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ABSTRACT

This report is incorporated into "Attaining Excellence: A TIMSS Resource Kit." Released in Fall 1997, the TIMSS resource kit was developed for educators and those interested in using TIMSS data to improve teaching, curricula, and student achievement in state and local communities. This module presents information about the grade 8 mathematics assessments conducted by the National Assessment of Educational Progress (NAEP) and the Third International Mathematics and Science Study (TIMSS). The first section of the mathematics assessment module compares the frameworks underlying the NAEP and TIMSS grade 8 mathematics assessments and the distributions of test questions across content areas, focusing on the areas of geometry and algebra. The second and third sections of the module describe the geometry and algebra portions of the assessments in detail using actual test questions (often called "items") from the NAEP and TIMSS assessments to illustrate how areas of the frameworks became operationalized into test questions. Assessment results are provided for the geometry and algebra questions to give some perspective on U.S. performance in these two mathematics content areas. (Contains 14 references.) (ASK)

ATTAINING
EXCELLENCE

TIMSS as a Starting Point to Examine Mathematics Assessments

An In-Depth Look at Geometry and Algebra

U.S. Department of Education
Office of Educational Research and Improvement

ATTAINING
EXCELLENCE

TIMSS AS A STARTING POINT TO EXAMINE MATHEMATICS ASSESSMENTS

AN IN-DEPTH LOOK AT GEOMETRY AND ALGEBRA

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INTRODUCTION

TIMSS as a Starting Point to Examine Mathematics Assessments: An In-Depth Look at Geometry and Algebra is designed to be incorporated into *Attaining Excellence: A TIMSS Resource Kit.* Released in fall 1997, the TIMSS resource kit is developed for educators and those interested in using TIMSS data to improve teaching, curricula, and students' achievement in states and local communities.

This module presents information about the grade 8 mathematics assessments conducted by the National Assessment of Educational Progress (NAEP) and the Third International Mathematics and Science Study (TIMSS). In particular, the first section of the mathematics assessment module compares the frameworks underlying the NAEP and TIMSS grade 8 mathematics assessments and the distributions of test questions across content areas, focusing on the areas of geometry and algebra. The second and third sections of the module describe the geometry and algebra portions of the assessments in detail using actual test questions (often called items) from NAEP and TIMSS assessments to illustrate how areas of the frameworks became operationalized into test questions. The assessment results are provided for the geometry and algebra questions to provide some perspective about U.S. performance in these two mathematics content areas.

TIMSS: THE IEA'S THIRD INTERNATIONAL MATHEMATICS AND SCIENCE STUDY

- **Provides international comparisons in mathematics and science achievement**
- **Involves 41 countries and more than 30 languages**
- **Provides results at three grade levels – 3/4, 7/8, and the final year of secondary schooling (grade 12 in the United States)**
- **Was conducted in 1995**
- **Will be conducted again at the eighth grade in 1999**

NAEP: THE NATION'S REPORT CARD

- **Has been tracking U.S. progress in academic achievement since 1969**
- **Conducts assessments in various subject areas, including mathematics, science, reading, writing, history/geography, and other fields**
- **Assesses student achievement at grades 4, 8, and 12**
- **Began providing state-level as well as national mathematics achievement results in 1990; assessments also were conducted in 1992 and 1996**
- **Provides results for 40 or more states; next assessment is scheduled for the year 2000**

MATHEMATICS ACHIEVEMENT AS A PRIORITY IN THE UNITED STATES

TIMSS comes at a time when achievement in mathematics and science has been designated as an educational priority. One of our eight current National Education Goals is that "by the year 2000, the United States will be first in the world in mathematics and science achievement." In addition, mathematics and science experts have issued major calls for reform in the teaching of these subjects. For example, the National Council of Teachers of Mathematics published *Curriculum and Evaluation Standards for School Mathematics* [1] in 1989, and *Professional Standards for Teaching Mathematics* [2] in 1991.

Considerable energy is being devoted to reform in mathematics education in many states, districts, schools, and classrooms, and the national data from the NAEP 1996 mathematics assessment shows that progress in student achievement has been made since 1990.[3] Average scores on the NAEP mathematics scale were higher in 1996 than in 1992 for all three grades (4, 8 and 12), and higher in 1992 than in 1990. However, the improvement has been concentrated at the basic and proficient levels, with few students (2% to 4%) performing at the advanced levels.

The TIMSS results show that U.S. students at grade 4 have reasonable levels of mathematics achievement when compared with other countries. In mathematics, U.S. fourth graders performed above the international average of the 26 countries participating in the TIMSS assessment at that grade level.[4] However, the international standing of U.S. fourth graders was stronger than that of U.S. eighth graders.

In mathematics, U.S. eighth graders performed below the international average of the 41 TIMSS countries.[5] Most of the 25 countries that participated at both grades 4 and 8 had a similar standing relative to the international average at both grade levels. Only one country—the United States—fell from above the international average at grade 4 to below the international average at grade 8.

From an international perspective, performance was even lower at grade 12.[6] In mathematics literacy (general knowledge), the U.S. students' performance was below the international average and among the lowest of the 21 TIMSS countries. Students in only two countries (Cyprus and South Africa) performed significantly below students in the United States. In addition to the general mathematics knowledge test for all students, 16 countries participated in the assessment of advanced mathematics given to students having taken advanced coursework in mathematics. No country scored below the United States on the advanced mathematics test.

The NAEP results indicate that mathematics achievement is improving, but taken together with the TIMSS results, it is clear that the United States has far to go to reach its goal of high mathematics achievement.

The U.S. Department of Education has several initiatives designed to continue monitoring U.S. students' mathematics achievement in the future.

- The National Center for Education Statistics (NCES) will be conducting a TIMSS follow-up study in 1999, when the students who took the mathematics and science achievement tests in grade 4 have reached grade 8. This will enable the participating countries to compare their performance with the 1995 eighth-grade results and assess the level of progress made by this group of students over the intervening 4 years.

- The NAEP mathematics assessment will be administered in the year 2000 to continue to monitor progress for the nation and states.

- In his 1997 State of the Union address, President Clinton challenged every community and state to adopt national standards of excellence in education. He called for voluntary administration of individual-level national tests in mathematics at grade 8 to monitor progress toward these standards.

Additionally, a substantial number of states and school districts either have mathematics assessments or plan to have such assessments in the near future. For example, of the 46 states with mathematics standards in place, specialists in 23 states report that a state mathematics assessment is based on the recently completed standards.[7] The remaining 23 states are developing or have plans for new assessments.

What is the Purpose of the Mathematics Assessment Module?

This module aims to contribute to the mathematics education reform efforts by inviting those engaged in developing, conducting, and improving mathematics assessments to compare their efforts to the NAEP and TIMSS mathematics assessments. This mathematics assessment module is intended to do the following:

- Provide an overview of the NAEP and TIMSS mathematics assessment frameworks.
- Show how the NAEP and TIMSS frameworks compare at grade 8 for the areas of geometry and algebra. The intention is to illustrate how each area in the mathematics frameworks includes a range of topics and to provide in-depth information about the topics covered in the NAEP and TIMSS assessments.
- Describe how the NAEP and TIMSS test questions are distributed across the various mathematics content areas (e.g., number sense and operations, algebra, geometry) described in the assessment frameworks.
- Illustrate how various topic areas in the NAEP and TIMSS assessment frameworks have been operationalized into test questions by providing examples of the geometry and algebra test questions included in NAEP and TIMSS.
- Describe how to use materials from those assessments to examine mathematics assessments in states, districts, schools, and classrooms.

Perhaps most importantly, beyond serving as a vehicle to examine mathematics assessments, this module can help bridge the gap between assessment and practice. The in-depth look at the geometry and algebra topics assessed by NAEP and TIMSS, and the types of test questions used, can be used as a lens through which to examine the mathematics curriculum and instruction in your state, district, school, or classroom. The assessment module provides concrete descriptions of the kinds of mathematics that students were expected to know and do in order to perform well in the NAEP and TIMSS assessments, and that students will be expected to know and be able to do on other upcoming mathematics assessments. The mathematics content areas deemed important in national and international tests can serve to highlight the types of mathematics that should be emphasized in curricula and classroom instruction across the United States.

Why Emphasize Geometry and Algebra in the Assessment Module?

This module focuses on geometry and algebra because the TIMSS results consistently show the need for greater attention to more advanced mathematics topics in the eighth-grade curriculum. As will be demonstrated in this module, the geometry and algebra areas account for almost one-half of the items in the 1996 NAEP mathematics assessment and one-third of the items in TIMSS.

- The results from the TIMSS international curriculum analysis confirmed that curricula of the high-performing countries tend to focus on geometry and algebra, while eighth-grade mathematics curricula in the United States consist largely of less advanced topics, with a preponderance of arithmetic.[8]

- In the TIMSS videotapes of eighth-grade mathematics classrooms, 40 percent of U.S. eighth-grade mathematics lessons included arithmetic topics such as whole number operations, fractions, and decimals, whereas these topics were much less common in Germany and Japan. In contrast, German and Japanese eighth-grade lessons were more likely to cover algebra and geometry. Japanese schools offer a single curriculum for all students through the end of grade 9. In mathematics, all eighth-grade students study a curriculum focused heavily on algebra and geometry.[9]

- Analyses of the mathematics content areas showing the most growth in achievement between grades 7 and 8 revealed that several of the high-performing countries showed the most growth in these two areas.[10]

- In the TIMSS assessment of students in their final year of secondary school, it was in geometry where the U.S. advanced mathematics students performed worst.[11] As noted by Neal Lane, Director of the National Science Foundation:

"This is consistent with performance in grades 4 and 8, but unexpected because these advanced students have all had formal geometry coursework. The results show that both geometry and algebra need to be key subjects of study throughout the curriculum."

The questions listed in the chart below are important considerations as you reflect on curriculum and assessment in your own state or district.

Curriculum and Assessment Considerations

- **Considering the variety of mathematics course options available to students in the United States, what percentage of eighth graders in your state or school district are studying geometry and algebra?**
- **Is the content of your students' geometry and algebra courses as rigorous as that envisioned by the NAEP and TIMSS frameworks?**
- **Are eighth graders in your state or school district given opportunities to solve geometry and algebra problems such as those included in NAEP and TIMSS as part of their classroom instruction?**
- **Do state and district mathematics assessments at the eighth grade include the types of geometry and algebra assessment questions included in NAEP and TIMSS?**

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How Is the Assessment Module Organized?

