process, and level the emphases of the tests did not vary. The percentages of the items on the individual tests fitting into the areas within each of the three categories were similar to the average percentages for all of the tests.

The results of this examination show that the six standardized mathematics achievement tests studied do not cover adequately the range of content called for in the NCTM *Standards* for Grades 5-8. A majority of items, 71 percent, fall into content area of Numbers and Number Relations. While 9 percent fall into the Measurement content area, between 3 and 6 percent fall into each of the remaining content areas. In the process category, a majority of items, 79 percent, are in the Computation and Estimation process area. While 20 percent fall into the Communication process, only 1 percent or less fall into the remaining process areas called for by the NCTM *Standards*, Problem Solving, Connections, Reasoning, and Patterns and Functions. An average of 89 percent of the items are classified as Procedural and 11 percent as Conceptual.

If tests are to reflect the new vision of the mathematics curriculum developed by Working Groups of the NCTM Commission on Standards for School Mathematics it will be necessary to vary content more than done at present; the processes of Problem Solving, Reasoning, Connections and Patterns and Functions will need to be added. An increase in the conceptual level of their items will also be needed. And if the NCTM *Standards* are to be implemented by schools, schools will need to select a set of standardized mathematics achievement tests different from those used by the majority of districts and states, or encourage the development of more adequate tests by districts and states or by the developers of the standardized tests they currently purchase.

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TABLE 1
PERCENT OF ITEMS FOR EACH CATEGORY

	CONTENT						PROCESS						LEVEL	
	nr	ns	alg	p&s	geo	mea	ps	com	rea	con	c&e	p&f	conc	proc
SRA	82	7	7	0	4	0	3	5	1	0	91	0	16	84
CAT	73	5	6	6	4	6	0	11	6	0	83	0	10	90
SAT	64	0	10	9	2	15	0	38	0	0	62	0	8	92
ITBS	62	1	7	3	4	13	0	9	1	0	89	1	4	96
MAT	66	6	0	5	8	15	0	21	0	0	79	0	12	88
CTBS	76	0	0	11	8	5	0	25	2	0	71	2	15	85
AVG.	71	3	5	6	5	9	1	20	1	0	79	1	11	89
RNG.	20	7	10	11	6	15	1	18	2	0	20	2	11	12

A Framework for Authentic Assessment in Mathematics

By Susanne Lajoie

Vast differences exist between the tasks learned in school mathematics and those mathematicians or users of mathematics actually carry out (Pollak, 1987; Resnick, 1988; Lampert, 1990). Much of how we learn inside the classroom is different from how we learn outside of the classroom (Resnick, 1987). The focus inside the typical American classroom is on what the individual learner can accomplish independent of the group, or of any tools for learning such as calculators. In contrast, outside-of-classroom learning situations often are group situations where knowledge must be shared and where tools are available to enhance or extend our knowledge. Inside the classroom students are taught to manipulate symbols and abstract principles, but outside the classroom learning often is concrete and situated in the context in which it will be used. The term "authentic" has been used to suggest that some classroom activities are lacking in realism and to conjure up an image of an alternate approach.

Requests for more authentic classroom activities have led to requests for authentic forms of assessment. These requests have come from several audiences. They range from student, teacher, district, and state sources to a national agenda on the integration of instruction and assessment. While the rhetoric is convincing, the images of authentic activities and assessment are still imprecise. This article is written to stimulate discussion on ways that authentic assessment can be operationally defined in the area of school mathematics.

The distinctions between in-school and out-of-school learning have implications for defining authentic forms of assessment. In considering these distinctions we must also consider whether a framework for authentic assessment should incorporate a guide for authentic instructional activities in the classroom. Should we be concerned with mathematical knowledge that transfers to everyday uses of mathematics or should we consider authentic mathematics as something mathematicians do in their domain?

My primary focus in defining authentic assessment in mathematics is to provide a robust perspective of the individual learner's understanding of mathematics. Several audiences are considered as I define worthwhile mathematical tasks from the mathematical educator's perspective, followed by a description of two interrelated theoretical perspectives on authentic activities as described in the literature on situated cognition and social constructivism.

Worthwhile Mathematical Tasks

The NCTM *Standards* represent goals for worthwhile or essential mathematics that are designed to make students mathematically powerful. These goals must be translated into tasks that exemplify authenticity. Only then can a framework for authentic assessment be developed.

A definition of an authentic mathematical activity emerges from the general assumptions of the NCTM *Standards.* One assumption is that knowing mathematics is doing mathematics. Doing mathematics refers to gathering and discovering knowledge in the course of solving genuine problems where knowledge emerges from experiences that are challenging but solvable. One way to increase such opportunities is to provide students with experiences in building mathematical models, structures, and simulations across multiple disciplines. Model building and discovering mathematical patterns are dynamic and constructive processes. Technology can be used to facilitate these cognitive processes as well as to record them. It can be used to assess developmental changes in reasoning, hypothesis formulations, verifications, and revisions. Technology can also serve as a medium for instructional manipulation where small changes in the instructional environment may account for changes in the learner's acquisition of knowledge.

The first four of the NCTM *Standards* were written as overarching goals that should be considered for all mathematics content at all levels. Any specific mathematical content, according to the four, should be designed to provide students with opportunities for mathematical problem solving, communication, reasoning, and connections.

Problem Solving

Activities that give students experience with problem solving can emerge from problem situations. These situations can be used to motivate students and serve as a context in which information is learned and knowledge is recreated across grades. Such situations imply complex, messy, and culturally-based problems that are open to multiple strategies and solutions (Zarinnia & Romberg, in press). Problems that are messy or ill-defined can provide more freedom for learners to pursue questions that reflect their personal interests. Such interests can be promoted by providing students with relevant, real-world applications. Real-world problems often include too little or too much information; they cannot be solved by applying a set of routine procedures.

Problem solving with mathematics involves modeling the problem and formulating and verifying hypotheses by collecting and interpreting data using pattern analysis, graphing, or computers and calculators. Technology is a powerful tool; it permits learners to manipulate data and see the consequences of their work in a few seconds.

Problem-solving activities need to include those that apply mathematics to the real world and those that arise from the investigation of mathematical ideas. Traditional curricula have emphasized mathematical ideas. The impetus for developing real and relevant problems stems from the need to contextualize mathematical concepts in a concrete rather than abstract manner. These real world problems may take on cultural biases depending on the students who work with them. In addition to including applied and pure mathematical problem types, problem representations should be varied to provide for individual differences, i.e., verbal, numerical, graphical, geometrical, or symbolic, and permit several ways of reaching a solution.

Communicating

Communicating about mathematical ideas permits students to synthesize information about the ideas. There are a variety of modes of communication including reading, writing, discussion, and listening or concrete, pictorial, graphical, or algebraic methods. Activities which require students to communicate about mathematics provide them with opportunities to reflect on and clarify their own thinking and to develop a communal understanding of mathematical ideas and notations.

Students need opportunities to present ideas using language to insure that they understand words and their definitions and meanings. Teachers who structure classes to encourage communication provide students with opportunities to validate their thinking about mathematics. They can foster communication by asking questions, posing problems, or asking students to develop problems. Different levels of communication can be obtained by interviewing individual students, by using small groups, or by classroom discussions. These levels permit students to ask questions, discuss ideas, offer constructive criticism, and summarize discoveries in writing. Cultural and gender differences should be considered by those structuring activities to encourage communication.

Reasoning

Mathematics involves both inductive and deductive reasoning. Inductive reasoning is associated with mathematical creativity or invention. Deductive reasoning involves understanding the premises of a mathematical problem and reasoning logically with the given information. Challenging problem situations can provide opportunities for students to develop mathematical reasoning in a variety of contexts. The maturation of mathematical reasoning is a long process. Special developmental differences in reasoning, especially in Grades 5-8 where students move from concrete to abstract reasoners, must be planned for. The development of mathematical reasoning could be facilitated in both instructional and assessment settings if appropriate prompts, Why is this true? What if you changed this? Do you see a pattern?, and others are made available to learners.

Making Connections

A curriculum that integrates a broad range of mathematical topics rather than treating each topic in isolation is a connected curriculum. Number concepts, computation, estimation, functions, algebra, statistics, probability, geometry, and measurement become more useful to students when treated in an integrated fashion. Students can be helped to make connections between the topics if they are provided with contexts that require their integration when solving problems. It is not enough, however, to provide connections among mathematical topics; the connection of mathematics with other topics and such disciplines as science, music, and business is also necessary (Bransford, et. al, 1988, Rosenheck, 1991). Teachers from other disciplines can help to identify the mathematical ideals that can be explored in their domains. Geography, for example, provides opportunities for the use of scaling, proportion, ratio, similarity, and other mathematical ideas. Using mathematics in contexts promotes attitudes of inquiry and investigation as well as sensitivity to the inter-relationships between formal mathematics and the real world.

Problem solving, communicating, reasoning, and making connections can be seen as curriculum goals that permeate the entire mathematics curriculum. Specific content areas also need to be addressed: number and number relations, number systems and number theory, computations and estimation, patterns and functions, algebra, statistics, probability, geometry, and measurement. In reviewing what the Standards deem worthwhile mathematical activities, it is important to realize that a single assessment of such activities will not provide a complete picture of a student's intellectual growth. Furthermore, different types of assessment are necessary to provide a complete picture of the learner's knowledge. In developing new forms of assessment, one must determine the types of assessment that are best for evaluating the various kinds of knowledge. Both individuals and small groups should be assessed, but for different skills. Small-group learning situations may be useful for measuring the ability to talk about and listen to ideas. Individual assessments may be better for assessing the learner's ability to synthesize knowledge.

Theories of situated cognition, social constructivism, and the influence of the group on the learning of individuals can be useful in defining authentic activities and authentic assessment. Although research on situated cognition is still in its infancy, there is evidence that certain activities described by its proponents are similar to those described as worthwhile by mathematical educators. Situated cognition refers to learning that takes place in the context in which one plans to use the knowledge. Problems must be realistic or authentic in the sense that the applications of knowledge are made apparent to the learner while the learning is taking place rather than outside of the context in which it could be used.

Situated Cognition

Situated cognition has developed out of the cognitive apprenticeship model of instruction (Collins, Brown, & Newman, 1989). The notion of a cognitive apprenticeship comes from traditional apprenticeships where novices learn their trade from a master. The masters share their knowledge with novices, assisting them in developing a product. Similarly, cognitive apprenticeships are designed around the notion that skilled learners can share their knowledge with less skilled learners to accomplish cognitive tasks. Cognitive apprenticeships, however, must model cognitive processes that are often difficult to externalize so that novices can observe or reflect upon the skills for a particular domain. In theory, the cognitive apprenticeship models offer suggestions for which skills to model for novices, how to provide scaffolding or assistance to less skilled learners, and when to fade such assistance when learners demonstrate they can construct their own meaning. Since the NCTM *Standards* call for an integration of instruction and assessment, the cognitive apprenticeship model has promise. It provides learners with ways to self-reflect and correct their performances based on assessment feedback. This theory does not provide specific guidelines for when and what type of feedback to offer or when to drop back on the amount of assistance provided. If this theory were used to define authentic assessment in an operational way for mathematics knowledge, then such criteria would have to be developed.

Scaffolding or adaptive feedback is important in instruction and assessment. Vygotsky (1978) proposed that assessment consider both an individual's actual development or performance on a task without feedback and their potential development or performance on a task with feedback during test taking. With traditional assessment where learners' actual development is assessed, it would be difficult to differentiate between two learners who have the same score. The two learners could look quite different from one another if they were assessed in situations where limited feedback was provided in the test context. Assessment with feedback could measure the learners' potential rather than their actual performance. Learners may not need feedback the next time they are tested; thus the test would have become a learning experience in and of itself. This is a dynamic and adaptive form of assessment. It is dynamic since learners can be retested; it is adaptive since learners can learn from the test. Dynamic forms of assessment can provide feedback to learners, giving them ways to improve themselves and opportunities to reach their potential. Tests that serve a learning function may also improve learners' motivation and sense of self-efficacy.

Social Constructivism

The cognitive apprenticeship model is similar to the theory of social constructivism (Vygotsky, 1978). Learning occurs, according to the theory, when one shares cognitions with more capable peers. The NCTM *Standards* emphasize learners construction, verification, and revision of mathemtical models. They also stress the importance of fostering problem solving, communicating, reasoning, and making connections through small group or whole class discussions. Situated cognition and social constructivist theories fit the NCTM *Standards* well.

Several researchers have examined the construction of mathematical meaning using small groups (Lampert, 1990: Resnick, 1988; Schoenfeld, 1985). Lampert discusses the importance of finding a common mathematical language for learners to use when communicating ideas. The group helps facilitate reasoning about mathematics. Resnick describes the importance of viewing mathematics as an ill-structured discipline where problem representations can be discussed and argued before mathematical procedures are employed. Resnick is particularly clear on the necessity of having a common core of knowledge in order to promote the types of dialogues that Lampert refers to in her work. Small groups can also foster reflection or the metacognitive skills necessary to evaluate mathematical problems (Schoenfeld, 1985).

The theories reviewed here provide great promise for building authentic activities as well as authentic assessments. There is a gap in the literature on how to operationalize these theories. It is difficult to design groups that will ensure the sharing of cognition and optimize learning for each group member. If more capable peers assist the less able learners by articulating their cognitive processes, we need to know how to design problem solving siutations that will allow for the articulation of such processes, yet provide opportunities for the less skilled to participate in the overall task.

Authentic Assessment

Authentic assessment must take place in the context of the learning process. It must consider both the learning and the situation in which the learner is assessed. Authentic assessment must provide information on what the learner knows or does not know and the developmental changes in such knowing. Repeated measures of appropriate learning indicators must be made in order to obtain a robust picture of the learner's knowledge. These indicators must include a range of cognitive and conative abilities so that multiple perspectives are available for a particular area.

Authentic assessment will require instruments that provide in-depth perspectives on learning. Collins, Hawkins, and Frederiksen (in press) have begun to address the best tools for obtaining these perspectives. They suggest that one picture does not mean a thousand words when assessing what learners know. At least three different assessment mediums, they suggest, ought to be used to obtain an integrated picture of the learners. The benefits of such mediums as paper and pencil, video, and computers jointly provide a more authentic picture of the learner than a single medium. Paper-and-pencil tests, the traditional form of assessment, are used to measure students' knowledge of facts, concepts, procedures, problem-solving ability, and text comprehension ability. Collins et al. (in press) suggest broader uses of these tests. Paper and pencil can also be used to record how students compose texts and documents of various kinds. Students traditionally have been assessed on their essays, but other writing tasks such as letters, reports, memos, drawings, and graphs can also be used to supplement compositions. Paper and pencil can also be used to assess how well students critique the quality of other documents.

Video can be used as a medium for assessing students' communication, explanation, summarization, argumentation, listening, and question-asking and question-answering skills. Video can also be used to assess student interactions in the context of cooperative problem-solving activities. Video records of dynamic interactions can be scored at a later time. They provide opportunities for scoring oral presentations, explanations provided in a small group setting, and joint problem-solving activities.

The computer can provide yet another view of the learner. It can effectively track the process of learning as well as a learner's response to feedback. It can also simulate realistic situations in the classroom. The computer provides opportunities for assessing the dynamic nature of problem solving and opportunities to systematically vary the instructional environment on the feedback dimension and observe the effects on learning outcomes. The feedback dimension provides us with a novel mechanism for assessing how well or how poorly individuals respond to certain learning environments. The ability to track student performance provides opportunities for assessing such strategic aspects of knowledge such as hypothesis formation, hypothesis verification, or motivational aspects of learning–how persistent students are at trying to solve the problem–as well as actual learning outcomes. Thus computers provide opportunities for dynamic forms of assessment of those indicators that are determined to be the criteria for a successful performance.

Collins et al. (in press) suggest that the use of these three mediums of assessment will provide a more robust picture of the learner. The assessment media, however, is only as authentic as the task that the learner is being tested on. Care must be taken to define the types of student records that will be collected with each medium, and to insure that such records reflect the performance indices that are most relevant to that medium.

Finally, the purpose or use of the assessment must be considered. If the use of assessment is by the learner and/or teachers, then such assessment tools must be available in the classroom on a regular basis, weaving together instruction and assessment. Learners should be able to use the tools to reflect on their strengths and weaknesses. Tests or assessment tools should be transparent in the sense that those who are being assessed understand the criteria on which they are being judged so they they can improve their performance (Frederiksen & Collins, 1989). Frederiksen and Collins suggest that one way to ensure that assessment criteria are transparent is to provide a library of exemplars for students to visit. This library provides copies of records of student performances that have been critiqued by master assessors in terms of the relevant criteria. Such a library would help students evaluate their own performance and perhaps provide landmarks of success for which to strive. In addition to self-assessment, feedback should be given to students after a test is taken to help them improve their performances. Teachers can be assisted in using the assessment tools to determine what concepts students have misinterpreted.

Principles for Operationally Defining Authentic Assessment

We seek to define and operationalize authentic assessment in order to improve learning. Thus, students should find undertaking an assessment task a learning experience. And teachers should learn what their students know or do not know as a result of the assessment task. Some tentative principles for operationally defining authentic assessment grow out of the theories and literature reviewed:

1. It must provide us with multiple indicators of the learning of the individual in the cognitive and conative dimensions that affect learning. The cognitive dimensions should include content knowledge, how that knowledge is structured, and how information is processed with that knowledge. The conative dimensions should address students' interest in and persistence on tasks as well as their beliefs about their ability to perform. Student interest in a topic often increases in conjunction with a deeper conceptual knowledge of that topic. Student choices may reflect their level of engagement and interest. These indicators must be examined repeatedly if they are to provide us with information on learning transitions or developmental maturity. Multiple mediums of assessment are necessary if we are to obtain valid indicators, i.e., that which we define as authentic. One measure, obtained by one medium, is unlikely to provide us with sufficient information on an individual. Varied types of procedures are necessary for gathering assessment information (Collins et al., in press; Romberg, in press).

2. It must be relevant, meaningful and realistic. It must be instructionally relevant, as indicated by its alignment with the NCTM *Standards*. It must relate to pure and applied tasks that are meaningful to students and that provide them with opportunities to reflect, organize, model, represent, and argue within and across mathematical domains.

3. It must be accompanied by scoring and scaling procedures that are constructed in ways appropriate to the assessment tasks.

4. It must be evaluated in terms of whether it improves instruction, is aligned with the NCTM *Standards*, and provides information on what the student knows.

5. It must consider racial/ethnic and cultural biases, gender issues, and aptitude biases.

6. It must be an integral part of the classroom. Because teachers appear more likely to teach the information to students that appears on tests, assessment tasks should be aligned with authentic activities such as those outlined in the NCTM *Standards*. Teachers need to be an integral part of the assessment loop so that they can learn from the assessment information and structure their instruction accordingly.

7. It must consider ways to differentiate between individual and group measures of growth and to provide for ways of assessing individual growth within a group activity.

Alternatives to paper-and-pencil multiple-choice tests do exist. Those listed here incorporate several principles of, and hold promise as authentic forms for, the assessment of mathematics learning:

Australian IMPACT Project

A set of studies were conducted in Australia to faciliate communication within the college level mathematics classroom (Clarke, Stephens, & Waywood, in press). Journals were kept by students and used by both teachers and students to foster a dialogue about what the students were learning. The quality of student journals progressed from simple narratives that described concepts to summaries that integrated mathematics knowledge, to dialogues regarding what questions should be addressed, what meaning could be constructed, as well as the connections of their work with other mathematics knowledge. These journals were beneficial to both teachers and students since they provided opportunities for dialogues that were not possible during a regular classroom session. They demonstrate that instruction and assessment can be integrated in the classroom. Student journals could provide us with new techniques for authentically assessing mathematical communication skills by providing the mechanism for examining transitions in developmental maturity in these skills.

Vermont Portfolios

Portfolios are promising as an assessment tool since they provide multiple examples of student work and provide students with experience in generating mathematical ideas, seeing mathematics as part of the culture, and being encultured into the mathematics experience. What is particularly intriguing about portfolios is the multiple audiences that can use them to obtain knowledge of the learners, teachers, and curriculum. Guidelines are needed, however, on how to score such materials.

California Assessment Program

The California Assessment Program (CAP, 1989) has addressed the concerns of the NCTM *Standards* with providing students opportunities to demonstrate their construction of mathematical meaning consistent with their mathematical development. Open-ended questions are provided that give students opportunities to think for themselves and to express their ideas. Communication is fostered in classroom discussions as well as in writing tasks. The data from this project provides a wealth of information regarding students' misconceptions and reasoning abilities.

Cognitively Guided Instruction

In the Cognitively Guided Instruction project (Carpenter, Fennema, Peterson, & Carey, 1987; Carpenter & Fennema, 1988) instructional decisions are based on careful analyses of student knowledge and the goals of instruction. Problems are selected that closely match the student's knowledge level. The assessment emphasis is on the learning processes of students. Individual and group data are collected.

Problem Situations

De Lange (1987) has designed mathematical problem situations comprised of multiple items with varied levels of difficulty. In his assessment of the Hewet Mathematics Project in The Netherlands, five different tasks were used to gather information: a timed written task, two-stage tasks, a take-home exam, an essay task, and an oral task. These provide a multifaceted evaluation of the learner. The two-stage tasks are especially interesting, in light of our principles of authentic assessment. Stage one includes open-ended questions and essay questions. These items are scored and returned to the student. In stage two, students are provided with their scores from stage one, allowed to take the stage one tests home, and given as long as three weeks to answer the same questions. The final assessment includes scores from stage one and stage two. Students can learn from their mistakes and from the feedback regarding their mistakes, making the testing process an interactive one that assists students in reaching their potential.

Superitems

Superitems are designed to elicit mathematical reasoning about mathematical concepts (Romberg & Collis, in press). The items are built to assess four different levels of mathematical maturity. At level four, the most mature level, the learner must articulate some understanding of the mathematical concepts either in words or symbols. The tasks can be used to obtain measures of developmental reasoning and serve as a first step in the identification of learning transitions in mathematical content areas.

I have laid out a tentative framework for the development of authentic forms of assessment. These and other alternative forms of assessment that incorporate new technologies hold promise for fitting within the operational definition of authentic assessment. Several parts of the framework require additional discussion or additional research. We will need to determine how cognitive and conative learning indicators can be operationalized in the context of an assessment task. We will need to study how to obtain frequent and valid measures of learners' performances. And we will need to define what we are assessing in individual and group situations. Finally, when we are considering the multiple audiences that may use measures obtained by authentic means, we must keep equity issues in focus.

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A Study on the Reform of Mathematics Education

The National Center for Research in Mathematical Sciences Education (NCRMSE) is beginning a study on the status of the mathematics education reform movement. The investigators need to identify schools that are engaged in substantial reform efforts. Help them identify schools by nominating one that you know is reforming its mathematics education program. **Return the completed nomination form to:** Implementation of Reform Working Group, National Center for Research in Mathematical Sciences Education, Wisconsin Center for Education Research, School of Education, University of Wisconsin-Madison, 1025 West Johnson Street, Madison, WI 53706.

I nominate the following school as one that is engaged in substantial reform efforts in mathematics education:

School name (please give us the FULL name): ____________________

School mailing address

Street and number: ____________________

City, state, and zip code: ____________________

School phone (with area code): ____________________

School contact person (please give us the FULL name): ____________________

Grades in the school: ____________ Number of students enrolled in the school. ____________

Description of school (please place one check in each column below).

______ Large-urban school	______ Public school
______ Small-urban school	______ Catholic school
______ Suburban school	______ Private (non-Catholic) school
______ Rural school	

Answer the following questions with a yes or no:

1. Are there substantial efforts to improve the teaching of mathematics? ______
2. Are there substantial staff development efforts on the teaching of mathematics? ______
3. Has new content been incorporated into the school's mathematics curriculum? For example, would you say that the content is substantially different from what was being taught in 1985? ______
4. Are there efforts to change how students are assessed in mathematics in the school or district? ______
5. Does the school have a mathematics department? ______
6. Does the school have a mathematics specialist on staff? ______
7. Do teachers in the school spend planning time together focusing on the teaching of mathematics? ______

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NCRMSE RESEARCH REVIEW

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In This Issue:

NCRMSE Research Review:
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Learning with Understanding

When students acquire knowledge that is connected—rich in linkages among ideas—they understand what they learn. When they understand what they learn, students can retrieve knowledge readily and use it to solve problems in a variety of situations. Over the last two decades, researchers have provided valuable information about understanding and how it develops.

The NCRMSE Working Group on the Learning/Teaching of Whole Numbers is one of seven Working Groups that make up the National Center for Research in Mathematical Sciences Education. Mathematics should be learned with understanding, the group believes, if it is to be useful to students. Starting from the premise that the development of understanding is a basic goal for instruction in mathematics, they reason, instruction should be designed so that students are able to build connections. While it may seem evident that mathematical ideas should not be taught as isolated bits of information, the kind of instruction that is most effective in fostering connections, and thus understanding, is less clear.

The Working Group seeks to learn how the knowledge about understanding which for the most part has been identified outside of classrooms, can be translated into effective instruction. In the next several years, the group will describe and analyze programs of instruction designed to develop understanding in children. Its focus will be on place value and multidigit concepts and procedures. Various instructional programs will be compared and contrasted to gain insight into how instructional components relate to various kinds of learning. The results of the group's work can be used to design new programs of instruction that enable students to learn with a greater degree of understanding. The group, chaired by Elizabeth Fennema and Thomas P. Carpenter, Professors of Curriculum and Instruction in the School of Education at the University of Wisconsin-Madison, also includes principal investigators and affiliated researchers: Karen Fuson, Northwestern University; James Hiebert and Diana Wearne, University of Delaware; Piet Human, Alwyn Olivier, and Hanlie Murray, Stellenbosch University in South Africa. These people are involved in research programs investigating place value and multidigit ideas in instruction.

Work started with the identification of a theoretical rationale which describes parameters that need to be considered as programs are compared and contrasted. The parameters include the goals and assumptions underlying the programs, scope and the

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Soon to be Released Book Carries Assessment Imperatives*

Are current testing practices consistent with the goals of the reform movement in school mathematics? If not, what are the alternatives? How can authentic performance in mathematics be assessed? These and similar questions about tests and their uses have forced those advocating change to examine the way that mathematical performance data is gathered and used by American schools. The chapters of *Mathematics Assessment and Evaluation: Imperatives for Mathematics Educators* provide recent views on the issues surrounding mathematics tests. The need for valid performance data, the implications of the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) for test development, the identification of valid items and tests in terms of the Standards, the procedures not being used to construct a sample of state assessment tests, gender differences in test taking, and the methods of reporting student achievement are topics carried in the book's 13 chapters:

- ✓ "Overview" by Thomas A. Romberg
- ✓ "Evaluation: A Coat of Many Colors" by Thomas A. Romberg
- ✓ "Implications of the NCTM Standards for Mathematics Assessment" by Norman Webb and Thomas A. Romberg
- ✓ "Curriculum and Test Alignment" by Thomas A. Romberg, Linda Wilson, 'Mamphano Khaketla, and Silvia Chavarria
- ✓ "State Assessment Test Development Procedures" by James Braswell
- ✓ "Test Development Profile of a State-Mandated Large-Scale Assessment Instruments in Mathematics" by Tej Pandey
- ✓ "Assessing Students' Learning in Courses Using Graphics Tools: A Preliminary Research Agenda" by Sharon L. Senk
- ✓ "Mathematics Testing with Calculators: Ransoming the Hostages" by John G. Harvey
- ✓ "Gender Differences in Test Taking: A Review" by Margaret R. Meyer
- ✓ "Communication and the Learning of Mathematics" by David Clarke, Max Stephens, and Andrew Waywood
- ✓ "Measuring Levels of Mathematical Understanding" by Mark Wilson
- ✓ "A Framework for the California Assessment Program to Report Students' Achievement in Mathematics" by E. Anne Zarinnia and Thomas A. Romberg
- ✓ "Evaluation—Some Other Perspectives" by Philip C. Clarkson

The book's editor, Thomas A. Romberg, is Sears Roebuck Foundation-Bascom Professor in Education at the University of Wisconsin-Madison.

Mathematics Assessment and Evaluation: Imperatives for Mathematics Educators
Thomas A. Romberg, Editor; 1992, 240 pages (tentative), July 1992, $14.95 paper, $44.50 hardcover. Available from: State University of New York Press, c/o CUP Services, P.O. Box 6525, Ithaca, NY 14851; phone (607) 277-2211

*The book was developed at the National Center for Research in Mathematical Sciences Education with funding from the Office of Educational Research and Improvement, U.S. Department of Education (OERI/ED).

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sequence of the mathematics to be taught, the role of problem solving, establishing meaning for symbols, the development of skills, coherence within and between lessons, students' articulation of their cognitions, the role of the teacher, teacher beliefs and knowledge, assessment and instruction for individual learners, and classroom climate and discourse (Carpenter, Hiebert, Fennema, Fuson, Olivier, & Wearne, 1991). Each set of investigators will provide a description of their instructional program in terms of the identified parameters. These descriptions will be synthesized so that commonalities and differences of the various instructional programs become apparent.

Measurement of student learning in the four programs will also be done, although this is not a simple matter. The goal is to describe the kinds of understanding that each program develops in students and to be able to relate instructional practice to outcomes. Assessment procedures will be developed to reflect differences in programs, measure the development of fundamental place value concepts and understanding of multidigit algorithms, and assess ability to apply knowledge in unfamiliar contexts. The goal is not to find the "best program" but to compare and contrast the various programs so that in-depth knowledge of how instruction is related to learning with understanding is gained.

Some of the most robust knowledge about the development of understanding is in the domain of whole number arithmetic and the research programs of the group members are investigating the translation of this knowledge into classroom practice. It is difficult to describe adequately the instruction in each program because it emerges and changes as the researchers gain information.

However, the instructional approaches share a number of common components. They utilize research-based knowledge about how understanding is developed and share a common emphasis on the development of learners' conceptual understanding. In each program there is an emphasis on problem solving, on children inventing procedures for solving problems, and on children discussing the strategies that they have used to solve problems. The programs differ in how much explicit instructional support they provide to encourage the development of more advanced understanding in students. Support is provided in some programs by the explicit use of structured physical representations of place value and by discussions of the potential linkages between symbols and objects. In other programs, teachers decide whether or not to use materials depending upon their perception of the child's understanding.

To illustrate the work of four different programs, consider how meaning is established for place value and for symbolic procedures or algorithms for addition and subtraction. One research program, directed by Karen Fuson of Northwestern University (Fuson, 1986, 1990) investigates linking children's developing conceptions of place value with operations on concrete materials. Teachers are given explicit instructional materials to use. In these materials, students are led to connect or link symbols to external representations, base-10 blocks, that embody the quantitative values that are only implicit in the written base-10 system. Thus, the blocks can help children understand and use the quantities in the written place value system. The blocks can be manipulated in a way that corresponds directly to the steps in standard addition and subtraction algorithms, or they can be used by children to invent their own procedures to solve problems (Fuson, Fraivillig, & Burghardt, in press).

When using base-10 blocks in a linking approach, meaning is given to computational algorithms by connecting each step in the algorithm to corresponding actions on the blocks. Consider the subtraction algorithm which requires regrouping. Blocks first are used to represent the larger number. To subtract, 10 for 1 regrouping or exchanging is required so that there are enough units within each place for the appropriate number to be removed. When the exchange is made with blocks, the corresponding regrouping marks are noted with the written symbols. Thus, quantitative exchanges with the blocks are linked directly to regrouping in the symbolic algorithm.

Teachers help students establish meaning for symbols by building connections . . .

Fuson currently is investigating culturally appropriate ways for instruction to support the conceptual development of urban Hispanic children as they develop abilities to solve multidigit addition and subtraction problems. This support involves the linking procedure described above as well as the use of "tens words" for all numbers larger than one digit. Tens words are English versions of Chinese number words; they make explicit the place value concepts embedded in spoken number words by specifically naming the tens, e.g., 53 is said as "five tens and three" and 14 is said as "one ten and four." The linking and use of tens words are embedded in a classroom environment of problem solving in which children individually or in groups solve problems and then discuss alternative solution procedures (Fuson & Fraivillig, in press).

A second instructional research program dealing with place value is that of James Hiebert and Diana Wearne at the University of Delaware (Hiebert and Wearne, 1992). They structure their teaching of place value as Fuson does, by helping students connect symbolic notation with base-10 blocks. Teachers help students establish meaning for symbols by building connections with the referents and then encouraging students to develop strategies for adding and subtracting using the referents and the written symbols. Strategies are worked out in the context of problem situations that involve both addition and subtraction.

In the Hiebert and Wearne project, procedures for manipulating symbols are built on the meanings students have established for the symbols. More complicated algorithms are worked out by elaborating previously constructed symbolic procedures. Teachers ask students to share and analyze alternative procedures, including the standard algorithm. Students defend and justify procedures based on the meanings of the symbols and on analogies to referents.

Two instructional research programs which are somewhat different from the previous two are the Cognitively Guided Instruction (CGI) project at the University of Wisconsin-Madison directed by Elizabeth Fennema and Thomas Carpenter (Carpenter and Fennema, in press), and the Problem Centered Primary Mathematics Program (PCM) at Stellenbosch University in South Africa directed by Piet Human, Alwyn Olivier, and Hanlie Murray (Olivier, Murray, and Human, 1990). In both of these programs, students are asked to use their existing knowledge about place value and solving one- digit problems to invent procedures for solving problems involving multidigit numbers. Teachers encourage students to extend their invented procedures to solve increasingly challenging problems. The emphasis is

When children in these programs invent symbolic procedures, they must draw on their own knowledge . . .

on inventing and modifying procedures rather than on drawing explicit connections between procedures using symbols and actions with physical materials. Consider a verbatim report of a PCM child's solution to an addition problem. The child had not been taught the procedure but had invented it. Asked to solve the symbolic problem 236 + 325, he wrote on his paper:

200 + 300 → 500 + 50 → 550 + 6 → 556 + 5 → 561. (Olivier et al., 1990)

For an example of a procedure invented by a CGI learner, see the article on CGI that appears later in this newsletter.

When children in these programs invent symbolic procedures, they must draw on their own knowledge of place value because they have no algorithms to fall back on. Also, children must build on the procedures they have developed in single-digit contexts. This encourages children to form connections between their previous knowledge of place value concepts and operations with multidigit numbers, and to extend their knowledge of single-digit numbers to build procedures with multidigit numbers.

The CGI and PCM programs differ somewhat on instruction with standard algorithms once children are proficient with invented strategies. In the PCM program, young children use only their own invented algorithms. A fundamental principle is that children should never believe that they are compelled to use any specific procedure. Standard algorithms are delayed until Grade 4 or 5. Because the CGI program is based on teacher decision making, once students are reasonably proficient in constructing their own procedures for solving multidigit problems, standard algorithms are sometimes introduced as more efficient procedures that provide a way of recording and keeping track of steps in a systematic way. Thus, the standard algorithms are connected to place value concepts through the invented procedures.

The Whole Number Working Group will not identify an optimal program, one that is most effective in teaching for understanding. The group believes that all instructional programs that promote understanding, although different in some ways, may include significant components which are similar across programs. Thus, it hopes to identify and describe these instructional components which are essential for the development of all students' understanding of mathematics.

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Cognitively Guided Instruction

> CGI is a philosophy versus a recipe ... You as a teacher have to take the knowledge that CGI is about problem types, about solution strategies, about how children develop cognitively and you have to apply that to your own teaching style. (Mazie Jenkins, First Grade Teacher)
>
> A CGI classroom is where you build on the math knowledge of your children according to what they know... You don't build objectives that say they should be doing this, this, this, and this. You sort of take what they know and build on from there. (Susan Gehn, First Grade Teacher)

Cognitively Guided Instruction (CGI) is captured in the above statements of experienced CGI teachers; it is about teachers making instructional decisions based on their knowledge of individual children's thinking. Currently in its seventh year of funding from the National Science Foundation, CGI was started by Elizabeth Fennema, Thomas Carpenter, Penelope Peterson, and Megan Franke as a research program to investigate the impact of research-based knowledge about children's thinking on teachers and their students. The project currently includes the investigation of childrens' and teachers' thinking in Kindergarten through 3rd grade, the study of CGI in urban schools, and the study of the impact of CGI in preservice teacher education. Teacher education materials are being developed for use with both inservice and preservice teachers.

What CGI Is

CGI IS NOT A TRADITIONAL PRIMARY school mathematics program. It does not prescribe the scope and sequence of the mathematics to be taught. Nor does it provide instructional materials or activities for children, or suggest that there is an optimal way to organize a class for instruction. Instead, knowledge about mathematics, in terms of how children think about the mathematics, is shared with teachers. The learning environments of CGI workshops are structured so that teachers learn how the knowledge about children's mathematical thinking can help them learn about their own children. With CGI support, teachers decide how to use that knowledge to make instructional decisions. As teachers implement and reformulate the plans they make, unique CGI classrooms emerge. Each teacher creates a teaching and learning environment that is structured to fit his or her teaching style, knowledge, beliefs, and children.

Even though CGI does not prescribe instruction, CGI classrooms do have similarities. Children in CGI classrooms spend most of their time solving problems, usually problems that are related to a book the teacher has read to them, a unit they may be studying outside of mathematics class, or something going on in their lives. Various physical materials are available to children to assist them in solving the problems. Each child decides how and when to use the materials, fingers, paper and pencil; or to solve the problem mentally. Children are not shown how to solve the problems. Instead each child solves them in any way that s/he can, sometimes in more than one way, and reports how the problem was solved to peers and teacher. The teacher and peers listen and question until they understand the problem solutions, and then the entire process is repeated. Using information from each child's reporting of problem solutions, teachers make decisions about what each child knows and how instruction should be structured to enable that child to learn.

Starting at the kindergarten level, CGI teachers ask children to solve a large variety of problems involving addition, subtraction, multiplication, or division. Children learn place value as they invent procedures to solve problems that require regrouping and counting by 10s. Little work is focussed explicitly on the mastering of counting, basic facts, or computational algorithms. Instead problems are selected carefully so that children count by 1s, 10s, or 100s depending on the child; discuss relationships between basic number facts; and invent procedures to solve problems involving two- and three-digit numbers.

The climate in a CGI classroom is one in which each person's thinking is important and respected by peers and teacher. Children approach problem solving willingly and recognize that their thinking is critical. Each child is perceived by the teacher to be in charge of his or her own learning as individual knowledge of mathematics is used to solve problems that are realistic to her or him. Mathematics is usually taught at least an hour a day; it is also integrated into the many other learning activities children do.

Author Notes

Work on children's thinking in geometry and measurement began with CGI teachers in January 1992; the results have not yet been implemented in classrooms.

