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ABSTRACT

Four students participated in a two year study (fifth- and sixth-grade) focused on the development of their understanding of multiplication of fractions. During the first year, all students received individualized instruction designed to encourage them to build on their informal knowledge of partitioning to solve problems involving multiplication of fractions. During the second year, all students received similar individualized instruction 4 times over a period of 9 months. They also received classroom instruction focused on algorithmic procedures for multiplication of fractions. In the long term, all students consistently drew on their informal knowledge of partitioning on their own to solve problems. However, students' thinking was also dominated by their knowledge of algorithmic procedures at times. (Contains 43 references.) (Author/ASK)

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LONG-TERM EFFECTS OF BUILDING ON INFORMAL KNOWLEDGE IN A COMPLEX CONTENT DOMAIN: THE CASE OF MULTIPLICATION OF FRACTIONS

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Abstract

Four students participated in a two year study (fifth- & sixth-grade) focused on the development of their understanding of multiplication of fractions. During the first year, all students received individualized instruction designed to encourage them to build on their informal knowledge of partitioning to solve problems involving multiplication of fractions. During the second year, all students received similar individualized instruction four times over a period of nine months. They also received classroom instruction focused on algorithmic procedures for multiplication of fractions. In the long term, all students consistently drew on their informal knowledge of partitioning on their own to solve problems. However, students' thinking was also dominated by their knowledge of algorithmic procedures at times.

LONG-TERM EFFECTS OF BUILDING ON INFORMAL KNOWLEDGE IN A COMPLEX CONTENT DOMAIN: THE CASE OF MULTIPLICATION OF FRACTIONS

For a number of years, researchers have been concerned with issues related to students' understanding of mathematics and the nature of its development (Brownell & Sims, 1946; Davis, 1984; Hiebert & Carpenter, 1992). In recent years, researchers have provided deep insights into the nature of students' mathematical understanding. They have also begun to provide insights into how students' mathematical understanding can grow (e.g., Mack, 1990; Olive, 1999; Streefland, 1993). However, there is still much to learn about how students can initially learn mathematics with understanding and the ways in which initially learning with understanding may influence students' learning of mathematics over time. This paper provides insights into the long-term effects of learning mathematics with understanding.

The purpose of this paper is to examine the long-term effects of building on students' informal knowledge of partitioning with respect to their understanding of multiplication of fractions. The long-term effects are examined from two perspectives: (a) the ways students did and did not draw on their knowledge of partitioning to solve problems involving the multiplication of fractions several months after the conclusion of instruction, and (b) identifying factors that contributed to, or hindered students from drawing on their knowledge of partitioning as they solved problems involving the multiplication of fractions in the long-term. This paper is based on data from the second year of a two-year study that examined the development of students' understanding of both multiplication and division of fractions as they built on their informal knowledge of partitioning during instruction.

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Conceptual Framework

Informal Knowledge and Mathematical Understanding

One view of understanding that has endured over time is that understanding depends on relationships the individual forms between pieces of her/his knowledge. These relationships can be between the individual's newly acquired knowledge and her/his existing knowledge, as well as between pieces of existing knowledge (Davis, 1984; Greeno, 1978; Hatano, 1996). With respect to mathematical understanding, the formation of relationships between pieces of knowledge can be stimulated in a variety of ways. One way is by the individual abstracting from her/his own activities and matching mathematical concepts, symbols, and procedures with other representations that are meaningful to her/him. These representations can be such things as real-life situations or actions on concrete or pictorial representations (Davis, 1984; Hiebert, 1988). Another way relationships can be stimulated is by the individual engaging in social interactions that encourage her/him to think about mathematical ideas from a variety of perspectives (Cobb, 1994; Sfard, 1998; Voigt, 1994; Vygotsky, 1978). However the relationships are stimulated, their actual formation occurs within the mind of the individual (Brownell & Sims, 1946; Davis, 1984).

In recent years, research has begun to provide insights into ways students develop understanding of mathematical concepts, symbols, and procedures by focusing on the knowledge students construct from their real-world experiences and bring to formal instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Lampert, 1986; Mack, 1990). This type of knowledge has been discussed under the guise of several names, one of which is informal knowledge (Ginsburg, 1982; Mack, 1990; Saxe, 1988). This type of knowledge can be characterized generally as applied, real-life circumstantial knowledge that may be either correct or incorrect. Additionally, the individual can draw on this knowledge in response to problems posed in the context of real-world situations familiar to her/him (Leinhardt, 1988).

Studies focusing on students' informal knowledge have documented that many students possess a rich store of informal knowledge related to a variety of mathematical content domains. They have also documented that students could draw on their informal knowledge to give meaning to mathematical concepts, symbols, and procedures when problems involving concepts and various forms of representation were closely related to problems that drew on their informal knowledge (Fennema, Franke, Carpenter, & Carey, 1993; Lampert, 1986; Mack, 1990). Thus, one way understanding can develop is by students building on their informal mathematical knowledge.

Recently, a few studies have suggested that building on informal knowledge not only helps students in their initial attempts to give meaning to mathematical concepts, symbols, and procedures, it also influences students' understanding in the long-term. These studies documented that when students were encouraged to initially draw on their informal knowledge to give meaning to mathematical concepts, symbols, and procedures, they often drew on this knowledge to deepen their understanding of mathematical ideas they encountered at later times. Furthermore, these studies documented that students drew on their informal knowledge on their own even when not encouraged to do so by instruction. Students also drew on this knowledge to solve problems in meaningful ways even when instruction focused on performing rote computational procedures (McNeal, 1995; Wood & Sellers, 1996).

Much of the research focusing on the role of informal knowledge in the development of mathematical understanding has examined students' understanding of whole number arithmetic or addition and subtraction of rational numbers. In each of these situations, researchers documented that students' informal knowledge initially reflected critical mathematical concepts underlying the content domain (Fennema et al., 1993; Lampert, 1986; Mack, 1990). Similarly, the few studies that have focused on the long-term effects of building on informal knowledge have examined students' understanding of whole number arithmetic (McNeal, 1995; Wood & Sellers, 1996).

A number of researchers question the viability of building on informal knowledge to develop students' understanding of a complex mathematical content domain, such as multiplication of fractions (Behr, Harel, Post, & Lesh, 1992; Fischbein, 1999; Greer, 1988; Tirosch, 1999). These researchers suggest students' informal knowledge does not initially reflect critical mathematical concepts underlying the content domain. They further suggest students' informal knowledge needs to be restructured in various ways before students can build on this knowledge to give meaning to complex mathematical concepts, symbols, and procedures associated with the domain (Greer, 1988; Moschkovich, 1998/1999). Uncertainty exists as to whether or not students can actually restructure their informal knowledge in a way that enables them to build on this knowledge to develop an understanding of a complex mathematical content domain (Fischbein, 1999).

Despite the uncertainty, recent research has begun to provide insights into the ways students might build on their informal knowledge to develop an understanding of complex mathematical ideas (Hatano, 1996; Mack, in submission; Pirie & Kieren, 1994). Furthermore, this research has begun to suggest that building on informal knowledge is a viable way to develop students' understanding of a complex mathematical content domain, particularly for a domain such as multiplication of fractions (Mack, in submission). However, the long-term effects of building on this knowledge in a complex content domain are not yet clear.

Building on Informal Knowledge of Partitioning to Understand Multiplication of Fractions

A number of researchers propose that for students to develop a deep understanding of multiplication of fractions, they need to be able to think about critical mathematical ideas associated with the domain in a variety of ways. In particular, Behr, Harel, Post, and Lesh (1992; 1993), Kieren (1988), Streefland (1991), and Vergnaud (1983) suggest students need to view the concept of "fraction" in multiple ways because situations involving multiplication of fractions involve various interpretations of the concept. For example, a problem such as "3/4 x 2/3 = ?" can be interpreted in terms of "finding" or "taking a part of a part of a whole," such as "finding three fourths of two thirds of one whole pizza." In this situation, "3/4" acts as an operator (e.g., a size transformation where a quantity is reduced to three fourths of its original size). Therefore, Behr et al., Kieren, Streefland, and Vergnaud suggest students need to view fractions not only as parts of wholes, but also as operators. These researchers also suggest students need to view fractions as quantities in and of themselves (e.g., 3/4 represents a single quantity with a specific value that is greater than one half of a whole and less than one whole) and in terms of division (e.g., 3/4 represents a quantity of measure three that is divided into four equal-sized parts as well as the result of this division) since these interpretations are also involved in multiplication of fractions. (Please see Behr et al., (1993) and Kieren (1988) for in-depth discussions of the various interpretations of fractions.)

Several researchers further suggest students need to think about the ideas of partitioning (i.e., the process of dividing a whole or unit into equal-sized parts) and unitizing (i.e., What is the whole or unit?) in a variety of ways so they can determine the appropriate units to be partitioned in a situation involving multiplication of fractions, as well as the various units upon which their partitionings are based

(Armstrong & Bezuk, 1995; Behr et al., 1992; 1994; Confrey, 1994; Empson, 1999; Kieren, 1988; 1995; Streefland, 1993; Steffe & Cobb, 1988). In particular, these researchers suggests students need to realize that a continuous whole (e.g., one whole pizza), a discrete whole (e.g., three whole pizzas), and various portions of these two types of wholes can serve as a unit. These researchers also suggest students need to realize that they can partition different types of units. Additionally, these researchers suggest students need to be able to reconstruct units as they draw on their knowledge of partitioning to solve problems involving multiplication of fractions in meaningful ways.