The assessment module is organized to illustrate the process that a state or school district might follow in examining its eighth-grade mathematics assessment and comparing it with those of NAEP and TIMSS. The first section of the assessment module describes the NAEP and TIMSS mathematics frameworks. It describes the purpose of an assessment framework and how frameworks are developed and gives an overview of the NAEP and TIMSS mathematics frameworks. It provides a comparison between the two frameworks, showing that at grade 8 the NAEP and TIMSS frameworks specify similar coverage of mathematics content areas.

The framework section describes six key topic areas within geometry based on an analysis of the topics emphasized at grade 8 in both the NAEP and TIMSS frameworks. The geometry topics covered in each of the frameworks are presented in detail, and a graphic is presented showing how the topics covered in the NAEP and TIMSS frameworks relate to the six key geometry areas. Similarly, five key topics within algebra are presented. The algebra topics covered in the NAEP and TIMSS frameworks are presented in detail and summarized in relation to the five key algebra areas.

The last two sections of the module present examples of the types of test questions used in NAEP and TIMSS to assess the geometry and algebra content areas, respectively. In each section, the example questions are presented by the key topic areas common to NAEP and TIMSS. As each topic area within geometry and algebra is discussed, in turn, the example test questions provide concrete examples that can help those responsible for state-, district-, school-, and classroom-level assessments appreciate the kind of challenging content that is an element of national and international assessments at grade 8.

The NAEP and TIMSS assessments use both open-ended and multiple-choice questions. Each of the key topic areas is illustrated by several test questions, together with the students' achievement results. For each TIMSS question, the achievement results are given for the United States, for the highest one or two performing countries on that particular item, and in terms of the average percent of correct responses across all of the 41 participating TIMSS countries at grade 8. The average across countries is called the international average.

While the majority of the example test questions are from the eighth-grade NAEP assessments conducted in 1992 and 1996 and the TIMSS assessment conducted in 1995, some grade 4 test questions are presented to show how fundamental concepts can be introduced and assessed for elementary school students and some grade 12 test questions are shown to illustrate future directions. Most of the TIMSS test questions are from the written assessment. However, TIMSS also included a performance assessment where subsamples of the fourth and eighth graders participating in the written assessment performed "hands-on" tasks using more elaborate equipment such as modeling clay, balances, scissors, and cardboard, and two of these tasks also are presented in this module.[12] Since a substantial proportion of the NAEP questions and about one-third of the TIMSS questions are being held secure for use in measuring trends in achievement across time, occasionally items were not available to illustrate important topic areas. In these instances, a sample question has been developed and presented, even though, of course, no assessment results are available.

As the module moves from section to section comparing NAEP and TIMSS, it serves to illustrate the process that a state or school district could perform to compare its mathematics assessment to the NAEP national assessment and the TIMSS international assessment. More specifically, as described below, the comparison begins at the global level and becomes more and more specific. While this module describes the complete process only for geometry and algebra at grade 8, a state or school district may find it worthwhile to examine their assessment frameworks in all content areas and at other grades.

Process for Examining Your Mathematics Assessments

One of the goals of this publication is to help educators examine their own assessments in relation to the content and performance expectations of national and international assessments. Below are important questions to help guide you through the process of analyzing your mathematics assessment frameworks.

1. *What content areas and performance abilities are covered in your mathematics frameworks?* Compare your mathematics assessment to those of NAEP and TIMSS at a global level, including the content areas (e.g., number sense,

and operations, geometry, and algebra) and performance abilities covered. Understanding is more than simply knowing, and the NAEP and TIMSS frameworks devote considerable attention to the performance dimension of assessment.

2. *What is the relative emphasis of each of the content areas and performance abilities?* Compare your assessment framework to NAEP and TIMSS in terms of the distribution of items across the areas designated in the framework. It may be that the content areas and performance abilities are similar, but that the emphasis of your assessment is different.

3. *How do other aspects of your assessment approach compare to NAEP and TIMSS (such as the mix of item formats as well as the use of manipulatives and calculators)?* NAEP, for example, includes manipulative materials such as geometric shapes and TIMSS has a separate performance assessment component.

4. *What geometry and algebra topic areas are included in your framework?* Compare the detail about the topics covered within each content area. In geometry at grade 8, for example, the NAEP framework specifies eight separate topics each with several subtopics and TIMSS presents two major areas each with subareas. Try to identify similarities and differences among the NAEP and TIMSS topic areas and those specified in your mathematics assessment framework.

5. *Does your assessment include a range of item types and challenging test questions?* Examine the items in your assessment compared with those in NAEP and TIMSS for each topic area. Give special attention to how challenging the test questions are in terms of both the content and thought processes required to produce a successful answer.

In examining your mathematics assessment, you may discover that it has many desirable features not found in NAEP and TIMSS. Like any assessments, the NAEP and TIMSS frameworks and test questions can be improved, and some of the particular emphases may not be those stressed in your state or school district. Still, as the TIMSS experience has shown, the process of comparison provides a basis for much insightful discussion and consideration about the goals of mathematics education and the various ways to achieve those goals.

There are, of course, many ways to approach assessment. The information about frameworks as well as the geometry and algebra items presented in the next sections of this module are intended to promote better understanding of the assessment process, provide a catalyst for examining aspects of state and local mathematics assessments, and inspire others to create high-quality measures of mathematics learning. The achievement results for the items will help readers better understand the strengths and weaknesses students have in some mathematics topic areas and the importance of helping students improve their understanding of and skills in mathematics.

COMPARING THE NAEP AND TIMSS MATHEMATICS ASSESSMENT FRAMEWORKS

PURPOSE OF AN ASSESSMENT FRAMEWORK

One of the major challenges in assessing student proficiency in a school subject is deciding just what questions to ask. Subjects such as mathematics, science, and language cover a wide range of topics and are taught in a variety of ways. Any assessment of student achievement that aims to do justice to student learning must ensure that the topics and approaches that are the major focus of teaching and learning efforts are adequately addressed.

In order that interested parties such as parents, students, teachers, subject-matter experts, administrators, legislators, and community representatives may understand the coverage of an assessment, it is necessary to have some way to encapsulate and delineate the subject area to be assessed. An ***Assessment Framework*** provides a conceptual structure within which the content, nature, and extent of the assessment may be presented, together with the specifications for the number, type, and content of the assessment questions to be developed.

Since the assessment framework determines what will be assessed, it is very important for the credibility of the results that the framework be accepted by all interested parties as an adequate representation of the subject matter as it is taught and as it should be learned. It should be noted that the NAEP and TIMSS assessments contain items that cover a range of difficulty, from basic knowledge that one would expect all students to have mastered at that age to conceptually complex items in algebra and geometry. To ensure general acceptance, both NAEP and TIMSS frameworks were developed through an extensive and widely inclusive process of consensus building.

In NAEP, the assessment frameworks are developed under contract to the National Assessment Governing Board (NAGB). Created by Congress in 1988 to formulate policy for NAEP, NAGB is specifically charged with developing assessment frameworks and test specifications through a national consensus approach, identifying appropriate achievement goals for each age and grade, and carrying out other NAEP policy responsibilities.

The framework for NAEP's 1996 mathematics assessment[13] was developed by the College Board through a process that incorporated input from a wide range of interests and built on recent national reform efforts in mathematics education and assessment, evaluations of previous NAEP frameworks and assessments, and recommendations from the mathematics education and assessment communities. This exhaustive process of consultation and review ensured that the resulting frameworks were responsive to the major issues in mathematics education and addressed as far as possible the concerns of those working to improve student achievement in this area. The College Board also developed specifications for the assessment items, with particular attention to the mix of item formats, the item distribution over content areas within mathematics, and the conditions under which items are to be presented to students (e.g., use of manipulatives, use of calculators, and time to complete the items).

The NAEP mathematics framework also serves as the basis for NAEP's voluntary state-level assessments conducted in conjunction with the national assessments. State-level assessments in mathematics have been conducted at grade 8 in 1990, 1992, and 1996, as well as at grade 4 in 1992 and 1996. Participation has been widespread, with 40 or more states participating in each assessment.

How was your state or district mathematics framework developed?

How is your framework tied to national and state standards?

Ensuring consensus on a framework for an international assessment like TIMSS provides additional challenges. Different traditions of mathematics education, different approaches to curricular design, and different pedagogical philosophies all have to be reconciled in a framework that is acceptable to all. In TIMSS, national research coordinators worked with the international coordinating team, and with the assistance of a subject-matter advisory committee of international mathematics experts, to develop a framework for the international assessment.[14] This work was informed by the preliminary results from an analysis of the mathematics curricula of the participating countries.

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While no single framework can match exactly the curriculum of every country, the international assessments built on the TIMSS framework for eighth-grade mathematics achieved a high degree of acceptance among mathematics educators around the world.

Overview of the NAEP Mathematics Framework

The framework for the 1996 NAEP mathematics assessment considers the curriculum in terms of strands of mathematical content drawn from five broad areas of mathematics (Figure 1). These content strands reflect the content standards of the *National Council of Teachers of Mathematics' (NCTM) Curriculum and Evaluation Standards for School Mathematics.* The content strands are not intended to separate mathematics into discrete elements, but rather to provide a useful classification scheme that describes the full spectrum of mathematical content.

What major content areas are covered in your mathematics assessment framework?

How does your framework emphasize problem solving as well as content?