By Elizabeth Fennema, Thomas P. Carpenter, and Megan L. Franke

Knowledge about Children's Thinking

Recent research has provided a reasonably clear picture of how many basic mathematical ideas develop in children. This research has shown that children enter school with well developed informal or intuitive systems of mathematical knowledge that can be used as a basis for the further development of their understanding of mathematical concepts, symbols, and procedures. Even before children are introduced to formal notions of addition, subtraction, multiplication, and division—they can solve a variety of problems involving the actions of joining, separating, comparing, grouping, partitioning, and the like.

To understand children's intuitive problem solving processes, it is necessary to understand the different problem situations that characterize addition, subtraction, multiplication and division. Therefore we start by helping teachers develop a taxonomy of problem types. Some of the distinctions among subtraction problems and division problems are illustrated by the problems in Table 1.

The distinctions among the problems in Table 1 are critical because they reflect the ways that children think about and solve the problems. Initially children directly model the action or relations in the problem. They solve the first problem in Table 1 by making a set of 12 counters (to represent the stamps) and removing 8 of them. They solve the second by first making a set of 8 counters and then adding more until there are 12. The fourth problem is often solved by matching a set of 12 counters with a set of 8 counters. Many children cannot solve the third problem because it is difficult to model. They have no place to start because the initial set, i.e., how many stamps Sybil had to start with, is not known. The fifth problem is solved by putting 12 counters into groups of 4 and counting the number of groups. The last problem is solved by first dealing 12 counters into 4 groups, and then by counting the number of objects in each group. With the possible exception of the third problem in Table 1, all these problems can be solved by many children who are as young as Kindergartners or 1st graders, if they are given opportunity to model the problem situations.

These modeling or concrete strategies provide a foundation for the development of more abstract ways for solving problems and thinking about numbers that involve counting. For example, children come to recognize that it is not necessary to make a set of 8 objects to solve the second problem. They can find the answer just by counting from 8 to 12 and keeping track of the number of counts. Similarly, the first problem can be solved by counting back 4 from 12, and the fifth problem can be solved by counting by 4s (4, 8, 12). The sixth problem, on the other hand, is more difficult to solve by counting. Since the number of objects in each group, i.e., the number of stamps each child is to receive, is not known, children do not have a specific number to count by. Thus, this analysis of problems and children's strategies for solving them provides a principled basis for understanding differences in the difficulty of the problems and why children may have more difficulty in using particular strategies to solve certain problems.

In the process of solving problems, children learn number facts, not as isolated bits of information, but in a way that builds on the relationships between facts. Certain facts like doubles, e.g., 6 + 6, are learned earlier than other facts, and children use this knowledge to solve problems and to learn other facts. For example, consider how one child figured out that 8 + 9 is 17, "Well, 8 and 8 is 16 and 8 and 9 will be just one more. So it's 17."

Knowledge of place value and computational algorithms also can be developed through problem solving. The counting and modeling solutions that children use with smaller numbers are extended naturally to problems with larger numbers. Rather than using individual counters and counting by 1s,

Table 1

Addition, Subtraction, Multiplication and Division Problems

1 Sybil had 12 stamps. She gave 8 of them to George. How many stamps did Sybil have left?

2 Sybil had 8 stamps. George gave her some more and then she had 12. How many stamps did George give her?

3 Sybil had some stamps. George gave her 8 more and then she had 12 stamps. How many stamps did Sybil have before George gave her any?

4 Sybil had 12 stamps. George had 8 stamps. How many more did Sybil have than George?

5 Sybil had 12 stamps. She put 4 of the stamps on each page of a book. On how many pages will she put stamps?

6 Sybil had 12 stamps. She wants to divide them so that she and 3 friends have the same number of stamps. How many will each person get?

children use physical representations for 10s and 100s to model two- and three-digit numbers. They learn to use symbols by inventing procedures for solving two- and three-digit problems without counters as illustrated in Table 2.

In summary, children start school with a conception of basic mathematics that is much richer and more integrated than that presented in most traditional mathematics programs. For example, in most textbooks subtraction is presented as only a separating or take-away action. However, subtraction can also be represented by the comparing, joining, and part-whole problems illustrated in Table 1, and young children can solve such problems. Symbols then, are learned not as abstractions, but as a way of representing situations that children already understand. Rather than expecting children to learn skills in isolation and then to learn how to apply those skills to solve problems, the learning of computational procedures is facilitated by problem-solving experiences that permit children to invent ways to calculate answers to problems.

Table 2

Children's Invented Procedures for Two-Digit Problems

Susan had 27 stickers. She bought 34 more stickers. How many stickers did she have then?

Megan: Mmmm 27, 37, 47, 57. Now I need 4 more. Well, 58, 59, 60, 61. (Megan mentally separated 34 into 30 and 4, counted 30 by 10s, and then counted 4 more.)

Todd: Well 20 and 30, that's 50; and the 4 and 7, that's 11. So it's 61. (Todd combined the 10s to get 50, combined the ones to get 11, and then combined the two sums.)

Roberto had 41 candies. He ate 23 of them. How many candies did Roberto have left?

Juan: Well 40 take away 20 is 20. But it was 41, so that's 21 take away 3, that's 20, 19, 18. He had 18. (Juan changed the numbers into 40 and 20 and then added the one he had taken away from the 40. He then took away the 3 he had taken away from the 23.)

Janice: 40 take away 20 is 20, take away 3 more is 17, but we have to put one back, so it's 18. (Janice had changed the numbers into 40 and 20. She then took away the 3 she had taken away from the 23 and added in the 1 she had taken away from the 41.)

Adam: 40 - 20 is 20, and 3 - 1 is 2; so I take 2 away from 20. That's 18. (Adam separated the 10s and then the 1s and then took the remaining 1s from the total.)

What Research has Shown About CGI

Previous studies have investigated whether the CGI knowledge that we shared in the workshops had an impact on teachers and on students; the results have been reported in a variety of publications. The studies have used a variety of methodologies to study teachers including precise observations of teaching, paper and pencil assessments, individual interviews, and indepth case studies. To assess children's thinking, standardized tests, self-developed paper and pencil tests, and individual interviews have been used. The majority of the studies that have been reported to date have been concerned with the learning and attitudes of 1st grade children and with the thinking and instruction of their teachers. The findings from a number of studies that have been conducted over the last seven years are synthesized here.

Teachers and CGI

In general, teachers can learn the knowledge about the mathematical domains and children's thinking within those domains. The knowledge has proved useful to them. They are able to use it as they plan for and implement instruction and to assess what individual children know. Teachers can use this knowledge to make instructional decisions, both before and during instruction.

The instruction of teachers who have been through a CGI workshop is different than the instruction of teachers who have not been exposed to CGI, and children in CGI classrooms do different things when compared to those in non-CGI classrooms. When compared to non-CGI teachers, CGI teachers assess their children's knowledge more often and use a larger variety of procedures to gain knowledge about children. Much assessment is integrated into ongoing instruction, when the teachers gain knowledge of children by asking questions and listening to their children's responses. Some teachers supplement this informal assessment with individual interviews. Mathematics is integrated throughout the day and problems are situated in a variety of contexts which have meaning to children. Teachers find mathematics becomes more fun to teach when CGI principles are used.

Successful Implementers of CGI

While all teachers who have participated in a CGI workshop appear to change their instruction, some teachers are better able to implement CGI than others. A number of studies have identified teachers who appear to implement CGI better than others using some kind of inter-rater judgment or by measuring the learning of children in the classrooms. The two sets of teachers' characteristics have been compared and contrasted and the relationships between these characteristics and their children's learning examined.

One characteristic critical to any implementation of CGI is the knowledge that teachers have: knowledge of content analyses and children's thinking in general, as well as knowledge of the thinking of specific children in their classrooms. Before any exposure to CGI, many teachers have an intuitive knowledge of content analyses and how children solve problems. However, that knowledge does not appear to be particularly well integrated and organized. It is not particularly useful to them as they make instructional decisions. After participating in CGI workshops and using the knowledge as they teach, the knowledge becomes integrated into a more coherent network and used as a basis for making instructional decisions. The knowledge of the better implementers of CGI is more highly integrated than the knowledge of those who implement it less well. The degree of knowledge that teachers have about CGI and their children's learning is correlated with what their children learn in mathematics.

Teachers' beliefs about mathematics instruction, i.e., their role and students' role in learning mathematics, is another important characteristic. Those teachers who hold beliefs more closely aligned with the philosophy of CGI are better able to implement CGI. The degree to which these beliefs are held is positively correlated with their children's learning.

More successful CGI teachers believe more strongly than less successful CGI teachers that: 1) children's learning should be considered as they make instructional decisions; 2) children have informal knowledge that enables them to solve problems without instruction; 3) the teacher's role is to build a learning environment where children can construct their own knowledge rather than where the teacher is a transmitter of knowledge; and 4) the learning of procedural skills does not have to come before children can solve problems.

Becoming a CGI teacher is not done overnight, nor is it accomplished by the end of a workshop. It takes time and interaction with children to learn CGI knowledge, and to incorporate it into a classroom. The more the knowledge is used to gain an understanding of individual children's thinking and ability, the more important it becomes to teachers. They increasingly ask questions that elicit children's thinking, listen to what children report, and build their instruction on what is heard. Teachers increasingly come to believe in the importance of children's thinking as they see what children are able to do and what they are able to learn when given the opportunity to engage in problem solving appropriate to their ability.

Children and CGI

The learning and beliefs of children who spent one year in a classroom taught by teachers who had attended a CGI workshop have been compared with those of teachers who had no CGI education. Children in the CGI teachers' classrooms spent more time solving problems and talking about mathematics with their peers and teacher and less time working on computational procedures than did children in non-CGI teachers' classrooms. They reported more confidence in their ability to do mathematics and a higher level of understanding than did non-CGI students. When compared to non-CGI students, children in CGI classrooms were better problem solvers; in spite of the fact that they spent only about half as much time explicitly practicing number fact skills, they actually recalled number facts at a higher level than did non-CGI students.

CGI children became more flexible in their choice of solution strategies and increased their fluency in reporting their mathematical thinking. Children in CGI classrooms learned much more than has been expected of children in traditional classrooms. They learned to solve a larger variety of addition/subtraction and multiplication/division problems; their understanding of place value increased; and they learned to be flexible in their use of invented strategies to solve multidigit problems.

CGI In 1992

The Longitudinal Study of CGI in the Primary School

The purpose of current CGI research is to study the impact of providing primary teachers with access to a structured, coherent body of knowledge about children's thinking in mathematics on teachers' knowledge and beliefs, their instruction, and their students' learning over a three-year period. Research based knowledge about children's thinking in addition/subtraction, multiplication/division, place value, early ideas of fractions, geometry, and measurement has been identified. Workshops have been developed and taught to most of the Kindergarten through 3rd grade teachers in participating schools. Currently how teachers come to understand their students' thinking, how teachers use children's thinking to develop and provide instruction, the impact of the knowledge of children's thinking on teachers' knowledge and beliefs, and the cumulative effect of being in CGI classrooms for three years on students' mathematics learning are being studied. Additional studies are looking at the scope, sequence, and pedagogical presentation of mathematical ideas by two different, expert CGI teachers per year to obtain rich descriptions of CGI in Grades 1-3.

The Development of CGI Educational Materials

The materials that were written to enable CGI to be implemented are being revised into a coordinated program which can be used with either preservice or inservice teachers. These materials will include chapters detailing CGI philosophy; the content analyses of addition/subtraction, multiplication/division, place value and multidigit algorithms, functions, and geometry; children's thinking; and video tapes that illustrate children's thinking and prototypic classrooms. If feasible, hypermedia will be made available to help teachers interact with CGI ideas. These CGI educational materials currently are being tested using a variety of procedures to organize workshops. Descriptions of the procedures will be made available to assist future workshop developers.

CGI in Urban Settings

Under the direction of Deborah A. Carey, a further study of CGI's impact on 1st grade children and their teachers is underway in six magnet schools with 60 percent or more racial/ethnic minority populations in Prince George's County, Maryland. Of particular interest is the change in teachers' expectations as they learn to assess their children's knowledge. The change in instructional behavior which happens as teachers learn to use children's knowledge is being documented. Both quantitative and qualitative research methodologies are being used. Any modifications of workshop materials which are necessary when working in multicultural settings will be noted and incorporated into the teacher education materials that are being produced.

CGI and Preservice Education

A further investigation is studying the conditions under which CGI can be incorporated into preservice teacher education programs. Of interest is whether and under what circumstances successful inservice teacher education programs can have similar effects on preservice teachers. This study, directed by Donald Chambers, is a collaboration with teacher education faculty at Queens College, City University of New York; San Diego State University; and the University of North Carolina at Greensboro, and the primary-grade teachers in schools used by those institutions for field-based preservice teaching experiences.

The first phase of this project, beginning in February 1991 and lasting about 18 months, is designed to help the teacher education faculty and primary-grade teachers at each of the three sites develop sufficient expertise in CGI knowledge and its application in primary-grade classrooms so that they can serve as experts in the education of preservice teachers. Activities during this period will include workshops at each of the three sites, a two-week summer conference in Madison, Wisconsin, and visits to the classrooms of the site team teachers. Not only are the project members at each site becoming familiar with the application of the principles of CGI, but they are also planning for implementing CGI into their preservice teacher education programs through university course modifications and field assignments. Some piloting of instructional activities is also taking place at this time. Workshops are being conducted at two sites to expand the cadre of CGI mentor teachers available to supervise the field experiences of preservice teachers.

The actual intervention will begin in the fall of 1992. Team members at each site are developing the site's intervention plan which will be finalized during the second summer conference in July 1992. Plans for the evaluation of the impact of CGI in the preservice teacher education program are also being developed for each site. The evaluation will look at changes in the beliefs and knowledge of preservice teachers and their ability to use CGI principles in their interactions with students. Interventions will be documented and modified over a two-year period in an attempt to achieve the maximum possible impact.

The results of the study will be published in a monograph that will include an overview of CGI, descriptions of the activities and results at each of the three project sites, an evaluation of each site, a cross-site synthesis, and suggestions for the incorporation of CGI into teacher education programs at other institutions. This monograph should become available in the spring of 1995.

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Review of NCRMSE Research

The mathematics education community reached the conclusion several years ago, "that all students need to learn more, and often different, mathematics and that instruction in mathematics must be significantly revised" (NCTM, 1989). Under the aegis of the National Council of Teachers of Mathematics (NCTM), a cross section of that community developed standards for a reformed school mathematics curricula in 1987-1988. Their standards can be found in *Curriculum and Evaluation Standards for School Mathematics,* a 1989 publication of NCTM.

NCRMSE invited scholars who were developing or working with innovative mathematics programs or materials that possessed features related to the Standards to a 1988 meeting. Participants at that meeting prepared proposals to study the programs or materials when used by teachers in United States classrooms. A proposal of the Freudenthal Institute, University of Utrecht, The Netherlands, was funded by NCRMSE. The proposal involved the preparation of a prototypic unit by Dutch mathematics educators, and a trial of the unit with six teachers and their algebra classes at Whitnall High School, Greenfield, Wisconsin.

This review summarizes a publication on the Whitnall study, *Learning and Testing Mathematics in Context: The Case of Data Visualization* (de Lange, van Reeuwijk, Burrill, & Romberg, in press). It covers the basis for the unit, the actual unit, the method used for the study, and the results of the study. The unit, *Data Visualization,* has been published (de Lange, Verhage, 1991).

Realistic mathematics education is the term used by the Freudenthal Institute, University of Utrecht, to describe the type of textbooks adopted by most Dutch schools. According to the director of the institute, Jan de Lange, more than 50 percent of primary level textbooks could, in 1985, be classified as realistic and, at present, four out of five newly selected books are realistic in content. The unit on data visualization was developed for the study from the realistic view that students need to experience real-world situations or problems. In the view of its Freudenthal Institute developers, concrete situations allow students to apply the mathematics they know, and in effect, to mathematize situations. It is from such experiences that students develop an understanding of mathematical concepts.

The unit on data visualization is designed to assist students in the development of the skills required to use critically the statistics presented by the media. Today's adults encounter visual representations of data in newspapers and on television on a daily basis. The Standards state that knowledge of statistics is necessary if students are to become intelligent consumers who are capable of making informed and critical decisions. The unit includes activities designed to teach students about describing a data set numerically, representing data graphically, and examining representations of data critically. Using tables and graphs found in current newspapers and magazines, the unit examines presentations and conclusions about population demographics and presidential elections, the relation between education and income, running speed and oxygen consumption, crime, and cholesterol.

The unit exposes students to a series of problems built around a context or theme. Students interact with the teacher and other students as they begin work on the problems. The teacher does not tell the students how to solve the problems in a traditional sense, but leads or guides the students through the instructional activities, monitoring their progress and creating opportunities for them to share their approaches or to discuss the relative merits of their solutions. Assessment activities are designed to motivate students by providing them with feedback on their progress. The activities fit the guidelines provided in the *Standards* (NCTM, 1989, p. 191).

The unit on data visualization was developed for the study from the realistic view that students need to experience real-world situations or problems.

The unit is constructed so that it elicits students involvement. Early activities are designed to foster students' intuitive exploration of the field but later activities require a more reasoned approach. Each section's activities give students new techniques while requiring the use some basic arithmetic skills. The topic is treated in a general way in the first four chapters, but in the last three chapters, students begin to learn more specific content such as box-plots, stem-leaf diagrams, and scatter diagrams. Each of the unit's chapters addresses lower order cognitive goals, but also includes open questions that address higher order cognitive goals. The interplay between intuitive and reasoned, new and

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old, general and specific, and lower and higher goals give all students the opportunity to experience success with some portion of the content throughout the unit.

Seven Whitnall High School algebra classes were involved in a five-week trial of the unit on data visualization. The seven included one honors, four 9th grade, and two 10th grade classes. An observer from the Freudenthal Institute met with the six teachers of the classes before they began to use the unit and—on a weekly basis—while they were using the unit. During these meetings, the teachers and the observer worked through and discussed the activities. The observer also sat in on three of the classes on a regular basis and the remaining four on an intermittent basis during the trial period. Students completed an attitude questionnaire during their initial session with the unit. They also completed several assessment tasks during the study.

The intent of the study was to identify the kinds of problems schools, teachers, and students would face when implementing the vision of school mathematics implied by the Standards. While the unit was designed to encourage students to explore a mathematical domain under the guidance of a supportive teacher, the observers concluded that the unit was a necessary but not a sufficient element for implementing the vision. Those carrying out the study underestimated the kind and the amount of teacher preparation necessary before the start of the trial. It appeared that, regardless of how conscientious and well educated teachers are, they must know, understand, and accept the philosophy on which the reform of mathematics is based if they are to implement the new pedagogy it envisions. At least one of the teachers in the study treated the unit as a traditional textbook, assigning students tasks and using class

. . . regardless of how conscientious and well educated teachers are, they must know, understand, and accept the philosophy on which the reform of mathematics is based if they are to implement the new pedagogy it envisions.

time to tell them how to do the tasks. All teachers in the study found it difficult to move from the role of classroom authority to student guide. In their words, it was hard ". . . not to teach." Yet, all said they had changed their conception of teaching after their use of the unit.

Some teachers felt insecure at the beginning of the unit. Teachers who had viewed themselves as teaching statistics at the outset of the trial, began to view themselves as helping students learn how to use statistics to solve problems in the next few weeks. Suddenly there were no right answers; instead, there were several possible solutions to problems. To help students learn to solve problems, teachers learned to probe students about their understanding, to listen to what students had to say, and finally to interpret students' explanations; communication became an essential part of the classroom enterprise.

Students had little difficulty adapting to the new materials and new ways of learning mathematics. While all students indicated they enjoyed the unit on data visualization, some did not view it as "real" mathematics. Students who flourished in mathematics classes that emphasized practice and replication found themselves challenged in new ways when working through the unit. Some who had been considered "poor in math" were able to succeed and some who had been considered "good in math" suddenly found themselves less successful.

Earlier assessment research by the Dutch had suggested that the correlations among scores for written restricted-time tests, oral tests, and take-home tests were low. Different assessment strategies, they concluded, were measuring different capabilities. Whitnall teachers also found that students who were best at traditional tests were not always best at the essay or similar product-development tests; different forms of assessment presented more equitable views of all students' achievement.

Students did experience some frustration as their teacher's behavior changed. One student remarked, "How can I do something if you don't tell me what to do?" Students who were used to taking short cuts to avoid reading questions found that key words were less useful, and students who were used to answering yes or no questions found they were now forced to justify their conclusions. The attitudes of students changed during the five-week trial: they increased their appreciation of common sense, discussion, and creativity, and they developed the view that mathematics is more than the memorization of rules.

Both the level of student engagement and the quality of their learning activity increased during the study. Observers noted that students engaged in "animated and purposeful discussions about its activities." One teacher remarked that "most of the time, in most of the classes, all the students were engaged in doing mathematics." While teachers reported that changing their teaching approach was a genuine struggle, they were eager to try a second unit. They also concluded the project had value—in their words (de Lange, van Reeuwijk, Burrill, & Romberg, in press):

The students were excited about ideas—they were thinking and interpreting problems that were real and not contrived. No one said, 'When will I ever need this?' They were learning to listen to each other as some raised valid points others had not considered . . . The creativity and ingenuity of many of the students was exciting. All of the students had the opportunity to succeed and, in doing so exceeded our expectations. Consistently there was evidence of higher-order thinking and analysis in all of the classes, not just in the honors class. (p. 194-195)

The authors of the study conclude that, based on their experiences while carrying out the study, there is a danger that schools, teachers, educators, and publishers will make only "nominal" changes in their practices and then consider themselves in line with the NCTM *Standards* (1989). There is a pervasive view—held by teachers, their students, and the public—that equates teaching with telling students what to learn and how to learn it rather than with guiding students' learning process. Changes in instructional practices, according to this study, must accompany changes in materials.

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The bibliography contains only a partial list of CGI publications. A complete list of CGI publications can be obtained by writing NCRMSE.

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NCRMSE RESEARCH REVIEW

The Teaching and Learning of Mathematics

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Focus is Middle School Mathematics

Middle grade students—those in Grades 5-8—learn mathematics that is very different from that learned in earlier grades. Students in early elementary grades work with the whole-number system as they learn to count sets of objects or to measure attributes. As students move to middle school mathematics, they begin to work with other number systems to represent quantities and magnitudes. Quantities such as common fractions, decimal fractions, ratios, percents, directed numbers, and integers form the core of this content.

The National Center for Research in Mathematical Sciences Education's (NCRMSE) Working Group on the Learning/Teaching of Quantities is examining the growth of student competence in the representation of quantities and magnitudes with systems other than what is considered whole number. It has developed a research program that will provide a foundation for instructional and curricular change at the middle school level in the areas of rational number sense, operations on fractions and decimals, and proportional reasoning. Each of the five content-oriented Working Groups of NCRMSE deliberately has chosen to build its research around a content domain rather than grade level or some other feature of schooling. The content-oriented Working Groups are concerned with how students construct an understanding of the major concepts and procedures within a domain, how they apply this knowledge to solve problems, and how instruction influences these processes.

Several present members of the Learning/Teaching of Quantities Working Group held a series of meetings and discussions on the quantities content domain during the period 1987-1990. A summary of the collaborative work of that earlier network of scholars is in the soon-to-be-published book, *Rational Numbers: An Integration of Research* (Carpenter, Fennema, & Romberg, in press). One member of that network, Professor of Education Judith Sowder, San Diego State University, reconvened that group and other educators with the same interest in 1991 under the auspices of NCRMSE and now chairs this Working Group.

In addition to Chair Sowder, the Learning/Teaching of Quantities Working Group includes principal investigators responsible for work directly related to the group's research agenda, affiliated researchers who carry out activities that contribute to the group's research agenda such as the preparation of a major paper, and affiliated members who may contribute to Working Group meetings or share their work with group members. The group's principal investigators are all from San Diego State University: Barbara Armstrong, Alfinio Flores, Randolph Philipp, and Judith Sowder. Affiliated

Judith Sowder, Chair
Learning/Teaching of Quantities
Working Group

researchers of the working group are: Robbie Case and James Greeno, Stanford University; Guershon Harel, Purdue University; James Kaput, University of Massachusetts-Dartmouth; Thomas Kieren, University of Alberta; Carolyn Krohn, Stanford University; Susan Lamon, Marquette University; Richard Lesh, Educational Testing Service; Nancy Mack, University of Pittsburgh; Robert Orton, University of Minnesota; Marty Simon, Pennsylvania State University; and Nadine Bezuk, Mary Koehler, Douglas McLeod, Larry Sowder, Alba Thompson, and Patrick Thompson, all of San Diego State University. Affiliated Working Group members are four San Diego-area teachers, Becky Breedlove, Heidi Janzen, Steve Klass, and Sally Movido who assist the group in formulating its research agenda and have an active role in its research; Edward Silver of the University of Pittsburgh, who is a liaison with NCRMSE's National Advisory Panel; and Thomas Carpenter of the University of Wisconsin-Madison, who oversees its work.

Research Priorities were developed in early 1991 by Working Group members. Their list included five priorities:

1. To interpret classroom activity, content analyses of the various aspects of quantities must be completed; such analyses must be carried as part of ongoing classroom-based research.
2. To carry out integrated research on teaching and learning, issues regarding a collaboration between researchers and teachers need to be addressed; teachers should be treated as full participants in the development and completion of such research rather than as its potential consumers.
3. To articulate the shared point of view of the group about the teaching and learning of quantities, group members may develop commissioned papers.
4. To clarify their role in the teaching of quantities, research must address materials and activities used by teachers in their classrooms.
5. To develop a more specific focus within the domain, case studies of teachers could offer valuable direction.

The conceptual framework prepared for the research of this Working Group was guided by the list of priorities and earlier research reports of group members. While the paper, A Conceptual Framework for Research on Teaching and Learning in the Multiplicative Conceptual Field, was developed by Chair Sowder in collaboration with group members located at San Diego State University, it was submitted to all Working Group members for review. Members reports that received special attention in the development of the framework were chapters in *Number Concepts and Operations in the Middle Grades* (Hiebert & Behr, 1988); *Establishing Foundations for Research on Number Sense and Related Topics* (Sowder & Schappelle, 1989); *Rational Numbers: An Integration of Research* (Carpenter, Fennema, & Romberg, in press); and a paper by Greeno (in preparation), *Notes Toward Semantics of Rational Number.*

Research Projects

Principal investigators Sowder, Armstrong, Flores, and Philipp have sought answers to a series of research questions using a case-study approach. Their questions focus on the interaction between teacher knowledge and teacher decisions as well as the connection between teacher decisions and student progress in learning of quantitative reasoning, rational numbers and operations, and ratio and proportional reasoning. In their research they will attempt to answer the following:

A. How does teachers' understanding of content influence the manner in which they teach and what their students learn?

B. As they become more familiar with the mathematics involved and come to understand how students learn the mathematical content,
   - What changes and shifts can be noted in the way content is treated by teachers?
   - What changes can be noted in the way teachers make instructional decisions?
   - What types of topics do teachers view as important for testing?
   - How do teachers determine whether student learning is significantly enhanced?
   - How do teachers' priorities—e.g., time allocation—change?

Teachers involved in preliminary case studies took part in a series of seminars designed to increase their content knowledge. The content of the seminars was developed around materials prepared by group member Patrick Thompson. Seminar discussions focused on teachers' reactions to items on a content analysis instrument and on frameworks for studying quantities and problems involving quantities. Reports on the preliminary case studies were used as a basis for a program presented at a Psychology of Mathematics Education meeting in August 1992.

The preliminary case studies provided researchers with an opportunity to refine interview and content knowledge instruments. The interviews focus on teachers' planning and decision making, the types of resources they use in teaching, their mathematical and pedagogical backgrounds, and their beliefs about mathematics and the learning of mathematics. Information from these preliminary studies is being used to plan for later studies on measuring teacher knowledge, classroom observation techniques and record keeping, and planning for the professional development of future teachers.

In its next phase, the project will work with a group of middle school teachers who have an inadequate mathematics knowledge base and as a result experience difficulties when teaching middle school mathematics. In the two-year project, content seminars designed for this group of teachers will focus on rational numbers and quantitative reasoning during the first year and proportional reasoning and the development of multiplicative structures that move from additive to multiplicative reasoning during the second year. Measures of changes in students' learning will be obtained from the students taught by project teachers during the two-year study. Expert teachers who are part of the Working Group will share planning, teacher development, and classroom observation activities with principal investigators. Members of the Working Group will prepare papers that describe the results of their research efforts.

Affiliated researchers contribute to the Working Group's research by developing papers, by reading and reacting to papers from the principal investigators, and by sharing their work when it relates to Working Group goals. Patrick Thompson's (in press) work on quantitative reasoning, for example, included papers, a microworld, and videotapes. It was used with teachers who were part of a preliminary study in the preparation of a series of seminars on mathematical content. The work of Alba Thompson (1991) provided the categories used in the analysis of data on the characteristics of exemplary teachers that is reported in the next pages of this issue of the *NCRMSE Research Review*. Documents by researchers Case, Krohn, and Greeno will be used to plan mathematics content work for future Working Group research.

Affiliated members meet with other Working Group members to discuss theoretical issues and research applications and to review research priorities. National meetings of educational researchers provide opportunities for such meetings. One meeting recently was planned to coincide with the annual meeting of the American Educational Research Association in San Francisco during April 1992; members of the Working Group who were attending the AERA meeting met to discuss the progress of Working Group research and reacted to the papers that were prepared from its data.

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Reflective Practitioners Reform School Mathematics

by Randolph Philipp, Alfinio Flores, and Judith Sowder

For more than a decade, the National Council of Teachers of Mathematics (NCTM) has guided the reform of school mathematics. Early in 1989 it appointed a commission that produced a set of standards for teaching mathematics. The standards, said the commission, rest on two assumptions (NCTM, 1991, p. 2):

- Teachers are key figures in changing the ways in which mathematics is taught and learned in schools.
- Such changes require that teachers have long-term support and adequate resources.

The assumption that teachers are linchpins in the fundamental reform of school mathematics is widespread. Little discussion, however, has focused on the characteristics of teachers who use classroom approaches that embody the vision originally set out by the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989).

Members of the NCRMSE Working Group on the Teaching/Learning of Quantities began research in 1991 to identify the teacher characteristics associated with individuals considered by their peers as leaders or early adopters—whose mathematics teaching already exemplifies the spirit of the NCTM *Standards*. This article reports on preliminary work with middle school mathematics teachers. During the first year, researchers obtained data on teacher characteristics by conducting extensive interviews, an assessment of content knowledge, and observations during a series of three-hour seminars over a three-month period. They sought information on teachers' conceptions of teaching, their pedagogical and content knowledge, and their teaching behaviors.

While the study set out to identify the characteristics of teachers associated with reform-oriented teaching, early findings also support the assumption that teachers require long-term support and adequate resources. Teacher educators and teacher leaders should find these experiences replete with ideas for designing programs that would support the efforts of teachers ready to change their teaching behavior.

Teachers selected for the study had often been asked to participate in local and state-level projects for mathematics teachers in curriculum and assessment, research, leadership, and professional development areas. They were active in professional development programs and had established reputations as reform-oriented teachers within the local mathematics education community. They described documents such as the *Mathematics Framework for California Public Schools* (California Department of Education, 1985) and the NCTM *Standards* (NCTM, 1989) as having a profound and liberating effect on them and an incalculable influence on their teaching. For these teachers, the publications seemed to validate their teaching practices.

Conceptions of Mathematics Teaching

In designing their study, the researchers sought a framework for their examination of teachers' conceptions of teaching. They used a recently proposed framework consisting of five categories, each with a developmental perspective (Thompson, 1991). The five categories of conceptions include: (a) what mathematics is; (b) what it means to learn mathematics; (c) what it means to teach mathematics; (d) what the roles of the teacher and the students should be; and (e) what constitutes evidence of student knowledge and criteria for judging correctness, accuracy, or acceptability of mathematical results and conclusions. Data on teachers' conceptions of mathematics teaching were analyzed according to the five categories.

What Mathematics Is

While teachers in this study were not asked specifically what mathematics is, researchers used their answers to indirect questions to obtain evidence of their thinking. Among the indirect questions were those on the kind of mathematics teachers believed should be taught in schools. Rules were not the focus of the mathematics taught by these teachers; they were adamant that school mathematics should not focus on the learning of algorithmic skills. One of their "toughest" jobs, according to these teachers, was structuring their classes so their students learned to discover solutions on their own and to develop their own rules. Many students, they said, thought mathematics involved learning what rule to apply.

Problem solving was considered an extremely important aspect of mathematics by all of the teachers. At times they seemed to view problem solving as one of many topics in the mathematics curriculum, and at other times they considered it an approach they used to teach mathematics. One teacher commented: "I think problem solving kind of drives what's going to happen. I try to keep my teaching involved around some kind of problem situation."

Another teacher spoke of mathematics as a language that was connected to the real world:

To me, to learn mathematics means to get to a point when you understand that there is no such thing as learning anything in isolation. I tell my kids that mathematics is like a foreign language, and that it's just sort of a language that explains the way things work in the world. . .to learn mathematics to me is to understand the applications, the usefulness of that language in describing the world around us.

What it Means to Learn Mathematics

Teachers believed that students learn mathematics by being immersed in doing mathematics. They said students learn a great deal of mathematics outside of school that is not recognized in classrooms, "I think the kids have a really pretty good understanding sometimes of quantities, but that doesn't get explored; it doesn't get emphasized at all; it gets almost quashed in the race to learn how to add four-digit numbers. . . ."

Teachers encouraged students to explore and make conjectures and they had them discuss and evaluate the solutions put forward by students during discussions. When teachers prepared activities for their classes they used problems suggested by newspapers or happenings in their schools or they approached traditional topics in unusual ways. One teacher was amazed that the text devoted only three pages to perimeter and area; this, he thought, indicated students were expected to learn these difficult concepts by looking at examples of regular figures or learning formulas. Yet in his experience:

> When students are learning mathematics. . .what I see is the students who are willing to explore and make mistakes without getting all flustered, students who are willing to not depend on formulas or numbers or things like that in order to solve a problem, students that are willing to talk to each other and with each other about what might work and what might not work.

What it Means to Teach Mathematics

The teachers in this study felt that the development of an understanding of conceptual relationships was an essential focus of their teaching. They were dissatisfied when they, as learners, did not understand the deeper conceptual relationships. Their comments are represented by the following:

> My primary focus is understanding. I want those kids to know not just how to do it, but why it works. And sometimes they get tired of that and they want the shortcut, but I really want them to know why it works. I think if you understand why, then you're not going to forget. And you're going to have a background to pursue further.

Teachers felt constrained by curriculum guidelines and textbooks. While their conceptions of appropriate content were tied to their district's curriculum guidelines, some interpreted the guidelines liberally. Either they rarely used textbooks or they used textbooks to cover certain topics at certain points in the year. Most selected other materials and ideas to enrich texts. The goal, according to one teacher, was a "concept-oriented program where I'm putting together units of concepts rather than just chapters in a book and still covering concepts that I'm supposed to be covering, but hopefully in a more meaningful way for the kids."

Testing programs also affected, to some extent, what teachers taught. Only one of the teachers did not let the standardized tests requirements influence her curriculum choices, "I'm doing some other potentially more meaningful things with these children, and maybe we just don't know yet how to measure these things. . . . And maybe the impact of what I do won't even be known for ten years." Another teacher expressed the conflict he felt in the effort to "feed into what we are trying to do, the direction we're going [in the reform movement]," yet knowing students "are going to be assessed in some kind of standardized, multiple-choice thing at the end of the year."

Teachers who are reflective practitioners, according to this study, may be more likely to move away from the traditional model of mathematics teaching.

Dealing with diversity and individual differences were concerns of all teachers. They did not accelerate their brighter students. The schools where some taught had made conscious attempts to avoid tracking strategies and to mainstream all students. One teacher believed that accommodating differences—cultural, learning, and behavioral—was his biggest challenge, "I kind of keep my expectations at a level where kids are going to be successful if they just do what I ask them to do. I challenge them, but I don't make them frustrated." Another focused on providing students with multiple entry levels:

> And, so what I try to really concentrate a lot on is: Am I presenting this information in ways that allow multiple entry levels and allow

accessibility by kids of all abilities? And am I trying to teach in a variety of ways? And am I aware? How do I know? What do I do, and how do I know if the kids are not getting it?. . . .I try to be very, very conscious of the reality that all of these people in the room have an incredible variety of prior knowledge and experience, and they're all coming into this learning that I think I want to have happen at exactly the same time with this diversity.

Teachers working in private schools with students from upper socio- economic levels also accommodated student differences. To do so, they avoided closed and narrow mathematics lessons.

The Roles of Mathematics Teachers and Students

The roles of these teachers were affected by their beliefs. To them, the goal of mathematics teaching ought to be the development of children's conceptual understanding of mathematics. They believe that children develop mathematical understanding when teachers provide them with experiences that allow them to do mathematics and ask them probing questions about their work.

Most of these teachers see themselves as facilitators who lead children as they "figure things out on their own." As one put it, a teacher "can either be—and this is a very old cliche—a sage on the stage or a guide on the side, and when the teacher is a sage on the stage, that's a teacher who's going to have students who depend on the teacher for answers." Students of the sage, he said, grow to depend on the teacher for answers and hints when stuck and students of the guide grow to see that mathematics problems exist all around them and that there are other sources of mathematical authority than the teacher. He offered the advice, "What the teacher needs to do—and this is really important—is to back off a little bit."

Students' roles can be inferred from these teachers' views of their roles. One put it more explicitly, "I don't have to then tell them that a centimeter is . . . , I mean 2.54 centimeters makes an inch, because they'll just sort of pick that up. I guess I trust more kids' abilities to be inductive about things than what I was ever taught how to do."

What Constitutes Evidence of Student Knowledge

Student responses to verbal questions were the primary means used by these teachers to obtain evidence of student knowledge. An atmosphere of acceptance, they said, is a necessary component of their questioning strategy. This atmosphere also permits students to be honest when they "don't get it." Said one teacher, "I try to have an atmosphere in my classroom of it being wonderful if you're able to articulate what you don't know. Students know that 'why' is going to be the next thing I ask because I really want them to understand" said one: "Can you explain why this works? Why did we do this? How does this happen? How would you do this and why?"

Observations of students provided these teachers with crucial information about students. In their words, they watched their students' work habits when given problems, their use of manipulatives to model mathematical processes, their selection and use of calculators, and their approaches to problems.

Journal writing was also used by teachers as a means of assessment. Students were asked to keep mathematics journals. The quality of the explanations in their journals provided teachers with additional information.

Although the teachers in this study seemed to attend to the thinking processes of their students, descriptions of their use of assessment information seldom mentioned individual students. Mathematics educators stress using knowledge of students' understanding to make curricular decisions (Carpenter, Fennema, Peterson, & Carey, 1988). Group members appeared to use the information gained through assessment activities to guide their classes as a whole; as one teacher put it, "There's a lot of constant monitoring of where they're at and adjusting what I have to do—kind of based on the way they react and based on the kinds of work I see them do."

