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ABSTRACT

The rationale of this study is based on the premise that the outcomes of children's learning are specifically shaped by the interaction of their individual cognitive structures with the presentations of curricular content in the classroom. Therefore, since the interpretive tools that children apply to instructional content may vary substantially, even within the same classroom, the representations they construct about curricular content are unlikely to have uniform meaning. Moreover, some of these representations may not necessarily be synchronous with the conceptual learning goals intended by curriculum developers. It falls to teachers, then, to be cognizant of these potential pupil variations in order to adapt curriculum to take into account the ways in which the same seemingly objective content will be interpreted to mean different things to students with different preconceptions. This study examined the ways in which first grade children's spontaneous concepts informed their understanding of school-taught ideas at addressed elements of algebraic reasoning. In particular, the study assessed children's responses to written activities in the school's developmentally sequenced spiraling curriculum as a function of differences in their previous knowledge of equivalence and related concepts. It was predicted that despite common instructional opportunities designed to "draw on the children's rich store of mathematical understanding and information," learning outcomes and modes of representing that curricular content would not be uniform. Rather, they would vary with the quality of spontaneous concepts individual children brought to the learning situation. This study suggests that first grade students with immature spontaneous notions about number tend to attribute static, specific, non-quantitative linguistic meaning to relational concepts in the mathematics curriculum. (ASK)

First Graders' Responses to School-Taught Mathematics as a Function of Their Spontaneous Conceptions of Basic Algebraic Relationships

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The rationale for this study is based on the premise that the outcomes of children's learning are specifically shaped by the interaction of their individual cognitive structures with the presentations of curricular content in the classroom. Therefore, since the interpretive tools that children apply to instructional content may vary substantially even within the same classroom, the representations they construct about curricular content are unlikely to have uniform meaning. Moreover, some of these representations may not necessarily be synchronous with the conceptual learning goals intended by curriculum developers. It falls to teachers, then, to be cognizant of these the potential pupil variations in order to adapt curriculum to take into account the ways in which the same seemingly objective content will be interpreted to mean different things to students with different preconceptions.

The intent of this study was to examine the ways in which first grade children's spontaneous concepts informed their understanding of school-taught ideas dealing with elements of algebraic reasoning. In particular, the study assessed children's responses to written activities in the school's developmentally sequenced spiraling curriculum as a function of differences in their previous knowledge of equivalence and related concepts. It was predicted that despite common instructional opportunities designed to "draw on the child's 'rich store of mathematical understanding and information'" (Fraivillig, Murphy, & Fuson, 1999, p.150), learning outcomes and modes of representing that curricular content would not be uniform. Rather it was predicted that they would vary with the quality of spontaneous concepts individual children brought to the learning situation.

Theoretical Framework

The broad theoretical framework guiding this research addresses questions about the extent to which first graders' spontaneous notions of equivalence and related concepts are characterized by mature or immature logical reasoning structures. This line of investigation emanates from Piaget's (1965) cognitive-structuralist stage theory and rests on the importance of the development of concrete operational thinking and the appreciation of conservation of quantity and the coordination of perspectives as the basis for understanding mathematical expressions of equivalence. It examines the extent to which underlying immature concepts might guide children's interpretations of school-taught conventional procedures so that they assimilate new information to old structures to build the foundation for inaccurate conceptual understanding. The study also relies on the distinctions that Vygotsky's (1986) socio-cultural theory makes about the initial separation of spontaneous and scientific thought in young children as part of the developmental process. In this view, successful learning experiences need to provide children with enough, but not too much new information, that can serve to transform spontaneous concepts into conventional culturally accepted concepts. A basic assumption of this study, therefore, was that some children's spontaneous concepts were not well matched with the starting points of instruction in the formal mathematics curriculum and that, therefore, effective scaffolding during instruction would not occur. As a consequence, assimilative distortions would take place and these children would develop idiosyncratic rather than conventional ideas about school-taught concepts and procedures.

Specifically, in the present study, it was expected that children would develop very different representations of first grade mathematics curricular content as a function of whether or not they had fully developed concepts about logical principles related to reversibility, compensation, equivalence, and appreciation of relative magnitude that underlie the development of concrete operational thinking. It was predicted that, very much like young children who deploy gestures while counting for a labeling rather than for a quantitative function (Kaplan, 1985) cognitively immature first graders would tend to attribute static, specific, non-quantitative linguistic meanings to relational concepts in the mathematics curriculum. Thus, it was hypothesized that because they were not using the more abstract levels of thinking that the some tasks demanded, their representations of the tasks would confine themselves to one way solutions processes, often employing counting on by ones, as the basis for their answers.

Methods of Inquiry

Previous investigations with somewhat older children on the effects of the curriculum used in this study, suggest that children who learn with it do not perform less well than students who learn with more traditional approaches and may even do better in some areas. These investigations, however, were based primarily on standardized test results that did not consider the nature of conceptual knowledge as much as the correctness of answers through which reasoning may be inferred (Fraivillig, et al, 1999). On a smaller scale, the present study attempted to probe the thinking processes of participating children in addition to the answer outcomes of those processes. It did this through a series of individual interviews with children that examined both spontaneous notions of equivalence and nonequivalence and the ways in which the children translated these ideas into applications in the school curriculum.

The participants of this study were 12 first grade students from one class who were interviewed at the mid-point and at the end of the school year. The children who were still in the school at the mid-point of second grade were briefly interviewed again on related second-grade curriculum content. **(Note that this presentation addresses only the data from the first grade interviews.)** The children attended a middle-class suburban community school composed of primarily Caucasian families. Participants included 8 boys and 4 girls ranging in age from 6-6 to 7-6 at the beginning of the study. Six other students in the class who were labeled as in need of resource room or special education classes were not included in the sample. The school district used the Everyday Mathematics curriculum (University of Chicago School Mathematics Project, 1995) in grades K - 4.

During first grade, the children were seen individually in their classroom and were interviewed for about 2 hours spread out over 2-4 sessions depending upon the ability of each child to remain on-task. All interviews were videotaped. In the first set of interviews the children were asked to do a series of informal tasks including oral counting, counting of objects, some conservation tasks, combining small numbers mentally, and predicting how to make a scale balance. In the second interview, each child was asked to do some curricular worksheet problems involving fact families, figuring out number patterns, writing numbers in relation to place value conventions, predicting a number outcome based on applying an addition or subtraction rule, and constructing and representing word problems using number fact families. A list of all tasks used appears in Figure 1.