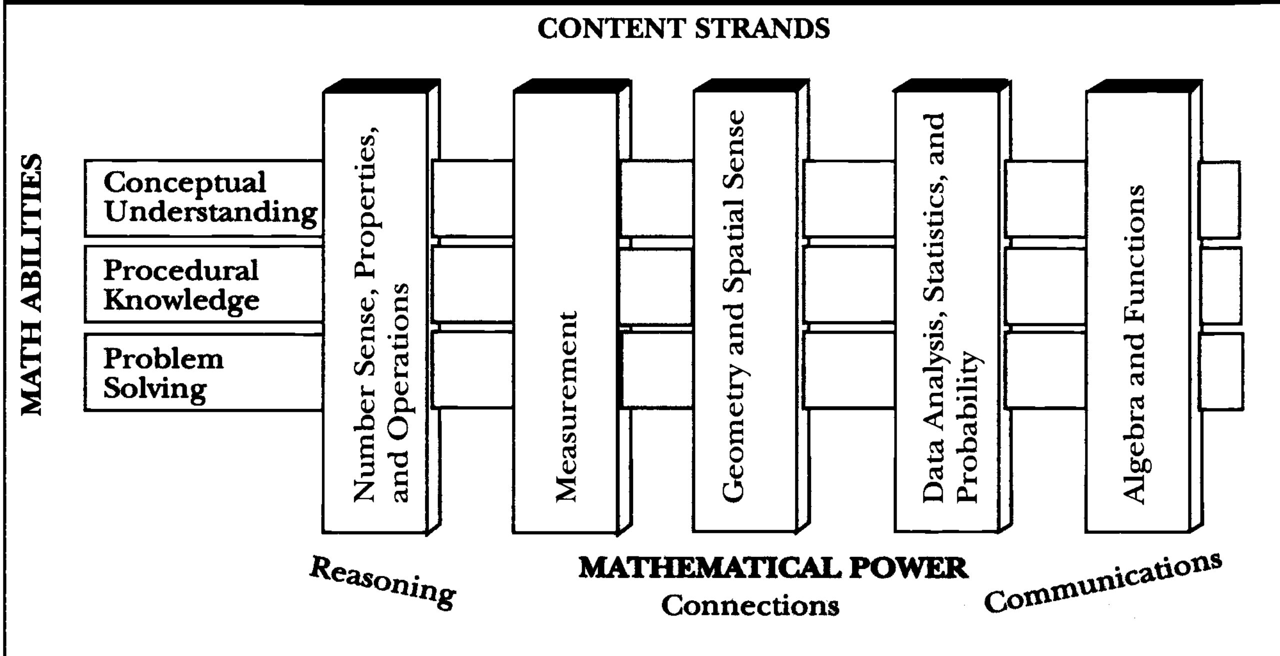
Can you think of specific examples?

How do these examples compare with test items in this document?

Content descriptions alone cannot fully express the kind of mathematical thinking described in the NCTM Standards and suggested by research in effective mathematics education. In real life, few mathematical situations fall clearly into one content strand or the other, and few naturally reflect only one facet of mathematical thinking. To ensure a broad scope in test construction, items must be classified not just in terms of content, but also in a way that captures a wide range of mathematical abilities. To address this issue of item classification, the NAEP framework focuses primarily on the mathematical content strands, but with additional specifications related to an assessment dimension referred to as "mathematical power" (see Figure 1).

Mathematical power is conceived as consisting of mathematical abilities (conceptual understanding, procedural knowledge, and problem solving) within a broader context of reasoning, and with connections across the broad scope of mathematical content and thinking. Communication is viewed as both a unifying thread and a way for students to provide meaningful responses to tasks. In the NAEP mathematics assessment, the conception of mathematical power as reasoning, connections, and communication plays an

FIGURE 1. MATHEMATICAL FRAMEWORK FOR NAEP 1996 FRAMEWORK



SOURCE: National Assessment Governing Board, Mathematics Framework for the 1996 National Assessment of Educational Progress.

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important role in measuring student achievement. Through the use of extended constructed-response questions, students are required to communicate their ideas, demonstrate the reasoning used to solve problems, and connect their learning across mathematical content strands and to other disciplines.

Overview of the TIMSS Mathematics Framework

For the purposes of TIMSS, the curriculum consists of the subject-matter content, the intellectual processes (performance expectations), and the affective outcomes (perspectives) of school mathematics and science that are intended for, implemented in, or attained during students' schooling experiences. Any element of the curriculum may be characterized, therefore, in terms of three parameters: subject-matter content, performance expectations, and perspectives or context. These three parameters constitute the aspects or dimensions of the TIMSS frameworks (see Figure 2).

Each aspect of the framework consists of a hierarchy of categories and subcategories. In mathematics, the **content** aspect of the framework is partitioned into 8 major categories, with each composed of 2 to 28 subcategories. The content categories and subcategories were agreed upon following an extensive process of consultation among mathematics educators, curriculum specialists, and teachers from around the world. The content aspect of the TIMSS framework corresponds to what NAEP terms content strands.

The **performance expectations** aspect of the TIMSS frameworks describe the kinds of performance that students are expected to demonstrate while engaged with the mathematics content. The five main categories are knowing, using routine procedures, investigating and problem solving, mathematical reasoning, and communicating. The performance expectations aspect is closely aligned with the NAEP concepts of mathematical abilities and mathematical power, and has antecedents in the NCTM *Standards*. Like the content aspect, the performance expectations aspect has a hierarchical system of categories and subcategories, which may be used to provide more or less detail in characterizing any particular element.

The **perspectives** aspect of the curriculum frameworks is intended to depict curricular goals that focus on the development of students' attitudes,

Figure 2. The Three Aspects and Major Categories of the Mathematics Framework – TIMSS 1995

CONTENT
- **Numbers**
- **Measurement**
- **Geometry**
- **Proportionality**
- **Functions, relations, and equations**
- **Data representation, probability, and statistics**
- **Elementary analysis**
- **Validation and structure**

PERFORMANCE EXPECTATIONS
- **Knowing**
- **Using routine procedures**
- **Investigating and problem solving**
- **Mathematical reasoning**
- **Communicating**

PERSPECTIVES
- **Attitudes**
- **Careers**
- **Participation**
- **Increasing interest**
- **Habits of mind**

SOURCE: Robitaille, D.F., McKnight, C., Schmidt, W., Britton, E., Raizen, S., and Nicol, C. (1993). *TIMSS Monograph No. 1: Curriculum Frameworks for Mathematics and Science.* Vancouver, BC: Pacific Educational Press.

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interests, and motivations in mathematics and science teaching. This aspect makes it possible to describe learning outcomes or curriculum materials that are intended to promote positive attitudes, as well as goals that encourage students to consider careers in mathematics, science, and technology.

Distribution of Items Across Framework Areas in NAEP and TIMSS

The distribution of assessment items among the various mathematical content areas defined in a framework is a critical feature of any assessment design, because it reflects the relative importance and value given to each of the areas. The minimum percentages of items distributed among the content strands in the NAEP mathematics assessments are presented in Figure 3. (Because many test questions measure more than one content area, the actual percentages are greater than these numbers with a total of more than 100%.) In TIMSS, the "blueprints" specifying the allocation of testing time to content categories were based on expert opinion from the participating countries and on a preliminary analysis of the curriculum analysis data. The percentage of the items in each of the six content categories of the framework used for the eighth-grade assessment is shown in Figure 4.

In terms of content coverage, the two frameworks have much in common, with the main difference being one of emphasis. In the NAEP item pool, 45 percent of the items are devoted to algebra and geometry, whereas

Figure 3. Minimum Percentage Distribution of Items by Content Strand in Grade 8 – NAEP 1996

Number Sense, Properties, and Operations	25
Measurement	15
Geometry and Spatial Sense	20
Data Analysis, Statistics, and Probability	15
Algebra and Functions	25

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in TIMSS the corresponding figure is 33 percent. TIMSS, on the other hand, places more emphasis on number sense, and particularly on fractions and proportions. The TIMSS categories of fractions and number sense and proportionality account for 41 percent of the item pool, compared with 25 percent for the NAEP content strand of number sense, properties, and operations. The areas of measurement and data analysis, statistics, and probability contain approximately equal percentages of the item pools in both frameworks.

While the scope of NAEP's 1996 item pool was specified primarily in terms of mathematics content coverage, it was important also to maintain a balance across items measuring the central aspects of mathematical power. The 1996 specifications were designed to incorporate the overarching *NCTM Standards* for problem solving, communicating, reasoning, and connecting, as well as the NCTM assessment categories of conceptual understanding, procedural knowledge, and problem solving. The intention was to ensure that the assessment reflected a balance among items that measured "knowing that" or "knowing about," "knowing how," and "solving problems," within an overall demonstration of mathematical thinking in a variety of situations. At each grade level, the framework called for approximately one-third of the items to be classified as conceptual understanding, at least one-third as procedural knowledge, and at least one-third as problem solving. These guidelines applied only to the item pool as a whole and were not to be applied separately within each content strand. In developing the TIMSS blueprints, balance across the categories of the performance expectations aspect also was an important concern.

What are the relative emphases of content areas and processes in your framework?

What kinds of test questions does your assessment include?

FIGURE 4. PERCENTAGE DISTRIBUTION OF ITEMS BY CONTENT CATEGORY IN GRADE 8 – TIMSS 1995

Fractions and Number Sense	**34**
Measurement	**12**
Data Representation, Analysis, and Probability	**14**
Geometry	**15**
Algebra	**18**
Proportionality	**7**

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CALCULATOR USE IN NAEP AND TIMSS

Increased emphasis on calculator usage in carrying out mathematics tasks and solving problems is a feature of the NAEP 1996 assessment framework. This is in line with the philosophy of the *NCTM Standards* that the availability of such tools should provide students with an opportunity to demonstrate a higher level of mathematical thinking than they would otherwise be able to exhibit. Increased calculator use also is consistent with practices in most states, regarding both classroom work and assessment. Permitting students access to calculators as part of the assessment process can provide a more realistic picture of the mathematics that students know and are able to do in their real world. In the NAEP 1996 assessment, students were provided calculators (four-function at grade 4 and scientific at grades 8 and 12) to use with approximately one-third to one-fourth of the items.

For reasons of equity and uniformity across countries, TIMSS did not permit calculator usage as part of the assessments at grades 4 and 8. This was a matter of lengthy debate because some countries were working to incorporate calculator usage into their curricula at these grades and supported calculator use as part of the testing, while other countries did not include calculator use as part of mathematics instruction, or even forbade it. The issue was resolved by resource limitations. Many of the fourth and eighth graders in a number of the TIMSS countries did not own calculators, and these countries simply did not have the resources to provide calculators for students during the testing. Since the testing of high school students involved advanced mathematics and physics, students were permitted to use calculators for all assessments during the TIMSS testing if they wished.

USE OF MATHEMATICAL TOOLS AND MANIPULATIVES IN NAEP AND TIMSS

As well as recommending increased calculator usage in the assessment, the 1996 NAEP framework called for greater use of other mathematical tools such as rulers and protractors and for the inclusion of manipulable geometric shapes where feasible.