Knowledge of Middle School Mathematics

Middle school teachers in this study had a comprehensive knowledge of the mathematics they were teaching and a good understanding of the relevant concepts and procedures. The study had acquired evidence of the mathematical knowledge of these teachers using a written test with a subsequent discussion of responses, interviews, and discussions of problems during research seminars. The written test covered concepts related to rational numbers, fractions, percents, ratios, and rates. Teachers demonstrated a thorough knowledge of the school mathematics appropriate for the grades they were teaching. They had constructed their knowledge in ways that enhanced making connections and using alternate representations and meanings.

Mathematical knowledge, for these teachers, was interwoven with pedagogical knowledge. They knew of numerous ways to teach concepts but made a decision to teach a concept or topic in a certain way because they thought it the best approach for their students. Those concepts and tools that were accessible to their students were of greater interest to them than concepts that required other more advanced mathematical tools. As an example, when presented with the problem,

> A 6th-grade student says that $\frac{3}{8} + \frac{5}{12}$ is $\frac{8}{20}$ and justifies her reasoning as follows: "If I made 3 out of 8 free throws in the morning, and 5 out of 12 free throws in the afternoon, then altogether I made 8 out of 20 free throws." How would you respond to that student?,

the teachers had mathematical knowledge that would enable them to explain the problem using algebraic notation and variables, and the concept of weighted averages. They thought such an approach was pedagogically inappropriate for 6th -grade students and sought to develop approaches that would be suitable for students at this level.

Teaching Behaviors

Teaching requires planning and on-line decision making. These complex cognitive skills are applied in the relatively poorly structured yet dynamic environment of classrooms (Leinhardt & Greeno, 1986). Teachers' descriptions of their planning activities and of a typical day in their classrooms were used to obtain data on teaching behaviors.

Long-term planning, undertaken by all of the teachers, was related to their perceptions of their obligation to follow curriculum guides or texts. A majority began with a top-down approach, considering the "big picture" and breaking it down into units and then into individual lessons. One of the teachers approached planning by identifying the "big ideas" and finding a theme into which she could fit the big mathematical ideas. One teacher mentioned using both the NCTM *Standards* and the *California Framework* as well as her experience as a teacher of 3rd and 4th graders to develop broad units of study for her 5th-grade class. These units were then divided into weeks of study. Only one teacher used a text-oriented approach while trying to develop a concept-oriented rather than a chapter-oriented focus.

Planning involved sequencing that related to students' understanding of previously learned material. According to one of the teachers, "What is important is that the teacher sees how whatever specifically they are teaching, how it fits into what came before it and what comes after it, and how the child is progressing."

Short-term or day-to-day planning, according to these teachers, often occurred shortly before class or "right on the spot." One teacher described daily planning as "thinking stuff—stuff that I think at night and I think in the morning on my way to school." She wrote out lesson plans but she described the process as "more of a brainstorming thing for me; I'll just write a whole bunch of activities that I've seen, or remember, or think I could do." While they create daily lesson plans, these teachers do not feel an obligation to follow them. A basic outline may be erased, "based on what the reaction to a lesson is on one day. Nothing is set in concrete, more like jello."

Resources are essential to the planning processes used by the teachers. They are professionals and have gained broad knowledge of their field from reading, gathering materials, and taking advantage of professional development oppor-

A Response to Readers

Readers of the *NCRMSE Research Review* have asked about the rationale behind the choices of the Center's title, The National Center for Research in Mathematical Sciences Education (NCRMSE), and of the Center's logo. According to NCRMSE Director Thomas A. Romberg, they were selected for the following reasons:

The word "national" was used in the title to represent two realities: the research program of the Center focuses on the national reform effort in school mathematics, and its work is carried out by a network that includes approximately 200 of the nation's mathematics educators. The term "mathematical sciences education" reflects the notion shared by these scholars that the school curriculum should include topics such as statistics and discrete mathematics. It also reflects the fact that when NCRMSE was created in 1987, it had a formal relationship with the Mathematical Sciences Education Board of the National Academy of Sciences.

The NCRMSE logo consists of a red torus surrounded by eight linked black orbits. The torus portion of the design represents the reform effort in school mathematics. The black orbits represent the linkages among the Center's seven Working Groups and its management staff. The combination of the torus and its orbits conveys the mission of the Center, which is to provide a research base for the reform movement in school mathematics. (The logo appears on the *NCRMSE Research Review's* first page, in the top left corner.)

tunities. One described the materials she had accumulated:

> I have been known for a long time as having an incredible resource library. It's too bad I didn't buy stock in Dale Seymour and Creative Publications. And I read all the time, like when the *Arithmetic Teacher* or the *Mathematics Teacher* comes, I read it that day. . . . So if you told me to put together some stuff on volume, I could have a year's worth.

A second teacher described how she used her resources:

> At first I just sat down and off the top of my head thought about, "What do I want them to learn when it comes to geometry? "—and I just wrote down some ideas. Then I start piling all sorts of resources together. I read through the section on geometry in the *Standards;* I read through a lot of curriculum guides on geometry, and through textbooks, to see what they cover. I look at different resource materials, for example, from Marilyn Burns. And I think of ideas for activities. So I have a broad plan throughout the unit of the concepts I want covered, possible activities, and then I sort of plot them in. I throw all my files and all my resource materials on geometry together in a box, and I keep that and look through it for lessons and ideas and take out what I need.

A Typical Day

Teachers in this study described their early teaching as resembling descriptions found in the literature (Romberg & Carpenter, 1986). In the words of one of them:

> The first several years of teaching I really was into "This is the section of the book that we're doing today, and here's the practice problems, and now we'll go over homework, and then I'll teach you how to do it, then you'll practice, and then you'll have some to try before you go home," and that kind of thing.

The teachers teach very differently now, when compared to the model described above. Activities and journals were used by all members of the group. A description of their previous day's class included journal writing toward the end of the class, and it was considered a critical part of their mathematics lesson. One of them commented:

> I have [my students] write for a certain length of time and write everything they can think of. Lots of times, if I'm walking around and see some interesting things in their writing, then we'll discuss those as a way to develop the idea we're working on or reviewing from the previous day.

Activities making up lessons are typified by an extensive use of manipulatives. The teachers choose manipulatives carefully, depending on how they would incorporate them into the learning experience. They were familiar with all commonly used manipulatives and indicate they had good classroom supplies, but they sometimes had to order specific types of manipulatives for their schools.

While whole-class instruction is used by all of the teachers, only half of them used cooperative groups as a regular part of the lessons they described. The other half did not think whole-class instruction was optimal and spoke of using more group work in the future. In the words of one, "I would say that most of my teaching is to the class as a whole, which is kind of dangerous. I have been assessing whether that's the best way to do things."

Typical lessons also include problem-solving discussions that often are linked to activities or reports. Going over homework assignments is not perceived by these teachers as an important component of a typical class. They tended to begin a class with some problem or situation related to a previous lesson in order to assess student understanding and then decide whether to give a previous topic more attention before proceeding with the lesson they had planned for the day.

Asked about the major barriers they face when teaching mathematics the way they believe it should be taught, these teachers listed a lack of time and materials, pressure from parents and administrators to make their mathematics instruction conform to traditional methodologies, and the weak mathematical backgrounds of their students. Said one who took a broader perspective, "School is probably the biggest barrier to teaching mathematics. . . . School is the most bizarre place. There is nothing else like it in the whole wide world." Her comments followed a discussion on the ways in which schools partition time, fragmenting the school day.

Descriptions of long-term and day-to-day planning and of typical classes provided by these teachers are reminiscent of the descriptions of preparation and improvisation provided by Yinger (1990, p. 88), "Preparation expects diversity, surprise, the random, and the wild. To prepare is to get ready, to become equipped, and to become receptive." By collecting many ideas and activities they can draw upon, these teachers seem to feel that they are ready to face the diversity and surprises of the day-to-day lessons that are then, to some degree, improvised. Improvisation, according to Yinger, is a highly responsive act that calls for skill and sensitivity to the moment and place. It is structured by actions and dependent on knowledge, beliefs, and goals. To this group of teachers, being well-prepared seems to mean being ready for whatever the day holds, being able to be both proactive and reactive, to be actors without lines who are responsive to their audiences' needs.

The teachers in this study viewed their profession as involving constant growth and change. This view is revealed by their participation in professional conferences and inservice programs, their completion of graduate studies, and their approaches to instructional planning. One teacher said that she and her colleagues had considered putting together a three-year program so that after three years teachers could cycle back through a curriculum. But she and her colleagues decided, "No, we never want to do that. Every year should be different from every other year." Change for these teachers was gradual and ongoing. While they faced barriers in their attempts to reform their teaching, they belonged to strong advocacy groups and continued to seek encouragement and support for their efforts.

The quality of reflectiveness appears to undergird the changes made by these teachers as they acquired classroom experience. A reflective practitioner of mathematics teaching, according to Houston and Clift (1990), uses knowledge that is pedagogically based, knowledge of students both as groups and as individual learners, an understanding of the milieu of the school and of the community, and an understanding of how all of these interrelate. Reflectiveness was illustrated in the thinking and practices of these teachers. They frequently had ready answers to questions during interviews, giving the impression that they had thought about the issues. They listened carefully and were able to help others articulate their thoughts during seminars. One spoke about lying awake and thinking about the discussions that occurred during seminars. Others indicated they thought about the discussions long after the seminars.

Because teachers can be viewed as key figures in the reform of school mathematics, this study set out to identify the characteristics associated with teachers whose mathematics teaching already exemplified the spirit of the NCTM *Standards*. A preliminary study designed as a precursor to a larger study, it found that these teachers' beliefs were congruent with those expressed in the NCTM *Standards*. It also found that the teachers were still struggling to revise their teaching approaches so that they provided problem-solving opportunities for students and incorporated effective ways of responding to their individual differences. Teachers who are reflective practitioners, according to this study, may be more likely to move away from the traditional model of mathematics teaching. In some circles, these teachers would be called heroic in that they sought out their own resources and their sources of support to help them reform their teaching approaches. This finding reinforces the second of the two NCTM assumptions (NCTM, 1991, p. 2), that support and resources will need to be provided if larger numbers of teachers are to change the ways mathematics is taught in the nation's schools.

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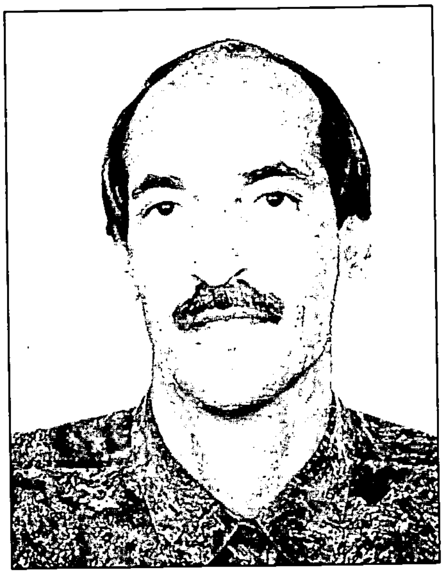
Review of NCRMSE Research



Steven Kirsner, Affiliated Researcher Implementation of Reform Working Group

The prevailing traditionally-oriented practices in mathematics classrooms contrast dramatically with those envisioned by contemporary reformers of mathematics teaching and learning. A major premise underlying reform efforts is that students are active learners: They do not passively absorb new knowledge, but actively build on their intuitive and informal knowledge and construct new meanings and understandings. Classrooms that foster active learning involve students in problem-solving activities in a learning community that permits communication among students. Teachers in these classrooms must assume a new and more challenging but also more rewarding role than they previously played.

NCRMSE's Implementation of Reform Working Group has fostered the development of research that describes reform efforts and their impact at classroom, school, district, and state levels. Steven A. Kirsner, an affiliated researcher of the Implementation of Reform Working Group, who is a project manager at the National Center for Research on Teacher Learning, Michigan State University, recently completed one of these studies. The study explored the difficulties faced by teachers who seek to change their roles in order to develop classroom environments that foster active learning.

This review describes and includes excerpts from a paper, Creating A Flexible and Responsive Learning Environment for General Mathematics Students, which reports a portion of the research by Kirsner. An account of one teacher's attempt to create a learning environment such as that portrayed in the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989), the paper was prepared by Steven A. Kirsner and Sandra Bethell for the April 1992 meeting of the American Educational Research Association held in San Francisco.

Sandy, a high school mathematics and foreign language teacher, has taught four years and completed three years of graduate education in educational policy and mathematics education areas. Before she resumed her graduate study, she was a fairly traditional teacher. She changed her beliefs about teaching, learning, curriculum, and equity significantly in the course of her work.

Believing that all students are capable and deserve to learn powerful mathematics, she volunteered to teach a general mathematics class of 16 students, 10 of whom had been labeled "special education" students. Her students who had encountered varying degrees of failure and frustration during their previous school mathematics experiences, "had learned that they were not supposed to succeed in mathematics classes. If they were expected to be successful, they certainly would not have been assigned to general mathematics, where they typically would be expected to rehearse the same low-level computations for the third, fourth, or fifth consecutive year."

In an effort to establish an environment where her students could be successful, Sandy attempted the creation of a learning community in which students were encouraged to engage actively in mathematical problem solving and to reason mathematically, orally, and by writing. She employed an eclectic blend of teaching activities, emphasizing manipulatives, students' discussion and communication about mathematical ideas, and groupwork and cooperative learning, as well as individual student seatwork and teacher demonstrations. Rather than using a textbook, she relied on materials she had collected from courses and workshops and wrote her own activities, worksheets, and tests for the 10-week unit on probability and statistics. She justified her unit on the grounds that it would improve computational skills as required by the district curriculum.

Sandy perceived that other teachers in her school saw her as an outsider due to her nontraditional views of teaching mathematics. Other teachers relied on textbooks. They rarely discussed or solicited advice about their own teaching and were unlikely to spend an entire class period exploring one or two problems. Yet Sandy, a Professional Development Teacher, was located at one of the Professional Development School sites that had agreed to work with teacher educators and researchers from Michigan State University to implement a program that stresses collaborative working environments for schools.

Like most of her coworkers, Sandy did not have much time to consult with mathematics educators, given the time constraints on her regular teaching load. While she occasionally consulted with Researcher Kirsner and a special education teacher who was then a doctoral student in teacher education, she thought that other forms of support were lacking and is confident that increased collaboration and support among her coworkers could greatly enhance her and her students' learning.

Researcher Kirsner observed Sandy's general mathematics class on a regular basis over a four-month period; he also completed interviews with four of her students. A first problem for her 16 students dealt with the probability of having an automobile accident when a driver is either (1) sober or (2) intoxicated, a problem that has significance for most high school students. Students were given data on the number of sober people who have accidents compared to the number of intoxicated drivers who have accidents. Students explored the problem but left the classroom confused about its solution. Sandy suggested that when the goal is learning with understanding, students' may be allowed to puzzle over problems for a longer period of time.

At the end of the activity, according to the study, some students demonstrated facility with fractions in the context of solving a meaningful probability problem "that would not have been evident had Sandy confined herself to teaching fractions traditionally." This finding has been supported by Romberg (1992) who suggests, "Thus, present strategies for teaching mathematics by first teaching skills and then exposing students to stylized application problems need to be reversed; concepts and skills should emerge from experience with problematic situations" (p. 37).

Interviews with students suggest that they too sensed the effects of the community of learners established by Sandy. One student, Penny, reflects on group work in an excerpt from an interview. Some of her words have been emphasized by the researchers (S=student; I=interviewer):

I: What's that been like to work in groups instead of what you're used to?

S: Well, first of all, when I worked in, when I didn't work in groups, it was harder to get to know people. It was hard to do math because the people who know how to do it, we could learn from each other's ideas. But we didn't do it over there. *We just worked separated and we actually didn't learn practically anything but what we learned from the teacher. Here we learn from everybody. We learn how they do it, how they understand it and we share our ideas with each other.*

Students in Sandy's class had seldom been engaged in academic activities during classes. Observations of Gene, a sophomore student, showed movement from being a disengaged student to being one who became creatively engaged. In the researchers' words:

> Gene astounded us by becoming enthusiastically engaged about the mathematical content of a geometry unit. He observed a pattern about polygons, formulated a hypothesis, and tried assiduously to make sense of his observations during the last three weeks of class. What came to be discussed as "Gene's theorem" became the subject matter of at least four class discussions and demonstrations. Although we cannot explain this change in Gene's attitude and behavior with any degree of certainty, we do assert that the classroom environment was (a) supportive of this type of mathematical inquiry and reasoning and (b) flexible enough to give Gene multiple opportunities to pursue this theorem and refine his thinking with the entire class.

...some students demonstrated facility with fractions in the context of solving a meaningful probability problem "that would not have been evident had Sandy confined herself to teaching fractions traditionally."

An interview with Gene recounts his views of the effect of a classroom environment that promoted inquiry and learning and encouraged his inclination to "look for a pattern":

I: All right. How do you explain that all of a sudden you were more interested and more engaged? Is there is an explanation?

S: It's just—I didn't think that. . . I didn't like it before because it just wasn't interesting. And then I found something that puzzled me and I wanted to keep at it.

I: And what puzzled you was the. . .

S: Every time you add an angle or a side you add a 180 degrees. You go from a triangle to a quadrilateral, you add 180 degrees.

I: Is that something that you just thought of yourself or was it—

S: Well, it was on one of our—see were doing polygons and all that. And we had those funny sides for each one of them. And then we had to add up the angles. And she went around the class and they went from 160 to 190, you know around that range. And they went up like about 180 degrees each time. And I just wanted to see if there was a pattern . . . .

I: So for a triangle it is 180, then a quadrilateral was 360, then a pentagon is 180 more than that. 540.

S: 540. Then 720.

While Gene had been reluctant to participate in previous classes, in an environment that promoted problem solving in which he felt safe enough to participate, he showed enthusiasm for a mathematical task.

This study demonstrates that, while there are significant obstacles, a high school teacher can develop a learning environment that promotes learning with understanding. While changing classroom practices is challenging, in this case the rewards of increased student understanding, increased learning-oriented discourse among students, and increased student engagement in class activities occurred in a brief four-month period. While the teacher in this study thought additional support would have been helpful, her activities developed an environment that increased learning for a group of students, nearly two-thirds of whom had been designated special education students.

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The Teaching and Learning of Mathematics

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An Algebra for All Students

Reformers of school mathematics would teach a substantial body of mathematics to all of this nation's students. One course, algebra, however, creates a major barrier to reform efforts. Students find algebra one of the most alienating parts of their school curriculum. For a large proportion of 9th-grade students, algebra becomes a filter. Based on whether they are permitted to enter this course or their experiences in it, they come to view themselves as having little potential for involvement in further mathematical endeavor.

Most of the significant others in students' lives—parents, teachers, counselors, and peers—help to create and reinforce their limiting views about mathematics. American schools uniformly ask students to develop arithmetic skills before exposing them to algebraic concepts. A student's performance in algebra is the primary criterion used by parents and teachers to determine the fitness of 13- or 14-year-olds for a sequence of college preparatory mathematics courses. And the same criterion is used by counselors to route these students into–or out of–mathematics-related academic and career choices.

Other industrialized nations treat algebra differently, and they move larger numbers of students on to more advanced mathematics when compared to the United States. While educators here delay the study of algebraic concepts until secondary school, these concepts are systematically included as early as the 3rd grade in Japan (Fujii, 1992). Educators here treat algebra as an independent subject, isolated from the mathematics students have encountered in earlier grades and not integrated with that they will encounter in later grades. Some nations view algebra as a language; in that view, students require repeated exposure over an extended amount of time to become efficient in its use. In addition, algebraic concepts are taught in a manner that connects them with other mathematics content during elementary and secondary grades.

The National Center for Research in Mathematical Sciences Education's (NCRMSE) Working Group on Learning/Teaching of Algebra and Quantitative Analysis is reexamining the place of algebra in a core quantitative mathematics curriculum. According to the chair of the Working Group, James Kaput of the University of Massachusetts-Dartmouth, the nearly 80 group members include leaders of the major algebra-related research and development projects here and abroad, mathematics education researchers from this and other nations, doctoral students in mathematics education, and classroom teachers. Many members worked together earlier as part of a symposium on the graphical representation of functions or were participants in an interest group on algebra and technology.

Students find algebra one of the most alienating parts of their school curriculum.

Over the long term, the Working Group seeks to develop coherence in research activity and foster a common sense of "programme" regarding algebra research. It plans a series of research syntheses that describe commonalities among research findings and interrelationships among members' findings. Finally, it will develop a range of practical recommendations for teacher educators and for curriculum and tool developers.

Activities of the Working Group

There are three major interrelated activities of the Working Group. The total group is rethinking the school algebra experience in a fundamental way. Its members take the position that algebra is the study of only one kind of mathematical object–the function–and a small set of operations (both unary and binary) that can be performed on such objects.

Rethinking the Algebra Experience

Through its continuing discussions, group members are examining the character and purposes of algebraic reasoning, its relationship to other forms of mathematical reasoning, and other related topics. They are also discussing how the development of algebraic reasoning can be fostered in all students and how algebra could be translated across middle-school/secondary-school boundaries. Electronic networks are being used to insure that all members can be part of these group discussions.

Several group members met early in 1992 to plan for a series of research syntheses and empirical studies that would relate to the reform of algebra. While the series will be completed in a collaborative fashion, it was begun around six topics. They include an examination of the literature and a synthesis on the cognitive "shift" from arithmetic and quantitative thinking to formal, algebraic thinking by Jack Smith of Michigan State University; an analysis of the factors underlying the representational and symbolic manipulative power of new technologies by David McArthur of the Rand Corporation; a paper that clarifies the notion of symbol sense by Abraham Arcavi of the Weizmann Institute–Israel; a synthesis of the various conceptions and representations of function and an evaluation of their treatment in the literature by Alan Schoenfeld and his colleagues at the University of California–Berkeley; and a study of the evolution of the concept of function and the role the concept played in developing new forms of mathematics by Patrick Thompson of San Diego State University.

James Kaput, Chair, Learning/Teaching of Algebra and Quantitative Analysis Working Group

When the initial series of syntheses has been completed, several small group meetings or focus groups will be developed to address the series. These groups will include six to eight members and their deliberations will lead to the identification of the need for additional syntheses or empirical studies. Such topics as equity, tracking, assessment, and teacher development as these relate to algebra, will receive increased attention from group members during 1993 and 1994.

The Teachers and Algebra Project

The Teachers and Algebra Project, a second Working Group activity, has developed the content of a Grade 6-12 curriculum based on the concept of function. It is gathering evidence about the teachability of the curriculum through extended clinical discussions and observations with teachers and their students. The project is directed by Judah Schwartz of the Harvard Educational Technology Center in Cambridge, Massachusetts. Director Schwartz is a principle investigator of the Working Group on the Teaching/Learning of Algebra and Quantitative Analysis. A longer article that describes the characteristics of the new curriculum and provides findings from a preliminary analysis of data obtained when project teachers began to use the curriculum appears in this issue of the *NCRMSE Research Review*.

Analogues From Which Project Framework Developed

	ARITHMETIC	ALGEBRA
Primitive Object:	**quantity**	**function (number recipe)**
Primary Graphical Representations:	**number line**	**graphical-Cartesian plot**
Symbolic:	**numerals**	**symbolic-expressions**
Unary Operations:	**exponentiation greatest (least) integer**	**translation dilation/contraction reflection in coordinated axes**
Binary Operations:	**+ − * /**	**+ − * / composition**
Comparison Operators:	**=, < >, >, <**	**=, < >, >, ≥, <, ≤**

Steps Toward a Revised Curriculum

Developing recommendations for curricula and tools is the third activity of this Working Group. Its efforts are framed by four fundamental assumptions about algebra that relate to quantitative relationships, functions as a central theme, new modes of representation, and algebraic thinking:

Algebra must be seen as part of a larger curriculum that involves creating, understanding, and applying quantitative relationships.

Algebra must now be seen as growing from quantitative reasoning at a fairly early age and extending to ideas and applications that have traditionally been viewed as the province of calculus. It must engage a wider set of analytical tools, especially graphical ones; and it must include connections to the wide variety of domains, both practical and theoretical, that use quantitative analysis.

The algebra curriculum should be organized around the concept of function.

Putting the idea of function, hence variable, at the center of the algebra curriculum has not been accomplished in the popular school algebra curriculum despite nearly a century of mathematically informed recommendations (MAA, 1923; NEA, 1969). Changed technological circumstances provide renewed urgency and opportunity for operationalizing these first two assumptions. The third and fourth assumptions reflect the new circumstances.

New modes of representation, graphical and otherwise, need to complement the traditional numerical and symbolic views of functions and the relations among them.

Historically, algebra has been identified with a set of formal propositions which evolved within static media to serve the scholarly interests of a small knowledge-producing elite. The new, dynamic, and highly flexible electronic media allow visual, graphical representations of quantitative relationships that are likely to be more learnable and applicable by the greatly enlarged segment of the population who must now learn and use them (Kaput, 1990). These representations include, but are not limited to, traditional coordinate graphs of functions.

Algebraic thinking, which embodies representation of patterns, deliberate generalization, and most importantly, active exploration and conjecture, must be reflected throughout the quantitative analysis curriculum.

The interactive capability of computers, coupled with their ability to perform routine computations, can greatly facilitate algebraic thinking particularly given the representational pluralism they make possible. Students can now test conjectures fluently, using different combinations of representations as appropriate, ranging from concrete, physically oriented systems (Greeno, 1989), to more traditional coordinate graphs (Schwartz & Yerushalmy, 1990), to standard symbolic formulas.

The framework for this project grows from a revised content analysis that draws explicit analogues with arithmetic. These analogues are shown in the box. Two strands will be woven together to develop the initial content for the course. One will deal with the formal mathematical structure, independent of its application to the modeling of the worlds of nature and people and their activities. The other will deal with the use of mathematical analysis as a language for modeling the students' world experience.

Moving important ideas in algebra and quantitative analysis to lower grade levels puts new demands on teachers. Many may not have had backgrounds that exposed them to this new content; nor did their education prepare them to teach it. Thus, whether all teachers can learn and teach the content of the new course must be studied.

According to Chair Kaput, while some pioneering work was undertaken earlier, the bulk of research on students and algebra was completed in the 1980s. The research findings indicate that students lack an understanding of the idea of variable, are unable to model quantitative situations, and have difficulties in parsing and operating on symbolic expressions (Booth, 1984; Clement, 1982; Kieran, 1983; Matz, 1980; Sleeman, 1984; Wagner, 1981). Later research showed that three fourths of

Moving important ideas in algebra and quantitative analysis to lower grade levels puts new demands on teachers.

15-year-old students, when given a task, avoided algebra, while only 10 percent used algebra correctly (Lee & Wheeler, 1987, 1989). The shortcomings of current algebra curricula, according to Working Group Chair Kaput, seem to result from a disregard of the function concept. He identifies the five shortcomings as: a lack of clarity about mathematical objects involving an equal sign; not making graphical distinctions between equations, relations, and functions; presenting students with expressions rather than functions; providing students with too many variables in stressing the algebraic manipulation of symbols; and ignoring the differences between coefficient and variable.

While the goal of this Working Group is to rethink the way schools teach algebra, it has had to rethink the sequencing of algebra content, with a view of the changes that are possible when technology is utilized in classrooms. A list of the members of the Working Group and a list of references on the reform of algebra and related topics can be obtained from Working Group Chair James Kaput, Department of Mathematics, University of Massachusetts-Dartmouth, North Dartmouth, MA 02747.

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The Teachers and Algebra Project

by Judah L. Schwartz

This nation's students will need a substantively reformed mathematics curriculum if they are are to face the challenges of the 21st century. The Teachers and Algebra Project is studying ways in which the traditional curriculum used in high school algebra can be reformed. Its three-year research plan builds on the belief that two factors are essential to the success of a reformed curriculum. Such a curriculum will need to contain coherent mathematical ideas, and teachers will need to learn the reformed content and find it acceptable. Thus the project began by reforming an algebra curriculum. It is now exploring teachers' reactions to the reformed curriculum, working with a small group of middle school and secondary teachers. This report describes the characteristics of the reformed curriculum in its first section. In the second section, a preliminary analysis of data obtained when project teachers began to use the curriculum is described. The Teachers and Algebra Project is directed by Judah Schwartz of the Harvard Educational Technology Center in Cambridge, Massachusetts, a principle investigator of the NCRMSE Working Group on the Teaching/Learning of Algebra and Quantitative Analysis.

The Reformed Curriculum

The mathematics of function acts as the unifying idea for the reformed curriculum. Two dimensions, mathematical objects and mathematical actions, provide it with structure. Mathematical objects such as numbers or functions are aspects of mathematics that are collectively understood and come to be viewed as "things" by those experienced in the knowledge and use of mathematics. This curriculum builds from the position that the function is mathematically and pedagogically the primary and fundamental object of the subjects of algebra, trigonometry, probability, and statistics, pre-calculus, and calculus. Existing algebra curricula may confuse students because they confound equations and expressions, functions and relations, unary and binary operations, and variables and unknowns. The reformed curriculum clarifies these aspects by relating them to the central notion of function.

Key Mathematical Ideas

The "big ideas" of mathematics are not ordinarily apparent in early mathematics courses. They appear when students take more advanced courses or became committed to mathematics or a science that uses mathematics extensively. The big ideas included in the reformed curriculum are: representation, transformation, symmetry, invariance, scale, continuity, order and betweenness, boundedness, uniqueness, relaxation and constraint, successive approximation, and proof and plausibility. In this curriculum, computer graphics permit the introduction of important mathematical ideas early in the mathematical education of all students.

Functions

Functions traditionally have been represented in several ways (Harel & Kaput, 1991; Sfard, 1991): (a) the numerical-tabular, which can aid students in making the transition from arithmetic thinking to mathematical thinking; (b) the symbolic, which stresses the process nature of functions, and (c) the graphic, which stresses their integrity as entities as well as processes. In the traditional course in algebra, students may learn very little about the notion of function as an entity. This cripples students when they begin to learn calculus, which fundamentally is about the unary operations of differentiation and integration on functions as objects.

Certain properties and behaviors of functions appear more naturally and are more readily grasped by students through the use of one of the representations. Students comprehend the binary operation of composition more readily using an algebraic symbolic representation, but the unary operation of dilation/contraction more readily using a graphical representation.

Unary Operations

The design of the project's reformed curriculum began with the selection of a small but sufficient set of operations that can be applied to functions. Students seem to grasp unary operations (horizontal and vertical translations, horizontal and vertical dilations and contractions, and reflections in the coordinate axes) rapidly if they encounter them first as graphical representations. It is also important to treat such operations in an appropriate modeling context using cognitively and pedagogically appropriate questions such as: Given the graph of a person's height as a function of age, what do each of the transformations shown on the board correspond to?

Binary Operations

Students seem to grasp binary operations readily if they encounter them first as symbolic representations. The binary operations on functions, which include the four arithmetic operations and composition, can be expressed both symbolically and graphically. Software environments that permit users, whether they be students or teachers, to manipulate functions in these ways have been designed and tested. A list of them can be obtained from the author.

Comparisons

Functions can be compared to one another; a binary comparison could involve $R = S$, $R < S$, or $R > S$. The use of an equal sign produces an equation or identity, while the use of a greater-than, or less-than sign produces an inequality. If functions are compared in a graphical environment, the solution set of the comparison is both evident and immediately accessible by identifying intervals where one function lies on, above, or below the other. Such an environment, linked to symbolic representations, also provides a medium for the exploration of the manipulations that may be performed on an equation or inequality–for example, What operations on the functions being compared leave the solution set invariant? The use of computer graphics to support such inquiries leads quickly to a rather large set of syntactically complicated algebraic activities showing inequalities that can be reduced to a small set of graphical activities possessing easily recognized quantitative content.

Context

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) advocate activities that grow out of problem situations. Functions can be treated as mathematical objects—patterns worthy of interest in their own right—or as representations of patterns that exist in situations or phenomena. Regardless of the treatment, functions act to contextualize mathematics. At an elementary level, functions contextualize numerical computations and provide both a means and an end for the examination of quantitatively rich situations. Unlike the traditional word problems made up of a single set of numbers with a single numerical solution, functions draw students to explore situations for their regularity, to build insight rather than to compute single right answers.

Mathematical Actions

Transforming

All problems of the form, "simplify," "factor," "expand," and "collect similar terms," are instances of transformations of the symbolic form of a function and reexpress it in a different, but equivalent form. Transformed mathematical objects may represent some new aspect of the situation being modeled. If the average speed at which a car travels during a trip and the total time the trip takes are known, for example, a new mathematical object can be generated by multiplying the two quantities to obtain a quantity that describes the distance covered by the car during the trip. The many types of transformations carried out on mathematical objects include arithmetic operations; various topological and metric changes; differentiation, integration, and composition of functions; and vector sums and products. Operations carried out on mathematical objects also come to be thought of as objects that permit the definition of higher order operations.

Modeling

Expressing, in mathematical form, the relationships between quantities in the real world is called modeling. In the reformed algebra curriculum, the mathematical expressions of these relationships are in the form of functions. Modeling allows the student to use his/her understanding of mathematical structures and their allowable transformations to reason about the situation being modeled. For a mathematical model to be useful, the mathematical elements and relationships must reflect those elements and relationships of the situation that the student regards as essential to what is being described, as well as the purpose for which it is being described.

. . . the use of functions draws the student to explore situations for their regularity, to build insight rather than to compute single right answers

Conjecturing

The search for and exploration of patterns in the interrelationships among mathematical objects can be called conjecturing. This aspect of mathematics has been sorely neglected by school mathematics, yet it lies at the heart of mathematical creativity. The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) suggest that students can begin to make and explore conjectures even before they can understand and use a wide variety of mathematical objects and the transformations that can be performed with them. This new curriculum conveys to students and teachers an implicit message that mathematics is not a closed and finished discipline, that they can go beyond learning and teaching the mathematics made by others to the making of new mathematics.

Teachers and The Reformed Curriculum

To insure that the content of the reformed algebra curriculum would be learned and accepted by teachers, the staff of the Teachers and Algebra Project began to work with core groups of three middle and three secondary teachers from suburban and urban schools in the Boston area. The teachers are volunteers who are motivated to change the way they teach, but who have not been especially equipped or encouraged to change. Several dozen additional teachers have participated in week-long workshops and followup activities. On the basis of more than 1,000 hours of interview and observational data collected from these teachers, it is clear that their knowledge of the algebra domain has the same constraints as the curriculum they were educated to teach and the materials they presently work with in their classrooms. Some of these constraints are discussed in the lead article of this issue of the *NCRMSE Research Review.*

Five themes seemed to recur with regularity in an analysis of the data collected from project teachers during interviews, meetings, and workshops. These include (a) teachers as users of the verbal language of algebra, (b) teachers as learners of algebra, (c) teacher as users of graphical and symbolic representations, (d) teachers' rationale for teaching algebra, and (e) teachers as curriculum designers. The first three themes will be the focus of the next three sections. The teachers involved in this project did not possess a well-developed rationale for why students should learn algebra. While their understanding of algebra expanded dramatically, the process of applying that understanding to their classrooms was a slow and complicated one.

Teachers As Users of the Language of Algebra

Project teachers made noticeable changes in their use of mathematical language during their involvement with the project. They increased the precision with which they used mathematical terms and vocabulary. As they gained experience manipulating graphical representations of functions that paralleled their traditional manipulations of symbolic representations of functions, the mathematical language they employed included more visual metaphors.

During early meetings, teachers rarely used the term "function" and made few distinctions between such words as "equation" and "expression." After the meanings of these words were discussed, a self-consciousness emerged as teachers caught themselves using the word equation when they meant function. In later meetings, the mistakes became a source of humor, with careful listeners correcting their colleagues. Their confusion with the terms probably represented their confusion with the role of x when used as a variable or when used as an unknown.

The meanings of the words used in teaching algebra and the meanings of the words in relation to other topics or ideas were explored systematically in project discussions. Teachers considered whether they used certain words in ways that inherently lead to confusion rather than clarity for their students, and whether there is a logic regarding word meaning that could enhance rather than undermine teaching. As teachers and project staff increased their understanding of the mathematics that lies behind mathematical terms, they concluded that the traditional algebra curriculum does not possess a coherent language for its concepts.

Judah L. Schwartz, Principal Investigator of Algebra and Quantitative Analysis working group.

Teachers were encouraged to become familiar and comfortable with software environments during project meetings. The software environments allowed teachers to manipulate functions in graphic and symbolic ways. Teachers began to use a greater number of words and phrases about seeing and moving: "Now I can see it." "If you get far enough away it looks like a parabola." "I can just compare the pictures." "Where did that hump come from?" "It's a quartic, but see, it is really flat." "We have to change the fatness factor." "Parabolas are 'fat and thin'." "Cubics 'float up here over the axis'."

Words about motion and action also became commonplace at project meetings. As teachers began to manipulate and modify functions graphically as well as symbolically, they described their actions verbally, "Let's drag this one over here." "All we have to do is stretch it and then push it up." Distinctions between functions that seemed compelling when represented symbolically could be seen in a quite different way when represented graphically.

Teachers commented that it is difficult to pull themselves away from the use of symbolic representations. They are now looking at the graphical representations of the constructs for which they used only symbolic representations in their classes. If they are assured that they will not have to forego the language of symbols in favor of the language of graphs, teachers admit that they feel empowered by graphical possibilities. While this use of language indicates movement, the extent and depth of this movement varies from teacher to teacher.

Teachers as Learners of Algebra

Teachers moved from their roles as "knowers" into those of questioners and learners during the series of project meetings. They began with an understanding of algebra that was "locally firm and globally fragmented and incoherent," in the words of one project observer. They had difficulty talking about the value of algebra, in part perhaps because most of them had had little opportunity to use algebra outside of their classrooms. Teachers were taught new ways of thinking about and organizing the subject and provided with software that illustrated these new ways. Project researchers saw dramatic changes in teachers as they explored and discovered algebra from these perspectives.

When presented with a series of algebra problems in an early session, teachers began working on them with a firm belief and a sense of security in their knowledge of the subject. Most of them skipped introductory exercises or problems and went straight to those that were the most challenging. They soon had to retreat and rethink their approaches, however, as they realized that their understanding of the problems and concepts was based solely on a symbolic representation perspective. Teachers also spent early meetings exploring new software and the options provided in each of the packages.

The teachers' learning was, at first, a private matter. Their voices were low as they worked to solve problems, asking quiet questions of their partners or the project staff. They rarely commented about learning something new, seeing a new dimension of algebra, or understanding something that they had never before comprehended. Some would say, "How could a student infer that?" or "Maybe a really savvy student would get it, but I'm not sure about my classes." These remarks indicated that the teachers were learning new things; their conjectures about whether the materials could be learned by their students may, in fact, have been a projection of their own uncertainty about the material. There was a growing openness as group discussions evolved from individual questions. By the second session, teachers had become uninhibited, making remarks or asking questions of anyone near them: "Well this certainly isn't in my bag of tricks!." "I never had this understanding of algebra." "I never saw it this way until now." "I learned something new today." "I don't have any answer for that and that is why this is new and exciting territory for me."

