DOCUMENT RESUME

ED 443 820		TM 030 299
AUTHOR	Barrett, Jeffrey E.; Clements, Douglas H.	
TITLE	Quantifying Length: Children's Developing Abstractions for Measures of Linear Quantity in One-Dimensional and Two-Dimensional Contexts.	
PUB DATE	1999-00-00	
NOTE	64p.	
PUB TYPE	Reports - Research (143)	
EDRS PRICE	MF01/PC03 Plus Postage.	
DESCRIPTORS	*Abstract Reasoning; *Cognitive Development; Constructivism (Learning); *Elementary School Students; *Geometric Concepts; Intermediate Grades; *Measurement Techniques	
IDENTIFIERS	*Length	

ABSTRACT

The measurement and description of polygons and paths by elementary school students was studied from a constructivist point of view. A teaching experiment was devised to promote understanding of length based on the hypothesis that as children coordinate their number concept and their one-dimensional/two-dimensional spatial concepts they gain understanding of, and build more abstracted schemes for, length and perimeter. Four fourth graders were studied in the second semester of their academic year. Four themes were encountered in the study. Children quantified length by partitive operations and later by iterative operations, gradually restructuring their internal images. They coordinated their number sequence with their spatial images as they kept shifting from making visual comparisons between objects to quantifying the extent of partitioning operations. The children represented length when they curtailed their own movements through linear space, and they coordinated several linear quantities for an object by disembedding the linear aspects of the object in space. The children appeared to progress along four levels of strategy for length, and it is suggested that progress through these levels follows from increasing integration between conceptual knowledge and figural knowledge related to the measure of length. Children gained abstraction for length and perimeter concepts as they increased the correspondence between their counting scheme, their partitive scheme, and their iterative scheme. Six instructional strategies that supported the children in creating and recognizing relevant length structures and in reflecting on relations among measures of length taken along a complex figure are identified, and implications for instruction are discussed. (Contains 1 table, 8 figures, and 54 references.) (SLD)

Quantifying Length: Children's Developing Abstractions for Measures of Linear Quantity In One-Dimensional and Two-Dimensional Contexts

Jeffrey E. Barrett
Illinois State University
jbarrett@math.ilstu.edu

Douglas H. Clements
State University of New York at Buffalo
clements@acsu.buffalo.edu

There is a continuing need for research that integrates the study of teaching and learning across divergent lines of inquiry in mathematics education (Cobb & Bowers, 1999; Tzur, 1999). There are currently several divergent theoretical models of children's knowledge and of the ways children gain further knowledge about geometry and space, a particular cognitive domain that we examine in this paper (Lehrer & Chazan, 1998; Mammana & Villani, 1998). Investigating children's learning of length and perimeter offers researchers the opportunity to examine the relation of ideas and practices by relating children's evolving geometry/space concepts to their understanding and practice of measurement. This combination of topics allows both a psychological analysis of the constructive/interpretive practice of individuals and a socio-cultural analysis of tool development, especially the relation between tools and those who use them.

This report provides a content-specific examination of a geometric topic--the measurement and description of polygons and paths--that affords an opportunity to integrate multiple theoretical concerns: we describe children's mental models for length within a constructivist theory, directing our attention to the interactions between the teacher/researcher and the learner, and recognizing the role of tools and representational media. While we attend to the constructive processes of individuals building up knowledge of space and number, we do not assume that such knowledge is fixed in their own persons, but acknowledge an epistemological tension between individuals and their situation within social organizations of classrooms and schools, and between individual knowledge builders and the collection of tools they acquire from their broader social settings in learning communities (Simon, 1995; Simon, 1999; Steffe & Thompson, 1999; Steffe & D'Ambrosio, 1995; Tzur, 1999).

Children are expected to learn to measure length and perimeter before entering grade 8, usually by grade 5 (cite P&S Working Draft 1998, NCTM 89). Measurement procedures should not be taught in isolation from real contexts, nor should they be separated from conceptual knowledge of length and perimeter. But the linkage between procedures and concepts is difficult to prescribe (Hiebert, 1986); it is not yet clear how to promote the growth of length knowledge that is at once procedural (syntactic) and conceptual (semantic) in instructional settings. Indeed, children in the U.S. perform poorly on geometry and measurement tasks according to the recent Third International Mathematics and Science Study (Peak, 1996), and children in the U.S. are generally poor at solving problems of any sort that require multiple steps (Campbell, Voelkl, & Donahue, 1997), especially length and perimeter tasks with multiple-steps (William Schmidt, personal communication, July, 1998). Further, the results from the Sixth Mathematics Assessment of the National Assessment of Educational Progress report that only 18% of fourth-grade children were able to draw simple rectangles given a ruler and protractor, although more than half of these students were able to measure the dimensions of existing drawings, suggesting that children may struggle to represent measurements, but not to identify lengths. These findings suggest that present instruction is not adequate: there is a need for developmental research to offer instructional approaches based on theoretically coherent models of children's learning of length and perimeter concepts. Therefore, it seems critical that we build a detailed model of children's understanding of length, and describe ways children learn to coordinate measures along multi-dimensional paths if we expect to promote more robust understanding that would support problem solving on complex geometric tasks involving multiple steps.

Lehrer and Chazan (1998, pp. 105-106) question the adequacy of the theoretical work of van Hiele as a sufficient basis for describing children's geometric concepts and spatial understanding. They note the incompleteness of van Hiele's theoretical framework for describing children's conceptual development, arguing that it lacks a practical focus on students' strategies, actions and various representational approaches. In particular, Pegg and Davey argue (1998) that the SOLO taxonomy of learning developed by Biggs & Collis overcomes several weaknesses of van Hiele theory: uni-dimensionality where multiple dimensions of operations are active, difficulty of prescribing instructional interventions due to oversimplification of learning processes (just target the right "language" of the students), and the generalization of structures that are not sensitive to content foci, or to realistic contexts in which mathematization is (1998, pp. 113-116). These researchers have attempted to forge a more comprehensive and practical theoretical framework by synthesizing the work of Piaget (1960) van Hiele (1986), and Biggs & Collis (1982).

There is a tension between uni-dimensional and multi-dimensional accounts of ways children understand geometric tasks. This tension becomes evident as researchers try to characterize interactions between children's mental models for geometric objects or relations (conceptual knowledge), their patterns of strategies for expressing or extending those mental models (strategic or procedural knowledge), and the productions of drawings, manipulations of tools, or verbal children exhibit as they apply their strategies and signify their conceptual structures in socio-cultural forms (perceptual and spatial-motor sequences of operations) (e.g., Chiu, 1996). These tensions appear weighty enough to produce cracks in the widely-held theoretical frameworks that have been used to explain development of geometric thinking (Clements & Battista, 1992), suggesting a need for a context-sensitive theory describing children's understanding of length.

Researchers have identified several critical theoretical issues for developing a context-sensitive theory on children's knowledge of length: 1) the need to account for interactions between children's drawings and correspondent mental (Ferrari, 1992; Fischbein, 1993; Gagatsis & Patronis, 1990) , 2) the tension between a teacher's representation and a students' representation (Bowers & Doerr, 1998; Kamii, 1995; Olive & Steffe, ; Schoenfeld, Smith, & Arcavi, 1993; Steffe, 1991), 3) the tension between a real object and geometric ideals indicated by that object (Clements & McMillen, 1996; Cockcroft & Marshall, 1999), and 4) the tension between logical and naive sequences of reasoning regarding geometric objects (Schifter & O'Brien, 1997; Simon, 1995).

To provide a platform for addressing these four areas of tension, we adopted a constructivist paradigm focusing on abstraction (movement from a perceptual level, to an internalized level, to an interiorized level) in keeping with the perspective of Steffe and Cobb (1988). These researchers suggest that the most abstract outcome is for children to isolate structures, patterns and actions out of the experiential stream of their actions, observations and interactions with others (Steffe & Cobb, 1988, p. 337). As children abstract spatial and numerical knowledge structures, they gain increasingly conceptual iteration schemes for length, eventually forging nested, coordinated images of units of units of length (Clements, 1997). Thus, children's representations of real objects become the object of analysis for the researcher, while analyzing at once their own models of the children's mathematical thinking. These models are inferred by observing the children's drawings, and by examining the fit between their drawings, their verbal accounts, and the products of their reasoning expressed in logical chains of cause and effect.

Within a constructivist account of learning and development, we hypothesize that children gain cohesive, relevant images for length and perimeter by reflecting on sub-dividing actions and iterating actions needed to step along paths or count iterations of a small object along another linear object (Carpenter & Lewis, 1976; Hiebert, 1981; Kamii, 1995; Piaget et al., 1960). Thus, children learn to measure segments in increasingly accurate and precise ways (Steffe, 1976; Steffe, 1991). But what learning mechanisms enable children to construct more robust structures from less robust precursors? Are there common cognitive plateaus at which children may rest, pausing before they construct more robust structures and schemes? In this paper, we investigate how children connect knowledge structures of number and space by examining sequences of strategies and sequences of representations for measuring path length and perimeter.

While children are able to quantify collections of discrete items quite early, perhaps before the age of two (Fuson, 1992, p. 248), it is much later that children begin to quantify length by counting up the number of iterations of a unit object along the extent of a longer object, as an advance in precision over undifferentiated perceptual comparisons, "...for a long time children are content with visual comparison, and only later do they think of putting objects alongside each other to check their estimate, while the concept of a measuring rule which can provide a common measure arises later still." (Piaget et al., 1960, p. 28). Piaget suggests that children are typically seven or eight years of age before they advance to metrical measures of space (Level IIIB by Piaget's scheme for development of length measurement) (Piaget et al., 1960, p. 394). In particular, we see quantifying length as a particular manifestation of a broader scheme for generating quantity (cf.: Case, Okamoto, Griffin, Siegler, & Keating, 1996; Steffe, 1991)). Further, Steffe's account of the genesis of quantity as a process of gradual abstraction beginning from regularity within attentional patterns to the eventual abstraction of collections of sub-collections of units offers a powerful interpretive framework for analyzing children's often divergent and apparently inconsistent strategies for enumerating length. We also draw on the work of Fischbein (1993), characterizing a tension between children's external, perceptual images of objects or movements and their interiorized, conceptual representations of related geometrical ideas. By postulating the most effective kind of knowledge as that which he terms a "figural concept", Fischbein articulates the value of *connectedness* for a child who gains predictive advantage from their theoretical-conceptual knowledge of geometric objects or, explanatory advantages from making multiple observations of a class of experiential events (figural perceptions organized into a sequence).

Extant studies on length suggest that children struggle to unitize the parts of objects they are quantifying (Cannon, 1992; Carpenter & Lewis, 1976; Hiebert, 1984; Piaget, 1952), that they gain from incorporating prevailing socio-cultural perspectives either through negotiated norms of practice for stepping and iterating, or building units of units (Stephan, Cobb, Gravemeijer, & McClain, 1998, April), or through the heightened sense of value associated with standard rulers as acceptable (though heavily concept-laden) tools for quantifying objects (Boulton-Lewis, Wilss, & Mutch, 1996; Nunes, Light, & Mason, 1993), and that they may overcome deficits in iterative and subdivisive processing to meet number line tasks by an inexact process of assigning numerical value to portions of line segments being labeled for their length in relation to the whole (Pettito, 1990). Chiu suggests that children's ability to compare path length depends heavily on several intuitions for path length such as compression, detours, complexity and straightness, arguing that measures of length are critically dependent on children's perceptual and motor operations (Chiu, 1996). On this view, children process their knowledge of length through complex procedural filters, intuitive schemes that children use to interpret their empirical and perceptual images. Unfortunately, this view fails to address the relation of number schemes to these intuitive schemes.

Only recently have researchers begun grappling with the need for a comprehensive account of learning geometry and spatial thinking as it relates to length (Clements, 1997; Clements et al., 1998; Lehrer, Jacabson, Kemney, & Strom, 1999; Pegg & Davey, 1998; Stephan et al., 1998, April). Recent work by Stephan and Cobb provides a careful elaboration of two cognitive shifts: 1) between *counting* footsteps along a path and *measuring* by iterating a foot unit, and 2) between iterating *single units* and iterating a *unit of units.* Lehrer et. al. provide a critical account of the role of discourse, and of the role of the teacher in promoting tools for representing length concepts. Still, there is a need for a comprehensive theory that extends to describe complex, multi-directional path length, and one that is compatible with recent advances in structuring theory proposed by Battista and Clements (1999; 1998).

Clements et. al. observed 3 strategies for children working on path length and measure tasks: (strategy 1) children did not connect the number for a measure with the perceptible extent of the line segment, but tended to estimate or guess as a way of assigning quantity to line segments, nor did they sub-divide continuous linear objects but guessed at operations, (strategy 2) children connected numbers to path length by relating movement to their number sequence, but struggled to coordinate these iterative structures along complex paths, attending primarily to marks indicating boundaries between pseudo-unitary parts of paths, (strategy 3) children quantified length by reflecting on interiorized images of nested sequences of iterative units that they imposed on linear objects (1997, p. 90-91). These researchers hypothesized a learning trajectory in which children construct more flexible, extensible schemes by reflecting on physical, external experiences of movement of shorter objects along a linear object. When children begin to re-present motor items, they introduce hash marks on drawings. This would mean a shift from strategy 1 to strategy 2. As children using strategy 2 are operating with figurative images they meet constraints: they notice a need for equivalence among sub-divided pieces, a need to avoid gapping or overlapping to preserve the equivalence. As children anticipate meeting those constraints, they tend to construct an anticipatory scheme, recognizing the crucial structure of iteration and establishing the value of imposing a mental image that would "guarantee" the preservation of the iterative structure. This would involve a conservation of length through measures of perimeter. The children in this study often failed to conserve length along multi-directional paths, although they were conserving length measures along uni-directional paths. We address this anomaly later in this report.

