DOCUMENT RESUME

ED 454 053 SE 064 908

AUTHOR	Taber, Susan B.
TITLE	Making Connections among Different Representations: The Case of Multiplication of Fractions.
PUB DATE	2001-04-00
NOTE	48p.; Paper presented at the Annual Meeting of the American Educational Research Association (Seattle, WA, April 10-14, 2001). Supported in part by a grant from Rowan University.
PUB TYPE	Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	Cognitive Processes; *Fractions; Grade 5; Intermediate Grades; *Mathematics Instruction; *Multiplication; Teaching Methods
IDENTIFIERS	*Representations (Mathematics)

ABSTRACT

The purpose of this study was to document and trace how students' understanding of situations that involve finding a fractional part of a fraction developed during a course of instruction in which students explored and represented various kinds of situations using each of the five representational systems. A class of 5th grade students (n=22) participated in this study which involved 13 days of instruction spread over 4 school weeks. The results of this study suggest that understanding the process of making connections among representations is of crucial importance. Examining student solution methods in various representational systems for problems involving the multiplication of a fractional quantity by a fractional operator and attempting to tease out the connections among those representations suggests that students can develop the double partitioned model as a meaningful physical representation for the product of two fractions. (Contains 22 references, 9 figures, and 2 tables.) (ASK)

MAKING CONNECTIONS AMONG DIFFERENT REPRESENTATIONS

THE CASE OF MULTIPLICATION OF FRACTIONS

Susan B. Taber
Rowan University

Presented at the Annual Meeting

American Educational Research Association

April, 2001

Running head: Representations and Multiplication of Fractions

The research reported in the paper was supported in part by a grant from Rowan University. The opinions expressed in the paper are those of the author and not necessarily those of the university.

Introduction

It is the ninth day of a thirteen day investigation of multiplication of fractions in a fifth-grade class of 22 students. The day before students had worked in groups of three to solve each of five word problems in "two different ways" (See Figure 1); today I am leading a whole class discussion of their solution methods.

After reading the first problem, "Alice is five feet tall. Find out how tall she'll be if she eats a cake that makes her one-fourth as tall," Carol[1] said, "I did five times one-fourth[2] equals five-fourths and that simplifies to one and one-fourth." (See Figure 2a)

Explaining the diagram he made on the board, Bill said, "I made that divided into five parts, and then I did five divided by four equals one and one-fourth times one equals one and one fourth. So then I colored in one whole box and divided the second box into four pieces and colored in one of those four." (See Figure 2b)

When I asked how the picture could help them find the answer, another student replied, "The answer is one-and one fourth so you color in one box and you divide another box into fourths."

I persisted, "But what if you didn't know the answer. How could you find the answer from the picture?"

Karen responded, "You could just use lines to break it up instead of using multiplication." When I asked her how she would break it up, she said she would break the second block up into fourths "Because if you have one and one-fourth, you can't just color the whole box."

I had noticed while circulating among the groups the day before that all students had written "5 × 1/4 = 5/4 = 1 1/4," or "5 ÷ 4 = 1 1/4," or both on their papers. Those who had

---

[1]All student names are pseudonyms

[2]Throughout the paper I indicate spoken fractions by spelling out the word, "one-fourth". Written fractions are represented numerically, "1/4." Even though I am utilizing slashes to represent the fraction bars in this paper, I utilized the horizontal fraction bar for all numerical representations of fractions during instruction, interviews, and in written problems.

partitioned the original representation of 5 feet had made an illustration of the answer 1 1/4 either by estimating the location of 1 1/4 in the second foot or by partitioning the second foot into four parts and shading one of them as described by Bill and Karen. One of my goals for the discussion of students' solutions was to press them to make connections between the numerical expressions they had written and partitioning the diagrams of starting quantities to show the results of the operations on those quantities.

I asked, "But what if you didn't already know you have one and one-fourth?" Lisa raised her hand.

"You could divide the whole thing into fourths." She then explained that three vertical lines could be drawn through the entire rectangle. When I asked her how much of the repartitioned rectangle would represent Alice's new height, she said, "One of those columns. . . It is one and one-fourth." When I asked her to explain how the picture showed 1 1/4 as the answer, she said, "Because a whole foot is four of those little columns. You put five x's. Four of those five are a whole; each represents 1/4. . . five-fourths equals 1 and one-fourth." (See Figure 2c.)

When I asked for additional comments, Karen said, "You could also have done five divided by four is one remainder one and the one would be the one-fourth."

For problem two, Rebecca stated," I drew two-eighths." (See Figure 3a.) I asked Rebecca to read the story "Alice is five-eighths of an inch tall. She nibbles some of the mushroom but it makes her shrink to two-fifths as tall. How tall is she now?" and then to explain why she colored in two-eighths.

When Rebecca stated that she couldn't remember, Hillary volunteered, "Because five-eighths times two fifths is--"

I stopped her. "That's the part that Mark is going to talk about. We don't know that yet. What we're trying to do is understand how this picture works. Why would we color in two-eighths?"

Bill raised his hand, "You could do what Lisa did for number one and divide that into five parts." Under Bill's direction I drew four vertical lines through the representation; then Angela directed me to put x's in the five shaded boxes in two of the new columns. (See Figure 3b). When I asked for how much we had marked, Tyler said, "Ten-eighths?"

I asked, "Are these little pieces eighths?" Tyler responded, "Ten-sixteenths?" I suggested we count what we have, and called on TJ to help us count. Instead TJ announced, "Ten-fortieths . . . because there are forty boxes." As a class we counted the pieces. After TJ stated that ten-fortieths is equivalent to one-fourth, I asked the class whether what we had done agreed with the answer Rebecca had shown us. They chorused, "Yes."

To my query of why? Karen answered, "Because ten-fortieths is the same as two-eighths." Angela added, "Two-eighths is the same as one-fourth."

I then asked Mark to explain his solution "$5/8 \times 2/5 = 10/40 = 1/4$." He said, "I did five-eighths times two-fifths and [that equals] ten-fortieths and that simplifies to one-fourth."

I continued the discussion by asking the students to explain how the numerator and denominator of the Mark's product 10/40 were shown in the picture that I had constructed under Bill and Angela's direction.

After they did so, TJ asked to explain how he had solved the original problem. "On the picture there (Figure 3a) there are five of them and I did two of the five of them." Because the original sketch from the problem was not on the board, I redrew it and said, "Here's the five-eighths. (See Figure 3c) TJ, please tell us what you did."

TJ said, "I didn't really use the top part. I pretended it wasn't there. . . And I just used the five, and since there is two-fifths, I just did two."

I had planned this instructional investigation as an opportunity to explore how students make sense of a persistently difficult topic in the school curriculum, multiplication of fractions. In addition to trying to learn more about the content of students' understanding of the meaning of multiplication of fractions, I wanted to explore the ways in which students utilized and made connections among various representations-- manipulatable models, pictures, "real-life" examples and contexts (story problems), number symbols, and verbal explanations--to make sense of this complex and complicated domain.

I have selected this conversation as an example of students' making sense of a persistently difficult topic in the school curriculum, multiplication of fractions. The study that I describe in this paper investigates changes in fifth-grade students' understanding of fractional quantities and operators as they explore finding a part of a quantity by manipulating physical models, drawing pictures, and describing the problem situations and their solutions with spoken and written language and mathematical expressions and symbols. The class discussion on Day Nine was a pivotal event in the development of students' relating the product of two fractions to the story problem they had been asked to solve and to the diagram that I had suggested they utilize in finding the solution to the problem. Students made connections that I had not expected at this point of the instruction, and in fact, developed a manipulatable model of repartitioning the model of 5/8 to show 2/5 of 5/8 as 5/40 and connected it to finding the product of 2/5 and 5/8.

Theoretical Perspectives

Although using physical representations such as manipulative models or drawings during mathematics instruction is believed to help students learn mathematics more meaningful, the reasons for using them and the benefits have been the subject of some debate. Various rationales for using such models have been proposed: "multiple embodiments" (Dienes, 1964), "mapping" (Resnick, 1982), as a"thinking tool" for modeling and solving problem situations (Van de Walle, 1997). Lesh (1979) describes five representational systems (e.g. experience based "scripts," manipulatable models, pictures or diagrams, spoken languages, and written symbols) and describes understanding as being able to recognize an idea as it is embedded in each of the systems, manipulate the idea within a given system, and translate the idea from one system to another. Lesh, Post, and Behr (1987) argue that translations among representational systems are not automatic and that the process of translation must not be taken for granted. "Translations are often assumed to be easy. Our research shows that concrete problems often produce lower success rates than comparable 'word problems' or written symbolic problems" (p. 56). Although Lesh, et al. (1987) described the translation process as involving being able to "strip away" extraneous details from the situations being represented, Ball (1993) points out that representations are metaphorical, closely corresponding to some elements of a situation while obscuring other elements, and claims that meaning for representations is negotiated in the classroom.

The mathematical context for this study--multiplication of fractions-- is a complex and complicated domain. As both Streefland (1991) and Greer (1992, 1994) have pointed out, learning to multiply with fractions involves not only learning to multiply new kinds of numbers, but also learning to recognize new kinds of situations that can be expressed as multiplication. Students' knowledge of whole-number multiplication does not easily extend to multiplication by fractions less than one. Indeed, constructing an understanding of multiplication of fractions appears to require reconceptualizing number and operations (Hiebert & Behr, 1988). Kieren (1988) argues that operations with fractions are not simply extensions of whole number operations, but are built up from experiences with situations involving fractional quantities and operations on those quantities. Taber (1991, 1993, 1999) found that students, including both those who had learned to multiply of fractions and those who had not, divided to solve problems in which the operator or multiplier was a fraction and multiplied to solve problems which had whole number operators and fractional quantities. Behr, Harel, Post and Lesh (1993), Kieren (1988), Streefland (1991), Lamon (1999) have identified the ability to think of a fraction as an operator that can change the size of other quantities as one of the crucial understandings of fraction number knowledge. Taber utilized the story of *Alice's Adventures in Wonderland* (1994, in press) as a basis for classroom instruction that resulted in students' learning to represent situations such as finding 3/4 of a quantity as multiplication by the composite operator 3/4 rather than as division by 4 followed by multiplication by 3.