For example, consider the above problem "3/4 x 2/3 = ?," which can be interpreted as finding three fourths of two thirds of one whole pizza. One way students could solve this problem is by initially viewing "two thirds" as a quantity resulting from the partitioning of one unit, an original unit that is one whole pizza. Students could then view two thirds as a new unit (i.e., a two-thirds-units, which can be represented as 2/3-unit) that needs to be partitioned. Students could partition the 2/3-unit by dividing the entire unit into a total of four equal-sized parts. They could then consider three fourths of the 2/3-unit in relation to the original unit, or consider one fourth of the 2/3-unit three times in relation to the original unit to name the resulting amount of one half (Behr et al., 1992; 1994; Olive, 1999; Steffe, 1988). Thus, one way students can reconstruct and partition units as they solve problems is by first viewing one whole as the unit to be partitioned, then viewing a fractional part of the original whole as the unit to be partitioned, and lastly considering the resulting quantity in relation to the original unit.

In addition to thinking about critical mathematical ideas associated with multiplication of fractions in a variety of ways, a number of researchers suggest students' knowledge of partitioning plays a crucial role in the development of their understanding of the domain. Behr et al. (1992; 1994) claim that knowledge of partitioning lends itself to understanding the concept of fractions as "operators," because operators involve a size transformation that is achieved by both partitioning a unit and duplicating portions of the unit. Behr et al. further claim that knowledge of fractions as operators lends itself to understanding multiplication of fractions when situations involve a transformation in the size of the unit, such as those found in the interpretation of multiplication of fractions focused on "finding" or "taking a part of a part of a whole." Additionally, Confrey (1994), Kieren (1995), and Olive (1999) suggest students' knowledge of partitioning can provide a foundation for the development of their understanding of multiplication of fractions.

Mack (1990) suggests students come to instruction with a rich store of informal knowledge related to partitioning. She further suggests students' knowledge focuses on the "number of parts" in a whole as though each part represents an independent whole number quantity (e.g., 3/4 means "three of four pieces") rather than a fractional quantity. Although students' informal knowledge of partitioning may not fully reflect the conceptual complexities researchers claim are needed for understanding multiplication of fractions (Behr et al., 1992; 1993; Kieren, 1988; Streefland, 1991; Vergnaud, 1983), Mack suggests it may be possible for students to build on this knowledge to develop a deep understanding of the domain.

Recent research has examined the viability of building on students' informal knowledge of partitioning to develop their understanding of multiplication of fractions. Results from the first year of a two year study conducted by the author documented how six fifth-grade students were able to draw on their informal knowledge of partitioning to reconstruct units and solve problems involving the multiplication of fractions in meaningful ways even though this knowledge initially focused only on the "number of parts" in a problem situation (Mack, in submission). (Please see Table 1 for a summary of the mental processes that occurred as students built on their knowledge of partitioning to reconstruct and partition units.) The results also documented how four of the six students were able to draw on their knowledge of partitioning to give meaning to number sentences involving the multiplication of fractions. Thus, the results of the first year of this two year study suggest that building on informal knowledge of partitioning is a viable way to develop students' understanding of multiplication of fractions.

Although building on informal knowledge of partitioning appears to aid students in initially developing an understanding of multiplication of fractions, it is not yet clear what the long-term effects are of building on this knowledge. In particular, will students continue to draw on their knowledge of partitioning to deepen their understanding of multiplication of fractions, as students often do for whole number arithmetic? If so, how specifically might students draw on this knowledge to deepen their understanding of the domain? If not, why do students cease drawing on their knowledge of partitioning as they attempt to understand complex mathematical ideas associated with multiplication of fractions? This paper examines these issues by focusing on data from the second year of the author's two year study described above.

Methodology

Sample

The sample consisted of four sixth-grade students (1 girl & 3 boys) of average mathematical ability. All students participated in the first year of this two year study. During the first year, all four students received instruction on the multiplication and division of fractions over a three-month period. Each student received this instruction in a one-to-one instructional session where the author (hereafter referred to as "I") functioned as the teacher.

At the beginning of the first year, all four students came to instruction with informal knowledge of partitioning that focused on partitioning only units they considered to have a "measure of one" into specific numbers of parts. The students' responses suggested these units could only exist in the form of one continuous whole (e.g., one whole pizza), a unit portion of a continuous whole (e.g., one third of one whole pizza), or a discrete set where the number of elements in the set was equal to the naming portion of the fraction of concern (e.g., one <u>third</u> of <u>three</u> whole pizzas). All students were able to draw on their knowledge of partitioning to solve a few problems involving the multiplication of fractions in meaningful ways at the beginning o the study. These problems involved the interpretation of multiplication of fractions focused on finding or taking a part of a part of a whole and were limited to finding a unit fraction of one half of another unit, such as finding one half or one third of one half of one whole pizza.

At the end of the first year, all four students were able to draw on their knowledge of partitioning to reconstruct units and solve a variety of problems involving the multiplication of fractions that focused on finding or taking a part of a part of a whole. Two students, Lisa and Lee, drew on this knowledge to reconstruct and partition units by considering what it means to partition any unit into a fractional amount, by realizing that at times they did not need to repartition a composite unit, by realizing that at times they could repartition a composite unit, and by realizing that at times they could group pieces of a composite unit. Consequently, these two students were able to draw on their knowledge of partitioning to solve problems corresponding to 1/b x 1/d (e. g., 1/3 x 1/4), a/b x b/d (e.g., 1/4 x 4/5), a/nb x b/c (e.g., 3/4 x 2/3), and a/b x nb/d (e. g. 2/3 x 9/10). The other two students, Adam and Sam, drew on their knowledge of partitioning to reconstruct and partition units in the same ways as Lisa and Lee. However, Adam and Sam also realized that at times they could both repartition and group resulting pieces of a composite unit. Consequently, Adam and Sam were able to solve the same types of problems that Lisa and Lee were able to solve. However, they were also able to solve problems corresponding to a/b x c/d, where b and c are relatively prime (e.g., 3/4 x 7/8). Additionally, three students (Adam, Lee, & Sam) were able to draw on their knowledge of partitioning to give meaning to number sentences involving the multiplication of two proper fractions (e.g., 2/3 x 3/4).

Prior to the start of the second year of the study, none of the four students received instruction on multiplication of fractions in their fifth- or sixth-grade mathematics class. During the last four weeks of the first year, all student received instruction on fractions in their regular fifth-grade mathematics class. This instruction focused on identifying fractions represented pictorially and algorithmic procedures for adding and subtracting fractions. All four students received instruction of fractions in their regular sixth-grade mathematics class approximately mid-way through the second year of the study. Additionally, all four students came from a middle school that draws students from predominately lower to middle socio-economic backgrounds in the western Pennsylvania region of the United States.

General Characteristics of Follow-up Instructional Sessions

Each student was regarded as an independent case study. Each student received instruction four times in a one-to-one instructional session where I functioned as the teacher. All follow-up instructional sessions lasted 30 minutes and occurred during regular school hours.

For each student, the first follow-up instructional session occurred approximately five months after the first year of the study concluded. Additionally, the first follow-up session occurred at the beginning of the school year before the student received instruction on fractions in her/his regular sixth-grade mathematics class. The second follow-up session occurred at the end of the first week in which the student received instruction on fractions in her/his regular mathematics class. The third session occurred one week after the conclusion of classroom instruction on fractions. The fourth follow-up session occurred at the end of the school year, approximately four months after the conclusion of formal instruction on fractions. All follow-up instructional sessions were audio taped and video taped.

The purpose of all follow-up instructional sessions was two-fold: (a) to gain further insights into ways the students did and did not draw on their knowledge of partitioning to give meaning to and solve problems involving multiplication of fractions, and (b) to provide each student with opportunities to continue drawing on her/his knowledge of partitioning to give meaning to concepts, symbols, and procedures associated with multiplication of fractions. Therefore, all follow-up instructional sessions were conducted in the same manner as the instructional sessions of the first year. More specifically, all follow-up sessions combined clinical interviews with instruction based on the guiding principles of viewing students' learning and teachers' instruction as problem solving and the student and teacher as cooperative problem solvers (Carpenter et al., 1989; Ginsburg, 1982). The student's primary role during each follow-up session was three-fold: (a) to attempt to solve the problems s/he received in whatever ways were meaningful to her/him, (b) to verbally communicate her/his thought processes related to solution strategies or their attempts, and (c) to ask questions related to problems and for instructional assistance as s/he thought needed.

My primary role as the teacher was also three-fold: (a) to present the student with appropriate problems that were based on the student's thinking and a rational task analysis I conducted for situations involving the multiplication of fractions, which will be discussed in a later section, (b) to encourage the student to draw on her/his knowledge of partitioning to understand and solve problems in meaningful ways when the problems were presented verbally in contextual situations or represented symbolically, and (c) to provide instructional assistance upon request or when I thought needed.