In the present teaching experiment, we expected children to initiate novel schemes and improve existing schemes for length while meeting questions, prompts and tasks posed by the researcher/teacher to guide them toward increasingly internalized, mathematicized, and justified knowledge structures [Hershkowitz, 1998 #198]. We examine the process of unitization as it occurs in the measure of length and perimeter, attending to the children's own logical processes for quantifying continuous space along the edges of polygons or the parts of complex, multi-directional paths. In summary, we work from a close, psychological frame of reference. As such, this study complements classroom-based studies of socio-cultural processes and curriculum-development research on children's understanding of length, promoting a substantial integration of several theoretical frames (Fischbein, 1993; Piaget et al., 1960; Steffe, 1991; Steffe & Cobb, 1988; van Hiele, 1986) needed to construct a comprehensive, domain-specific model of children's understanding of length and perimeter.

METHOD

Theoretical Framework and Design of Study

We devised a teaching experiment to promote understanding of length based on the hypothesis that as children coordinate their number concept and their 1D/2D spatial concepts they gain understanding of, and build more abstracted schemes for length and perimeter. Nevertheless, in keeping with an emic perspective (Strauss & Corbin, 1990) we sought to encounter and make salient only themes that were generated by our observations. Thus, we forged tasks and task sequences in anticipation of themes identified in previous research and in response to children's actions throughout the process of the teaching experiment. By attending to children's processes of abstraction, we sought to make the length iteration scheme an object of reflection both for the researchers and for the children learning about length. Thus, we worked recursively to promote reflection, especially whenever we noted perturbations or disequilibrium [Piaget, 1985 #8; Steffe, in press #81].

As researcher/teacher, the first author studied four children, Anna, Paul, Alex and Natasha through a teaching experiment focusing on length and perimeter, part of a broader study of children's geometric thinking (cite the grant here). In this report, we examine fourth-grade children's growth during the second semester of the academic year. We conducted nine videotaped teaching episodes with each of the four children. We presented the children with fixed-perimeter tasks on triangles and rectangles, often with the requirement that they move back and forth between drawings, physical objects such as plastic straws, and verbal explanations of their work in both.

A typical teaching episode focused around two or three tasks requiring the student to describe a complex path or perimeter by working with a physical model (e.g.: set of toothpicks, a plastic straw, a string, or within a tile floor grid) and asked the child to describe a complete figure, its parts and the measures of these parts both singly and in relation to the entire figure. We typically asked them to generate several possible polygons to fit a constraint (e.g., "find all the triangles having a perimeter of 24 units"), requiring descriptions of the constituent side lengths in addition to having them describe the sum of those lengths, and the perimeter.

We generated these tasks on the basis of (i) our previous observations and analyses of their developing length schemes and (ii) our hypothetical expectations of the openness of their schemes to modification and extension in response to probes and perturbations involving more complex task extensions or elaboration. By attending to their work in earlier episodes, I (first author) would design and pose tasks that would either fall within or just beyond their strategies and understanding for length, sometimes responding spontaneously to unexpected struggles or accomplishments by extending or alternatively, by compressing a task.

By selecting these four children from our sample of twelve from a given classroom, we examined a broad range of strategies associated with three previously-identified levels of conceptual growth on path length and perimeter concepts. We use the term "levels" as it has been used by Karmiloff-Smith (1990) and by van Hiele (1986) to indicate the hierarchical nature of knowledge that is increasingly integrated and comprehensive, and not to indicate a "water mark" along a continuous vertical scale of aptitude. We sought to forge sequences of tasks promoting children's growth through a sequence of increasingly abstract strategies for measuring length. This paper describes the emergence of the importance of correspondences between counting *collections* and counting *connected sequences of length items*, what we came to identify as the need to coordinate iteration along several directions within a 2D plane. Complete descriptions of the tasks and of the participants can be found in Barrett (1998).

FINDINGS

We met four themes in this teaching experiment. First, children quantify length by partitive operations, and later by iterative operations, gradually restructuring their internal images to record either partitive (additive) or iterative (multiplicative) images. Second, children coordinate their number sequence with their spatial image for objects as they shift from making visual comparisons between objects to quantifying the extent of partitioning operations. Third, children represent length when they curtail their own movements through linear space (traversals), exchanging experiential records of movement for symbols that represent completed actions, thereby reifying length as a punctuated collection of movements, leading to an iterative scheme. Fourth, children must coordinate several linear quantities for an object by disembedding the linear aspects of a 2D or 3D object in space.

Initial Levels of Strategy For Anna, Paul, Natasha and Alex

At the outset of our study, we found Anna used strategies that conformed most closely to the visual-guessing strategy, requiring perceptual objects and imposing guesses or estimates of quantity. Paul exhibited a strategy that was often flexible, reflective, and based on iterative strategies involving nested sequences of length units. Alex and Natasha both appeared to represent an in-between strategy of making and using perceptual marks to indicate movement while partitioning a line segment and maintaining partial connections between number schemes and length schemes.

[Insert Table 1 here]

Analysis of Anna's structures for length

At the beginning of the teaching experiment, Anna exhibited a perceptual strategy for understanding perimeter and length concepts. She required instant perceptual access to objects to find length. Anna often failed to construct her own units in situations where standard units were unavailable and an object was unsegmented. Likewise, she often failed to select appropriate standard units. Anna treated length as a verbal, number-based record.

We selected Anna as a participant in the teaching experiment because she was not yet able to operate on figural representations to measure compound linear quantities.

Anna's under-developed strategies for number operations and number sense constrained her work on length and perimeter tasks in comparison to the other twelve children interviewed from her fourth-grade class: she possessed a fair knowledge and practice with numbers less than 50, although she was prone to exhibit calculation errors on whole numbers greater than 20. She was also hesitant to think in terms of part and whole when operating on whole-number quantities. Although she exhibited mostly visual, perceptual strategies and not figural or representational strategies, we believed she would progress to figural representations, given adequate opportunities to reflect on records of her own motion, and on the process of creating those records of motion.

Perceptual Processing: Counting perceptual markers to partition length

When asked about length along a flexible plastic straw (of length 48 cm, notched at each 2 cm interval to indicate 24 parts), Anna counted sequences of perceptual markers along each section of straw. During the initial interview (2/07/97), I (the first author) bent a section of straw at several notch points and named the length for each indicated section. First, I told her that the entire straw had a length of "twenty-four". Then, I showed her a bent section that was one piece long, and then I bent it again, leaving just two pieces, taking care to bend the straw notch by notch and calling it two "straw pieces long". Next, I made four consecutive bends in the straw while counting aloud from one to four, describing the length of the resulting section a "length of four". She nodded in agreement. However, when I counted the pieces in a short section having a length of just two straw pieces a moment later, she interrupted. The following excerpt suggests her lack of an exact correspondence between perceptual markers (notches) and the parts of the straw:

R: For now we're just gonna say this is a length of two, or this is a length of one [bending the plastic straw to isolate first two segments, and then one segment].

Anna: Wait a second. Are you supposed to, like, count this as the end too? Like the end?

R: Well, let's see. Tell me what you mean. Let's talk about this for a second. [now I indicate a bent section just three straw segments long] How long is it between my fingers? How far is it between my fingers?

Anna: It's two. It looks like two.

R: Two what?

Anna: Two . . .

R: Those little marks, you mean? We could call them notches.

Anna: I mean, like, when you said it was four . . .

R: Like this? [holding up a section that is four segments long]

Anna: Yeah. I don't know how you got that. Cause there's only three notches here [pointing to the three notches and neglecting the notch at the bend indicating the terminal end of the straw section].

Anna focused on her perception of notches and not on her perception of gaps *between* the notches. Anna could not yet maintain an exact correspondence between perceptible notch points and figural intervals for length. Next, I pointed to the gaps between notches and explained that I counted the segments between notches rather than the notch marks. I introduced a narrative description of a small hamster taking steps that were just the length of the interval between successive notches on the straw:

R: I'll say that is a length of four hamster steps.

Anna: Okay. Because I didn't know if you were counting that [indicating the last notch at the vertex] ?

R: And I count, do you think you know why I'm counting it?

Anna: Probably because it looks like it's –part of that. Because, even though you bend it, it looks like its part of that, just because, like going straight across,

R: Um hmmm?

Anna: Like going straight across it seems that's the way you count it.

Anna's question about the last notch mark indicates her need for a way of identifying countable items. But she settles on counting notches to find length and not segments.

R: What'd you do with your eyes as you found the length?

Anna: I looked at the notches.

Next, I asked her to find the length of the entire straw, whether it would be 24 or not. She counted it by touching each successive notch and counting aloud up to 23. She did not count the end piece, but stopped when she touched the final notch along the straw.

This interchange with Anna typifies the struggle all four children exhibited when first describing the notched plastic straw; the straw device was at once difficult to interpret, and yet proved to be a powerful tool for modeling a class of polygons having one invariant feature--a perimeter of 24 units. While all four children were challenged by this set of misleading cues (the notches in the straw prove to be salient, tending to lead children to count the notches rather than the gaps between them), only Anna failed to shift her strategy to count gaps instead of notches.

Later during this first session, Anna adapted a new counting scheme for length that involved counting end points and notches. Since I had reported more "pieces" along that first section than she did, she apparently reconsidered her counting procedure. But how exactly did she change her counting scheme? When asked to describe perimeter or path length, she often allowed a perimeter that was inconsistent with the sum of the sides for a given polygon.

[insert FIGURE 1 ANNA: here]

For example, I asked Anna to tell me about the length of the individual sides for the first triangle she made with the straw. She described each of the three side lengths by counting both endpoints and the notches, reporting sides of lengths 8, 9 and 10, as in Figure 1. The sum of the side lengths she reported was 27, but she did not appear to sum them. She stated that the perimeter of this triangle would be 24, not 27. She allowed this same kind of inconsistency for a second straw triangle. She reported lengths of 7, 9 and 11 and said that it too would have a perimeter of 24. We believe she worked at a perceptual level without integrating the parts of straw figures into cohesive sets of "connected lengths" (cf.: Clements et al., 1998, p. 209).

Figural Operations: Re-presenting sequences of movement

During the first teaching episode (TE 1), Anna moved from a perceptual emphasis to a more figural understanding for length; she reorganized her length scheme, integrating her number scheme into her length measuring scheme. Specifically, she came to represent experiential records of movement around a rectangle shape by creating symbols that would stand for those completed action sequences. She segmented an entire path by taking steps and making permanent records of each successive stepping action.

At the beginning of this episode, Anna measured a path around a rectangular set of floor tiles by counting her steps altogether around the figure. She had been stepping around the perimeter of two adjacent tiles (two contiguous tiles have a perimeter of 6 tile edges) without finding a consistent number of steps to report (varying between 6 and 10 footsteps). Finally, after reporting several different numbers as the length of the path, Anna suggested pasting paper-notes down behind her heel each time she stepped. Thus, she established a set of records that could be re-presented and coordinated with her verbal counting sequence: she represented each of her footsteps around the edges of the tile rectangle by pasting one paper-note by her heel.

R: So what does that tell us?

Anna: Well, it tells us that 1, 2, 3, 4, 5 and 6. [She points to each of the pieces of paper around the rectangle pattern on the floor, pointing to the beginning of each footstep rather than the far boundary of each step] So that tells us that it's really six! [talking rapidly and more loudly now]

R: For sure then?

Anna: I had a good idea then!

R: So you draw me that picture now. Like the way the paper markers help.

Anna: I think it's a definite six.

Later, I asked her about just one side of the figure, pointing to a side that had four different post-it notes along its edge in the drawing she had made:

[insert FIGURE 2 ANNA, here]

R: How about this last side, do we know what [length] that is?

[Anna labels this side "3 steps" on her drawing]

R: How do you know it's three steps?

Anna: Cause you wouldn't count this four [here she points to the first piece of paper placed at the far corner of the tile figure from where she began measuring that side]. So it's one, two, three [pointing to the successive hash marks on the one side].

R: So you're counting actually, the pieces of paper that you used, and what do the pieces of paper mean? Why are you counting them?

Anna: Cause of the lines, so it would help my heels, like so I would know ... [the rest of this statement is inaudible]

R: So the pieces of paper are where your heel was.

Anna: Yeah. Um hmm.

R: So, when you say it's three steps, so how'd you know it's three steps? Cause of your heels?

Anna: [Yes]. Well, no, the pieces of paper, show my heels.

Anna no longer relied directly on counting perceptual items or immediate motions through paths. Instead, she employed symbols (hash marks along the lines of her drawings) to represent the already completed, but re-presentable action of stepping along a path. Notably, she did not step through the path again to support her argument, as she always had before. She relied on the figural operation of pointing to the sequence of paper notes to re-present her perceptual process of stepping around the tiles and counting her steps.