Mack's recent work (in press) focused on making connections between students' informal knowledge of fractions and partitioning on the one hand and the more formal mathematical operation of multiplication by fraction operators. She documented the ways in which six students thought about various kinds of partitioning problems over time, focusing on the ways in which they utilized their informal knowledge of fractions and of partitioning to find fractional parts of fractional quantities and how they coordinated that knowledge with the classroom instruction they received on how to multiply fractions. Although students were able to make pictures and diagrams showing how to find 2/3 of 3/4 (for example), they at first resisted the idea that 2/3 of 3/4 could be symbolized as 2/3 x 3/4. After they were taught to multiply fractions, however, they wanted to represent situations symbolically as products because it was easier and quicker than making diagrams. Still later, they were able to coordinate their informal knowledge of partitioning and reunitizing quantities to justify the results they obtained when finding the product of two fractions.

The purpose of the study reported here was to document and trace how students' understanding of situations that involve finding a fractional part of a fraction developed during the course of instruction in which students explored and represented various kinds of situations using each of the five representational systems of Lesh's translation model (1979).

Method

Subjects

A class of twenty-two fifth-grade students in a parochial school in an urban/suburban community participated in this study which involved thirteen days of instruction spread over four school weeks in the spring of the school year. (The thirteen days of instruction were spread over seven calendar weeks because of the Easter observances and holiday of the school. There was a two week gap between day 11 and 12 of the instruction.) The author (hereafter referred to as I) taught math during the regular mathematics period on Mondays, Wednesdays, and Fridays. On Tuesdays and Thursdays the regular classroom teacher taught geometry lessons from the math book. Prior to the instruction on multiplication of fractions students had been instructed by their regular teacher on the topics of equivalent fractions and addition and subtraction of fractions with like and unlike denominators using the regular textbook.

Data Sources

Information about students' ideas, understanding, and representations of their mathematical thinking were collected from the following sources. A pre-test consisting of 20 multiplication, two addition, and two subtraction problems, each containing either a whole number and fraction, or two fractions was given to all students prior to the beginning of instruction. The same test, with the addition of four more problems involving finding a fractional part of a fraction, was given two days after the end of the instruction.

All class sessions were audiotaped and transcribed, providing a record of classroom discussions. On days when students worked in groups to explore and discuss problem situations, I circulated among the groups and recorded my conversations with each group. The transcripts of these small group discussions were then correlated with the written work collected from each of the groups, and identified with each student's name and the names of other students in the same group. All written work, whether done in class or as homework, was collected and photocopied before being returned to the students for discussion in class.

Five students were interviewed on three occasions each: before the beginning of the instruction, at the midpoint of instruction (prior to the introduction of problems involving two fractions), and a week after the last day of instruction. All interviews were audiotaped and the transcription of the audiotapes was compiled with the written notes of each interview and the student's written work to obtain a complete record of each interview.

I selected the students after examining the solutions students used on the pre-test problems. I wanted to interview students whose strategies for solving the pre-test problems seemed to suggest different ways of understanding the problems. I selected TJ as an interview subject because he had solved the problems with fraction multipliers by making an equivalent fraction with the whole number as the denominator of the equivalent fraction and had solved 13 of the 16 multiplication problems correctly. I selected Sasha because she drew diagrams to indicate partitioning and regrouping for problems with fraction multipliers, but multiplied to solve problems with whole number multipliers and had solved 9 of the 16 problems correctly. Carol had attempted to multiply on problems with whole number multipliers and to divide on problems with fraction multipliers, correctly solving 6 of the 16 problems. Hillary had not solved any of the pre-test problems correctly, but she had attempted to multiply on the problems with whole number multipliers and had subtracted on the problems with fraction multipliers. Lisa had solved 3 of the pre-test problems correctly, and had subtracted on some of the problems with fraction multipliers while not attempting most of the problems with whole number multipliers.

I presented the interview questions to students one at a time. The questions were written on quarter sheets of paper; some included diagrams and some did not. I asked students to solve each problem and to explain to me what they had done and why. I then asked follow-up questions about what they had done until I was satisfied that they had given me as much information as they could about their solution strategie. I did not attempt to teach them during the interview, although on one of the problems when students reached an impasse using a numerical method, I asked them if they could draw something to help them think about the problem. I took written notes of the students' actions as they worked on each problem which I later combined with the transcript of the audio-taped interview and the students' written solutions as a record of what students said and did during each interview. I coded students' written solutions to pre-test, post-test, and interview problems both for correctness and for the strategy used. Strategies were defined according to a dictionary of strategies from earlier studies (Taber, 1991, 1999). I also examined each student's solutions for patterns of strategy use on the written tests and on each of the interviews. For example, one student always used the algorithm for multiplication of fractions, but another student drew diagrams when finding part of a part and multiplied when finding part of a whole number quantity.

Design of the instruction

I took Lesh's (1979) five representational systems as a starting point both for planning instruction and for investigating the ways in which students make sense of and represent situations that involve finding parts of whole number quantities and of fractional quantities. I designed instructional and data gathering activities that would involve students in utilizing each of the five systems as they explored and represented problem situations and developed methods for representing and solving them. Both instructional activities and data-collecting activities involving multiplication of fractions were introduced to students as real-life situations or word problems, incorporating either a fraction and a whole number or two fractions. These word problems described actions involving either Combining equal quantities, Comparing quantities of different sizes, Changing the size of a quantity through expansion or shrinking, or Partitioning the quantity by a fractional operator. Problems with the action of Comparing and Changing could have either a fraction or whole number multiplier. Examples of each problem type are listed in Table 1.

I also incorporated various kinds of number relationships in the problems that I selected for students to work on in class and as assessment tasks. Features that were varied included whether or not the whole number in problems containing a whole number and a fraction was a multiple of the denominator of the fraction. I also included situations in which the operator was a unit fraction as well as those in which it was a common fraction. Mack's (in press) rational analysis of the numerical characteristics of partitioning situations, suggests that students utilize different kinds of informal knowledge in representing and solving problems such as 2/3 of 3/5 (a/b × nb/c) than they use to represent the solution of problems like 2/3 of 5/6 (a/b × c/d). In the case of 2/3 of 3/4 the denominator of the operator 2/3 is a factor of the numerator of the fraction 3/4. In the situation 2/3 of 5/6, the 3 and the 5 are relatively prime. I therefore included both a/b × nb/c and a/b × c/d situations in the set of problems I designed for class discussion and assessment of students' thinking.

During instruction students explored a variety of situations involving multiplication of fractions, including multiplication of whole numbers and fractions, mixed numbers and fractions, two mixed numbers, and two fractions. Each day's instruction involved students in utilizing each of Lesh's five representational systems as they created solutions for the problems that I presented to them. The various problem situations were introduced via "real-life" or imaginary situations that were related to students' experiences. These included food, distances between home and school, comparing ages of family members, and the familiar story of *Alice's Adventures in Wonderland*.

Students utilized physical models such as fraction circles and counters to represent the situations and to find the results of the actions on the quantities. Most textbooks (for example, Macmillan, 1992) introduce multiplication of fractions via a rectangle partitioned and shaded in two directions to show each of the fractional factors and the product as the region in which the two shadings overlap (See Figure 4). Most textbooks present this as a model for multiplying a/b × c/d = ab/cd and then instruct students that it can be utilized to multiply whole numbers and mixed numerals by fractions, after changing the other numbers into improper fractions. Rather than attempt to represent the variety of problem situations listed in Table 1 by a single model, I suggested models to the students which afforded closer correspondences to the actions of the problems. For example, we used fraction circle pieces to represent Combining problem actions such as finding 6 servings each consisting of 3/5 of a pie. We used counters to represent Partitioning problems such as finding 3/4 of a set of 24 students.

Sometimes pictures served as physical models. For instance, although students could imagine Alice in Wonderland growing or shrinking or the enlarging or decreasing of a balloon, the numerical information in such situations is not part of the physical model. I, therefore, presented pictures of rulers on which units of length were clearly marked as models on which students could represent the results of shrinking or growing (See Figure 1). I also utilized diagrams showing distances in miles or fractional quantities of food such as 3/4 of a pan of lasagna or 5/6 of a carton of ice cream which students could partition in the process of finding a fractional part of the quantity. Pictures or drawings became manipulatable models when students added details to them to demonstrate mental or physical actions on the quantities in the problem. When students manipulated the quantities represented in such pictures, I considered them to be manipulatable models. Sometimes students created the drawings to represent the starting quantity and to demonstrate how the action of the problem transformed the starting quantity. At other times, they modified a given representation of 3/4 to show how they found 1/3 of it.

When students drew pictures to represent only the solution to a problem that they had solved by some other means, I considered the representation as a picture, a static representation of one of the quantities in the problem or of its solution. The Figures 2a and 2b described in the Introduction can be considered as pictures because students drew them as representations of the answers they had already found. The actions on pictures described by Lisa and Bill, however, served as manipulatable models for finding a solution.

Verbal representations included students' verbal descriptions of the action taking place in the problem, such as "he only has half as much" or "Alice is shrinking"; their verbal descriptions of their problem solutions, descriptions of discoveries or generalizations that they had discovered such as "to find a half of a fraction, double the denominator" or "to multiply a mixed numeral by a mixed numeral change both to improper fractions then multiply the numerators and the denominators." Verbal statements of ideas or concepts are also included in this category. Some of these were "When you multiply by a fraction, your answer is smaller because the fraction is less than one." "Multiplying by a fraction is the same as dividing by the denominator and then multiplying by the numerator; you're just doing both at the same time."

Written symbols include numerical expressions such as $4 \times 3/4 = 12/4 = 3$ that students wrote as they solved problems. Other representations in this mode include story problems that students wrote to go along with given multiplication expressions, and the discoveries that students wrote down as a record of the "rules" that they had generated while representing and solving problems.

Throughout the instruction I encouraged students to utilize each of the representational modes and to make connections between them. The vignette at the beginning of this paper illustrates the ways in which I asked students to relate their numerical methods for solving story problems to actions on the pictorial representations. When students represented situations and their solutions with models or pictures, I asked them to decide what sort of mathematical expression we could write as a record of the problem and its solution. I also asked them to describe verbally how the written expression related to the physical or pictorial model that they had utilized (and transformed). Not only are students' solutions and solution methods of interest, but this study also focuses on the ways in which students made connections between and among various kinds of representations.

As students had not had prior instruction in the multiplication of fractions, I sequenced the instruction to begin with situations involving whole number multipliers and fractional quantities before introducing fractional operators. Problems in which fractional operators multiplied whole number quantities were explored before problems involving fractional operators and fractional quantities. Topics were introduced in the following sequence:

1. Combining equal groups or sets of fractional quantities

   3 servings of pie if each serving is 3/4 of a pie (W × f)

2. Partitioning a set into equal size groups and connecting the whole number procedure of dividing by the denominator followed by multiplying by the numerator to multiplying by a fractional operator.

