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ABSTRACT

This paper discusses a pedagogical technique called the discovery approach in teaching remedial and college algebra. Examples of how to introduce and enhance some algebra topics using this approach in mathematics classrooms are illustrated. It is argued that student performance would improve if more topics were presented in a discovery fashion. (ASK)

Algebra with a Discovery Approach

by

Harvey Yarborough

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Structure.

To us, mathematics is structured. Topics are connected. Exploration of one topic leads to fascination with other related topics. The more we learn, the more we want to know. True knowledge is based on understanding, and we're not satisfied with simply accepting facts; we must see the relationships between the new knowledge and the knowledge we already have.

To many of our students, in contrast, mathematics is a collection of discrete unrelated topics. Each problem is a problem on its own, unrelated to other studies. Topics are not connected. The learning that occurs has no structure. As a result, the learning is short lived. Unless the knowledge is being constantly used, it quickly decays.

"Why must I study mathematics?" is a legitimate question for these students. Being so unstructured, their mathematics is of little value to them. They see no use of it in their world. Thus, it is of little interest to them. They view required math courses as obstacles placed in their path by educational bureaucrats. Their primary objective is to get past these obstacles in any possible way. Credit for the course, not knowledge of the material, is their primary goal.

If mathematics to become real and useful to our students, they must learn to structure their learning.

Concepts must be understood. Imaginations need be developed and used to explore concepts. Creativity should be recognized and rewarded. Blind acceptance of instruction without understanding should be discouraged (a reasonable level of skepticism is healthy). Basic principles should be illuminated, not obscured with detail.

I had my awakening in 1988 when I met and visited with Dr. Charley Pine[1]. He taught using a discovery approach. To him, understanding was everything. Drill and practice and artificial memory aids, he felt, were of little value. His study of the performances of New Jersey high school students who took a standardized algebra test months after their completion of an algebra course convinced him that students who learned with a discovery approach retained algebra far better than did students who learned with any other method.

I became aware of mistakes I was making. I was trying to program my students as I would a computer. I gave precise instructions. I made sure they were complete. I tried to anticipate any modification of a problem a student might encounter and address those

[1] Dr. Charles Pine was a professor of physics at Rudgers University and chairman of "The Algebra Project" – a consortium of New Jersey math teachers committed to teaching high school algebra with a discovery approach.

variations. I proved theorems on the board since, surely, this would impress upon students that I knew of what I spoke. "Good" students would ingest my lectures without complaint. When tested, they would regurgitate what I had given them. If too much time hadn't elapsed, one could still discern part of the content of my lectures.

The purpose of an algebra course, I thought, was to condition students to perform algebraic manipulations on command. I was trying to turn students into manipulating machines. It was frustrating to my students. It was frustrating to me. No matter how careful I was with these instructions, no matter how precise my directions, most students would forget some detail, which would cause them to miss a problem.

You can now buy a hand held computer, at a modest price, that is basically what I was trying to turn my students into. This product, however, is vastly superior to one of my students. It not only stores more in memory, it has perfect recall of these facts and procedures. One can set it aside and leave it for a month and it still remembers how to solve an equation, factor a polynomial, simplify an expression, or do any of a myriad of other manipulations. My students didn't have this perfect retention.

I now realize my instruction concentrated on the weaker part of the human mind, the memory. At the same time I was ignoring a stronger part of the human mind, the imagination. I was pretending my students were computers, not people.

Students aren't computers. They haven't perfect recall of facts and instructions.

The weak performance of my students convinced me I should do something different. I decided to try Dr. Pine's suggestion. I tried teaching with discovery.

I immediately noticed an increased interest level in my students. Classes became lively discussions. The heightened interest continued after class as was demonstrated by students choosing to remain in the classroom, after dismissal, to continue the discussions. Algebra changed from being a class that I dreaded to one that I actually enjoyed teaching.

I wrote problem sets to illustrate the concepts discussed in class. As crude as they are, these problem sets seem more effective than textbooks. Perhaps the organization of the material is more natural. Perhaps students are more inclined to use their imagination and devise their own strategies for solving the problems when a strategy is not spelled out for them. Since the summer of 1990 I've been teaching remedial and college algebra without a textbook.

I avoid mechanical instructions. Emphasis, I think, should be placed on concepts – not mechanics. Let me contrast two sets of instructions for addition with positive and negative numbers.