Changes in Anna's drawings during this episode illustrate a shift from immediate perceptual strategy toward more figural, reflective strategies. After initially representing her footsteps as ellipses that varied in size and shape, she began marking the end of her own steps with post-it notes and representing these notes on diagrams with small, numbered pictures. These later diagrams did not have the mid line through the rectangle that her earlier diagrams did (thus she omitted an irrelevant contextual attribute of the tile figure), and did not use pictures of a footstep. This trend toward simplicity in her drawings suggests the gradual attention she afforded the essential attribute of length, and decreasing attention she afforded irrelevant details (such as foot shape, or width of footprint, as in Figure 2).

Constrained by Absence of Perceptual Material

During teaching episode 3, Anna was shown a partially obscured triangle figure and asked to find the length of a hidden side. A child's jump rope (made with 2 inch long "macaroni-style" plastic sections) was arranged on the floor in the shape of a triangle and then obscured by placing a rug across one side and one vertex. She was not able to find, estimate, or guess the length of the obscured portions of the side, whereas each of the other three children offered estimations of it. Anna's struggle was consistent with our expectation at this stage of the teaching episodes. We believed she based her knowledge of length on perceptual images or motor re-presentations of traversals of objects. She was not yet able to extend her strategy for counting to non-perceptible objects.

[INSERT Figure 3 about here]

In trying to help Anna find a measurement that would relate to a unit--believing that she was not using numbers to represent iterated units but merely to report estimates of quantity that she accepted but did not understand—I helped her iterate a single pen cap (approximately 1.5 inches long) along the visible edge of the jump rope triangle on the floor. Anna did not really know how to do this and count without overlapping or gapping.

I noted in my field notes that her struggles to describe measures along objects seemed unremarkable, given her inability to carry out a basic iteration along an object.

Integrating Sequences Of Length Items By Iterating a Figural Collection

During the fourth teaching episode, Anna was given three 1-inch pieces of toothpick and asked to measure the length and width of an 8.5 by 11 inch sheet of paper. She used the three toothpick pieces only once, arranging them end to end (see Figure 4).

[insert FIGURE 4 ANNA here]

Having established a set of three items as a composite item, she treated the set of three as an image that could be drawn repeatedly. While she had used hash marks to record her own movements through a stepping process on earlier tasks, this was the first time she exhibited an iterated sequence of a unit consisting of three sub-units. Thus, Anna began iterating a unit of units through a notational advance: she represented a small collection of sticks along the edge of the paper she was measuring instead of counting each successive iteration of single sticks. Unlike her work with the post-it notes as records of her own footsteps, she now made drawings of a unit of units (the collection of 3 sticks), iterating a figurative object rather than the actual stick pieces. She used the image much like the post-it notes. It is critical to note that Anna recounted her strategy for using the sticks by pointing to the group of 3 sticks, drawing it separately, and reporting that she repeated the pattern by pointing to successive instances of the three stick image.

Restructuring Units of Length By Overcoming a Perceptual Limitation: Using the "Smallest Thing In the World" as Symbolic Markers

During teaching episode 6, Anna continued to exhibit a figural level of representations for length, yet she exhibited iterative operations. Anna drew and counted hash marks along paths to establish consistent-sized intervals. She eventually argued that hash marks could be placed close enough together along a line segment that they would finally represent very small pieces of length, "the smallest thing in the world". Two important shifts occurred:

Anna developed a local integration for units along specific dimensions, and she moved to subitize length objects for iteration (cf.: Fischbein, 1993).

[insert FIGURE 5 ANNA here]

The first shift in strategy consisted in using very closely-packed collections of hash marks to measure two edges of a circuit between four cities on a map (see Figure 5). She gradually filled up one entire edge by making hash marks and filling in between hash marks with more and more marks until she reached the limit (apparently she meant covering that edge). The other important shift in strategy was her development of local, sequential integrations of length items along more than one side of a figure. She said the length of the southern and western legs of the circuit (in a ratio of approximately 3:7) could have corresponding length values of 100 and 1500, a ratio much smaller than the actual ratio of 3:7. Later, Anna revised her statement to suggest that the shorter side would be 100, but the longer side must not be 1500, but 285 instead. Later yet, she suggested that perhaps if the shorter side were 500, then the longer side might be some multiple of 500. She was gradually coordinating perceptual comparisons of length along two sides. We believe this suggests she was imposing figural unit pieces onto her perceptual image of the map, pieces that she could manipulate through several iterations along each side of the map figure.

Coordinating corresponding properties between a measuring unit and linear collections of objects (tips & longs)

Anna later moved from a perceptual counting strategy to a figural-partitioning strategy, reflecting on verbal counting acts rather than perceptual images. During the seventh teaching episode, Anna attended to what she called the "properties" of the straw by insisting that she count both of the endpoints of any straw object being measured, since endpoints were part of the straw just like the notches. During this episode, she took her verbal counting actions as objects for reflection. Anna began to insist that she must include both endpoints whenever she measured along a straw since the endpoints would be an essential aspect of length. She spoke of the "property" of the straw necessarily including its endpoints:

Anna: One, two, three, . . . twenty-four, twenty-five. So this straw really has a perimeter of 25.

R: Okay.

[now Anna goes back and shows me that if she doesn't count the end, the first end, then you get 24.]

R: So which is right?

Anna: Twenty-five.

R: Why?

Anna: Cause you want to get like, you wanna definitely count these [pointing at the two ends of the straw], right? You don't wanna forget them.

R: Why?

Anna: Because um,

R: What do they mean?

Anna: They are a 'property',

R: Are they part?

Anna: Yes. They are a part of the straw and the length. And its more helpful, cause. Because if we didn't have these, its kinda harder!

For Anna, the endpoints of the straw were an essential part of the straw as an object, and the measurement of length needed to account for all the parts of the object. Anna tried to distinguish between two aspects of linear objects (the plastic straws): traversibility and the composition of a whole from its parts. She struggled with this distinction as she tried to isolate the essential length attributes of the straw. In the next vignette, Anna tried to isolate the property of length, while disregarding its other properties (being a plastic, tubular, striped, flexible object):

[Insert FIGURE 6 ANNA here]

R: What would you call that? [pointing to a 3 unit length of straw]

Anna: I'm not gonna take it as long. . . this whole property here is one long. Then if you put 'em together, you would get 4 long, because they're all properties.

Anna attempted to describe the *length* units by identifying "longness" and the endpoints. She described two countable parts of length:

R: Show me 4 longness on here [the plastic straw]

Anna: one, two, three, four. [stopping with three pieces]

R: Bend it so I can see just exactly four sticking out.

Anna: [she isolates a section of straw that is only 3 segments long]

R: But when I do this, it only looks like...

[interrupting the Interviewer:]

Anna: It's not three. You know why? Cause I told you we're counting this, right? This is a tip. The very tip.

R: Do you count tips when you're counting longness?

Anna: Umm hmm.

R: You do?

Anna: You're supposed to!

Anna explained that she must count both long sections between notches, as indicated by her use of the word "longs" in noun form, and the endpoint of the straw that would be attached to the first "long", and she called this endpoint a tip. She argued that what I took to be a section of length three must in fact be a section of length four, since I failed to count the tip (the endpoint of the straw). We note that Anna developed some increasingly descriptive and concise terminology for long objects; she also isolated two critical "properties" that one must attend to in measures of length, "longness" and "tips".

Discussion of Anna's Shifts in Strategy: Learning to Identify the Length Attribute, Partition, and Count Iterations

Anna was constrained by her struggle to identify and disembed length attributes from among other attributes in figures. She was also constrained by a more general lack of connections between her scheme for number and her scheme for space. Nonetheless, she was developing substantial figural strategies for partitioning objects: she forged increasingly consistent relations between the parts of a geometric figure and the whole, especially as she reflected on numerical labels for related (part/whole) aspects of figures and spatial objects represented by those figures. She struggled to coordinate her knowledge for length along different dimensions: she did not iterate a unit of length through a geometric path involving turns, nor did she abstract the length of an object from its other characteristics. Rather, she required immediate perceptual objects to measure length. Anna struggled to define units of length as iterable units: Anna continued to count the delimiters for line segments along linear objects (e.g., counting hash marks) rather than counting collections of iterable units along those objects.

Despite these constraints, Anna made several advances toward a figural structure for length based in iterative operations. She maintained some pre-proportional connections among the parts of several polygons she drew. At times she was able to investigate the length of polygonal figures in terms of connected units that were consistent around the vertices. She was able to use a figural partitioning process to construct a line segment of prescribed length in terms of units defined by a partitioning of a longer segment.

Re-presenting Schemes By Curtailing Pointing Motions

Anna advanced from a dependence upon counting motor-unit items (her own stepping motions) to a figural strategy of counting markers representing iterative motion along a path. During the first teaching episode Anna isolated critical length attributes and excluded non-critical details by stripping drawings of footprints of irrelevant detail, leaving only a line segment to replace each footprint. She had been constrained from constructing a one-dimensional measure around the two tiles by attending to the width of her foot. When the drawing media enabled her to shift to an emphasis on the distance along the edges and between post-it notes, she learned to treat the edges of the figure differently. We attribute this gain to her development of an abstracted image of the object, an image she built up by discarding the footprint image as a graphic tool, and curtailing her motor actions (Steffe, 1991). She began counting a collection of directional vector symbols (the sticky notes arrayed along each of the four sides), reporting a number to represent each sequence of iterative stepping motions along each side of the figure.

Nesting Figural Units Of Length To Restructure Length Images

By re-presenting the set of three sticks into a single, composite mental image that she could iterate along a linear path, Anna established figural material for a powerful nesting metaphor. This was an important advance. During Teaching Episode 4, Anna measured along an edge of a paper with a set of three perceptual, unit items (broken pieces of toothpick, congruent in length). She asked if she might have enough toothpick pieces to cover the length of the sheet of paper but I declined her request. She began drawing several groups of three segments in the same arrangement and shape of the three pieces she had already placed end to end at one corner of the sheet of paper. Anna iterated the set of three sticks by drawing several images in close succession, making a figural re-presentation of the composite of three and treating it as one item that could be counted as either 3 or 1.

By re-presenting the set of three sticks as a single, composite image, Anna established figural material for a nesting metaphor, suggesting both three units and one unit of measure by the same symbol (Steffe, 1991). This nested image provided operational efficiency: she could draw this set of three more reliably (and she could count reliably by sets of three). Further, nesting a composite unit of three and counting by three anticipates a unit of units, a process that Clements et. al. (1997) hypothesized in an earlier paper would lead to more abstracted units of units and more practical strategies for measuring length (Clements, 1997; Steffe & Cobb, 1988).

Finally, Anna's attempts to cope with the assignment of a very large number (1500 units) to a line segment less than nine inches long (TE 6) led her to search for a minimal unit of length, a pursuit that led her to reflect on the tension between perceptual and figural unit items (Pettito, 1990). By meeting a perturbation that suggested the possibility of unit items smaller than her own perceptually minimal unit item, she developed a verbal "tag" to mark such a transitional instance, calling these smaller unit items "the smallest things in the world" (Fischbein, 1993). We believe she was attempting to coordinate her notion of shrinking as an operation with her own prior experiences of measure based on visual scanning motions. Here she met and named a kind of unit item for length that would be too small to scan perceptually, appearing to the eye only as an image of overlapping hash marks without gaps. Gagatsis and Patronis (1990) describe a similar insight by children as they gradually shrank one dimension of a rectangle, keeping the perimeter constant, producing an apparent 'loss' of that dimension as the rectangle approached the visual "limit" of two line segments gradually merging into one. Incidentally, by defining the width of the iterable item, Anna gained a relatively constant unit of length, something she had not established earlier (cf.: Hiebert, 1981).

We claim that Anna's work with the minimal-size units precipitated a further gain: she soon developed the terms "longs" and "notches", and later, "longs" and "tips" to describe parts of the straw (TE 7). Anna modified her practice of attending to all the notches or hash marks, and began attending to the span between delimiters, which she called longs, and included the extreme ends of the straw, calling them tips. Her new strategy brought her closer to the essential unitization of the straw object by making its length features evident by her verbal representations (cf.: Lehrer and Strom (in press)). Anna did not conserve length along the perimeter although she was attentive to the conservation of length in single-directional contexts. Her partitive measuring scheme (count-all-partition-marks-and-endpoints) lacked integration with her representations and schemes for arithmetical operations.

Restructuring Length Schemes: Connecting Continuous Linear Space and Number

When Anna first met the straw task, she disregarded endpoints whenever she measured length. Later, she came to include both endpoints while measuring straw length. During one of the last teaching episodes, she met further perturbations when asked to draw decreasing length sections of straw: first four, then two and lastly one straw piece. Anna responded to a closely sequenced set of questions that brought her counting number sequence into conflict with her perceptual scanning sequences, prompting a figural representation of the straw in her drawings. To meet the perturbation, Anna invented the term "tip" for either endpoint of a measured object and sought to distinguish it from the unitary segments along the straw she called "longs". She recorded tips and longs with distinctive written symbols and counted both aspects of the straw to find length. She curtailed her length measurement operations to an enumeration of longs and tips.