   Give 2/3 of 15 cupcakes to Mark (f x W). 2/3 × 15 accomplishes the same result as 15 ÷ 3 × 2.

3. Expanding or shrinking a quantity

   Alice is 54 inches tall, if she drinks from a bottle that makes her three times as tall how tall will she be then? (W × f)

   Alice is 54 inches tall, if she drinks from a bottle that makes her 2/3 as tall, how tall will she be then? (f × W)

4. Comparison of two quantities

Mark is 11. If his father is 4 times as old as Mark, how old is his father? (W × f)

Mark is 11.Mark's brother is 3/4 as old as Mark. How old is his brother? (f × W)

5. Shrinking a fractional quantity

Alice is 5/8 of an inch tall. If she drinks from a bottle that makes her 2/5 as tall, how tall will she be then? (f × f)

6. Partitioning a fractional quantity

There is 3/4 of a cake on the table. If your mother gives you 1/3 of it, how much of a cake does she give you? (f × f)

There is 3/4 of a pound of M&Ms in the bag. If Jill takes 1/2 the M&Ms in the bag to school, how many pounds of M&Ms does she take? (f × f)

There is 5/6 of a carton of ice-cream. If Jane, Jack, and Jim each eat 1/3 of the ice cream, how much ice cream does each person eat?

7. Comparing a larger fractional quantity to a smaller one.

Emma ran 3/4 of a mile. If Janice ran 2/3 as far as Emma, how far did Jill run?

In a previous study I had introduced an area model (suggested by Bezuk & Armstrong, 1992, Graeber, 1993) for students to use in finding the product of two fractions (Taber, in press). Although students had successfully utilized it to find the product of two fractions, I had since come to realize that the area model represents just the computation of two fractions. It is not closely linked to the problem actions of partitioning, shrinking, or comparing two quantities. Furthermore, as the two factors play similar roles in computing area, the concept of a fractional operator is not represented in the area model. As before, I planned to introduce the product of two fractions via a story in which an already very small Alice drinks a liquid that makes her even smaller. I would also have students explore the fractional operator in stories that involved partitioning fractional amounts of food. Based on Mack's discussion of how students partition quantities and of the ways in which relationships among the numbers affect their partitioning methods (Mack, in press), I decided to provide diagrams representing the starting quantities in the problems, encourage students to partition the diagrams, then develop multiplication of the two fractions as a way of symbolizing the actions they had performed in partitioning the models.

I also planned to begin with examples in which the fractional parts of the original quantity could be regrouped to find the resulting quantity-- for example 1/3 of 3/4, 2/3 of 3/4, 2/5 of 5/8-- before introducing situations in which the original whole would have to be repartitioned, as in for example 2/3 of 5/6.

Throughout the instruction and assessment activities, I looked for evidence of connections that students made within and among the various forms of representation. I wanted to find out how or whether using a representation from one system, for example a manipulatable model, helped students create and understand a representation in another system, such as written symbols. Rather than thinking of this process as "translation" (Lesh, et. al, 1987), I began to see it as a complex process of students' moving back and forth among representations as they developed and connected ideas. Each form of representation contains features that may be extraneous to the mathematical idea a student is trying to understand or may even conflict with the mathematical concept that the model or picture is expected to portray. Like a language, each system has its own conventions, idioms, strengths, and limitations that afford varying ways in which ideas can be expressed and communicated. An idea or concept cannot be translated point by point or feature by feature from one representation into another. By attending to the ways that students utilized various representations to form ideas and to make connections among ideas, I hoped to discover ways in which students employed the various systems and connected ideas in one system to ideas in another as they made meaning within the domain of multiplication by fractional operators.

Figure 5 depicts some of the mathematical ideas and representations and the pathways connecting them that students might express as they explore and solve various problem situations in the domain of multiplication of fractions.

The four kinds of problem actions that I included in the "real life" or imagined stories Combine, Compare, Change, and Partition are represented as diamonds, circles, pentagons, and squares respectively. The fact that the mathematical meaning of the situations is invariant even when the types of numbers in the problems are changed from whole numbers to fractions is expressed by representing the four possible Compare problem versions in a vertical arrangement, one over the next and doing the same with the four Change problem versions. (See Taber, 1991 and 1999 for a description of each of the four problem types).

Mathematical ideas are represented by ovals. These include ideas such as repeatedly adding or iterating a quantity, comparing a small quantity to a larger quantity, comparing a large quantity to a smaller quantity, expanding a quantity, shrinking a quantity, and partitioning a quantity. These ideas can exist without being attached to numerical referents. They describe actions that can be imagined or physically enacted to transform quantities. Notice that, for example, both Change problems in which whole number quantities are enlarged and in which fractional quantities are enlarged are connected to the idea of expansion. Interviews with students (See Taber, 1991, 1999, in press for details) reveal that students think of these problems as describing the same kind of action whether a whole number quantity or a fraction quantity was enlarged.

Other ideas include the commutative property of multiplication and the concept of a fraction as a composite operator--that multiplying by a fraction has the same effect as simultaneously dividing by the denominator and multiplying by the numerator. The commutative property of multiplication can provide a pathway for students to find the product of a fraction multiplier times a whole number quantity, for example, recognizing that $3/4 \times 24$ gives the same result as $24 \times 3/4$. When students develop an understanding of a fraction as a composite operator rather than as just a part of a whole, they are able to recognize situations involving finding 3/4 of a quantity as representable by 3/4 times the starting quantity -- instead of as the quantity divided by 4 and multiplied by 3.

The shaded rectangles in Figure 5 represent physical models, either manipulatable models or pictures that can be transformed to illustrate the ideas found in the ovals. Repeatedly adding or iterating a quantity can be physically represented by making multiple representations of a quantity and combining them. It can be represented symbolically as repeated addition. The concepts of expanding and making a comparison of a small quantity to a larger quantity are connected to the repeated addition physical model through the idea of repeated addition because the only way to quantify a physical representation of expansion or comparison is through a repeated addition model. Of course, these ideas can also be connected directly to the concept of multiplying. Students often state that the presence of the word "times" in verbal descriptions of Multiplicative

Change and Comparison situations lets them know that they should multiply to find the answer (See Taber, 1991, 1999)

The shaded rectangles in the lower half of Figure 5 represent two ways in which students may physically represent the action of partitioning quantities. Partitioning and grouping may involve physically separating a collection of counters into three equal groups and selecting two of them, making a sketch of objects organized into three equal sized groups and circling two of the groups, or selecting two of three parts in a representation of 3/8 of a circle. Mack (in press) and Taber (1991, 1999, in press) describe students using these kinds of representations to represent partitioning problems. When the denominator of one factor and the numerator of the other factor in a partitioning problem are relatively prime, students must repartition the starting quantity in order to obtain enough parts -- as, for example, in finding 1/2 of 5/8. To model this students must repartition each of the eighths in the original whole in order to determine the answer as 5/16 (See Mack, in press, and Taber, in press). In order to quantify situations involving comparing larger quantities to smaller quantities or shrinking quantities with a physical model, students often use partitioning representations and language.

It is not my purpose in this paper to thoroughly discuss each aspect of Figure 5. I have included it here as a means of showing how various concepts, actions, and understandings of multiplication, division, and multiplication of fractions may be connected via physical representations, pictures, language, story problems, and written symbols to each other in this complex domain of multiplication of fractions. In the study reported here, I focused on how students interpreted various kinds of problem situations (partitioning, multiplicative change, and multiplicative comparison) involving two fractions, and how they represented them with models, drawings, in spoken language and with written symbols.

Results

Instruction

The description of the class discussion on Day Nine at the beginning of this paper captures some of the ways in which students made sense of the meaning finding the product of two fractions. Prior to Day Eight students had solved a variety of problems that contained a whole number and a fraction or a whole number and a mixed number--with whole number and fractional operators. I had planned the five problems (See Figure 1) that students worked on in small groups as a preliminary exploration of situations in which a fractional part of a fractional quantity was to be determined. The two Alice problems were included to encourage students to make connections between Multiplicative Change situations involving a fraction and a whole number and those involving two fractions. I also wanted to see what methods students would use to find two-fifths of five-eighths. Would they notice that two fifths of five eighths could be found simply by selecting two of the shaded five eighths? Problem three was like those we had done before--partitioning a whole number quantity to find a fractional part of it. Problem four involved partitioning a fractional quantity to find half of it. The numerator 5 was relatively prime to the denominator of one-half so that students would have to repartition each of the five-eighths in order to find half of five-eighths.

As the students worked in groups to solve the problems on Day Eight I noticed that most of them were solving the problems by numerical means rather than utilizing the pictures in their solution methods. When I urged them to show how they could use the pictures to show how they found the answers, they partitioned the picture as an illustration of the answer they had already found. Examination of all their papers after class confirmed my observations during class.

All students had written a multiplication expression, either 5 × 1/4 or 1/4 × 5 to solve the first problem. For the "second way," 19 of the papers showed a version of 5 ÷ 4 = 1 1/4. Three students, all in the same group, utilized multiplication and marked 1 1/4 on the diagram of the five feet. On the second problem, the papers of the students in one group showed as their two solution methods 1) a subtraction of 2/5 from 5/8 and 2) the product of 2/5 and 5/8 along with a darker shading of two of the five-eighths on the diagram. All other papers from the class showed the product of 2/5 and 5/8 computed correctly along with the two-eighths shaded more darkly. The third problem also showed an illustration of the answer on the diagram rather than a solution. All students multiplied 16 × 2/3 correctly. Some papers also contained the equations, "16 ÷ 3 = 5 1/3; 5 1/3 × 2 = 10 2/3." All the diagrams showed the answer shaded in correctly, but only two of the papers indicated a partitioning of the diagram of the 16 miles (See Figure 6). One of my goals, therefore, for Day Nine was to engage students in a discussion of how the diagrams could be used to solve the problems and then to connect solutions that they found by physically partitioning the diagrams with the solutions they found numerically.

I was also surprised that students had invented a way to find the product of two fractions. Up to this point we had only utilized multiplication expressions with a whole number and a fraction. Although we had found the products of mixed numerals and fractions, students had done so by using the distributive property--that is they multiplied the whole number times the whole number part of the mixed numeral and the whole number times the fractional part, then added the two partial products together. The students' invention/adoption of a method for multiplying two fractions was a surprise to me, and I felt it was important that we work together to develop an understanding of the meaning of this operation.