Here is a set of mechanical instructions that are similar to those in a text we once used.

To add numbers with the same sign:

a) *Find the absolute value of each of the numbers.*
b) *Add these absolute values.*
c) *Prefix the sum with the common sign of the original numbers.*

To add two numbers that differ in sign:

a) *Take the absolute value of each number.*
b) *Find the difference of these absolute values.*
c) *Prefix the sum with the sign of the number that has the larger absolute value.*

Follow these instructions precisely, don't make a mistake, and you will always get the correct answer to an addition problem. These instructions are good for computers but not so good for humans. [2]

Now consider this conceptual description of addition:

> *Imagine the number line. Start at the first addend. Move the appropriate distance to the right if you're adding a positive number, left for a negative number.*

I assume this would be difficult to program into a computer. A student, on the other hand, has little trouble following these instructions. The student has an imagination. The instructions describe a concept – a mental image one can visualize. It's a natural way of viewing the material and it's much simpler than the detailed mechanical instructions. Further instructions aren't necessary. A student can now "figure out" how to add any set of positive and negative numbers.

Furthermore, with this concept a questions such as "What must you add to -3 to get 7?" is reasonable. Subtraction, then, is related to addition in a very natural fashion. A mental image comes to mind. Picture the number line. How far and which direction must you go to get from -3 to 7. What, then, do you add to -3 to get 7. This is subtraction as a concept – not a mechanical trick.

For a discovery approach to work, the material needs be organized in a natural fashion. Let me illustrate with quadratic equations with irrational roots.

> It would be unreasonable, I think, to expect a student to discover how to solve an equation such as $x^2 - 6x + 1 = 0$ without coaching or guidance. It is not unreasonable, however, to expect a student to arrive at solutions of the equation $x^2 = 7$.
>
> It is now reasonable to expect a student to find solutions of an equation such as $(7x - 3)^2 = 5$.

[2] Late in the semester I would see a student add -3 and -5 and get positive 8. "Why <u>positive</u> eight?" I would ask. "Because two negatives make a positive" would be the reply.

I write the equation on the board and ask *"How can you change this to an equation you know how to solve?"* I wander around the room looking over shoulders. Most students expand the squared binomial and attempt to solve the equation by factoring – not a bad strategy except it doesn't work.

I repeat *"How can you change this to an equation you know how to solve?"*

Eventually someone will notice that a simple substitution[3] gives $a^2 = 5$ an equation we can solve. A solution to this equation leads to solutions of the original equation.

Now the equation $x^2 - 4x = 1$ is reasonable.

Again *"How can you change this to an equation you know how to solve?"* Someone will ascertain that the equation can be written as $(x-2)^2 = 5$, an equation we can solve.

Now $x^2 - 6x + 1 = 0$ is reasonable since one need only figure *how to change this to an equation we know how to solve.*

What if the coefficient of x is odd, or a fraction? What if the coefficient of x^2 is different from one? *How can you change the equation to one you know how to solve?*

I'm convinced that, in many cases, the difference between a weak student and a strong student is that the weak student views each new problem as entirely new while the strong student ask *"How can I change this to a problem I know how to solve?"* The weak student thinks of each problem is independent of others and makes a massive attempt to memorize methods for solving every problem. The strong student looks for structure. I've seen weak students become strong after they learned to look for structure and make connections. Certainly some students have more talent than others, but, in many cases, the weak student isn't lacking in talent – he simply doesn't know **how** to learn math.

For material to make sense, for students to be able to discover concepts, material needs be organized in a natural fashion. Similarly, definitions should be stated in a form that is natural and conceptual rather than mechanical. Absolute value is a good illustration of this point.

[3] I object to "taking the square root of both sides of the equation." If $p = q$, then $p^n = q^n$. Taking the square root of both sides of an equation assumes the converse of this theorem, which we know isn't valid.

Absolute value is usually defined as $|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$ or $|a| = \sqrt{a^2}$.

These are mechanical definitions. They give convenient instructions for finding an absolute value or writing a program to assign absolute values, but, they provide little insight into the significance of absolute value or for what the concept is used.

Why was absolute value invented? I'm no historian but from the applications I've seen I assume that absolute value was invented to represent distance[4].

Since absolute value is so often used to represent distance, why not define absolute value as a distance? I define $|a-b|$ as "the distance between a and b." The following questions are reasonable:

Picture 7 and 13 on the number line. How far apart are they i.e. what's $|7-13|$?

