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ABSTRACT

There is a popular view that arithmetic is only a collection of dull algorithms containing no interesting ideas. However, if one goes a little below the surface, it will become apparent that there are many fascinating ideas waiting to be discovered. The paper tries to show that mathematical ideas can be recognized and understood without the use of "high tech," graphing calculators. Calculators are fast and accurate, and when properly used they allow students to bypass tedious arithmetic and get to the important concepts in a problem. However, educators must be careful not to let the new technology become the driving force in the curriculum. The focus must always be on conceptual mathematical understanding. This paper presents some key "mini-lessons" from arithmetic, algebra, and geometry that will illustrate that elementary mathematics rests on a set of key ideas that are best illustrated using simple calculations. Lesson 1 explores fractions and factor trees. Lesson 2 teaches repeating decimals. Lesson 3 looks at infinite sums. Lesson 4 discusses Gauss's early discovery in finding the sum of the first 100 integers. Lesson 5 explores solutions to the mixture problem, or what students dread--word problems. Lesson 6 explores Pythagorean triples. (VWC)

TWO MODES OF MATHEMATICS INSTRUCTION

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In recent years mathematics education has witnessed a dramatic increase in student use of hand-held calculators. Students are being encouraged to use calculators in many areas of mathematics, particularly in graphing. Many faculty have revised their method of class presentation and now give lectures that are calculator dependent. In this new style of presentation the instructor uses a projected image of a calculator on a screen to lead students through solutions to a set of problems. New texts have been published that depend heavily on the use of the calculator. A typical such text begins with a chapter explaining the use of standard calculator. The remainder of the book stresses problems that involve the use of a calculator. The net effect is to leave the student with the false impression that mathematics can't be done without the use of a calculator.

There can be little doubt that calculators make it possible to do difficult calculations quickly and efficiently. When used for this purpose they must be viewed positively since they allow students to move on to the more interesting aspects of a mathematics problem. However, educators should always keep in mind that much of elementary mathematics rests on a set of key ideas that are best illustrated using simple calculations. In fact doing these elementary calculations is often the necessary ingredient for a true understanding of the concept.

In this paper I will present some key "mini-lessons" from arithmetic, algebra, and geometry that I hope will illustrate my point of view. The first three lessons come from basic arithmetic. There is a popular view that arithmetic is only a collection of dull algorithms containing no interesting ideas. However, if we go a little below the surface we will see that there are many fascinating ideas waiting to be discovered.

Lesson 1: Fractions and Factor Trees

Let's consider the technique necessary to add two fractions, say $\frac{1}{6}+\frac{1}{8}$.

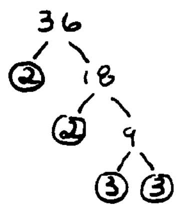
Many students quickly fall into the obvious trap and respond

$\frac{1}{6}+\frac{1}{8}=\frac{2}{14}$ (incorrect)

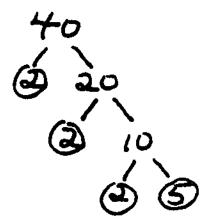
$\frac{2}{14}$ can't be the correct answer because $\frac{2}{14}=\frac{1}{7}$ and $\frac{1}{7}$ is smaller than $\frac{1}{6}$, so the right side would be smaller than the left. With a little coaching the student realizes that the numerators of the fractions can be added only when we have a common denominator. Thus the problem reduces to finding a common denominator for the two fractions, and then expressing them as equivalent fractions using this denominator.

For a common denominator we must find a number that is a common multiple of 6 and 8. Also we want it to be as small as possible so it will be the lowest common denominator. Most students easily see that 24 is the lowest common denominator. Then $\frac{1}{6}=\frac{4}{24}$, $\frac{1}{8}=\frac{3}{24}$, so $\frac{1}{6}+\frac{1}{8}=\frac{4}{24}+\frac{3}{24}=\frac{7}{24}$.

But what if the denominators were large numbers? Would it still be easy to find the lowest common denominator? The answer is yes. We can factor each denominator into a product of primes using factor trees. For example consider $\frac{1}{36} + \frac{1}{40}$. First we factor 36 and 40 using factor trees. Each prime factor is circled in the tree.



$36 = 2 \cdot 2 \cdot 3 \cdot 3$



$40 = 2 \cdot 2 \cdot 2 \cdot 5$

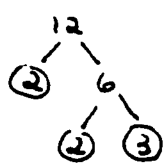
From these factorizations it is clear that the smallest common multiple of 36 and 40 is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 360$.