The Data

The data for this presentation consisted of the videotaped interviews and written work of the children in first grade. Both formats were scored for accuracy of response and coded for strategies used to obtain answers. Subsequently, particular informal and formal tasks items were matched for underlying logical principles of compensation, reversibility, and equivalence needed for sensible responding (see Figure 2). In addition, a case study examination was conducted in which patterns of responses were analyzed across tasks for individual children. One of these cases will be presented today.

With the exception of the seriation and class inclusion tasks, all other tasks had between 4 and 20 parts to them with each part scored on a scale that ranged between 0 and 4 in most cases. For the most part, a score of 4 indicated that the child was correct and correctly used a logical process while a score of 0 indicated that the child's answer was incorrect. Incorrect answers that also contained a clear distortion of logic (e.g., overgeneralization of the commutativity principle to include subtraction such as 7 - 10 = 3 because 10 - 7 = 3) were scored as -1. Scores in-between reflected corrected responses after probing or correct responses based on the execution of mechanical routines without demonstration of an appreciation of quantitative relationships (e.g., finding equivalent values for a series of combinations on either side of an equals sign by using a down one, up one pattern in the rows instead of determining the equality of values on either side of the equals sign). Total possible scores on all informal tasks could range from -26 to 192, with actual informal scores ranging from 86 to 173. Total possible scores on the formal curricular tasks could range from -26 to 342., with actual scores ranging from 150 to 321.

<u>Quantitative Results</u>

As indicated in Table 1, there was a significant and substantial correlation between overall scores on the informal tasks and formal tasks ($r(10) = 0.896$, $p < .001$). A more detailed examination of the correlations between each informal task and its structurally related formal curriculum tasks indicated a bit more clearly where the strength of the overall correlation was situation. As indicated in Table 2, we see that successful performance on six of the formal tasks was associated with more highly developed spontaneous concepts. In particular, working with number fact families, doing the Frames and Arrows task, and using equations and concrete models to represent quantitative relationships in simple word problems were consistently positively associated with more fully developed spontaneous concepts. Valuing and comparing coins, using a Function Machine model, and renaming numbers were inconsistently associated with the development of spontaneous concepts. Performance on three other formal tasks showed no relationship to the children's development of spontaneous concepts with anticipated structural links. These tasks involved completing a written series of numbers that increased or decreased by ones, recording numbers in designated place value positions, and creating sensible stories using three numbers in a number fact family.

The results of all these correlations, though, whether significant or not, do not address the important issue of how specific forms of representation on informal tasks inform the reasoning behind the answers obtained on the formal curricular tasks. Therefore, an even more refined examination was made between performance on specific items within the informal tasks and their relationships to particular outcomes on specific formal curriculum tasks on a case by case basis. These findings focused on particular kinds of mistakes and reasons that students shared in their interviews and that could be observed in the videotapes or in their written work. Specifically applications (or lack of applications) of the dynamic principles of reversibility, compensation, determination of equivalence, and relative magnitude were reviewed in detail as they were manifested first in children's spontaneous concepts revealed on the informal task items and then in relation to the matching formal concepts. This presentation will focus on one such case analysis.

The Case of Karen: Spontaneous Concepts

Karen was 6 ½ years of age when first interviewed. She was slow and careful in responding to questions and used counting on or counting down with her fingers as her primary problem solving strategies. Her performance on the informal tasks indicated that she had not yet fully formed structures for understanding the reversibility of operations, the idea that a change in one aspect accompanied a reciprocal change in a related aspect (i.e., compensation), or that equivalence was not determined by appearance but by underlying structural similarities of a quantitative nature.

Conservation. On the conservation tasks Karen was able to judge two initially equivalent arrays as remaining equivalent when one was spread out. Her justification, however, was simply that they were the same because "it was spread out." This justification did not involve any expression of the idea that movement back to the original position (reversibility) was the basis of equality or that the more spread out array was compensated for by the greater density in the other array. Thus, Karen's response, which was actually typical of almost all children in the class, showed that she was not concentrating on two properties at the same time.

This one-sided approach to the task was even more apparent in Karen's responses to the conservation of substance and liquid tasks. On these tasks, she lost sight of the quantitative equivalence of the materials after one part of the display underwent a physical transformation that changed its appearance but not its quantitative value. As seen in Figure 3, her initial judgment of equivalence was easily challenged as Karen asserted that the amounts did not remain the same after the transformation occurred. What would this mean in terms of Karen's applications of equivalence concepts to exercises using cardinality, number equations, fact families, or the function machine in the curriculum?

Class inclusion. When shown two groups of plastic circle chips, 8 blue and 4 white, and asked which was more, the plastic or the blue circles, Karen quickly judged the blue to be more. As seen in Figure 4, when questioned about how many plastic chips there were, she said "4" (actually the number of white chips). However, when asked specifically to count the plastic chips, she did count all the chips and obtained "12" as her answer. It was only at that point that Karen decided that there were more plastic chips than blue chips. Her initial response, however, demonstrated that on her own, she could not focus her attention on two dimensions at the same time and that she lost sight of the larger category (plastic) when she focused on a subcategory (color). What would this mean in terms of Karen's understanding of place value, fact families, or money value comparisons in the curriculum?

Balancing a scale. Karen was given a series of questions related to making a pan-scale balance by adjusting the weights on one side to match the weights on the other side. Each piece was of equivalent weight so that 6 pieces would balance 6 pieces and so the task was one about number. When given the chance to place additional pieces on the lighter side to balance the scale, the task posed no difficulty for Karen. By counting up from the smaller value until she reached the larger value, she was able to adjust the scale to balance 3 and 5, 5 and 7, and 3 and 6. However, when the task was complicated so that the balance had to be obtained within the set of weights that were presented (e.g., 8 and 4 - balance without adding any extra pieces, but just by moving them around from one side to the other), Karen's first try was to take 4 from the 8 and add it to the 4. Thus, as seen in Figure 5, she focused only on one side, the side she wanted to increase, by creating another 8 on the opposite side. After making this shift, however, she was surprised to see that she now had 8 on one side but only 4 on the other side. She then took off the 4 pieces that had tipped the balance, leaving 4 on each side and 4 in her hand. Then she split her 4 into two parts equal parts and then put 2 additional pieces back into each pan. She saw it balanced. When asked how many pieces were on each side, she then counted up the pieces on one side and said that she had 6 and 6. From her strategies, we can see that her initial goal was not to make 6 and 6, but just to reduce the amount of "damage" done by removing the 4 pieces that had tipped the scale in the other direction. In this way, Karen could remain focused on a one-sided effect rather than the relational aspect of considering the loss and gain simultaneously. Thus, she essentially used an adding-on trial and error approach until she got it right. What would this mean for Karen's appreciation of number fact families, word problems involving basic addition and subtraction relationships, and the function machine exercises in the curriculum?