By the fourth meeting, teachers worked with an escalating excitement, delight, and rigor. Many telling remarks were made during a discussion about imaginary roots in the complex plane. Participants in this conversation stretched very hard to visualize a notion that had only been familiar to them in symbolic form. Their remarks included: "Until today I didn't have a model to deal with complex numbers." "Here we all are, trying to see what we really only know symbolically. This is a real struggle." "I have never had this visually." "Now I know why complex numbers are in pairs." "How do I picture complex zeros? I don't!" "Are we in 3D?" "This is very exciting. That is a hard visual to get." "So that's where the imaginary numbers have been! Wow! I see. I mean imaginary is imaginary, so I never knew I could see them."

Teachers' Use of Graphical and Symbolic Representation

Teachers identified an interesting tension early in the series of project meetings. They noticed that they had developed a "symbolic fluency," or comfort with and reliance upon symbolic representation, early in their mathematics education. Most said they were still thinking in terms of numbers and symbols when they looked at graphical representations of functions. Project staff wanted to know whether teachers held a symbolic bias that could affect their learning and teaching of algebra, and when the teachers' symbolic and graphical approaches would become interwoven. The teachers discussion provided answers to these questions. One teacher said, as her students were studying slopes, "What they don't know because of memorizing symbols with no picture in mind!" Teachers finally began to use both the graph and the symbolic representation as they explored some new dimension of algebra, "It took a few days to get used to reading the pictures, then to tack it back on to what we know." Midway through the series, teachers showed a tendency to abandon one perspective for the other, the old for the new or the familiar for the unfamiliar, rather than a combination of the two. In the later sessions, teachers said

continued on back page

Review of NCRMSE Research

Generations of algebra students have found abstract algebraic expressions difficult to understand. There is a consensus within the mathematics community that functions–one form of algebraic expression–are among the most powerful and useful of mathematical notions. Most advanced mathematics courses are built around the use of symbolic representations of functions. First courses in algebra introduce content on the symbolic representation of functions and the manipulation of these symbolic representations.

Graphical representations of functions receive little attention in algebra classes. While research is sparse on how the use of graphical representations increases student understanding, those working with graphical representations believe that, for most students, such representations make functions easier to learn about and use. Creating graphical representations was once a time-consuming and cumbersome task for teachers. With today's computers and graphing calculators, teachers and students can create graphical representations of functions and transformations of these functions quickly and easily. These technological tools have the potential both to reshape the ways in which algebraic concepts are taught and to restructure the levels at which they are taught.

The National Center for Research in Mathematical Sciences Education began an effort to synthesize research on teaching, learning, curriculum, and assessment that focused on graphical representation of functions in 1988. It initiated its efforts with a conference that brought together researchers who were concerned about current issues in the domain, the impact of technology on the domain, and the integration of research on teaching, learning, curriculum, and assessment. Participants at the conference developed a common vocabulary with which to describe their work and a common agenda for future work. This led to the development of a series of papers on the content of the domain and what is known about student thinking, teacher thinking, teacher knowledge, classroom instruction, and curriculum that relates to the domain. Conference participants have continued their efforts as members of the NCRMSE Working Group on the Teaching/Learning of Algebra and Quantitative Analysis. The Working Group, chaired by James Kaput, University of Massachusetts–Dartmouth is described earlier in this newsletter. This review summarizes the papers of original conference participants, which consider the implications of graphing technology for mathematics educators and their students.

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Three authors, Frank Demana of Ohio State University, Harold Schoen of the University of Iowa, and Bert Waits of Ohio State University, believe that graphing is a first step for learners as they build an understanding of the representation of functions. They examined the content that involved graphs in a typical mathematics textbook for each of the Grades 1-8. According to their analysis, about 3 percent of the textbook pages for Grades 7 and 8 contain graphing content; this amount is twice the amount found in any of the textbooks for the earlier grades. They concluded that students in Grades 1-6 have almost no experience constructing a graph of any kind. Nor do students encounter activities that ask them to make a global interpretation of a graph. The authors note that the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) recommend the introduction of appropriate graphing activities very early (Grades K-4) and make these activities a major part of the curriculum at middle and secondary levels.

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Author Sharon Dugdale, University of California-Davis, reviews her own and others' research on students' conceptions of functional relationships. Her review cites research evidence that students who view transformations of functions graphically and in the form of algebraic symbols perform better on non-standard questions than those who only view them in the form of algebraic symbols. She concludes that instructional methods that use multiple approaches need to be designed to improve students' development of graphical skills. While function-plotting software permits visual representations of algebraic functions to play a more substantial role in classes, according to Dugdale, any revised instructional approaches need to consider how students' perceptions change and how their ideas about graphical representations evolve as they acquire broader experience.

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Skills, connections, and coordinations that are difficult for students to develop, according to Judit Moschkovich and Alan Schoenfeld of the University of California-Berkeley and Abraham Arcavi of the Weizmann Institute of Science-Rehovot, Israel, may seem trivial to those who have mastered them. Their chapter reports on a research and development program that is mapping the way students come to understand the complex domain of different symbolic representations of linear relations so that curricula can be constructed that help students deal with that complexity. Their work is based on an approach that defines understanding as making connections and analyzing "content domains" to determine the kinds of connections that competent practitioners make. In their view, pedagogy needs to move from its emphasis on procedural skills to an emphasis on making connections and developing an understanding if it is to plan algebra content for students that is mathematically interesting.

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A historical account of the evolution of teacher and textbook thinking about functions is provided by Thomas Cooney and Melvin Wilson of the University of Georgia. They conclude that there has been a movement in secondary mathematics during the 1900s toward an increasing emphasis on the concept of function. In their words, "The emphasis on functions as a unifying mathematics concept, as a representation of real-world phenomena, and as an important mathematical structure remains central to contemporary discussions." They note that the definition of a function has changed over the last decade and that technology may become a major factor in determining how school mathematics will treat functions in the future. Their review of relevant research leads them to conclude that little is known about the interaction between teachers' knowledge and beliefs about functions and the extension of these interactions to their classroom practice. The authors caution future researchers who would examine teachers' knowledge and beliefs to appreciate the importance of "the context in which knowing and believing occurs."

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While knowledge about what teachers know and how they think about algebra is fragmentary, according to F. Alexander Norman of the University of Texas at San Antonio, some inferences can be drawn from research on students' understanding of algebra. He suggests that these inferences can be used by researchers to begin a comprehensive investigation of teachers' knowledge of graphing and functions. His review of research includes categories on students' understanding of functions, graphs, and multiple representations. It concludes with a series of research questions that need to be addressed and a description of a long-term study that deals with some of the questions. A few of the questions are: Are the cognitive learning processes of teachers and the specific knowledge required for them different from those of students? How important do teachers perceive the function concept to be? In view of the different emphases suggested by the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989), how might teachers' beliefs about functions and graphs affect their acceptance of a mathematics curriculum with a different orientation? What are teachers' views on the importance of introducing the notion of functions via multiple representations?

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In her chapter on classroom instruction, Carolyn Kieran of the University of Quebec at Montreal traces work on students' perceptions of functions. Relying on research completed in an international sample of countries, she emphasizes the differences in perceptions among age and ability groups of students and the implications these hold for teachers. Sketching and interpreting graphs, while routinely taught in The Netherlands and England, receive little attention in North American schools. Further, when U.S. students tabulate, plot, or read values from graphs, they seldom have the opportunity to apply their skills to a practical situation. She points out, in a historical narrative, that graphs were used to represent early conceptions of functions. Author Kieran concludes her chapter with descriptions of recent projects using technology-supported environments that involve graphs and functions. The descriptions fit into three categories: (a) activities that do not require a knowledge of algebra; (b) activities that are included in first-year algebra courses; and (c) activities with students who have completed at least one course in algebra. She concludes that a "tide of change" is evident and that the "newer" approaches to graphing have three major focal points, interpreting global features of graphs, using basic families of graphs to explore the roles of the parameters of the algebraic representation, and using graphs as a tool for problem solving in applied settings. The teachers involved in the projects she described reported positive effects of the new approaches on student motivation.

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In their chapter on the curricular implications of the graphical representation of functions, Randolph Philipp of San Diego State University and William Martin and Glen Richgels of the University of Wisconsin–Madison make the point that "for most of this century, curricular decisions regarding the use of graphs and functions have been made without the benefit of research." In their historical perspective, they trace the introduction of algebra during the first half of the 19th century at Harvard, Yale, and Princeton. Algebra was moved from a college to a high school course with few changes in its content later during the same century, according to their account. They cite a 1926 survey in which "all high school teachers, except mathematics teachers, believed that more students ought to be enrolled in their courses" and a recommendation of the same era that would have admitted only 25 per-

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cent of high school students to algebra classes. They state that a main implication of the "graphical representation capabilities of computers and calculators is that solutions to interesting and realistic problems are accessible, even without well-developed manipulative skills." With examples, they show that students who have dropped out or been excluded from mathematics may have greater access to mathematics with a graphically-oriented curriculum. Such a curriculum, they note, would require changed approaches to teaching, classroom materials, and assessment.

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While students have found algebra difficult to understand, "our only educational solution to date has been demographic–to eliminate ... students from the responsibility for learning it," says James Kaput of the University of Massachusetts-Dartmouth. He calls for proleptic research, research that deliberately anticipates the future, to remedy what "we [are now seeing as] a severe conservative bias in mathematics education research relative to technological change... that is likely to endure." He covers alternatives to present-day approaches that would place algebra in the middle grades, that would teach algebra concepts to students who have not yet mastered arithmetic in its several aspects, that would pay significantly more attention to linking experiences to formal mathematics, that would incorporate simulations that provide "continuous, real-time feedback in several experiential dimensions." Finally, he points to additional research that should be undertaken to anticipate the mathematics of the future.

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In his summary chapter, Steven Williams of Washington State University concludes that good teaching, "the careful choice of mathematical tasks coupled with efforts to establish an environment for exploration, conjecture, and dialogue, [has] been shown in a number of studies to be successful in leading students to understanding functions and their graphs." Assuming the view of learning as enculturation, he calls for an in-depth analysis of the content domain or the specification of the conceptual field for functions and graphs. Technological tools, he cautions, must be seen as tools and not "as creators of knowledge." What technology can do, he says, must be linked to human activity and human concerns; careful anthropological studies of how functions and their graphs are used by various cultures in their everyday practices could help to define the content domain. While the complexity of the domain makes doing research on the cognitive processes used by students difficult, he points readers to research strategies that effectively are uncovering the understanding of students. As a basis for further modeling of the understanding, learning, and teaching of graphs and functions, however, he points out that we need more of these careful analyses. Researchers, he says, have yet to assess which functions and graphs are "put to use in the mathematical cultures ... our students will need to join."

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The papers from which the above comments were drawn form the basis for *Integrating Research on the Graphical Representation of Functions*, a book edited by Thomas A. Romberg, director of NCRMSE, and Elizabeth Fennema and Thomas P. Carpenter, associate directors of NCRMSE. The editors are professors of mathematics education at the University of Wisconsin–Madison. The book will be published by Lawrence Erlbaum Associates, Incorporated, during the spring of 1993.

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An Annotated Bibliography of Publications

The National Center for Research in Mathematical Sciences Education (NCRMSE) produces reports that summarize its research. Some of these reports are in press or have been published as journal articles, as chapters in books, or as research reports. The Center maintains an annotated bibliography of its publications. It also prepares a more abbreviated annotated list of pre-publication reports–working drafts–that are being prepared for publication. Readers of the *NCRMSE Research Review* can obtain copies of the pre-publication reports for the cost of their reproduction and mailing. For copies of one of the lists or further information, contact Donald Chambers, Director of Dissemination, National Center for Research in Mathematical Sciences Education, University of Wisconsin-Madison, 1025 W. Johnson Street, Madison, WI 53706, FAX (608) 263-3406, telephone (608) 263-0761.

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their work with the graphs had given them insight into their work with symbolic representations, "It makes all the sense in the world, it feels right–its just another representation."

Teachers seem to have gone through a succession of stages. At first they were skeptical about anything that they were unfamiliar with–the symbolic was valid and central to the subject. As they began to learn about the mathematics and could understand more broadly and deeply when using graphical representations, they began to think that all of algebra should be done graphically. It was only after additional reflection that they realized they were using the two forms of representation in complementary ways; they began to move toward a more balanced position, treating the two representations as complementary, each with strengths and weaknesses.

The Teachers and Algebra Project set out to reform the mathematical content and pedagogy of algebra. To achieve its goal, it built a reformed curriculum around the mathematical idea of function. Project staff are now examining whether teachers find the reformed content both learnable and acceptable. In the third and final year, project data will be analyzed and a final report on project activities completed. Preliminary analyses of project data suggest that while teachers do not learn mathematics from their coworkers, teachers occasionally learn new techniques and approaches to the teaching of mathematics from their coworkers. Those who were motivated to change their classroom practices often encountered skepticism and hostility from their less change-oriented coworkers. Where there were evident shifts in teachers' practices, teachers received strong administrative support and encouragement for their efforts. The project is supported by the Teaching/ Learning of Algebra Working Group of the National Center for Research in Mathematical Sciences Education.

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NCRMSE RESEARCH REVIEW

The Teaching and Learning of Mathematics

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In This Issue:

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Research Charts Change in School Mathematics

Leaders of the mathematics education community in 1989 adopted a series of goals to ensure that young Americans will enter their workplaces and the democratic institutions of this society with the mathematical knowledge and skill they require (NCTM). The goals launched a national initiative, the second in the last half of this century, to reform school mathematics. The first such initiative, which was labeled New Math, took place in the 1950s. Retrospective studies of that earlier initiative showed it had not been properly implemented, nor had its effects been widespread (CBMS, 1975).

The Working Group on the Implementation of Reform at the National Center for Research in Mathematical Sciences Education (NCRMSE) has developed a far-reaching research agenda related to the mathematics education community's 1989 goals. Walter Secada, University of Wisconsin-Madison, chairs the group, which is studying implementation efforts within the culture of the school to reform the learning and teaching of mathematics. The Working Group, according to Chair Secada, will identify innovative processes, obstacles to reform, and ways to overcome obstacles. The group seeks to redefine reform, shifting toward the view that it is a complex set of relationships that require ongoing change and renewal.

The NCRMSE Working Group on reform includes researchers from across the U.S. who work with or study educational reform. Group members' interests include equity, curriculum development, authentic assessment, the study of teaching, teacher change, and school-wide change. The group includes scholars and practitioners who range in experience from those who are beginning their careers to those who are seasoned researchers. It maintains a network for its members, facilitating their study of reform in mathematics education as well as topics related to reform. Because this Working Group is set up in a manner that permits its activities to cut across or interrelate with all of the other NCRMSE Working Groups, it is able to apply quickly or to refine the findings of the other groups' research. This issue of the *NCRMSE Research Review* carries descriptions of the activities of the Working Group, an article about one of its major studies of school-based reform, and a review of portions of a monograph on equity developed by several group members.

New Book Considers Domain of Rational Numbers*

This book presents a content analysis of the domain of rational numbers. The analysis considers the mathematical complexity and applications of the domain. The book describes the integration of research on teaching, learning, and assessment and explores the implications of such an integration. The last section of the book describes, in detail, a school-based curriculum project where principles gained from research on rational numbers have given direction to the development of instructional activities and assessment procedures.

Part 1: Overview

Toward a Unified Discipline of Scientific Inquiry
T. P. Carpenter, E. Fennema, T.A. Romberg.

Part 2: Content Analyses of Rational Numbers

Rational Numbers: Toward a Semantic Analysis–Emphasis on the Operator Construct,
M. J. Behr, G. Harel, T. Post, R. Lesh.

Rational and Fractional Numbers: From Quotient Fields to Recursive Understanding
T. E. Kieren

Part 3: Students' Thinking

Learning Rational Numbers with Understanding: The case of informal Knowledge,
N. Mack.

Protoquantitative Origins of Ratio Reasoning,
L. Resnick, J. A. Singer.

Ratio and Proportion: Children's Cognitive and Metacognitive Processes,
S. J. Lamon.

Part 4: Teachers and Teaching

Halves, Pieces, and Twoths: Constructing Representational Contexts in Teaching Fractions,
D. L. Ball.

A Critical Analysis of Teaching Rational Numbers,
C. A. Brown.

Benefits and Costs of Research that links Teaching and Learning Mathematics
J. Hiebert.

Using Principles from Cognitive Psychology to Guide Rational Number Instruction for Prospective Teachers,
J. T. Sowder, N. Bezuk, L. K. Sowder.

Part 5: Assessment

Assessment of Rational Number Understanding: A Schema-Based Approach,
S. P. Marshall.

Part 6: Curriculum and Instruction

Fractions: A Realistic Approach.
L. Streefland

Curriculum Implications of Research on the Learning, Teaching, and Assessing of Rational Number Concepts,
T. R. Post, K.A. Cramer, M. Behr, R. Lesh, and G. Harel.

Rational Numbers: An Integration of Research was edited by Thomas P. Carpenter, Elizabeth Fennema, and Thomas A. Romberg, professors in the School of Education at the University of Wisconsin-Madison. It is 372 pages in length and was published by Lawrence Erlbaum Associates, Inc. Publishers, Hillsdale, NJ (201) 666-4110

*The book was developed at the National Center for Research in Mathematical Sciences Education with funding from the Office of Educational Research and Improvement, U.S. Department of Education (OERI/ED).

The Networks

The Working Group on the Implementation of Reform supports a series of networks that foster collaborative efforts. For their first meeting, group members prepared written summaries of their research on reform, identified areas they appeared to share with other members, and suggested areas for future collaboration. The discussion of the summaries held at that meeting led to the development of a symposium, Multiple Perspectives on School-Based Reform of Mathematics, presented at the 1993 AERA conference. At the second meeting of the Working Group, members discussed case studies of reform, including the Urban Mathematics Collaboratives Project funded by the Ford Foundation, the Interactive Mathematics Program funded by the National Science Foundation, and the role of teacher collaboration in school level-reform at QUASAR sites.

The Working Group also supports a series of research projects undertaken by networks of Working Group members. These groups meet to discuss their work, to share ideas, and to plan for research related to the Working Group's goal. One subgroup of members, for example, met to discuss plans for and react to a study on the innovative use of technological tools in geometry that was directed by Martha Stone Wiske, a Principal Investigator of the Working Group.

The Working Group identified a network of scholars who are interested in research methods for the study of teacher change. This subgroup, described under the heading Teachers and Teacher Change, developed a set of preliminary concept papers and distributed it to others with the same interests. Some of the members are expanding their concept papers so they fit into a projected monograph on the study of teacher change.

Staff members of this NCRMSE Working Group work with or maintain communication with research projects that have similar interests. Examples of such projects are The Network Incorporated's Case Studies for the Study of Reform in Science and Mathematics Education, and the National Council of Teachers of Mathematics's Exxon Foundation Task Force on reform in mathematics.

The School Level Reform of Mathematics

A major study of the mathematics education reform movement and its implementation was begun by this Working Group in late 1991. The nationwide study has identified a sample of schools that are involved in substantive efforts to enhance their mathematics programs. The research, directed by Working Group Chair Secada, seeks to document how sample schools have become sites that support teaching for understanding, engage students in challenging mathematical content, and support the professional development of their teaching staff. The conceptual framework on which the study is based is on pages 5-7 of this newsletter.

For the initial phase of the study, Working Group staff members solicited nominations of schools that were engaged in substantive reform efforts. The more than 7,000 individuals from whom they solicited school nominations include the supervisors of mathematics who are members of the National Council of Supervisors of Mathematics, teacher educators who are members of the National Council of Teachers of Mathematics (NCTM), persons who are members of the American Educational Research Association's (AERA) Special Interest Group on Research in Mathematics, the nominees and awardees for the Presidential Award for Excellence in Science and Mathematics Teaching for the Years 1990, 1991, and 1992, and participants at NCTM regional and national conferences.

During its first year, the staff members obtained survey data on the reform efforts undertaken in more than 200 schools. Five respondents– one designated a key informant and four teachers– from each of the 200 schools completed questionnaires about their schools that included detailed questions about student experiences and the professional lives of teachers–An additional 300 schools will be mailed survey forms during the Spring of 1993. A smaller sample of 40 schools were selected as subjects for follow-up telephone interviews. Of the 40 schools, 20 will also be visited by researchers. In-depth case studies of 12 schools will be completed. These activities are part of an effort to document the breadth and depth of reform that exists in these schools, and to understand social and organizational features in the schools that nurture or impede reform.

Equity

One of the most pressing challenges facing the reformers of school mathematics is ensuring that reform efforts result in increasing all students' access to high quality mathematics curriculum and instruction. The issues of access and equity are

A major study of the mathematics education reform movement and its implementation was begun by this Working Group

deeply embedded in the Working Group's study on school-level reform. The Working Group has developed a series of papers on innovation and equity that will be published in the book, *New Directions for Equity in Mathematics Education.* It is also assisting the Women in the Mathematics Education (WME) Special Interest Group (SIG) in its support of a small task force that is exploring emerging research and development needs.

Teachers and Teacher Change

If all students are to learn more mathematics, most teachers will need to change the way they teach mathematics. A subgroup of scholars in this Working Group who are reviewing research methods for the study of teacher change began work on a monograph, *Methodologies for Studying Teacher Change in the Reform of School Mathematics,* at their first meeting. The chapters were designed to provide this network of scholars with a common background from which to plan future research.

Working Group principal investigators have carried out studies related to teacher change. A two-year study investigated the influence of technological innovations–among them, the *Geometric Supposer*–on the teaching of high school geometry. The study was

The issues of access and equity are deeply embedded in the Working Group's study on school-level reform.

conducted by Martha Stone Wiske under a NCRMSE subcontract with Harvard University. Investigator Wiske documented the changes made by teachers with the data she collected during visits to three research sites and interviews with nearly 50 teachers: Their courses became more open-ended than previously; they included opportunities for students to conduct investigations and inquiries; and they encouraged students to make conjectures and to support the conjectures with justifications. Dr. Wiske also documented the isolation that the teachers felt in their workplaces and the ways they sought support from like-minded colleagues outside their schools, in their districts, or at state and professional meetings. In the follow-up action-research phase of her study, Dr. Wiske shared a summary of her findings with various groups of stakeholders. She then sought their advice on policy changes that could be implemented to develop greater support for teachers (Wiske, M. S., Levinson, C. Y., Schlichtman, P., & Stroup, W., 1992; Wiske, M. S., & Levinson, C., 1992).

Another study looked at changes in teachers' conceptions about mathematics and teaching practices that resulted from teachers' long-term participation in a Professional Development School. The study was conducted by the late Steven Kirsner under an NCRMSE subcontract with Michigan State University. Investigator Kirsner found that those teachers who are amenable to changing their practices require time and patience. They must learn to let students struggle as they engage in problem-solving activities. According to the research, students often leave mathematics classes confused about class tasks. Teachers who once provided answers–a form of closure–must now learn to allow each student to construct his or her own understanding of the content. Imperatives and challenges for improving the professional lives of teachers, according to the research, include providing them with time to work collaboratively, to reflect on their practice, and to observe others' teaching practices. Portions of one of the publications prepared at the end of this study (Kirsner, S. A., & Bethell, S., 1992) were summarized in the Fall 1992 issue of *NCRMSE Research Review.*

Researchers who are examining the reform of school mathematics or other areas within the Implementation of Reform Working Group research agenda are invited to share their work with group members. Two of the papers cited in the article are available from the National Council of Teachers of Mathematics. Copies of the other cited papers can be obtained from NCRMSE. For further information on the Implementation of Reform Working Group, readers may contact its chair: Dr. Walter Secada, National Center for Research in Mathematical Sciences Education, University of Wisconsin-Madison, 1025 W. Johnson Street, Madison, WI 53706.

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SCHOOL-LEVEL REFORM and the TEACHING/LEARNING of MATHEMATICS

Walter G. Secada and Lisa Byrd

An impressive knowledge base was developed during the last decade that increasingly informs efforts to enhance the teaching of school mathematics. That knowledge base is given visibility in such documents as the NCTM *Standards* (1989, 1991) and the *California Mathematics Framework* (1992), state and national policy initiatives for teacher training and curriculum development, and high school graduation requirements that demand additional mathematics courses of students.

Schools as Units of Analysis

Policy initiatives typically have focused on the nation, state, or district levels, and recent research has been designed to inform policy development at these levels. The research relates the levels to a classroom, a teacher, or a student. Few research efforts have considered a school or a school's mathematics department as a meaningful unit of analysis. In their studies of elementary-school mathematics in California (Prawat, Remillard, Putnam, & Heaton, 1992; Cohen & Ball, 1990), for example, members of the research team from the Elementary Subjects Center focused on how individual teachers interpreted and enacted the mandates of the *California Mathematics Framework* in their classrooms. The school–as an intervening unit–was not included in their analyses.

The work of several researchers provides some excellent ideas of how good teaching or teachers might look under the aegis of reform (Lampert, 1990; Ball, 1993; Fennema, Franke, Carpenter, & Carey, in press). But these analyses do not take into consideration the place where teachers work and where mathematics is taught–the school.

Some work on schools as organizations and on the larger processes of schooling has considered mathematics. Work on tracking (Oakes, Gamoran, & Page, 1992), on effective schools (Good & Brophy, 1986), on school restructuring (Newmann, 1993), on high school departments including mathematics departments (Little & McLaughlin, in press), and on cooperative groups (Cohen, 1992) provides insight on how schools are organized and how their organization and operation affect the teaching and learning of mathematics. The mathematics content that is taught and how the quality of that content may constrain the teaching/learning process typically have not been considered.

There are sound reasons why researchers should be concerned about the school as a site for enhancing the teaching of mathematics. Teachers and policy makers alike face challenges when they try to implement change in the larger context of the school. Unless efforts to improve mathematics teaching are understood in settings that are more expansive than an individual teacher's classroom, these challenges will not be understood. Curriculum development efforts, for example, often rely on individual-teacher volunteers to test materials. School-wide efforts to adopt changes or test new materials seldom take place. Without an understanding of how materials are treated within a school and related issues such as cross-grade articulation and within-school variability in adoption, it is unlikely that the full effects of the preliminary curricula will become apparent. Suppose a class were to receive outstanding mathematics instruction on worthwhile content for a year during its schooling. Would the class be returned to instruction-(business)-as-usual after that year? Or would the school make programmatic or school-level efforts to ensure continuity across several grade levels?

The preceding paragraphs define the issues that inform the research efforts of the Working Group on The Implementation of Reform at the National Center for Research in Mathematical Sciences Education. Their study of school-level reform is focused at the intersection of two major lines of research: improving the teaching and learning of mathematics, and the restructuring and reorganization of schools.

Framework for the Study of School-Level Reform

Four key constructs provide the framework used by Working Group staff members for their study of school-level reform. The constructs are derived from multiple fields of inquiry: effective schools, school restructuring and organization, and teachers' professional lives. Mathematics is viewed as one of many aspects of the study set within the larger environment of schools. The key constructs of the framework are collective action, student experiences in mathematics, equity, and ideal practice.

Collective action

This study distinguishes between collective action and the work of the individual teacher who endeavors to enhance the mathematics learning of his or her students, but who often works in isolation, without the support of the collective efforts of a group of teachers or a program, department, or school. The construct, collective action, is used to convey the notion that the school, or some significant unit of it, may adopt a particular mission. In this case, the mission would be the enhancement of mathematics teaching and learning. The school or a unit of it may then take concerted, coordinated actions to achieve a particular mission.

Student experiences in mathematics

The second key construct, student experiences in mathematics, refers to the constellation that students may experience in school mathematics that: (a) supports reasoning, (b) contains worthwhile content, and (c) forms a coherent whole. Aspects of these experiences include the curriculum, the teaching and assessment students encounter, the technologies and other tools students use, the oral and written communication in mathematics that students take part in, the locus of mathematical authority in students' classrooms, the attention given to student beliefs about and attitudes toward mathematics, and the access to mathematics courses provided all students. The construct, student experiences in mathematics, is used to obtain information about the importance of connections both across disciplines and within mathematics, of cross-grade coherence and program articulation, and of the gestalt–how students experience mathematics as a discipline and as an entity that is dynamic and useful to them.

Equity

Equity is used in the study to refer to the range of concerns and actions that schools, teachers, and districts take when they act on the belief that all students–regardless of race, gender, social class, or language ability–can learn mathematics. It includes educational opportunity (to learn mathematics) and educational achievement (in mathematics). Are there, for example, systemic, school-wide efforts to ensure that diverse student populations are encouraged to take mathematics and that their experiences in mathematics are worthwhile? The research seeks knowledge about how schools construe equity in mathematics education and how they act upon that meaning.

The study is designed to provide an understanding of how each of the

. . .the study of reform seeks information about how ideal practice gets translated into school-wide practice.

constructs; collective action, student experiences in mathematics, and equity are constituted and interact at a school. What are the parameters, for example, of (a) collective action, (b) student experiences in mathematics, and (c) equity at a given school? How did each of these parameters develop over time? What trade-offs took place during their development? How are the parameters maintained and nourished? What are the obstacles that had to be overcome? How are the various parameters related within and across the major constructs?

Ideal practice

Documents such as the *Standards* (NCTM, 1989, 1991), represent ideal practice. Other visions of ideal practice can be derived from work on learning with understanding (Hiebert & Carpenter, 1992), constructivist prescriptions for teaching (Davis, Maher, & Noddings, 1990), teaching as a profession (Lieberman, 1988), reflective teachers (Zeichner, in press), and other theoretical analyses of teaching. Descriptions of ideal practice are being derived for the key constructs.

Finally, the study of reform seeks information about how ideal practice gets translated into school-wide practice. An ideal seldom is encountered in actual contexts. Substantial variability occurs in actual contexts because of the constraints imposed by the competing forces of school environments. Schools, in addition, may choose to pursue different avenues for reform or focus on different issues. They may also be at different points in their development of reform activities. Hence, in order to implement practices that move in the direction of reform, most schools make trade-offs.

The constructs of collective action, student experiences, equity, and ideal practice with its variability, provide structure to this study of reform. They are also useful guides in a search for schools whose efforts to enhance mathematics programs are substantive. Each of the constructs has multiple dimensions that will undergo further elaboration. Schools may find this beginning framework useful as they begin or increase their efforts to implement reform. They will need to consider balancing their efforts and resources in an optimal way as they change their mathematics programs. While schools may find it necessary to make trade-offs in the process of seeking ideal practice, they should be conscious of the impact of the trade-offs that relate to the four constructs.

Some Initial Results

A series of survey forms and structured interview formats were developed to collect research data from the sample of schools that were nominated as sites where substantive reform in mathematics education has been achieved. Both the survey form designed for a school's teachers and that designed for a key informant include questions that relate to the four constructs. Individual questions are worded in nonjudgmental ways to avoid eliciting responses based on social desirability. Some questions were included on both teacher and key-informant survey forms. The answers to the questions that the two forms have in common permit researchers to reach–by triangulation–a perception of the school's efforts. Brief written descriptions of a school's efforts to improve its mathematics program provided additional support for the school-wide perceptions.

Initial tabulation of data from 200 schools includes responses from 715 teachers: 85.9 percent of teachers in these schools indicated strong support for their school's efforts to reform its mathematics program. When asked whether the goals and priorities for their school's mathematics courses were clear, 38.6 percent of teachers strongly agreed and 50.3 percent agreed. More than 31 percent of teachers did not think that mathematics teachers in their school made conscious efforts to coordinate assessment practices or the manner in which they structured and taught their mathematics classes. The largest percentage of teachers preferred planning sessions devoted to the coordination of content that also suggested materials and activities to guide instruction. Secondary analyses revealed high school mathematics teachers do more team teaching and collaborative planning for curriculum and assessment than do their elementary and middle school colleagues. The initial results are taken from an extensive data summary prepared for the 200 schools in March 1993. Additional information on the 5-year study can be obtained from Dr. Walter Secada, National Center for Research in Mathematical Sciences Education, University of Wisconsin-Madison, 1025 W. Johnson Street, Madison, WI 53706.

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NCRMSE Research

Elizabeth Fennema, an associate director of NCRMSE and one of the organizers of the series of seminars on equity.

Walter Secada, an associate director of NCRMSE and one of the organizers of the series of seminars on equity.

Making school mathematics more equitable is one of the most complex issues facing educators. One objective of The National Center for Research in Mathematical Sciences Education is to inspire new approaches to equity. To that end, the Working Group on The Implementation of Reform in 1991-1992 sponsored a seminar series, New Directions for Equity in Mathematics Education. The presenters for the series offered innovative ideas on equity. Of the presenters, sixteen have prepared papers enriched by the content of seminar discussions and the presentations of other participants. The series of seminars was organized by Professors Elizabeth Fennema and Walter Secada, School of Education, University of Wisconsin-Madison. This review provides brief summaries of several of the papers.

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In his overview of the papers, Walter Secada observes that equity efforts must be interpreted within a context of social, political, and symbolic processes and beliefs. Such processes and beliefs create and sustain a mathematics education community, but they may also act to limit or prevent change. Some of the mathematics education community's shared beliefs and assumptions about reform and research are problematic from an equity perspective. Secada cites, for example, conceptions of good teaching and suggests that teaching standards may restrict our view of teaching to what happens inside classrooms and focus only on the teacher's content and pedagogical knowledge. A more appropriate description, according to Secada, would encompass teaching that keeps diverse students in the mathematics pipeline. This alternate criterion would include teachers whose knowledge, beliefs, and classroom interactions with students seem to align with the descriptions provided in NCTM's Professional Standards for Teaching Mathematics (1991), who are also concerned about equity. It would also include teachers who would not otherwise be included if only the criteria of the *Standards* were applied. To Secada the primary equity issue is access.

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Edward A. Silver, Margaret Schwan Smith, and Barbara Scott Nelson describe the approach taken to promote equity and mathematics education reform by the QUASAR Project. An educational reform project, it fosters and studies the development and implementation of enhanced mathematics instructional programs. QUASAR targets students who attend middle schools in economically disadvantaged communities. There are two central equity issues addressed in the QUASAR Project: (1) the need to increase access for all students to high-quality mathematics instruction that challenges them to think and reason, and (2) the need to increase the relevance of mathematics to their lives. The authors provide vignettes of classrooms in QUASAR schools where the emphasis is placed on the development of understanding fostered through both oral and written communication of ideas. Students are active participants who are involved in the construction of their own new knowledge and understanding. Thinking and reasoning are valued and argument and justification are supported. While the curriculum has expanded beyond the topics and skills of typical middle school instruction, the authors note that changes in instructional practice have proceeded slowly and unevenly.

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In the past, mathematics programs promoted as new or better have not, in fact, alleviated existing inequities–even though these programs were based on a large body of research on the learning of mathematics that suggests that all children, regardless of their cultural backgrounds, learn mathematics in similar ways. When classrooms are structured to take into account this knowledge of how children learn, learning is enhanced. Cognitively Guided Instruction is based on the principle that teachers who possess research-based knowledge about children's thinking in general are able to apply that knowledge to the thinking of individual children. Teachers are able to assess accurately a child's knowledge and to make decisions about instruction based on that child's thinking, rather than on expectations related to the characteristics of the learner such as race, gender, or ethnicity. **Deborah Carey, Elizabeth Fennema, Thomas P. Carpenter,** and **Megan Franke** describe the impact of Cognitively Guided Instruction on teachers and students in schools with student populations that are greater than 70 percent African American. According to the authors, "Students' potential for engaging in relevant, thoughtful problem-solving tasks was realized and this challenged existing norms about what might be considered appropriate mathematics content for 1st-grade children in urban classrooms, who traditionally have been subjected to low-level drill-and-practice type activities."

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Girls and women are underrepresented in mathematics and mathematics-related careers. Their underrepresentation traditionally has been described as a problem between girls and mathematics, a "girl problem." When girls are viewed as causing the problem by something they do or, more often, by something they lack, it follows that girls must solve the problem. **Patricia B. Campbell** recommends that the problem be redefined to include the role of education. The educational process affects girls' learning of mathematics as well as their attitudes toward mathematics. Individual classrooms and research on gender differences must change. Campbell calls for research on the question, "Why are gender differences in math/science majors and careers so much greater than cognitive math/science differences?" and for attention to educational, rather than biological causes, and their solution. She recommends efforts to prevent the "girl problem" from developing in the first place and devotes a special section in her paper, Special Programs: What Works, to reinforcing these efforts

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Suzanne K. Damarin reviews recent work of feminist epistemologists and philosophers. Then, guided by the critiques of science developed by feminist scholars, she critiques research on gender and mathematics. She notes that the feminist critique of science is part of a general movement that challenges notions of objectivity, reality, and truth and seeks to redefine science. To women scientists, some scientific findings, particularly those about women, have been unacceptable and two primary streams of research that counter the findings have emerged. Feminist empiricism, she explains, begins from the premise that, while scientific methods may be sound, some unsound practices or procedures lead to findings that are biased. Among the examples she recounts are drawing conclusions about the general population from studies of male samples and relating research results to male norms. She concludes that the empirical research on "gender and mathematics has . . . reframed the scientific study of women and mathematics . . ." But the Feminist Standpoint Epistemology, says Damarin, would have research question how women experience mathematics, and it offers a direction for future research on women and mathematics. The research that fits into this philosophic stream examines women's confidence in their mathematical knowledge and ability while at the same time examining the messages women receive that affect their sense of confidence. Individual women (and their teachers), she says, must struggle with these issues if "they are to learn and to teach mathematics."

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Marilyn Frankenstein argues that education in general and mathematics in particular will become more equitable only when the class structure in society becomes more equitable. Teaching about class structure, she maintains, can contribute to equity in mathematics education. In her teaching, she covers socio-economic issues in the context of a business and consumer mathematics course, with the assumption that students' increased mathematical understanding will enable them to examine the class structure of their capitalist society. She asserts that participants in her class are empowered when they realize they understand more mathematics than their test results indicate. The class uses economic data to support mathematics learning. When students participate in discussions about the structure of society, they probe behind data questioning, for example, the categories created to sort data, the persons placed in the categories, and the information that has been obscured by mathematical transformations of data. While her students are developing tools for understanding and performing statistical analyses, they also expose the myth of a classless society.

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The University of Minnesota Talented Youth Mathematics Program (UMTYMP) provides 140 to 150 mathematically talented students with an intense, accelerated mathematics program. The students cover four years of high school mathematics in two years, and two years of university-level calculus while still in high school. According to UMTYMP Director **Harvey B. Keynes,** program statistics on early enrollment, retention, and achievement once favored males. A series of interventions such as social, cultural, and counseling activities and changes in recruitment and UMTYMP classroom structure now achieve more equitable results. Keynes describes the interventions and the outcomes related to them in his paper, "Can Equity Thrive in a Culture of Mathematical Excellence?" He concludes that the interventions developed to improve female involvement in the program also enhanced the involvement of students of color and the rest of the students. Based on the UMTYMP experience, he says, "Equity has strengthened excellence."