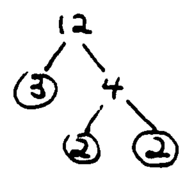
Then $\frac{1}{36} = \frac{1}{36} \cdot \frac{10}{10} = \frac{10}{360}$

$\frac{1}{40} = \frac{1}{40} \cdot \frac{9}{9} = \frac{9}{360}$

and $\frac{1}{36} + \frac{1}{40} = \frac{10}{360} + \frac{9}{360} = \frac{19}{360}$.

The factor tree method leads to the following question. A number can have more than one factor tree associated to it. For example





Will different trees for the same number always lead to the same prime factorization? The answer is yes and is known as The Fundamental Theorem of Arithmetic.

Lesson 2: Repeating Decimals

Consider the problem of finding the decimal expansion for a fraction. This involves the repetitive process of long division, and the natural temptation would be to do all such computations with a calculator. However, when the expansions are infinite in length, the calculator can only give approximations which do not reveal certain interesting patterns.

For example,

$\frac{1}{2} = .5$	$\frac{1}{8} = .125$	$\frac{1}{14} = .0\overline{714285}$
$\frac{1}{3} = .\overline{3}$	$\frac{1}{9} = .\overline{1}$	$\frac{1}{15} = .0\overline{6}$
$\frac{1}{4} = .25$	$\frac{1}{10} = .1$	$\frac{1}{16} = .0625$
$\frac{1}{5} = .2$	$\frac{1}{11} = .\overline{09}$	$\frac{1}{17} = .\overline{0588235294117647}$
$\frac{1}{6} = .1\overline{6}$	$\frac{1}{12} = .08\overline{3}$	($\overline{}$ indicates repeating digits)
$\frac{1}{7} = .\overline{142857}$	$\frac{1}{13} = .\overline{076923}$	

We immediately notice that each value is either a finite decimal or an infinite repeating decimal. If we look more carefully, we see that finite decimals are obtained when the denominator has factors only of 2 or 5. This should not be so surprising since our number system is based on 10 , and 10 = 2 x 5. However, there is still the question of why all the other fractions lead to <u>repeating</u> decimals. Let's use the division algorithm to evaluate a typical case, say $\frac{1}{7}$. We will circle each remainder in the division process starting with the initial numerator of 1. This should make the pattern clearer.

$$\begin{array}{r} .142857 \\ 7\overline{)1.000000} \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

Starting with the initial numerator of 1, the sequence of remainders is 1, 3, 2, 6, 4, 5, 1. But once this second 1 occurs, we are back in the original starting position, and now the same cycle will repeat itself until a remainder of 1 is again obtained, and so on. Therefore the value of $\frac{1}{7}$ is $.\overline{142857}$.

In fact what this argument shows is that any decimal expansion of a fraction must be either finite terminating or infinite repeating. This is true because if the division algorithm does not terminate after a finite number of steps, then at some point a remainder must repeat. Once a remainder repeats, we go through an infinite cycle in the division. This interesting and non-trivial result could not be understood if the calculations were done with a calculator.

Lesson 3: Infinite Sums

Having developed a technique for adding a finite set of fractions, we can now consider the problem of how to add infinitely many fractions. For example, what is the value of the following infinite sum:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Of course we know that it is only possible to add a finite set of numbers, so how can we assign a value to the above expression? We can do it by creating a sequence of finite sums, with each new sum containing one more term than the previous one, then taking the limit of these sequence values as we include more and more terms. For our example the sums are

$$\frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{8}{16} + \frac{4}{16} + \frac{2}{16} + \frac{1}{16} = \frac{15}{16}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{16}{32} + \frac{8}{32} + \frac{4}{32} + \frac{2}{32} + \frac{1}{32} = \frac{31}{32}$$

$$\cdots$$

Notice that in each calculation we had to find a common denominator for the fractions being added. Now we can determine the value of the infinite sum. As the number of terms being added increases, both the numerator and denominator of the resulting finite sum increase in size. However, at each stage the numerator is always one less than the denominator.

It follows that the infinite sum has a limiting value of 1 since the finite sums get closer and closer to the value of 1, even though they never reach the value 1.

What we have done in this problem is to introduce the concept of a limit: something that can be approached but never quite reached. This concept is the foundation for calculus and mathematical analysis.

It would seem that the above example is one in which the calculator would be an effective tool for computing the finite sums to see if a limit is being approached. However, there are infinite sums for which the calculator will give a misleading result. This will happen if the finite sums grow larger and larger towards infinity, but do so at a very slow rate. One such example is the infinite sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

It can be shown that we can always find a finite sum that is larger than any given specified value; in particular we can find a finite sum that exceeds the value 100. However, in order to exceed this value our finite sum would need to contain more than 1,000,000,000,000,000,000,000,000,000,000 terms. Obviously no calculator could be used to process so many terms.