Because the second year of the study focused in part of the viability of building on informal knowledge of partitioning, I played a major role in focusing the student on ideas of partitioning and reconstructing units (Brown, Collins, & Duguid, 1989; Vygotsky, 1978). More specifically, when the student did not consider the need for equal-sized pieces when stating answers to problems as simple fractions, I asked questions such as, "Are these pieces (the four pieces that resulted from partitioning two of three 1/3-units in half) the same size as this piece (the remaining unpartitioned 1/3-unit). Similarly, when the student was unsure of how to describe her/his procedure for solving problems represented symbolically in terms of her/his solution with self-generated diagrams, I asked questions such as "What did you partition your [picture] into first? How does what you did match the meaning of this denominator (for the multiplier)?"

I presented all problems to each student verbally. I encouraged the student to think aloud as s/he solved problems, or if s/he failed to think aloud, to explain what s/he had been thinking during the solution process. Additionally, I asked the student whether s/he wanted each problem presented in the form of a real-world problem situation or a number sentence. When the student requested the problem be presented in the context of a real-world situation, I asked her/him to suggest the specific context for the problem (e.g., pizza, cookies, etc.). When the student requested the problem be presented in the form of a number sentence, I asked the student to write a number sentence for a problem s/he thought s/he could solve.

I provided concrete materials in the form of fraction circles and strips for each student to use. Paper and pencil were also available. I encouraged the student to use the concrete materials, draw pictures, or use other ways that were meaningful to her/him to solve the problems. The student was free to solve problems by using the traditional algorithm for multiplication of fractions that s/he learned in her/his regular sixth-grade mathematics class. However, I did not explicitly encourage or discourage the student from using this algorithm. I took this approach for several reasons: (a) to determine if the student possessed only rote knowledge of the traditional algorithm or if this knowledge was focused on concepts in some way, (b) to avoid any possible interference I might impose by suggesting the student use or not use the traditional algorithm, and (c) to determine if the student's knowledge of the traditional algorithm influenced her/his solution processes in any way (Hiebert & Wearne, 1986; Mack, 1995).

The instructional content during the follow-up sessions focused both on topics related to multiplication of fractions and on topics related to division of fractions. Additionally, the instructional content focused on the multiplication and division of fractions less than one. The focus on fractions less than one was primarily due to the students' struggle to give meaning to number sentences involving the multiplication or division of two proper fractions. Although the content focused on both multiplication and division of fractions, this paper focuses only on students' knowledge of partitioning in relation to both number sentences and problem situations involving the multiplication of two fractions that corresponded to finding of taking a part of a part of a whole. This narrow focus is due to the fact that the major changes that occurred with respect to the ways students drew on their knowledge of partitioning to reconstruct and partition units were initially observed as the students solved problems involving the multiplication of two proper fractions during the first year of the study.

After each follow-up instructional session, I planned a lesson for the student's next session. I considered several factors when planning each lesson - the student's knowledge related to number sentences for multiplication of fractions, any misconceptions the student communicated, the student's responses to and solution strategies for problems presented in previous sessions, my knowledge of how students think about fraction concepts and operations on fractions (Fennema et al., 1993; Shulman, 1986), and my knowledge of concepts underlying multiplication of fractions. To reflect the purpose of the study, I designed the lessons to be flexible with respect to the different types of problem situations students encountered, when and how students encountered number sentences for multiplication of fractions, and movement back and forth between various types of problem situations and number sentences for multiplication of fractions.

Regular Classroom Instruction on Fractions

In addition to working with each of the four students during the follow-up instructional sessions, I met with each student's regular sixth-grade mathematics teacher (N=2) individually two times during the study. The first meeting occurred approximately two weeks before instruction on fractions began in her/his classroom. The second meeting occurred approximately one week after the conclusion of classroom instruction on fractions. All meetings with the teachers were audio taped.

During each meeting, the teacher shared information about what specific fraction topics would be, or were covered in class (i.e., identifying fractions represented pictorially, equivalent fractions, and addition, subtraction, multiplication and division of fractions). The teacher also discussed how the specific topics would be, or were covered and what s/he wanted each student to learn about multiplying and dividing fractions with respect to both concepts and procedures. During the second meeting, the teacher also shared samples of the student's written homework and tests related to multiplying and dividing fractions. During this meeting, the teacher also shared what s/he thought each student understood about multiplication of fractions.

All teachers' responses, as well as the questions and responses on samples of the students' written work suggested that classroom instruction focused on computational problems that were represented symbolically (e.g., 2/3 x 3/4 = ?) and helping the students learn traditional algorithms for operations with fractions in a rote manner. As one teacher explained,

> I think it's important for them just to know how to do the problems at this time. They can learn why they solve the problems in these ways when they're older. That's too much for them to understand right now...Word problems are hard for them. They do better with computational problems, so that's what I teach them.

Thus, each student received instruction on multiplication of fractions in her/his regular mathematics class that focused on utilizing the traditional algorithm for multiplication of fractions to rotely multiply numerators together and multiply denominators together to solve problems such as 2/3 x 3/4 = ?

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Assessment Tasks and Problem Situations for Multiplication of Fractions

Each question given to the student during any follow-up instructional session was regarded as an assessment task. In general, the tasks were based on five central ideas: (a) the more parts a unit is partitioned into, the smaller the parts become, (b) a fraction can have various meanings (e.g., part of a whole, operator, division, single number with a specific value), (c) selected ideas of equivalence can be related to concrete, pictorial, and symbolic representations, (d) selected ideas associated with the concept of units (e.g., a unit can take the form of one continuous whole, a discrete set, or a portion of these different types of wholes, more than one unit can exist in a problem), and (e) units can be reconstructed and partitioned in a variety of ways.

During the first follow-up instructional session, the tasks focused on determining what meaning the student had for a fraction, such as three fourths, when the fraction was stated verbally and represented symbolically (e.g., 3/4). The tasks also focused on the student's ability to reconstruct and partition units as s/he solved problems presented verbally in the context of real-world situations. Additionally, the tasks focused on determining what connections, if any, existed between the student's solutions for real-world problems and number sentences for multiplication of fractions.

In following follow-up sessions, the tasks focused primarily on the student's ability to reconstruct and partition units as s/he solved problems in meaningful ways. These tasks took the form of different types of problem situations involving the multiplication of two proper fractions. The tasks also focused on the student's ability to write appropriate number sentences for different problem situations and on the student's ability to generate problem situations from number sentences for multiplication of fractions.

During the first year of the study, all four students encountered five different types of problem situations for the multiplication of fractions. These different types of problem situations emerged from a rational task analysis I conducted prior to the beginning of the study. My analysis was based on the unit analyses of Steffe (1988) and Behr et al. (1992; 1994) who suggest that understanding multiplication of fractions depends on the formation and reformation of different types of units of quantity. My analysis was also based on Steffe's and Behr et al.'s suggestion that one way to view conceptual differences between situations involving multiplication of fractions is by focusing on the relationship between the denominator of the multiplier and the numerator of the multiplicand when the multiplier is considered to be an operator and the problems are represented symbolically (e.g., a/b x c/d where a/b is an operator and the relationship between b and c).

My rational task analysis focused on the following three types of relationships that could exist between the denominator of the multiplier and the numerator of the multiplicand: (a) the denominator and numerator are equal, (b) one is a multiple of the other, and (c) the two terms are relatively prime. My analysis also focused on Kieren's (1995) and Olive's (1999) suggestion that problems involving the multiplication of two unit fractions (e.g., 1/3 x 1/4) are a special case. Thus, my analysis yielded the following five different types of problem situations: 1/b x 1/d (e.g., 1/3 x 1/4), a/b x b/d (e.g., 1/4 x 4/5), a/nb x b/d (e.g., 3/4 x 2/3), a/b x nb/d (e.g., 2/3 x 9/10), and a/b x c/d (e.g., 3/4 x 7/8). (Please see Mack (in submission) for an in-depth discussion of this rational task analysis and different types of problem situations for multiplication of fractions.)

The student's ability to solve problems involving these different types of problems during the first year of the study provided the basis for the specific types of problems s/he encountered during the second year. For example, each student was able to solve problems corresponding to a/nb x b/d during the first year. All students solved these problems by realizing that a composite unit in the form of a non-unit portion of a continuous whole (e g. two thirds of one whole pizza) could be repartitioned. However, none of the students came to this realization on their own. They all required instructional assistance that focused them on considering what it means to partition any quantity into a fractional amount. Therefore, each student encountered this same type of problem situation during the second year to determine if the student's prior realization related to ways to reconstruct and partition units would endure over time.

Data Analysis

The data consisted of transcriptions of critical protocol segments from the student's taped instructional sessions, the student's written work, and detailed notes I made after each of the student's instructional sessions. This data was generated from both the first and second year of the study. The data also consisted of transcriptions of critical protocol segments from the two meetings with the student's regular sixth-grade mathematics teacher.