TIMSS included a performance assessment where subsamples of the fourth and eighth graders participating in the main survey also did "hands-on" tasks using materials easily accessible to all participating countries. In one task, Rubber Bands, students investigated what happened to the length of a rubber band as more and more weights were hung on it. They graphed their data, described the trend in their data, and extrapolated beyond their data to make predictions or to develop general rules. In another task they were given a balance, 20g and 50g weights, and modeling clay. They were asked to form lumps of clay weighing 20g, 10g, 15g, and 35g in weight (in that order), and to explain their strategy for forming each one. For the Around the Bend task, students were supplied with a wooden model of a corridor having a right-angle bend, and several pieces of cardboard cut to represent pieces of furniture. The task, in general, was to make judgments about the real-world furniture represented by the models and generalize a rule so that the furniture would always go around the bend. Two TIMSS performance tasks are included in the section presenting geometry items. One involves constructing nets of boxes and the other requires folding and cutting to make geometric shapes.

Does your assessment include the use of mathematical tools and manipulatives?

Geometry and Spatial Sense in NAEP and TIMSS in Grade 8

This section looks specifically at the geometry content area in the NAEP and TIMSS mathematics frameworks. Within each content area, the NAEP and TIMSS frameworks contain detail about the topics and subtopics that should be assessed. As shown in Figure 5, the NAEP framework for geometry is quite detailed, including eight major topic areas at grade 8 and a ninth area that can be assessed at an introductory level. Most of the topic areas contain subtopics. Usually the NAEP topics and subtopics include a process dimension. That is, students are required to "describe," "draw," or "apply" knowledge and understanding about geometric shapes, including their location, relationships, and properties.

As shown in Figure 6, the TIMSS framework also contains detail about the content that should be covered in geometry. The listing, however, focuses solely on content because the TIMSS framework assumes that the performance expectations (e.g., using routine procedures, investigating and problem solving, and mathematical reasoning) apply to all specified content. TIMSS has two major topic areas. The first topic area involves position, visualization, and shape in two- and three-dimensional geometry. The second area covers symmetry, congruence, and similarity.

An examination of the NAEP and TIMSS geometry frameworks indicates substantial overlap in the geometry topics covered. The following are six major topic areas included in both frameworks.

- AREA ONE: Describe, Visualize, Draw, and Construct Geometric Figures.
- AREA TWO: Combine, Subdivide, and Change Geometric Figures.
- AREA THREE: Identify Relationships Between Figures in Terms of Symmetry, Motions and Transformations, and Congruence and Similarity.
- AREA FOUR: Apply Geometric Properties and Relationships in Solving Problems.
- AREA FIVE: Represent Geometric Figures and Properties Algebraically Using Coordinates and Properties of Lines.
- AREA SIX: Solve Real-World Problems Using Geometric Models and Explain Relationships Involving Geometric Concepts.

Figure 7 shows how the six areas relate to the geometry content specified in both the NAEP and TIMSS mathematics frameworks. Geometry items from NAEP and TIMSS illustrating each of the six areas are found in the next major section of this document.

Are these six areas of geometry included in your geometry framework?

Are any of these areas more emphasized than others?

Geometry and Spatial Sense in NAEP

NAEP's geometry content area at grades 4 and 8 extends well beyond low-level identification of geometric shapes. The NCTM *Standards* stress that spatial sense must be an integral component of the study and assessment of geometry, and the NAEP assessment is consistent with this view. The focus is on drawings and constructions, as well as on mathematical reasoning about combinations of shapes and transformations. Using proportional thinking in geometric contexts is important, as is making some initial links between geometry and algebra.

As previously noted (see Figure 1), mathematical abilities (conceptual understanding, procedural, and problem solving), connections across mathematical content, and communication skills are viewed as unifying threads across all NAEP content areas.

The topics and subtopics within the geometry content area have been excerpted from the *NAEP 1996 Mathematics Framework* and presented in Figure 5. The topic list was intended to be illustrative rather than inclusive. However, some level of specificity was needed to guide item writers and ensure adequate coverage of the content areas and abilities to be assessed. Also, test questions that involved synthesizing knowledge across topics and subtopics were considered desirable. In everyday life, individuals are expected to solve problems that naturally involve more than one specific mathematics topic.

For each of grades 4, 8, and 12, the following symbols were used in the NAEP framework:

"●" Indicates that the subtopic could be assessed at that grade level.

"⊙" Indicates that the subtopic may be assessed at that grade level, probably using manipulatives or a pictorial model.

"○" Indicates that the subtopic should not be assessed at that grade level.

Even though the emphasis in this document is on grade 8, with some attention on the types of understandings assessed at grade 4, Figure 5 presents the complete set of geometry topics specified, including grade 12. This will help users of this document place the progression of geometric learning into perspective across the school years.

FIGURE 5. NAEP 1996 MATHEMATICS FRAMEWORK: GEOMETRY AND SPATIAL SENSE

	Grade 4	Grade 8	Grade 12
1. Describe, visualize, draw, and construct geometric figures			
a. Draw or sketch a figure given a verbal description	●	●	●
b. Given a figure, write a verbal description of its geometric qualities	❍	●	●
2. Investigate and predict results of combining, subdividing, and changing shapes (e.g., paper folding, dissecting, tiling, and rearranging pieces of solids)	●	●	●
3. Identify the relationship (congruence, similarity) between a figure and its image under a transformation			
a. Use motion geometry (informal: lines of symmetry, flips, turns, and slides)	●	●	●
b. Use transformations (translations, rotations, reflections, dilations, and symmetry)			
i. Synthetic	❍	⊙	●
ii. Algebraic	❍	❍	●
4. Describe the intersection of two or more geometric figures			
a. Two dimensional	❍	●	●
b. Planar cross-section of a solid	❍	●	●
5. Classify figures in terms of congruence and similarity, and informally apply these relationships using proportional reasoning where appropriate	❍	●	●

● Subtopic can be assessed at this grade level.
⊙ Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).
❍ Subtopic should not be assessed at this grade level.

FIGURE 5. CONTINUED

	Grade 4	Grade 8	Grade 12
6. Apply geometric properties and relationships in solving problems			
a. Use concepts of "between," "inside," "on," and "outside"	●	●	❍
b. Use the Pythagorean relationship to solve problems	❍	●	●
c. Apply properties of ratio and proportion with respect to similarity	❍	⊙	●
d. Solve problems involving right triangle trigonometric applications	❍	●	●
7. Establish and explain relationships involving geometric concepts			
a. Make conjectures	●	●	●
b. Validate and justify conclusions and generalizations	●	●	●
c. Use informal induction and deduction	❍	●	●
8. Represent problem situations with geometric models and apply properties of figures in meaningful contexts to solve mathematical and real-world problems	●	●	●
9. Represent geometric figures and properties algebraically using coordinates and vectors			
a. Use properties of lines (including distance, midpoint, slope, parallelism, and perpendicularity) to describe figures algebraically	❍	⊙	●
b. Algebraically describe conic sections and their properties	❍	❍	●
c. Use vectors in problem situations (addition, subtraction, scalar multiplication, dot product)	❍	❍	●

● Subtopic can be assessed at this grade level.
⊙ Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).
❍ Subtopic should not be assessed at this grade level.

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Geometry in TIMSS

The geometry content section of the TIMSS Detailed Mathematics Framework is reproduced in Figure 6. Of the three aspects in the TIMSS mathematics framework (content, performance expectations, and perspectives), the content aspect consists of a breakdown of the subject matter into varying levels of specificity. As noted in the TIMSS framework, the task of determining an appropriate number of categories and subcategories was a challenging one. The categories used in the TIMSS framework were discussed extensively in many international settings, and there was consensus that the ones used were appropriate for their intended purpose. In developing frameworks, there is always a tension between providing the detail necessary for test developers and becoming so atomized in the presentation of details that the essential themes of the content area are lost.

The performance expectations aspect of the TIMSS framework describes the kinds of performance that students will be expected to demonstrate while engaged with the content, including knowing, using routine procedures, investigating and problem solving, mathematical reasoning, and communicating.

FIGURE 6. TIMSS DETAILED MATHEMATICS FRAMEWORK: GEOMETRY

1. GEOMETRY: POSITION, VISUALIZATION, AND SHAPE

a. **Two-dimensional Geometry**

Coordinate Geometry (line and coordinate graphs, equation of line in the plane, conic sections and their equations);

Basics (points, lines, segments, rays, angles; parallelism and perpendicularity); and

Polygons and Circles (triangles; quadrilaterals: their classification and properties; Pythagorean theorem and applications; other polygons, circles, and their properties)

b. **Three-dimensional Geometry**

(three-dimensional shapes and surfaces and their properties; planes and lines in space; spatial perception and visualization; coordinate systems in three dimensions; equations of lines, planes, and surfaces in space)

c. **Vectors**

2. GEOMETRY: SYMMETRY, CONGRUENCE, AND SIMILARITY

a. **Transformations** (patterns, tessellations, friezes, stencils, etc.; symmetry [line and rotational symmetry, symmetry in three dimensions, symmetry in algebra and number patterns]; transformations: symmetries and congruence, enlargements [dilations], combinations of geometric transformations, group structure of transformations, matrix representation of transformations)

b. **Congruence and Similarity** (congruences [congruent triangles and their properties; SSS, SAS], congruent quadrilaterals and polygons and their properties, similarities [similar triangles and their properties])

c. **Constructions Using Straight-Edge and Compass**

FIGURE 7. GEOMETRY AND SPATIAL SENSE AT GRADE 8: THE RELATIONSHIP BETWEEN THE CONTENT DIMENSIONS OF THE 1996 NAEP AND TIMSS MATHEMATICS FRAMEWORKS AT GRADE 8