Consider the equation $|x-5| = 3$. Picture the number line. Where is a number if it is 3 units away from 5? Have all possibilities been considered?

What about the open sentence $|x-5| > 3$? Draw a number line. Shade in the region(s) on the line where x must be to make this a true statement. What's the solution set? (I insist on interval notation.)

Now consider $|x+7| < 10$. *How can you change this problem to one you know how to solve?* $|x+7|$ represents the distance from x to what number?

$|x|$ is the distance from x to what number? What is the solution set of $|x| \leq \sqrt{419}$? Where is x on the number line if $|x| > \frac{3\pi}{7}$?

Think about the inequality $|23x+19| > \sqrt[3]{5}$. *How can you change this problem to one you know how to solve?* Substituting "a" for "$23x+19$" makes the problem easy.

Look at the inequality $|5x-31|+7 < 10$. *How can you change this problem to one you know how to solve?*

Consider the inequality $|13x+9|+15 \geq 7$. *How can you change this problem to one you know how to solve?* What is it's solution set?

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[4] "$\ldots |f(x)-L| < \varepsilon$ whenever $|x-a| < \delta$" is part of a commonly accepted definition for the basic calculus concept of limit. The expression is usually read "the **distance** from $f(x)$ to L is less than ε whenever the **distance** from x to a is less than δ."

"$|a-b|$ is the distance between a and b" is a valid definition of absolute value. It's consistent with any other definition– but it is natural and conceptual.

Several times I've recommended substitutions. I'm obsessed with substitutions. A substitution is a valuable and too frequently overlooked technique for making a complex problem appear simple. It often helps make a connection between a new problem and one that is familiar.

How can you change this problem to one you know how to solve? Often the answer is "make a substitution."

Strong students intuitively see substitutions. They look at $(7x-3)^2 = 37$, for example, and see $(\quad)^2 = 37$. They may not realize they've even made a substitution. Frankly, some students keep the substitutions secret. That way they can pretend superiority.

Weak students don't intuitively see substitutions. They're overwhelmed by the complexity of the problem $(7x-3)^2 = 37$. They see a complicated mess. Most likely, they'll square the binomial because that's something they've done before and it's comforting to be doing something. They expect the problem to be hard. When encouraged to spell out a substitution, they're often surprised at how simple the problem becomes. They expect difficulty. They make the problem difficult. They've not learned a secret we know –once you understand something it becomes simple. They should be encouraged to look for simplicity. They should learn to search for substitutions. **A weak student who learns to make substitutions will become a stronger student.**

> "Factor $(2x+3)^5(x-4)^6+(2x+3)^6(x-4)^5$."
>
> Most students are overwhelmed by this problem but with substitutions the problem becomes $a^5b^6+a^6b^5$, an easy problem.
>
> "Simplify $\left(x^{1/2}+2x^{-1/2}\right)^2$."
>
> Simple substitutions give either $(a+b)^2$ or $(a+2b)^2$. Either is a more basic algebra problem. After expanding this simpler problem, the original problem is much easier.
>
> "Solve $\left(x^2+3x+1\right)^2-4\left(x^2+3x+1\right)-5=0$."
>
> A substitution is obvious when one knows to look for it.
>
> "Write an equation for a parabola which opens upward and has x-intercepts $\frac{7-\sqrt{3}}{2}$ and $\frac{7+\sqrt{3}}{2}$."

This problem stumps many students. Yet these same students, when instructed to write an equation for a parabola which opens upward and has x-intercepts a and b, will respond $y = (x - a)(x - b)$. They can, then, solve the original problem.

"Solve $5^{3x+2} = 7^{2x-1}$."

I was surprised when after getting the equation "$(3x + 2)\ln 5 = (2x - 1)\ln 7$" a student said: "Lets substitute. Let $a = \ln 5$ and $b = \ln 7$." The students in the class had less trouble solving $(3x + 1)a = (2x - 1)b$, for x, than they had with $(3x + 2)\ln 5 = (2x - 1)\ln 7$.

Why use discovery? Why not simply lecture through the material in a natural and logical fashion?