Counting objects. Karen was presented with 30 cubes and asked to count them. Her first count was by ones and accurate. As shown in Figure 6, when asked to count them by twos, she did take two cubes at a time and said the first few numbers from memory (2, 4, 6, 8, 10, 12). Then she proceeded to count numbers in her head essentially attempting to skip the appropriate number of names between each utterance. This worked until she got to number 20. After that she skipped over to 24 and then went on with 26, and 28. After that she jumped to 33 and then said 35. When asked how many cubes there were counting by twos, Karen said there were 35. Her skip counting by fives and tens was accurate and used appropriate number names. Nevertheless, she accepted that there were 30 cubes if counted by ones, fives, or tens, but that there were 35 if counted by twos. What would this mean for Karen's understanding of the Frames and Arrows routine, for her sense of cardinality in dealing with number fact families, and her ability to apply skip counting to coin values found in the curriculum?

Mental arithmetic for orally presented values. Karen was presented with two single digit numbers at a time and asked either "How much are # and #?" or "How much is # take away #?" The presentation of the numbers was ordered so that two doubles combinations preceded the combination between the doubles. The combination between the doubles was presented in AB and BA order. The doubles were expected to be memorized and so based on that knowledge, the question about the between values was intended to assess if the child could adjust or derive new quantitative information from the doubles fact, i.e., a form of compensation. After the addition combinations were completed, the sum of the combinations was presented in terms of a subtraction counterpart for that combination. The subtraction variant was used to assess whether the child understood the nature of inverse operations, i.e., reversibility. As shown in Figure 7, Karen was asked "4 and 4" followed by "5 and 5." She immediately and correctly answered "8" and "10." I then reiterated that "4 and 4 was 8; 5 and 5 was 10; so how much is 4 and 5?" To this, Karen responded by counting on her fingers from 5, saying ;"6, 7, 8, 9. So 9." After this, I again reiterated the previous series of numbers combinations and their answers, adding, "so how much is 5 and 4?" Karen immediately responded with "9" and said she knew this because it was a turnaround fact. The combinations were repeated and next Karen was asked, "How much is 9 take away 5?" For this she began to count down from 9 using her fingers and saying "9, 8, 7, 6, 5, 4, 3. Three?" I did not correct her, but reiterated the combinations as she stated them and then asked her, "How much is 9 take away 4?" For this she again counted down using her fingers and saying, "9...8, 7, 6, 5. Five?" From her responses it was clear that the numbers were not seen as related to one another, but rather each combination was seen as an occasion to use counting up or down, usually accurately, but sometimes not. The logic of the quantitative relationships in terms of compensation and reversibility was not behind Karen's actions. Her mention of the turnaround fact was no more than a syntactical rule that really meant to her that it didn't matter which way you said it, but that you can count on from the larger of the two numbers regardless of which number was said first. Based on this conception, what would number values, numerical equations, and function machine exercises in the curriculum mean to Karen?

The Case of Karen: The Relationship of Spontaneous Concepts to Curricular Tasks

Karen's one-way and limited spontaneous responses on the informal tasks were next explored for evidence of how they might be informing her judgments and approaches to curricular tasks with similar underlying structures. It was noted that some of the curricular tasks did not rely on such structures and on these Karen performed consistently well. For example, she had no trouble place numerals in the correct place value position based on directions to, "Write a 3-digit number with 6 in the hundreds place, 7 in the tens place, and 8 in the ones place." For such items Karen knew which position each term referred to and was performed as a rote skill, much like reciting the fives and tens skip counting lists. She also understood that smaller numbers would require larger numerals to be positioned in the ones place and smaller numerals to be positioned in the hundreds place whereas larger numbers would follow the opposite pattern. Placement of digits, however, does not imply understanding of their relative quantitative values any more than recitation of number names infers understanding of cardinality.

Frames and arrows exercises. The frames and arrows curricular exercises require the child to identify and continue a pattern of addition or subtraction of a constant value in a series of numbers. To perform this task effectively, the child needs to have a sense of number as part of a series in a two-way relationship such that for any given number, the value before the that place follows the rule in one direction and the number after that place follows the rule in the opposite direction. For example, in a simple pattern where "+2" is the rule and the list is given as, " 24, 26, 28, 30, □," the child should be able to recognize that 30 represents 2 more than 28, and 2 less than the number following it. However, in fact, to fill in this blank, the child needs only to count on 2 from 30 to get 32. A child can perform this level of the frames and arrows exercises without having to invoke any two-way logical principles. As shown in Figure 8, Karen, in fact, was able to do this type of format successfully. However, when the patterns required some element of appreciation of reversibility or compensation, Karen's performance retained its one-way structure. Thus when Karen had to use the rule "count up by 10s" in a series that began with the number 8, she could just count up to get the next number (i.e., 18). Then using her rote knowledge of place value, she was able to pick out a pattern in which the tens place increased by 1 for each continuing number. So Karen could errorlessly write, (8), 18, 28, 38, 48. However, when the rule just below this one was "count down by 10s," (see Figure 8a) she could not apply this strategy. That was because in order to advance, she also had to move backwards at the same time. In this case the number given was 56 with four blanks following and arrows pointing forward. To get the correct answer, Karen needed to track the tens place digit backward while moving forward. This proved to be too much to juggle for Karen and so she just wrote, (56), 55, 54, 53, 52. Apparently going down by ones was rote enough for her to move forward and backward at the same time. The fact is, though, that she did not understand the nature of the operation that was required and so assumed that her approach was fine. The question really is, then, what does this teach Karen about the meaning of counting by 10s backwards? Does this have the same meaning for her as a child who is not limited to one-way functions?