After each follow-up instructional session, and several times after the conclusion of the study, I reviewed the data to describe the ways the student did or did not draw on her/his knowledge of partitioning to solve problems involving the multiplication of fractions. More specifically, my review focused on describing the different problems the students encountered (e.g., number sentences or different types of problem real-world problem situations), describing whether the student solved the problems by drawing on knowledge of partitioning or applying an algorithmic procedure for multiplying fractions, and describing why the student chose to solve the problems in these ways. If the student drew on her/his knowledge of partitioning, my review further focused on describing what the student considered to be the unit or units in each problem situation and the specific ways in which s/he partitioned the various units in each situation. If the student applied an algorithmic procedure, my review further focused on describing the various steps of the algorithm and describing what meaning, if any the student had for the algorithm. Additionally, my review focused on describing situations where the student needed assistance in understanding and solving a problem and the nature of the assistance that proved helpful. Lastly, my review focused on comparing the student's solution strategies, explanations, and questions about specific problems during the follow-up sessions with those related to similar problems s/he received during the first year of the study, as well as comparing the student's solution strategies and explanations to the nature of instruction s/he received in her/his regular sixth-grade mathematics class.

Results

Four themes characterized the long-term effects of building on informal knowledge with respect to students' knowledge of partitioning and their understanding of multiplication of fractions: (a) students drew on their knowledge of partitioning on their own to reconstruct units and solve problems in meaningful ways, (b) weak connections between knowledge of partitioning and symbolic representations quickly disappeared and took time to rebuild, (c) students' reliance on their knowledge of partitioning changed after receiving classroom instruction focused on algorithmic procedures for multiplication of fractions, and (d) students drew on their knowledge of partitioning to justify their solutions when using algorithmic procedures. The results are organized into four sections based on these themes. (Please note: Fractions written in words (e.g., three fourths) denote fractions stated verbally. Fractions written as a/b denote fractions represented symbolically. Additionally, although problem situations may seem contrived, the students themselves suggested the specific contexts for all problems they received.)

Drawing on Knowledge of Partitioning to Reconstruct Units and Solve Problems

During the first year of the study, all four students built on their informal knowledge of partitioning to reconstruct and partition units to solve problem situations involving the multiplication of fractions in meaningful ways. However, with the exception of one instance, none of the students initially drew on their knowledge of partitioning on their own when they first encountered each different type of problem situation. All students required that I ask them to consider what it means to partition any quantity into a fractional amount. They also required that I ask them to consider equal-sharing situations, such as "Share 10 cookies equally between four people," and the ideas that at times each person received more than one whole cookie while at other times each person received less than one whole cookie when a specific number of cookies were shared fairly. By the end of the first year, all students drew on their knowledge of partitioning on their own to solve problem situations involving the multiplication of fractions.

At the beginning of the first follow-up session, all four students immediately drew on their knowledge of partitioning to solve the first problem situation they encountered. All students drew on their knowledge of partitioning on their own by considering what it means to partition any quantity into a fractional amount. As they did, the students reconstructed and partitioned appropriate units to solve problems in meaningful ways.

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The following protocol, which was taken from Lee's first follow-up instructional session, illustrates how the students drew on their knowledge of partitioning on their own to solve problems involving the multiplication of fractions. The protocol also illustrates how this knowledge was connected to a deep understanding of partitioning quantities into fractional amounts.

(Please note: Less suggested the specific context for this problem.)

NM: You have three fourths of a pizza. You eat five sixths of the amount of pizza that you have for lunch today. How much of the whole pizza did you eat?

Lee: (Drew a circle and partitioned it into fourths by first splitting the circle in half horizontally then vertically. Shaded the one-fourth piece he did not have. See Figure 1a. Partitioned each of the remaining three one-fourth pieces in half. See Figure 1b. Marked five of the pieces.) I had three fourths of a pizza, then I divided each one (of the three one-fourth pieces) in two because I needed sixths. I had three pieces but I needed six to get sixths. It doesn't matter what this (three fourths of the pizza) is. See when I need sixths, I just take what I have and make it into six pieces, sixths. I ate five sixths. That's five of these pieces. (Drew curve connecting pieces. See Figure 1c.) That's two fourths and half (of a fourth).

NM: Well, after you partitioned each of these pieces (the three one-fourth pieces) in two, were all the pieces the same size including this one (the shaded one-fourth pieces that did not have)?

Lee: Oh, they all need to be the same size. (Partitioned the shaded one-fourth piece in half. See Figure 1d.) There's eight pieces (in the whole pizza). So I ate five eighths.

(Please insert Figure 1 here.)

Lee's drawing of a whole circle suggested he viewed one whole pizza as the initial unit to be partitioned. Lee's drawing of three fourths of a circle, his partitioning of each of the three one-fourth pieces, and his explanations related to sixths suggested he viewed three fourths of the pizza as the unit to be partitioned. Lee's partitioning actions and his comment, "I divided each one in two because...I had three pieces but I needed six further suggested he realized that he could repartition the unit to obtain the desired fractional amount. Additionally, Lee's answers of "two fourths and a half (of a fourth)" and "five eighths" suggested he viewed one whole pizza as the unit reference when stating his answers. Lee's actions and explanations throughout the protocol suggested he realized these ideas on his own.

The other three students solved problem situations involving the multiplication of fractions in a manner similar to Lee's. The students quickly identified the various units in each problem situation. They then focused on what it means to partition any unit into a fractional amount and partitioned the units in appropriate ways on their own.

All students were able to solve the same types of problem situations at the beginning of the second year that they were able to solve at the end of the first year. They solved the different types of problems in the same ways during both years of the study. Thus, if the students repartitioned a unit to solve problems corresponding to a/nb x b/d during the first year, they repartitioned the unit on this same type of problem the second year. Similarly, if the students grouped pieces of a composite unit to solve problems corresponding to a/b x nb/d during the first year, they grouped pieces of a composite unit to solve this same type of problem the second year. For example, during the first year, Lee solved a problem corresponding to a/nb x b/d that involved finding three fourths of two thirds of a cookie. He solved the problem by first partitioning a circle into three parts to find thirds. He then focused on two thirds of the circle. Next, Lee partitioned each of two one third-pieces in half to obtain four pieces, or fourths. This action of repartitioning the composite unit was similar to his partitioning actions in the example above, which also involved a problem corresponding to a/nb x b/d (i.e., 5/6 x 3/4).

Although all students were able to draw on their knowledge of partitioning to solve a variety of problem situations involving multiplication of fractions, they all initially stated their answers as complex fractions at the beginning of the second year. This was illustrated in Lee's example above. Additionally, all students initially considered these answers to be appropriately stated. However, after I asked only one question about the size of the pieces in their diagram, all four students stated their answers as simple fractions, such as five eighths. The students then continued stating their answers as simple fractions throughout the remainder of the second year.

For most of the second year, all four students consistently drew on their knowledge of partitioning to solve real-world problem situations with no assistance from me. In so doing, the students were able to reconstruct and partition appropriate units to solve problems involving the multiplication of fractions in meaningful ways.

Weak Connections To Symbolic Representations Quickly Disappeared

At the beginning of the first year, all four students viewed the meaning of multiplication only in terms of repeated addition. For example, all students explained the meaning of the expression "3x5" as "five plus five plus five." During the first year, none of the students viewed the problem situations they encountered in terms of multiplication of fractions on their own. All four students initially thought of these problems as division problems. The students focused on their partitioning actions as they utilized self-generated diagrams to solve the problems and explained, "You gotta divide it up so it's division."

Approximately half-way through the first year, I introduced all students to the idea the "multiplication" can have several meanings, one of which is "finding" or "taking a part of a part of a whole." I also suggested a number sentence such as "2/3 x 3/4 = ?" could be used to represent a problem corresponding to "finding two thirds of three fourths of one whole pizza." None of the students readily accepted these ideas. All of the students continued to focus on their partitioning actions with diagrams and struggled with the idea that partitioning could be involved in multiplication. As Adam explained,

> This doesn't make sense. You gotta divide it up and that doesn't seem like it should be multiplication. When you divide it up, that's division but sometimes it's subtraction. It's never been multiplication before. I just don't get it.

I attempted to help the students view the problems they encountered in terms of multiplication by guiding them to consider that the problems involved finding or taking a part of a part of a whole. I also attempted to help the students see how their partitioning actions corresponded to various elements in number sentences involving multiplication of fractions, such as when finding two thirds of one fourth of one whole pizza, or 2/3 x 1/4, the one fourth is partitioned into thirds and two of these thirds are considered in relation to the whole pizza to obtain the answer.

By the end of the first year, three students (Adam, Lee, & Sam) viewed the problem situations they encountered in terms of multiplication of fractions. These three

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students were able to write appropriate number sentences for the problems. They were also able to explain the number sentences in terms of their partitioning actions with self-generated diagrams. However, the students often asked for my assistance when writing number sentences and explaining solutions. Thus, the connections Adam, Lee, and Sam made between their knowledge of partitioning and symbolic representations for multiplication of fractions appeared to be tenuous at the end of the first year.