While she did not establish an exact correspondence between countable length items--she failed to establish and maintain a part-whole scheme consisting of "connected lengths" that would correspond to the overall path length or perimeter--she came to recognize the critical aspects of subdivision and sequencing. In summary, Anna learned to identify and count subdivided parts (connected lengths) distributed along a linear dimension of an object and to recognize markers delimiting iteration along a segment: by curtailing her pointing motions, by constructing and projecting figural representations of composite units of length, and by pressing for connections between continuous linear space and number.

Analysis of Natasha's structures for length

Natasha exhibited figural strategies for partitioning continuous line segments and for measuring along segmented objects at the outset of the teaching experiment. During the first clinical interview, Natasha created and counted perceptual marks on her own line drawings of polygons. She also counted her own pointing movements and represented those by labeling line segments with number labels. But Natasha was constrained by a need to count motor-unit items as the basis for her partitioning strategy, being unable as yet to segment and quantify a path or perimeter without such perceptual material. When she met tasks requiring linear measures of groups of unpartitioned line segments (e.g.: in a path, or around a polygon), she often failed to maintain a consistent unit, but relied instead on sequences of motor items to find length along successive segments. Nevertheless, Natasha often coordinated her number schemes with her more perceptual spatial schemes for length.

Connecting Linear Space to Numerical Quantity: Measuring A Countable "Chain" Of Connected Length Units

Natasha often exhibited spontaneous connections between her spatial schemes and her numerical schemes, leading to effective strategies. For example, she was given a piece of string much like the plastic straw, a 48 cm loop of string partitioned into 2 cm intervals by a sequence of evenly-spaced marks (January 31: initial interview) and asked to find several possible rectangles having a perimeter of 24. Having formed and made drawings of two rectangles with perimeter of 24, one with measures of 4-8-4-8, and another of 3-9-3-9, Natasha stated that there must be another, one with "sides of 2" units each. This statement typifies how many children rely on arithmetic schemes for perimeter tasks, merely following a number pattern. But Natasha tended to shift between arithmetic schemes and spatial schemes. Natasha had been measuring by pointing to successive dots along the string and counting along each side, but now she checked her spatial scheme (of point counting) by invoking her arithmetic scheme for perimeter. She drew a rectangle without partitioning its sides, and labeled the shorter sides each "2". She counted to nine while pointing iteratively along one of the longer sides:

N: It doesn't equal 24.

R: But we know that it does, right?

[Natasha writes "9" next to one long side, but with some hesitation]

R: What would equal 24?

N: Ten and ten.

Natasha moved easily and purposely between her numerical scheme for finding "rectangular sets" of numbers and her spatial scheme for traversing edges of rectangles.

Structuring Figural Images Of Length: Curtailing Motor-Item Counting Along a Uni-directional Linear Path

When asked to measure the perimeter of a 3 by 5 rectangular tile figure embedded in a tile pattern on the floor, Natasha counted her own footsteps several times, producing several different measures, ranging between 15 and 19 steps. She eventually settled on the use of post-it notes (like Anna) to mark the path. This helped to stabilize her measuring activity by curtailing her physical traversals and perceptual scanning operations. She reported a length of five and one half steps, and a width of four. Although the tile grid of the floor was visible, she measured by counting her own footsteps and not by iterating tile edge lengths. Nonetheless, when asked to measure in terms of the length of a tile she quickly reported it as sixteen, easily incorporating the new unit within her length measuring scheme. This suggests a flexible scheme at a figural level of reflection.

Counting On Through Only One Direction: A Figural Length Scheme Impeded by Changes in Direction At the Vertices

Natasha established a practice of counting one-by-one around a figure with reference to motor or verbal items she intended as units of length--in which case she *accumulated* length items without respect to vertices along the perimeter--or, of counting units along sides of figures--in which case she *truncated* length items at each vertex. But she was not able to *count-on and integrate* length items along a given side and simultaneously include those items in the perimeter prior to teaching episode 2.

For example, I asked Natasha to find some rectangle shapes that could be formed from a 48 meter piece of garden hose, exhausting the entire length of the hose. The task required concurrent measures of parts and whole along a perimeter:

N: Umm. --- Umm. Like, eight. 20 and 20 is 40, and 4 and 4.

R: How'd you think of that? Does that work?

N: 'Cause, 20 plus 20 is 40, plus 4 is 44, plus 4 is 48....

R: Are there any other possible ways besides the 20 and the 4? Can you find one? What would it be?

N: Umm. 24 and 24.

R: Can it be 24 on a side?

N: It won't fit. Then there would only be two sides.

R: Two sides! So, what do you think?

N: I can't think of any others.

Natasha used a find-sum-of-sides-to-match-perimeter strategy~~.~~ I took this as evidence that she had not yet developed the ability to reflect on a figural image and re-organize the length of the sides around the figure to create a complex unit of units of length.

Anticipating integrations of length units along complex two-dimensional paths

While measuring rectangles (TE 3: April 23), Natasha counted along each separate side and later combined the four lengths to find the perimeter without expecting to integrate her counting acts through the sides of the figure; although she seemed convinced that rectangles could be formed from collections of line segments, Natasha did *not* anticipate that length would be preserved through the perimeter of rectangles, being made up of collections of collections of line segments. For example (TE 2: April 18), she said she doubted that one rectangle of perimeter 26 could be transformed into a different rectangle of perimeter 26 without affecting a change in the *sum* of the constituent sides. Thus, I was convinced she was not operating on a composite "unit of units" of length, but merely on several different length "units" juxtaposed into a rectangle.

To examine this hypothesis, I asked her to imagine wrapping a rope around two different rectangular shapes. Would a rope that wraps exactly around one wrap exactly around the other? First, I asked Natasha to form a 5 by 8 rectangle shape and mark it on the floor grid. She used three jump ropes to do so. Then, I asked whether she could form any other rectangles using just this much rope. She formed two sides, 7 tiles by 6 tiles, and then completed a rectangle. I asked her whether the rope surrounding a 5 by 8 rectangle in another room would wrap exactly around this 7 by 6 rectangle:

R: How big is this? Let's see. [I motion around in a circular motion with my hand]

N: [Walks around the interior path on the tiles, counting from 1 up to 22]

R: I want you to do a different thing. I want you to count…

N: On the lines?

R: Yeah, like you were walking on the lines.

[She counted from 1 up to 26, stepping twice around each corner tile and labeling the ordinal position of each successive tile edge.]

N: Twenty-six.

R: Okay. Now the other one was five by eight. Now how big is this one?

N: [pausing] Seven by ... six.

R: Now here is my question: If the other person in the other room made this [7 x 6] one and then cut [enough rope to wrap it], and then sent it to you, would that wrap around your five by eight "one" or not? What do you think?

N: Oh. Ummm. [long pause] ....

N: Maybe.

Although she had used the same length of physical rope to wrap around the two different rectangles, the one 5 x 8 and the other 6 x 7, she was not sure. After drawing the two figures, she still wanted to use perceptual materials (i.e., rope) to compare. For Natasha, measuring perimeter and measuring collections of side lengths were not necessarily related as part-part and whole. We believe that while Natasha expected to conserve length, she was not able to reconcile the subdivision of length through multiple dimensions, around a bent path.

Reflecting on the Rope as a Set of Flexible-Connected Lengths in Multi-Directional Space

Later, Natasha modified the 6 x 7 rectangular figure on the floor to shorten one side to 5 and then lengthened the next to 8. After doing so, I asked her to draw this new figure. She drew the shape, and labeled the side lengths. I asked her the perimeter and she said it would be 26. When asked to prove it, she said:

N: because, this, these two sides,

[pointing to the two longest sides]

that one and this one are eight, and that's sixteen, and then each of these are five

[pointing to the shortest sides],

and that's ten. So twenty-six.

Natasha was restructuring her scheme for perimeter: she represented the records of her previous figural partitioning without enacting motor items along each of the sides when she drew and labeled a 5x8 rectangular figure without working from perceptual material, but only by reflecting on her movement of the vertices of the rope figures. While she had counted the lengths of those sides earlier to make the first two drawings, she did not recount the side lengths. We believe she treated the sides as composite units of units which could be taken as material for reflection. She transformed the 6 x 7 figure into a 5 x 8 figure by merely changing the height from 6 to 5 and adjusting the width by pulling the ropes to yield 8 instead of 7 and thereby reflected on the dynamic connection between the component sides of the rectangles.

Coordinating Length Measures Along Several Directions: Curtailing and Integrating Figural Strategies By A Stepping Metaphor

Later still during episode 3, I demonstrated a way of counting-on around a vertex while examining a rectangle figure embedded in the tile floor:

R: So how many tile-lengths is it all the way around the whole thing? From here to --- [sweeping my arm in a circle]?

N: Twenty-six. [pause] No, wait. [now she looks and sweeps her vision through the four sides, apparently counting individual tile edges as she looks at each of the four sides in turn] Twenty-six.

R: What were you doing with your eyes as you counted?

N: Like, in my mind, this [placing her foot in the tile, between the two edges of a tile] would be one, like, from here to here [touching one corner of a tile and then the next corner of that tile along the edge beneath the jump-rope] and then,

R: Did you go all the way around and count 26?

N: Yes.

Here she integrated her strategies of counting-and-summing sides and stepping-around-a-shape-to-count-on-around to find perimeter, reconciling sides and perimeter in a part-part to whole relation that would preserve length through additive operations within two dimensional space.

Resilient Structuring For Length

During TE 7 (May 21, 1997) Natasha began counting points again, rather than connected lengths. I set out to perturb this recurrent and weaker strategy by having her draw and label some one-dimensional line segments in a sequence I expected to lead to a logical inconsistency. I asked her to show me a line segment that would have a length of four, then another of length three, then two straw pieces. When asked for a length of two straw pieces, she drew a line segment from the extreme end of the straw to the first notch, producing a segment only 1 straw piece in length.

[INSERT FIGURE 7 here]

R: How did you decide how to make it two straw pieces [long]?

N: Like in between two [motioning to two adjacent notches on the straw]?

R: So draw me a line that is one straw piece long.

N: [draws, again exactly the same length as before, one straw piece in length, but right away looks up as if puzzled]

R: Which one, the same one?

N: Wait. This would be two [now drawing two straw pieces, beginning at a notch, passing through one notch and stopping at the next one] and this would be one [pointing to the previous line segment that she had previously labeled as "2"]

I took her process of changing the labels and revising her explanations as evidence of a reorganization of her scheme for finding units of length in the context of the plastic straw; this was the return to a correspondence scheme for length I expected. I gradually revisited several earlier tasks, moving back through several inconsistent responses given earlier during the episode. In this instance, Natasha improved her length measuring scheme by drawing and labeling a careful sequence of decreasing length segments (see Figure 7) down to a length of zero, meeting the asymptotic case (cf.: Gagatsis & Patronis, 1990).

Limitations: Relying Upon Perceptual Strategies To Coordinate Measures of Parts and Whole

Late in the Teaching Experiment (June 11: TE 9), when asked to measure an unsegmented straw, Natasha first described it as having a length of 24 cm, and then formed it into a triangle, having side lengths she described as 10, 10 and 8. It is noteworthy that she drew and labeled the triangle-shaped straw and the unbent straw before finding an inconsistency between the two drawings (see Figure 8). She said the triangle contained "too much" in comparison with the length of the unbent straw labeled 22 cm.

[Insert Figure 8 here]

Natasha demonstrated a partially-integrated conceptual strategy for finding perimeter and path length. However, she reverted to figural strategies to verify her conceptual judgments of length during the final teaching episode (see TE 9, June 11, 1997, dissertation). Thus, we believe she was developing a sequential integration for linear units, but she was not yet able to coordinate multiple figural images of linear objects in a comprehensive way.

<u>Discussion of Natasha's Increasingly Integrated Schemes: Counting Sub-division Markers vs Iterating Unit-Segments Along Continuous Linear Objects</u>

Natasha exemplifies the need for children to maintain simultaneous attention to space and number as a means of establishing more cohesive and consistent models of actual objects, and thus more integrated knowledge of length (refer to the string incident, 9-2-9-2). Natasha moved easily and purposely between her numerical scheme for finding sets of four numbers that would sum in two pairs to 24, and her spatial scheme for traversing the edges of drawings, scanning with her own hand and eye to connect motions, images and quantity along a continuous dimension. While Natasha was attending to points along the object, she was not establishing an exact (1:1 and onto) correspondence between her observed points and the connected lengths of string delimited by those same points. However, she expected that her counting operations with those points would fit consistently with the arithmetic decomposition of 24 into two pair of doubles (e.g.: 9 and 9, and 2 and 2 did not satisfy her expectation since it yielded 22 rather than 24).

Natasha's apparent expectation of continuity and connectivity between numerical schemes and spatial schemes gained strength as she produced increasingly detailed drawings of polygons and paths. She was able at one and the same time to perform perceptual scans to segment and count along a continuous length, and to evaluate that perceptual/empirical length measure by checking whether it would be consistent with the rectangular sets of numbers she expected by way of her knowledge of rectangles (opposite sides are congruent). Natasha gained increasingly complex and integrated knowledge of length as she came to reflect on her own motions along objects and as she reflected on the relation between number labels and the length of the sides they described.

<u>Alex' Structure for Measuring Length</u>

Like Natasha, Alex often expressed length as the number of perceptual markers (*pseudo-units*) counted along an object. Unlike Natasha, Alex did not maintain exact correspondence along even a single dimension. Instead he employed visual estimation and guessing strategies. We sought to understand Alex' attempts to coordinate his measures of length along the composite sides of polygons with respect to his measure of the length along the perimeter and with his analysis of the length of the unbent straw.

