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ABSTRACT

Elementary number theory is a standard topic in the mathematical preparation of preservice elementary teachers. To understand elementary number theory, a student must be comfortable with the representation of natural numbers as the product of primes. This paper discusses methods for accomplishing this goal in a methods course. It also considers the role of measurement (quotient) division in justifying divisibility results and demonstrates one way that this connection can be exploited to strengthen student understanding of multiplication and division. The appendix contains three worksheets for students to complete. (ASK)

APPLYING INSIGHTS FROM RESEARCH ON LEARNING: TEACHING NUMBER THEORY FOR PRESERVICE ELEMENTARY TEACHERS

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Elementary number theory is a standard topic in the mathematical preparation of preservice elementary teachers. The study of divisibility, divisors, multiples and prime factorization provides essential opportunities to enrich one's understanding of multiplication, division, and the multiplicative structure of natural numbers.

The course in which I teach a unit on number theory is a foundations of arithmetic course that is the first in a three course, nine credit-hour sequence of mathematics courses for elementary teachers. This first course carries a minimal prerequisite of high school algebra. Most of the students are firm in their claim that their previous knowledge of arithmetic is generally "rules without reasons"; results of a pretest given at the beginning of the course consistently support that claim.

In this course, students re-examine the topics of elementary mathematics from an adult perspective; through this process, they identify and fill many gaps in their conceptual understanding. The course work is devoted to exploring how the ideas of elementary mathematics are tied to its informal and formal foundations. Students are expected to make sense of arithmetic: this means being able to explain and illustrate mathematical truths, in a personally and mathematically meaningful way.

Number theory is a topic that can be taught at many levels. It is clearly unreasonable to expect most preservice elementary teachers to engage with this topic in the same way that a mathematics major might. In designing instruction, I seek to establish an appropriate level of rigor that will support making sense of basic number-theoretic results, encourage the development of pedagogical content knowledge of number theory, and help students to integrate their evolving knowledge of number theory with their informal and formal understandings of arithmetic in general.

In this article, I will focus on two aspects of my instruction that have been especially influenced by research on the learning of collegiate mathematics, specifically that reported in references [3] and [6]. To understand elementary number theory, the student must be comfortable with the representation of natural numbers as the product of primes. This includes constructing prime factorizations, performing arithmetic on prime factorizations, and using the structure embedded in the factorizations to recognize and justify divisibility relationships. Methods for accomplishing this goal will be discussed. I will also consider the role of measurement (quotitive) division in justifying divisibility results and show one way that this connection can be exploited to strengthen student understanding of multiplication and division.

Understanding exponentiation

For the past several semesters, I have given a short pretest to students entering my course. The most recent version of the test was given to sixty students during this school year (in August 1998 and January 1999). The use of calculators is not allowed on this test. Most relevant to the present discussion are the following results that indicate their understanding of exponentiation as they enter the course.

All of these students had either completed an intermediate algebra course or had demonstrated algebraic proficiency on a placement exam. Paradoxically, it seems that this proficiency does not extend to exponential expressions that involve numbers only. Here are the results of two questions, in which students were asked to express the indicated product as a single base raised to an exponent larger than 1.

Q: $12^{18} \times 12^{13} =$	percent who gave each answer
12^{31}	38%
144^{31}	40%
other incorrect answers	22%

Q: $8^{20} \times 6^{20} =$	percent who gave each answer
48^{20}	23.3%
48^{40}	48.3%
other incorrect answers	28.3%

It is evident from this data (and from their own statements during class discussion of the pretest problems) that students entering the course tend to operate on exponential expressions by trying to remember rules, rather than by reasoning about the meaning of the notation. Naturally, an inability to work in a meaningful way with exponential expressions will adversely affect one's ability to recognize and explain divisibility relationships for numbers expressed in prime-factored form. Thus, it was clear that any revised instruction on number theory topics would have to be preceded by some work focused on the meaning of exponentiation. The design of instruction was informed by a general theoretical perspective as well as my own specific experiences conducting research into the learning of number theory.

How research influenced the design of instruction

The study reported on in [3] was initially inspired by the researchers' curiosity about student understandings of least common multiple, particularly our observation that students used the "prime factorization approach" for finding the least common multiple without having any idea as to why it produced the correct answer. This led us to examine how students view the concepts of multiple, divisor, and the divisibility relation itself, particularly when the context included the use of the prime factorization of a number. Our interest in the prime factorization also led to the design of the pretest items mentioned in the previous section.