Assessment Module Area Geometry and Spatial Sense	1996 NAEP Framework	TIMSS Framework
AREA ONE: Describe, Visualize, Draw, and Construct Geometric Figures	**Subtopic 1:** Describe, Visualize, Draw, and Construct Geometric Figures	**Subtopic 1:** Position, Visualization, and Shape: B) Two-dimensional (points, lines, angles) C) Two-dimensional (polygons and circles) D) Three-dimensional (shapes and surfaces)
AREA TWO: Combine, Subdivide, and Change Geometric Figures	**Subtopic 2:** Investigate and Predict Results of Combining, Subdividing, and Changing Shapes **Subtopic 4:** Describe the Intersection of Two or More Geometric Figures	**Subtopic 1:** Position, Visualization, and Shape: C) Two-dimensional (polygons and circles) D) Three-dimensional (shapes and surfaces)
AREA THREE: Identify Relationships Between Figures in Terms of Symmetry, Motions and Transformations, and Congruence and Similarity	**Subtopic 3:** Identify the Relationships (Congruence, Similarity) Between a Figure and Its Image Under a Transformation **Subtopic 5:** Classify Figures in Terms of Congruence and Similarity	**Subtopic 2:** Symmetry, Congruence, and Similarity: A) Transformations B) Congruence and Similarity

FIGURE 7. CONTINUED

Assessment Module Area	1996 NAEP Framework	TIMSS Framework
AREA FOUR: Apply Geometric Properties and Relationships in Solving Problems	**Subtopic 6:** Apply Geometric Properties and Relations in Solving Problems (Pythagorean Relationships; Ratio and Proportion with Respect to Similarity) **Subtopic 5:** Apply the Relationships of Congruence and Similarity Using Proportional Reasoning	**Subtopic 1:** Position, Visualization, and Shape: C) and D) Properties of Two- and Three-Dimensional Shapes; Application of Pythagorean theorem
AREA FIVE: Represent Geometric Figures and Properties Algebraically Using Coordinates and Properties of Lines	**Subtopic 9:** Represent Geometric Figures and Properties Algebraically Using Coordinates and Properties of Lines	**Subtopic 1:** Position, Visualization, and Shape: A) Two-dimensional (Coordinate Geometry)
AREA SIX: Solve Real-World Problems Using Geometric Models and Explain Relationships Involving Geometric Concepts	**Subtopic 7:** Establish and Explain Relationships Involving Geometric Concepts **Subtopic 8:** Represent Problem Situations with Geometric Models and Apply Properties of Figures in Meaningful Contexts to Solve Mathematical and Real-World Problems	Depends on the Problem-Solving Task

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Algebra and Functions in NAEP and TIMSS in Grade 8

As with the geometry frameworks for NAEP and TIMSS, the detailed algebra frameworks contain considerable overlap. The NAEP detailed framework for algebra is shown in Figure 10. It contains seven major areas, with another six areas to be assessed at an introductory level. The TIMSS detail relating to the area of algebra is shown in Figure 9. Primarily, the TIMSS framework focuses on functions, relations, and equations. However, it is organized differently from the NAEP framework in some respects.

Comparing the important areas encompassed in the NAEP and TIMSS detailed frameworks at grade 8 yielded five major areas within algebra and functions. These five areas include:

- AREA ONE: Recognize and Extend Patterns and Relationships
- AREA TWO: Use Symbols to Represent Situations Algebraically
- AREA THREE: Evaluate and Perform Operations on Algebraic Expressions (Includes Exponents and Roots)
- AREA FOUR: Solve Linear Equations and Inequalities (Includes Graphical Representations of Equations)
- AREA FIVE: Solve Real-World Problems Using Algebraic Models and Explain Relationships Involving Algebraic Concepts

The relationship between the five areas and the topics specified in both the NAEP and TIMSS frameworks is shown in Figure 10. Algebra items illustrating each of these five areas are presented in the last section of this document (see "Sample Test Items: Algebra and Functions").

Are these five areas in your algebra framework?

Which areas are more emphasized than others?

ALGEBRA AND FUNCTIONS IN NAEP

In the NAEP mathematics framework, the algebra and functions content strand extends from work with simple patterns at grade 4 to sophisticated analysis at grade 12, involving precalculus and some topics from discrete mathematics. Algebra can be thought of in many ways, including the study of patterns, functions, and relations; the interconnection between language and representations; building structures based on generalized arithmetic; or modeling phenomena. The topics and subtopics within the algebra and functions content area have been excerpted from the NAEP 1996 Mathematics Framework and presented in Figure 8.

To show the range across the grades, Figure 8 presents the complete set of topics specified for algebra and functions. However, as can be seen, topics 8 through 14 are primarily for high school students. On the figure a "●" indicates that the subtopic could be assessed at that grade level, a "⊙" indicates that the subtopic might be introduced in the assessment at a very simple level, probably using a manipulative or pictorial model, and a "○" indicates that the subtopic should not be assessed at that grade level.

At grade 8, this content area includes the concept of variable and the development of symbol sense. The assessment includes items about the meaning of variable and an understanding of symbolic representations in problem-solving contexts. Students are asked to use variables to represent a rule underlying a pattern. Students also are expected to demonstrate fluency with equivalent algebraic representations. They should be able to use a variety of methods to solve linear equations and inequalities, and have a beginning understanding of using linear functions, equations, and other representations as modeling tools. Students also should be able to apply algebraic thinking and notation to solve problems and justify their conclusions. They should be expected to relate representations of problem situations arithmetically (e.g., using a table), algebraically (e.g., using an equation or function), and geometrically (e.g., using a graph).

FIGURE 8. NAEP 1996 MATHEMATICS FRAMEWORK: ALGEBRA AND FUNCTIONS

	Grade 4	Grade 8	Grade 12
1. Describe, extend, interpolate, transform, and create a wide variety of patterns and functional relationships			
a. Recognize patterns and sequences	●	●	●
b. Extend a pattern or functional relationship	●	●	●
c. Given a verbal description, extend or interpolate with a pattern (complete a missing term)	❍	●	●
d. Translate patterns from one context to another	⊙	●	●
e. Create an example of a pattern or functional relationship	●	●	●
f. Understand and apply the concept of a variable	⊙	●	●
2. Use multiple representations for situations to translate among diagrams, models and symbolic expressions	●	●	●
3. Use number lines and rectangular coordinate systems as representational tools			
a. Identify or graph sets of points or lines on a number line or in a rectangular coordinate system	●	●	●
b. Identify or graph sets of points in a polar coordinate system	❍	●	●
c. Work with applications using coordinates	❍	●	●
d. Transform the graph of a function	❍	⊙	●
4. Represent and describe solutions to linear equations and inequalities to solve mathematical and real-world problems			
a. Solution sets of whole numbers	●	●	●
b. Solution sets of real numbers	⊙	●	●

● Subtopic can be assessed at this grade level.

⊙ Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

❍ Subtopic should not be assessed at this grade level.

FIGURE 8. CONTINUED

	Grade 4	Grade 8	Grade 12
5. Interpret contextual situations and perform algebraic operations on real numbers and algebraic expressions to solve mathematical and real-world problems			
a. Perform basic operations, using appropriate tools, on real numbers in meaningful contexts (including grouping and order of multiple operations involving basic operations, exponents, and roots)	●	●	●
b. Solve problems involving substitution in expressions and formulas	○	●	●
c. Solve meaningful problems involving a formula with one variable	○	●	●
d. Use equivalent forms to solve problems	○	●	●
6. Solve systems of equations and inequalities using appropriate methods			
a. Solve systems graphically	○	●	●
b. Solve systems algebraically	○	○	●
c. Solve systems using matrices	○	○	●
7. Use mathematical reasoning			
a. Make conjectures	●	●	●
b. Validate and justify conclusions and generalizations	●	●	●
c. Use informal induction and deduction	⊙	●	●

● Subtopic can be assessed at this grade level.

⊙ Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

○ Subtopic should not be assessed at this grade level.

FIGURE 8. CONTINUED

	Grade 4	Grade 8	Grade 12
8. Represent problem situations with discrete structures			
a. Use finite graphs and matrices	●	⊙	●
b. Use sequences and series	○	○	●
c. Use recursive relations (including numerical and graphical iteration and finite differences)	○	○	●
9. Solve polynomial equations with real and complex roots using a variety of algebraic and graphical methods and using appropriate tools.	○	○	●
10. Approximate solutions of equations (bisection, sign changes, and successive approximations)	○	⊙	●
11. Use appropriate notation and terminology to describe functions and their properties (including domain, range, function composition, and inverses)	○	○	●
12. Compare and apply the numerical, symbolic, and graphical properties of a variety of functions and families of functions, examining general parameters and their effect on curve shape	○	⊙	●
13. Apply function concepts to model and deal with real-world situations	○	⊙	●
14. Use trigonometry			
a. Use triangle trigonometry to model problem situations	○	○	●
b. Use trigonometric and circular functions to model real-world phenomena	○	○	●
c. Apply concepts of trigonometry to solve real-world problems	○	○	●

Algebra and Functions in TIMSS

The algebra and functions content section of the TIMSS Detailed Mathematics Framework is reproduced in Figure 9. Just as NAEP stresses the same cognitive abilities across content areas (conceptual understanding, procedural knowledge, and problem solving) and emphasizes reasoning, communicating, and making connections, TIMSS uses the same performance expectations across the content areas, including knowing, using routine procedures, investigating and problem solving, reasoning, and communicating.

Because the content of TIMSS is arranged somewhat differently from that of NAEP, several TIMSS mathematics content areas are referenced in Figure 9. The first is the area of functions, relations, and equations. However, TIMSS covers exponents, roots, and radicals within the area of numbers under a section designated as "other numbers and number concepts," and linear interpolation and extrapolation under proportionality (an area covered separately in TIMSS but not in NAEP).