<u>Expressing cardinal values by different number names</u>. On another task was one in which Karen was asked to write several cardinal values in different ways as for example, list five different names for "10." A sample of 5 + 5 was given. Karen did not use the sample to guide her in any way, but instead came up with several + 1, -1, and +0 responses suggesting that she did not see how a cardinal value could be broken up into components and still maintain its value (see Figure 9). This clearly related to her inability to understand that the amount of substance does not change because its shape changes. For Karen the wholeness of 10 does not represent several possible smaller values that represent components of "10ness," but rather that "10ness," does not really exist beyond the last number word said at the end of a count. The number does not represent the whole and a part simultaneously. This is also related to Karen's failure to understand the nesting of categories in the class inclusion task and in her failure to appreciate subtraction as an integrated and inverse process in dealing with orally presented small number combinations. In fact, one of her responses to the number names task further demonstrates that Karen uses the "turnaround" fact term not as a true exemplar of reversibility, but simply as a rote and probably overgeneralized idea. This shows up when she says that 8 + 2 is 10 and 8 - 2 is also 10 (see Figure 9a). In this context, seems to work with visual patterns, not quantitative relationships. This is again demonstrated when Karen was asked to cross out any combinations in

a given group of combinations that did not produce the value "12." For this task, Karen counted up and down to test every example including 10 - 2 and 4 +5 as well as 3 + 9 after having just counted 9 + 3. Thus, while she got the answers all correct, her procedure demonstrated that she did not appreciate the cardinal value simultaneously as the sum of its parts.

Fact family combinations. Fact family combinations are those single digit combinations using 3 numbers that exhaust all possible addition and subtraction combinations for those numbers. These combinations can be executed as a visual pattern or they can be executed using the logical relationships of reversibility and inverse operations. Karen was successful on some combinations and not on others. Those items on which she was successful do not reveal how she derived her answers. However, on the items in which she produced quantitatively impossible equations or failed to confine her responses to the three numbers in the family, we see that Karen used a rote and mechanical one-way operation. As Figure 10 shows, On the 4, 5, 9 combination presented with a domino containing 4 dots and 5 dots, Karen did not even consider the subtraction or inverse part of the relationship, but rather produced a series of four partially related addition combinations and even misused the operation sign that was provided. She wrote that 4 + 5 is 9; 5 + 4 is 9; 9 - 5 is 14, and 9 - 4 is 13, thus converting the subtraction pairs into addition and introducing two more numbers from outside of the system. Similarly, in recording the 7, 3, 10 family from the three numbers already given, Karen filled in blank spaces by not repeating any combinations, but also by not making sense of all combinations. She wrote that 7 + 3 = 10; that 10 = 7 + 3; that 10 - 7 = 3; but that 7 = 3 - 10. This confusion of reversibility with a visual pattern seems to be closely related to both Karen's failure to conserve substance and liquid in that the notion of equivalence in regard to two sides of an equation is lost. It also appears to be related to her count-on and count backwards strategies for dealing with orally presented number combinations in that the relationships among the numbers is lost to her.

Function machine exercises. On this task the addition or subtraction of a constant becomes the rule for operating the machine (see Figure 11a). The child first needs to figure out the rule by looking at the relationship between numbers already given in an "in column" and an "out column." Once the rule is recognized, it must then be applied to one side or the other of the "in" and "out" columns. Completing the "out column" when the "in column" value is given is fairly straightforward and requires only a uni-directional approach, i.e., count up or count down. However, when a combination of "in column" and "out column" values are missing, then the child needs to use inverse operations and reversibility knowledge in relation to number values. Still, though, if the numbers are small enough, the child can still use a counting up and down strategy, coupled with a rote procedural rule that goes something like: "if your 'in' column number is missing and the rule is subtract something, then add instead. Or if your 'in' column number is missing and the rule is add something, then subtract instead, i.e, do the opposite when the in-column is missing." Using this rote procedural rule and a consistent strategy of counting up and down by ones regardless of the rule and how much counting was required, Karen was generally successful in getting mostly correct answers on this task. However, if we examine those items on which Karen did not obtain correct answers, we see that her errors are not small counting mistakes, but broad conceptual mistakes.

Figure 11 indicates some of the misconceptions that Karen brought to this task based on her incomplete understanding of reversibility and inverse relations as indicated in her performance on the conservation, oral number facts, and balance scale tasks. We see here that while Karen can generally apply any rule through counting by ones, but her answers do not always make numerical sense. For example (see Figure 11b), when she has to count down from larger numbers, she cannot continue to count by ones and so she tries to count by tens (e.g., to go from 130 to 80, she guesses that -30 is the rule. Then she fails to apply the rule accurately on all subsequent parts of the chart and even uses the same number [60] three times in the out column.) Also when the numbers are in place and the rule must be determined, she forgets about the relationship between the in-box and out-box numbers and instead just adds them together by counting on by ones. So she takes a -2 rule and changes it into a +8 rule by adding 5 **and** 3 instead of subtracting 3 **from** 5. This tendency to disregard the relationship between the numbers in favor of applying an arbitrary counting on by ones procedure is clearly related to Karen's uni-directional performance on the informal balance scale and conservation tasks.

This seems to suggest that while a child like Karen can obtain many correct answers, the processes she uses to obtain these answers have little to do with the underlying curricular objectives of the task. To do this task properly a child needs to understand that the rule can work in either direction *because the operations are related to one another,* not because this is *what you are supposed to do.* That is a subtraction or addition rule needs to be replaced by its inverse if the object is to identify the "in column" number when the "out column" number has been given. This is not the same as saying to oneself, "if it says subtract, then add and if it says add, then subtract."

Equations and models of one step number stories. Karen's curriculum also provides opportunities for learning to write standard number sentences or equations. The form of these equations is usually mastered by all students, but the question is really what do these equations actually represent to the children who write them? To assess this meaning, Karen was given several short basic "story problems" depicting addition and subtraction relationships. She was asked to write down the answer to the question, then write an equation or number sentence that tell what happened in the story, and then to show with some blocks what the equation was describing. Karen easily counted up or down to get her answers and had no trouble writing an equation to express the addition or subtraction action in the story. She also had no problem in modeling the addition stories. Where she did have difficulty was in showing with blocks what the equation and/or the story meant for subtraction. For example, as shown in Figure 12, for the story "Sarah had 9 pennies. She gave away 3 pennies. How many are left?" Karen first took 9 blocks. Then she took away 3 blocks. Then she put back 3 blocks. Then she took away 3 blocks. And finally, she put back 3 blocks. This depicted the equation: 9 - 3 + 3 - 3 + 3 = 9. This vacillation between addition and subtraction models of the equation suggest that Karen cannot hold both the question/action and the answer in her head at the same time. Unlike addition, subtraction destroys the set with which it started. It appears that for Karen she needs to keep the set intact in order to see where the subtraction took place. But if she does this, then the subtraction cannot take place. This dilemma is very much tied to her difficulty with class inclusion. The set of plastic circles disappears when her attention is focused on the color of the circles. In order for the plastic circles to maintain their integrity as a full set, they need to be disassociated from their blueness. In the same way, Karen cannot see what is taken away (3 pennies) and what is left (6 pennies) yet still keep in mind the 9 pennies with which she started. Thus, she continues to replace the part that was lost. The question then really is about the meaning of the written symbolic equation for a child who cannot see the forest and the trees at the same time. Like so much of the mathematics that Karen encounters in first grade, equations are just another procedural routine to be followed and replicated without consistent quantitative meaning.