Examining the interview data, we determined that many students avoid computing and reasoning about prime factorizations because they do not see the expressions as "actual numbers" until they are returned to decimal form. We believe that this lack of experience with manipulating and interpreting prime factorizations explains why many students are unable or reluctant to use the structure of the prime factorization of a number to settle questions of divisibility or indivisibility (first reported in [6]).

My colleagues, Karen Thomas and Georgia Tolias, and I use the action/process/object theoretical perspective in our research (for details, see [1]). It postulates that an understanding of mathematical concepts originates in actions on (mental) objects, and that such actions simultaneously help solidify one's understanding of the objects to which they are applied. In view of the theory, I decided to try to give students more experience with arithmetic actions on numbers in prime-factored form -- not the operations that return the expressions to decimal form, but rather operations that transform two prime factorizations to a new one.

In the revised instruction, the discussion of the concepts of multiplication, division, and exponentiation leads immediately into work with prime factorizations. Group tasks assigned at this point appear on Worksheet 1 in the Appendix to this paper. The exercises in part 1 highlight the fact that the prime factorization of a product of two numbers is the product of their prime factorizations. More to the point, students perform arithmetic on prime factorizations as objects, rather than returning the numbers to decimal form. In part 2, the idea is to carry out the action of division in prime-factored form, and to observe that the result may be either a natural number or a fraction. Reflecting on how the division is carried out leads to articulating conditions that predict the results through comparison of the multiplicative structures of the divisor and dividend.

A few weeks later, after division and its relationship to the divisibility relation has been examined (in the way discussed later in this paper), there is an effort to encourage students to reconstruct their understandings of divisors, multiples, and divisibility within the prime-factored representation, and to use the structure provided by the representation to settle divisibility questions. This is the purpose of Worksheet 2 in the Appendix. These problems present integer pairs A and B, and the task is to determine whether or not A is divisible by B. In the case that A is divisible by B, the student writes the multiplicative sentence that expresses that fact.

Watching students work on these exercises is interesting. Typically, they initially rely on changing the numbers to decimal form, but by the fifth problem, they realize that they need a method that does not require decimal form. Recalling algebraic procedures for simplifying rational exponential expressions, they first carry out the division explicitly. Writing the divisibility result in multiplicative form highlights the role of the divisor (B) and resulting quotient as factors of A; this appears to support the development of the ability to use structure instead of direct division to reason about divisibility. The activity culminates in a discussion on how it can be determined by inspection of prime factorizations whether A is divisible by B.

Next, depending on the amount of time available, the more advanced exercise on Worksheet 3 might be solved. An extended discussion of various aspects of the solution to this problem is the focus of [1]. Students work individually on the problem for a few minutes and then discuss it in their groups. This leads to a whole class discussion about the issues of divisibility and representation that arise.

It is fair to ask whether any improvement in understanding has come from this revised instruction. While no formal study has been done, one instrument that documents some improvement is a final exam question (see below) that I use each semester. The revised instruction was implemented for the first time in Fall 1997, and "PF" refers to "prime factorizations":

Q: Is 26325 a factor of the number $2^{12} \times 3^{80} \times 5 \times 13^{10}$? Explain.

class	correct conclusion with explanation	used PF, incorrect conclusions	did not use PF, incorrect conclusions
Spring 97 (n=29)	17.2%	13.8%	69.0%
Fall 97 (n=32)	40.6%	37.5%	21.9%

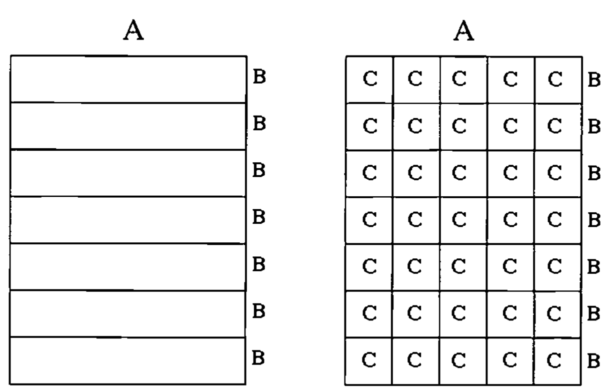
Comparing the results on this problem suggests that the changes in instruction did indeed have a positive effect on student understandings. Most important is the fact that 69% of the students in the first class did not see the possible utility of finding the prime factors of 26325, but almost 80% of those who experienced the revised instruction did.