At the beginning of the second year, there was no indication that any kind of connection had previously existed between any of the students' knowledge of partitioning and their knowledge of symbolic representations for multiplication of fractions. There was also no indication that any of the students had previously viewed multiplication in terms of finding of taking a part of a part of a whole. For example, during the first follow-up session I asked each student to write a number sentence for a problem situation s/he had just solved, which involved multiplication of fractions. One student, Lisa, wrote a division problem (i.e., 5/6 ÷ 3/4) and explained, "I divided it up." Two students, Lee and Sam, responded, "I don't know" and wrote only the fractions involved in the problem and the answer they obtained without writing any operation sign. One student, Adam, responded, "I don't remember" and did not write anything on his paper. When I suggested we could write a multiplication number sentence such as 5/6 x 3/4 to represent the problem situation, all four students quickly responded in a manner that suggested they did not think the problem could be viewed as involving multiplication. None of them were certain what operation was involved; however, they were certain it could not be multiplication. As Adam explained, "I don't think that right's. It doesn't seem like it should be multiply because it just doesn't."

To rebuild connections between the students' knowledge of partitioning and symbolic representations for multiplication of fractions required a directed effort and much time during the second year. Following the students' initial responses, I reintroduced them to the idea that multiplication could be viewed in terms of finding or taking a part of a part of a whole. I also tried to help the students see how their partitioning actions with self-generated diagrams corresponded to various elements in number sentences for multiplication of fractions. I tried to help all four students understand these ideas in a manner similar to that undertaken during the first year of the study.

By the end of the second follow-up session, Adam, Lee, and Sam had once again made very tenuous connections between their knowledge of partitioning and symbolic representations for multiplication of fractions. With assistance from me that focused on first writing problem situations in English phrases such as "find 3/4 of 2/3 of one whole pizza", these three students were able to write appropriate number sentences for the problems. Additionally, with my assistance, two of these students (Lee & Sam) attempted to generate problem situations from number sentences for multiplication of fractions.

The connections Adam, Lee, and Sam made grew stronger over the final months of the second year. However, only Sam developed fairly strong connections between his knowledge of partitioning and symbolic representations by the end of the study. Sam was able to write appropriate number sentences for any problem situation he encountered involving multiplication of fractions. He was also able to generate appropriate problem situations from number sentences with only minimal, if any assistance from me. Additionally, Sam was able to recognize and solve problem situations involving multiplication of fractions outside of the instructional sessions. Sam explained this at the end of his fourth follow-up session.

> I understand fractions a lot better than I used to. Now I know what these numbers mean. Now I help my mom with some of the groceries. We went shopping and my mom bought something for a fraction of the cost and she asked me what's the fraction of two hundred twenty six. I kinda of drew a picture in my head and I knew it was multiplication. I multiplied it in my head and I got the right answer. It was fun.

One student, Lisa, was unsuccessful in making connections between her knowledge of partitioning and number sentences for multiplication of fractions. Whenever Lisa attempted to make connections, she focused on her partitioning actions with diagrams and her view of multiplication as repeated addition. Thus, Lisa was unable to determine or accept the idea that partitioning and multiplication could be related in some way. Lisa consistently explained,

> If I do this, I divided it up. That wouldn't be times because in multiplication, if like you had one times two that would be one circle and you put two dots in it, not divide it in two.

Consequently, Lisa's thoughts that the problem situations she encountered could not involve multiplication remained unchanged throughout both years of the study.

Reliance on Knowledge of Partitioning Changed After Formal Instruction

Prior to the second follow-up session, all students had been encouraged to draw on their knowledge of partitioning to solve problems in whatever ways were meaningful to them. Consequently, all four students drew on their knowledge of partitioning to reconstruct and partition units as they utilized self-generated diagrams to solve problems presented in the context of real-world situations. The students also drew on their knowledge of partitioning to solve problems represented symbolically, such as 2/3

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x 3/4 = ? More specifically, whenever the students encountered a multiplication problem represented symbolically, they all requested that I "make it a word problem so I can draw a picture and get the answer," which they did by drawing on their knowledge of partitioning to reconstruct and partition various units in the problem. None of the four students ever suggested that a problem represented in the form of a/b x c/d could be solved by multiplying numerators together and multiplying denominators together. Thus, the students' solutions to all problems were characterized by a reliance on their knowledge of partitioning prior to receiving formal classroom instruction on multiplication of fractions.

All four students received formal instruction on multiplication of fractions in their regular mathematics classes between their second and third follow-up session. After receiving this instruction, the students' knowledge of the traditional algorithm for multiplication of fractions dominated their thinking as they solved problems. At the beginning of the third and fourth follow-up sessions, all four students requested that their problems be presented in the form of number sentences for multiplication of fractions. None of them wanted any word problems. When problems were presented as number sentences, all students quickly solved the problems by referring to the traditional algorithm for multiplication of fractions and ideas of "canceling" and "reducing." None of the students made any reference to their knowledge of partitioning or asked that I "make it a word problem," as they had done before. Their focus was solely on using the traditional algorithm to solve the problems.

The following protocol, which was taken from Lisa's third follow-up session, illustrates how students drew on their knowledge of the traditional algorithm for multiplication of fractions as they solved problems represented symbolically. The protocol also illustrates how students did not refer to their knowledge of partitioning as they focused on the traditional algorithm. Additionally, the protocol illustrates how the students' knowledge of the traditional algorithm appeared to be rote knowledge.

NM: Show me any problem you know how to solve that has fractions in it and solve the problem.

Lisa: (Quickly wrote 2/3 x 10/20 =. See Figure 2.) I learned how to do cross products. First, I see if I can divide two and twenty by the same thing. That's two. Two divided by two is one and twenty divided by two is ten. So I cross these out and put the one here (by 2) and ten here (by 20). I'm not sure if I can divide three and ten by anything. I have to think about it. (Pause.) I don't think I can. One times ten equals ten and three times ten equals thirty. Ten-thirtieths. (Wrote 10/30). That can't be reduced. Wait. (Pause.) I'm thinking about it. It can be reduced. Ten divided by three equals ten (wrote 3 next to numerator in 10/30). I mean, (pause). Wait. Oh no, divide by ten (crossed out ÷3 and wrote ÷10 next to both the numerator and denominator in 10/30). Thirty divided by ten because you have to divide by the same number, is three. That's one third. (Wrote = 1/3).

(Please insert Figure 2 here.)

Lisa's comment "I learned how to do cross products" and her many references to division suggested she focused on the traditional algorithm for multiplication of fractions in conjunction with the idea of "canceling", or factoring fractions of one (e.g., 2/2, 10/10) as she solved the problem. These comments, as well as the comment "one times ten equals ten" suggested Lisa's knowledge of both the traditional algorithm and the idea of canceling was rote knowledge. Additionally, Lisa made no references to partitioning various units as she solved the problem. Thus, her solution was characterized by a focus on the traditional algorithm for multiplication of fractions.

The other three students responded in a manner similar to Lisa's when they encountered number sentences for multiplication of fractions. The students made no references to the idea of partitioning various units. Instead, the students first focused on the idea of canceling. After they "canceled" factors, they then multiplied numerators together and multiplied denominators together and "reduced" their answers as needed. Whenever the students utilized the traditional algorithm, they obtained correct answers to computational problems involving multiplication of fractions.

The students' knowledge of the traditional algorithm for multiplication of fractions not only dominated their thinking as they solved problems represented symbolically. It also dominated the students' thinking as they solved problems presented in the context of real-world situations during the third follow-up session. Regardless of how problems were presented, all four students wanted to solve the problems by using the traditional algorithm rather than by creating diagrams and drawing on their knowledge of partitioning.

More specifically, after the students solved problems represented symbolically, I created corresponding word problems for the number sentences. I then asked the students to use diagrams or fraction materials to prove that their answers to the number sentences were correct. None of the four students wanted to solve the problems in ways other than by utilizing the traditional algorithm. They all responded to my request in a similar manner, "I know this (the answer to the number sentence that was obtained by utilizing the traditional algorithm) is right because that's what my teacher taught me." When I again asked the students to use pictures or materials to prove their solutions were correct, all four students drew only the beginning portion of their solution. The students then referred to their solution to the number sentence and their use of the traditional algorithm. For example, Sam "proved" his answer "1/2" was correct for 2/3 x 3/4 by drawing only three fourths of a circle. He then pointed to his answer to the number sentence and explained,

> Half, 'cause it's right there (pointing to 1/2). Two times three is six. Three times four is twelve. Six-twelfths. I can reduce that. Six goes into six one time and six goes into twelve two times. That's what I learned in math class.

When I asked the students why they preferred to solve problems by using the traditional algorithm, they all responded in a similar manner. As Sam explained, "It's fast and it's easy. The numbers are right there. You don't have to think about what they mean. You just can do it." For all four students, the dominance of their knowledge of the traditional algorithm occurred primarily during their third follow-up session.

One major consequence of students focusing on the traditional algorithm rather than on their knowledge of partitioning as they solved problems was that they often failed to determine the appropriate unit reference for each fraction in a problem situation. Three students (Adam, Lee, & Lisa) focused on the fractions individually in terms of "the number of parts" of a whole. Consequently, these students viewed each fraction as referring to the same whole rather than viewing the multiplying fraction in relation to the multiplicand. For example, during her third follow-up session, Lisa encountered a problem involving feeding her horse two thirds of three fourths of a bucket of oats. Lisa drew a circle, partitioned the circle into fourths, and shaded three of the one-fourth pieces. She then partitioned the entire circle into thirds and explained, "I need two thirds so I have to divide the whole bucket into three pieces." After I commented that she needed to find two thirds of only three fourths of the bucket, Lisa pointed to her partitioning of the circle into thirds and said, "This is how you do thirds."