Figure 9. TIMSS Detailed Mathematics Framework: Algebra and Functions

1) Functions, Relations, and Equations
    a) Patterns, relations, and functions
    b) Equations and formulas
2) Number Concepts
    a) Exponents, roots, and radicals
3) Proportionality
    a) Linear interpolation and extrapolation

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FIGURE 10. ALGEBRA AND FUNCTIONS AT GRADE 8: THE RELATIONSHIP BETWEEN THE CONTENT DIMENSIONS OF THE 1996 NAEP AND TIMSS MATHEMATICS FRAMEWORKS

Assessment Module Area Algebra and Functions	1996 NAEP Framework	TIMSS Framework
AREA ONE: Recognize and Extend Patterns and Relationships	**Subtopic 1:** Describe, extend, interpolate, transform, and create a wide variety of patterns and functional relationships	**Subtopic 1:** Functions, Relations, and Equations A) Patterns, Relations, and Functions
AREA TWO: Use Symbols to Represent Situations Algebraically	**Subtopic 2:** Use multiple representations for situations to translate among diagrams, models, and symbolic expressions	**Subtopic 1:** Functions, Relations, and Equations B) Equations and Formulas
AREA THREE: Evaluate and Perform Operations on Algebraic Expressions (Includes Exponents and Roots)	**Subtopic 5:** Perform algebraic operations on real numbers and algebraic expressions, including substitution in expressions and formulas	**Subtopic 1:** Functions, Relations, and Equations B) Equations and Formulas **Subtopic 2:** Number Concepts A) Exponents, roots, and radicals
AREA FOUR: Solve Linear Equations and Inequalities (Includes Graphical Representations of Equations)	**Subtopic 4:** Represent and describe solutions to linear equations and inequalities to solve mathematical and real-world problems **Subtopic 6:** Solve systems of equations and inequalities using appropriate methods (e.g., graphically) **Subtopic 3:** Use number lines and rectangular coordinate systems as representational tools	**Subtopic 1:** Functions, Relations, and Equations B) Equations and Formulas **Subtopic 3:** Proportionality A) Linear Interpolation and Extrapolation
AREA FIVE: Solve Real-World Problems Using Algebraic Models and Explain Relationships Involving Algebraic Concepts	**Subtopic 7:** Use Mathematical Reasoning **Subtopic 5:** Interpret Contextual Situations and Solve Real-World Problems	Depends on the Problem-Solving Task

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Sample Test Items: Geometry and Spatial Sense

AREA ONE: Geometry and Spatial Sense

Describe, Visualize, Draw, and Construct Geometric Figures

As described in the *NCTM Standards*, identification, description, comparison, and classification of geometric figures and their properties are important skills in the content area of geometry. The NAEP framework requires students to describe, visualize, draw, and construct geometric figures, and the TIMSS framework covers familiarity with a wide variety of geometric figures and their properties, including both two- and three-dimensional shapes. TIMSS also specifically mentions points and lines, angles, parallelism, and perpendicularity.

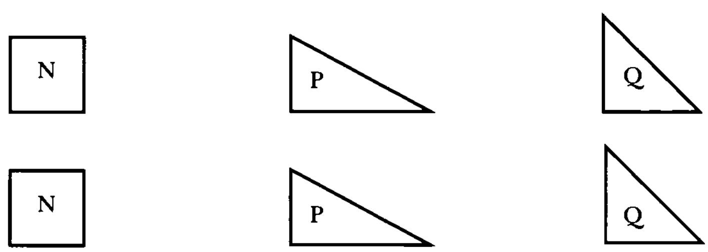
In summary, students should be able to describe, visualize, draw, and construct a wide variety of geometric figures. These figures include points and lines; line segments, rays, and angles; and coordinate systems. The two frameworks mention a broad array of two-dimensional shapes, including triangles, quadrilaterals, other polygons, and circles. Three-dimensional shapes also are emphasized, such as cubes, rectangular solids, and spheres.

There are many opportunities for students to provide descriptions of geometric qualities, draw or sketch various figures, or to visualize relationships. For example, students might be asked to:

- Compare geometric shapes, giving examples of how they are similar and different.
- Identify the two-dimensional shapes making up the faces of a three-dimensional model.
- Write a description of a figure or picture composed of geometric figures.
- Use a protractor and a ruler to draw and construct a variety of one-, two-, and three-dimensional geometric figures.

The following questions from the NAEP and TIMSS assessments illustrate a few of the types of items included in this area.

To answer Example 1, fourth graders participating in NAEP's 1996 assessment received a packet of six cardboard pieces: two each of shapes N, P and Q shown below. They were asked to describe how shape N differed from shapes P and Q.



Nearly two-thirds of the fourth graders successfully described a difference. Students could have provided any of the following information:

- N is a square, but P and Q are triangles.
- N has four sides, vertices, angles, but P and Q each have three.
- All of the angles of N are right angles, but not for P and Q.

NAEP 1996

Example 1: Describe Triangles and Squares

Laura was asked to choose 1 of the 3 shapes N, P, and Q that is different from the other 2. Laura chose shape N. Explain how shape N is different from shapes P and Q.

Answer: N has 4 sides P+Q only have 3
P+Q.S sides are different lengths
Ns are all equal

Example 1	**Grade 4**
United States	64
	% correct

49

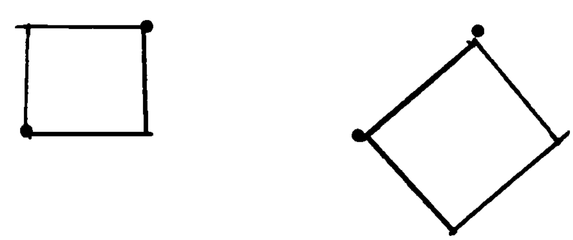
Some test questions in NAEP and TIMSS assessed students' ability to draw simple geometric figures, such as squares, rectangles, and triangles, while other questions assessed more complex concepts such as drawing quadrilaterals other than rectangles and drawing perpendicular lines. As expected, performance varied by grade level and according to the complexity of the drawing required. As shown in Example 2 from NAEP's 1992 assessment, 40 percent of the fourth graders and 67 percent of the eighth graders were successful when asked to draw a square given two corner points. Correct figures included squares with the given dots as either adjacent vertices or as diagonal vertices.

Example 2	Grade 4
United States	40
	% correct
	Grade 8
United States	67
	% correct

NAEP 1992

Example 2: Draw a Square

In the space below, use your ruler to draw a square with two of its corners at the points shown.



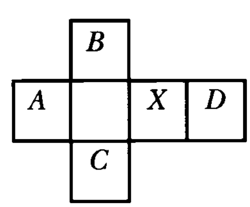
Two Possible Solutions

As would be anticipated, NAEP and TIMSS results suggest that as geometric figures become more complex, students have more difficulty visualizing them. One 1992 NAEP test question (see Example 3) asked fourth- and eighth-grade students to visualize how a flat object will appear if it is folded into a cube. Eighth graders were more adept at this task than fourth graders (55% correct at grade 8 and 22% correct at grade 4). At grade 4, 28 percent of the students simply selected B as the letter on top of the diagram. However, the most popular incorrect answer was D (30% at grade 4 and 22% at grade 8), indicating that some students could visualize the cube folded up with the blank square as the base, even though they had difficulty visualizing the folded cube resting on face X. As suggested in the NCTM analysis of the results of NAEP's 1992 mathematics assessment, seventh grade students may need more practice and experience with different types of nets (flat shapes that will fold up to make a solid). Through folding nets students will not only improve their visualization skills but will also gain a better understanding of the relationship between three-dimensional and two-dimensional shapes. (Example 33 on page 78 presents a TIMSS performance task based on constructing nets.)

Example 3	**Grade 4**
United States	22
	% correct
	Grade 8
United States	55
	% correct

NAEP 1992

Example 3:
Visualize Three-dimensional Cube



The squares in the figure above represent the faces of a cube which has been cut along some edges and flattened. When the original cube was resting on face *X*, which face was on top?

(A.) A

B. B

C. C

D. D

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Example 4 assesses students' understanding of the relationship between points on a line, as a precursor to more complex concepts necessary for analytic geometry such as distances between points on a line, locations of midpoints, and slopes. To receive full credit, a student needed to provide a diagram illustrating the idea that B is halfway between A and C. Thus, B could be shown as the midpoint between A and C as demonstrated by the answer below. (Of course, C and A could have been reversed in the drawing, and the line does not have to be horizontal.)

Example 4: Visualize and Draw Points on a Line

NAEP 1996

Jaime knows the following facts about points A, B, and C.

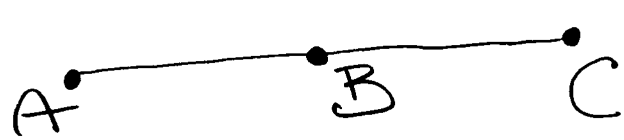
- Points A, B, and C are on the same line, but might not be in that order.
- Point C is twice as far from point A as it is from point B.

Jaime concluded that point C is always between points A and B.

Is Jaime's conclusion correct?

☐ Yes ■ No

In the space provided, use a diagram to explain your answer.



Example 4	Grade 8
United States	23
	% correct

Can the students in your classroom, school, district, and state visualize and draw a variety of one-, two-, and three-dimensional geometric shapes?

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AREA TWO: Geometry and Spatial Sense

Combine, Subdivide, and Change Geometric Figures

Under this area, students were asked to build on their drawing, visualizing, and constructing skills to create new shapes with certain characteristics. This area includes NAEP's Subtopic 2, investigating and predicting results of combining, subdividing, and changing shapes, as well as Subtopic 4 where students are asked to describe the intersection of two or more geometric figures. From the TIMSS framework, this area includes familiarity with two- and three-dimensional shapes. In particular, students could:

- Use cardboard shapes to create new shapes with certain characteristics.
- Draw all the possible figures that could be formed from two (or three or four) given shapes.
- Draw the figures that could result from making one (or two or three) straight cuts in a given solid.
- Determine how many figures are within a figure (e.g., squares in a rectangle, cubes in a rectangular solid, triangles in a trapezoid).

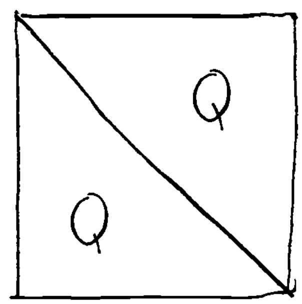
Examples 5 and 6, given in NAEP's 1996 assessment, also were based on the six cardboard shapes described under Example 1. In Example 5, fourth graders were asked to draw a square using these manipulatives, while in Example 6, eighth graders were asked to outline figures embedded in a larger shape made of triangles. Interestingly, nearly three-fourths of the fourth-grade students could combine the two triangles (two pieces labeled Q) to draw a square and the line showing where the two pieces met. This is in contrast to the 40 percent of fourth graders able to draw a square when given two vertices (see Example 2). As indicated, eighth-grade students had two options in subdividing the figure given in Example 6. Sixty-four percent were successful in drawing at least one of the solutions.

NAEP 1996

Example 5: Combine Triangles to Draw a Square

You will need the 2 pieces labeled *Q*. Please find those 2 pieces now.

Use the 2 pieces labeled *Q* to make a square. Trace the square and draw the line to show where the 2 pieces meet.