Create a story problem. Another indicator that the numbers do not have a strong cardinal meaning for Karen and that they are not seen as being part of a balanced relationship comes from her responses to a task in which she was asked to make up a story problem for several sets of three numbers. The sets of numbers all constituted a number family relationship. In general, while she used all and only those numbers that were provided in each story, the stories did not depict any connection between the numbers (see Figure 13). For example, for the set of numbers 6-3-9, she said, "I saw 6 dogs walking. Then I came back and I only saw 3. How many did I see altogether?" While the number 9 could answer the question of how many altogether, the fact that the story is framed in subtraction language, does not make sense. It appears that Karen responded to the order of the numbers, i.e., 6 is more than 3 so that means less, and then reversed the story line because the third number 9 was also more than 3. Her general sense of number magnitude for small numbers seems to be operating here, but it is operating in isolation because she has no logical framework for conserving the set as a related series of small numbers. As would be expected, she seems to be taking a syntactical rather than a semantic approach to these stories and the sets of numbers incorporated in them.

When larger numbers were used, even though they were "comfortable numbers," Karen's story became even less representative of connections between the numbers. For the numbers 100-200-300, for example, she said, "I saw 300 people then I saw 200 people. 100 were only left. How many people did I see altogether?" Here she seems to be connecting the 300 to the 200 in a subtraction direction, but does not carry through and instead of having 100 be the result, she includes it as another number to be counted and changes the story into neither an addition nor a subtraction model. In fact, this time she seems to be introducing the possibility of another number from outside the system.

Only one of Karen's stories made mathematical sense and this one allowed her to tell an addition story by adding the value "2." For that story she made numerical sense. She said, " I read 24 books and I had nothing to do, so I read 2 more. How many books did I *have* altogether?" Notice though that even here she uses the term "have altogether" instead of "read altogether" which would make contextual sense. Even in this better format, Karen still seems to be following a linguistic pattern approach rather than a mathematical one. Overall, then, Karen's construction of simple word problem stories seems to reflects her inability to focus on more than one aspect of a quantitative situation at a time, her approach to numbers from a syntactical rather than a semantic framework, and her uni-directional understanding of quantity rather than an understanding of quantity through the principles of reversibility, compensation, and equivalence.

What will all this mean for Karen in second grade? That is a question for another presentation. But I can share with you now, that it doesn't get much better from a conceptual point of view.

Discussion and Conclusions

Very much like young children who deploy gestures while counting for a labeling rather than a quantitative function (Graham, 1999; Kaplan 1985), this study suggests that first graders with immature spontaneous notions about number tend to attribute static, specific, non-quantitative linguistic meaning to relational concepts in the mathematics curriculum. Thus, they often get correct answers on content that does not call for relational reasoning, but fail to obtain correct answers on structurally more advanced items that demand the application of reversibility or compensation principles. Their paperwork assignments, then appear as "C" or even "B" work because of the unevenness of task demands within the curriculum and that based on their sometimes correct answers, it appears that they "get it" but are just "making some mistakes." In reality, children who are not using the abstract level of thinking that key elements of some tasks demand, confine themselves to one way solutions, often employing counting on by ones, as the basis for these solutions. This strategy works for getting some answers, but it does not help build knowledge of higher level equivalence relationships, the precursors to algebraic thinking. As a consequence the children who are locked into the one-way syntactic solution process are likely to develop enduring misconceptions or only partial conceptions about the mathematics they are using in first grade.

My strong intuition is that just as lower level informal concepts create lower level approaches to dealing with more abstract equivalence concepts in the curriculum, so will lower level approaches to the first grade curriculum carry over to impair understanding of later curricular concepts that should be building on clear understanding and applications of reversibility and compensation principles encountered in the earlier curriculum. It is further suggested that these misconceptions interfere with subsequent school mathematics learning because children use them to inform their understanding of new material. As a result second grade children who functioned at a syntactic level on first grade curriculum are likely to continue to function at this level on second grade curriculum and beyond. Thus, we often see the rote application and parroting of rules with numbers rather than the development of a strong and flexible number sense as children move up in the grades. My initial inspection of the second grade interviews that I conducted last month seem to suggest that this is the case. Some of the children seem to be developing ritualized approaches using rule-based number manipulation techniques. They are overgeneralizing a good thing in some cases such as using 10 as the basis for addition with 9s and then subtracting 1. At least in one case this technique has become standard operating procedure and the child's justification for all answers even when this procedure does not make sense. My interest in the future is to develop some specific intervention strategies to prevent these misunderstood processes from carrying over from grade to grade.

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First Grade Informal Tasks

Eight categories of informal tasks were used, including:

1)Conservation of number, substance, and liquid
2)Oral Counting by ones, twos, fives, and tens
3)Counting of 30 objects by ones, twos, fives, and tens
4)Class inclusion
5)Seriation
6)Mental arithmetic using single digit number facts presented orally
7)Orally presented single digit combinations in the context of a simple story
8)Balancing a pan scale by adding or removing same-weight objects (totals from 6-12)

First Grade Formal Tasks

Nine categories of formal curricular tasks were used, including:

1)Place Value - Place value identification of 2- and 3-digit numbers
2)Number Series -Writing numbers in linear sequence forward and backward by ones
3)Money - Determining and comparing the values of coin combinations presented physically and with symbolic notations
4)Frames and Arrows (the label the curriculum developers used for this task) - Identifying and continuing written number patterns using addition and subtraction of ones, twos, threes, fives, tens, hundreds
5)Function Machine (the label the curriculum developers used for this task) - Identifying and completing a chart with one, two and three digit numbers following a rule of addition or subtraction with 1, 2, 3, or 10
6)Fact Families - Completing four-equation number fact families with number combinations totaling between 6 and 10
7)Number Names - Renaming numbers between 10 and 40 by breaking the number into two or more parts and adding or subtracting the parts
8)Equations and Models - Solving simple addition and subtraction word problems using an equation and modeling with blocks using numbers between 2 and 20
9)Creating Fact Family Stories - Creating a sensible word story for three numbers in a fact family using numbers combinations of single and double digits with one example using 100, 200, 300

Figure 1. A list of all informal and formal tasks used in first grade.