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Gloria Ladson-Billings' premise is that all students can be successful in mathematics if their understanding of mathematics is linked to cultural referents that are meaningful to them. The instruction students receive must convey to them the teacher's belief in their ability to master the subject matter. Two vignettes of 6th-grade classrooms, one taught by a very experienced female teacher in a low income, predominantly African American school district, and another taught by a young male student teacher in an upper middle class community, are contrasted. Students in the woman's class remain engaged and excited throughout the mathematics class. Students in the man's class refuse to settle down and attend to the lesson. Ladson-Billings identifies the characteristics that distinguish the classrooms: 1) Students treated as competent are likely to demonstrate competence, 2) instructional scaffolding for students allows them to move from what they know to what they do not know, 3) the major focus of the classroom must be instructional, 4) authentic education extends students' thinking and ability beyond what they already know, and 5) effective pedagogy requires in-depth knowledge of the students as well as of subject matter.

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Large numbers of Hispanic students have limited English proficiency, yet only 3.2 percent of the nation's teachers are prepared to teach limited-English proficient (LEP) students. **Lena Licon Khisty** notes that Hispanic students consistently perform well below the United States average in mathematics. There is, she notes, nothing inherent in Hispanic students, their culture, or their families that accounts for the discrepancy in performance. She further argues that teachers' use of language in mathematics instruction is a major factor. Simply increasing the number of bilingual teachers or improving their clarity of speech, however, is not the answer. At issue are teachers' conceptual explanations and their ability to use questions and cues to extend students' talking and thinking in both languages. In the learning context minority students must be engaged in higher level critical thinking, and in their dialogues, teachers and students must come to understand the different "cultural language" of each other. To improve instruction in mathematics for Hispanic students, teachers will need to create learning environments that, rather than ignoring or devaluing students' home language and experiences, capitalize on them.

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The current science teaching model does not work for the nation as a whole and is even more unsatisfactory for minority students. That model of teaching requires that students assimilate textbook information, recall facts to answer questions, and make abstract connections out of context or in contexts that are unrealistic. This type of teaching marginalizes students who do not possess mainstream ways of knowing and communicating. **Beth Warren** and **Ann S. Rosebery** describe a collaborative project in which they, with bilingual, ESL, and science teachers, teach science from a sense-making approach. They create a community of scientific practice in language-minority classrooms because they view scientific practice as a socially and culturally mediated process of meaning construction and criticism. Warren and Rosebery describe the activities of a classroom organized around students' own questions and inquiries. They also describe the changes in the students' discussion and thinking about mathematics in such a classroom, and the changes in teachers and the teacher education process. Teachers experience participation in sense-making activities themselves as they are treated, not as persons who are learning to teach science, but as persons who are learning to be scientists. Creating science education programs that include rather than exclude linguistic minorities demands more than new curricula and new teaching strategies. Egalitarian sense-making science teaching practice will, according to these authors, transform teachers and students alike. Both will be empowered to think, talk, and act scientifically.

The papers summarized in these columns will appear in *New Directions in Equity for Mathematics Education,* a book edited by Walter Secada, Elizabeth Fennema, and Lisa Byrd. Drs. Secada and Fennema are Associate Directors and Lisa Byrd is a Project Assistant at the National Center for Research in Mathematical Sciences Education, The University of Wisconsin-Madison, Madison, WI 53706.

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Assessment and School Mathematics

Thomas A. Romberg, chair of the Models of Authentic Assessment Working Group and director of the National Center for Research in Mathematical Sciences Education.

Assessment is a necessary part of school mathematics curriculum and instruction. Traditional mathematics assessment procedures test a limited set of procedural skills. Such assessment procedures do not reflect the expanded view of school mathematics that the reform movement seeks. If reform goals are to be reached, they must be changed.

Researchers at the National Center for Research on Mathematical Sciences Education (NCRMSE) began to examine issues related to mathematics assessment nearly a decade ago. They looked at the assessment procedures used by United States teachers as well as at their test-preparation practices. Using actual test items, they studied the alignment of widely used tests with the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989). Their results demonstrated that the assessment procedures commonly used in schools were not only inadequate but should be viewed as a major barrier to the reform of school mathematics. This work is reported in *Mathematics Assessment and Evaluation: Imperatives for Mathematics Educators* (Romberg, 1992) a book published last year.

In 1990, several of the same researchers formed one of the NCRMSE Working Groups, Models of Authentic Assessment. The group, led by NCRMSE Director Thomas A. Romberg, made plans to develop criteria for authentic assessment models and to identify, test, and implement alternative assessment practices for school mathematics. The Working Group designed its research agenda to include a few major research investigations and a series of smaller explorations, as well as several collaborative activities.

Major Investigations
The major investigations of the Working Group include an adaptation of a Dutch curriculum-embedded testing model for use by teachers during instruction, the use of "super-items" that are based on a developmental model of reasoning, and the development of technology-based (video and computer) assessment models.

Curriculum-Embedded Model
Jan de Lange of the Freudenthal Institute in The Netherlands and Thomas A. Romberg are preparing an assessment plan for a middle school mathematics curriculum. Their goal is to build closer connections between assessment and instructional activities. They plan to develop techniques for scoring, aggregating, and reporting information on student performances. They, with the assistance of other staff members, will design, develop, and test assessment activities in varied formats, gather data on the relation between different formats and learning goals, and develop reliable scoring rubrics.

Superitems
Kevin Collis, with other group members, is completing a set of superitems based on the Structure of Observed Learned Outcomes (SOLO) Taxonomy for Grade 8. The items are being field tested and another group member, Mark Wilson, will analyze the data obtained during the field test using his design for a psychometric model. Working Group members are beginning to develop assessment items in the areas of probability and statistics. The items are designed to obtain data that will answer research questions on the teaching and learning of statistics. The data will be used to portray student understanding as it develops over the course of K-12 schooling, using the framework of the SOLO Taxonomy.

Assessment and Technology
Susanne Lajoie, one of the group's principal investigators, has used both video and computer technologies to prepare a library of exemplars of student products that would convey expectations to students and to devise ways to use the computer to record data on students' problem-solving activities. She is continuing to develop scoring rubrics for computer-derived data on student work and to collect verbal protocols that illustrate student development in the areas of problem solving, communication, reasoning, and connections.

Classroom Explorations
Two of the smaller explorations of the Working Group were carried out in classrooms. Linda Wilson studied the way in which a high school mathematics teacher, in a school that has stressed assessment reform, implemented reform ideas in her classroom. Her findings are reported in the next article in this newsletter, What Gets Graded Is What Gets Valued. M. Elizabeth Graue and Stephanie Smith conceptualized assessment from an instructional perspective. They took the position that assessment should grow from the "dimensions of learning that are the focus of instruction." They then tested the utility of their model in a middle school classroom. Their findings are reported in the last article, Review of NCRMSE Research, in this issue of the newsletter. David Clarke and Max Stephens along with Margarita Wallbridge (1993) checked to see whether changing the assessment practices at the senior secondary grades had an impact on instruction and assessment in the earlier grades. They carried out an analysis of curriculum documents, work samples from teacher and student interviews, and a questionnaire. From these data, changes in policies, curriculum and teaching, assessment, and reporting practices could be inferred. Seven members of the Working Group presented papers about their assessment research at a symposium, Moving Beyond the Rhetoric About Authentic Assessment in Mathematics, held at the annual meeting of the American Educational Research Association in April 1993 in Atlanta, GA.

Collaborative Activities
Working Group members have developed collaborative relationships and provided information on assessment to national, state, and local groups. Richard Kitchen is collaborating with The Consortium for Mathematics and

Its Applications (COMAP) to test high school units developed for the ARISE Project. He will focus on the assessment practices of several teachers who use the units in actual classrooms to extend knowledge of the ways assessment can inform instruction. Other group members have met with several state education agencies, including those in Wisconsin and Delaware, that are developing frameworks for alternative assessment programs, participated in New Standards Project meetings, and made presentations to participants attending conferences on assessment at the national level.

The assessment activities of NCRMSE have led policy makers and legislators at state and national levels to request assistance from the Working Group chair or its members. They were asked, in 1991, to prepare a document for the National Center for Education Statistics, *Improving Mathematical Performance: Reflections and Suggestions Based on the Results of NAEP's Twelfth-Grade Assessment.* The document is designed to provide teachers and other educators with an analysis of a report on the mathematics portion of the National Assessment of Educational Progress (NAEP). The first section of the document provides an overview of the NAEP report, including its purpose, framework used for developing test items, and a description of the 1990 level of performance for Grade 12. The remaining portions of the report take a closer look at student performance and at released test items for each of five content-reporting categories. The final section provides suggestions for improving the mathematical performance of secondary students. This report was followed by another that contains an examination of the NAEP items and their relation to the NCTM *Curriculum and Evaluation Standards, In Light of the NCTM Standards, Does the NAEP Mathematics Assessment Measure Up? A Look at the 1990 NAEP 12th-Grade Items.* The authors (Romberg, Smith, Smith, & Wilson, 1992) found that the NAEP items did not relate to the traditional high school curriculum nor did they relate to the NCTM *Standards.* Large numbers of the items, they concluded, covered content typically included in a traditional Grades 5-8 curriculum.

They took the position that assessment should grow from the "dimensions of learning that are the focus of instruction."

The National Assessment Governing Board (NAGB) asked the authors of the earlier reports to reflect on setting achievement levels for the NAEP mathematics assessment and to examine the feasibility of using existing international data to validate the derived levels. On the basis of this research, using both 1990 and 1992 NAEP mathematics items, the authors concluded that efforts to set achievement levels of the NAEP assessment were flawed, primarily because of the lack of validity of the items, but also because of the level-setting procedure. After additional analyses of items from the international examination, their study concluded it was not feasible to calibrate scores from either of the international tests with the NAEP.

In 1992 NCRMSE Director Thomas A. Romberg, chair of the Models of Authentic Assessment Working Group, was appointed to the United States Department of Education's Advisory Committee on Testing in Chapter 1 by the Secretary of Education. The committee was asked to examine the adequacy of standardized tests in measuring the academic achievement of Chapter 1 students. The committee developed a lengthy report on Chapter 1 testing and concluded that a new policy balance was necessary, that "Chapter 1 testing and evaluation regulations should become less concerned with large-scale evaluation . . . and more oriented toward enhancing effective classroom instruction and elevating student achievement." After the committee completed its work, its members, including NCRMSE Director Romberg, were asked to testify before the United States House Subcommittee on Elementary, Secondary, and Vocational Education on the Testing Committee's recommendations.

Some of the findings of the members of the Models of Authentic Assessment Working Group are being prepared for use by teachers and will be included in a 1994 book, *Reform in School Mathematics and Authentic Assessment,* to be published by SUNY press.

Assessment Standards

Early in 1993 NCRMSE Director Romberg and Working Group members Jeremy Kilpatrick, Susanne Lajoie, Sandra Marshall, Marvin Smith, Norman Webb, and Linda Wilson were asked to assist with the

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What Gets Graded Is What Gets Valued

by Linda Dager Wilson

Linda Dager Wilson, member of the Models of Authentic Assessment Working Group.

Rhetoric About Authentic Assessment

The term authentic assessment arose from educators' need to distinguish their new ideas on assessment from more traditional ideas on testing and testing instruments. When educators describe the current status of a person's capabilities within a specific conceptual, procedural, or developmental domain, they are assessing that person.

Assessing has been confused with testing, measuring, or evaluating. A teacher's assessment practices include all of the ways she/he determines what students know or can do. Testing can be contrasted with assessment in that it involves creating a situation that will inform decisions. Measurement involves specifying "how much" capability exists, and evaluation involves assigning a value to it (Lesh & Lamon, 1992).

Today the term authentic assessment encompasses a broad range of criteria, but performance remains an essential aspect. Proponents of authentic assessment argue that to gain a true description of what students know or can do in a discipline or domain, they need the opportunity to perform work in that discipline. A traditional standardized achievement test item would have asked, What is the most appropriate unit of measure for the length of a room: centimeters, meters, or kilometers? A more authentic task would ask the student to measure the length of the room while the teacher observes the student's performance.

Authentic assessment must build from authentic assessment tasks. To be considered authentic, assessment tasks will, according to Archbald and Newmann (1988), meet the criteria of: 1) disciplined inquiry; 2) integration of knowledge; and 3) value beyond evaluation. Disciplined inquiry depends on prior conceptual and procedural knowledge but it develops an in-depth understanding of a problem and "moves beyond knowledge that has been produced by others" (Archbald & Newmann, 1988, p. 2). Its objective is the production of new knowledge such as that created by scientists or historians. If assessment tasks are to reveal students' integration of knowledge, they must address the content as a whole rather than as a collection of fragments. Students are "challenged to understand integrated forms of knowledge . . ." when they are "involved in the production, not simply the reproduction of new knowledge," because production requires the integration of knowledge (p. 3). Features that characterize tasks that possess a value beyond evaluation produce either discourse, material results, or performances, and require the flexible use of time and collaboration with others.

Examples of assessment tasks that meet the three criteria specified above help to clarify the concept of authenticity. A paper-and-pencil task from the National Assessment of Educational Progress's Pilot Study of

Higher-Order Thinking Skills Assessment Techniques in Science and Mathematics is cited by Archbald and Newmann as meeting their three criteria (1988). The task asks children to examine data about five children competing in three athletic events and then to decide which of the five would be the all-around winner. The task is "open-ended" in that students must devise their own solution strategies and justify their answers rather than seeking one "right" answer. The exhibitions of mastery required by some high schools as a diploma requirement are also cited by Archbald and Newmann. Students demonstrate competence in multiple disciplines and produce projects that reveal their integration of knowledge with the exhibitions.

Grant Wiggins (1989) develops a similar set of criteria for authentic tasks: A task is deemed authentic if it requires "the performance of exemplary tasks" (p. 703) and is "responsive to individual students and to school contexts" (p. 704). Authentic tasks can, according to Wiggins, "reveal achievement on the essentials" (p. 704). In the portfolio-based assessment program in writing and mathematics that currently is being implemented in Vermont, open-ended tasks such as the one described earlier, and an oral history project are examples of tasks that fit his criteria for authenticity.

While the two sets of criteria described above deal with authentic assessment in several content areas, some educators have focused more specifically on assessment in mathematics (Webb & Romberg, 1988; de Lange, 1987; Lajoie, 1991). These educators detail criteria that are specific to mathematics but echo the broadly based criteria of Archbald and Newmann, and Wiggins. Lajoie (1991), for example, calls for assessment that will:

- Obtain multiple indications of individuals' knowledge, performance, and disposition.
- Use tasks that are instructionally relevant, meaningful to students, and realistic for the discipline.
- Use scoring and scaling procedures that are appropriate to the tasks being assessed.
- Align tasks with both curriculum and instruction and design them to show what students know.
- Consider racial/ethnic, cultural, gender, and aptitude fairness.
- Be integral to the classroom environment.
- Provide for assessing individual growth when part of a group activity.

Classroom teachers are central to the implementation of these new approaches to assessment.

Classroom teachers are central to the implementation of these new approaches to assessment. A recent case study of a high school mathematics teacher's assessment practices was designed to examine how teachers are interpreting these approaches to assessment in their classrooms. The remaining portions of this report are developed from the case study.

Assessment Reform at Valley High

Valley High is a typical small-town United States high school. Its nearly 800 students choose from a wide variety of courses from the English, mathematics, foreign language, science, art, music, and social studies departments. The mathematics department offers classes from a general mathematics track and a calculus/college preparatory track.

While, on surface, Valley High appears similar to other middle class schools in this country, it is beginning to feel the winds of change. Two years ago, the Valley School District began to initiate a series of district-wide reforms. It developed a Strategic Planning Commission of 150 members from the community and the school faculty, administration, and board and charged it with deciding what shape the reforms should take. The commission decided that its first goal would be implementing authentic assessment, based on its belief that assessment reform held promise as a catalyst for broader school reform.

A 1992 television show that focused on assessment highlighted Valley High as a "school that works." Members of the Valley School District board and the principal of Valley High present the school system as embracing authentic assessment as a central tenet. During the TV program, they claim the district is reevaluating its curriculum in light of its assessment philosophy.

Valley School District clearly has begun an ambitious assessment reform effort. At the elementary school, teachers have the option of using portfolios in place of report cards and grades. Extended parent-teacher conferences allow teachers to demonstrate the work in the portfolios to parents. Some teachers have shown parents videotapes of students at work. Valley High is more cautious about eliminating letter grades. According to its plan, by 1996 graduation will depend not on Carnegie units but on an autobiography, a thesis, a commencement project, and a portfolio.

Ms. League, the subject of this case study, has taught at Valley High for five years. She teaches a total of six classes per day with preparations in geometry, algebra II, and technical mathematics. In her overcrowded school, she has not yet been assigned a classroom of her own but carts her teaching materials down corridors to a variety of rooms. Her teaching credential and bachelor's degree in mathematics were completed after an earlier 20-year career in business.

While other members of her mathematics department were working to maintain the status quo, Ms. League, who likes to think of herself as a maverick, fought to keep mathematics a central part of her district's new vision and to change to a less traditional mathematics textbook series. She endorsed the reforms and was the only representative of her department on the district's Strategic Planning Commission and on the Gateway Planning Committee, which was directing the Valley High graduation plan.

Assessment in the Classroom

During the three spring months of 1992, Ms. League was observed two to four days each week as she taught a class in algebra II. She was also observed a few times during the same interval as she taught geometry and technical mathematics classes. In addition to the observations, data were collected during four formal interviews with Ms. League as well as during daily informal discussions and interviews with 10 of her 22 students. Field notes and transcripts of audiotapes of the observations and interviews were analyzed to describe the ways this teacher assessed what her students knew and could do in mathematics and to examine the relationship between her beliefs and the assessment practices she used. To set up a contrast, a series of six observations of algebra I classes taught by a second Valley High teacher, two formal interviews, and several informal discussions were completed. Both teachers shared their student materials and grade books with the researcher.

A Daily Routine

On an ordinary day, Ms. League began her class by asking students about their homework assignment. A few responded with questions about particular textbook problems.

Ms. League demonstrated solutions to the linear and quadratic equations in about 20 minutes. She then assigned a problem for the students "to try" in their "teaching pairs," which are the prearranged small groups or pairs that work together on this type of assignment. Members of these groups sat near one another, pulling their desks closer to work. This problem was more abstract and complicated than the problems the students had completed for homework. Ms. League suggested that they do it by sketching a graph and "making decisions" from the graph.

As students worked on the problem, Ms. League circulated and gave help where it was needed. At one point, she paused and interrupted their work by saying, "If you're having trouble with this, you need to write down, I need to study how to graph hyperbolas. Be sure to write that down."

Before she demonstrated the solution to the problem, she said, "When I was walking around, I saw all levels of thinking. Some of you were struggling with the graph. Others had trouble with the procedures. Remember that the test is on Friday, and while some of you did OK working together today, remember that on the test you will have to work alone. Be sure you know how much you can do alone before the evaluation on Friday."

Students continued working on problems in pairs until the end of the class, when Ms. League gave them their homework assignment for the next day (six more problems from the textbook similar to the problems they had worked on in class). Then she added, "Your other assignment is to look at what you wrote down today about what you need to do. For example, 'I need to review parabolas.' So do that tonight also."

Two aspects of this classroom are especially interesting from an assessment perspective. The first is that Ms. League routinely observed and interviewed her students as they worked in their teaching pairs, and she also emphasized ways students could self-assess. Ms. League obtained knowledge about what her students know and can do in mathematics by interviewing them and observing them as they worked in pairs. She knew her students and could discuss their strengths and weaknesses at length. During one interview, Ms. League explained the marks in her grade book for the current term. She spoke at length about one student whose name had been selected at random from those in the grade book.

This informal method of assessment is not uncommon among high school mathematics teachers, but in the case of Ms. League, it is especially well-developed. It is also central to her beliefs about teaching, that is, she believes that knowing her students is fundamental to her teaching. She views this practice of observing and interviewing her students as they work as one of the more "authentic" aspects of her assessment practice. Indeed, according to Lajoie's criteria for authentic assessment (1991), this practice could be seen as one of the "multiple indicators of knowledge."

Ms. League asked her students to remind themselves of their strengths and weaknesses as learners. She frequently asked them to write comments to themselves about the concepts on which they needed work or the procedures on which they needed practice. An increasing number of sources agree that self-evaluation is an aspect of authentic assessment and advocate that students keep journals on their progress, or invite students to be active participants in the assessment of their learning.

Ms. League would often ask students to respond to higher-order thinking questions that went beyond the procedural aspects of a lesson. In the first lesson on logarithms, for example, she asked students to think about why there can be no log of a negative number. She made it clear that she expected students to write individual explanations. Sometimes these higher-order explanations were to be finished in class, and sometimes they were to be part of a regular homework assignment

Students counted as worthy of their energy in this class only those tasks or activities that were reviewed and then recorded in the grade book.

that included textbook problems. It was evident, from the frequency of these assignments, that Ms. League valued her students' conceptual knowledge, not solely their ability to carry out algebraic procedures.

An Interpretation

Ms. League used a number of authentic assessment practices, including observing, interviewing, and asking students to write about mathematical concepts and their learning of them. One would expect her students to gain the sense that both carrying out routine algorithms and talking and writing about mathematics were valued. Interviews with students indicated, however, that was not the case, as the following excerpt suggests:

LW: Have you ever had to write anything in mathematics class? Write a paragraph or explain something?
Sara: Yes. A report. I did a report once. That was for extra credit, though.
LW: For what class was that
Sara: Algebra I. That was a while ago.
LW: So you never had a situation where a teacher said, "Here's a problem. Tell me what the answer is. Now explain how you did it."
Sara: Yes. I have had that on a couple of tests before, too. Like—
LW: In this class?
Sara: Well, not in this class.

Why is it that students in this class did not recognize that they routinely were given writing assignments? The answer seems to rest in the system Ms. League used to develop grades for her students. At the end of each quarter, students received a traditional A, B, C, or D grade based on the quizzes, tests, and examinations they took during that quarter. Homework was checked occasionally and points were given for its completion. Homework checks, however, did not include the "extra" assignments designed to foster self-evaluation or higher-order thinking. Thus the "extra" assignments did not receive points.

The tests and quizzes written by Ms. League were based on textbook problems. Items that were identical to the textbook problems except that they included different numbers were assigned for homework. The tests were used to determine whether students could carry out the procedures or algorithms they had been doing in class for the previous week or ten days. They did not include the higher-order thinking questions, nor did they include questions related to self-evaluations. Students were not allowed to collaborate on the tests and quizzes, but completed them individually.

The techniques described earlier as components of an authentic assessment system were not part of the grading system in this class. Students, for the most part, ignored any activities that did not "count" toward their grade. On any given day, roughly half

Her initial efforts toward more authentic assessment practices were in large part failures because they were not incorporated into her grading system.

of them did not complete their homework. When Ms. League asked them to do writing assignments in class or for homework, most of them ignored her. They did not do the self-evaluations nor did they answer the higher-order thinking questions. In fact, they did not recognize that writing was a part of what they did in their algebra classroom.

Students counted as worthy of their energy in this class only those tasks or activities that were reviewed and then recorded in the grade book. They paid little attention to any assessment activities other than quizzes, tests, examinations, and an occasional homework assignment. Because they were graded, these activities were valued by students.

What counted as mathematical knowledge–for Ms. League's students–was correctly carrying out procedures for solving decontextualized problems such as a system of equations. Ms. League had a different notion of what counted as doing mathematics, illustrated by her higher-order questions and her exhortations to students to think, write, and work collaboratively. The non-traditional activities had more to do with reasoning, reflecting, and communicating than they did with routine procedures. Since the non-traditional activities were not incorporated into the grading system, they were not valued by students. In effect, this teacher's expectations for her students were lowered by her grading system.

Ms. League was teaching in a school that publicly has embraced notions of authentic assessment and has initiated district-wide reforms to that end. The principal of her high school is a visible proponent of reform. And Ms. League was and continues to be involved extensively in those efforts. She has, unlike many of her co-workers, served on several key reform committees and openly endorsed their cause. She did implement several authentic assessment practices, despite her challenging working conditions–e.g., six classes with a total of 150 students in three different courses, and no classroom of her own. She was given no assistance by her traditional text for implementing non-traditional teaching techniques; nor was she given any time for developing supplements.

Her initial efforts toward more authentic assessment practices were in large part failures because they were not incorporated into her grading system. This account suggests that the issue of grading systems will become more critical as the assessment reform movement makes progress. Grades, especially at the secondary level, are a powerful force that define what it means to know and do mathematics in a classroom.

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Review of NCRMSE Research

A Model for Classroom Assessment

Is measurement a good model for classroom assessment? In their paper, Conceptualizing Assessment From an Instructional Perspective, researchers M. Elizabeth Graue and Stephanie Z. Smith (1993) take the position that classroom assessment should use models that are more like instruction than measurement. The basic differences between classroom practice and the theory of educational measurement have been explored in recent work in the areas of cognitive science, curriculum, policy, and teaching. Teaching, learning, and cognition increasingly take a perspective that emphasizes local activity and the making of meaning. Educational measurement grew from psychological models with roots in behaviorist theory and the distributional characteristics of large numbers of individuals (Shepard, 1991).

Graue and Smith have proposed a dynamic and multifaceted model of classroom assessment. Their model is based on the premise that assessment should grow from the dimensions of learning that are the focus of instruction. This model builds from three assumptions:

1) Assessment should be seen as a system of complementary parts that are considered in concert. An integrated approach that incorporates data from a variety of sources provides a more stable picture of student learning than single-administration data sources.
2) Information gathered during instruction is the closest to student learning and should be given the most weight in assessment. Contextualized evidence of learning across time is of particular interest.
3) Values for instruction should guide the development of assessment activities. In the case of mathematics, the NCTM *Standards* (NCTM, 1989, 1991) can be used as a guide.

According to the model, three dimensions of learning are of interest to instructional assessment in mathematics: content, process, and dispositions. Content includes concepts, procedures, representational strategies, and problem situations. Process includes problem solving, communication, reasoning, and connections. Dispositions includes beliefs, autonomy, and excitement. A more detailed description of the model will appear in a future issue of *Educational Assessment.*

Examination of the Model

This article summarizes an examination of the utility of the Graue/Smith model made with four mathematics teachers. The data were collected from four middle school teachers as they implemented Expressions and Formulas, a unit from the 40-unit curriculum, *Mathematics in Context.* The curriculum is designed to focus on mathematical content that is situated within specific contexts. The contexts provide meaning and inspire activities. The unit is designed for 6th-grade students and introduces the concept of a formula and the need for an agreed-upon order of operations. Students are provided experiences with notation and formulas and begin to develop an understanding of conventions.

Using data collected through classroom observations and one-on-one interviews, Graue and Smith sought to understand how four teachers learned what students do (content), what students learn (process), and what students feel (dispositions). They asked how teachers' thinking about assessment would change if teaching and learning were their focus and they began to move away from a traditional testing perspective. Excerpts from their examination include comments from teachers that illustrate the evidence of change.

Content

Content typically receives the most attention during mathematics assessment. Students, according to the Graue/Smith model, should show growth in situational, conceptual, and procedural understanding and develop representational strategies that move from the simplistic to the more sophisticated. In general, the four teachers learned about their students' understanding of content by asking them questions–one-on-one, through instructional tasks, or in quiz situations. During the course of the unit, teachers talked about these sources of information. One of the teachers described his best source of information–talking with students in a one-on-one format (Graue & Smith 1993)–as follows:

You just have them exchange papers and if there are 50 long-division problems or whatever, it's pretty easy objectively to correct those things and then the kids hand them in and all you've got is a number...

It's easier if I can sit down or rove around the room and talk with kids individually and really ask them specific questions. It's hard for me to look at a piece of homework, because you don't really . . . sometimes you don't know how much help they've had at home and that sort of thing. Or they might arrive at an answer, but how did they arrive at it? Did they really understand what's going on here?

Another teacher thought his assessment task had been easier when students did more computational activities. When he described his approach, he realized the he may previously "have fallen down on the job where math was concerned." He hadn't figured out how to gather information about students' learning with the new instructional format. He thought that portfolios of student work could be the answer (Graue & Smith).

Maybe this was an advantage, I don't know, but in doing math traditionally, what often happens is there's a homework correcting time built into the period so what you do is you take advantage of the kids. You just have them exchange papers and if there are 50 long division problems or whatever, it's pretty easy objectively to correct those things and then the kids hand them in and all you've got is a number. . . . So you thump that down and you're done. Now, if I have a kid do anything remotely creative or anything that involves actual thought, then somewhere I have to look at it and, to be perfectly honest, I've fallen down on the job there where math is concerned. I mean, I've struggled and I've stayed somewhat abreast, but I have a number of things that I just haven't looked at and now, in this week before grades, I've got a number of things at home that, maybe I collected this a month ago, and I'm thinking, "Oh God, what was this?" I'll figure it out and I'll do the kids justice, but I would be more comfortable if I were able, directly after the kids turn something in, to sit down and think about it. But to be perfectly honest, in the day of a 6th-grade teacher, that's rare. I mean, you can sometimes, but other days you just have to put it on the back burner, and then what happens, happens. (p. 11)

He tied looking at student work to a common constraint of middle school teachers–finding enough time to do the kind of evaluation that new forms of curriculum require.

Process

The notion of process in the Graue and Smith examination included problem solving, communication, reasoning, and connections. The *Mathematics in Context* unit was structured to provide experience in the four processes. Teachers were able to track students' thinking as the student's communicated their ideas about mathematical processes. When asked how a quiz went, one teacher told the researchers:

I think it went real well. Again it was the sort of thing there I could look at it and figure out essentially what kind of thinking kids were doing. They weren't just rote answers where you really had no clue as to what the thinking process was.(p. 13)

Another teacher agreed:

They had to really show proof on quite a few of them, and you could sort of see, by their work, how they were thinking–certain steps that they were taking–which was easier than some of the assignments we were trying to give. And that helped me too. (p. 13)

But another teacher had concerns:

The kids did better than I thought, because it's a hard test. . . . There were too many words in the test. I mean, you had to read too much to get to the meat and potatoes, and some of the kids, like Jeffrey, that was too much of an effort for him. You know, given a formula and just plug in these numbers to this formula, he can knock it out cold. They've got those formulas down. They really do . . . . But hiding the numbers in a story problem–particularly as detailed as Tim's story problems were–that was an effort for some of them. (p. 14)

Situating mathematical content in real-life contexts has allowed students to build understanding. The stage for problems typically is set by language-rich descriptions of situations. For one teacher, the context provided a window, but in the view of another, the language used to build the context became a barrier for some students. The students who were used to substituting numbers for letters in formulas did not want to make their way through words, and some gave up. In terms of evidence of learning, traditional forms of teaching and assessing mathematics rewarded students who were skilled at completing non-contextual items. In the view of the authors, the real world does not come in simplistic problem formats–in fact, half the battle may be framing the problem rather than coming up with an answer. Adding additional language to mathematical activities is a tool for some and a hurdle for others.

Dispositions

The NCTM *Curriculum Standards* (1989) emphasize developing positive dispositions toward mathematics. The Graue/Smith model used three components to track students' learning in this area: beliefs about mathematics, autonomy, and excitement. Beliefs, in their view, were indicated by what students think mathematics is, as well as the value they place on it in various contexts. They theorized that autonomy, which develops as students become more expert and sophisticated in their mathematical thinking, can be evidenced by the students' feelings of perseverance, control, and confidence. Excitement can be seen in the reactions of students in the classroom and their willingness to try new tasks.

Teachers gained information on student's dispositions in a variety of ways: by observing their reactions to instruction, their problem-solving activities, their written work, and their interaction with others. One of the teachers described how the new approach to teaching used in the *Mathematics in Context* project provided rich information about students (Graue & Smith, 1993):

> With the sort of more innovative approaches that we have with math now, linking with language and doing a lot of writing, there's more of an effort component that's possible to measure than there used to be. I can watch a kid in class here and see if he cooperates with other people, see if he's on task, whereas the traditional seat work kind of stuff, the only thing you could tell if you gave a kid 50 division problems is whether you had the answers right. Here you see the kid operating in a greater variety of ways and so it's easier to get some sort of feeling for what he's putting out.

If teachers are to monitor student dispositions, they need access to evidence of those dispositions. These teachers frequently gathered this information through observation and by looking at written work. When the issue of participation in class discussions came up, teachers found that they could not rely on calling on the students who volunteered in class because this group tended to include only a small number of the students. One teacher mused that she had more information on her "high flyers" because they participated more (Graue & Smith, 1993):

> There's probably a third who always have their hands up. You really have to watch that as a teacher. You can't always be calling on those kids, and I think the more experience you have as a teacher you just don't do that. You say, I'll give you one chance to answer something kid, but not another one. It's going to have to be someone else in the room. So I try to rotate that a lot if I can. (p. 17)

These teachers realized that their access to information was related directly to student participation and they found that they could not leave that up to the students. Student participation became a different kind of activity, a source of assessment information. In addition to observing, the teachers also tracked students dispositions through self-evaluations during units.

The teachers struggled against student and parental beliefs about the nature of mathematics and 6th-grade mathematics instruction. Some parents seemed to feel that the curriculum did not challenge the most capable students. Elementary schools that fed into the middle school had accelerated some groups of students by placing them into the next grade level of mathematics classes–e.g., 4th-grade students were placed in 5th-grade mathematics classes. In this study, students were not ability-grouped and almost every student participated in the *Mathematics in Context* project. Teachers discussed their concerns regarding student beliefs about mathematics that related to assessment:

> The problem of course, too, with math is that there's this notion that kids have of whether they're good in math or not. And if they are, they are supposed to get A's. And if they aren't, they're not. So, all of a sudden you take into account a kid's work habits and some of these other subjective things and he's not doing so well, and then you give this kid a B, and the kid comes and says, "Wait, I'm

These teachers realized that their access to information was related directly to student participation and they found that they could not leave that up to the students.

the smartest math student in here." Well, whatever that means, you do well on a standardized test, I suppose. But you do run into that. Especially with 6th graders who come from elementary schools that have been ability-grouped.

One of the problems we've had here, really, is convincing kids that it was OK to be in a math class with everybody, and often times what they'll do, they'll claim that your curriculum is too demeaning or the 6th-grade work course is boring. . . . But what I found is that if you take kids and pull them out of class and put them in a small group with 3 or 4 other kids whom they consider to be mathematically bright and give them the same things to do that we do in here, then it's OK. Then the stuff is great. It's only if they're in this group and they've been used to this special treatment that they feel that they're being somehow ill served. (p. 18)

Students felt that their math ability, as shown on tests given earlier and their elementary school placements in the high-ability group, was their ticket to an A. One teacher corroborated that view and added that the elitist attitude seemed to be fostered by the practices in some elementary schools:

Some of this attitude is coming from some of the elementary schools, I think, where they're sitting off in the choir; they're a little elite group and that type of thing. "Go ahead and do the 6th-grade and the 7th-grade math book." . . .The parents push that. . . . Some of them don't care and they just say, "Hey, my kid is ready for 7th-grade math and I want him in that. I want him in a special group. No matter what you say you don't get anywhere. (p. 19)

Teachers felt parents and students think that working in heterogeneous groups diminishes the quality of the classroom experience. One teacher in the study was especially troubled by mixing students of different abilities during assessment activities:

I feel real guilty if I take a brighter student who's catching on and pair him up with a kid whose head is somewhere else and then try to assess them. And they don't like that either. They sometimes never get past the fact that I'm sitting next to so and so and I don't want to be sitting next to so and so, and no matter what you say, I really don't want to be doing that. And so, I try to avoid that in the classroom. Those are problems I think I can control. . . . you might have noticed, Doug and I do less grouping now than we did earlier in the year. (p. 19)

The teachers who explored this new approach to teaching mathematics found that their information gathering had changed and so had their evaluation of that information. Two of the teachers found they were looking at their students' learning very differently from the way they had viewed it previously, and that translated into a major change in their grading practices:

What I have had to do is sort of change my grading system. You know how all math teachers want to have numbers? I don't do that any more with those tests. I grade it like it was a history exam, so I'm more subjective, anyway, and I like that better actually. I don't have to put a number. I don't have to put 40% on the top of it. It allows a lot of freedom that way if I have some kid that I know is struggling, I know doesn't handle it conceptually, but has done something right–I can give that kid a letter grade and not have to justify it on the terms of the percentage. I might give a kid a C who got a 30. I don't have to put the 30% on there. I can just put this letter grade and I can talk to them about what they did well and what they didn't do so well. So in some ways it's kind of freed me from the shackles of number and I like that a lot. (p. 20)

Another teacher described what he considered an overly simplistic view of evaluation that lulled people into thinking that grades mean more than they actually do:

. . . they were looking at their students' learning very differently from the way they had viewed it previously, and that translated into a major change in their grading practices: . . .

I've never relied completely on numbers. I think that's a very comfortable thing to do, especially if math teachers do it, because you can reduce almost everything to a number and you can kind of kid yourself into thinking that the number means more than it actually does. Maybe it's because I've been around math enough to know that. . . . I've seen teachers do things like give points for assignments. Even math assignments. Points weren't necessarily even based on how many problems were right. They might have been based on whether the kid's name was on the paper and whether they were neat or not, and they give every assignment 50 points or something, and rate these assignments. And some of the assignments may have been long, some short, some easy, some hard, very diverse things. They all counted the same. . . . And then, they'll apply those numbers to some sort of preconceived grading scale where they'll simply say, well, 93-100 is an A, 86-93 is a B, and all of that is, of course, completely arbitrary, but people don't know that, or they don't think about it, or they don't care. (p. 21)

Conclusion

This examination of classroom assessment from the perspective of classroom instructional practice identified several themes for further study. Strategies for gathering contextualized information such as observations, interviews, and project work were not yet fully developed or integrated into the work of the four teachers, although they were recognized as valuable sources of information about student growth. These teachers readily used observations to obtain information about student dispositions, but they were not comfortable using the more informal techniques to find out about content or process growth. In the future, informal strategies for obtaining information on content and process growth could be included in mathematics units. Rather than providing a short quiz at the end of units, teachers could be reminded of the kinds of learning and thinking they should be looking for as students progress through unit activities that relate to instructional goals.

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continued from page 3

preparation of assessment standards by the National Council of Teachers of Mathematics. The new set of standards is designed to complement the *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Professional Standards for Teaching School Mathematics* (1991). The first draft of *Assessment Standards for School Mathematics* was completed by a group of mathematics assessment specialists during the summer of 1993. It will be circulated for comment to members of the mathematics education community during the 1993-1994 academic year. The draft will be revised for publication during the summer of 1994.

For additional information about the Working Group, contact Thomas A. Romberg by telephone at (608) 263-3605. NCRMSE has prepared an annotated bibliography of its working papers. Readers can request a copy of the bibliography from Donald Chambers, Director of Dissemination, NCRMSE, 1025 W. Johnson Street, Madison, WI 53706, phone (608) 263-0761, E-mail dlchmbrs@vms.macc.wisc.edu.