Adam and Lee also responded in a manner similar to Lisa's as they solved problems presented in the context of real-world situations during their third follow-up session. These students focused on the fractions in the problems and thought of all fractions as originating from the same whole rather than thinking of the multiplying fraction in terms of the multiplicand. Consequently, the students were confused as they attempted to solve the problems. They once again returned to their knowledge of the traditional algorithm for multiplication of fractions and asked for problems to be presented as number sentences, because as Lisa explained, "I know how to solve those That's what I learned."

Drawing on Knowledge of Partitioning to Justify Solutions

The dominating influence of students' knowledge of the traditional algorithm for multiplication of fractions diminished greatly with time and became limited to students' work with number sentences (e.g., 2/3 x 3/4 = ?). As this happened, all four students relied on their knowledge of partitioning once again. The specific situations in which the students relied on this knowledge differed somewhat from before. As in the first year of the study and the first and second follow-up sessions, all students drew on their knowledge of partitioning to reconstruct and partition units to solve problems in the context of real-world situations in meaningful ways. However, unlike before, all four students were more confident in the answers they obtained to problems by drawing on their knowledge of partitioning than in those they obtained by employing the traditional algorithm for multiplication of fractions. Consequently, all students also drew on their knowledge of partitioning to justify their solutions for number sentences. All four students drew on their knowledge of partitioning in these different situations on their own initiative. For all students, this happened at the beginning of their fourth follow-up session, which occurred approximately four months after the conclusion of formal classroom instruction on fractions.

The following protocol, which was taken from Sam's fourth follow-up session, illustrates how students drew on their knowledge of partitioning to justify their solutions to number sentences for multiplication of fractions. The protocol also illustrates how students' knowledge of the traditional algorithm continued to influence their thinking in limited ways several months after the conclusion of formal instruction.

NM: What kind of problem would you like to start with today? Do you want just numbers in your problems or do you want word problems?

Sam: Numbers. A multiplication problem.

NM: Three fourths times two thirds, what's that equal to?

Sam: I'm gonna have to work that out. (Wrote 3/4 x 2/3 =. See Figure 3a.). First you cancel. Three and three, that's one (crossed out both 3's and wrote 1's). Two goes into four two times, so that's (the 2) a one and that's (the 4) two. One times one is one. Two times two is two. (Wrote 1/2.) It's half.

NM: Can you make up a word problem that goes with that?

Sam: You feed a penguin three fourths of one fish and you feed another penguin two thirds of another fish.

NM: You have to have a question.

Sam: How much fish did you feed in all? Wait. (Pause.) That wouldn't be right. That would be add. (Pause.) I need some help on this.

NM: This (number sentence) says three fourths of two thirds. You start with two thirds of a fish. You feed one penguin three fourths of the fish that you have. How much of the whole fish did you feed the penguin?

Sam: Half. That's what I got here (pointing to number sentence), but I'd better make sure it's right. (Drew fish.)

NM: You have only two thirds of the fish.

Sam: (Partitioned fish into three supposedly equal-sized parts.)

NM: Of that two thirds, you're going to feed the penguin three fourths of it. How much of the whole fish is that?

Sam: Just one section. (Shaded one third of the fish.) Wait, and half of this. (Partitioned middle third in half. See Figure 3b.) Because two fourths is one half, so I need another half to equal three fourths. See (partitions first third in half, points to each one fourth of the two thirds one-by-one, see Figure 3c) that's the first quarter there, the second, the third, and that's the fourth. That's four fourths (indicating all four parts of the partitioned two thirds). If I go like this (partitions last third in half, see Figure 3d) there's six pieces. So I give the penguin three sixths of the fish. That's half of the fish, so that (pointing to answer to number sentence, "1/2") is right.

Sam's explanation for his solution to "3/4 x 2/3" suggested he drew on his knowledge of the traditional algorithm for multiplication of fractions rather than his knowledge of partitioning to solve the problem. Sam's partitioning actions with the fish diagram and his comments related to fourths suggested he drew on his knowledge of partitioning rather than his knowledge of the traditional algorithm to solve the problem in a contextual situation. More specifically, Sam's drawing of the whole fish suggested he viewed the entire fish as the initial unit to be partitioned. His focus on two thirds of the fish suggested he then viewed this portion as the unit to be partitioned. Lastly, Sam's partitioning of the remaining one-third piece and his answer "three sixths" suggested he viewed the entire fish as the unit when stating his answer. Additionally, Sam's comments, "I'd better make sure," "so this is right," and his partitioning actions with the fish diagram suggested he realized he should obtain the same answer regardless of the form in which the problem was presented. These comments and actions further suggested Sam was more confident of his answer to the fish problem than he was of his answer to the number sentence. Additionally, Sam's comments and actions suggested he realized that he could determine the correctness of his answer to the number sentence by drawing a diagram and partitioning the various units involved in the problem.

The other three students drew on their knowledge of partitioning to justify their answers to number sentences in a manner similar to Sam's. After drawing on their knowledge of the traditional algorithm to solve number sentences for multiplication of fractions, the students reconstructed and partitioned units as they utilized self-generated diagrams to solve corresponding problem situations. The students then compared the answers they obtained for the different problem representations, which happened to agree in all instances. The students concluded that their answers to the number sentences were correct not only because the answers agreed with the ones they obtained by partitioning diagrams, but also because as Lisa explained, "I can see it here (in the diagram)."

When I asked the students why they used their solutions with diagrams to determine whether or not their answers to the number sentences were correct, all four students responded in a similar manner. The students suggested that drawing diagrams and partitioning various units helped them understand the meaning of, or the unit reference for the fractions involved in the problems. The students further suggested that knowing they understood this idea promoted confidence in their answers. As Sam explained,

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> In the number problems, the numbers are right there. You don't have to think about what they mean. You just do it, but sometimes you're wrong, but you don't know it 'cause you don't know what the numbers mean. You have to think about the word problems. You have to think about what the numbers and words mean. You think about it and you draw a picture and then you can tell if you're thinking about it right. Then you know you're answer is right.

Thus, the students drew on their knowledge of partitioning in the long-term not only to solve problems involving multiplication of fractions, but also to justify their answers to problems represented symbolically.

Discussion

This study provides insights into the long-term effects of students building on their informal knowledge to understand a complex mathematical content domain. This study also provides further insights into students' understanding of multiplication of fractions. The results document that students continued to draw on their knowledge of partitioning on their own initiative when they encountered problems involving the multiplication of fractions several months after the conclusion of instruction. The results illustrate that as students drew on this knowledge, they reconstructed and partitioned appropriate units to solve problems in meaningful ways. The results also document that students drew on their knowledge of partitioning to justify their answers to computational problems. Thus, the results suggest that in the long-term, it is both possible and probable that students will draw on their informal knowledge to deepen their understanding of mathematical ideas (McNeal, 1995; Wood & Sellers, 1996), even ideas associated with complex content domains.

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Although students may draw on their informal knowledge to deepen their understanding of a complex domain in the long-term, the results caution that students' ability and willingness to do so may be influenced by rote knowledge of computational procedures. The results document that after students received instruction focused on the traditional algorithm for multiplication of fractions, students' rote knowledge of this algorithm dominated their thinking as they solved problems. The results illustrate that this dominating influence was due in part to students' perceptions that they could solve problems more efficiently by utilizing the traditional algorithm than by drawing on their knowledge of partitioning. The results also illustrate that this dominating influence was related to tenuous connections that existed between the students' knowledge of partitioning and their knowledge of symbolic representations for multiplication of fractions. Thus, the results of this study add further evidence in support of the argument for teaching concepts prior to procedures (Hiebert & Wearne, 1986; Mack, 1990). However, they also suggest that issues of efficiency should be addressed when teaching concepts prior to procedures. Additionally, the results suggest that strong connections need to be developed between concepts associated with students' informal knowledge and mathematical procedures for students to continue to draw on this knowledge to deepen their understanding of a complex domain.

How to effectively help students develop strong connections between their informal knowledge and mathematical procedures is not yet clear for a domain such as multiplication of fractions. Nor is it yet clear how to effectively help students develop connections that endure over time. The results document that matching partitioning actions with diagrams to number sentences and engaging in social interactions to discuss various meanings of multiplication helped three of the four students develop connections between their knowledge of partitioning and symbolic representations for multiplication of fractions. However, these connections were initially short-lived and required much time and a directed effort to rebuild. Even after these efforts, only one of the three students developed fairly strong connections. The results suggest that students' ability to develop connections between their informal knowledge and symbolic representations was greatly influenced by their view of multiplication as repeated addition. How might students expand their view of multiplication beyond repeated addition in meaningful ways? How might students draw on these expanded views to build connections between their informal knowledge of partitioning and number sentences for multiplication of fractions? Further investigations are needed to gain insights into ways to help students develop strong connections between concepts and procedures associated with multiplication of fractions that will endure over time. Results of such investigations may also provide insights into ways to help students develop connections between seemingly dissonant ideas associated with other complex content domains.