I had also not planned to introduce the double partitioned rectangle as a model for multiplication of two fractions at this point of the instruction. I had planned to spend days 9 and 10, at least, exploring problems such as 1/3 of 3/10, 2/5 of 5/8, and 2/3 of 9/10, with the starting quantities represented by rectangles or circles. I hoped that students would develop an understanding of 1/3 of 3/10 as one of the three tenths by seeing that 1/3 of 3/10 meant one of the three parts in three tenths. I hoped they would make connections between their methods for finding 2/5 of 5/8 with their previously developed understanding of the meaning of and methods for finding 2/5 of 5 and connect 2/3 of 9/10 with finding 2/3 of 9.

I planned to introduce the double-partitioned rectangle later as a means of repartitioning the starting quantities in situations like 1/4 of 3/4 where the denominator of the first fraction and the numerator of the second fraction are relatively prime. The unexpected invention and use of the double-partitioned model for representing the product of two fractions on Day Nine indicated to me that this model can be meaningful for students and they can connect the physical representation to both a story involving partitioning a fractional quantity and to computing the product of two fractions. Indeed, we spent the next two days partitioning cakes and rectangles to represent parts of fraction quantities, computing the products of the two fractions, and discussing the answers obtained by the two methods. Students were quite comfortable with the idea that multiplying by a fraction decreased the size of the starting quantity. When I had introduced Alice in Wonderland and explained that the caterpillar had told her how to control her size, they quickly made the connection to multiplying by a number less than one to decrease a quantity. Students frequently referred to that idea both during instruction and during interviews.

Students' performance on the pre-test and post-test

Students solved the problems on the post-test much more successfully than they had on the pre-test. There were 11 or more correct answers to only two of the questions on the pre-test. Eleven of the 22 students correctly solved the following Cinderella problem, eight of them by dividing by the denominator and multiplying by the numerator, two by showing 2/3 = 20/30 and indicating that 20 was the correct answer, and one by drawing a diagram showing 20 as two of three groups with 10 in each group.

> Cinderella's stepsister was allowed to invite 30 guests to a party. But Cinderella was allowed to invite only $\frac{2}{3}$ as many guests as her stepsister. How many guests could Cinderella invite?

Thirteen students correctly solved this pre-test problem:

> There are 20 students in the class. Three-fourths ($\frac{3}{4}$) of them are 11 years old. How many students are 11 years old?

Five of the students with correct solutions to this problem divided 20 by 4 and multiplied the result by 3; one student divided 20 by 4 and subtracted the result from 20; two students showed 3/4 = 15/20 and circled the 15; two students made sketches showing 20 objects partitioned into four equal groups with three of the groups selected; and three students did not show how they solved the problem.

The only problem that no student was able to solve correctly on the pre-test was the following partitioning problem accompanied by the drawing seen in Figure 7.

> There is $\frac{5}{6}$ of a carton of ice-cream in the freezer. If Jill, Jack, and Jamie each eat $\frac{1}{3}$ of the ice cream, how much of a carton of ice-cream does each person eat?

On the post-test the number of students correctly solving each of the multiplication problems ranged between 17 and 22. On the post-test most students multiplied to solve the problems, whether the multiplier was a fraction or a whole number. Students rarely used drawings to illustrate problem solutions except when a drawing representing the beginning fractional quantity was included as part of the problem. Table 2 lists the four problems which included pictures and shows the number of correct solutions utilizing each of the strategies students used on these problems. Table 2 suggests that many students made connections between partitioning the pictures and the algorithm for multiplying fractions. Thirteen of the students partitioned the diagrams that accompanied the first of these two problems. On the last two problems 11 and 12 students, respectively, who created a correct picture of a rectangle partitioned by both denominators also correctly multiplied to obtain the answer. Table 2 further suggests that students who utilized both the algorithm and the picture were more apt to be correct than students who only partitioned the picture.

Some students' solutions to the two addition and two subtraction problems appearing on the post-test (each containing one whole number and one common fraction) suggest that some of students may have been hammering every problem on the post-test with their new tool: multiplication of fractions. Eight students multiplied to solve two of the four addition / subtraction problems; two students multiplied on three problems and three students multiplied on all four. In order to understand the changes in students' thinking over the course of this unit, it is therefore important to consider changes in the strategies of the interviewed students and the reasons they gave for solving problems as they did.

<u>Interviews</u>

I interviewed five students on three occasions: prior to the start of instruction, after the sixth day of instruction (prior to instruction on problems with two fraction factors), and a week after the last day of instruction. The representations that students created to solve two of the interview questions provide insights into their use of different representational systems in their solution processes.

A partitioning problem in which students were asked to find 2/3 of 3/8 appeared on both the pre-test and the interview unaccompanied by a diagram. None of the five interviewed students solved the following problem correctly on the pre-test:

> Mark's house is $\frac{3}{8}$ of a mile from school. If Mark ran $\frac{2}{3}$ of the way to school, how far did he run?

Three of the students omitted it, Lisa subtracted, and Hillary gave the operator, 2/3 as the answer.

On the post-test four of the students solved the problem correctly. Sasha drew a diagram partitioned into eight parts, shaded three of the parts and then bracketed two of the eighths and also shaded them darker, writing "2/8 of a mile" for her answer. TJ, Hillary, and Carol multiplied 2/3 times 3/8 to find the answer "6/24 = 1/4 mile." Lisa attempted to find the answer by subtracting.

On the first interview the problem appeared as:

There is 3/8 of a pound of M&Ms candy in the bag. Paul takes 2/3 of the candy to school. How much candy does he take to school?

Both Carol and Sasha drew diagrams to show 3/8 of a rectangle and then indicated that two of the eighths they had drawn and shaded represented the candy Paul had taken to school. TJ, Hillary, and Lisa attempted to subtract the two numbers. When Hillary and Lisa stated that they couldn't subtract because 16/24 is more than 9/24, I asked them if making a drawing would help them. Sasha made a drawing and then indicated that two of her shaded three-eighths represented the amount of candy Paul took to school.

By the time of the second interview, students had solved partitioning, multiplicative change, and multiplicative comparison problems with fraction operators and whole number quantities. To solve the same M&Ms problem as before three of the students, Carol, Sasha, and Tom drew rectangles, partitioned their rectangles into eighths, represented the three-eighths and then two of the three eighths as the answer. Lisa attempted to subtract as she had in the first interview, realized she was at an impasse, but did not draw a picture. The following excerpt from Hillary's interview shows how she was able to create a diagram representing the problem and then interpret her diagram to solve the problem.

S[tudent]: Two-thirds

I[nterviewer]: Two thirds of what?

S: The bag of candy.

I: Yes, but how much is in the bag of candy?

S: three-eighths.

I: Ok, so what would two-thirds of three-eighths be?

S: [attempts to solve the problem by writing 9/24 - 16/24] it won't work out

I: Why isn't that working out?

S: I can't take sixteen from nine.

I: Ok, because?

S: Nine is less than sixteen.

I: Ok. Let's try this. Show me what three-eighths of a pound of candy would look like.

S: [Draws a rectangle, partitions it into eight parts]

I: Ok, so how much would three-eighths be?

S: [shades three of the partitions]

I: All right, so how much of that three-eighths does she get?

S: Um, oh, two-thirds.

I: Ok, she gets two-thirds of that. So, what would that be?

S: [circles two of the shaded eighths]

I: Ok, so those two things right there?

S: Yes.

I: And how much of the whole pound is that?

S: 2, um two-eighths.

At the time of this interview, before any classroom discussion of problems with fraction quantities and fraction multipliers, four of the five interviewed students were able to find 2/3 of 3/8 by drawing, partitioning, and identifying the relevant parts of a diagram.

On the third interview (after the post-test) four of the students solved the problem by multiplying the two fractions together. When asked if they could "solve the problem another way," Carol said that she could not solve the problem another way and Hillary said that multiplication was the correct way to solve it "because subtraction would give the wrong answer." TJ and Sasha also drew diagrams to show 2/3 of 3/8 and connected their representations to their written solutions. As the following excerpt shows, TJ described the diagram prior to drawing it.

I: Could you find the answer another way?

S: If I had a picture I could, I guess.

I: Ok.

S: Because I would take, I would take two of, because there would be like three of whatever you have, and you just take two of them and that would be two-eighths, and you simplify that to one-fourth.

I: Oh, ok. Why don't you just draw me a little picture.

Sasha's explanation of why she multiplied also includes a verbalization of her understanding that multiplication by a fraction makes a quantity smaller.

S: 1/4 of the pound.

I: Ok, now tell me why you multiplied.

S: Because I, it told me that he took a certain amount from the bag, so I decided to multiply.

I: Ok. Why did you decide to multiply?

S: Because the numbers are less than a whole, so the answer could come out less.

I: Ok, would there be another way you could solve it?

S: You could try to subtract it, but then, I, um,

I: Ok, could you maybe draw a picture or something that would show the answer?

S: [Student draws a rectangle partitioned into eight parts with three shaded]

I: Ok, so you made 8 parts and you colored in three, right?

S: Mm hmm. If he takes two of the thirds,

I: Ok.

S: And that was two-eighths.

Students' solution strategies suggest that making a diagram enabled the students to represent and solve the problem prior to any classroom discussion of the problem or the introduction of the multiplication algorithm. In fact, the diagram especially helped Hillary in understanding the situation as "not being subtraction." The excerpts from the third interviews indicate some of the connections that Sasha and TJ formed between the pictures they drew, the multiplication algorithm, and the principle that multiplying by a fraction results in a smaller answer, and, in TJ's case, a mental representation of finding 2/3 of 3/8.

As discussed earlier, no student was able to solve the ice-cream problem [There is 5/6 of a carton of ice-cream in the freezer. If Jill, Jack and Jamie each eat 1/3 of the ice-cream, how much of a carton does each person eat?] on the pre-test, although 19 of the 22 did so successfully on the post-test. Analysis of the interview protocols provides evidence of changes in students' understanding of how to partition the diagram that was presented along with the written problem and of the connections they made between the diagram and multiplying 5/6 by 1/3.

On the first interview one student tried to repartition the diagram representing 5/6 of a carton of ice-cream into three equal portions (see Figure 8a) while another student partitioned the entire carton into three equal portions and stated that each would be one-third (see Figure 8b). The other three students considered each of the five sixths as a whole, stated that each person would receive one part plus part of other parts as indicated in the following excerpt from TJ's interview.