I'm convinced that true conceptual learning is personal and private. A teacher cannot make the connections for a student; the student must do it. The teacher cannot erect the structure; it must be done by the student. The teacher can model, the teacher can encourage, the teacher can coach, but the teacher cannot force the learning to occur. There is a bumper sticker that is popular with math teachers which says "math is not a spectator sport." The reason for its popularity is its truth. We all know that you don't learn math by listening to lectures, you don't learn math by viewing film clips or listening to tutorials; you learn math by doing it.

Many times over the years I've left class feeling good because of the clear explanation I knew I had presented of a topic only to find that students had acquired little understanding of the subject.

On the other hand, a student who explains a topic surely understands it. To explain subject matter, one must make connections between the individual aspects of the topic. As every teacher knows, you have a much better understanding of a subject after you have explained it.

Teaching with discovery involves the student in this aspect of learning. While drill and practice place emphasis on the individual parts of a subject, discovery emphasizes the connections.

Teaching by discovery isn't new. I didn't invent it. Dr. Pine didn't invent it. The Socratic method refers to that used by Socrates who taught by asking questions and allowing the student to supply the answers. It's long been recognized as an excellent way to teach.

What are the drawbacks of teaching with discovery? Why isn't the method more commonly used?

First, it's a challenging way to teach. It requires more intensity of the teacher than simply presenting a lecture. It demands flexibility. The teacher must prepare, even more than for other classes, but the preparation can't be overly rigid. Surprises often occur.

Second, it requires well-developed classroom management skills. Private conversations among students, even about the material, cannot be allowed in class or chaos results. An aggressive individual mustn't be allowed to dominate the class discussion for other students will lose interest. Students must not feel threatened but each individual must be encouraged to participate in the class discussion. The teacher cannot allow himself to be manipulated into supplying answers to questions the students should answer for themselves. Students must be taught to make guesses without fear of being wrong. A wrong answer isn't a sin but an essential part of the learning process. All these things must be maintained while the focus on the material is constantly sustained.

Third, the teacher is more obviously accountable for the shortcomings of students. With other approaches failure can be blamed on the text or the tutorials or some other aspect of the instructional package. With a discovery approach, responsibility rests with the student and the teacher.

Fourth, discovery classes aren't popular with some students. I think I share both distinctions of being the most loved and most hated math teacher at Brazosport College. Type four learners, the rascals who resent authority, relish my classes since the material is presented as common sense and logic. Type two learners feel cheated because I don't reward those skills they've spent so much of their academic lives cultivating. I have a reputation of being hard despite the fact that the success rate of my students is higher than that of most of my colleagues. Lets face it, most math students aren't really interested on learning math, they want to get credit for a course while expending as little effort as possible. Students are continually looking for ways to get credit for material without having to trouble themselves with understanding that material. It is the teacher's responsibility, I think, to assure that a student who passes a course indeed understands the material. Many students seek teachers whom they perceive as being less demanding.

Fifth, and perhaps the most significant, is the teacher's fear of failure. I had always suspected that a discovery approach would be a superior way to teach but for twenty years, because I feared failure, I was reluctant to try it. What if I was unable to pull it off? I expected the other problems. I knew this method would be challenging for me. Did I have the ability to teach in this fashion? Might I damage my students by trying a method for which I didn't have the talent? Even after talking with Dr. Pine I didn't immediately change my way of teaching. I experimented. It was the reactions of students that convinced me I could and should use this approach

Despite these problems, I consider my experience with discovery teaching to be the most satisfying and rewarding of my teaching career. Frankly, I feel the students in my other classes are being shortchanged because the material would be more interesting and relevant for them had I the time and perseverance to organize those courses for discovery learning. With discovery teaching, I see deeper understanding I see heightened interest, and I see weak students become stronger. Each day is an adventure for both my students and me. I get to know my students better than I do when I use another method. Sometimes I see frustration and anxiety on the faces of my students. Mixed in, however, are expressions of pure uninhibited delight when a student gets a first taste of true mathematical thinking and I have the vicarious pleasure of sharing that student's excitement.

I'm convinced that performance of students would improve if more topics were presented in a discovery fashion. Organizing the material in a natural, logical fashion benefits both the teacher and the student. Concepts inspire the imagination and create interest while rules reduce mathematics to mindless drudgery. Definitions that are conceptual rather than mechanical are meaningful to the student. When new topics are related to previous learning and the connections are made, mathematics becomes structured. When it is structured, it becomes a real, worthwhile, enduring part of the student's education.

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