As the results document, students' knowledge of partitioning had a profound effect on their understanding of multiplication of fractions in the long term. This knowledge not only helped students solve problems in meaningful ways, it also helped students focus on critical mathematical ideas associated with the domain. Additionally, this knowledge helped students overcome the ambiguity that they often associated with symbolic representations for fractions. Thus, the results of this study suggest that building on informal knowledge of partitioning may be a viable way to develop students' understanding of multiplication of fractions not only in the short-term, but also in the long-term.

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References

Armstrong, B. E. & Bezuk, N. (1995). Multiplication and division of fractions: The search for meaning. In J. T. Sowder & B. Schappelle (Eds.). Providing a foundation for teaching mathematics in the middle grades (pp. 85-119). Albany, NY: SUNY Press.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, proportion. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 296-333). New York: Macmillan.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis-emphasis on the operator construct. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), Rational Numbers: An Integration of Research (pp. 13-47). Hillsdale, NJ: Lawrence Erlbaum.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.), Multiplicative Structures (pp. 123-176). Albany, NY: SUNY Press.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. Educational Researcher, 18 (1), 32-42.

Brownell, W. A., & Sims, V. M. (1946). The nature of understanding. The Forty-fifth Yearbook of the National Society for the Study of Education: Part 1. The Measurement of Understanding (pp. 27-43). Chicago: National Society for the Study of Education.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. American Educational Research Journal, 26, 385-401.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. Educational Researcher, 23 (7), 13-20.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.) The development of multiplicative reasoning in the learning of mathematics (pp. 293-330). Albany, NY: SUNY Press.

Davis, R. B. (1984). Learning Mathematics: The Cognitive Science Approach to Mathematics Education. Norwood, NJ: Ablex.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a First-grade classroom. Cognition and Instruction, 17, 283-242.

Fennema, E., Franke, M. L., Carpenter, T. P., & Carey, D. A. (1993). Using children's mathematical knowledge in instruction. American Educational Research Journal, 30 (3), 555-583.

Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. Educational Studies in Mathematics, 38, 11-50.

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Ginsburg, H. P. (1982). Children's Arithmetic. Pro-Ed, Austin.

Greeno, J. G. (1978). Understanding and procedural knowledge in mathematics instruction. Educational Psychologist, 12, 262-283.

Greer, B. (1988). Nonconservation of multiplication and division: Analysis of a symptom. The Journal of Mathematical Behavior, 7, 281-298.

Hatano, G. (1996). A conception of knowledge acquisition and its implications for mathematics education. In L. P. Steffe et al. (Eds.), Theories of mathematical learning (pp. 197-217). Mahwah, New Jersey: Lawrence Erlbaum.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. Educational Studies in Mathematics, 19, 333-355.

Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), Handbook of Research on Mathematics Teaching and Learning (pp. 65-97). New York: Macmillan.

Hiebert, J. & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 199-223). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. J. Behr (Eds.), Number Concepts and Operations in the Middle Grades (pp. 162-181). Hillsdale, NJ: Lawrence Erlbaum.

Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. Schappelle (Eds.). Providing a foundation for teaching mathematics in the middle grades (pp. 31-65). Albany, NY: SUNY Press.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. Cognition and Instruction, 3, 305-342.

Leinhardt, G. (1988). Getting to know: Tracing students' mathematical knowledge from intuition to competence. Educational Psychologist, 23, 119-144.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. Journal for Research in Mathematics Education, 21, 16-32.

Mack, N. K. (1995). Confounding whole number and fraction concepts when building on informal knowledge. Journal for Research in Mathematics Education, 26, 422-441.

Mack, N. K. (in submission). Building on Informal Knowledge in a Complex Content Domain: Partitioning, Units, and Understanding Multiplication of Fractions.

McNeal, B. (1995). Learning not to think in a textbook-based mathematics class. Journal of Mathematical Behavior, 14, 205-234.

Moschkovich, J. N. (1998/1999). Students' use of the X-intercept as instance of a transitional conception. Educational Studies in Mathematics, 37, 169-197.

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. Mathematical Thinking and Learning, 1, 4.

Pirie, S. & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? Educational Studies in Mathematics, 26, 165-190.

Saxe, G. (1988). Candy selling and math learning. Educational Researcher, 17 (6), 14-21.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. Educational Researcher, 27 (2), 4-13.

Shulman, L. S. (1986). Paradigms and research programs in the study of teaching: A contemporary perspective. In M. Wittrock (Ed.), Handbook of Research on Teaching Third Edition. Macmillan, New York.

Streefland, L. (1991). Fractions in realistic mathematics education. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. A. Fennema, & T. A. Romberg (Eds.), Rational numbers an integration of research (pp. 289-325). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.) Number concepts and operations in the middle grades (pp. 119-140). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. & Cobb, P. (1988). Construction of arithmetical meanings and strategies. New York: Springer-Verlag.

Tirosch, D. (1999). Introduction. Educational Studies in Mathematics, 38, 1-9.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), Acquisition of mathematics concepts and processes (pp. 127-174). New York: Academic Press.

Voigt, J. (1994). Negotiation of mathematical meaning and learning mathematics. Educational Studies in Mathematics, 26, 275-298.

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Vygotsky, L. S. (1978). Mind and Society: The Development of Higher Psychological Processes. Harvard University Press, Cambridge.

Wood, T. & Sellers, P. (1996). Assessment of a problem-centered mathematics program: Third grade. Journal for Research in Mathematics Education, 27, 337-353.

Table 1

Mental Processes and Corresponding Situations When First Communicated

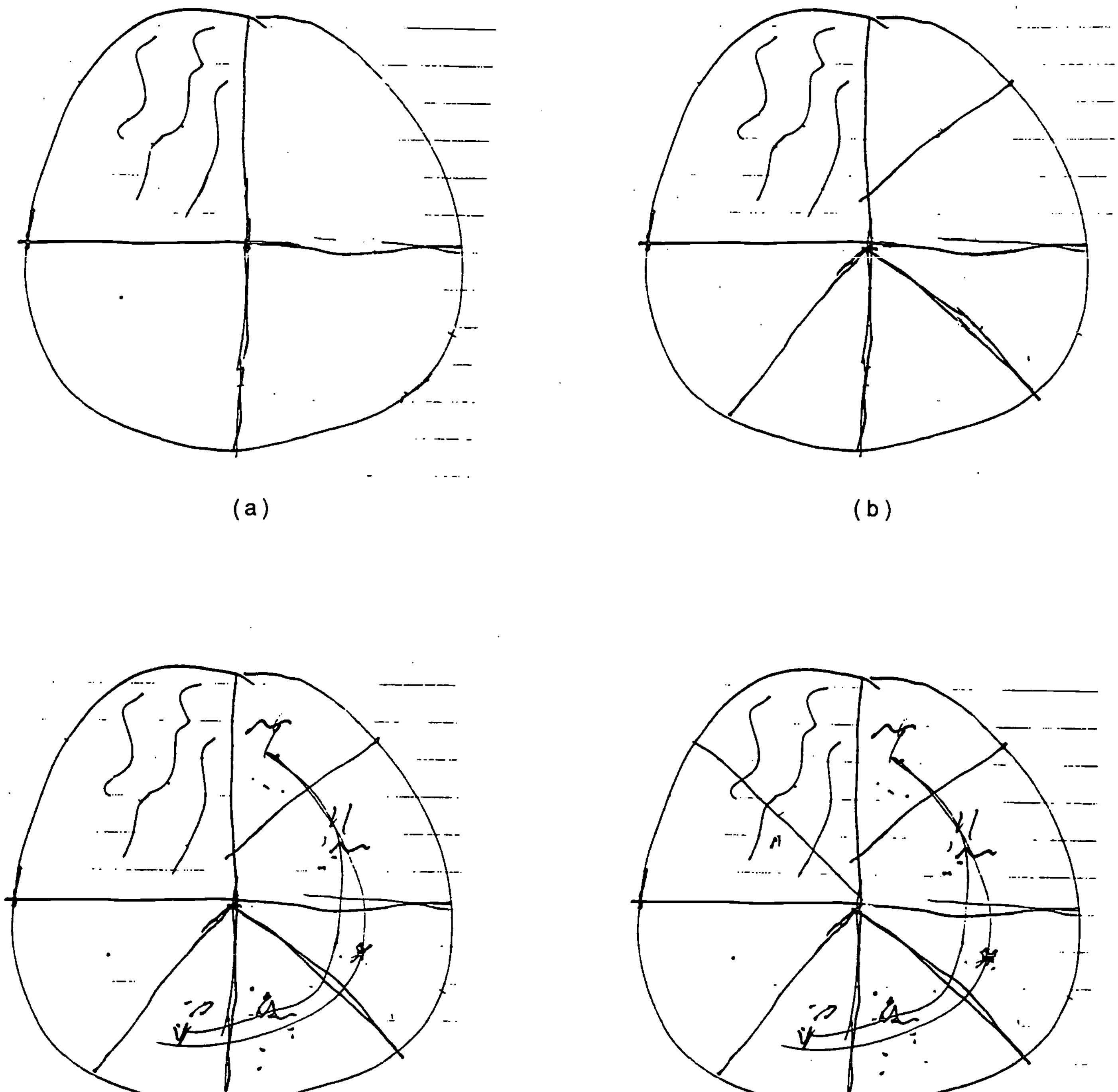
Mental Process	Corresponding Mathematical Expression	Corresponding Contextual Situation
Focusing on fractional portions of units	a ÷ b = n c/d, where n 1	Share 10 cookies between 4 people.
Considering what it means to partition a unit into a fractional amount	a/b x nb	Find one third of of 12 cookies.
Realizing a composite unit need not be partitioned by seeing one unit embedded within another	a/b x b/d	Find two thirds of of three fourths of one whole pizza.
Realizing a composite unit can can be repartitioned	a/nb x b/d	Find three fourths of of two thirds of one whole pizza.
Realizing pieces of a composite unit can be grouped	a/b x nb/d	Find two thirds of of nine tenths of one whole pizza.
Realizing a composite unit can be repartitioned and resulting pieces can be grouped	a/b x c/d, where b and c are relatively prime	Find three fourths of seven eighths of one whole pizza.