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In This Issue:

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Reforming Geometry

Visual reasoning is central to mathematics. It is an integral part of mathematical and scientific inquiry. Geometry also has another aspect that it shares with the broader field of mathematics–its potential use for modeling situations. Because of this aspect, geometry should–those who seek a reformed school mathematics curriculum claim–be integrated into the mathematics courses children study throughout their K-12 years. The NCTM *Standards* (1989) and other reform documents (MSEB, 1989, MSEB; 1990) call for increased content in geometry and spatial reasoning across Grades K-4, 5-8, and 9-12.

Geometry, a course devoted to visually guided thought, is taught to students in the United States at the 10th grade level. Most elementary school curricula cover the descriptive aspects of geometry briefly. Students learn to identify and name conventional shapes in elementary and middle school grades. Their 10th-grade course is built around two-column proofs of theorems in Euclidean geometry. Current instructional practice isolates the content of geometry from the content of other mathematical areas, a practice that begins with algebra in 8th- or 9th- grade classrooms. A discussion of algebra is carried in the Winter 1993 issue of the *NCRMSE Research Review*. The isolation of geometric content to a single high school course and the emphasis within this course on proofs contrasts with international practices–which give geometry and spatial reasoning more central roles in K-12 curricula. In The Netherlands, for example, children begin in the primary grades with such realistic geometry tasks as sighting and projecting, locating and orienting, and drawing and measuring (de Moor, 1991).

Research Program

The National Center for Research in Mathematical Sciences Education (NCRMSE) is undertaking a research agenda that will support the reform of school mathematics. One of its seven working groups focuses on the Learning/Teaching of Geometry. The research of this group is designed to facilitate a transition to a more integrated and more comprehensive approach for the teaching of geometric content. The working group on the Learning/Teaching of Geometry was formed in late 1991; it plans to carry out five major activities:

1 Investigate the informal notions that students bring to their understanding of geometric concepts and to their understanding of geometry as a whole. Researchers will explore children's intuitions about such fundamental concepts as angle, shape, and area. They will extend this research to include children's intuitions about the corresponding attributes of solids and motion. Their findings will form the cornerstone of their efforts to improve teaching by making teachers more aware of their students' thought processes in the area of geometry.

2 Investigate the ways in which visually-guided thought influences mathematical thinking about number and quantity. Instruction in rational numbers, for example, is often based on the assumption that children have a working understanding of area. Similarly, instruction in the interpretation of graphs of functions often assumes that students can recognize the relationships between magnitude and position or between change and position.

3 Explore alternatives for envisioning environments for the study of geometry. The computer provides the primary tool, but computer-based exploratory environments such as the *Geometric Supposer* (Kaput, 1990) or Logo (Battista & Clements, 1988; Lehrer, Randle, & Sancilio, 1989) can be used to encourage children's explorations of the foundations of traditional geometry. The software tools make it possible to explore geometry in new ways. In a similar vein, some computer-based cognitive tools use geometric concepts to clarify the nature of scientific and related mathematic concepts. Chemists, for

The isolation of geometric content to a single high school course and the emphasis within this course on proofs contrasts with international practices—

example, build polygons and polyhedra to understand molecules. Some members of the working group will examine systems that link the simulation of physical motion to tables, graphs, or functions.

4 Study how research on students' intuitions about geometry and computer-based tools for the learning of geometry can effectively be merged to improve the teaching of geometry. There is evidence that Cognitively Guided Instruction (CGI) is effective with primary-grade children. How can teachers be provided with new knowledge about student cognition in geometry as well as new instructional tools? Will CGI instructional effectiveness be enhanced when teachers are provided with new tools as well as new knowledge about student cognition in geometry?

5 Critically examine the current geometry curriculum with the goal of suggesting change as a result of the research findings.

Working Group Activities

The members of the working group meet to discuss research related to the reform of K-12 geometry twice each year. They also communicate via an E-mail network. At their 1993 meetings they began to identify and integrate the multiple views about geometry teaching and learning. The meetings give group members an opportunity to evaluate current research on geometry and to plan collaborative activities that extend the thrusts of current research projects. While all group participants are conducting research in areas related to the learning/teaching of geometry, some have backgrounds in mathematics or mathematics education, and others in cognitive or educational psychology, curriculum development, or software design.

The five major program activities are carried out by Working Group Chair Richard Lehrer and NCRMSE staff members or by other members of the working group. The working group members are located predominately at higher education institutions. While several members pursue research activities related to geometry with minigrants funded by NCRMSE, others combine these grants with funding from other sources. This is especially true of the group members who are designing and developing technologies to enhance visualization activities.

Preliminary Reports

Some of the investigations of the informal notions that students bring to understanding geometric concepts and to their understanding of geometry as a whole have been completed. Chair Richard Lehrer and staff member Cathy Jacobson have prepared a preliminary report on their findings, which appears in this issue under the title Reform in the Primary Grades. In this article, they report on their work with primary-grade teachers, the teacher workshops, CGI Geometry curriculum, and their classroom observations on the ways in which geometry content was implemented in primary-grade classrooms. Brief summaries of the work of other members of the working group are contained in the next section of this article. Much of their work incorporates a computer-assisted technology component. Brief descriptions of their research activities that relate to geometry follow:

... students became more motivated when learning geometry in computer-assisted classrooms.

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Daniel Lynn Watt is a senior curriculum developer and geometry team leader for the Elementary Mathematics Project at the Education Development Center in Newton, Massachusetts. He and other staff members are developing a technology-rich geometry curriculum for elementary students. The geometry component of the curriculum, *Math and More,* builds on four assumptions: 1) Students learn geometry by using it for constructive and creative purposes; 2) Students learn geometry effectively when it is connected to their everyday lives and to the cultural artifacts they see around them; 3) Students benefit from an in-depth exposure to a few geometric ideas and concepts; 4) Student learning will be supported effectively when both highly directive structured and constructive or open-ended software activities are available.

Several *Math and More* units for Grades 1 and 2 were completed in late 1993. The units use computers, computer software, and video as tools to support students in understanding and using geometric ideas.

Specialized software environments or microworlds were developed for each of the units. In the Grade 1 unit, Maps and Movement, students use software to move a button-driven turtle as they build elementary maps from rectilinear path pieces, such as straight lines, corners, and intersections, with 25 landmark icons. Initial activities require that students move a pointer around pre-developed maps. Students give and follow directions, compute and compare distances, and find different routes to the same location. They then build their own maps and pose problems for other students. Later software environment activities include predicting the results of following a set of instructions on a map, solving treasure hunts using directions for clues, and making a treasure map. The video related to the maps unit tells a story of two animals who roam a neighborhood looking for human friends, Jesse and Raquel, who have gone off to attend school.

In the Grade 2 unit, Geometry in Design, students use software to build geometric quilt designs. Users begin by building a core square and retain the square in a special area of the screen; they then use their core square design to build larger quilt design elements. While they complete the design, they learn about patterns and shapes, congruence, symmetry and asymmetry, and visualizing or predicting the effects of such transformations as flipping or rotating a design element. The video related to the unit shows the richness of the geometric design found worldwide in arts and crafts. It animates the process by which geometric transformations can be used to build complex patterns from simple elements, and provides examples of quilt designes for students to copy and modify.

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Martha Wallace has been working with two of her colleagues, Richard Allen and Judith Cederberg at St. Olaf College in Northfield, Minnesota, to help secondary teachers use computer-assisted software. They hope to make the teaching and learning of geometry in secondary schools a more engaging and dynamic activity with their project activities. Inspired by the NCTM *Standards,* their objectives are 1) to help teachers develop the knowledge, skills, and confidence necessary in using computer-based tools to transform their geometry classrooms into a mathematical community where students explore, conjecture, verify, and communicate mathematically and teachers are their partners in inquiry rather than correct-answer authority figures; 2) to enable teachers to share their expertise with other pre-service and practicing teachers.

While their research is not yet complete, Wallace and colleagues note that 90 percent of their participants indicate increased confidence in knowledge of geometry, knowledge of pedagogy and computers related to geometry, ability to develop instructional materials that incorporate computer technology, and ability to mentor other teachers. A similar percentage believe that their students became more motivated when learn-

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Geometry in the Primary Grades

by Richard Lehrer and Cathy Jacobson

Richard Lehrer, an associate director of NCRMSE, is chair of the working group on the Learning/Teaching of Geometry

Cathy Jacobson is a staff member of the working group on the Learning/Teaching of Geometry.

Cognitively Guided Instruction (CGI) is based on the premise that primary-grade instruction in mathematics should begin with a teacher's understanding of how children think about mathematics. In the research reported in this article, the CGI principles have been used by the authors to develop collaboratively–through an ongoing dialogue with primary level teachers–an extension of CGI that includes geometry and spatial-visualization components.

CGI classrooms emphasize problem-solving tasks and communication between teacher and children and among children. CGI teachers create learning environments in which children actively construct their own knowledge. This contrasts with classrooms in which children are often passive recipients of knowledge from textbooks or teachers. The goal of CGI teachers is student learning that is based on understanding rather than on the memorization of mathematical rituals. For additional information on the CGI program, see the Spring 1992 issue of *NCRMSE Research Review.*

Knowledge About Children's Thinking

Teachers' knowledge of children's thinking about the area of number has been the key to the CGI program. A significant amount of research on how children's thinking about number develops has been completed. That is not true for geometry, particularly for primary-school-age children. This three-year longitudinal study of the development of children's thinking about shape; measurement which includes length, angle, and area; depiction which includes drawings and graphing; and visualization has been undertaken to expand knowledge about children's thinking in the areas that comprise geometry. It began in 1992-1993 with 1st-, 2nd-, and 3rd-grade students, and the evolution of their conceptions about geometry over a two-year period– e.g., as the original 1st-grade students progressed to 3rd-grade, 2nd grade to 4th, and 3rd grade to 5th.

This study is finding that young children's reasoning about shape and dimension is much more elaborate than some previous research has indicated. Earlier research, for example, suggested that children's ideas about shape were limited. Children, it suggested, considered only the overall appearance of a shape and its global similarity to other shapes. This study is finding that children use many attributes of shape to make comparisons. The children in this study have ideas about both two- and three-dimensional shapes that are rooted in qualities that have practical implications for the development of activities–e.g., a shape's orientation or its size. They rarely attend to conventional attributes or properties of shapes, perhaps because the conventions are useful for more formal geometry but have few implications for children's everyday use of geometric concepts.

While the children in the present study possess knowledge about is not yet organized into a system. In accordance with previous research, these children rarely articulate relationships among attributes of shape and, as a result, they seldom organize relationships into systems. They use an extensive vocabulary and show an awareness of attributes of shape in their descriptions of shape and pattern during the primary school years. Children, when asked to compare shapes, often talk about the acuteness of some of the angles using words such as "pointy," or the orientation of line segments using words such as "slanty," or the number of sides or faces. They also talk about what the shape resembles and other aspects of appearance. These research findings have been synthesized into a new model of learning informal geometry that blends cognition and perception.

The CGI Geometry Curriculum

These findings suggest that primary-grade children could benefit from a wider range of experiences in the area of geometry. To provide children with these experiences, an experimental curriculum was developed. Building on children's knowledge of the world and the types of knowledge they develop in the course of everyday activity, it contains the four major strands: (1) wayfinding, (2) depiction, (3) two- and three-dimensional shapes, and (4) measurement.

These findings suggest that primary-grade children could benefit from a wider range of experiences in the area of geometry.

The Wayfinding strand grows from children's learning about space by walking or moving in that space. Children, for example, find their way to a friend's apartment or house and as they do they develop knowledge about position and direction during preschool years. Wayfinding is connected to shape as well. The trace of the path taken while finding one's way can be thought of as a shape.

The Depiction strand grows from children's rich history of drawing before they enter school. Children's knowledge of depiction is extended in this curriculum as they develop drawings of different views of objects, integrate other children's drawings of views as they construct an object, or use or invent other graphical conventions to describe space such as maps and nets.

The Two- and Three-Dimensional Shapes strand grows out of children's play with blocks and other building materials. Rather than simple identification of shapes, the CGI Geometry curriculum emphasizes the uses of shape. Children build shapes from different nets, design quilts and gardens, and investigate transformations as physical motions during the process of design. These activities in corporate the use of such computer tools as Logo.

The Measurement strand grows out of but also unifies the first three curricular strands. It also makes connections between the geometry and number strands of the curriculum. The wayfinding component of the curriculum includes problems of length, the distance between two locations, and angle measurement, as well as the direction one must take to find a location. Students often find that their solutions to measurement problems in wayfinding apply to other problems involving shape. They describe similarities among shapes using measures of length or angle, or they use depiction to determine the appropriate scale for a map. Measurement problems such as finding the area of the floor of a classroom anchor descriptions of space to such number topics as multiplication and place value. As an aside, some historians think geometry initially was developed by those who surveyed or measured certain parts of the earth.

Teachers and CGI Geometry

Professional development. The primary-grade teachers who participated in this research were provided with a series of workshops that took place throughout the school year. The series–approximately 60 hours–was led by a CGI mentor teacher and a project researcher. Workshop participants solved problems similar to those designed for the students in their classrooms. The workshops emphasized the evolution of children's thinking about the conceptual issues that were raised by the problems. Often the geometry problems were novel and provided teachers with increased content knowledge. Since teachers taught in different buildings, an electronic network was developed to facilitate communication between teachers and with the CGI mentor teacher and researchers.

Teachers also used their growing understanding of children's thinking about geometry to make instructional decisions.

After the series of workshops and a year of CGI work with their mathematics classes had been completed, teachers–with project staff–began to create the CGI Geometry curriculum. They began to develop problems and tasks that would provide effective windows through which to view student thinking. Teachers reflected on how to get students to reveal their thinking about geometry while creating the curriculum. Such reflection served as a catalyst for refining or redefining teachers' views on what is important in a geometry curriculum for primary school students.

Changes in teachers' beliefs and practices. The changes in collaborating teachers beliefs and practices with regard to geometry were examined in several ways. The teachers participated in an interview at the beginning and end of the school year and teaching practices of selected teachers were observed three times a week throughout the school year. The observations permitted researchers to relate a teacher's perceptions of changing beliefs with changes in the same teacher's classroom practices. Over the course of the research, these data revealed dramatic changes in how teachers thought about geometry and in their teaching practices.

Before they began the series of workshops, teachers saw geometry and spatial reasoning as only a minor adjunct to number sense and viewed their primary role as one of helping children develop number sense. Their idea of an appropriate geometry task reflected traditional textbook exercises. After the workshops and classroom experience with the CGI Geometry curriculum, teachers had broader conceptions of the domain of geometry and gave student thinking a larger role in curriculum planning. Teachers also said conceptual tools that parallel those available for classroom work with number–e.g., Logo or polyhedrons–would help them promote the development of spatial reasoning in primary-grade students.

Teachers talked about students' range of understandings of space–discerned by teachers as children completed activities–as well as the importance of conversation and discussion to students in their development of spatial thought. Student thinking became a more prominent concern of teachers, as indicated by one who said, "For the student who struggles to explain his strategy with number problems, we have to remember to design geometry problems so that the same student can also find a way to demonstrate the strategy he used with geometry problems."

The observations of teachers' classroom practices suggested that changes in beliefs about the teaching and learning of geometry were echoed in three major ways in classroom practices. First, the number of problems involving geometry and spatial visualization increased noticeably during the school year. In addition to problems involving shape–those most familiar to teachers–teachers began to pose problems in wayfinding, depiction, and measurement that reflected the major components of the CGI Geometry curriculum.

The character of the problems posed by teachers within the four strands changed. At first, teachers posed problems about shape that were variations on recognition and identification activities, but over time the problems on shape engaged children in constructing shapes and using shape in design. Rather than simply asking students to identify squares and rectangles, students were asked to create compositions of tri-angles using "core squares," and then to transform the core squares using slides, flips, or turns on their compositions to create a quilt design. They selected some of the problems developed by Dan Watt, a member of the NCRMSE working group, The Learning/Teaching of Geometry, whose work is described in a previous section of this newsletter.

Teachers views on the learning potential associated with problems in geometry changed dramatically. At first, teachers posed problems and children solved the problems. There was little discussion of students' solution strategies and few connections were made between the activities students performed and their thinking about them. During the first few months, the teaching and learning of geometry was activity-based and emphasized a product. Teachers were more adept at focusing on process for number problems than they were with geometry problems. They were more familiar with the number curriculum, and they knew more about student

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thinking about addition and subtraction than about geometry. Over time, they began to focus on student thinking in geometry, and student conversations about different ways to solve the same geometry problems occurred with increasing frequency. The conversations emphasized similarities and differences among different solution strategies and highlighted the need for students to justify–verbally or with drawings–the spatial actions they were taking.

As the year progressed, teachers and students began to make connections among depiction, pattern, position, and measurement, as well as among these components and number components. While carrying out a wayfinding activity, children measured length and angle, depicted paths and coordinated their paths using a to-scale map, modeled their actions in a small-scale space using Logo, and described the shapes made by different types of paths that started and ended at the same point. Teachers used many opportunities to create bridges from other curricula to geometry. During a unit on the geography of islands, a teacher was observed posing problems involving depiction–e.g., draw an island; and measurement–e.g., order the drawings according to area. During this task, children talked about different strategies for counting fractional pieces and suggested possibilities for units to measure area.

Teachers also used their growing understanding of children's thinking about geometry to make instructional decisions. This informed their decisions to allow further discussion or to pose other types of problems. Classroom teachers began to see themselves as researchers who could document student thinking about geometry and design activities that would make students' thinking more apparent.

An example of the use of understanding occurred when a teacher posed a problem about drawing-to-scale. Students were presented with an adult hippo drawn inside a coordinate grid and were asked to draw a baby hippo inside a scaled-down grid. As the teacher watched their progress, she noted several strategies that ranged from artistic attempts that did not follow any of the grid scaffolding, to semi-grid work in which only tails and feet were coordinated with the grid squares, to free-hand drawing. Only one student used the coordinate structure to draw the baby hippo. The teacher had some of the students share their work and then developed a class discussion about how best to draw the baby hippo so that it looked like a smaller version of the adult.

Thus the teacher obtained data about how each of her students approached the problem. Based on this data, she designed a set of activities in which the scale of each of the axes differed. In one instance, the scale was "shrunk" for only one of the axes, which caused students to draw "flat" hippos. She provided grids of different sizes, with axes scaled with the same multiple, for students to use in creating hippos of different sizes. As students experimented with the different scalings, they talked with other students and the teacher, providing the teacher with opportunities to observe their growth in understanding. The teacher revisited the topic later in the year in other contexts and continued to explore the growth of students' understanding of scale.

Students and CGI Geometry

The performances of children participating in CGI Geometry classes were compared with those participating in a longitudinal study of CGI in the primary grades. For additional information on the CGI longitudinal study see the Spring 1992 issue of *NCRMSE Research Review.* Comparisons were also made between 2nd-grade students' performances on a variety of geometry and number problems at the beginning and end of the school year.

The children participating in CGI Geometry did as well solving problems in number as their counterparts in the CGI longitudinal study. In addition, participating children, at the end of the one-year period, showed large differences in conceptions of geometry. They were able to think of shapes as paths as well as static figures, and they were much better than their counterparts in thinking about the relationships among properties of shapes. Their understanding of measurement also showed growth; they were able to reason about appropriate units of measure in the context of problems about length, angle, and area, and they could solve area problems involving irregular as well as regular figures. Marked improvements in their solving of wayfinding and depiction problems were also noted. Most were able to draw the top, side, and bottom views of a solid, and to integrate top, side, and bottom views in order to construct a solid.

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ing geometry in computer-assisted classrooms.

The researchers conclude that teachers new to their project require at least two years to integrate and implement their new geometry knowledge and teaching strategies. Trying new techniques with their traditional curriculum initially eased the transition to a revised curriculum for teachers. But equally important were extended school-year support services and networking opportunities that provided support when they were faced with change-resistant individuals, administrators, colleagues, or their students' parents.

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Both the NCTM *Standards* and the addenda series book, *Geometry in the Middle Grades* (NCTM, 1992), recommended that the middle school geometry curriculum provide many opportunities for students to explore their environments. Exploring their own familiar environments should, according to members of the Cognition and Technology Group (CTG), located at Vanderbilt University in Nashville, Tennessee, increase students' perception of the importance of geometry in their worlds. Members of the CTG group surveyed middle school students and teachers to determine their views of geometry. The responses indicated that both groups had limited views of the uses of geometry.

To provide opportunities for teachers and students to see, understand, and use geometry in the world around them, the CTG (1992) created the Jasper Adventure Series which consists of problem-solving videodisc adventure stories based on geometric content. Based on a theoretical framework that emphasizes the importance of anchoring or situating instruction in meaningful problem-solving contexts, the materials reflect the principles of NCTM *Standards*, particularly in building on the recommendation that activities emphasize complex, open-ended problem solving that connects mathematics to other subjects and to the world outside the classroom.

. . . these data revealed dramatic changes in how teachers thought about geometry

Two of the Jasper Adventure Series videodiscs highlight the nature of geometry in wayfinding and measurement. Each requires a minimum of three to five class periods to solve the problems they contain. They are designed with video-based extension problems that help students deepen and extend their knowledge of the mathematical concepts they used with the original videodisc problems. The extension problems are designed so that teachers can use them flexibly in order to meet the needs of particular students.

Teacher users of the two videodiscs participated in a two-day workshop before using the materials with their students. Surveys showed both teachers and students improved dramatically in their ability to identify uses of geometry. While teachers thought they could use the materials more effectively when they taught them again compared to their first year, they expressed a strong desire to meet at least annually to share strategies and information.

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Michael Battista of Kent State University and Douglas Clements of the State University of New York at Buffalo are examining student's learning of two- and three-dimensional geometry. Their work is part of a larger project to develop and test K-5 instructional units. They believe that children represent space based on their actions, so they designed Logo turtle and other non-computer activities to help students investigate the notions of length and rotation in the context of paths that are records of movements.

Three student strategies for solving two-dimensional length problems in the path contexts were apparent: 1) Some did not partition lengths, nor did they relate the number for the measure with the length of the line segment; 2) Most drew hash marks, dots, or line segments to partition lengths–they appeared to need these means to quantify length; 3) A few did not use a partitioning approach, but did use a quantitative concept when discussing the problem and drew proportional figures. It appeared that the third group created an abstract unit of length or a conceptual ruler–not a static image but an interior process of moving visually or physically along an object and segmenting it–that they projected onto unsegmented objects.

The researchers have begun developing descriptions of the errors students make when determining the number of cubes in a three-dimensional array of cubes. They have found that even though students, beginning in the 3rd grade, are taught a procedure for finding the number of cubes in a three-dimensional rectangular array, fewer than 40 percent of 5th graders can conceptualize such an array to enumerate its cubes in a meaningful way. This work supports the view that mentally constructing a three-dimensional cube array is a complex process involving numerical and spatial structuring supported by coordination and integration operations. To determine the number of cubes in a three-dimensional array, students must coordinate orthogonal views of faces, then integrate these views to construct one coherent mental model of the entire array of the cubes–an extremely difficult process for elementary school students.

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Kenneth Koedinger, Carnegie Mellon University, is developing software to provide high school students with technology-based opportunities to be creators rather than consumers of mathematics. The software will be developed to engage students in discovering and evaluating geometric conjectures aided by computer-based tools for performing experiments and writing proofs.

Employing the ACT theory of cognition, Koedinger and a project team are developing computer simulations of the reasoning processes involved in conjecture discovery and evaluation. The simulations or cognitive models serve as a basis for providing students with individualized computer tutoring. Interprocess communication technology will be used to add this tutoring support to such existing software as the *Geometer's Sketchpad. (1993).*

Tutor modules will be developed that support students as they construct and investigate geometric diagrams, propose conjectures about their diagrams, and prove or disprove their conjectures. A prototype of the investigation module currently is being tested in a Pittsburgh high school.

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For further information on the Learning/Teaching of Geometry Working Group, readers may contact its chair, Dr. Richard Lehrer, at the National Center for Research in Mathematical Sciences Education, University of Wisconsin-Madison, 1025 W. Johnson Street, Madison, WI 53706.

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Review of NCRMSE Research

The authors of a previous article in this newsletter stated that Cognitively Guided Instruction (CGI) is based on the premise that instruction in mathematics should begin with a teacher's understanding of how children think about mathematics. This article takes excerpts from a guide for primary-grade teachers by Richard Lehrer, Elizabeth Fennema, Thomas Carpenter, and Ellen Ansell. The guide, *Cognitively-Guided Instruction in Geometry*, was designed to provide collaborating teachers with introductory information on children's thinking about geometry. It was developed using the preliminary findings obtained in ongoing research. The guide–about the informal notions children bring to their understanding of geometric concepts and how these concepts develop–will be refined as additional data become available.

Recognizing Shapes Through Patterns

Patterns are an important part of geometry. The key to the formation of any geometrical pattern is the repetition of some unit or form. Artists make extensive use of repeated elements to create pleasing patterns. Quilts and rugs, for example, are often formed by creating a core design and then repeating this design. Similarly, if we look at nature, honeycombs form a recognizable pattern because a three-dimensional cell is repeated in all directions at once.

Although it may not seem obvious at first, shapes such as squares, triangles, and prisms are patterns too. A square is a pattern because it involves several kinds of repetition: The sides are congruent and are of equal measure; so, too, are the angles; and there are several lines of symmetry–e.g., folding along the diagonal results in two congruent pieces that, when superimposed, fit exactly

Reasoning About Shape

Just as children develop a number of strategies for solving arithmetic problems, children also develop a number of ways of reasoning about patterns that make up shapes. As children's reasoning about patterns and form develop, their understanding of the nature of shape and pattern changes. There are three major ways that children reason about shape and pattern: resemblance, attributes, and properties.

Resemblance. *Direct resemblance.* The most direct form of reasoning about shape is to identify a shape or pattern by its resemblance to previous shapes or patterns. Children often say things like "it looks like a ramp," "it's a squished-in rocket," or "it's a bit squarish." Here children are relating an unknown shape to one that is, given their experience, more familiar. A square, for example, is recognized as a type of box or as something that looks like the squares depicted in books. They relate a particular shape to typical examples of shapes that they know. When children are asked to find instances of shapes in their environment, they may use this form of reasoning to provide many examples.

Reasoning that relies on resemblance to visual prototypes makes the orientation and size of shapes especially salient. For instance, children often suggest that changing the orientation of a shape makes it a different shape. The two squares depicted in Figure 1 are often referred to by primary age children as a square and a diamond or "tippy" square. In a similar manner, some triangles may not be viewed as triangles because they have two sides that are longer in relation to the third and are not prototypical in children's experience. Two shapes may be viewed as belonging together because of their "skinniness" or "fatness." Skinny generally means that the shape's proportions are different from those of the prototypical figure. A child could conclude that the rectangle and triangle displayed in Figure 2 are more alike, by virtue of their being "skinny," than the two triangles displayed.

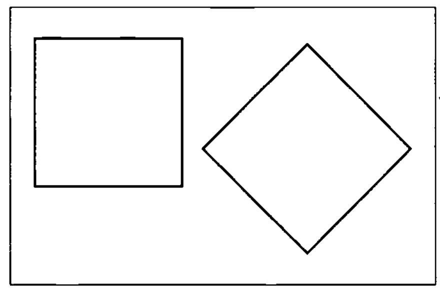

Figure 1: Two squares.

Indirect resemblance. Mentally changing a shape so that it becomes

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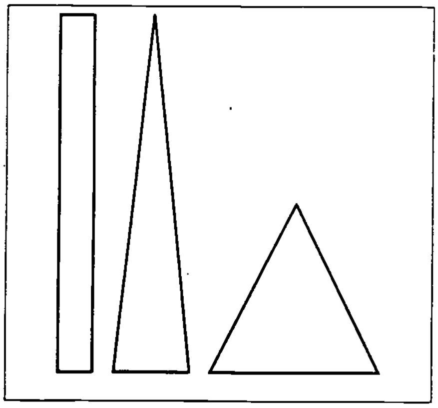

Figure 2: Skinny vs. fat shapes

one that is known is another form of resemblance-based reasoning. It is, however, an indirect form of resemblance because the student transforms an image and then compares it to her/his prototype. In this process, rectangles are described as "pushed-out squares," parallelograms as "bent rectangles," and pyramids are mentally shaved so that they come to resemble cones.

Resemblance-based thinking about shape is varied. Chiefly it relies on visual comparisons in a manner similar to children's first strategies for arithmetic problems, which rely on direct models of experiences. This is not surprising. Adults too will try to classify–by analogy or resemblance–something that is seen for the first time.

Attributes. At a second level of complexity, children reason about the attributes or characteristics of shapes. Attributes are elements that are noticeable and explicitly represented by a child: corners, edges, lines, and others. Children's perceptions of shape attributes are often very different than those of adults. Their descriptions of the attributes of shapes include words such as "pointy" for angles that are less than 60 degrees, "slanty" for line segments or faces on solids that are not oriented either vertically or horizontally, "points" for the vertices of shapes, and "edges" for the sides of plane figures.

When children first start to reason about attributes, they do so in a relatively uncoordinated way. The number of edges or sides does not necessarily have a relationship to the number of points or angles, for example. Seeing a six-sided shape, children often notice that there are six sides or "edges," but must count the number of "corners" (angles) to determine their number. At the next stage of reasoning, children develop coordination among attributes of a shape so that if a closed shape has four sides, then it has four angles. Counting the numbers of sides and then the number of angles and talking about them plays a key role in children's development of this stage of reasoning.

As children coordinate attributes of shapes, they also begin to recognize that some attributes are unaffected by other attributes. Rotating a triangle, for example, does not affect the number of sides or corners; large triangles have the same number of sides as small triangles and skinny triangles have the same number of corners or angles as fat triangles.

Properties. Children's thinking about attributes leads to the development of reasoning that views shape as a collection of properties–e.g. squares have four sides; the opposite sides are parallel, and there are four right angles. These properties of shape form an integrated system that defines a figure; taking one of the properties away changes the shape. In contrast, attributes are not yet organized into a system so that the critical attributes that define a shape are distinguished from the non-defining or the less critical attributes. Children's efforts to coordinate attributes leads them to develop properties. Measurement also plays a key role in students' development of property-based reasoning. If students can measure angles, then they have a basis for recognizing the definitional role of angles for many shapes. Right triangles are distinguished from other triangles, for example, because they possess one right angle.

Property-based reasoning is the key to students' increasing the ways they identify and think about shape and pattern. They learn that when the collection of properties changes, so does the shape: If a four-sided shape has one pair of parallel lines, it is a trapezoid; if it has two pairs of parallel lines, it is a parallelogram; if a four-sided figure has two pairs of parallel lines and at least one right angle, it is a rectangle and others. The collections of properties come to include increasingly abstract ways of looking at patterns, including lines of symmetry as well as differentiation of reflection and rotation symmetries, and scale.

Children eventually establish systems of relationships among the properties of figures. For example, if the opposite sides of a quadrilateral are congruent, then so are the opposite angles. These relationships among properties help children establish logical relationships among figures. A square becomes a kind of rectangle because it has all of the properties of a rectangle, even though it does not resemble a prototypical rectangle.

Table 1

Questions for a Triangle Classification Task

Problem Type	Example Question
Identification	What is this (number 1*) called?
Description	What makes it a triangle?
Comparison	How is this one (number 1) like this one (number 3)? Different?
Classification	Does (number 1) go with this triangle (number 4)?
Generation	How is this triangle (number 1) made?
Transformation	If I pull here (point to the top vertex), will it still be a triangle?
Conjecture	Someone put this one (number 4) with this one (number 1). Why do you think they did that?
Justification	Do you think they were right to put this one (4) with this one(1)? How can you be sure? Can you convince someone else? What makes a good argument?
Problem Posing	Can you make up some new problem about shapes that you would like to try out?

**Task inspired by Burger & Shaughnessy (1986)*

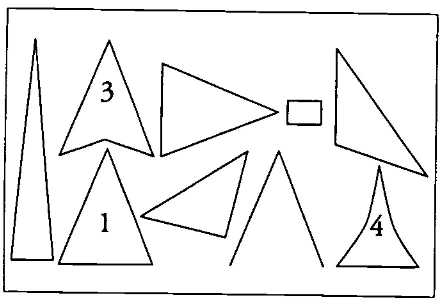

Figure 3: Put a T on every triangle.

Assessing Reasoning About Shape

Conversations are an important way to find out how children are thinking about shape. Here we focus on a few simple problems that serve as conversational aids. One type of problem has to do with shape classification. Children are asked to find all the triangles or rectangles or octagons or cylinders that they can in a collection of shapes. Children's justifications provide windows to their thinking, and their conversations generate new ideas. Shape classification can generate conversation and windows to thinking that go well beyond mere identification. In addition, for example, to asking children to decide if a shape is or is not a triangle, ask a question about a movement, "If I pull on this corner here, will it still be a triangle? Why?" Other conversational questions include "How did you decide that was a triangle?" "Why are these two figures both triangles?" "What goes with this (present a new shape)?"

Classification tasks can be used as windows to children's thinking. Table 1 displays a set of questions that can be used by teachers as probes to get more information about children's thinking. Each question generates a slightly different problem, with later questions that elicit an attribute-based or property-based reasoning. The following are excerpts from a conversation between a teacher and her 1st-grade children during a triangle classification task. Sixteen children in a 1st-grade classroom were seated on a rug. The teacher sat at the side of an overhead projector. Children faced a screen pulled over the blackboard. The teacher projected an overhead transparency with nine shapes (see Figure 3) onto the screen and asked children for their ideas with the words, "Put a T on every triangle."

Julia: (points to isosceles upright) Well, if it has one, two, three, corners, and three sides then it would be a triangle. (Julia reasons about the properties of number-of sides and number-of angles as defining a triangle.)

Teacher: Okay, this one (pointing) has a flat line on the bottom.
Ellen: Yah. Almost like this one except this one is on its side.
Teacher: Oh, so you're saying, what about if I do that (turns around so base is parallel to ground) to it? Now is that one on it's side still? (teacher rotates the figure)
Ellen: Well, that depends because it can have different looks even though it was on its side then. When it was on its side, it looked like it was going like this, but now it looks like it's straight.

Teacher: Oh, okay.

Ellen: Even though you didn't fully turn it. (Ellen reasons about the attribute of orientation and decides that it's not important here.)

Azim: It's half of a triangle.

Teacher: Half, why is it half of a triangle?

Azim: Cause the bottom is missing. (Azim compares the instance to a visual prototype. This is an example of resemblance-based reasoning.)

Ellen: I think it's a triangle because, well you know how there's like this tower of London, or something? (Ellen reasons about real-world prototypes, an example of reasoning by resemblance.)

Teacher: Ahuh.

Ellen: That's like a triangle and that looks a lot like it, except it's not the same.

Teacher: What do you mean it looks a lot like it but it's not the same?

Ellen: I think it's not the same as the tower of London.

Teacher: But it looks similar to something you know then, that you've seen?

Ellen: Yah. It's a triangle that's really stretched out that maybe if you put it into one of the pictures with the T's on it, if you went like this with it (pushes top down), then it would be a regular triangle again and it you stretched it out it would be like that again. (Ellen reasons about a movement that would preserve resemblance to a prototypical triangle.)

Teacher: Oh, so which one do you think is a regular triangle? You said it would be like a regular triangle, what do you mean by a regular triangle?

Ellen: A regular triangle is one that's not really stretched out. (Ellen compares the proportions of the example to the proportions of the prototype–the "regular" triangle. This is an example of resemblance-based reasoning.)

Teacher: Do you see a regular triangle up there?

Ellen: Yes.

Teacher: Which one?

Ellen: (Ellen points to the triangle with the number one in it.)

Teacher: Oh, so to you, when you're thinking about regular triangles, you're thinking about that kind of shape.

Ellen: Yes.

Peter: Well, a needle (cone shape) can be a triangle if you cut the like round thing off; then it looks like a triangle. (Resemblance–movement and comparison to a visual prototype.)

Helen: (goes up to screen with pointer) Even though the sides are stretched out, I think it still is a triangle because here's (points out three corners) one corner and here's another corner, and here's a third corner, and this (other triangle) has one corner, two corners, three corners. This has one, two, three sides, and this has three sides. (Helen reasons that the property of the number of sides and number of angles is important to classification, not the measure of the sides. This is a key step to seeing a shape as a collection of critical properties.)

Teacher: So you're thinking it doesn't matter what size the sides are. You're thinking it just has to have three? You're thinking it doesn't matter if it's two long ones and a short one, or if they are all kind of the same?

Ellen: I think it might be an ancestor.

Teacher: It might be an ancestor, what do you mean by that?

Ellen: I think it might of been like from the olden days. That's how they used to draw triangles. And they got smaller and wider and smaller and wider. (Ellen likens triangles to biological evolution, an unusual form of resemblance-based reasoning!)

Summary

The excerpts reprinted here are part of a 76-page guide prepared for teachers working with researchers investigating the development of a primary-grade curriculum CGI Geometry. Most of the article was taken from the chapter, Shape Through Pattern. Other chapters in the guide are Depiction, Thinking About Depiction, Direction, and Measurement, as well as an Appendix that discusses using Logo and a path perspective for shape, problem solving contexts, fundamental elements of dimension and shape, and a Glossary of geometric terms.

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continued from page 7

Participating students' involvement in and effort with geometry problems reflected their uniformly high level of involvement with mathematics in their classrooms. Concept maps of their ideas about the content and nature of mathematics were developed; the maps revealed that geometry and tasks involving visualization were increasingly perceived as part of mathematics, rather than as a separate subject, such as art. And, as with number problems, they saw geometry problems as having more than one method of solution.

Future research will continue to examine changes in teaching practices and changes in student conceptions of geometry. The same teachers who participated in the first year of the study are now beginning a second year using the CGI Geometry curriculum in their classrooms. The initial classroom observations made by researchers this year suggest that teachers with a year of experience with CGI Geometry are more effective in guiding the geometry activities of their students and that the student change observed during the second year is likely to be greater than that observed during the first year.

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In This Issue:

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Statistics and School Mathematics

Students encounter numerous statistical claims in their daily lives. Data are collected, summarized, analyzed, and transformed in most of this country's media, work places, and homes. The collecting, representing, and processing of data are assuming major importance in most nations. While statistics was once taught primarily to college students pursuing professional or academic careers, it is now becoming a part of the school mathematics curriculum.

Mathematics reform documents developed during the last decade stress teaching appropriate statistical content to students in elementary and secondary mathematics classes. This content should, according to the documents, develop students' ability to use statistics to understand their worlds, to create and interpret summaries of data and displays of information, and to be critical of claims and arguments that are based on data. The *Curriculum and Evaluation Standards for School Mathematics* developed by the National Council of Teachers of Mathematics (1989) and projects, such as the Quantitative Literacy (American Statistical Association, 1987) and Used Numbers and Reasoning Under Uncertainty (Technical Education Research Center, Lesley College, and the Consortium for Mathematics and Its Applications, 1989-1990), suggest ways data analysis could be implemented in the nation's schools.