Informal Task	Formal Task	Logical Underpinning
Conservation	Money	Equivalence, Relative Magnitude
	Fact Families	Reversibility, Compensation
	Number Renaming	Compensation
	Function Machine	Equivalence, Reversibility, Relative Magnitude
	Creating Fact Family Stories	Reversibility, Relative Magnitude
Oral Counting	Number Series Patterns	Rote skill
	Frames and Arrows	Reversibility, Relative Magnitude
Counting Objects	Money	Equivalence
	Frames and Arrows	Equivalence, Relative Magnitude
Mental Arithmetic (Oral Facts)	Fact Families	Reversibility, Compensation, Relative Magnitude
	Number Renaming	Reversibility, Compensation
	Function Machine	Equivalence, Reversibility
	Equations and Models	Equivalence, Reversibility
	Creating Fact Family Stories	Reversibility
Class Inclusion	Money	Equivalence
	Place Value	Compensation, Equivalence, Relative Magnitude
	Creating Fact Family Stories	Reversibility
Seriation	Number Series Patterns	Rote skill
	Frames and Arrows	Reversibility, Relative Magnitude
	Creating Fact Family Stories	Reversibility
Number Stories	Equations and Model	Equivalence, Reversibility
	Creating Fact Family Stories	Equivalence, Reversibility, Relative Magnitude
Balance Scale	Money	Equivalence
	Fact Families	Equivalence, Compensation, Relative Magnitude
	Function Machine	Reversibility, Compensation
	Number Renaming	Compensation, Equivalence
	Equations and Model	Equivalence
	Creating Fact Family Stories	Equivalence, Reversibility, Relative Magnitude

Figure 2. Structurally related ways of knowing underlying informal and formal tasks

Table 1

Overall Correlation Between Performance on Informal and Formal Curriculum Tasks

Informal Tasks (Mean Score)	Formal Tasks (Mean Score)	Correlation Coefficient
130.17	235.17	$r(10) = 0.896$*

*$p < .001$

Table 2

Correlations Between Specific Informal Tasks and Their Structurally Related Formal Tasks

Informal Tasks	Number Series	Place Value	Money	Fact Families	Number Names	Frames & Arrows	Function Machine	Equation Model	Create Stories
Conservation	-----	-----	0.561*	0.618*	0.632*	-----	0.605*	-----	0.280
Oral Counting	0.331	-----	-----	-----	-----	0.895**	------	-----	-----
Counting Objects	-----	-----	0.821**	-----	-----	0.895**	------	-----	-----
Mental Arith (Oral Facts)	-----	-----	-----	0.658*	0.536	-----	0.782**	0.585*	0.112
Class Inclusion	-----	0.319	0.128	-----	-----	-----	-----	-----	0.030
Seriation	0.385	-----	-----	-----	-----	0.702*	-----	-----	0.223
Number Stories	-----	-----	-----	-----	-----	-----	-----	0.914**	0.251
Balance Scale			0.499	0.696*	0.371		0.371	0.725**	0.363

Karen's Conservation Behaviors

Conservation of Number **(conserves)**	Conservation of Substance **(does not conserve)**	Conservation of Liquid **(does not conserve)**
"Same" "Just like moved them down."	1)Snake - (slowly) "Still the same" (Q) "Almost the same, but the snake has a little more."	1)Wider cup - "The small cup has more." [i.e., water comes up to higher level]
	2)Little balls - (slow and hesitant) "The pieces have less. No the same" (Q) "No, the pieces have more."	2)Narrow glass - "Not the same. The narrow one has more. The glass is bigger." [i.e., water comes up higher]
	3)Pancake - (slowly) "The pancake has more. More spread (gestures with hands)."	

Figure 3. Karen's conservation responses.

Karen's Class Inclusion Judgments

Question: Are there more plastic circles or more blue circles?	Initial Response	Response with Cues
●●●●●●●● ○○○○ 8 blue, 4 white, 12 plastic	"More blue"	Q1)How many plastic circles? "4" [actually white ones]
		Q2)Count the plastic circles. "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12"
		Q3)Which is more, plastic or blue? "Plastic"

Figure 4. Karen's class inclusion responses.

Karen's Balance Scale Behaviors

Initial Imbalance: ○○○○ ○○○○○○○○○	Initial Response	Correction Responses
Goal is to put the scale in balance without taking any extra pieces from outside the system and without eliminating any pieces from the system	(takes 4 from 8 and puts it in 4) ○○○○(●●●●) ○○○○ ○○○○ (●●●●)○○○○	R1)(takes the extra 4 out and holds them, leaving 4 and 4 in the scale) ○○○○ ○○○○ ●●●●
		R2)(puts 2 of the 4 in her hand into each pan - resulting in a balance) ○○○○●● ○○○○●●
		(Q)How many in each side? R3)(counts one side only) "1, 2, 3, 4, 5, 6....6 and 6"

Figure 5. Karen's balance-pan scale responses.

Karen's Skip Counting of Objects

Question: How many cubes are there? (30)	By ones	By twos	By fives	By tens
●●●●● ●●● ●●● ● ●●● ● ●●● ●● ●●●● ●●●● ●	1,2,3,4,5,6,7,8,9, 10,11,12,13,14, 15,6,17, 18, 19, 20, 21, 22, 23, 24, 24, 26, 27, 28, 29, 30 (30) ●●●●●●●●●●●●● ●●●●●●●●●●●●● ●●●●	2, 4, 6, 8, 10, 12...14...16...18.. 20...24...26... 28:.33....35 (35) ●● ●● ●● ●● ●● ●● ●● ●● ●● ●● ●● ●● ●● ●● ●●●	5, 10, 15, 20, 25, 30 (30) ●●●●● ●●●●● ●●●●● ●●●●● ●●●●● ●●●●●	10, 20, 30 (30) ●●●●●●●●●● ●●●●●●●●●● ●●●●●●●●●●

Figure 6. Karen's skip counting with objects.