I: It's one-third of the five sixths.

S: Ok.

I: Just try to do it with the picture. Can you do it with the picture?

S: I can't do, I'm coming up with one and one-third of each one of these [indicating a sixth in the diagram].

I: Ok, one and one-third of each one of these. So, show me.

S: Or, it's one and two-thirds of each one.

I: Ok. Do you want to draw on it to show that? (See Figure 8c)

S: So, this would be one and [partitions the second sixth about two-thirds of the way up]

I: Ok, you're saying that's two-thirds?

S: And this is one-third.

I: Mmhmm

S: I'll do the 1 first, so right here would be the 1.

I: Ok.

S: And the other two-thirds. So there's two-thirds left in this block.

I: Right. Ok.

S: There's 1 and two thirds.

I: Ok, so that looks even to me like everybody's got exactly the same amount, right?

S: Uh huh.

I: And it's one and two-thirds of a what? Is it one and two-thirds of a carton? Or one and two-thirds of the. . .

S: Of the, of this [pointing to a sixth]

I: What's this part called?

S: Of one-sixth

Hillary, however, had more difficulty in correctly naming the parts that she made when repartitioning the sixths in the diagram.

S: [Partitions two of the sixths in the diagram into halves. See Figure8d] That's what one person would get. The other person would get this much. The last person would get, wait! This person would get that much and the last person would get that much and there would be left over .

I: Oh! There would be left over? Ok, so, I'm just trying to draw. So this would be person 1, person 2, and person 3 and then you would have that left over part [one-half of one-sixth]?

S: Mm hmm

I: Now, if they shared that, how much would each one get of that?

S: Oh, they would each get a little piece right there.

I: Ok. So how much of the whole carton is that? Can you figure that out?

S: five-sixths?

I: Well, five-sixths is all of this, but I want the number for one person's share.

S: Oh.

I: So how much is that?

S: Oh, that's just this? Or all of it?

I: No, just this part here.

S: Um, one-half

I: uh. How much of the whole carton is this part right here?

S: one-sixth

I: Ok, and how much would that part be?

S: Half of one-sixth.

I: Ok, do you know what half of one-sixth is?

S: Um, this?

I: Ok, that's not the way we usually write it, but I know what you mean. We'll just leave it as half of one-sixth, right? And then, how much is that little bit there?

S: A quarter of one-sixth.

I: Is it a quarter of one-sixth? Here's the sixth. Ok, so you want to call it a quarter of a sixth because? Why are you calling it a quarter of a sixth?

S: Because it can't be a half, because it's not half.

I: Ok, show me where the half of the sixth is.

S: Right there.

I: Right there? Right. Ok. So, that's less than a half, is that what you're thinking?

S: Yeah.

I: Ok.

At the time of the first interview four of the five interviewed students recognized the need to repartition the five-sixths into three equal parts, although none of them was able to name the resulting quantity as a fraction. Carol's diagram (Figure 8a) showed that she estimated a repartitioning of the five-sixths into three equal portions. Sasha, TJ, and Hillary each treated the sixths as wholes and obtained equal shares of the five-sixths which they named, more or less successfully as sixths and parts of sixths. While Carol's repartitioning of the five-sixths by estimation indicated that she understood the 5/6 of a carton as a quantity to be partitioned evenly, the other three students seemed to view each sixth as a separate whole.

By the time of the second interview each of the three students was able to understand the problem as asking them to name the amount equal to one-third of 5/6. Sasha, Hillary, and Lisa at first considered the sixths as wholes and named the results of their partitions as one and parts of the whole (sixth). Both Carol and TJ were able to name the result as five-eighteenths as shown in the following excerpts.

S (Carol): This one was hard.

I: Well, maybe you've got a better way to think about it this time than last time.

S: It's, um, it's five-sixths. Would you do five-sixths times one-third?

I: Well, try it and see. Would that help? Why did you? Just work it out and then we'll see what--

S: 5/6 times 1/3 [writes 5/6 × 1/3 = 5/18] It would be eighteenths.

I: Ok, now what made you think of multiplying?

S: Um

I: It was just an idea?

S: Yeah.

I: Ok, so let's see if this answer makes any sense. Can you see a way to relate what you did to the picture?

S: Um, that's 5/6

I: Mm hmm

I: Suppose you were just going to divide this into three parts evenly, ok? How could you do that?

S: This would be one and 3/4, and the other one would be 3/4, and that would be 1/2.

I: Ok, so that doesn't quite do it, does it?

S: No, um,

I: What if you tried to put each of those sixths into three equal parts?

S: Um, you could do this [partitions the entire "carton" with two vertical lines. See Figure 9a].

I: Ok, so what would one person's part be.

S: One person's would be this part, another person would be there, another person would be there.

I: Ok, so how much would that one person's part be?

S: five-sixths?

I: five what? How many parts are there altogether?

S: Eighteen

I: Ok

S: Then five-eighteenths.

Like Carol, TJ was able to repartition the diagram to show the new parts as parts of the whole carton, rather than as parts of a sixth. At one point his diagram became so confused that I drew a new sketch of the 5/6 of a carton and asked him to show what he was doing on the clean diagram.

I: [responding to TJ's solution] It's 1 and two-thirds of one-sixth. Because you're giving each person one-sixth and then two-thirds of another sixth. Is that right?

S [TJ]: Yes

S: Can I do the lines?

I: Sure you can do anything that helps you.

I: Ok, so let's see, you put a line here. And one up here.

S: Are they like eighteenths?

I: Show me how you would do it in eighteenths.

S: I would do 1/6 is 4/18.

I: Well, here, ok

S: four-eighteenths because instead of doing thirds into sixths, three times six is eighteen.

I: Ok, so you're changing it into eighteenths. Ok, let's draw a new picture. Ok, so here's the 5/6 again. Now how are you going to change it into eighteenths?

S: I would just draw two lines through each one (Figure 9b).

I: Well, how about if you made your lines go that way.

S: Oh yeah.

I: So, show me that way. Ok, now how much does each person get (Figure 9c)?

S: So it's actually 15. Five?

I: Five what?

S: Eighteenths

I: Ok, so show me on the picture.

S: That's one person, that's what one person gets [indicating five-eighteenths]

The protocols from the third interview provide evidence that students made connections between the algorithm and the picture. Two of the students initially solved the problem by multiplying and then drew pictures to show the effect of repartitioning the picture of the whole carton into three parts by making vertical lines and coordinating the five parts in each one-third share with the five-eighteenths they obtained by multiplying. Two students at first treated the sixths as wholes and gave the answer as either "one and two-thirds parts" or as "one-sixth plus half a sixth plus a little bit of a sixth." When asked for another way to find the solution, these two students multiplied $1/3 \times 5/6$ to obtain an answer of five-eighteenths, but they could understand that 5/18 was the same answer as 1 and 2/3 parts only after answering a series of questions asking them to rename the 2/3s (of a sixth) parts they had found as in the following excerpt:

S[asha]: That would be two-thirds, so they would get one and two-thirds

I: So, they would get one and two-thirds what?

S: One and two-thirds of a carton?

I: Well, if you're getting one and two-thirds of a carton, that means they're getting more than a whole carton.

S: They're getting one-third here and they're getting,

I: Ok

S: two-thirds here

I: Right. But what is this thing that you have shaded in?

S: It's, oh, it's one-sixth.

I: Ok so they're getting one-sixth right there, right?

S: Mm hmm And then they're getting two, um, they're getting half, they're getting two-thirds of one of them.

I: Two-thirds of a?

S: Of one-sixth.

I: Ok, so they're getting, write that down, what you said, ok, two-thirds of a sixth. Ok, plus one-sixth, right?

[After much discussion of repartitioning a sixth, Sasha states --]

S: So, it would be two-eighteenths.

I: Ok. Ok, so two-thirds of a sixth?

S: two-eighteenths

I: So, that's this plus, what?

S: one-sixth

I: Ok

S: You would times, add them.

I: Ok, if you're going to add them, two-eighteenths plus one-sixth, right?

S: It equals five-eighteenths.

I. Could you think of another way to get it?

S: You could try to multiply.

I: And what would happen if you multiplied?

S: You would get the same answer.

I: Same answer, ok. Do you think you could do it on the, show it on the picture a different way? I mean, the way you did,

S: You would just,

I: Ok, show me.

S: Put it into sixes,

I: Ok

S: And do,

I: Now wait! This is gone, this is gone. They don't get any of that, right?

S: Yeah, they'd get a whole of one, and then two-thirds of one.

This suggests that when students think of the new partition as whole parts and parts of the original parts, it may make more difficult for them to recognize that 5/18, the answer obtained by multiplying, is equivalent to the answer they obtained by a piece by piece partitioning, resulting in 1 and 2/3 sixths.

The path from "one and two-thirds parts" to "five-eighteenths" is not a smooth one for the students nor is it easily connected to the product of $1/3 \times 5/6$. Students who repartitioned the whole model by making vertical lines to indicate thirds found it much easier to relate the model to their multiplication equation than did those who partitioned the model sixth by sixth and stated the answer as one and two-thirds parts or even as one and two-thirds sixths. The kind of partitioning representation that students constructed had a strong influence on the kind of connection they were able to make to their symbolic representation as the product of two fractions.

It is possible that multiplying first to solve the problem facilitated making a diagram showing the result as five-eighteenths rather than as one and two-thirds parts for the two students who multiplied before drawing a diagram on the third interview. In other words, knowledge of how to symbolize a situation using mathematical symbols may help students construct a more useful and meaningful physical representation of the problem situation.

Discussion

This study addressed two important aspects of mathematics teaching and learning. The first is how students utilize various types of representations and how they make connections among different representations. The results of this study suggest that understanding the process of making connections among representations is of crucial importance. As students develop meaningful ideas, concepts and procedures in one system, those mental constructs are not transferred to another system as "translations" with the same content, but are formed in the context and subject to the affordances and limitations of the representational system. Although all the connections among students' representations and understanding in the five systems of physical representations, pictures, verbal language, written symbols, and "real-life" problem situations could not be specified or traced, this study indicates that attending to them and pushing students to articulate their understanding in different representational media is crucial for the development and communication of their mathematical understanding.