Counting discrete items along an object: An inadequate perceptual strategy

In the first session Alex measured by pointing to the visible markers sub-dividing it. For the 48 cm straw marked with 2 cm notches, he counted 23:

R: And you are counting what?

Alex: How much there are.

R: How much what? 23 or 24 what?

Alex: That there's a length.

Alex always stopped at the last hole, failing to count the last interval along the straw. However, later during that same interview, when asked to find rectangles with a perimeter of 24, Alex drew a figure that he marked by making both hash marks and dots, placing a dot *in between* each set of hash marks, 24 dots in all around the perimeter of the rectangular figure. The dots corresponded exactly with the partitions created by the hash marks. Still, Alex persisted in stating that the straw was 23 long, and not 24. Apparently Alex allowed a loose fit between his perceptual images for length and his conceptual notions for iterative counting operations on length tasks, much in the same way that children in a study by Fischbein said a geometric 'point' would not have length or area in one setting, yet attributed length to that point in a different figural setting (Fischbein, 1993). Alex also had a strong tendency to depend on existing strategies or habits, combined with a willingness to approximate when no obvious conflict existed. This may explain his readiness to allow the straw to be 23 long.

Abstracting a 'wrapping' metaphor to iterate tile edges:
putting fringe on a rug

Alex had used tiles in two different ways by this point in the teaching experiment; he counted correctly through tiles to find the length of a row of tiles when asked to find a given distance between two grid lines in a tile floor, and as a way of covering a two-dimensional region (as in questions about area). During the second teaching episode on April 16, Alex restructured his figural scheme for length to accommodate a 'wrapping' scheme for finding perimeter. This marked a strategy shift: he began attending to edges of tiles rather than whole tiles to measure along shapes in the tile grid of the floor.

The interviewer asked him to consider how many "tiles" worth of fringe one would have if they were to wrap fringe around a rectangular-shaped rug in that room (the rug was roughly 2 tiles wide and 3 tiles long). After several rough estimates, he decided he needed "about 10 tiles" worth of string. After trying out a "10-tile" length of string, fitting it exactly around the rug, and then making a drawing of his work he began work on a related task: the interviewer asked him to imagine placing fringe material around another rectangular rug that would be 7 x 6 tiles:

Alex: [starts to walk around it, taking four steps inside the tiles but suddenly he stops, pauses, and begins talking:] 14, 12, ...[inaudible words here] 6 and 6 is 12 ---- 7 and 7 is 14 ---- 10 and 10 is 20 ---- plus 4 plus 2 is 6, ---- its 26.

R: You got 26?

Alex: Yeah.

He was invoking arithmetic schemes for numbers based either on the symmetry of the perceptual figure, or on his knowledge that rectangles have opposite sides equal (e.g. "5 and 5 is 10"). However, it was possible his invention of an arithmetic solution was not yet integrated into his figurative perception of the floor tile pattern so I asked him what 26 meant.

Alex: Twenty-six tiles?

R: Are there 26 square tiles? Where are they?

Alex: Seven across here [sweeping his hands along one edge], and six across there, [sweeping his hands along the next edge] and 7 across there . . .

At the time, I believed he was referring to the literal tiles, both rows and columns of tiles. His hand-sweeping motion toward rows and columns of tiles confirmed this belief.

As he stood looking at the tile floor and trying to describe how the 26 tiles could make up the perimeter of the rectangle pattern, I asked him once more to identify the 26 tiles. I expected that he would either count each corner tile once and report only 22 tiles, or count outside the figure and report 30 tiles. He paused, moving his hands and pointing along the rectangular figure, mouthing his words silently:

R: So show me the 26 tiles. Where are they? Can you step in one at a time and show me all 26?

[Alex walked around two sides and counted aloud up to 13, but seemed concerned. He halted immediately after stepping through the second side of the figure. I asked him to start over for the camera:]

Alex: One, 2, 3, 4, 5, 6, 7, [7 is the corner tile. He turns the corner, and says:] 8, but, no, it can't be, (with unusual vocal emphasis) [pausing]

R: What do you mean?

Alex: Because you gotta add an extra tile, or use it again.

R: Tell me?

Alex: 'Cause I already used this tile and I gotta use it again.

R: Why?

Alex: Because I will not get six unless I do.

Alex paused. He was still counting the corner tile as one unit of length, and so by the time he reached the final tile on this second side (6 tiles long), he had only counted on by five, reaching twelve, but he said "thirteen". His hesitation was apparently based on reaching 13 unexpectedly. Alex wanted to count the corner tile twice now, but lacked an effective justification for his new scheme.

Discussion of Alex' Extension of the Wrapping Structure

Alex needed to restructure his length measuring scheme to fit this tile-grid context; Alex was still constrained by his one-dimensional concept of perimeter. Alex depended upon arithmetical operations and his number scheme to extend his one-dimensional measures of length into the two-dimensional world of perimeters and polygons. Kamii (1996) pointed out the need for children to distinguish between discrete quantities of square tiles in contrast to quantities of area which consist of continuous regions. Her study emphasizes children's difficulties in distinguishing between uni-dimensional thinking and bi-dimensional thinking respecting area tasks. Alex's struggle to discriminate between the uni-directional length of one edge of a polygon and the bi-dimensional setting of the many sides of a polygon suggest that such discrimination may also be indicated for perimeter tasks even though they are understood in a more abstractly accurate way to be uni-dimensional objects (cf.: Schifter & O'Brien, 1997) .

Following this incident, Alex was asked to find the perimeter of a figure he had drawn having six sides, a rug with a "V"-notch cut from it [see figure]. Alex spontaneously imposed a tile grid structure onto the figure, and described its perimeter by appealing to the tile edges. For Alex, the newly adapted scheme of wrapping, and the now-familiar scheme for finding length along tile edges combined to enable him to analyze an irregular, six-sided figure and measure the perimeter of the non-standard figure he had drawn. His reorganization of his tile measuring scheme was flexible and extensible.

Discussion of Alex' Corner Mis-Counting

Even at the end of the teaching experiment, Alex was constrained by his inability to coordinate rows of conceptual units for length along a perimeter when he worked with squares as units of length within a grid. For example, Alex discussed a need for connections between two rows of tiles that he had used to show the outline of a seven by six rectangle shape as soon as he noticed the cumulative count of the two (12) was inconsistent with the sum of the two lengths of the edges (6+7). He resolved the immediate problem by counting the corner tile twice. When asked why, he was unable to account for his decision beyond explaining that he expected it to be 13. He had not yet isolated the characteristics of the two-dimensional tiles relevant to the measure of length. His inability to represent the coordination of units of units around two-dimensional paths and polygonal perimeters suggests that he was not able to anticipate such integrations in representing conceptual rows of intervals for length.

When he was asked to respond to another student who was double counting corner hash marks in a diagram of a straw triangle during the last episode (TE 9, June 12), he suggested explaining that you count the vertex notch two times since "corners count for more"; he explained that a path with many corners is longer since you must turn more as you traverse it. Thus, he allowed that the triangle he was looking at would have a perimeter of 27, with side lengths of 8, 8 and 8. We interpret this along two different frames of reference: turning as an angle measure, and coordination of line segments with a common endpoint. On the one hand, Alex was trying to explain how a path gained 3 in its length at the same time that it was bent into 3 parts forming a triangle. The argument Alex offers is consistent with an intuitive account described by Chiu as a deeply ingrained intuitive belief about path length based on children's informal experiences with compression (e.g.: objects like springs, or coiled ropes are apparently short, but potentially long (1996). Thus, Alex probably followed his intuition that paths with more "turns" in them are longer, since they are compressed (Clements, Battista, Sarama, & Swaminathan, 1996; Mitchelmore, 1997). On the other hand, Alex was representing figural images for rows of intervals, but he could not conceptualize the nested integrations required to coordinate successive measures for the unit of length, the length of a polygonal side, and the perimeter of the polygon.

Analysis of Paul's structures for length

Paul counted and coordinated continuous units of units as a basis for understanding and measuring length along one-dimensional paths, even multi-directional paths. Although he struggled on some occasions, he was not constrained by a need for perceptual items. Paul was able to reflexively operate on a figural image of a polygon, strong evidence of an interiorized image for the object. His actions and explanations suggest a powerful scheme for representing length in 2D. For example, when he was asked to find a way for someone to measure exactly enough rope to wrap around a rectangular tile figure without knowing anything about the exact dimensions of the rectangle figure (it was 5 x 8 tile edges) he suggested having them stretch out some rope along the floor to find how many tiles long it would be. This unfolding strategy suggests an interiorized representation for perimeter, a rare occurrence (Clements, 1997).

Paul exhibited conceptual strategies for identifying and iterating length units. For example, when he was asked to carry out several measurements along a figure embedded in a square grid of floor tiles, he explained that the tiles might be used to measure in *either* direction. He asked me whether to count the width, or the length of the tiles, implying that he understood either one could suit his purpose: he structured his own figural image of the tile by isolating relevant features of the tiles (length and width), and disregarding extraneous features. We hypothesize that his capacity to anticipate the iteration of tiles by stepping along an internal image of the tile-floor enabled him to evaluate the consistency between the tile as unit, and his need for a linear unit object.

CONCLUSIONS

In this teaching experiment, we reflected on the relationship of children's constructions of length and perimeter knowledge to teaching strategies emphasizing the use of tasks requiring closely connected imagistic and numerical descriptions of figures or objects. We worked toward the confluence of two kinds length and perimeter knowledge: the children's models for quantifying length knowledge, consisting of representations of length and perimeter situations in their role as student, and our emergent models of the children's thinking about length, consisting of representations of the children's structures and operations for quantifying length as teacher/researcher and researcher. As we planned tasks and interventions in the form of questions and extensions to earlier tasks, we reflected on ways the children had structured their own work, the language they used to explain or justify their strategies, and on constraints the children met. We formulated hypotheses about their knowledge structures (connections, relations, and internal representations) and about the ways of accessing and applying those structures to perimeter tasks. On the basis of informed hunches regarding their representations and schemes, we built up models of their ways of thinking about length and perimeter, concurrently subjecting the models to criticism and question on the basis of the children's strategies and changes in those strategies. Below, we discuss the overall shifts in strategy that we expect to generalize beyond this study, and we give an account of the mechanisms for such strategy development.

We believe the shift from describing length within a uni-directional context (as a partition of one line segment into unitary parts) to the abstract capacity for describing length in a multi-directional context is based on the development of an invariant, multiplicative relation between a unitary line segment symbol (a straw piece, or a tile edge) and the complex one-dimensional, multi-directional path set in a 2D plane (as is the perimeter of a polygon).[1] Children gained abstraction for length and perimeter concepts as they improved connections between their counting scheme, their partitive scheme and their iterative scheme. Lastly, we discuss the sequencing of teaching moves (issuing tasks, problems and questions) in relation to gains in this domain. This account of teaching and

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[1]In measurement theory, one always assigns length by setting up a correspondence between the natural numbers and a line (the number line), a correspondence that depends (Holder, cited in Narens, L.(1985), MIT Press) on the assignment of two distinct points along the number line which we name 0 and 1 (thereby avoiding the absurd case where there every real number maps to precisely the same point in space, producing an infinity of infinite values, and eliminating the instance of any one finite measure). Thus, in a logico-mathematical system, length depends on the proper mapping of the Real Numbers to a particular line in space. But in the cognitive-psychological realm, children appear to develop length knowledge much differently, via intutions gleaned from increasingly quantifiable sequences of iterated line segments in comparison to longer line segments. Thus, development depends not on a mapping, but on a spatially constructed sequence, an iterative building up of longer segments and collections of segments, or conversely, a partitioning of segments into a unitary segment. In either case, the assignment of number appears to be strictly a way of accounting for magnitude, rather than being a relation. Thus, it makes a great deal of sense to find that children's thinking gains abstraction and generalizability, helping them analyze multi-directional path length, when they begin to forge an invariant, multiplicative relation (an iterative operation based on the correspondence between unitary fragments and numerical value of a line segment resulting from the iteration). For children who are still decomposing and composing a whole from its parts, there is little sense of a mapping from real numbers to represent an iteration of a unitary segment.

learning clarifies a domain-specific tension between a teacher's identification of a child's mathematics (specific conceptual structures for a domain) and the generation of tasks that promote specific constructions based in the hypothetical structures of their existing knowledge (Tzur, 1999, p. 412) .

The children in this study appeared to progress along four levels of strategy for length: level 1, level 2a, level 2b and level 3. We believe progress through these levels follows from increasing integration between conceptual knowledge and figural knowledge relating to the measure of length. Geometric knowledge becomes most integrated and flexible when it consists of figural concepts rather than consisting either in mostly figural knowledge, or mostly conceptual knowledge (Fischbein, 1993).

For example, Anna relied upon figural images, to the exclusion of her own conceptual notions for measuring. Moreover, she failed to organize her work on tasks independently, relying often on "local" sequences that did not fit together in a conceptual framework: she employed three different ways to measure along the map path (TE6) without comparing them or integrating them. In contrast, Paul maintained close interactions between his spatial and numerical schemes that were more numerous and extensive than the connections of any one of the other three children. For example, he checked to see what type of unit would be appropriate as he approached several different tasks. He spent less time on perceptual interpretations and sequences of activities while working on the Map Task than the other three students (Barrett, 1998, pp. 293-297).