Few activities that involve statistics are now carried out in K-12 mathematics classrooms. While there is little information either from research or from practical experience to aid teachers who would introduce statistics at elementary- and secondary-grade levels, there is information on teaching and learning statistics at the college or university level. This research-based information suggests that a majority of higher education students do not understand elementary statistical concepts (Garfield & Ahlgren, 1988), even after completing several courses (Poston, 1981). In its current form, according to a consensus among educators and researchers (American Statistical Association, 1991; Mosteller, 1988; NCTM, 1989; and Shaughnessy, 1992), statistics education is inadequate.

Learning/Teaching of Statistics

The Learning/Teaching of Statistics Working Group of the National Center for Research in Mathematical Sciences Education (NCRMSE) is studying the ways in which statistical content can best be integrated into the school mathematics curriculum. While NCRMSE Director Thomas Romberg initiated the Working

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Susanne Lajoie, an associate director of NCRMSE, is chair of the Working Group on the Learning/Teaching of Statistics.

Group, Susanne Lajoie of McGill University now chairs the group. Sharon Derry and Richard Lehrer, University of Wisconsin-Madison, are the group's principal investigators. The group also includes 5 staff members and 17 affiliated members.

Five Working Groups were formed by NCRMSE early in 1991. They focused on the topics: whole number, quantities, algebra, implementation of reform, and models of authentic assessment. A sixth Working Group that focused on geometry was formed during late 1991. The Learning/Teaching of Statistics is the seventh and final NCRMSE Working Group. It began its activities early in 1993. The operation and research of each of the other Working Groups have been described in previous issues of *NCRMSE Research Review.*

While each of the members had completed independent research related to the learning/teaching of statistics before the creation of the Working Group, they began a collaborative program of research in June 1993. At their first meeting, these members summarized their current research and knowledge on learning/teaching statistics. They then identified the kinds of research they thought would be necessary to extend knowledge on the appropriate content, pedagogy, learning, and assessment of statistics for K-12 students.

Several studies by members of the Working Group that are designed to extend knowledge about statistics for K-12 students are now underway. The studies address a) the development of an international database of research literature that relates to the teaching and learning of statistics; b) the identification of the cognitive components of probability and statistics that are related to the K-12 curriculum; c) the development of a curriculum for statistics with Grade 7-8 teachers and the implementing of the curriculum in Grade 7-8 classes; d) the development of a technology-based authentic statistics project for Grade 8 mathematics students that includes the testing of new assessment models that use computer and videotape technologies.

Research Literature Database

Chair Lajoie and Affiliated Member Joan Garfield collaborated on the development of a computerized database of research, STAT-FILE, related to the teaching/learning of statistics. They combined their previous resources and eliminated entries that were duplicates. They then used the database to develop an annotated bibliography of the research literature on teaching/learning statistics. At the end of 1993, their bibliography included 110 entries that were relevant to K-12 research. The data base can be searched by title, author, key words, and publication or presentation information. Copies of the data base were sent to the members of the Working Group.

Cognitive Models Projects

Richard Lehrer and Jeffrey K. Horvath are examining cognitive models of less- and more-skilled Grade 4-5 students' problem-solving performances in statistics. They take the view that curriculum-based performances should be examined when information about student knowledge and understanding is sought, and students should be assessed in ways that provide opportunities for them to learn more about the curriculum or provide opportunities for them to receive assistance with their performance. Their work began with the spinner sum task, a probabilistic assessment task developed by the California Department of Education (Lehrer & Horvath, 1993). They are now extending their work on cognition with a study of students' developing understanding of the nature of chance.

Kevin Collis has begun a study of cognitive functioning in probability and statistics as it relates to the school curriculum. It addresses specifically the understanding of chance and data concepts in the wider social context. As a starting point, he is developing a questionnaire that uses the media–excerpts from newspapers, for example–to learn about students' understanding of the content. The questionnaire will include paper-and-pencil items, supplemented with graphical information when relevant, that are constructed to elicit statistical problem-solving skills. The four-part format (Collis & Romberg, 1992) and think-aloud techniques will be used in the construction of the items.

Statistics Curriculum Project

Sharon Derry and Helen Osana have developed instructional simulations that will stimulate students to acquire and use statistical concepts in the context of reasoning about realistic or non-trivial problem-solving situations. They began with a plan to design situations that would improve middle-school students' ability and propensity to reason statistically about the problems they encounter in their daily lives. During the summer of 1993, they worked with a small group of classroom teachers to develop and refine the design for instructional simulation activity that would be used by 7th- and 8th- grade students. The 3-4 week-long instructional game is called Vitamin Wars.

During the academic year 1993-1994, Derry and Osana and four classroom teachers began to implement the curriculum in 7th- and 8th-grade classrooms. As an introduction to the curriculum, a film on concepts about statistics and medical research was shown to the students. It provided a basis for discussion and analysis. Students then listened to presentations by local experts on statistics, argumentation, and scientific research and discussed them in small groups. During the last days of the activity, students were asked to assume the roles of researchers, lawyers, legislators, consumers, or business owners. Acting in these roles, students worked in collaborative groups conducting research and preparing and presenting testimony at a mock legislative hearing on government regulation of the vitamin industry.

The researchers collected field notes, video recordings, and student and teacher ratings of class activities. They also administered a reasoning task. The data will be analyzed to identify the specific events, features of materials, procedures, aspects of the classrooms, and aspects of mentorship that were useful in shaping the research activity or affecting the outcomes. Their classroom observations suggest that role-playing motivates most students, even some who are identified by teachers as having behavioral or motivational difficulties. Despite the use of the same instructional game, the use of identical procedures, and common inservice programs for teachers, each classroom evolved as a unique social and learning environment during the activity. Some of the roles students played also appeared to be more effective than others in promoting the development of reasoning, regardless of classroom context.

Investigators Derry and Osana are now investigating whether the simulations were more effective than the traditionally used instructional approaches in terms of the students' development of concepts, use of statistical concepts, quality of individual and social reasoning, and quality of individual and group products such as presentations, written interpretations, or others that were based on reasoning.

Authentic Statistics Project

Working Group Chair Lajoie is directing the Authentic Statistics Project. Designed to test new forms of statistics instruction and assessment, the project uses computer and videotape technologies. Its assessment and instructional models are based on situated learning theory and the *NCTM Standards*. Of special note is its attention to the problem solving, communication, reasoning, and connections emphasized by the *NCTM Standards* in both their instructional and assessment aspects. The project also designed instructional activities in statistics that would provide students with opportunities to reflect, organize, model, represent, and argue within and across other mathematical domains.

The Authentic Statistics Project developed an innovative approach to assessment, a computerized library of exemplars–text and video examples of student work–that convey to students both models of and criteria for average and above-average performance and can be used by students as benchmarks for their own statistics performances. When assessment criteria are made explicit to students, the researchers theorized, it can precipitate additional learning. A library-of-exemplars study was then undertaken to examine the effectiveness of the library as a tool for clarifying assessment criteria and for promoting statistical understanding. The Project also developed a second study to investigate whether tasks that require students to collect their own data are more effective for promoting understanding than tasks that require students to use preexisting data.

One portion of this project's study on the effects of text and video examples is reported in the next article in this newsletter. Other portions of the project's research were reported in papers presented at the April 1993 American Educational Research Association Conference in Atlanta: *New Ways to Measure Skills of Problem Solving, Reasoning, Communication, and Connectedness*, by Susanne Lajoie, John Lawless, Nancy Lavigne, and Steve Munsie (1993); and *The Use of Hypercard for Facilitating Assessment: A Library of Exemplars for Reifying Statistical Concepts*, by Susanne Lajoie, Nancy Lavigne, and John Lawless (1993).

In addition to the principal investigators Sharon Derry and Richard Lehrer, the Working Group includes Affiliated Members George Bright, Gail Burrill, Kevin Collis, Susan Friel, Ido Gal, Joan Garfield, Christopher Hancock, Victoria Jacobs, Clifford Konold, James Landwehr, Joel Levin, Kathleen Metz, Thomas Romberg, Andee Rubin, Richard Scheaffer, Ron Serlin, Leona Shauble, and Michael Shaughnessy. It also includes Staff Members Jeff Horvath, Nancy La Vigne, Steve Munsie, Helen Osana, and Tara Wilkie. Two staff members are located at the University of Wisconsin-Madison and three are located at McGill University in Quebec.

Conclusion

Future research on how statistical content can best be integrated into the school mathematics curriculum should include an integrated framework for the instructional and assessment process. Such a framework would involve looking at the statistical content, the learner's conceptual understanding of the content, and how instruction builds on the assessment of the learner in the context of an instructional situation. The Working Group members collaboratively will seek answers to the research questions they have identified.

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For further information on the Learning/Teaching of Statistics Working Group, readers may contact its chair, NCRMSE Associate Director Susanne Lajoie, Department of Educational and Counselling Psychology, Faculty of Education, McGill University, 3700 McTavish, Montreal, Quebec, Canada H3A 1Y2, telephone (514) 398-4240; FAX (514) 398-6968; E-mail (INSL@MUSICB.McGill.CA).

Integrating Statistics into the School Curriculum

Statistics is assuming a new role in the mathematics curriculum for K-12 students in the 1990s. Because statistics previously has not been taught to students in elementary or high school grades, little is known about students' experiences with statistics acquired outside of their classrooms and about instruction that can effectively enhance students' understanding of statistics in their classrooms. This section presents excerpts from papers on two studies that were carried out to develop a knowledge base that would enhance statistics instruction in middle school classrooms. The first paper, *Statistics in Middle School: An Exploration of Students' Informal Knowledge,* was completed by Victoria R. Jacobs and Susanne Lajoie. The second paper, *How Do Group Composition and Gender Influence the Learning of Statistics?* was completed by Susanne Lajoie and Nancy C. Lavigne.

An Exploration of Students' Informal Knowledge

Much of the recent research in mathematics education underscores the importance of understanding the extent of children's informal knowledge before preparing formal instructional programs. To some, students' prior knowledge is an essential starting point from which to build additional instruction. It can also provide teachers with a framework for thinking about their students' development in this content domain. An enrichment program was developed for middle school students who were identified by teachers as interested in challenges beyond their regular mathematics curriculum. The instructional activities that comprised the program provided the context from which students' statistical understanding could be studied. Both the content and the pedagogy of the enrichment program were designed to provide students with opportunities for extended thought and discussion about statistical problems. They were also designed to provide teachers with activities that elicit or encourage discussion about statistical content.

Content

The content of the enrichment program was designed to guide students in explorations of the concepts that make up inferential statistics. It was based on the assumption that chance and sampling–the use of results obtained from a sample in reaching conclusions about a larger population–are the core concepts of inferential statistics. Specific lessons focused on chance and its role in everyday decisions and on sampling and the logic underlying the factors that affect a researcher's ability to draw accurate conclusions about the population from the sample results. The inferential statistical content was selected because it is consistent with that recommended for the middle school age group in the curriculum *Standards* published by the National Council of Teachers of Mathematics and the guidelines prepared by The American Statistical Association. It was also selected because, while there is little research on students' understanding of any statistical content, many current projects emphasize descriptive rather than inferential statistics.

Pedagogy

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) recommend a constructivist pedagogy that is based on a new vision of mathematics learning. Learning mathematics is defined as doing mathematics. This view promotes an activity-based curriculum

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where students solve problems and emphasis is placed on their informal knowledge and their ability to communicate about the processes they use to reach solutions. Classes emphasize problem solving rather than computational skills and explanations of how solutions are reached rather than single correct answers. Assessment is incorporated into instruction and modified to reflect new beliefs about knowledge and learning.

Method

A researcher/teacher met with ten high-ability middle school students in a Wisconsin school once each week for 13 weeks during the students' regular mathematics class. The enrichment lessons were considered part of the students' regular curriculum and the teachers incorporated the students' performance in the program into their semester grades in mathematics.

The activities in the lessons were designed to promote statistical discussion. Measures of performance during the 13-week period included indices of class participation, weekly homework assignments, and a comprehensive examination administered as a pretest and posttest. All class dialogues were audiotaped, and protocol analysis provided the core of the analyses that were completed.

Results

At the conclusion of the 13-week enrichment program, students had developed a workable definition of chance, could recognize it in their daily lives, and describe how it affects the decisions they make. While most students were able to compute an exact probability, all were able to use the language of probability–e.g., more likely, less likely, impossible, etc.– to describe chance events. They had little trouble distinguishing more likely outcomes from less likely outcomes, but difficulty understanding chance events with outcomes that were equally likely. With regard to sampling, students easily identified its use and importance in their daily lives. They also were able to apply sampling logic to realistic situations. While all of the students participated in statistical discussions, some had difficulty in two areas. Students should be able to use the results obtained from the application of inferential statistics to inform their decisions about everyday situations; some of the students, however, had difficulty incorporating the results into their discussions. Students were also inconsistent in their ability to explain representative sampling procedures. This inconsistency seemed to be linked to the context of research questions. Opinion questions were treated differently from other questions by students who seemed to think that there should be an equal chance for each possible option–e.g., 50-50 or 33-33-33, and so forth.

Students' performance on a 50-point test suggested that their general level of statistical reasoning had increased as a result of their participation in the project. The test rewarded correct explanations as well as correct answers. In addition to their cognitive gains, students indicated both through verbal anecdotes and through individually written course evaluations that they enjoyed the course .

Useful Instructional Materials

Students were very articulate during classroom activities designed to stimulate discussion about statistical content. They responded verbally to open-ended questions, shared their strategies, and became comfortable with questions that could have multiple answers. They struggled, however, with open-ended written assignments. The quality of reasoning in their homework and essays on tests did not match the level they demonstrated during their oral interactions. This finding suggested to the researchers that students' conceptual understanding may be underestimated if evaluations are based only on written work.

The study found that two activities were particularly useful in promoting discussions that enabled students to learn from one another. A student-generated test and a student-directed class can provide educators with unique information about students' understanding. There is little research to guide the implementation of these activities.

The study also found that students had difficulty identifying why the content of statistics was classified as mathematics. When they reviewed the program, they viewed the division, multiplication, and fraction procedures they completed with numbers as mathematics. The students' views reflect society's view of mathematics as a nonconceptual field made up of numbers and their manipulation–a view that is destructive because it encourages the use of memorized algorithms without understanding.

Conclusions

This work provides some initial guidance for mathematics educators who must begin to implement statistics in the Grade K-12 curriculum. Its suggestions for instructional and assessment activities can provide an initial glimpse of students' informal conceptions of statistics. Although generalizing from the results is limited by the size and ability level of the sample, the methodology used in the study provides information that can be used by others interested in examining students' thinking about statistics.

The Influence of Group Composition and Gender

Theories of learning and instruction are increasingly considering learning that occurs in specific situations or contexts. The situations or contexts often involve small groups of individuals working together on a common task. Constructivist and situated learning perspectives emphasize the importance of social interaction that promotes thinking and the development of problem-solving skills in classrooms. These perspectives are based on the assumption that a cooperative or shared learning environment permits students to learn from others and can reduce students' anxiety about learning when they feel willing to share information. Group work encourages students to share their knowledge with peers and exposes them to multiple points of view. Small groups can enable students to develop mathematical power by developing higher-order thinking skills such as problem solving, reasoning, and communication, and behavioral attributes such as persistence.

Despite the success of small group learning situations, little attention has been paid to the nature of collaboration and the dynamics of small-group interactions that affect learning. Research shows that peer collaboration can cause students to shift perspectives but that joint decision-making is necessary for effective learning. Group interaction may result in some students relying on others, accepting little personal responsibility, and doing little independent thinking. Collaboration is a complex phenomenon and this complexity must be considered when collaborative classroom activities are used.

This study was designed to test the ways in which group composition influences learning. In particular, it looked at group composition in terms of gender to determine how gender influenced the learning of statistics by 8th-grade mathematics students. The goal of the research was to examine how best to instruct students and assess their learning in this new area of school mathematics.

According to the NCTM *Standards*, 8th graders should be provided with opportunities to do statistics–that is, to systematically collect, organize, and describe data; to construct, read, and interpret tables, charts, and graphs; to make inferences and arguments based on data and evaluate arguments based on data analyses; and to develop an appreciation for statistical methods as a powerful means for decision making. The study uses these guidelines by placing statistics instruction in an experimental context carried out in the classroom using computers that graph and analyze student data. Groups of students construct research questions, collect and analyze data, and display their findings and interpretations for the class. Both individual and group data are used to describe the transitions that occur in student learning.

To ensure that students were assessed fairly on their group projects, two conditions were developed: an exemplar condition–*video and text*–where a computer hypercard stack was designed to describe assessment criteria, and a *text-only* condition, also computer-administered, that provided a list of assessment criteria with textual descriptions of what these criteria meant and how they were weighted. The criteria included: quality of the research question–how clear and specific the research question is; data collection–how students go about gathering information that pertains to their question; data presentation–how data are summarized and presented and the types of tables, charts, and/or graphs constructed to represent the data; data analysis and interpretation–what statistics are selected to analyze a data set, and how students demonstrate their understanding and interpretation of the data analysis; presentation style–how the group explains the goals and findings of their project to the class; and creativity–how unique the project is. The *video and text* condition added a video component that allowed students to select one of the criteria and obtain a textual and digitized video demonstration of average and above-average performance by previous students who had designed statistics projects. The *text only* and *video and text* conditions were prepared to make the assessment criteria apparent to students.

Cognitive learning theories support small-group activities that have been found to facilitate learning for each individual by providing multiple perspectives as well as by negotiations during problem-solving activities (Blumenfeld, Soloway, Marx, Krajcik, Guzdial, & Palinscar, 1991; Cognition and Technology Group at Vanderbilt, 1990; Williams, 1992). There are few guidelines for assigning students to groups that are based on promoting learning for individual group members or on how gender affects group activities. An earlier pilot project found that group work did not always result in a positive learning experience for all group members. In the pilot, groups made up of mixed-ability levels completed a statistics project. Informal observations of group activity suggested that when females were in male-dominated groups they did not participate.

Other research that involved computer-based tasks (Underwood & McCaffrey, 1990) found that single-gender pairs improved participation and performance when compared to individual activity, but mixed pairs did not. In this research, single-gender pairs appeared to share task components and discuss possible solutions, while mixed-gender pairs tended to separate task components and then complete them separately. Other research (Webb, 1984) indicated that mixed-gender groups can be detrimental to females' mathematics achievement. Based on these findings, the current study was designed for single-gender groups to see whether gender differences would occur on measures of statistics problem solving, reasoning, or communication. All groups, regardless of gender, mixed students with varying ability levels. The predictions were that females would perform as well as males in single-gender groups on measures of statistical knowledge and that groups exposed to the *text and video* condition would outperform the groups in the *text only* condition.

Subjects for this study were from an 8th-grade mathematics classroom. Twenty-one students, 9 females and 12 males, were divided into 8 groups consisting of either two or three students. Students were grouped with same-gender peers, resulting in four groups of females and four groups of males. Teachers' rankings–e.g., high, medium, low–of each students' performance as measured by classroom assignments and examinations–were used to form groups of students with mixed mathematics achievement. When groups were made up of two students, a high- and low-ability student would be placed together. When groups were made up of three students, a high-, medium-, and low-ability student would be placed together. Each of the groups worked at one Apple Macintosh computer work station and were supervised by a researcher for a period of two weeks.

The eight groups were assigned randomly to the *text only* condition–in which textual descriptions of the criteria for assessing group projects were presented on the computer–or the *video and text* condition–in which the textual descriptions were supplemented with digitized video clips representing two levels of performance. The randomization process resulted in three groups of females and one of males being assigned to the *video and text* condition and three groups of males and one of female being assigned to the *text only* condition.

Instructional activities and assessment tasks were designed to promote learning and communication in an authentic learning environment. A pretest/posttest design was used to assess changes in student performance. Journals were used as group measures of statistical communication, problem solving, and reasoning. Structured journals contained specific prompts designed to encourage groups to define and explain concepts, to reason about data and graphs used in the tutorial, and to identify areas of difficulty. The prompts were designed as a means to foster learning. Journals served as ongoing measures of group performance and were analyzed in terms of the quality of statistical communication, problem-solving, and reasoning. All group interaction and presentations were audiotaped and videotaped.

Results

The prediction that females would perform as well as males in single-gender groupings on measures of statistical knowledge was not only confirmed but the results exceeded our expectations. The prediction that groups in the *video and text* condition would outperform those in the *text only* condition was not confirmed. The statistical knowledge of all of the groups increased from pre- to posttest, but groups of females working together benefitted more from the instruction than did groups of males. Analysis of group journals suggests they do provide useful information: Female groups responded to prompts more often than male groups. Based on their responses, it appears that females were more interested in planning and in understanding concepts through definitions, while males seemed somewhat more inclined to deal with questions that required interpreting the information that was presented in the data and graphs.

Conclusions

This study suggests that gender plays an important role in group problem solving using computer-based learning environments. While gender differences did not exist on a pretest of statistical knowledge, Grade 8 females outperformed males after the instruction. It appears that providing females with opportunities to work with other females on group projects that require computers has a more positive impact than single-gender cooperative learning situations for males. This finding is similar to that of Johnson and Johnson (1985), who found that a combination of cooperative learning and computer-assisted instruction had a positive impact on female students' attitudes toward computers.

There were also gender differences in the way students documented their group projects. Structured journals were given to each group and students were asked to document the statistical concepts they were learning, as well as their project ideas. Females tended to document information about statistical concepts and their plans for how to conduct their group projects, whereas the entries for the male groups were sparse in such categories, but robust where they were asked how they would apply statistical concepts in certain situations.

Both of the conditions used in the study, *video and text* and *text only*, produced significant changes in the statistics performance of students. The finding was not expected because it appeared that the *video and text* exemplars would make scoring criteria more apparent to students and thus serve as a more effective instructional tool when compared to *text only* exemplars. Further study of this phenomenon is needed before firm conclusions are drawn about the comparison. Future research on gender differences in learning statistics through same-gender groups could address the ability composition of groups. Students were assigned to mixed-ability groups to ensure that every group had a similar opportunity for success. There may, however, be a confounding between gender and ability composition in groups that must be considered in fostering group problem solving in statistics using computers. That multiple means of assessment provide a better profile of learning has been established in previous research. What needs additional examination is the relationship between individual and group assessment.

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Review of NCRMSE Research

In April 1994, NCRMSE Director Thomas A. Romberg delivered a paper, *The Mathematical Sciences Education Perspective of World Class Standards,* at the Annual Meeting of The National Council on Measurement in Education. Excerpts are abridged and the data provided in tables and figures removed in the version of the paper presented here. The entire paper is available from the Educational Resources Information Center (ERIC) System under the number TM 021 285.

Are NCTM's Curriculum Standards World Class?

The use of the term *world class standards* in the current political debates grows out of the rhetoric surrounding the National Educational Goals (U.S. Department of Education, 1990). Goals 3 and 4 mention mathematics, and mathematics is implicit in Goal 5. In the debates, the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) are often cited as the exemplar for establishing content standards for other core disciplines. This paper examines the problem of judging what is world class and then explains the mathematical sciences education views about world-class standards.

Since NCTM's *Curriculum Standards* have become our operational definition of a world-class mathematics program, questions about whether they are indeed world class need to be answered. To do this, my staff and I at the National Center for Research in Mathematical Sciences Education (Romberg et al., 1991) examined the mathematics frameworks of eight countries (Australia, France, Germany, Japan, The Netherlands, Spain, Norway, and The United Kingdom) and compared those frameworks with NCTM's *Curriculum Standards.* We found considerable variation in what is taught, in when ideas are introduced, and in what is emphasized. Thus, there is no international norm against which one can compare American views of what it is important to teach and learn in school mathematics. Nevertheless, we believe that the following statements about the vision of school mathematics presented in NCTM's *Curriculum and Evaluation Standards for School Mathematics* qualify the *Standards* as a world-class mathematics reform document.

✓ The NCTM *Standards,* when compared with the national curricula of other countries, do not represent a "radical" or "romantic" vision of school mathematics (p. 40).

✓ The manner in which countries build a detailed rationale for these reforms and, specifically, which changes are emphasized, depends on their past practices (p. 41). For example, "number sense" and "estimation" are specifically mentioned in NCTM's document for Grades K-4, but not in those of other countries. This does not mean the topics are unimportant in the other countries, only that they have been central in their curricula for decades–but not in ours–thus, no emphasis is needed.

✓ The four standards for mathematics teaching and learning, problem solving, communication, reasoning, and connections are reflected in all eight national curricula, not just in NCTM's *Standards.* The terms used for these standards may differ, but the underlying themes are consistent (p. 41).

✓ We are convinced that the variation in emphasis with respect to particular mathematical topics also is related to past cultural practices in different countries (p. 41).

✓ In Grades K-4, the *Standards,* while including topics new to the American curriculum, still put more emphasis on whole number arithmetic than other countries. This same finding applies to all work with numbers up to Grade 8. However, there is no standard on "number" for Grades 9-12. Other countries appear to be more balanced in their approach to this important aspect of mathematics (p. 41).

✓ While geometry is now included at all levels in the *Standards,* which, of course, is not reflected in most American classrooms, it still receives less emphasis than in the majority of other countries (p. 41).

✓ The *Standards* include more emphasis on statistics, probability, and discrete mathematics than do other countries (p. 42).

✓ Although the beginning principles of calculus are now being suggested for all students in the *Standards,* most other countries have long assumed this to be important and, in fact, expect much more than is advocated by NCTM (p. 42).

✓ Although different programs for students are common in other countries, their students are expected to study mathematics every year they are in school and are often offered several options. In the United States, the radical recommendation in the *Standards* that all American students study *real* mathematics for at least three years of high school falls short of the expectations of most other countries (p. 42).

The most striking difference between the NCTM *Standards* and the curriculum documents from other countries lies in the emphasis the other countries place on the social and attitudinal aspects of schooling. Schools need to be "a secure environment and place of trust," "social behaviors" need to be taught, "students should realize that mathematics is relevant," "students should gain pleasure from mathematics," and "personal qualities should be nurtured" are statements that appear often in these documents. Such statements put an emphasis on what happens in classrooms that is different from the focus on either cognitive learning or economic imperatives in the *Standards* (p. 42).

In conclusion, one cannot study the curricular documents from these countries without realizing that the current mathematics curriculum in the United States is far from being world class. On the other hand, NCTM's *Curriculum Standards* (1989) present a vision of content that is significantly in line with what other countries are now doing and with what they are planning to do. The expectations expressed in this vision, if realized in the schools, would bring all American students more in line with the expectations for students in the rest of the world.

Judging Whether Something is World Class

Given the difficulty of our attempt to judge whether NCTM's *Standards* are world class, it is apparent that the political rhetoric in the National Goals (1991) implies we know how to judge whether students' achievements in any country are indeed world class. In retrospect, to judge something implies either certifying that certain criteria have been met, or rank ordering a set-of-somethings on the basis of specific criteria. For National Education Goal 3, the somethings are students whose work is to be compared against the certification criterion, competency in challenging subject matter, for Grades 4, 8, and 12. Competency is to be defined so that all students learn to use their minds, are prepared for responsible citizenship, further learning, and productive employment–a difficult set of inferences. For Goal 5, similar certification criteria are implied for all adults. One would need to establish the connections between specific knowledge and skills in mathematics to be both employable and a responsible citizen. For National Education Goal 4, the something is a composite profile of U.S. students on achievement to rank order the American profile with those of other countries. Note that the judgments for Goals 3 and 4 are quite different judgments based on different criteria. Then, to judge whether something is world class involves either determining both that the certification criteria are comparable among nations and that the percentage of somethings that meet those criteria are comparable, or that the method of rank ordering on an attribute is reasonable across nations.

Thus, for one to argue that judgments related to Goals 3 and 5 are world class, one would have to build the case that the American certification criteria for all students (and adults) and goals are comparable with those of other nations, and also the percentage of students (adults) who meet those criteria is comparable. Evidence to build such a case for Goal 3 could be drawn by comparing the expectations in different countries for all students at these grade levels, along with comparisons of instructional programs and of school cultures. Evidence for Goal 5 for mathematics would involve a similar argument. For Goal 4, one could build a case for appropriate rank ordering of profiles if one could agree on how and when to assess mathematical achievement. In summary, to build such arguments about what is world class, one would need at least to examine the variations in student outcomes, and expectations for students, programs, and even school cultures across countries. The mathematical sciences education community assumes that any such comparisons should be based on our vision of school mathematics and not on current practice.

Compare Outcomes

A straightforward way of making comparisons between the mathematical achievement of students in several countries is to administer a common test to a sample of students at the same level of schooling in each country. This has been done several times in the recent past and another similar test is now in the final stages of planning. The central question that needs to be addressed is: How valid is the test as a measure of student achievement across nations?

To study the world-class validity of such tests, during the past few years my staff and I were asked by the National Center for Educational Statistics to examine the items administered in the two past comparative studies with respect to NCTM's *Curriculum Standards:* The test battery for the Second International Mathematics Study (SIMS) administered by the International Association for the Evaluation of Educational Achievement (1985) and the battery for the International Assessment of Educational Progress (IAEP) administered by Educational Testing Service (1990). The question we addressed was: Does the content of these tests reflect that expressed in the *Standards?* Of course, such a comparison is problematic, since these tests were developed before the *Standards* were written.

The results of our analysis of each battery at each grade level were similar (Romberg, Smith, Smith, & Wilson, 1992). From the categorization of the Grade 8 SIMS battery one can only conclude that the content coverage is out of balance. Performance profiles based on these tests cannot be used to make a valid judgement about world-class mathematics achievement for American students, let alone those of any other country.

Compare Expectations

For expectations one could examine the high-stakes examinations administered in different countries. For example, at Grade 12 one could examine the high school completion exams, such as those given in the different states in Australia, or in Norway, or in The Netherlands. But what would they be compared with in the U.S.? Chantal Shafroth (1993) recently made such a comparison. For the United States, she used the SAT Level I and Level II tests, and the AP Calculus Test. Her analyses are followed with an examination of the different types of questions posed. Even a cursory look at this report indicates that there are vast differences in mathematical expectations among these countries and between them and the United States.

A second type of high stakes examination is college admission tests. For example, the Mathematics Association of America recently published a set of admission examinations from several Japanese universities (Wu, 1993). The reason for publishing the examinations was to demonstrate to American mathematicians and mathematics educators the fact that the Japanese ask entering college students mathematical questions that many American college majors in mathematics would have difficulty answering.

Similar comparisons could be made with respect to examinations administered in some countries at the other levels of schooling. In fact, the New Standards Project (Resnick, Nolan, & Resnick, 1994) is currently studying examinations administered in The Netherlands and France for 8th graders to compare with tests they are developing.

In summary, while very incomplete, the evidence from these reports indicates that there are quite different mathematical expectations for students in other countries. It should be noted that the differences in both terminal and university entrance examinations are not just that some countries expect their students "to cover more" mathematics. This assumes that the discipline is a linear sequence of topics. The fact is that countries differ in what they consider to be the important mathematics all students should learn. Some of these differences may be overt, such as teaching transformation geometry or measurement with metric units; others are covert, such as a focus on number sense rather than on computational proficiency.

Compare Programs

To compare programs, one could examine curricular frameworks. Geoffrey Howson (1991) examined the national curriculum frameworks for mathematics for 14 countries. The United States was not one of these, since we do not have a centralized education system. Howson concluded "There is no 'easy' way of comparing what is done and what is achieved in various countries. Major differences in philosophy and structure make simplistic comparisons dangerous" (p. 7).

He goes on to state, "National school systems reflect social cultures and traditions, and are much influenced by economic considerations, past and present. Perhaps the simplest measure of the latter is the length of compulsory schooling. This can be as high as 12 years, but Portugal, for example, is only now moving away from a 6-year system" (p. 7). He also found that while politicians argue that there is a need to "forge a closer link between the national curriculum and assessment procedures than would appear to exist in any of the other countries below. . . .there are references to assessment in few national curricula. . . ." (p. 28).

Ken Travers and Ian Westbury (1989) reported on a more extensive, but perhaps less useful, analysis of mathematics curricula across the 22 countries that participated in SIMS. The focus of this study was on the relationships between the "intended curriculum," the "actual curriculum," and the "attained curriculum." The utility of their analysis is hampered by the focus on variation across countries, on specific features (making it difficult to get a sense of any one country's program), and on the relationships to the SIMS item pool.

Compare Cultures

The differences in school cultures across nations with respect to the teaching of mathematics has been systematically studied by Stevenson and Lee. They studied the *context of achievement* for a sample of American, Chinese, and Japanese children (1990) and concluded that the performance of American children in their study was due to several factors that were nither "elusive nor subtle."

> Some of the most salient reasons for poor performance appear to be the following: Insufficient time and emphasis were devoted to academic activities; children's academic achievement was not a widely shared goal; children and their parents overestimated the children's accomplishments; parental standards for achievement were low; there was little direct involvement of parents in children's schoolwork; and an emphasis on nativism may have undermined the belief that all but seriously disabled children should be able to master the content of the elementary school curriculum. (p. 103)

In a less formal study, Jan de Lange (1992), whose staff at the Freudenthal Institute in The Netherlands has been working with mine to develop instructional materials for middle school students, has reported on the vast differences in the culture of schools in America and The Netherlands. These differences include governance, the role of administrators, the role of parents, the daily rites and rituals of schooling, scheduling, interruptions, athletics, and so on. While schools have been established by all societies to educate their children, there are vast differences in how different cultures have actually created and defined schooling.

Mathematical Sciences Education View

The Mathematical Sciences Education community believes that comparative studies are very important. We can learn by understanding how different countries decide what mathematics their students should learn; how they teach that mathematics; how they expect their students to learn that mathematics; and how they determine student progress, proficiency, or achievement. Such comparisons can make our *commonplace* actions and beliefs problematic.

Second, the community does not see the study of what other countries do as an attempt simply to keep abreast. We believe in the old adage: "Anyone who just wants to keep abreast is bound to be second best." Comparative studies should be seen as opportunities for us to learn and reflect on our actions, and not simply as an attempt to copy the ideas of others.

Third, for achievement, the community recommends either the development of a more balanced examination system that is aligned with principles articulated in NCTM's *Standards*, or the selective adaptation of methods other countries use to judge achievement and compare our students using procedures based on the examination systems of those countries.

Fourth, the community doubts that the overall effort of attempting to develop a common test battery similar to those in SIMS, IAEP, or NAEP upon which one can validly compare student achievement across countries at any grade level is worth the cost and effort. We are particularly concerned that future studies (including TIMSS) will compare student achievements across countries in a horse-race fashion on examinations that fail to reflect world-class aspirations for the students in any country, let alone those in the United States.

Finally, given the differences in schools and cultures, the community doubts that there can be any agreed-upon criteria that could be called world class. To strive for such is merely political rhetoric. As such, the rhetoric may detract or undermine our efforts to make needed changes in schooling. Our work on mathematics and the teaching and learning of the subject in schools is only a small part of a much more serious need to restructure schooling in America.

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Equity and Mathematics Reform

Mathematics instruction has been more accessible to students who are members of society's dominant racial, cultural, social class, and gender groups than to those who are not. From a perspective of equity and fairness, all student groups need access to mathematics instruction and opportunities to excel in mathematics. The reform of school mathematics provides the educational community with possibilities for addressing the needs of an increasingly diverse student population. As they develop policy, research, and practice, educators will need to combine concerns for both equity and reform. If they fail to do so, students who do not come from dominant groups may, once again, be denied full participation. The options educators have and the choices they can make are in the areas of curriculum, pedagogy, assessment, and school contexts that facilitate student learning.

Curriculum

If students are to become fully prepared for the world they will inherit, according to a broad consensus in the mathematics education community, the goals of school mathematics must change. Students will require a higher level of mathematical, scientific, and technical literacy than they have in the past. To meet these new goals, the school mathematics curriculum will need to incorporate new content, de-emphasize other content, and draw heavily from real-world and scientific contexts that are linked to out-of-school opportunities.

Existing mathematics curricula ignore students' informal and pre-existing mathematical knowledge and may actively interfere with how they reason and learn. New curricula are being developed with conceptual coherence. These units will focus on learning-by-doing and include problem-solving activities, opportunities for students to justify their solutions to their peers, and other approaches that support students' development of skill in reasoning with mathematics.

While educators agree that school mathematics should prepare students for the opportunities they will encounter in later lives, the reality that these later-life opportunities are not equally distributed in the United States has not been integrated into the reform agenda. Which mathematical experiences will promote students' interests as individuals and as members of their respective social groups? If school mathematics, for instance, intends to promote full and unfettered participation in a democratic and multicultural society, then the curriculum that American Indian students encounter should help to empower them to manage the lands and resources that are in their trust and to pursue or protect their treaty-guaranteed rights. Students who live in urban areas should encounter a curriculum that will help them use the many resources available in cities and understand and deal with the issues and problems they will face.

In a society that stratifies opportunity on the basis of group membership, there are differential sanctions for the mastery of paper-and-pencil algorithms. New curricula de-emphasize such skills. Some minority communities view facility with paper-and-pencil algorithms as a two-edged sword: On the one hand, ease in using paper-and-pencil algorithms does not guarantee access to opportunity; on the other hand, the lack of mastery is often used to legitimize denying opportunities to people who have been stereotyped as lacking in mathematical competence. From this

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Today's students come from ethic and linguistic backgrounds that are unlike their teachers' backgrounds and experiences

perspective, the new curricula create a differential risk for students that is based on group membership. The larger society and educators as members of that society must address the differential sanctions but, until then, mathematics educators will need to convince parents of students of color that the new curricula are in their children's best interests.

How best to incorporate meaningful contexts that will support mathematics learning into curricula for diverse student groups is a third area of concern. The multicultural education and ethnomathematics literatures recommend using social and historical settings from around the world as contexts for doing mathematics. Unfortunately, some current efforts to develop a multicultural curriculum treat cultural and mathematical content superficially, recreating stereotypes, and subordinating one type of content to another. Researchers and practitioners are still experimenting and learning about the contexts that are appropriate for teaching mathematics to diverse student populations.

The NCTM *Curriculum and Evaluation Standards* (1989) and the NRC *Everybody Counts* (1989) were informed by knowledge about how students reason and learn, the discipline of mathematics, and the needs of society. The discussion of goals for school mathematics needs to be further informed by empirical knowledge about the social realities that students of diverse backgrounds will face, both as individuals and as members of social groups. Information about census and work force projections, and about the social stratification of opportunities in the larger society will also be necessary.

Pedagogy

After two decades, educational researchers have learned a lot about the teaching methods that help students learn basic number facts and computational skills (Romberg & Carpenter, 1986). There is widespread agreement among mathematics educators that the teaching of mathematics should support student reasoning and engagement in worthwhile content (NCTM, 1991). These pedagogical strategies have demonstrated their effectiveness in improving student performance on a range of instruments (Secada, 1992). New pedagogical approaches are now being developed to facilitate students' mathematical thinking and understanding (Davis, 1992; Putnam, Lampert, & Peterson, 1991; Schoenfeld, 1992).