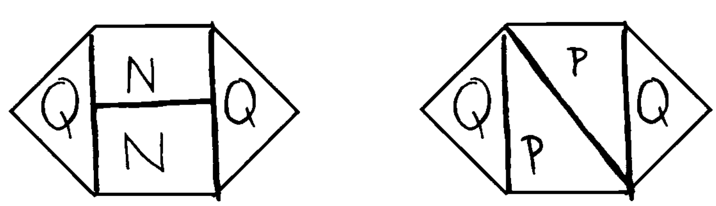
Example 5	**Grade 4**
United States	73
	% correct

NAEP 1996

Example 6: Subdivide Shape Made of Triangles

For this question you will need some of the pieces labeled *N*, *P*, and *Q*.

Use 4 of the 6 pieces labeled *N*, *P*, and *Q* to make the shape shown below. Draw the lines to show where the pieces meet and label the pieces.



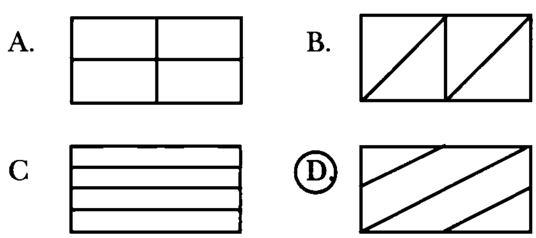
Two Possible Solutions

Example 6	**Grade 8**
United States	64
	% correct

TIMSS 1995

Example 7: Identify Rectangle Not Divided in Equal Parts

Which rectangle is NOT divided into 4 equal parts?

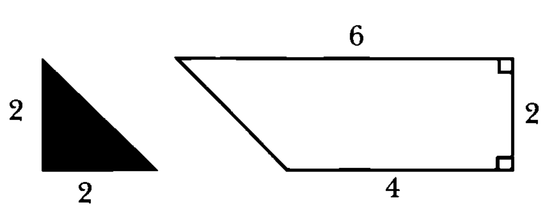


Example 7	Grade 4
United States	83
Singapore	85
Korea	90
International Avg.	73

% correct

TIMSS 1995

Example 8: Triangles in Trapezoid



How many triangles of the shape and size of the shaded triangle can the trapezoid above be divided into?

A. Three

B. Four

C. Five

D. Six

Examples 7 and 8 involve visualizing subdivisions of two-dimensional figures. Fourth-grade students around the world found Example 7 from TIMSS to be relatively easy. The best performance was by students in Korea where 90 percent of the students answered correctly followed by Singapore (85%) and the United States (83%). Across the 26 countries participating in TIMSS at the fourth grade, the international average of correct responses was 73 percent. Similarly, the majority of eighth graders in a number of countries were able to determine that the trapezoid could be divided into five triangles. The highest performing country on this item was Japan, where 84 percent of the students selected the correct answer. Across the 41 countries participating in the eighth-grade TIMSS mathematics assessment, the international average of correct responses was 53 percent. Eighth graders in the United States were below the international average and far from the mark of Japan with only 44 percent identifying the correct response.

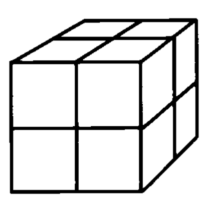
Example 8	Grade 8
United States	44
Japan	84
International Avg.	53

% correct

The NAEP framework includes students' ability to describe the intersection of two-dimensional figures as well as the planar cross-section of a solid. Under this area, students might be asked to solve problems using the intersection of geometric figures or explain what different shapes could be produced by different placements of a plane within a solid. For example, cross-sections of a cylinder might produce a circle, a rectangle, or an ellipse. Examples 9 and 10, both from NAEP, provide more complex situations in which students needed to visualize subdivisions of three-dimensional figures. Example 9 was relatively difficult for fourth graders. Just one-third demonstrated an understanding that the large cube contained eight small cubes.

NAEP 1996

Example 9:
Number of Small Cubes Within a Large Cube



In this figure, how many small cubes were put together to form the large cube?

A. 7

(B.) 8

C. 12

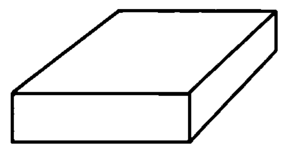
D. 24

Example 9	**Grade 4**
United States	33
	% correct

Example 10	**Grade 12**
United States	
All parts correct	42
Part (a).	90
Part (b).	68
Part (c).	74
Part (d).	56
	% correct

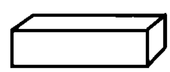
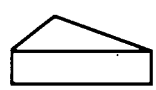
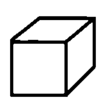
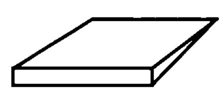
NAEP 1992

Example 10: Shapes Produced by a Single Cut of a Rectangular Solid



The piece of fudge shown above is in the shape of a rectangular solid. If a knife makes one straight cut through the fudge, which of the following can be the piece cut off?

Fill in one oval to indicate YES or NO for each shape.

		Yes	No
(a)		(A)	(B)
(b)		(A)	(B)
(c)		(A)	(B)
(d)		(A)	(B)

Example 11	**Grade 8**
United States	49
	% correct

NAEP 1992

Example 11: Find Area of Rectangle from Area of Triangle

If the area of the shaded triangle shown below is 4 square inches, what is the area of the entire square?

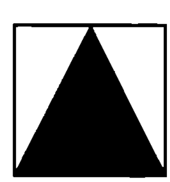
A. 4 square inches

(B.) 8 square inches

C. 12 square inches

D. 16 square inches

E. Not enough information given



57

Example 10 asked students to visualize a three-dimensional rectangular solid cast in a real-world context as a piece of fudge. Although this item was given only to twelfth-grade students in 1992, it is in alignment with the NAEP 1996 Mathematics Framework for eighth-grade students. Part A was the easiest. Ninety percent of twelfth graders recognized that a rectangular solid could be sliced from a larger rectangular solid. In contrast, only about one-half to two-thirds recognized you could slice off different shaped triangles by making diagonal cuts in the fudge.

Approximately three-fourths of the students appeared to understand that Part C would require both a horizontal and vertical cut (Parts B and D).

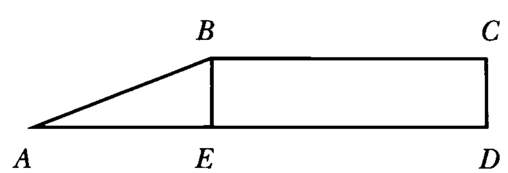
Example 11 represents an example of a problem assessing whether or not students understand that the area of any triangle inscribed in a square, rectangle, or parallelogram is equal to one-half of the area of the square, rectangle, or parallelogram. For example, as shown below, the top vertex of the triangle does not have to be at the midpoint of the top side of the square (rectangle or parallelogram).

Students also could have solved the problem by using the formula for computing the area of a triangle ($A = 1/2\ bh$). Application of this understanding also can be assessed by asking students to determine the area of a rectangle within a parallelogram (or of a parallelogram within a rectangle).

Example 12 involves using information about the intersection of a triangle and a rectangle as one of several steps in computing the area of a trapezoid. That is, BE is 60/15 = 4, which provides the height of triangle ABE. Students also needed to recognize that the area of ABE is one-half that of the rectangle whose area is 10 x 4 (or 40/2 = 20). Adding 20 for the area of the triangle to 60 (given area of rectangle) provides the correct answer of 80. This problem involves basic arithmetic and basic geometric understanding. Yet, in NAEP's 1992 assessment only 10 percent of the eighth-grade students solved this problem correctly.

NAEP 1992

Example 12: Area of Trapezoid



The area of rectangle *BCDE* shown above is 60 square inches. If the length of *AE* is 10 inches and the length of *ED* is 15 inches, what is the area of trapezoid *ABCD*, in square inches?

Answer: 80

Example 12	**Grade 8**
United States	10
	% correct

How would the students in your classroom, school, district, or state perform on items requiring them to combine and subdivide geometric shapes?

AREA THREE: Geometry and Spatial Sense

Identify Relationships Between Figures in Terms of Symmetry; Motions and Transformations; Congruence and Similarity; Ratios and Proportions

This is a very large area, encompassing NAEP's Subtopic 3, which includes motion geometry (symmetry, flips, turns, and slides), transformations, and congruence and similarity, all of which are emphasized in both the NAEP and TIMSS frameworks.

There are many student activities that could be included in this area. For example, students could be asked to:

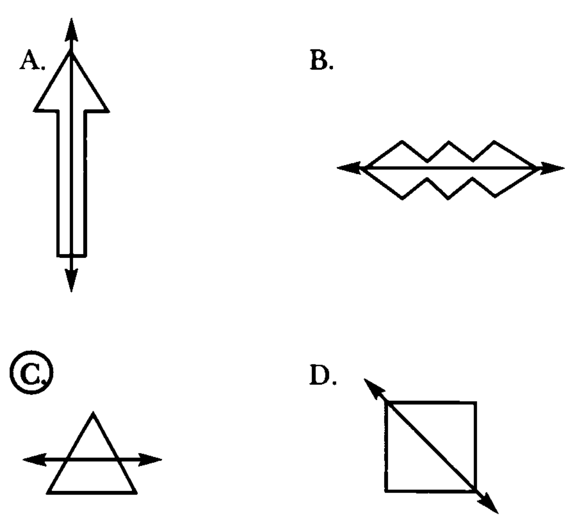
- Draw or identify the lines dividing a figure into two congruent parts.
- Cut or draw figures to render them symmetric about a line of folding.
- Draw or identify figures that could (or could not) result from a flip, slide, or turn of a given two- or three-dimensional figure
- Draw or identify the figure that would result from a given transformation of a given shape.
- Given a pattern of figures (e.g., each element is a 90-degree rotation of the previous element) determine the transformation, extend the pattern, and describe the rule.
- Determine which figures in a given set are congruent or similar.
- Describe why two given figures are similar (or not) and what is known about the figures because of the similarity (or lack of it).
- Determine the attributes of figures from their congruence or similarity, such as the measure of an unknown side of a figure, given that the figure is congruent (or similar) to a figure with known measures.

Examples 13 and 14 from TIMSS assess students' basic understanding of lines of symmetry at grades 4 and 8, respectively. On Example 13, fourth graders were asked to identify which figure did not show a line of symmetry. Fourth-grade students in the United States (74% correct responses) performed above the international average (64%) across the 26 participating countries. Yet, their performance was somewhat below that of top-performing Singapore (93%).