Now we will consider two problems from algebra. In a typical algebra problem we are given some information about an unknown quantity. This information must be used to determine an equation satisfied by the unknown quantity.

Lesson 4: Gauss's Early Discovery

One of the best known mathematical anecdotes relates how the great mathematician Carl Friedrich Gauss (1777 - 1855) made a remarkable discovery as a child. According to the story, Gauss's teacher needed some free time to do some paperwork so he assigned the class the difficult arithmetic of finding the sum of the first 100 integers. The teacher assumed that the problem would keep the class occupied for at least an hour. Imagine his surprise when the young Gauss appeared at his desk after a few minutes and presented him with the correct answer! How did Gauss do it?

Gauss's insightful solution is a simple demonstration of the power of algebra. The unknown quantity in the problem is the sum of the first 100 integers, i.e., $1+2+3+\cdots+98+99+100$. In algebra we always use a letter to represent an unknown quantity, so let S denote the sum of the first 100 integers. Then $S = 1+2+3+\cdots+98+99+100$.

Gauss realized that the same sum would be obtained if the numbers were added in the reverse order:

$$S = 100+99+98+\cdots+3+2+1.$$

If we write the second equation directly below the first, then add the equations by adding corresponding entries in each column, a simple result follows.

$$\begin{array}{rl} \text{first equation:} & S = 1 + 2 + 3 + \cdots + 98 + 99 + 100 \\ \text{second equation:} & S = 100 + 99 + 98 + \cdots + 3 + 2 + 1 \\ \hline \text{third equation:} & 2S = 101 + 101 + 101 + \cdots + 101 + 101 + 101 \end{array}$$

The third equation states that $2S$ is the sum of 101 listed 100 times (once for each column). Therefore $2S = (100)(101)$. If we divide each side of the last statement by 2, then we obtain the same result that Gauss found:

$$S = \frac{(100)(101)}{2} = (50)(101) = 5{,}050$$

Therefore the sum of the first 100 integers is 5,050.

This method can easily be generalized to find the sum of any number of consecutive integers staring with 1. Let S denote the sum of the first n integers. First we write the equation for S as an ascending sum, then we write the equation for S as a descending sum. If we add the two equations column by column, we obtain a value for $2S$.

$$\begin{array}{l} S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ \hline 2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \end{array}$$

How many $(n+1)$'s are there? There is an $(n+1)$ for each column, so there are n $(n+1)$'s. Therefore

$$2S = n(n+1)$$

and $S = \dfrac{n(n+1)}{2}$.

If we want the sum of the first 500 integers, we replace n with 500 in the formula: $S = \dfrac{(500)(501)}{2} = 125{,}250$.

Gauss's solution is a beautiful illustration of why it is important to think about what a calculation involves before actually doing it. Sometimes this is lost sight of when students rush to use their calculators.

Lesson 5: The Mixture Problem

Perhaps more than any other type of problem students dread algebra word problems. This is unfortunate because many of these problems require only elementary calculations and share a common structure that is easy to illustrate. Typical of this group of problems is the mixture problem, an example of which is given below.

A box contains a total of 40 nickels and dimes with a combined value of $3.60. How many of each type are in the box?

As in the previous lesson, we want to represent the unknown quantities by letters. This time there are two unknown quantities: the number of dimes and the number of nickels. We introduce letters d, n with d = the number of dimes,

n = the number of nickels.

Since we have two unknowns, we will need two equations to find the solution. Each equation will come from a factual statement in the problem. We can set up a box chart to represent the given information.

coin	number	value
nickels	n	5n
dimes	d	10d
total	40	360

Note that we have expressed the monetary values in cents so that we can avoid the decimals. Our two equations come from summing the columns.

The total number of coins is 40: $n + d = 40$

The total value of the coins is 380: $5n + 10d = 360$.

If we multiply the first equation by -5, rewrite the second equation, then add these equations we obtain

$$\begin{array}{r} -5n - 5d = -200 \\ 5n + 10d = 360 \\ \hline 5d = 160 \end{array}$$

Dividing by 5 now yields $d = \frac{160}{5} = 32$.

Since $n+d=40$, we must have $n = 8$.

There are 8 ;nickels and 32 dimes in the box. This checks out since the combined value is then $8(5) + 32(10) = 360¢$.

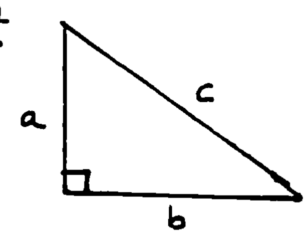
The interested reader should try to use the above box chart method to solve the following problems.