Measurement division and number theory

Studying number theory often compels students to re-examine their understanding of division and multiplication. As pointed out by Zazkis and Campbell [6], the confusion students have about number-theoretic concepts can often be traced to a lack of awareness of the importance of the inverse relation between multiplication and division: $a \div b = c$ if and only if $b \times c = a$.

Instruction intended to deepen their understanding of division and to emphasize its function as the reversal of multiplication was designed. Both equal-sharing (partitive) and measurement (quotitive) division are discussed, but the latter appears to be more closely related to an understanding of divisibility. In this approach, the division statement $A \div B$ is interpreted as asking how many portions of size B are in A. This relationship is represented and explored visually by asking questions such as the following.

<u>Example</u>: If $A \div B = 7$ and $B \div C = 5$, then what is $A \div C$?



There are 7 Bs in A: $A = 7B$.

There are 5 Cs in each B: $B = 5C$.

$A = 7B = 7(5C) = 35C$

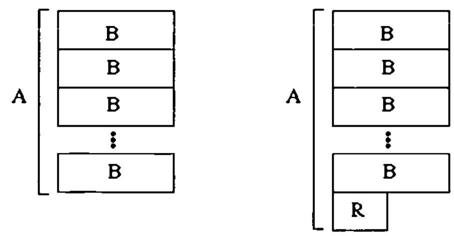
There are 35 Cs in A, so $A \div C = 35$.

Subsequent problems involve quotients that are not as easy to represent explicitly, such as "If $A \div B = 75$ and $B \div C = 132$, then what is $A \div C$?". This requires reasoning about operations carried out mentally, bringing thought about division to a higher level.

Mathematically, the measurement approach is precisely what is needed for the Division Algorithm, which guarantees existence of unique non-negative integers C and R such that A = BC + R, where C is the number of B's in A, and R is the remainder. In visual form, it is presented as:

When A is divided by B, two different things can happen:

the remainder is 0 | the remainder is not 0



if C is the number of Bs in A

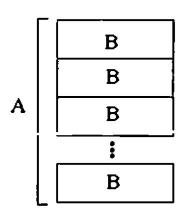
A = BC | A = BC + R

The fact that R can take on any of the values from 0 to B–1 is discussed in terms of the size of the bar representing it. Eventually, conceptual understanding of the Division Algorithm is assessed by posing questions about intended division such as the following:

1. What is the remainder upon division by 7 of $N = 7 \times 142 + 5$? Find your answer without doing any calculations.
2. How many numbers between 1253 and 1987 have remainder 11 upon division by 15?

The special status of the case of remainder zero is identified as the basis of the divisibility relation studied in number theory. Class discussion evokes a variety of colloquial expressions that students (as well as children) use to describe this situation:

When A is divided by B and the remainder is 0:



C is the number of B's in A

A ÷ B = C

A = BC

Colloquial expressions

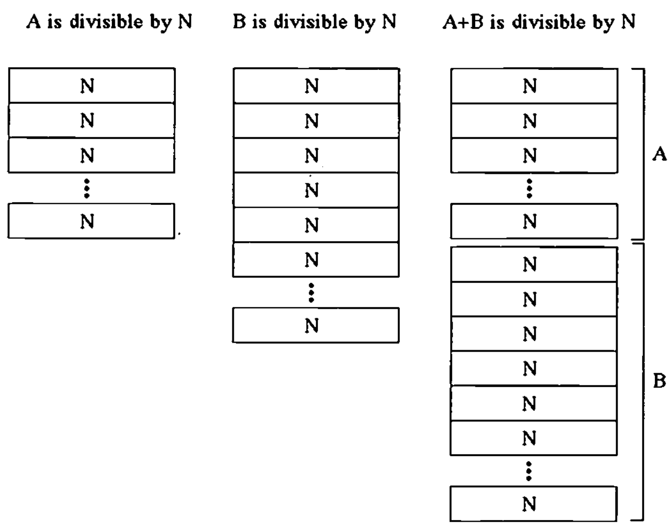
B goes into A evenly
B divides into A exactly
B goes into A perfectly
B fits into A perfectly

Formal expressions

A is divisible by B
A is a multiple of B
B is a factor of A
B is a divisor of A
(B divides A)

The emphasis on measurement division was a response to hearing students describe A being divisible by B as "B fits into A a whole number of times". This brought to my mind a sort of tactile approach to numbers, something that is characteristic of those who have good number sense (see examples in [4]), so it is certainly something to be encouraged. We see A as being made up of a certain number of copies of B. This also suggests the idea that A is a multiple of B, since A consists of multiple copies of B. Relating all of the formal expressions to a single image might help in understanding and remembering their equivalence.