Figure Captions

Figure 1: Lee's solution to a problem involving finding 5/6 of 3/4 of one whole pizza.

Figure 2: Lisa's solution to 2/3 x 10/20 = ?

Figure 3: Sam's solutions to problems corresponding to 3/4 x 2/3.



(a) (b) (c) (d)

Figure 1

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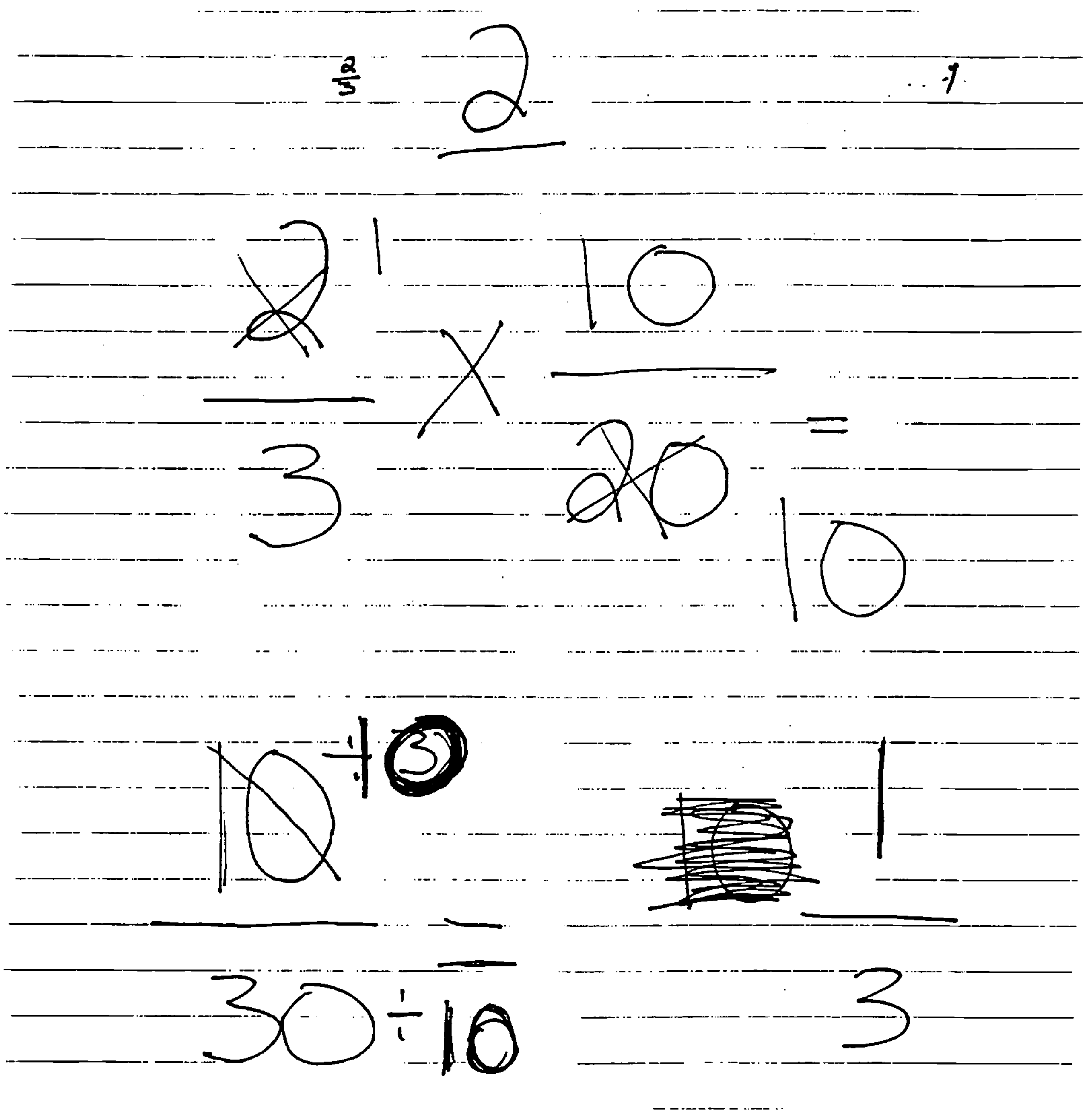

Figure 2

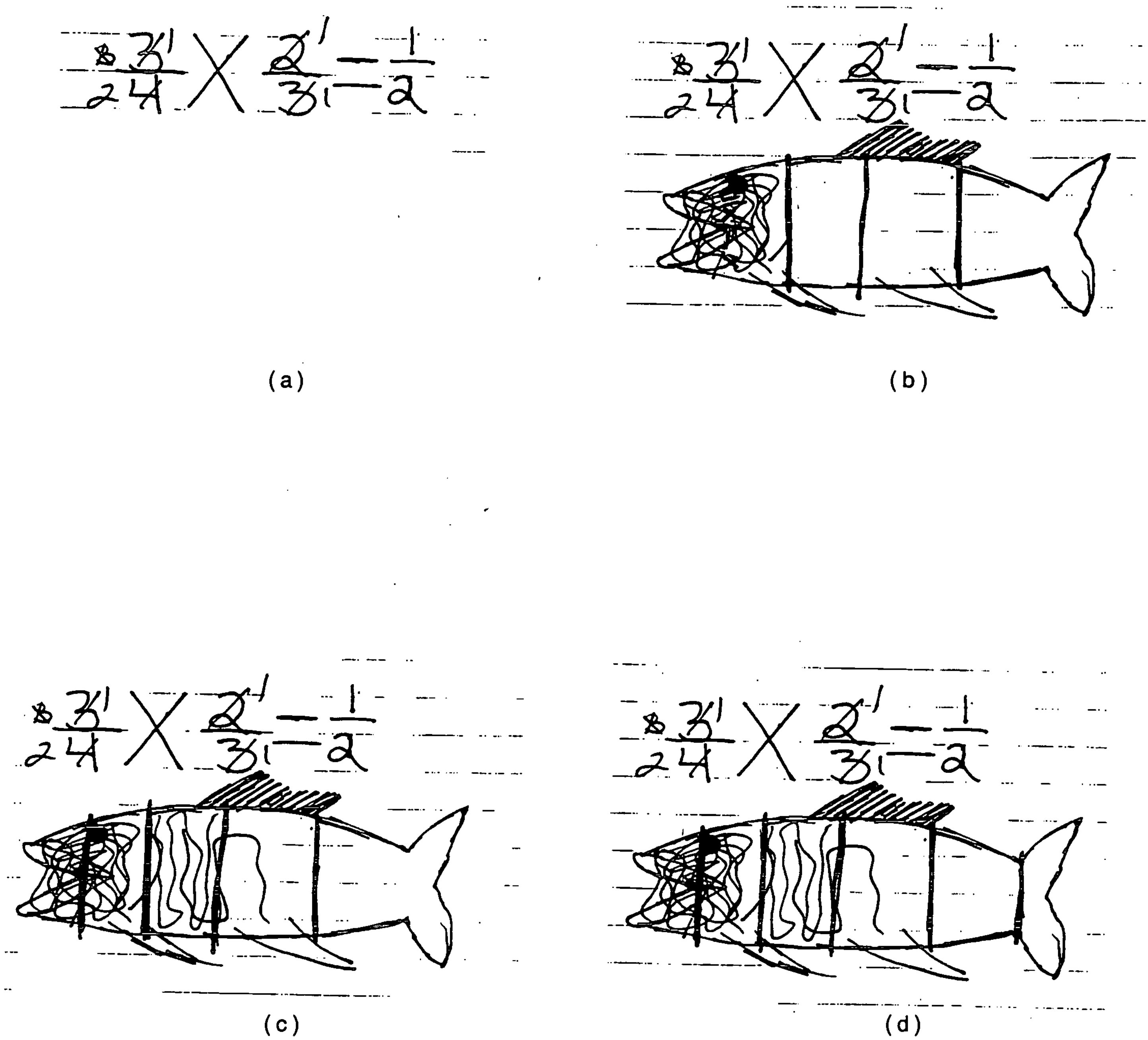

Figure 3

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