Paul and Natasha evinced more sophisticated strategies for length than the other two in the study. It was Natasha and Paul who connected their arithmetic and pre-proportional reasoning (for number) to their spatial reasoning most often, and with the most success. Alex and Anna tended to allow disconnections to exist between their schemes for number and space: Alex by attending to number schemes to the exclusion of spatial schemes, and Anna by attending to spatial schemes to the exclusion of number schemes. Using more sophisticated strategies for measuring length depends on connecting arithmetic/pre-proportional reasoning with spatial understanding/reasoning. This conclusion is also consistent with previous research investigating the connections of spatial and numeric thinking (Barrett & Clements, 1996; Clements, 1997; Clements, Swaminathan, Hannibal, Sarama, & Battista, in press; Ferrari, 1992; Hiebert, 1981; Pettito, 1990) .

Children gained abstraction for length and perimeter concepts as they increased the correspondences between their counting scheme, their partitive scheme and their iterative scheme. These three operations are critical for children as they work to understand measures of complex paths or polygons. Anna, Natasha and Alex all demonstrated growth when they coordinated these schemes. For example, when Alex was asked to find the measure of the perimeter of the tile floor with a rectangular-shaped set of floor tiles in a 7 by 6 arrangement, he found that the perimeter was 26 by linking his iterative scheme for number--doubling 13 (a sum of 7 and 6) and finding 26-- with his partitive scheme for images of the four related edges of the rectangle. He decomposed 26 into collections of sub parts (5, 5, 8 and 8) and related those to the 7 x 6 decomposition (7, 7, 6 and 6), maintaining all the while his correspondence between numbers and tile edge lengths, a correspondence between his number scheme and his iterative scheme for counting tile edges.

This kind of interconnected correspondence between related schemes is critical to mathematical modeling and provides an example of constructed--rather than imposed--connected representations. The value of connecting representations has recently been discussed (e.g.: Bowers & Doerr, 1998) in an effort to clarify the critical role of the student who must be an agent who chooses to connect several representations and not merely a recipient of some connected set of representations. These researchers suggest that whenever connected representations of mathematical ideas are given to the child rather than created by the child there is a necessary loss of relevance and meaning. Our findings support this claim. Tzur's (1999) identification of a developmental sequence of rational-number structures for operating on fractional numbers highlights the importance of guiding children to coordinate schemes for iteration and number. Children may forge composite units that supersede a single unit whenever they shift from a partitive scheme for combining parts to make a whole to an iterative scheme where rational numbers are taken as unit items that can be integrated into composite numbers that have a referent whole that can be exceeded. Tzur described children's early, partitive approaches to understanding improper fractions as *additive strategies*.

In our study, the children who evinced a partitive scheme were not inclined to expect that counting around an entire figure would be the same as counting along the several sides and summing the results. For example, Anna did not notice an inconsistency in reporting side lengths that summed to more or less than the length of the straw that was used to form the triangle from its three sub-lengths. Anna's focus was not on the iterative strategy of moving a unit along a sequence of positional shifts, but rather on the immediate collection of line segments defining any given side. For example, the concurrent membership of any particular straw piece or line segment along a side of a straw to its side and to the perimeter of the straw was not coordinated. This follows from the necessity of maintaining a part-whole reference to each of the separate sides, in contrast to other children who succeeded in setting up an abstracted reference to any particular side length, emphasizing the iterable unit of length rather than the unit of a side length.

Alternatively, whenever children in our study iterated a unitary segment by setting up a relation between the unit and another longer segment, they were able to measure the polygon as a collection of segments represented by numerical relations to the iterable unit of length, finding correspondences between measures of sides, sums of sides and the cumulative "counting on" around the figure. For example, Alex began to expect consistent values for the whole and for the sum of the parts when he adopted a wrapping metaphor. Likewise, Paul used wrapping to enumerate length during the first interview.

Implications. Six instructional strategies supported children in creating and recognizing relevant length structures, and in reflecting on relations among measures of length taken along a complex figure. While the list is not prescriptive as a sequence, nor is it comprehensive, it provides a broad overview of the teaching moves that characterized successful prompts and interventions in this study: 1) Give students opportunity to explore perimeter by extending their knowledge of length before introducing formal approaches to measuring area. Students appear to need specific practice in isolating features and properties of objects they employ as units to focus on the particular dimension they are measuring. 2) Prompt students to coordinate length measurement in two dimensions by having them report on perimeter both as a single number and as a composite sum of side lengths. Have them make statements relating the lengths of the sides to the perimeter by counting on around corners. Students experience particular difficulties when finding the length *around* the corners of figures. There is a tendency to single-count the corners as if they were only a single-edged object instead of a two-edged corner piece. 3) Help them to identify relevant and irrelevant attributes of objects for measuring. 4) Help students move physically (walking or tracing) along the objects they are measuring. Be quick to differentiate between counting steps along a linear object and counting the parts of the object once it has been partitioned. 5) Have students make drawings as records of their measurement activity to develop part-whole connections between the overall perimeter and the sides of a figure. 6) Contextualize students' measuring activities by creating narratives or stories to situate their measuring activity within meaningful contexts. Measurement takes its meaning from comparisons of real objects, and as such, children's schemes for measuring linear objects become more sophisticated when they are grounded in realistic situations that motivate comparison.

Whenever the children in our study were able to interpret the figural, imagistic information they gathered and coordinate it with more richly structured mental models of geometric ideas (cf.: Battista, 1999), they were able to resolve perturbations and create iterative images or models of length measures. By attending to children's ways of enumerating length for specific objects and relating several measurement actions within the framework of an account of an entire complex path or polygon, we generated tasks that extended the children's enumeration schemes through iteration of unit items: I promoted shifts toward iterative rather than partitive conceptions of length and perimeter by setting tasks that emphasized relations between several side lengths, and between the lengths of entire collections of side lengths in relation to the lengths of their constituent sides.

Developmentally, the process of measuring length appears to support children's understanding of quantity and number sequences, complementing children's experiences in the discrete world of counting and number (Piaget, 1952). How do children learn to distinguish between continuous and discrete quantity to describe length? In this study, the children often tried to coordinate their experienced iterations along a linear object and the reified markers along that object symbolizing a previous movement through continuous space. Whenever these two sequences did not correspond exactly, there was a perturbation involving the symbolized *unit* (a continuous unit) and the symbolic *item* (a discrete item). To meet this perturbation, the children worked to distinguish between parts of a continuous quantity (the un-segmented linear object) and the delimiters used to partition that continuous quantity (the hash marks).

Steffe suggested in his description of the generation of the concept 'quantity' that children understand counting fundamentally in terms of measurement activities: "My contention is that although the involved experiences may differ, the elementary operations that generate measurable quantities do not differ from the operations that generate countable quantities and, from this perspective, counting is a special form of measuring." (Steffe, 1991, p. 77). This being so, how do children understand the different kinds of counting? At what point do children learn to distinguish an extensive quantity, being expressible as a numerical quantity, from an intensive quantity which has its expression only in terms of its relation to another quantity? From a mathematical standpoint, a ruler represents the fusion of the discrete and the continuous in that it stands for the iteration of one object along another as a means of describing the ratio between the two (see Gravemeijer et. al. 1999, and Moss & Case, 1999). Perhaps linear measurement is better understood as a special case of rational number concepts than as a process for comparing dimensions of length or perimeter.

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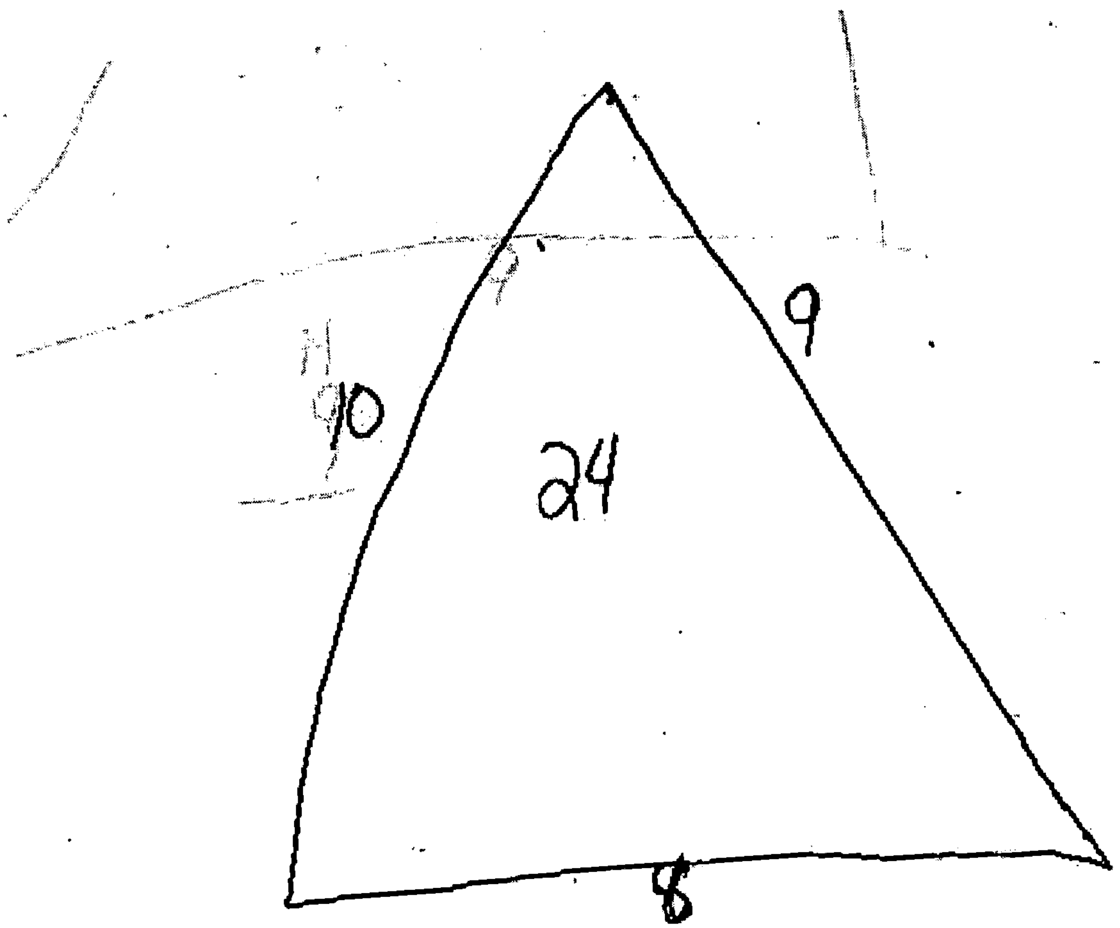


Figure 1. Anna's Drawing of the Straw Triangle During the Initial Clinical Interview, February 7.

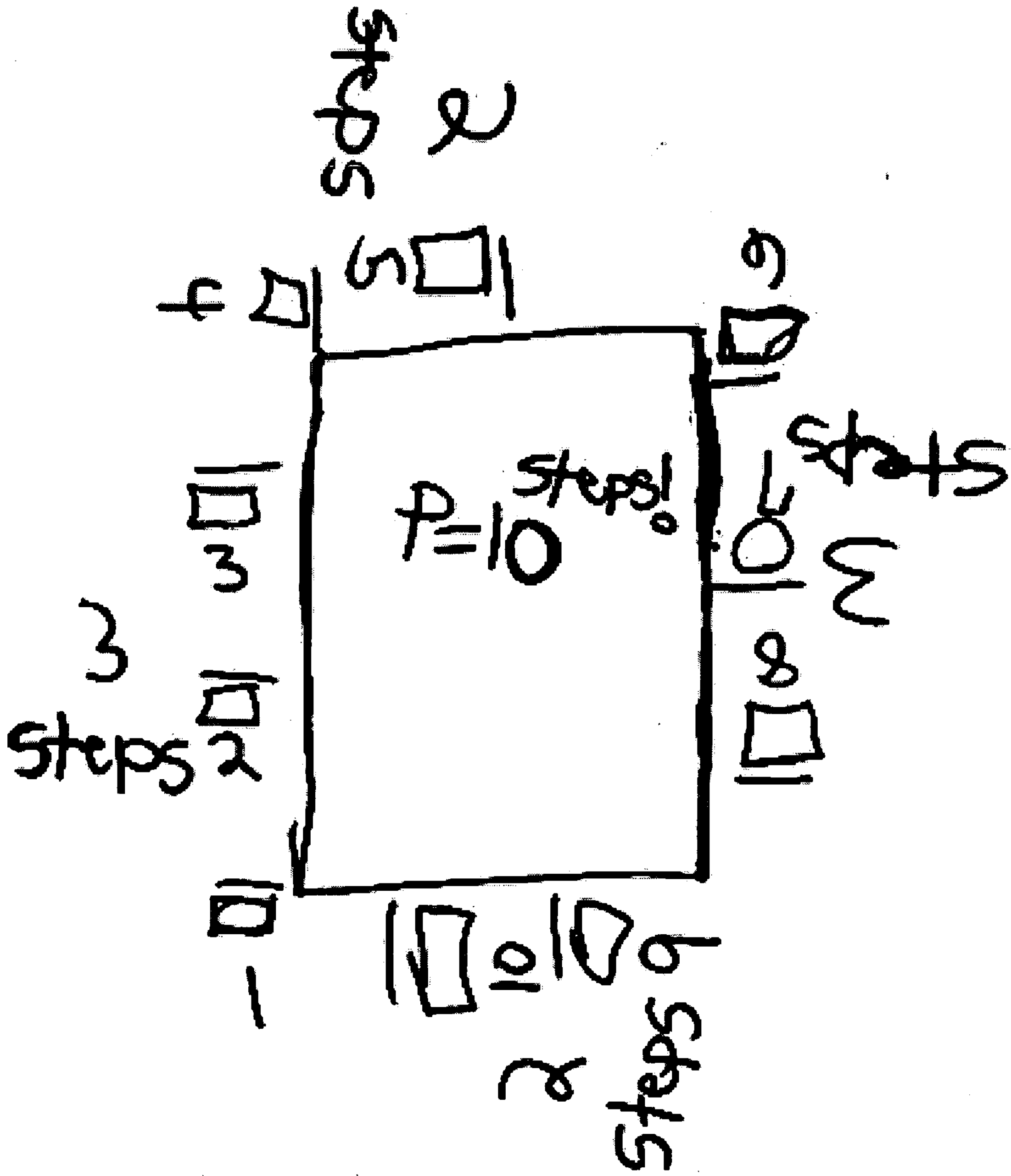


Figure 2. Anna's Drawing of Her Sticky-Note Measure of the Perimeter of Two Floor Tiles During Teaching Episode 1, March 7.