The second aspect is the mathematical domain of multiplication involving fractions. Students' misconceptions about these kinds of problems have been shown to be persistent over many years (Kouba, Zawojewski, & Strutchens, 1997). Examining students' solution methods in various representational systems for problems involving the multiplication of a fractional quantity by a fractional operator and attempting to tease out the connections among those representations suggests that students can develop the double partitioned model as a meaningful physical representation for the product of two fractions. Students' understanding of the model must not be taken for granted or assumed as to be a generalization that students can apply to all such problems. Ironically, Lisa, the girl who invented the double partitioning model on Day Nine of instruction, persisted in believing that problems like determining 2/3 of 3/8 were solved by subtracting. When she was pushed to make a diagram that showed the problem situation and the action of the problem, she could do so correctly, but she chose to solve problems using computation unless directed to use other methods. It is important that students create and transform representations with physical models (including manipulatable models and pictures), that they talk about their solution strategies and ideas as well as write about them, and especially that they be encouraged to examine how their representations in various formats are connected or not connected.

Finally, the diagram that I created (Figure 5) has given me new insights into the ways in which students' ideas about combining equal quantities, partitioning, comparing, and multiplicative change, the meaning of multiplication and division, and ideas about fractions are connected to the ways in which students represent their understanding in each of the representational systems.

References

Ball, D. (1993) Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, and T. A. Romberg (Eds.), Rational numbers: An integration of research (pp. 157-98). Hillsdale, NJ: Lawrence Erlbaum Associates.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis -- emphasis on the operator construct. In T. P. Carpenter, E. Fennema, and T. A. Romberg (Eds.), Rational numbers: An integration of research (pp. 13-47). Hillsdale, NJ: Lawrence Erlbaum Associates.

Bezeuk, N. S. & Armstrong, B. E. (1992) Understanding fraction multiplication. Mathematics Teacher, 85:729-33, 739-44.

Carroll, Lewis. Alice's Adventures in Wonderland. New York: D. Appleton & Co., 1866.

Dienes, Z. P. (1964). Building up mathematics .(second edition). London: Hutchinson

Graeber, A. O. (1993). Misconceptions about multiplication and division. Arithmetic Teacher, 408-411.

Greer, B. (1992). Multiplication and division as models of situations. In Douglas A. Grouws (Ed.), Handbook of Research on Mathematics Teaching and Learning (pp. 276-95). New York: Macmillan Publishing Company.

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel & J. Confrey (Eds.), Mulitplicative Structures (pp. 61-85). Albany, NY: SUNY Press.

Hiebert, J., & Behr, M. (1988). Introduction: Capturing the major themes. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle grades (pp. 1 - 18). Reston, VA: National Council of Teachers of Mathematics.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. J. Behr (Eds.), Number Concepts and Operations in the Middle Grades (pp. 162-181). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kouba, V. L., Zawojewski, J. S., & Strutchens, M. E. (1997). What do students know about numbers and operations? In P. A. Kenney & E. A. Silver (Eds.), Results from the Sixth Mathematics Assessment of the National Assessment of Educational Progress. (pp. 87-140).

Lamon, S. J. (1999). Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Lesh, R. (1979). Mathematical learning disabilities: Considerations for identification, diagnosis, and remediation. In R. A. Lesh, D. Mierkiewicz, and M. G. Kantowski (Eds.), Applied Mathematical Problem Solving. Columbus, OH: ERIC/SMEAC

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum Associates.

Mack, N.K. (in press). Building on informal knowledge in a complex content domain: Partitioning, units, and understanding multiplication of fractions. Journal for Research in Mathematics Education.

Macmillan/McGraw-Hill (1992). Mathematics in Action, Grade 5. New York: Macmillan/McGraw-Hill Publishing Company.

Resnick, L B. (1982). Syntax and semantics in learning to subtract. In T. Carpenter, J. Moser, & T. Romberg (Eds) Addition and subtraction: A cognitive perspective. Hillsdale, NJ: Lawrence Erlbaum Associates.

Streefland, L. (1991). Fractions in realistic mathematics education. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Taber, S. B. (1991). Students' understanding of multiplication by 1/N. Unpublished doctoral dissertation. The University of Delaware.

Taber, S. (1993, April). Cognitive obstacles in developing an understanding of multiplication of fractions. Presented at the Annual Meeting of the American Education Research Association, Atlanta, GA.

Taber, S. B. (1994, April). When worlds collide: Extending students' knowledge about partitioning to multiplication of fractions. Presented at the annual meeting of the American Education Research Association. New Orleans, LA.

Taber, S. B. (1999). Understanding multiplication with fractions; An analysis of problem features and students' strategies. Focus on learning problems in mathematics, 21 (2), 1-27.

Taber, S. B. (in press). Go ask Alice about multiplication of fractions. 2002 Yearbook: Making Sense of Fractions, Ratios, and Proportion. Reston, VA: National Council of Teachers of Mathematics.

Van de Walle, J. (1997) Elementary and Middle School Mathematics: Teaching Developmentally (third edition). New York: Longman.

Table 1

Extending Four Multiplication Problem Types to Include Fractional Operators and/or Quantities.

Problem Type	Whole Number Operator and Quantity	Whole Number Operator and Fractional Quantity	Fractional Operator and Whole Number Quantity	Fractional Operator and Fractional Quantity
Combine Equal Groups	**6**[1] trays of pies with 3 pies on each tray.	**6** trays of pies with 2/3 pie on each tray.		
Compare	Paul has 3 pies. Joe has **6** times as many pies as Paul.	Paul has 2/3 of a pie. Joe has **6** times as much pie as Paul.	Paul has 3 pies. Joe has **3/5** as much pie as Paul.	Paul has 2/3 of a pie. Joe has **3/5** as much pie as Paul.
Change	Alice is 3 feet tall. After eating the cake, she is **6** times as tall.	Alice is 2/3 of a foot tall. After eating the cake she is **6** times as tall.	Alice is 3 feet tall. After eating the cake, she is **3/5** as tall.	Alice is 2/3 of a foot tall. After eating the cake, she is **3/5** as tall.
Partitioning			Paul has 6 pies. He gives **3/5** of them to Joe.	Paul has 2/3 of a pie. He gives **3/5** of the pie to Joe.

[1]**bold** typeface indicates the factor that is the operator

Table 2
Students' Solution Strategies for Selected Problems from the Post-Test

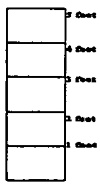
	Strategy					
Problem (Number of Correct Solutions)	Algorithm		Partitioned Picture[1]		Both	
	C	IC	C	IC	C	IC
There is 5/6 of a carton of ice-cream in the freezer. If Jill, Jack, and Jamie each eat 1/3 of the ice cream, how much of a carton of ice-cream does each person eat?(19)	6	0	9	3	4	0
There was 3/4 of a pan of lasagna left after dinner. If Mr. Jones took 1/2 of it for his lunch the next day, how much lasagna did he take? (18)	5	0	6	2	7	0
Shade the rectangle to solve this problem. Also write the answer. Mrs. Brown's yard has 3/5 acre of grass. She mowed 2/3 of the grass before dinner. How many acres of grass did she mow? (18)	0	0	7	3	12	0
Shade the rectangle to solve this problem. Also write the answer. Paul bought a bag of M&Ms. 1/3 of the M&Ms are red. He ate 3/8 of the red M&Ms and none of the others. How much of the bag of M&Ms did he eat? (20)	0	0	8	2	12	0

[1]The pictures for the ice cream and lasagna problems showed a rectangle with 5/6 and 3/4 shaded, respectively. The pictures for the grass and M&Ms problems were rectangles without any partitions.

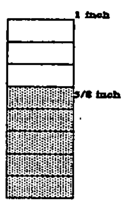
Figure Captions

1. Alice is 5 feet tall. Find out how tall she'll be if she eats a cake that makes her 1/4 as tall.

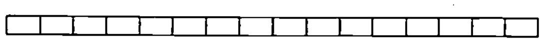
Show two ways to find out.



2. Alice is $\frac{5}{8}$ of an inch tall. She nibbles some of the mushroom, but it makes her shrink to $\frac{2}{5}$ as tall. How tall is Alice now? Show two ways to find out.

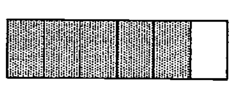


3. Mike bicycled 16 miles on Friday. On Saturday he bicycled just $\frac{2}{3}$ as far as he did on Friday. How far did he ride his bicycle on Saturday? Show two ways to find out.



16 miles

4. There was $\frac{5}{6}$ of a pound of candy in the cupboard. Millie took 1/2 of it to school. How much candy did Millie take to school? Show how to find the answer in two ways.



5. There was $\frac{5}{6}$ of a pound of candy in the cupboard. Jason took $\frac{1}{2}$ pound of the candy to school. How much candy was left? Show how to find the answer in two ways.

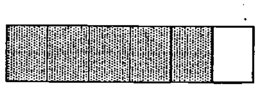
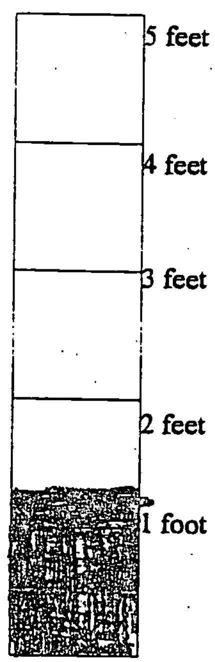


Figure 1: Problems for group work day 8.