Karen's Responses to Orally Presented Single-Digit Number Combinations

4 and 4	5 and 5	4 and 5	5 and 4	9 take away 5	9 take away 4
immediate "8"	immediate "10"	on fingers, "5, 6, 7, 8, 9. So 9"	immediate "9" (Q) "turnaround fact"	on fingers, "9, 8, 7, 6, 5, 4, 3. Three?"	on fingers, "9...8, 7, 6, 5. Five?"

Figure 7. Karen's mental arithmetic responses.

Karen's Frames and Arrows Reasoning

[+2 rule] Application of a one-way rule (correct)	[Count up by 10s rule] Application of a one-way rule (correct)
"add 2" 24 → 26 → 28 → 30 → **<u>32</u>**	8 (given) "9,10,11,12,13,14,15,16,17,**<u>18</u>**" ...... [18] → **<u>28 → 38 → 48</u>**

[Count down by 10s rule] Misapplication of a two-way rule (distortion to down by 1s)	[- 10 rule] Misapplication of a two-way rule (mixing of counting by ones, tens, and fives)	[- 1 rule] Misapplication of a two-way rule (reversal of minus one to plus one rule)
56 (given) ..... **<u>55→ 54→ 53→ 52</u>**	650 (given)→ 640 (given)→ ***630→ 620→ 615→610→ 600→ →550***	***652→ 653→*** 654 (given)→ ***655***

Figure 8. Karen's one-way responses to the frames and arrows task.

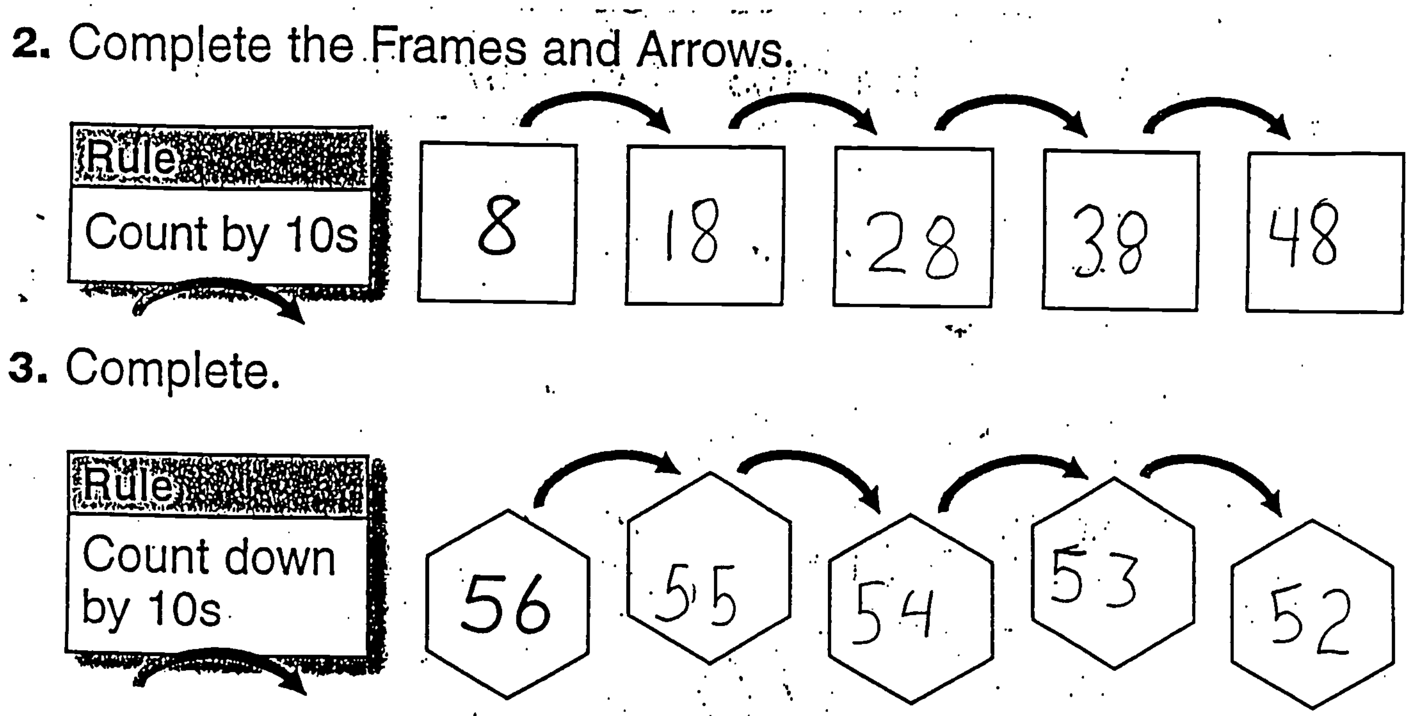


Figure 8a. Karen's misapplication of the counting down by 10s rule.

Karen's Responses to Number Names Tasks

Make five more names for "10"	Write five more names for "25"	Cross out the wrong names for "12"
(sample: 5 + 5) 10 + 0 9 + 1 11 - 1 8 + 2 8 - 2	(sample: 30 - 5) 24 + 1 25 - 0 0 - 25	*[14 - 3]* "14, ..13, 12, 11-no" *[4 + 5]* "4..5, 6, 7, 8, 9- no" *[9 + 3]* "9..10, 11, 12- yes" *[3 + 9]* 9..10, 11, 12 -yes" *[2 + 11]* "11..12, 13 - no" *[11 + 2]* "11..12, 13 - no" *[10 - 2]* "10..9, 8, - no"

Figure 9. Karen's appreciation of cardinal values and number components.