This visual approach is used throughout the unit in providing proofs and plausibility arguments for various results. For example, consider the useful theorem that when A and B are divisible by N, then A+B is as well:



This result is used repeatedly in number theory; for example, we use it to give plausibility arguments for the divisibility rules for 3, 9, 11, and the powers of 2. It also is used in justifying the primitive form of the Euclidean Algorithm: GCF(A, B) = GCF(A–B, B), when B is smaller than A, since the proof rests on the fact that the common factors of A–B and B are the same as the common factors of A and B.

The issue of divisibility rules is another for which my instruction has been influenced by the research. Zazkis and Campbell suggested that it might be appropriate to delay the introduction of divisibility rules until the property of divisibility is fully understood (see [6], page 561). I do delay their introduction, which also preserves the practice of logically connecting new knowledge to previous knowledge. To explain (not just apply) the divisibility rules, one needs to fully understand divisibility, including the result just illustrated. I have found, through classroom observations, that allowing the use of divisibility short-cuts before adequate understanding of divisibility has developed can result in confusion about what the operational underpinnings of divisibility actually are -- for example, divisibility by 3 becomes associated with the process of adding the digits and dividing by 3, rather than with seeing the number as made up of a certain number of portions of size 3. Moreover, early introduction of the rules seems to exaggerate their importance in the mind of the student. By introducing the rules late in the unit, we can discuss in a meaningful way why they are true by thinking about representative examples. This has resulted in more appropriate use of the rules and less confusion about their significance.

As a final comment, this focus on measurement division in number theory also supports the later course work on division with fractions. One interesting connection that came up this year was in our discussion of children's invented methods for division with fractions in [5]. We noted that the children featured in the article were spontaneously using the distributive property $(a+b)/c = a/c + b/c$ to simplify their work. Relating this situation back to the theorem discussed above resulted in a satisfying instance of integration, and a welcome opportunity to emphasize that elementary teachers need deep and broad knowledge about numbers and operations in order to make sense of, and to help build on, children's expressed understandings.

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Appendix

WORKSHEET 1

1. Write the prime factorization of each of the following numbers in standard form. Re-use your answers as is convenient!

(a) $75 =$ (b) $132 =$

(c) $1{,}000{,}000 = 10^6 =$ (d) $243 =$

(e) $600 =$ (f) $75 \times 132 =$

(g) $75 \times 243 =$ (h) $132 \times 600 =$

(i) $243{,}000{,}000 =$ (j) $600 \times 600 =$

2. Carry out the following two division problems in exponential form.

$$\frac{3^8 \times 5^4 \times 7^3}{3^4 \times 7^2} = \qquad \frac{3^8 \times 5^4 \times 7^3 \times 11^4}{3^{10} \times 5^3 \times 7^2} =$$

Notice that one has a quotient that is a natural number and the other does not. Why did that happen? Explain how you can predict from the structure of the original dividends and divisors which division will result in a natural number, without doing any calculating.

WORKSHEET 2

In each case, determine whether A is divisible by B. If A is divisible by B, write an equation $A = B \times C$ that expresses divisibility in multiplicative form.

(1) $A = 17200$ and $B = 43$ (2) $A = 468$ and $B = 17$

(3) $A = 2^5 \times 5^6 \times 11^3$ and $B = 5$ (4) $A = 2^5 \times 5^6 \times 11^3$ and $B = 121$

(5) $A = 3^{20} \times 5^{36} \times 13^8$ and $B = 65$ (6) $A = 3^{20} \times 5^{36} \times 13^8$ and $B = 49$

(7) $A = 3^{10} \times 11^8 \times 13^{20}$ and $B = 3^3 \times 13^{18}$ (8) $A = 6^{50}$ and $B = 2^{15} \times 3^{20}$

(9) $A = 5^{20} \times 11^{18} \times 17^{15}$ and $B = 5^{22} \times 11^{10} \times 17^6$ (10) $A = 2^{40} + 3^{82}$ and $B = 2$

WORKSHEET 3

There is a pattern in the sequence shown below.

$$2^2 \times 3^4,\ \ 2^3 \times 3^4,\ \ 2^2 \times 3^5,\ \ 2^4 \times 3^4,\ \ 2^2 \times 3^4 \times 5,\ \ 2^3 \times 3^5, \ldots$$

1. Write the next 6 entries in the sequence (all in prime-factored form).
2. What is the 200th term in the sequence, in prime-factored form?
3. Give an algorithm for finding the nth term in the sequence, in prime-factored form.

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