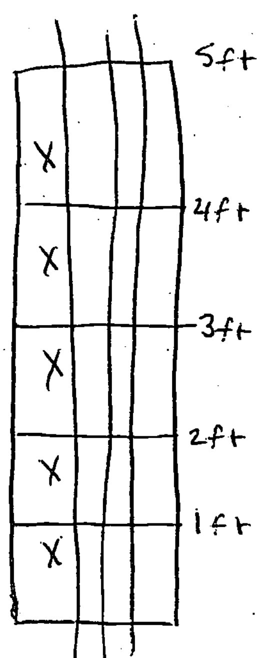
$5 \times \frac{1}{4} = \frac{5}{4} = 1\frac{1}{4}$ ft.

a



$5 \div 4 = 1\frac{1}{4} \times 1 = 1\frac{1}{4}$

b



c

Figure 2

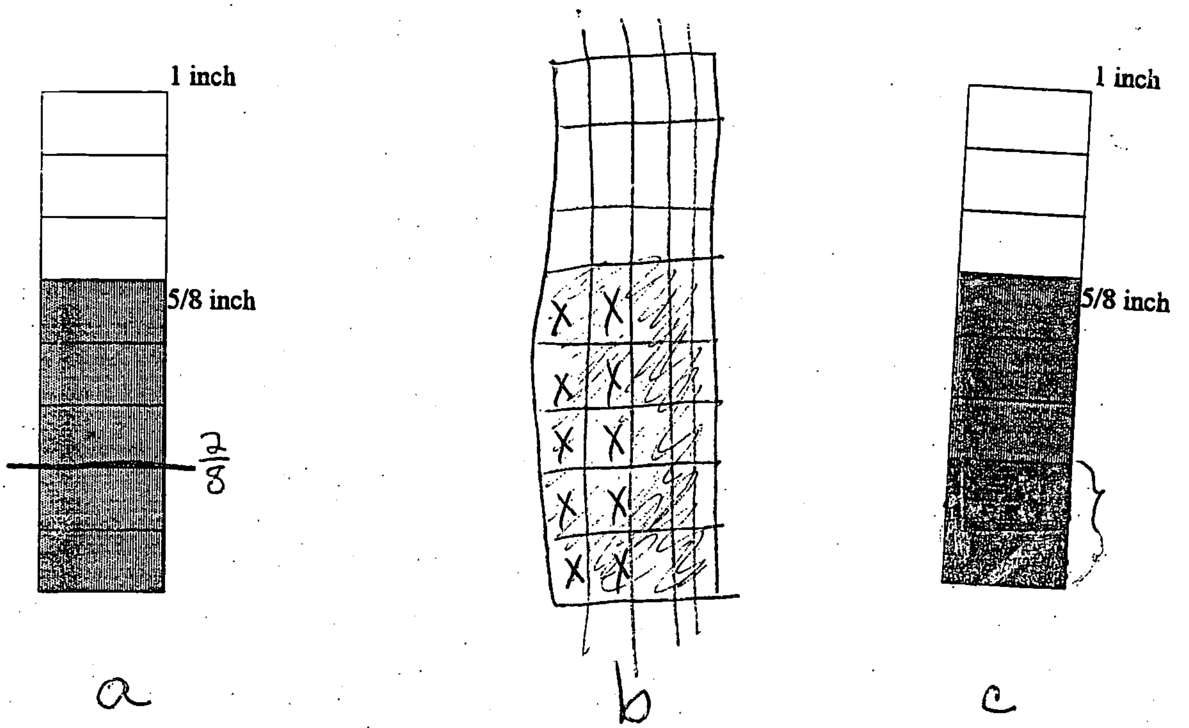
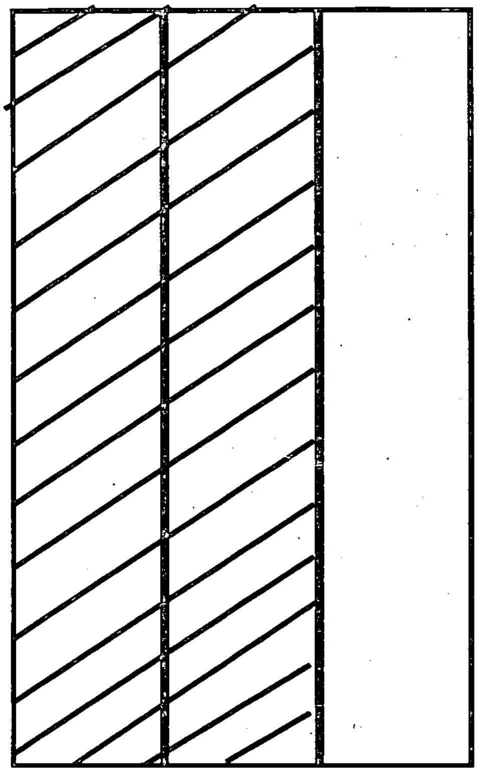
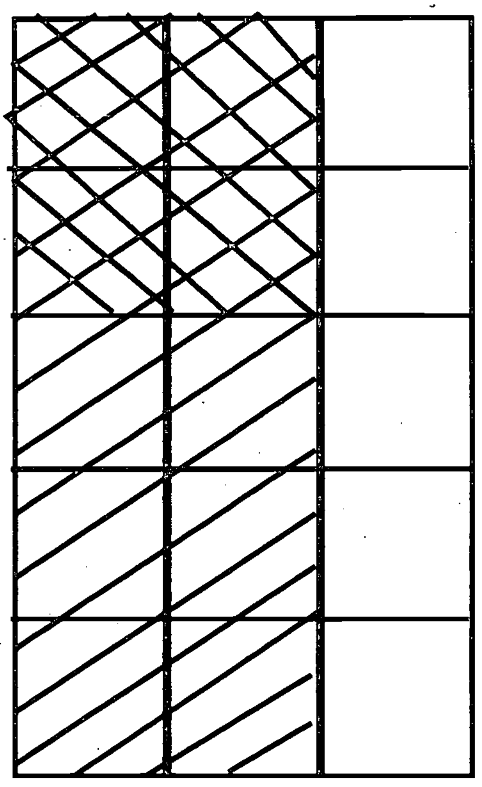


Figure 3



2/3



2/3 x 2/5 = 4/15

Figure 4

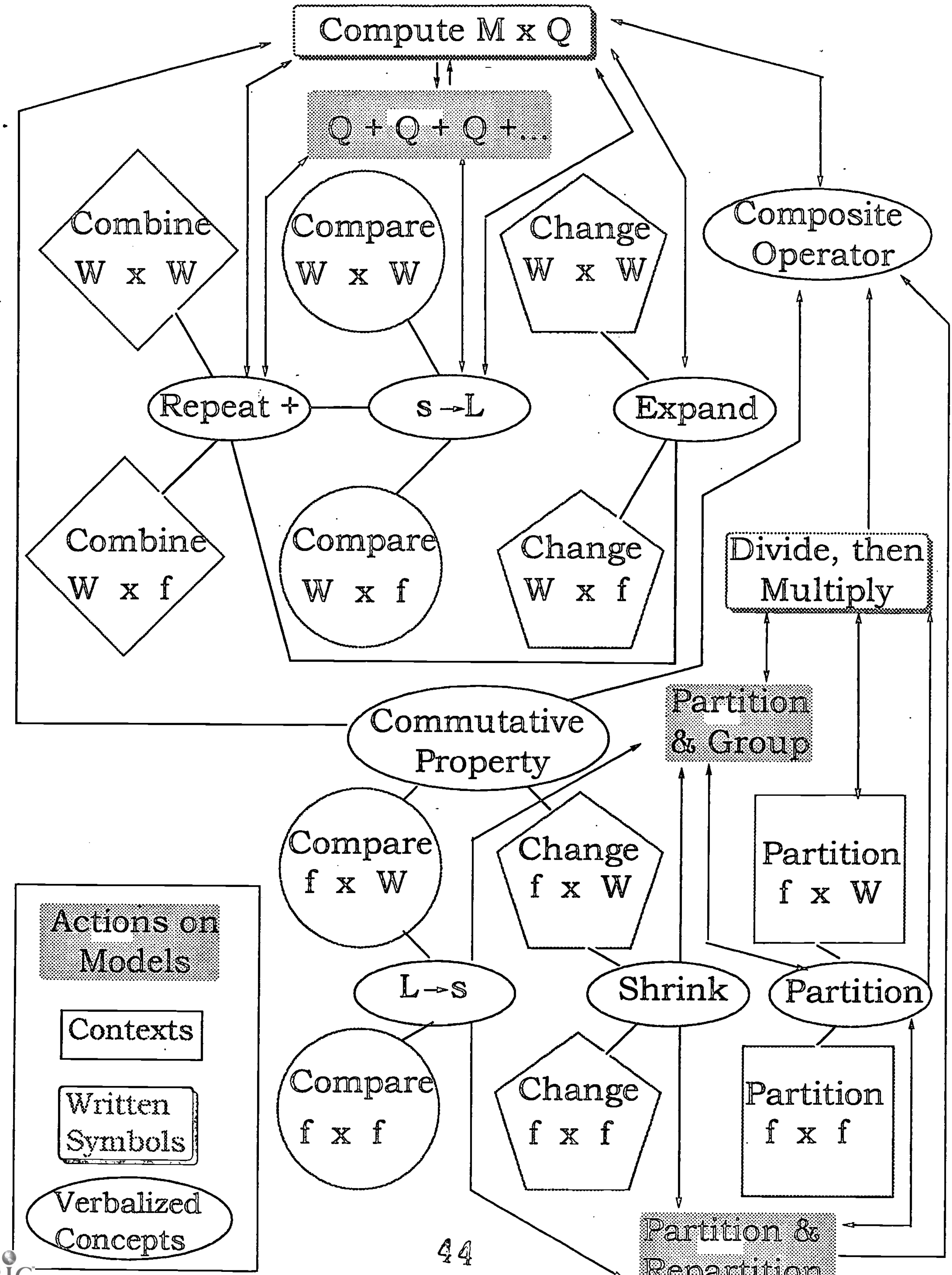
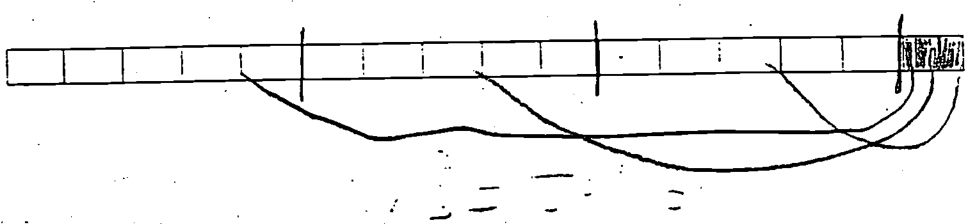

Compute M x Q
Q + Q + Q +...
Combine W x W
Compare W x W
Change W x W
Composite Operator
Repeat +
s → L
Expand
Combine W x f
Compare W x f
Change W x f
Divide, then Multiply
Commutative Property
Partition & Group
Compare f x W
Change f x W
Partition f x W
L → s
Shrink
Partition
Compare f x f
Change f x f
Partition f x f
Partition & Repartition
Actions on Models
Contexts
Written Symbols
Verbalized Concepts

Figure 5

3. Mike bicycled 16 miles on Friday. On Saturday he bicycled just $\frac{2}{3}$ as far as he did on Friday. How far did he ride his bicycle on Saturday? Show two ways to find out.