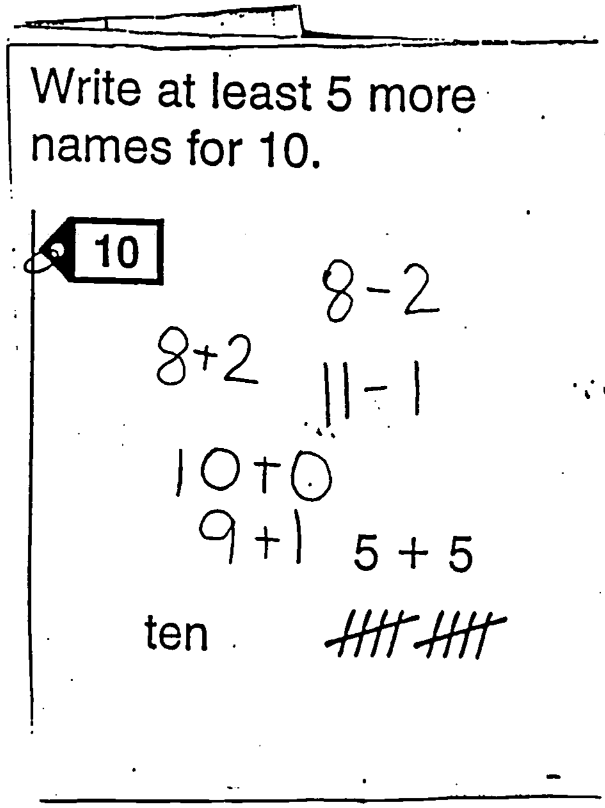


Figure 9a. Karen's overgeneralization of the commutativity principle.

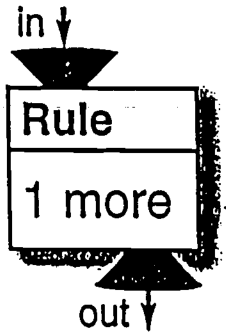
Karen's Responses Fact Family Combinations

2, 4, 6 family (all numbers given in fact triangle; need to fill in numbers in horizontal form; operations signs given)	4, 5, 9 family (9 had to be derived; operation signs and addends and subtrahends given; vertical form)	7, 3, 10 family (all numbers given in fact triangle; and operations signs given; need to fill in numbers in horizontal form)
4 + 2 = 6 6 - 2 = 4 2 + 4 = 6 6 - 4 = 2	$\begin{array}{r}4\\+5\\\hline 9\end{array}$ $\begin{array}{r}5\\+4\\\hline 9\end{array}$ $\begin{array}{r}9\\-5\\\hline \mathbf{14}\end{array}$ $\begin{array}{r}9\\-4\\\hline \mathbf{13}\end{array}$	7 = 3 = 10 10 = 7 + 3 ***7 = 3 - 10*** 10 - 7 = 3

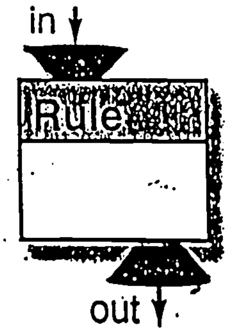
Figure 10. Karen's appreciation of the visual rather than logical features of number fact families.

What's My Rule?

19.



in	out
17	18
27	
37	
47	
57	



in	out
13	7
10	4
12	
8	
	6

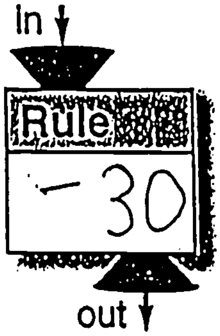
Figure 11a. Examples of function machine exercises requiring one-way and two-way reasoning.

Karen's Approach to the Function Machine Task

Mechanical Counting by Ones (-4 rule)	**Misidentifies Rule**	**Add 10 Rule**	**Subtract 10 Rule**	**+50 Rule** (have to complete in-boxes and out-boxes)
uses opposite operation to get in-box - counts up by ones *10*...6 *13*...9 *17*...13 *18*...14	inbox is 5 and outbox is 3 Says rule is +8	60 .....70 (uses counting by tens) 83....93 Counts up from in-box number on fingers by ones: 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93 Counts up in same way for: 47...57 162...172 But on 215 ....225 uses a count by tens strategy	Counts down by ones for first three numbers, then sees a pattern, but misuses the pattern. 90........80 (90, 89, 88, 87, 86, 85, 84, 83, 82, 81, 80) 76...66 (counts down by ones) 102...92 (use place value rule) 210....200 (uses place value rule) 238...218 (misuses place value rule)	Does not want to count down by ones and tries to count down by tens, but fails. 130....80 (given) 80.....*60* (off by 10) *40*.....60 (does incorrect opposite operation and off by 10) 180.....*60* (copies from 80 and 60 above) 100....40 (off by a lot) **[Note that same outbox number of 60 is used three times.]**

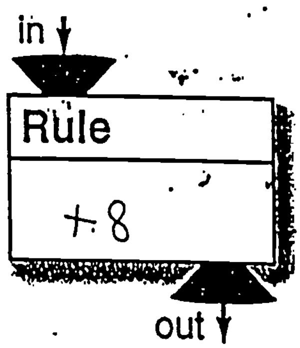
Figure 11. Karen's rote application of a procedural rule and counting up and down by ones strategies for dealing with the function machine task.

2. What's My Rule?



in	out
130	80
80	60
40	60
180	60
100	40

1.



in	out
5	3
7	5
9	7
12	10
8	6

Figure 11b. Karen's numerically unreasonable answers using a rule for the function machine.

Karen's Equations and Models of Number Stories

Tom has 6 pennies. Debbie has 4 pennies. How many pennies do they have together?	Sarah had 9 pennies. She gave away 3. How many are left?	18 lions were lying in the sun. 2 more lions joined them. How many lions is that altogether?	20 lions are lying in the sun. 5 go inside. How many lions are left in the sun?
10 6 + 6 = 10 □□□□□□↦□□□□	6 9 - 3 = 6 1)□□□□□□□□□ 2)□□□□□□ □□□ 3)□□□□□□↦□□□ 4)□□□□□□ □□□ 5)□□□□□□↦□□□	20 18 + 2 = 20 □□□□□□□□□□□□□ □□□□□↦□□	15 20 - 5 = 15 □□□□□□□□□□□□□ □□□□□□□↦□□□□□ □□□□□□□□□□□□□ □□□□□□□ 20 □□□□□ 5

Figure 12. Karen's representations of number stories through equations and through modeling the relationships with blocks.

Karen's Story Problem Constructions

6 - 3 - 9 story	8 - 4 - 12 story	24 - 2 - 26 story	**300 - 200 - 100 story**
I saw 6 dogs walking. Then I came back and I only saw 3. How many dogs did I see altogether? (6 - 3 = 9?)	I found 8 pennies and I came back and I only saw 4. How many did I see altogether? (8 - 4 = 12?)	I read 24 books and I had nothing to do so I read 2 more. How many books *did I have* altogether? (24 + 2 = 26)	I saw 300 people. Then I saw 200 people. 100 were only left. How many people did I see altogether? (300 + 200 - 100 = X?)

Figure 13. Karen's conception of stories about fact family combinations.

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