Today's students are more diverse ethnically and linguistically that those whom most of their teachers taught in previous decades. They come from ethic and linguistic backgrounds that are unlike their teachers' backgrounds and experiences (Grant & Secada, 1992). They may not be middle-class, they may not be White, and they may not speak English fluently. There is little research knowledge about how new pedagogies will affect learners who have been underrepresented in mathematics classes when compared to knowledge about pedagogies used with White middle-class students.

Much knowledge about pedagogy and minority students comes from the literature on multicultural education. James Banks (1993), for instance, argues that quality education for minority students should include integration of content across disciplines, opportunities for students to build their own knowledge, proactive efforts in the classroom and school to reduce prejudice, pedagogical practices that promote equity and inclusion, and an empowering school culture. Studies of teachers who are effective with African American students are showing teachers who care for the students and communicate that caring as part of their teaching (Ladson-Billings, 1994). Effective bilingual teachers use cultural referents to establish the norms for classroom behavior and to relate content to students' backgrounds (Tikunoff, 1985). They also use both languages and attend to English language development within the context of class content.

Work on gender differences in mathematics and on how status differences among students affect their instruction have added insight about the nature of teaching that is good for all students. Elizabeth Fennema and her colleagues have shown that often teachers pay less attention to girls than boys; ask boys questions that require higher order thinking and reasoning; praise boys more than girls for right answers; and plan competitive learning activities that place girls, who are more likely than boys to be socialized to cooperate, in an uncomfortable position (Fennema & Peterson, 1984). Work on status differences shows that the ideas of students who have high status are given more importance in small group settings; a group often assigns the substantive content of a task to its more capable members (Cohen, 1994). Teachers can take steps to increase their students' access to the mathematics they are taught by paying attention to their questioning of students, by using a balance of cooperative and competitive tasks, but monitoring groups to ensure that all members are participating in the work of the group, and by assisting students who are not participating.

As teachers begin to modify their classroom practices so that they support student understanding, they will need several kinds of support. Research on the school-level reform of mathematics is showing the support teachers provide each other reduces some of the stress or uncertainty that comes when teachers experiment with a new curriculum or try to change a teaching practice. While scholars have realized the importance of teachers' beliefs, assumptions, knowledge, and thinking, along with teachers' behavior and pedagogical practices (Clark & Peterson, 1986; Thompson, 1992), they are only beginning to understand the complex relationships between knowledge, beliefs, and practices. They have not explored the complexity that is added to these relationships when teachers are teaching mathematics to ethnically- and linguistically-diverse learners (Secada, 1991).

Assessment

Assessment and evaluation, according to the mathematics education community, need to be changed to support evolving views of curriculum and teaching. Traditionally, tests have been used to sort persons for educational and life opportunities. A 1993 draft of *Assessment Standards for School Mathematics* (NCTM) proposed standards that would promote equity by giving students optimal opportunities to demonstrate mathematical power, focusing on worthwhile content, including items that are open ended and require complex forms of thought, and providing teachers with information that can be used to make instructional decisions.

There are lingering questions about bias in assessment. With the possibility that new forms of assessment can make worthwhile mathematics accessible to larger numbers of students, the question of bias takes on added urgency. Will new forms of assessment bring with them new forms of bias that are linked to student demographics. A common strategy for identifying biased items looks for items that fail to predict total test scores or for items with error patterns that are differentially distributed among student groups. As a result, items on which low-achieving students could outperform high-achieving students tend to be thrown out early in the test development process because they do not predict overall test performance.

Contexts for mathematics problem solving are likely to be limited to those that already favor high-achieving students, since contexts that favor low-achieving students may be considered biased and dropped from the test. A pilot version of the California Assessment Program, for example, contained an item that asked students to determine what is wrong with someone assuming that he or she will be accepted to college, given that both College A and College B would accept half of a graduating class. Items set in a college context are unlikely to interest noncollege-bound students; hence, the test that includes them could underestimate these students' performance. Those who develop testing programs will need to determine whether to delete such items because of their potential for bias, or whether to create a smorgasbord of items that includes the same mathematics in different contexts. In the later case, contexts and items that appeal to different groups of students would be included.

The linguistic requirements of new assessments may place mathematics content in a subordinate role. Increased language requirements are likely to dampen the performance of limited English proficient students or others who are not familiar with conventional forms of English. The scoring of student work in mathematics will need to distinguish between mathematical and linguistic competence.

As forms of assessment change, states, districts, and schools will have to reexamine how they use the results they obtain in their accountability or evaluation systems. Differentials in student performance have often been explained by socioeconomic variables. States that report student achievement at the school level have allowed the removal from annual reports of the scores of students who participate in special programs or who possess limited proficiency in English. With increases in the diversity of student populations, schools may feel increased pressure to exclude more students from accountability systems or to attribute changes in scores to factors beyond a schools's or district's control. How to strike a balance between holding individual students, their teachers, programs, schools, districts, and even states accountable, or using external, non-school and non-changeable factors to explain differential performance is the dilemma.

With the possibility that new forms of assessment can make worthwhile mathematics accessible to larger numbers of students, the question of bias takes on added urgency.

Social Organization of School Mathematics

Social forces and economic considerations influence how classrooms and schools are organized. They also determine how schools create programs to remediate student performance deficiencies and how resources and personnel are distributed. Students have different experiences with mathematics that result from the track or ability group to which they are assigned during their school years. Ability groups are common to the lower-elementary grades; the name implies that students are assigned to them on the basis of ability. Careful analysis shows that, at least in elementary school, ability groups in mathematics often are the same groups developed for reading. Research on the formation of ability groups in reading shows that they are based on teachers' judgements of students' educability, a construct that, in addition to performance, includes students' classroom behavior, and social and emotional maturity. Ability groups can be created within classrooms or between classrooms of students. Their creation is likely to incorporate cultural bias

Educators need to know the historical, social, and cultural factors affecting the mathematics education of American minorities.

and, according to Slavin (1989), provides no academic benefit in mathematics.

In the upper elementary grades, between-class ability groups become a formal tracking system. While tracks are often defined by student demographic characteristics such as social class, race, or language ability, they can affect the quality of the mathematics students receive. Less experienced and less able teachers may be assigned to the lower tracks in a system. The mathematics program for these tracks focuses repetitively on procedural as opposed to problem-solving skills and on maintaining classroom order. Students in the lower track are unlikely to encounter the content that students in high tracks receive (Oakes, 1990). The benefits of tracking for high-achieving students may, in fact, be due to differences in the pacing of content or the quality of instruction (Gamoran, 1991).

Chapter I programs that provide bilingual and special educational services are making sweeping changes to align them with curricular, assessment, and systemic reforms. Schools will have leeway to fit these changes into their systems. If students are excluded from a school's mathematics program because they receive categorical services at that time, the program will fail to promote equity in mathematics. Categorical programs will need to balance their efforts between focused attention for students who are most in need and efforts that address all students.

Schools are microcosms of the society that surrounds them, yet they have unique identities, cultures, and customs. Teachers' efforts to change the way they teach may be limited by the culture and organizational structure of the school in which they work (Little, 1990). While individual teachers work very hard, they often work in isolation from others. Some find support for professional growth and development in their departments or groups of teachers within departments, while others must seek support through extra-school associations and professional organizations (Little & McLaughlin, 1993).

Within a school's walls, its atmosphere, its classes, and its daily routines influence teaching and learning. Do the teachers in that school and department work together to provide a coherent program of instruction where each year builds on each previous year of instruction? One teacher who teaches mathematics in accordance with reform documents can make mathematics come alive for a group of students. Will these students experience a less inspiring teacher who focuses on the repetition of content using drill and practice methods the next school year? If there are sanctions for students who fail to adapt to different content and approaches, the lack of program coherence will not lead to equity. Persons working with categorical programs are often unwilling to move their students to mainstream settings because mathematics programs lack coherence and thus differ in their accessibility for students.

The roles of parents and the larger community in supporting or impeding equity and reform have received little attention. The New Math was implemented in only a cursory way because, in part, parents turned against it. Family and home environments contribute to the academic success of children, and, by extension, to their study of mathematics. How a school addresses parental concerns about their children's education will influence the shape of reform in that school and whether all students take part in its efforts. Research is needed on the role that these factors play in diverse learners' success in school mathematics and the strategies that would enable racially, ethnically, and linguistically diverse parents to participate in their children's study of mathematics.

Educators need to know the historical, social, and cultural factors affecting the mathematics education of American minorities. The relationship between the culture of mainstream White America and the cultures of American minorities affects the degree to which members of minority groups are willing and able to cross cultural boundaries, including learning school mathematics and expressing that knowledge in the domains of their lives that are controlled by White Americans. Scholars have revealed the coping strategies African American and Latino students adopt to deal with stereotypic school knowledge, in particular, with mathematics (Fordham, 1988). Now educators must come to understand that the ways they promote the study of mathematics may have the unintended outcome of placing school mathematics outside the cultural frame of reference held by students.

Cross-cultural and historical studies show cultural differences in the mathematical knowledge, attitudes, and behaviors of various minority groups. At present, more is known about the ethnomathematics of schooled and unschooled peoples outside of the United States than of the peoples in this country (Nunes, 1992). It has been assumed by researchers and policy makers that the same mathematical knowledge, attitudes, and behaviors are shared by everyone. Those who do not hold these attitudes and behaviors are considered deficient or in need of intervention programs or activities. As a result of these assumptions, minorities' pre-existing cultural knowledge of mathematics, their mathematics attitudes, and their mathematical behaviors within their cultures or communities have been ignored.

Information about the indigenous systems and beliefs, the situations in everyday life or the activities to which minorities apply and practice their mathematical knowledge, and how the beliefs and practices differ from those of school mathematics is needed. As educators become sensitive to how girls and members of minority groups interpret the mathematical knowledge and practices they encounter in school mathematics classes, they can plan and implement interventions, school based, out of school, or in the larger society, that mitigate against these groups' beliefs that mathematics does not belong to them.

A concern for the mathematics education of all students is grounded in the core American values of the development of social and intellectual capital, and a consideration for fairness or justice. As the reform of school mathematics proceeds, researchers, practitioners, and policy makers will learn new things about the teaching and learning of mathematics and new things about the nature of equity. The complexity of the intersection of equity and reform will require an indepth and careful review of options and actions by these groups and by members of the mathematics education community.

Note

An earlier version of this article was prepared by the Study Planning Group for Diverse Racial, Ethnic, and Linguistic Groups for the Mathematical Sciences Education Board (MSEB) of the National Research Council (NRC). The study group included Walter Secada, chair, John Ogbu, Penelope Peterson, Lee M. Stiff, and Stuart Tonemah. While this version was written by Walter Secada, it has not been reviewed by members of the study group and neither their endorsements nor the endorsements of MSEB or NRC should be inferred.

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Gender and Mathematics Education Research

Five educators who have studied gender and mathematics education over the last two decades, presented a symposium at the April 1994 meeting of the American Education Research Association. The symposium, Research on Gender and Mathematics: Perspectives and New Directions, included Elizabeth Fennema, University of Wisconsin-Madison; Suzanne K. Damarin, The Ohio State University; Patricia B. Campbell, Campbell-Kibler Associates; Joanne Rossi Becker, San Jose State University; and Gilah Leder, La Trobe University, Victoria, Australia. The five prepared brief papers reflecting their perspectives on the new directions that research on gender and mathematics should take. This article provides abridged versions of the five papers that have been edited to smooth transitions between them.

Elizabeth Fennema

Most research on gender and mathematics during the last 25 years has been conducted from a positivist perspective and has provided powerful and rich information. While, in general, gender differences in mathematics appear to be decreasing, differences between males and females are still found in the learning of complex mathematics, in personal beliefs about mathematics, and in the selection of university majors or careers that involve mathematics. These differences vary by achievement level, socioeconomic status, ethnicity, school, and teacher. Since, in general, teachers tend to structure their classrooms to favor male learners, some interventions that help female learners have been identified.

From 1985-1994 there has been tremendous growth and change in research methodologies. Often grouped together as qualitative, the new methodologies have provided new insight into the complex phenomenon of education. Many researchers have begun to utilize these methodologies and it is sometimes suggested that no more studies of a positivist nature should be carried out. In order to continue the documentation of gender differences in participation and achievement in mathematics, however, some positivist research will be needed. National and international assessments must continue to include the sex of students as a variable, and individual schools must determine gender patterns in the election of mathematics courses and careers. We will not deepen our understanding of gender and mathematics, however, until the scholarly efforts conducted within a positivist framework are complemented with efforts that utilize other methodologies.

While scholarship on gender and mathematics could take many directions, two could help in the identification of important emphases for further research and ensure that women's voices will be adequately represented in educational scholarship: cognitive science perspectives that emphasize the irrelevance of female/male differences, and feminist perspectives that emphasize which female/male differences are critical to the learning of mathematics. Cognitive science deals with mental activity and processing. It is based partially on the major assumption that most behavior is guided by mental activity or cognition. The mainstream of current mathematics education research uses the cognitive-science perspective. Studies of teachers' knowledge and beliefs and of learners' thinking within specific mathematical domains are examples of this perspective. The complexity of the mental processing that occurs as teachers make instructional decisions was revealed using this perspective to study teachers, and the universals of problem-solving behavior that exist across cultures, races, and socioeconomic levels were identified using this research perspective to study learners' thinking. Little research on gender and mathematics has used this research perspective; most of the studies do not include the variable of sex or gender.

Research on teachers and teaching conducted from a cognitive-science perspective could enrich what is now known about gender and mathematics. Since there is evidence that teachers interact more with boys than with girls during mathematics classes, many assume that if the number of teachers interactions with girls and boys were equalized, gender differences in mathematics would disappear. There is evidence that just counting the number of interactions has resulted in an overly simplistic view about teachers and their relation to gender differences. Studies using the cognitive-science perspective could provide insight into whether teachers make a conscious decision to interact differently with girls and boys.

Studies investigating teachers' thinking have collected evidence that conflicts with commonly-held beliefs. One such study

Cognitive science and feminist perspectives, while sharing surface similarities, are based on dramatically different assumptions about females and males.

(Fennema, Peterson, Carpenter, & Lubinski, 1990) found that teachers thought the attributes of girls and boys who succeeded in mathematics were basically similar, yet their knowledge about which boys were successful was more accurate than their knowledge about which girls were successful. These teachers also attributed the boys' successes to ability more than to other sources and the girls' successes to effort more than to other sources. A second study (Weisbeck, 1993) found that while teachers reported they thought more about boys than about girls during instruction, they used similar characteristics when describing girls and boys. Although research that uses a cognitive science perspective is still in its infancy, at least as far as gender and mathematics are concerned, it could provide knowledge about the underlying mechanisms that have resulted in gender differences in mathematics, adding to existing knowledge of overt behavior.

The feminist perspective includes the multiple approaches of feminist methodologies, feminist science, feminist epistemologies, and feminist empiricism. These multiple approaches all focus on interpreting the world and its components from a feminine gender point of view, and the interpretations that result differ dramatically from those that accompany research carried out with other perspectives. Feminist scholars argue that scholarship, including that which developed the fields of mathematics and science, has been carried out by men from a masculine gender point-of-view and incorporates values that are shared by men but ignores those shared by women. While mathematics and science appear to be value free and to report universal truths, both, in reality, are based on masculine values and perceptions.

Feminists in mathematics education are struggling to define what a feminist approach to the study of mathematics might be. Some are examining the ways that females and males think and how they learn mathematics. Some are concerned with using women's voices and their histories to identify important questions. Others are examining the language of mathematics to determine whether it is gendered.

Cognitive science and feminist perspectives, while sharing surface similarities, are based on dramatically different assumptions about females and males. The assumptions dictate the questions that are developed, the design of studies, and the interpretation of findings. The assumptions are far-reaching and influence how the issue of gender and mathematics is viewed. Are males and females fundamentally different, so that all decisions about mathematics and knowledge about gender and mathematics need to grow from these differences? Or are males and females fundamentally the same, with the exception of their biological differences, and are these differences irrelevant with respect to mathematics teaching and learning? The research community on gender and mathematics must continue to examine the questions that are asked and the research methodologies that currently are used as they go about their business of scholarly inquiry.

Suzanne K. Damarin

Although all feminist research and theorizing begins with the goal of improving the lot of women in the world, feminism is not singular in its underlying assumptions, beliefs, methods, and goals. Feminists work within a range of perspectives and frameworks–liberal feminism, socialist feminism of several sorts, radical feminism, Black womanist theories, and postmodern feminism among them. Most recent research on gender and mathematics is carried out under the assumptions and using the methods associated with liberal feminism which assumes, basically, that the larger structures–e.g., capitalism, the scientific establishment, educational systems, and concepts such as the nature of mathematics, literacy as an essential, and the civilization itself, of current society are stable, essential, and appropriate. Liberal feminists "work within the system," attempting to improve the lot of women within a society that is otherwise left unchanged. Other branches of feminism seek to change the larger system or, in the case of postmodern feminism, to change the ways in which we understand how the system operates.

Several feminists are engaged in an effort to bring the insights, findings, and theories of feminism other than liberal feminism to bear on an understanding of gender and mathematics. This work is characterized by a multiplicity of approaches and the circulation of conclusions that are already tentative and suspect. Like all new paradigms, however, it begins with a rupture from the established ways of conceptualizing, conducting, and interpreting research regarding gender and mathematics. At the root of this movement is a disenchantment with the potential of liberal scientific study to yield solutions to the educational problems and dilemmas of women in relation to mathematics.

Like the field of Women's Studies, the study of gender and mathematics through the lenses of multiple feminism is multidisciplinary; it calls upon approaches and finds studies in fields that range from the "hard sciences" to cultural studies and the arts. Just as a Women's Studies approach to the study of the family might bring together research from sociology, social work, human ecology, medicine, population studies, literature, and religion, a feminism-based study in the area of gender and mathematics might bring together research and scholarship from sociology, educational measurement, history of mathematics in the U.S., studies of the representation of mathematics in the popular culture, mathematical biographies of great mathematicians and of young students, the philosophy of mathematics, and other fields. New data may be gathered from persons, documents, or cultural artifacts, chosen in light of the existing discourses and interpreted to yield new "stories" related to the topic of study. The multiplicity of studies, findings, and stories are read in relation to feminist theory and "against the grain" of each other and of the dominant discourses of gender and mathematics. The purpose is

Virtually all concepts, practices, and assumptions of mathematics education are open to examination through these feminist lenses.

to create a "new reading" of the dominant discourse, a reading that exposes hidden assumptions, unwarranted conclusions, contentious implications, paradoxes, questionable practices, and most importantly, interesting questions.

A feminist analysis of the belief that "math is a male domain," for example, leads to the following: Historical studies reveal that mathematics has long been claimed by men as their domain and philosophers have justified this claim; mathematicians' biographies often make strong masculinized claims about the demands of the field. Sociological studies of women in engineering and related fields of mathematically-based work reveal a high level of sexism in the workplace. Studies of science journalism uncover differences in the representations of scientists, presumed male, and of female scientists that suggest a paradox implicit in the very ideal of a female scientist. Similarly, studies of the popular press document that women and girls are frequently presented as incompetent in mathematics and as aliens in that domain. Even reports of equivalent mathematical performances by the sexes are couched in language suggesting the maleness of mathematics. Taken together and read in light of feminist thought, these findings lead to the conclusions that as a socially constructed area of activity, mathematics is indeed a male domain. In contrast, applications of educational and psychological research treat the belief among women that mathematics is a male domain as a personal attribute, indeed a defect, insofar as it interferes with the desired behavior–e.g., pursuing a mathematical skill. Based on this research, educational efforts are launched to convince young women that mathematics intrinsically is not a gendered field.

In effect, curricula designed and implemented to change this belief ask young women to overcome and/or deny the social realities around them. Would it not be more appropriate to acknowledge that reality in instruction and work to change it? The current goal of enticing individual young women to study mathematics would be replaced by claiming for all women the right of entry to and recognition within the domain of mathematics. This change would move women's claim to the right to mathematics education into the tradition of claims by women to the rights of economic independence, to many arenas of employment, to support for research on women's health, and to many other rights. In gaining each of these rights, education of women to recognize the social issues and education of the general public have been critical. The question, researchable through traditional methods, becomes, Would it work?

This is but one example of the ways that an investigative approach grounded in feminism might change our understanding of gender and mathematics. Virtually all concepts, practices, and assumptions of mathematics education are open to examination through these feminist lenses. Just as mathematics teaching is moving away from the "one right answer" mentality, with the acceptance of qualitative research methods and theories of constructedness and situatedness of knowledge, mathematics education research must also abandon the search for single solutions to complex multidimensional problems. The theories and practices of multiple feminisms, grounded in multiple contructions of sex and gender, offer to the study of gender and mathematics much knowledge and analysis regarding the workings and functions of sex and gender in society. Careful articulation of the feminist "knowledge explosion" with the knowledge created in two decades of outstanding mathematics education research into questions of sex differences and gender equity promises to yield many new insights, areas of study, and directions for educational research and change.

Patricia B. Campbell

Patricia B. Campbell

Compared to the 1970s, there has been improvement in the number of girls and women studying mathematics. There are indications, however, that girls' interest in mathematics and their participation in mathematics courses are again beginning to decline. At the same time, the gender gap between top students in science is actually increasing. The apparent stalling of progress has implications for research and perspectives about research. My research looks at programs and strategies designed to encourage or involve girls in mathematics. The assumptions that undergird my research include the following:

Societal and behavioral factors combine to make mathematics unwelcoming to and uncomfortable for many girls and an increasing number of boys. These factors can be identified and changed.

There are weaknesses in both qualitative and quantitative research methods; a single perspective cannot stand on its own.

Bias influences all research methods; the use of multiple methods to answer the same research questions is a reasonable way to reduce bias.

In general, the effects of well thought-out programs designed to increase the participation of girls and women in mathematics are unclear. Once control groups are added, interpretations of results change. In one study, students who completed a summer program, for example, increased their mathematics and science course-taking plans, but so did the students in the control group. Their higher scores after the program on measures of engineering and science-related career interests did not differ from the scores of students in the control group. Results obtained on measures of attitudes were mixed. Responses to open-ended questions showed that after attending the summer programs, girls saw the mathematics they encountered as fun and challenging. A mathematics attitude scale was given for several years, however, and no pre/post changes were found.

In a college study, those applying for a special program were quite different from nonapplicants even though they were enrolled in a similar science major. Some of these differences were fixed, for example, applicants became interested in science at an early age; others were not fixed, for example, applicants had stronger study habits. After two years of college and special-program participation, the groups became more similar and there were no differences in their retention in mathematics/science majors or in their mathematics/science grades. It appears that the college experience had a much stronger impact on these students than the special program experience.

The methods or assumptions used in these studies influenced their results in several ways: The change found in open-ended responses and the lack of change found in attitude scales suggest that subtle effects may not have been observed. By collecting relatively structured data, a number of outcomes and effects were not permitted. Any programmatic results that are obtained without the use of a randomly-assigned control group, which often is not used in applied research, must be questioned.

Gilah Leder

This paper explores the influence of choice of a research paradigm on the format and scope of the questions posed for investigation in particular studies: it addresses the issue of coeducational versus single-sex schooling as the means for doing it. Regardless of the research paradigm selected, the findings of a single study can be put into a broader context or their generalizability determined in a variety of ways. Evidence can be assessed: historically–e.g., through comparisons with data gathered across time; cross-nationally–e.g., through comparisons with findings from the same or comparable institutions in different societies; or cross-institutionally–e.g., by examining data from different institutions in the same society.

Gilah Leder

Early in 1993, staff at coeducational high school in Victoria asked for help in assessing the effectiveness of an experimental program they had recently introduced: teaching mathematics to Grade 10 students in single-sex settings. At other grade levels, mathematics would continue to be taught in mixed groups. By selecting Grade 10 for this intervention, the school hoped more females would elect to continue with the more rigorous mathematics courses in succeeding years. No plans were made, either formally or informally, to modify instructional strategies or curriculum materials previously used, or to examine the prevailing culture of the school.

At least two assumptions were implied by the intervention strategy described: that curricula and teaching methods traditionally are geared to the needs of males rather than females, and that content and strategies that facilitate the learning of mathematics for females are more readily achieved in a single-sex setting. Whether broader measures were needed to address the overall context in which learning took place was not considered.

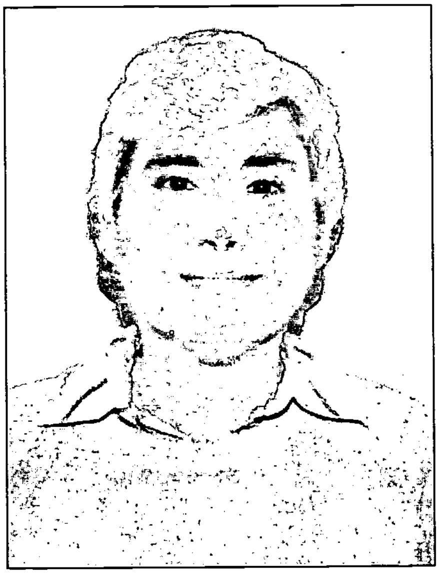
Both quantitative and qualitative techniques were used to gather the data. Attempts were made to document group characteristics as well as the nature of the mathematics traditionally taught in the school, and to determine whether there were any changes in these characteristics over the course of the intervention.

There is no doubt that the design and methods for the research were shaped by the socio/psychological framework adopted in earlier work. There was considerable emphasis on students' perceptions of coeducational and single-sex classrooms but very little on what these organizational settings implied about gender roles or cultural stereotypes. Students' perceptions about their teachers' preferences for single-sex or coeducational settings were sought; the antecedents or implications of these preferences were not explored. While the nature of the mathematics course was discussed, how or why its content had evolved as it had was not considered.

The segregationist perspective reflected in the design of the experiment appeared to have been adopted by many of the students. They indicated that males and females often behave differently in mathematics classes and that society accepts and possibly encourages these differences. It was generally accepted that teachers would probably interact differently with students in coeducational and single-sex settings. No student speculated spontaneously whether these differences were appropriate, whether the male/female roles they implied should be examined or challenged, or whether the rationale given by the school authorities for mounting the "experiment" should be questioned. These concerns might well have been addressed explicitly if the research had been framed in a different–a feminist–perspective.

Joanne Rossi Becker

This paper takes one feminist perspective on the subject of gender and mathematics derived from *Women's Ways of Knowing: The Development of Self, Voice, and Mind* (Belenky, Clinchy, Goldberger, & Tarule, 1986). *Women's Ways of Knowing* describes a series of stages in knowing that differ in fundamental ways from how men come to know. These stages represent a progression from dependence to autonomy. Learners move through five stages: silence, received knowing, subjective knowing, procedural knowing, and constructed knowing. In the silence stage, the learner accepts authority's verdict of what is true. In the received knowing stage, the learner learns by listening and returns the words of authority. As a subjective knower, the learner depends on what looks or feels right and what comes from her experience rather than an external source.



Joanne Rossi Becker

This seems to be a critical stage for women. The procedural knowing stage has two parts: separate and connected knowing. Separate knowers look to propositional logic to validate arguments, while connected knowers focus on the context and other people's knowledge. The final stage, constructed knowing, represents an effort to integrate what is known intuitively and what others know. Here the learner appreciates the complexity of knowledge.

Given a characterization of separate knowing as embodying logic, deduction, and certainty, and connected knowing as embodying intuition, creativity, and induction, it appears that mathematics has traditionally been taught to conform more to the former. Could that help account for the relatively small number of women pursuing mathematics-related fields? Research can provide evidence to support or refute the hypothesis derived from *Women's Ways of Knowing* and, if supported, show how this theory can provide an understanding of and improve women's participation in mathematics. The danger exists that acknowledging women's different ways of knowing will serve those (biological determinists) who wish to reinforce stereotypes that demean women's strength and limit their roles, a misuse and misinterpretation of research findings.

To test the hypothesis, in-depth interviews were conducted with 31 graduate students, 17 men and 14 women, in the mathematic sciences. They focused on factors influencing women and men to pursue graduate education. Later these were analyzed through the lens of the *Women's Ways of Knowing* model. One interesting finding was how similar the men and women were in the reasons expressed for liking mathematics and when their interests developed. Of course all liked mathematics because they were good at it. They were attracted to the analytical problem-solving aspects of the subject and particularly liked starting with certain assumptions and, through logic, solving the problem. The "objective" nature of the subject also appealed to these informants; they liked being able to determine whether a problem was solved or a proof was correct. This raises a question: Are women in mathematics more likely than female non-mathematicians to be separate knowers and thus to be attracted to the subject because, at least at the student stage, they perceive mathematics to be an objective subject in which they can find absolute truth? Do their views of mathematics evolve as they pursue further study and actually do research themselves?

Nearly all informants developed their liking of mathematics early, in elementary or junior high school. Frequently a teacher was mentioned as one who piqued their interest by providing an enriched curriculum that went beyond arithmetic to problem solving or algebra topics. Thus, it does seem possible for teachers and instruction to make a difference in students' ultimate career choices. Could more extensive use of connected teaching affect more students in this positive way? To better explore whether the model represents how women come to know mathematics, we need further research designed specifically to test the model. This research can be informed by work from several other perspectives. In particular, the segregation perspective makes a case for all-female classes. Does the change in instruction that occurs in such a setting provide a more connected learning environment, which might bring more women into the study of mathematics? Does it enhance students' performance in and attitude toward mathematics? Can a change to connected teaching at all levels recapture women who have been turned off to mathematics and science (Tobias, 1990)?

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Review of NCRMSE Research

Equity in Restructured Schools

by Walter Secada

A majority of people would consider schools equitable if there were no differences on highly valued educational outcomes among students grouped by race, ethnicity, gender or social class. Even if differences were found between such groups, a school could still be considered equitable. If a school is reducing the gap between groups on outcomes, for instance, it can be argued that this school is achieving some degree of equity. Similarly, reducing the differences in how groups of students or their parents perceive their educational experiences can be viewed as a positive step toward achieving equity. Equity, as an idea or a concept, means different things to different individuals or groups. Most people base their definitions of equity on a core notion of fairness, or in the case of law, on the core notion of justice. In general, they view as inequitable that which appears to be unfair. Hence, a school can work toward equity by applying efforts to remedy an injustice or by increasing the appearance of fairness or the sense of fairness felt by individuals or groups.

Equity, as an idea or a concept, means different things to different individuals or groups.

Scholars at the National Center for Research in Mathematical Sciences Education (NCRMSE) and the Center on Organization and Restructuring of Schools (CORS) continue to define equity as the absence of differences between groups on some important outcome measure. They are enlarging this conception of equity to consider the other conceptions of equity that exist in schools. The collaborative research on equity currently underway at NCRMSE and CORS is looking at five sets of interrelated questions:

1. What conceptions of equity do school people use to talk about issues of fairness and how are these conceptions articulated? When they talk about equity, which students do they talk about and how do they talk about them? Which people in these schools hold different conceptions?
2. How do schools, as organizations, enact these different conceptions of equity? Do individual teachers or other school personnel act alone or do they act in concert with others? Are particular conceptions of equity more likely to result in collective activity within a school when compared with other conceptions?
3. What organizational features in the restructured schools–cultural or structural–support or impede efforts to promote the different conceptions of equity? How do they operate?
4. How do external agencies influence the conceptions of equity found among school people and the actions school people take to promote equity?
5. How are conflicts or actions involving different notions of equity identified and resolved in schools? Do schools have the resources or other organizational features to support the resolution of such conflicts?

Conceptions of Equity

According to initial analyses of data, six conceptions of equity seem to be held by school personnel. These six form "starting points" for data collection and analysis. It is likely that other conceptions will emerge or that some of the original six will be modified during the course of the research. The six conceptions of equity are labeled: equity as a concern for the whole child; equity as a safety net for individual differences; equity as the same treatment for everyone; equity as compensation for social injustice; equity as triage; and equity as the maximum return on a minimal investment. While the six conceptions of equity have roots in commonsense notions of fairness, there are strengths and weaknesses in each of them.

Equity as a Concern for the Whole Child

The first conception of equity grows from a larger idea wherein education is viewed as concerned with the whole child. According to this perspective, each student is an individual who has unique and distinct educational, socio-emotional, and physical needs. In the research, most elementary teachers have expressed concerns about each of the children in their classroom. They feel a deep sense of responsibility and hold themselves accountable for the children's welfare. These teachers are able to give detailed, often heart-wrenching, examples of the actions they take to help a child with academic, emotional, or physical needs.

Equity as A Safety Net for Individual Differences

The conception of equity as a safety net recognizes that a single program cannot meet the educational needs of every student. Hence, teachers and other school personnel who hold this perspective create back-up programs, differentiated curricula, and other resources so that when one program does not work for a particular student, other options are available. Their notions about student and program mismatches often are couched in terms of psychological traits and include learning styles or ability.

Equity as The Same Treatment for Everyone

The conception of equity as the same treatment for everyone seeks to ensure that all children are treated the same way. This view could be used to justify giving all students a common core curriculum, providing them with similar opportunities to succeed, and holding them to the same performance standards, including those for classroom and school behavior. The argument that everyone should be treated the same is based on the belief that there should be one set of standards for high performance in an area and that society at large demands performance or mastery that meets those standards. All students should be treated the same way so that they have an equal chance to meet the standards and an equal opportunity to succeed in the society at large.

Equity as Compensation for Social Injustice

The conception of equity as compensation for social injustice argues that specific groups of students, for instance a specific ethnic group or females, have not received fair treatment in the larger society or that the groups are not receiving a fair share of the school's resources. From this perspective, the school should actively redistribute resources to remedy the larger social or the more specific school-level inequities.

Equity as Triage

According to the equity as triage conception, schools should divide students into three groups: those who are beyond help; those who, because of special skills or access to special resources, do not "really need the school, since they will make it anyway;" and those for whom the school could provide input that could make a critical difference in whether a student will achieve success or failure in the future. Following the triage model, school resources would be invested in only the last group, those who fall along the middle of the distribution of the criteria being used, for example, college admission.

Equity as A Maximum Return on Minimal Investment

According to the maximum return on minimal investment notion of equity, schools and teachers should invest in the students who are most likely to benefit from their investment. Given the scarce resources available to schools and the stresses that schools, their staffs, and their students face every day, attention and resources should be focused on those students who are most likely to succeed. A school's teachers, for instance, would look for the students whom they view as worth educating. The students would be seen as those who "could be saved." From this perspective, additional resources would be provided to the students at the top of a school's distribution on some indicator of achievement.

Dealing With Multiple Notions of Equity

It is possible for a person or for a school to hold what appear to be competing notions about equity. An individual teacher or a group of teachers may believe that, as far as standards for school discipline are concerned, students should be treated in the same way. Yet the same person or group may believe that the school should provide a range of academic and non-academic programs in order to address the educational aspirations of a diverse student body. Decision making becomes more complex when multiple notions of equity are applied to the same situation. A school may offer different mathematics courses in an effort to address student interests and abilities in a fair and equitable manner—equity as a safety net. Such an effort, if taken to an extreme, could result in an ever-increasing number of overlapping courses or the fragmentation of programs. If others in that school believe that the fairest way to educate students is to give them all the same core mathematics curriculum—equity as treating everyone the same—taken to the extreme, it could result in needless rigidity. Such a school would need to find ways to balance the interests of these groups. It would, for example, need to create courses around the same core curriculum but provide students with options as to how they encounter the mathematical ideals and how they demonstrate that they understand what they have learned.

External Influences

External agencies sometimes pressure a school staff to work toward equity in a particular way. One of the schools studied by NCRMSE/CORS was a magnet school created to help a district's desegregation efforts. As its staff developed innovative and highly visible programs over the years, the number of affluent students seeking and gaining entry into the school increased. The parents of the students in this school are now pressuring the school to create programs for gifted and talented students. Teachers who are resisting these pressures say that such programs would take attention and resources away from the school's original mission and lead to differences in the quality of instruction for groups of students.

Another school that was studied is also part of a cluster of magnet schools designed to help desegregate a district. This school's focus includes a program for gifted students that enrolls a disproportionate number of white students. The school says it provides high quality instruction to all students, regardless of the program they are enrolled in. While teachers' claim that students in regular classrooms receive the same high quality instruction as students in gifted classrooms, they have been unable to do anything about the size of classes. The average gifted and talented class has a pupil-teacher ratio that is in the high teens while other classes have a pupil-teacher ratio that is in the mid-twenties.

The first magnet school successfully resisted parental pressure because the staff shared the belief that creating a differentiated program for gifted students would lead to inequity. The second magnet school tried to insure a uniformly high quality of instruction across classes; teachers did not talk about the re-segregation that was occurring or class size–issues that they acknowledge implicitly could split the faculty into factions.

Managing Dilemmas

Mathematics reform seeks mathematical power and high and rigorous standards for all students. The faculty of both of the magnet schools in the study were familiar with the NCTM *Standards*, yet they encountered dilemmas. Preliminary analyses of research data reveal features about conceptions of equity in schools. Schools are pulled by competing notions of equity and of actions to promote equity. What a school does that promotes or works against equity is situated in that school's context. Schools and their personnel–even when they are acting in ways that can be considered equitable–are often unaware of the competing principles that undergird their efforts because they do not "talk about it." Can schools achieve equity when teachers act individually to promote their respective visions of equity? Or are there benefits when staff and teachers hold shared conceptions of equity? Schools need knowledge and resources to manage the dilemmas they encounter. The NCRMSE/CORS research is gathering information about how schools that are similar think about equity and how they manage the dilemmas they encounter. It will then identify the variety of options available to schools and the ways that schools can use the options to insure outcomes that are equitable.

This article is based on research conducted at the National Center for Research in Mathematical Sciences Education (NCRMSE) and the Center on Organization and Restructuring of Schools (CORS). Both Centers are funded by grants from the Office of Educational Research and Improvement (OERI), U.S. Department of Education. They are administered through the Wisconsin Center for Education Research (WCER), School of Education, University of Wisconsin-Madison.

to be released

Book on Equity in Mathematics Education

*New Directions for Equity in Mathematics Education** begins with an overview of broad cultural issues, such as how social class and notions of merit affect education. The second section of the book analyzes gender issues in mathematics learning, and the third and final section looks at language and mathematics. A number of themes cut across the three sections. These include a critique of the reform movement, teachers and the dynamics of the classroom, and the re-skilling and empowerment of teachers.

New Directions for Equity in Mathematics Education was edited by Walter G. Secada, Elizabeth Fennema, and Lisa Byrd. Drs. Secada and Fennema are professors in the School of Education at the University of Wisconsin-Madison. Ms. Byrd is a project assistant at the same institution. The book will be published by Cambridge University Press, 40 West 20th Street, New York, NY (212) 924-3900, in February 1995. Copies can be purchased in hardcover ($49.95) or paperback ($16.95) from 110 Midland Avenue, Port Chester, New York, 10573-4930, FAX (914) 937-4712.

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