1. A box contains a total of 24 dimes and quarters with a combined value of $3.30. How many of each type are there?

2. A 10 pound mixture of nuts has a value of $36. The mixture consists of cashews selling at $4 per pound and peanuts selling at $3 per pound. How many pounds of each type are in the mixture?

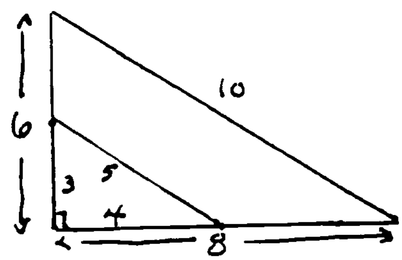
Lesson 6: Pythagorean Triples

The Pythagorean Theorem is one of the most well-known results in Euclidean geometry. It expresses the relationship that exists among the three sides of a right triangle.If we denote the two shorter sides by a and b, and the long side by c, then the theorem states that $a^2 + b^2 = c^2$.



Are there any solutions to the above formula with a, b, and c all positive integers? You may recall from high school that a = 3, b = 4, c = 5 is such a solution since $3^2+4^2=5^2$. We will denote this solution by the triple (3,4,5) and refer to it as a Pythagorean triple.

It's easy to see that we can use the triple (3,4,5) to generate other Pythagorean triples. If we double the sides of the 3-4-5 right triangle, we will obtain a 6-8-10 right triangle.



Therefore $6^2+8^2=10^2$, so (6,8,10) is a Pythagorean triple. Similarly we see that (9,12,15), (12,16,20), (15,20,25),... are all Pythagorean triples.

Since these new triples are multiples of the basic (3,4,5) triple, we are not too interested in them; instead we are more concerned with listing the "primitive" Pythagorean triples. The term "primitive" means that the numbers a, b, and c don't have a common factor. It is not too hard to show that each primitive triple (a,b,c) must have c = odd and exactly one of a, b even and the other odd. Let's assume we label so that b is the even value. Then it is possible to obtain a complete description of all primitive Pythagorean triples.

All primitive Pythagorean triples can be found by the following method. Choose positive integers x and y satisfying the following three conditions:

(1) x is greater than y.

(2) x is even and y is odd or x is odd and y is even.

(3) x and y are not divisible by a common factor.

Then $a=x^2-y^2$, $b=2xy$, $c=x^2+y^2$ yields all primitive triples.

As an illustration, the table below lists all primitive triples obtained for values of x up to 5.

x	y	$a=x^2-y^2$	$b=2xy$	$c=x^2+y^2$
2	1	3	4	5
3	2	5	12	13
4	1	15	8	17
4	3	7	24	25
5	2	21	20	29
5	4	9	40	41

Just as surprising as the formula is the fact that the ancient Babylonians had a method for generating laree primitive triples. A cuneiform tablet dating from 1500 B.C. contains a list of triples including (3,4,5) and (4961, 6480, 8161).

The French mathematician Pierre de Fermat (1601 - 1665) generalized the Pythagorean triple problem by looking for positive integer solutions to the equation $a^n+b^n=c^n$ when the exponent n is greater than 2. Sometime in the 1630's he conjectured that for all such values of n not one single solution exists. Fermat claimed to have a proof of his conjecture, but he never published it and no written proof was ever found in his surviving papers. His conjecture became one of the great problems of mathematics and remained unsolved for more than three centuries. Finally in 1995 Professor Andrew Wiles of Princeton University proved that Fermat's Conjecture was true.

At this point I would like to summarize my view towards calculators and mathematics. I have tried to show in this paper that mathematical ideas can be recognized and understood without the use of "high tech" calculators. On the other hand, I use a calculator, and in most of the courses that I teach I allow my students to use a calculator. Calculators are fast and accurate, and when properly used they allow students to bypass tedious arithmetic and get to the important concepts in a problem. However, we as educators must be careful not to let the new technology become the driving force in the curriculum. The focus must always be on conceptual mathematical understanding.

REFERENCES

H. Davenport, The Higher Arithmetic, Cambridge University Press (1952)

C. Ogilvy and J. Anderson, Excursions in Number Theory, Oxford University Press (1966)

A. Goodman and L. Hirsch. Understanding Elementary Algebra, West Publishing Company (1991)

H. Rademacher and O. Toeplitz, The Enjoyment of Mathematics, Princdton University Press (1951)

H. Rademacher, Higher Mathematics from an Elementary Point of View, Birkhauser (1983)

W. Reichmann, The Fascination of Numbers, Oxford University Press (1963)

J. Silverman, A Friendly Introduction to Number Theory, Prentice - Hall (1997)

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