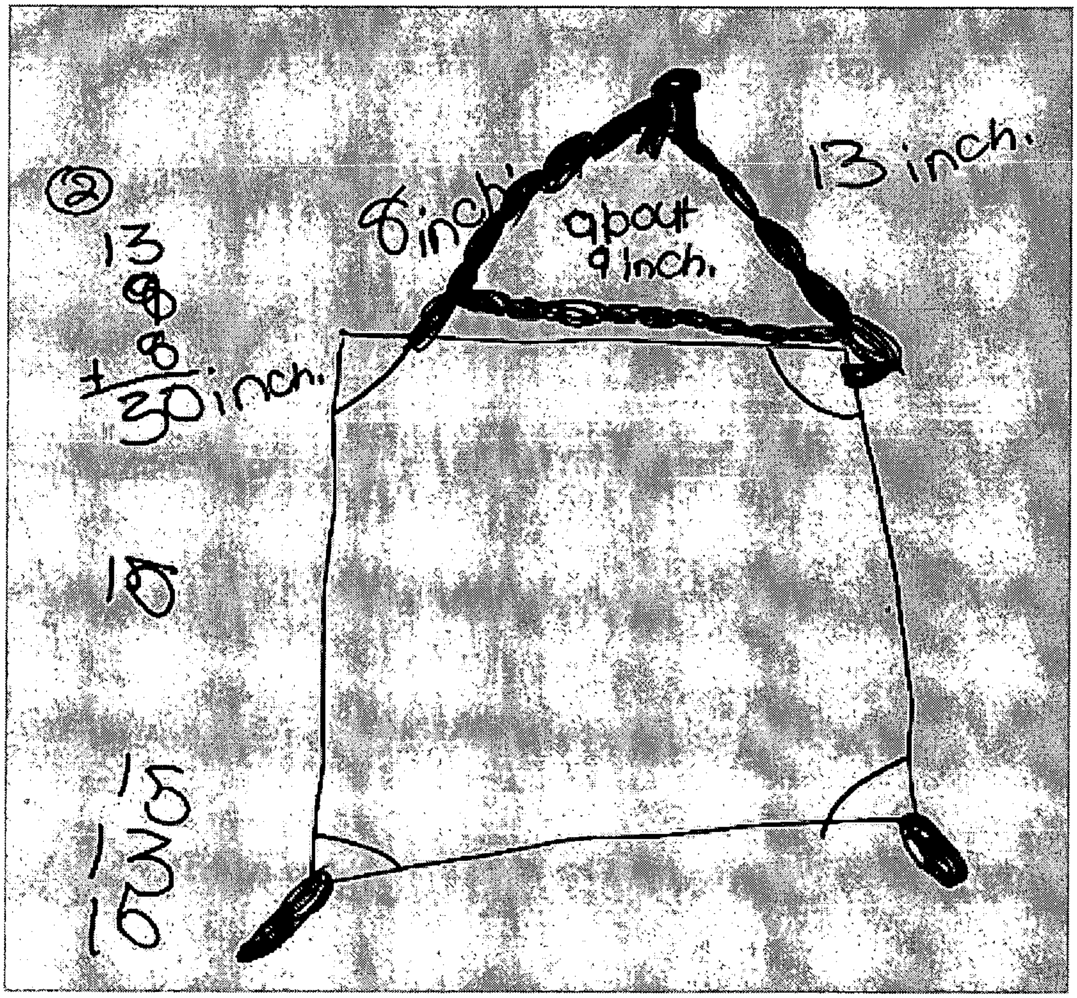
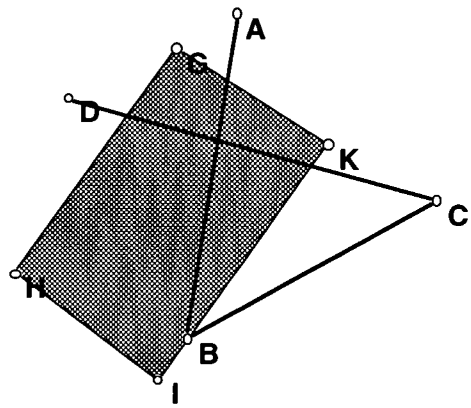


Figure 3A. Anna: Obscured Triangle Shape and Rug, as Drawn by Anna



ABCD represents the jump rope as it was configured, with A and D the handles. GHIK represents the rug that was lying on top of the jump rope figure, obscuring much of the triangle, but also giving an impression of another triangle, close to KBC.

Figure 3b. Anna: Obscured Triangle Shape and Rug, As It Was Actually Configured On the Floor.

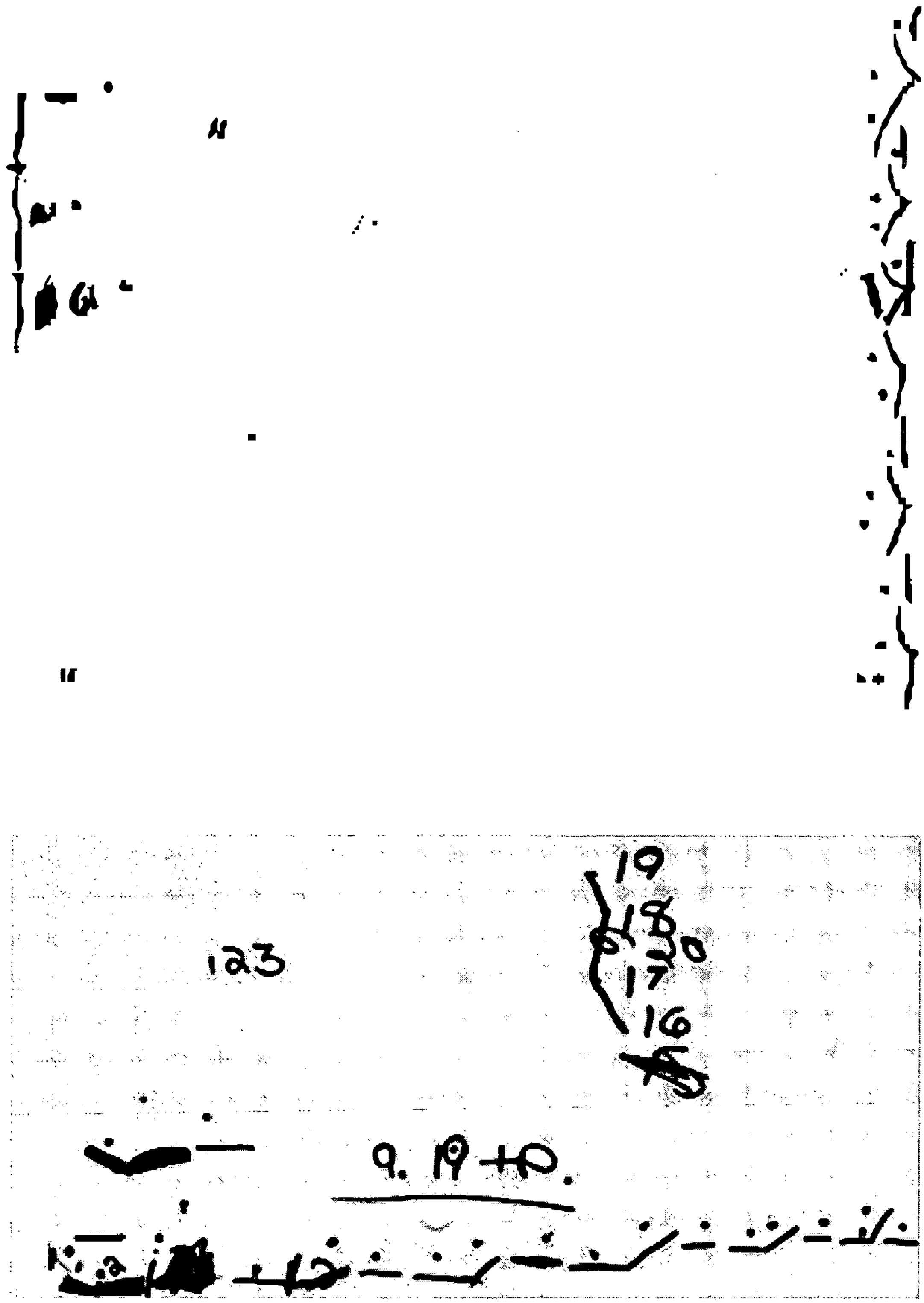


Figure 4. Anna, Using Three Toothpicks for Measuring Along A Sheet of Paper.

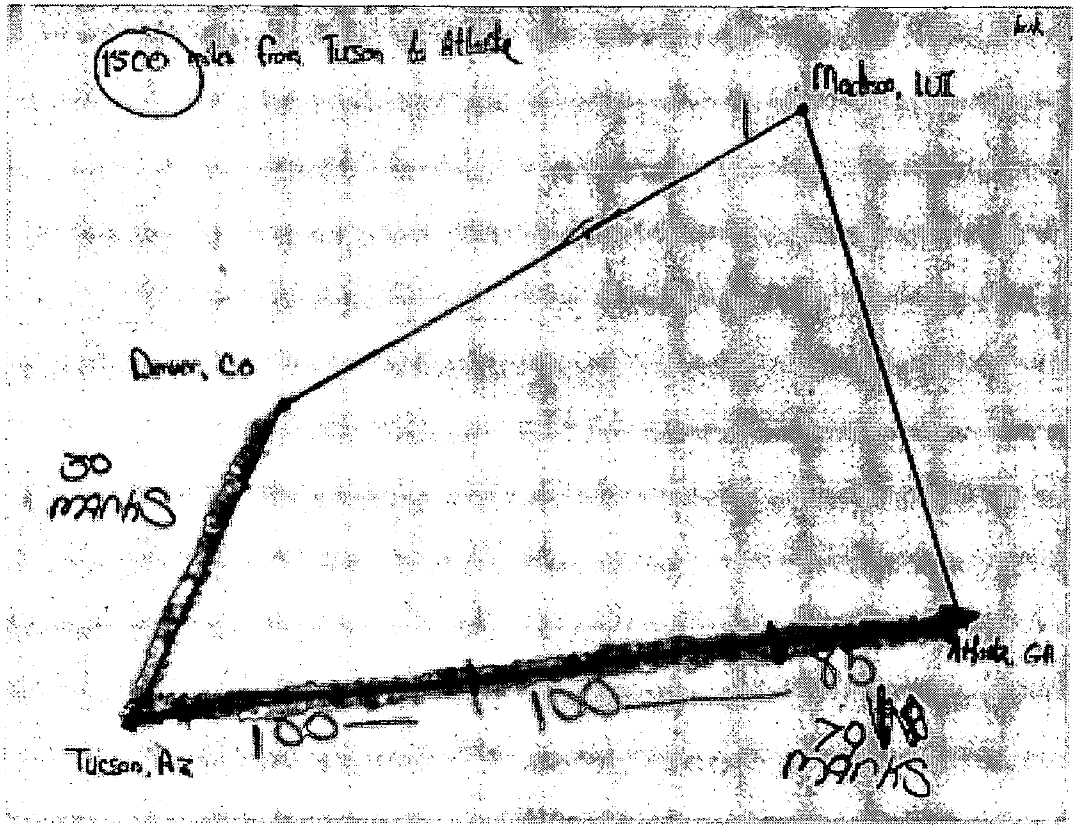


Figure 5. Anna, finding the length of a trip around a circuit of 4 cities on a map, having the length of the segment from Tucson to Atlanta marked as 1500 miles.