$16 \times \frac{2}{3} = \frac{32}{3} = 10\frac{2}{3}$ miles

Figure 6

10. There is $\frac{5}{6}$ of a carton of ice-cream in the freezer. If Jill, Jack, and Jamie each eat $\frac{1}{3}$ of the ice cream, how much of a carton of ice-cream does each person eat?

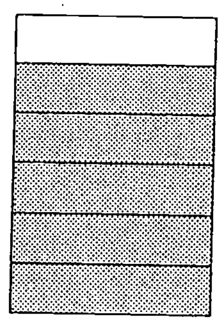


Figure 7

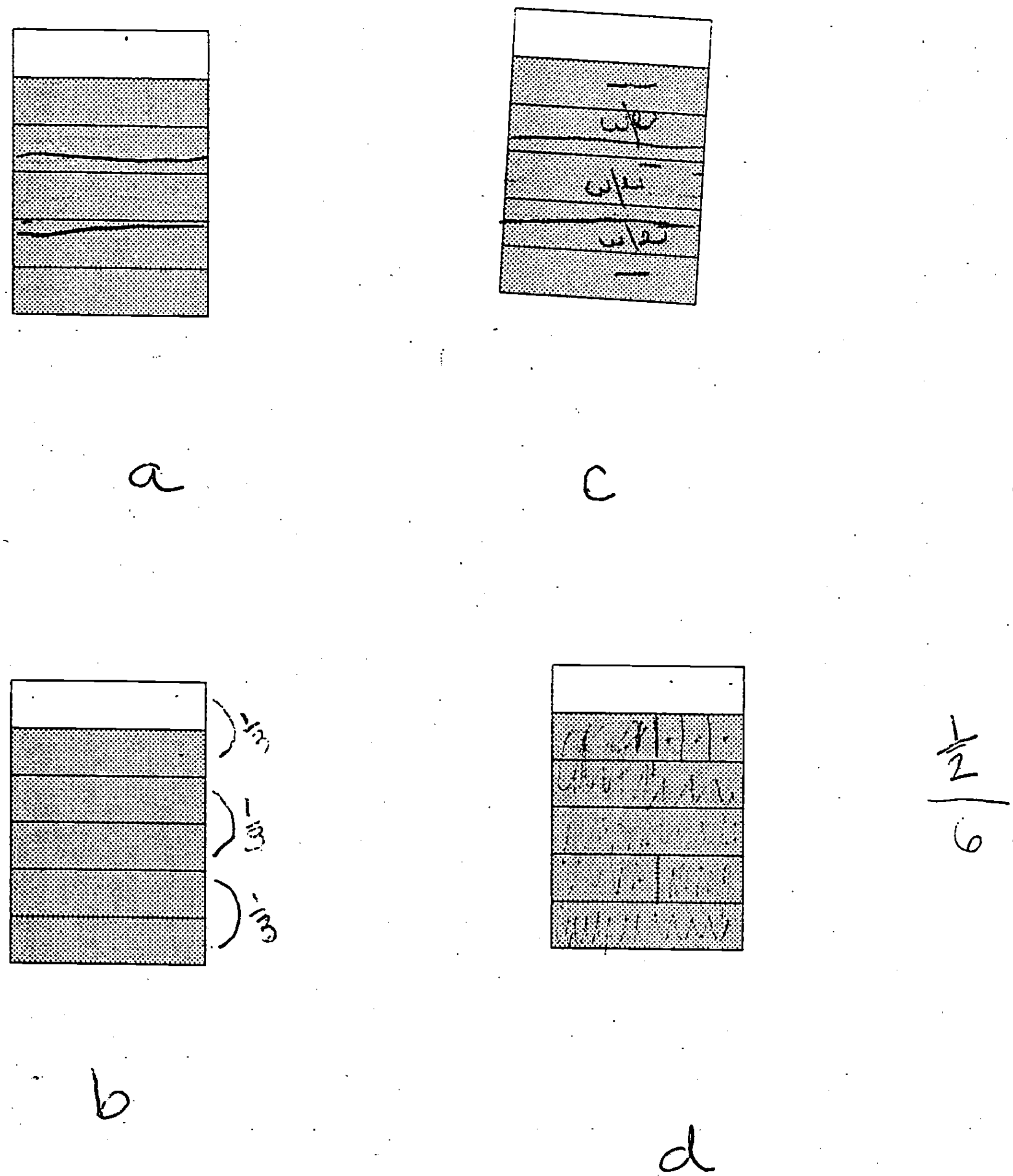
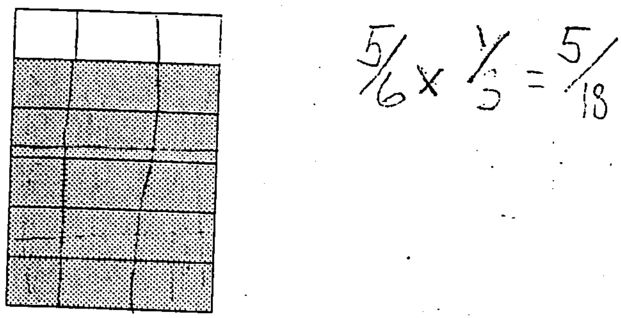
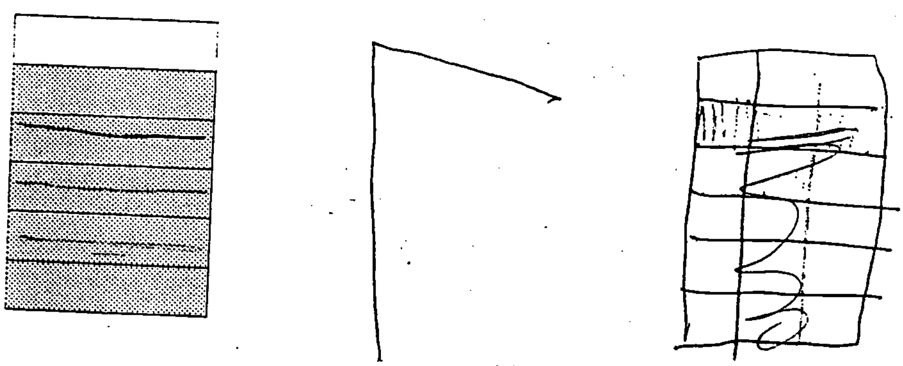


Figure 8



a



b

Figure 9

SE064908

ERIC®

U.S. Department of Education

Office of Educational Research and Improvement (OERI)
National Library of Education (NLE)
Educational Resources Information Center (ERIC)

REPRODUCTION RELEASE

(Specific Document)

I. DOCUMENT IDENTIFICATION:

Title: Making Connections Among Different Representations: The Case of Multiplication of Fractions

Author(s): Susan B. Taber

Corporate Source:	Publication Date:

II. REPRODUCTION RELEASE:

In order to disseminate as widely as possible timely and significant materials of interest to the educational community, documents announced in the monthly abstract journal of the ERIC system, *Resources in Education* (RIE), are usually made available to users in microfiche, reproduced paper copy, and electronic media, and sold through the ERIC Document Reproduction Service (EDRS). Credit is given to the source of each document, and, if reproduction release is granted, one of the following notices is affixed to the document.

If permission is granted to reproduce and disseminate the identified document, please CHECK ONE of the following three options and sign at the bottom of the page.

The sample sticker shown below will be affixed to all Level 1 documents	The sample sticker shown below will be affixed to all Level 2A documents	The sample sticker shown below will be affixed to all Level 2B documents
PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL HAS BEEN GRANTED BY ____ Sample ____ TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) 1	PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL IN MICROFICHE, AND IN ELECTRONIC MEDIA FOR ERIC COLLECTION SUBSCRIBERS ONLY, HAS BEEN GRANTED BY ____ Sample ____ TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) 2A	PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY ____ Sample ____ TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) 2B
Level 1 ☑	Level 2A ☐	Level 2B ☐
Check here for Level 1 release, permitting reproduction and dissemination in microfiche or other ERIC archival media (e.g., electronic) *and* paper copy.	Check here for Level 2A release, permitting reproduction and dissemination in microfiche and in electronic media for ERIC archival collection subscribers only	Check here for Level 2B release, permitting reproduction and dissemination in microfiche only

Documents will be processed as indicated provided reproduction quality permits.
If permission to reproduce is granted, but no box is checked, documents will be processed at Level 1.

I hereby grant to the Educational Resources Information Center (ERIC) nonexclusive permission to reproduce and disseminate this document as indicated above. Reproduction from the ERIC microfiche or electronic media by persons other than ERIC employees and its system contractors requires permission from the copyright holder. Exception is made for non-profit reproduction by libraries and other service agencies to satisfy information needs of educators in response to discrete inquiries.

Sign here, please

Signature: Susan B. Taber	Printed Name/Position/Title: Susan B. Taber / Asst. Professor	
Organization/Address: Dept of Elementary Education, Rowan U. 201 Mullica Hill Rd Glassboro, NJ 08028	Telephone: 856-256-4500 x3812	FAX:
	E-Mail Address: taber@rowan.edu	Date: 4/12/01

(over)

III. DOCUMENT AVAILABILITY INFORMATION (FROM NON-ERIC SOURCE):

If permission to reproduce is not granted to ERIC, *or*, if you wish ERIC to cite the availability of the document from another source, please provide the following information regarding the availability of the document. (ERIC will not announce a document unless it is publicly available, and a dependable source can be specified. Contributors should also be aware that ERIC selection criteria are significantly more stringent for documents that cannot be made available through EDRS.)

Publisher/Distributor:
Address:
Price:

IV. REFERRAL OF ERIC TO COPYRIGHT/REPRODUCTION RIGHTS HOLDER:

If the right to grant this reproduction release is held by someone other than the addressee, please provide the appropriate name and address:

Name:
Address:

V. WHERE TO SEND THIS FORM:

Send this form to the following ERIC Clearinghouse:

University of Maryland
ERIC Clearinghouse on Assessment and Evaluation
1129 Shriver Laboratory
College Park, MD 20742
Attn: Acquisitions

However, if solicited by the ERIC Facility, or if making an unsolicited contribution to ERIC, return this form (and the document being contributed) to:

ERIC Processing and Reference Facility
1100 West Street, 2nd Floor
Laurel, Maryland 20707-3598

Telephone: 301-497-4080
Toll Free: 800-799-3742
FAX: 301-953-0263
e-mail: ericfac@inet.ed.gov
WWW: http://ericfac.piccard.csc.com

EFF-088 (Rev. 9/97)