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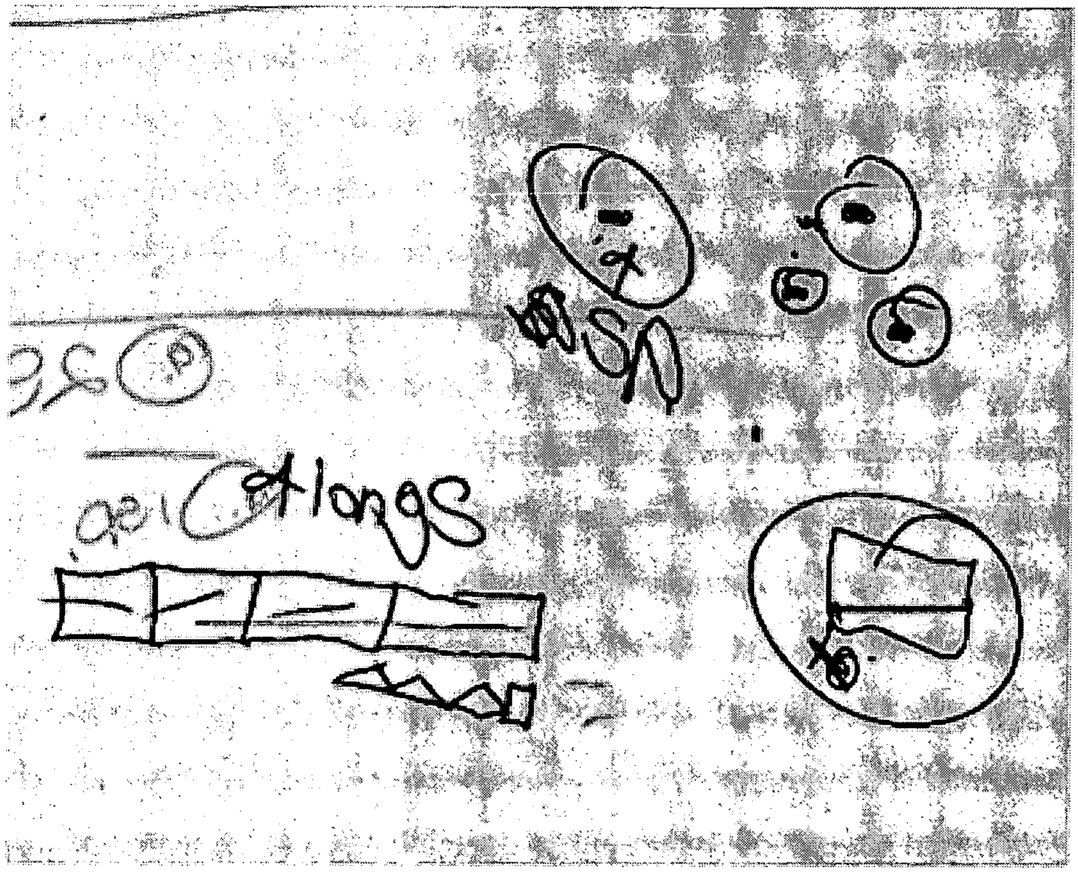

Figure 6a. Anna: Showing the Meaning of Straw Pieces and Tips and Longs During Teaching Episode 7

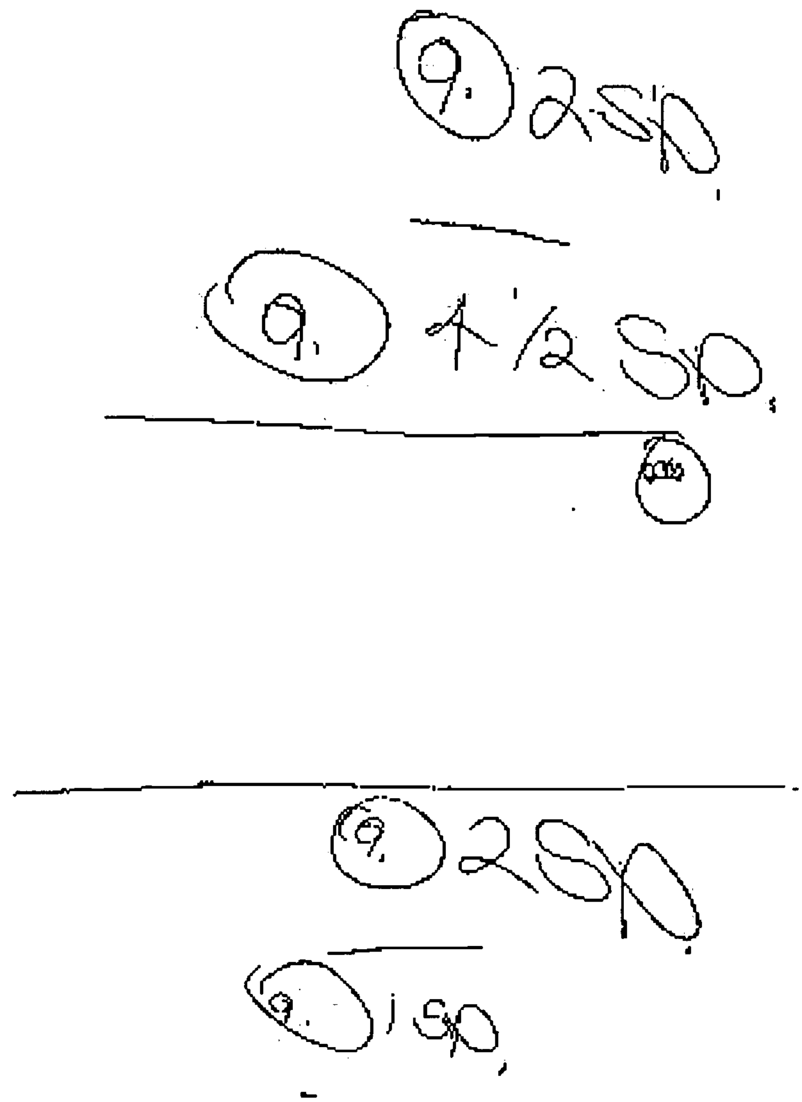

Figure 6b. Anna: Showing the Meaning of Straw Pieces and Tips and Longs During Teaching Episode 7

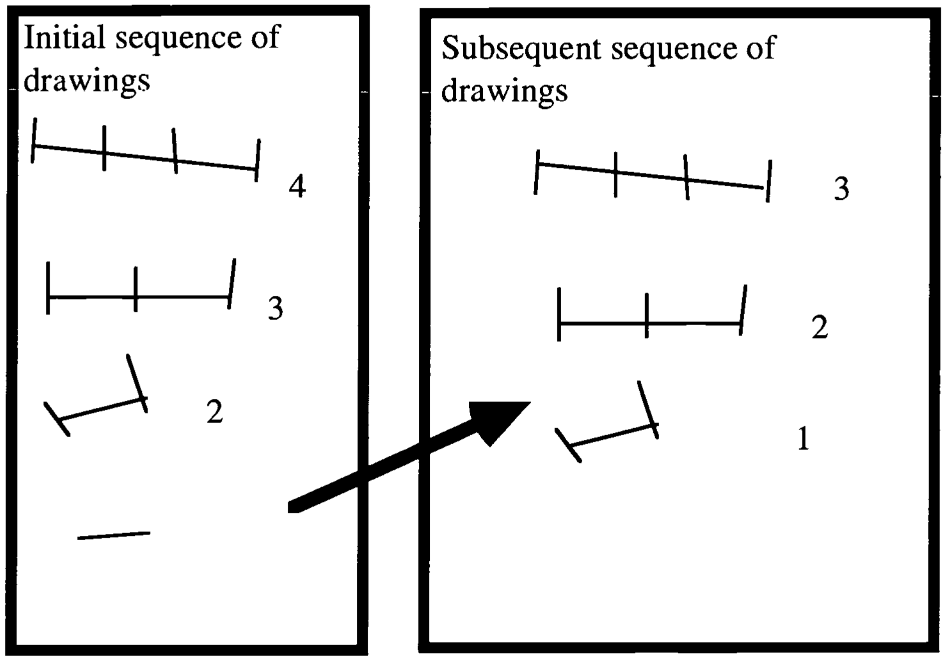

FIGURE 7. Natasha's parallel sequences with measures of incremental straw lengths.

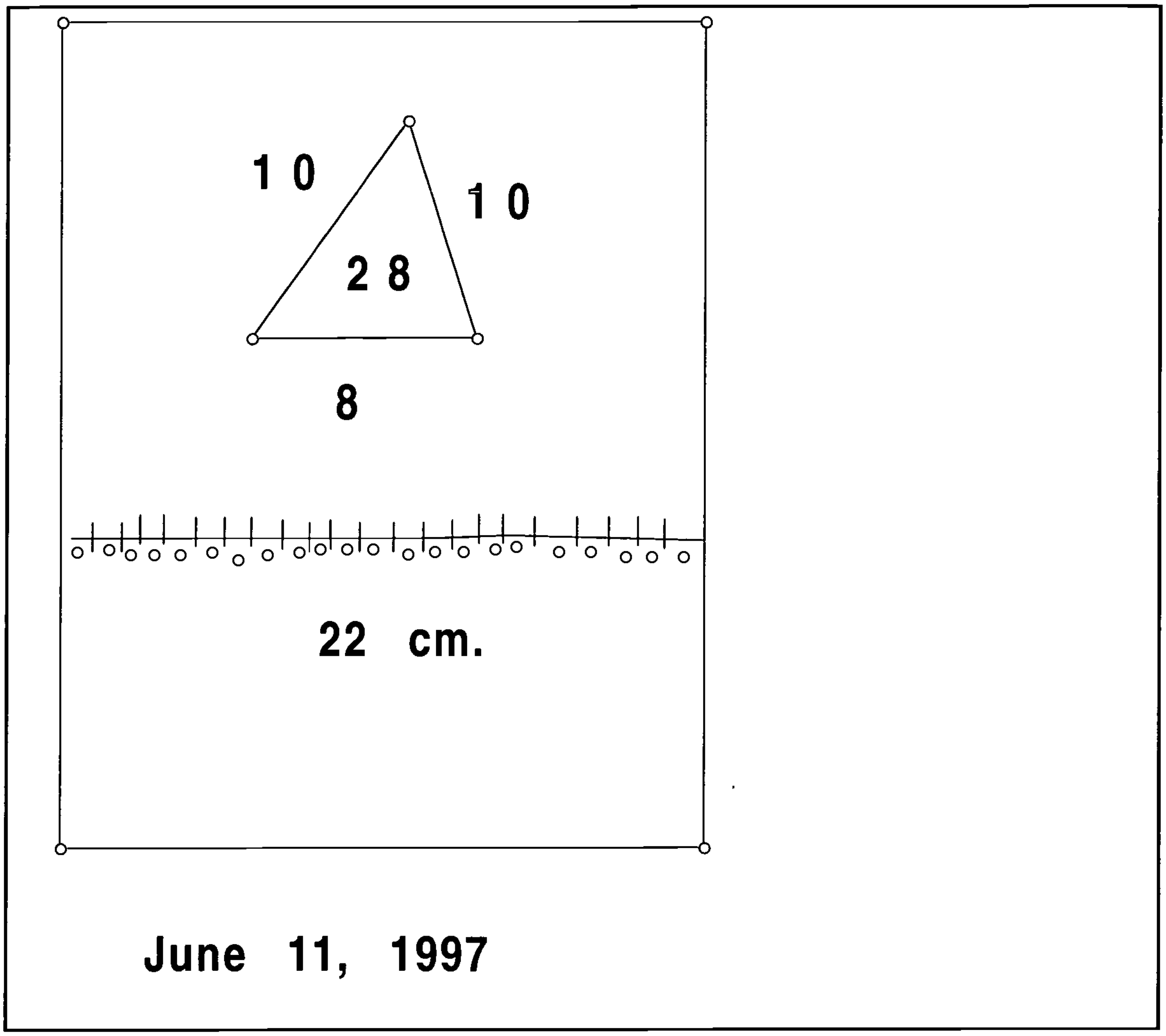

Figure 8. Natasha's drawing of a straw in both a folded and unfolded position.

Date	Session #	Topic
February 7	Clinical Interview (part A)	24-Part Straw used to form Triangles and Rectangles.
February 21	Clinical Interview (part B)	Draw/label Triangles and Rectangles of given perimeter.
March 7	Teaching Episode 1 (TE 1)	Working to find perimeter of rectangles marked by corners within a floor grid of tiles (standard 1' by 1' square tiles).
April 16	Teaching Episode 2	Using ruler to measure a particular length of string, use string to find girth along two dimensions of a cardboard box.
April 22	Teaching Episode 3	Find perimeter and side lengths of a triangle formed by a jump rope, lying partly beneath a rug on the floor.
May 2	Teaching Episode 4	Use 3 toothpick pieces to find length of various objects on a table top.
May 8	Teaching Episode 5	Use Geometer's Sketchpad to find several examples of triangles having a particular perimeter (30 cm).
May 16	Teaching Episode 6A	Map Task: Find the total distance along a quadrilateral shaped travel route, where only one side length is given.
May 19	Teaching Episode 6B	Use Geometer's Sketchpad software to explore triangles with a fixed perimeter.
May 21	Teaching Episode 7	Clinical Interview revisited (Straw task)
May 23	Extension of TE 7	Using Straws and Geometer's Sketchpad to describe straw figures.
June 6	Teaching Episode 8	Finding hexagons of given perimeter, making drawings and labeling.
June 12	Teaching Episode 9	Points vs Segments task (Cannon) and triangle drawings from straw figures.

Table 1. Schedule of Sessions with Anna

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