TIMSS 1995

Example 13:
Which Does Not Show a Line of Symmetry

Which of these does NOT show a line of symmetry?



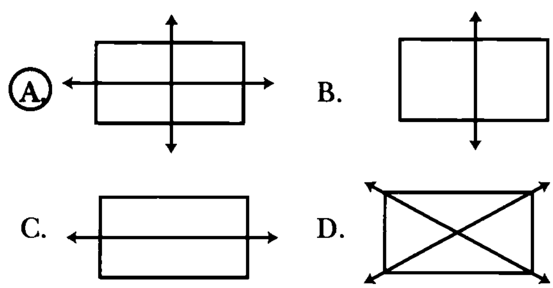
Example 13	**Grade 4**
United States	74
Singapore	93
International Avg.	64

% correct

TIMSS 1995

Example 14:
All the Lines of Symmetry for a Rectangle

Which shows all of the lines of symmetry for a rectangle?



Example 14	Grade 8
United States	70
Scotland	86
International Avg	66

% correct

On Example 14, eighth graders were asked to identify all of the lines of symmetry for a rectangle. The results paralleled those at fourth grade. Eighth graders in the United States performed above the international average, but not quite as well as those in some countries. Scotland had the highest achievement (86%), but 80 percent or more of the eighth graders in a number of other countries answered this question correctly (Belgium-French, England, France, Hungary, New Zealand, Singapore, and Thailand).

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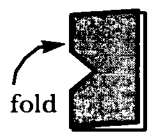
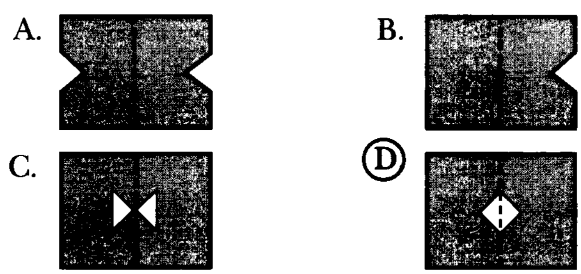
NAEP 1992

Example 15: Visualizing Unfolded Paper



A sheet of paper is folded once and a piece is cut out as shown above. Which of the following looks like the unfolded paper?



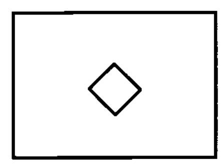
Example 15	Grade 4
United States	65
	% correct
	Grade 8
United States	87
	% correct

Examples 15 and 16 also assess understanding of symmetry, which is implicit in these examples involving visualization of geometric shapes. Example 15 is a multiple-choice question from the 1992 NAEP assessment that nearly 66 percent of the fourth-grade students and 87 percent of eighth-grade students answered correctly. In the TIMSS performance assessment, students were asked to actually perform folding and cutting tasks. In one such task, students were asked to fold sheets of paper so that a single cut would make the diamond shape, then to actually make the cut (see example 16). Students could try the task up to a total of three times.

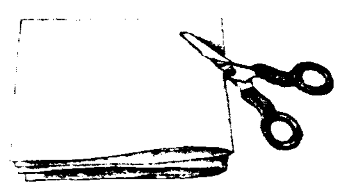
TIMSS 1995

Example 16: Visualizing Folded Paper

Look at the shape below. Fold a sheet of paper as many times as necessary and make ONE STRAIGHT CUT so that when the paper is unfolded it has the same SHAPE as the shape below. The SIZE of your paper and cutouts do not have to be the same as those shown here. If you are unsuccessful, you may try again with another sheet of paper. You may try this task a total of THREE times.



cut



result

Modified from a series of 3 folding and cutting tasks.

Example 16	**Grade 4**
United States	29
Slovenia	51
International Avg.	33
	% correct

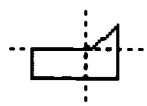
	Grade 8
United States	71
Romania	88
Slovenia	83
International Avg.	72
	% correct

Less than one-third of the fourth-grade students in the United States and 71 percent of the eighth-grade students were successful in making the cut that produced the diamond shape. This achievement was slightly less than the international average at grade 4 (based on 9 countries) and similar to the international average at grade 8 (based on 19 countries). The Slovenian students had the best performance on this task at grade 4 (51%) and among one of the best performances at grade 8 (83%). Eighty-eight percent of the Romanian eighth-grade students made a diamond shape with a single straight cut.

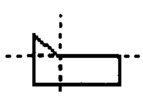
TIMSS 1995

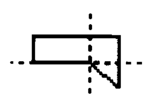
Example 17: Visualizing Half-turn Rotation

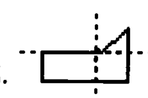
A half-turn about point *T* in the plane is applied to the shaded figure.

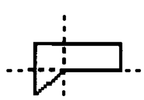


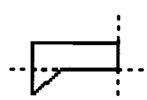
Which of these shows the result of the half-turn?

A. 

B. 

C. 

(D) 

E. 

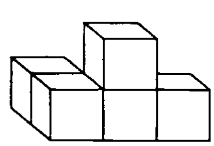
Example 17	**Grade 8**
United States	42
Hong Kong	73
Netherlands	70
International Avg.	53

% correct

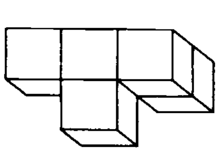
TIMSS 1995

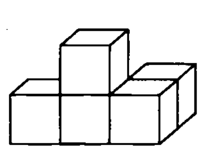
Example 18: Visualizing Three-dimensional Rotation

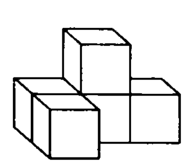
This figure will be turned to a different position.

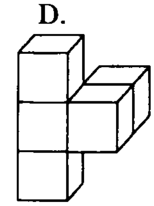


Which of these could be the figure after it is turned?

(A) 

B. 

C. 

D. 

Example 18	**Grade 8**
United States	62
Czech Republic	87
International Avg.	68

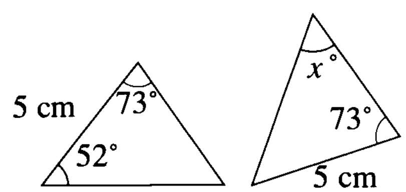
% correct

Examples 17 and 18 are TIMSS rotation problems, asking students to identify how figures will look turned. In Example 17, students were asked to visualize the result of a half-turn of a two-dimensional figure. On average, about one-half the students internationally answered this question correctly. Hong Kong had the best performance with 73 percent correct, although the Netherlands also did very well with 70 percent correct. In comparison, 42 percent of the eighth graders in the United States answered this question correctly. Interestingly, achievement both internationally and in the United States was somewhat higher on Example 18, even though it was a three-dimensional rotation problem. Sixty-two percent of the eighth graders in the United States answered this question correctly compared to 66 percent internationally, on average, and 87 percent in the Czech Republic.

TIMSS 1995

Example 19: Unknown Angle in Congruent Transformed Triangle

These triangles are congruent. The measures of some of the sides and angles of the triangles are shown.



What is the value of *x*?

A. 52

(B.) 55

C. 65

D. 73

E. 75

Example 19	Grade 8
United States	17
Singapore	69
Japan	69
International Avg.	36

% correct

How well do the students in your classroom, school, district, or state understand motion geometry, transformations, and congruence and similarity?

Example 19 asked students to determine the unknown angle in a transformed congruent triangle. Thus, students needed to understand the concept of congruence, be able to visualize the flipped and rotated figure, know the properties of angles in triangles (sum of angles is 180°), and perform the correct computation. Perhaps an open-ended format would have been a better approach to determine whether students could apply the 180 degrees fact. In the multiple-choice format, it also is possible for students to realize that the triangles are close to being isosceles, hence 55 degrees is the only viable alternative. Still, this task was difficult for eighth graders internationally, with an average percent correct of 36 percent. Students in the United States had particular difficulty—only 17 percent correct. An overwhelming majority of the eighth-grade students in the U.S. (61%) selected option A, revealing that they failed to notice that the figure was flipped as well as rotated. In comparison to the generally low levels of performance internationally, students in four participating Asian countries had relatively high achievement on this item. Eighth graders in Singapore and Japan had the best performance (69%) and those in Korea (66%) and Hong Kong (61%) also did well.

AREA FOUR: Geometry and Spatial Sense

Apply Geometric Properties and Relationships in Solving Problems

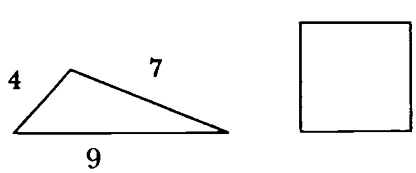
This area corresponds directly to NAEP and TIMSS expectations for problem solving, reasoning, and communicating. The emphasis is primarily on using mathematics to solve everyday problems and on demonstrating the types of practical reasoning skills expected by business and industry. To give students practice in problem solving using the properties of geometric figures and help them understand the relationships between figures, they can be asked to perform the following tasks:

- Apply properties of triangles, squares, and rectangles to solve problems.
- Apply properties of angles to solve problems (e.g., supplementary, complementary, bisectors).
- Apply properties of circles to solve problems.
- Solve problems involving the use of the Pythagorean theorem.
- Solve problems involving similarity, using ratios and proportions.

Example 20	**Grade 4**
United States	26
	% correct

NAEP 1996

Example 20: Perimeter of Triangle Related to Sides of Square



If both the square and the triangle above have the same perimeter, what is the length of each side of the square?

A. 4

(B.) 5

C. 6

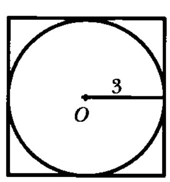
D. 7

Example 20 from NAEP's 1996 assessment involves an understanding of perimeter and the properties of the sides of a square. Essentially, fourth graders needed to find the perimeter of the triangle and divide by 4. However, they found this item very difficult(26% correct responses).

For Example 21, an open-ended format (or having options closer to the correct answer) would have been a better assessment approach. As the item stands, students can use simple reasoning to determine that the area of the square is 36, and that the area of the circle must be smaller than that, but not too much smaller. Thus, only options A and B can even be possible responses. Nevertheless, only 29 percent of the eighth graders in NAEP's 1992 assessment correctly identified the difference between the area of the circle and that of the square. It should be noted that students were provided with a scientific calculator containing the value of pi.

NAEP 1992

Example 21:
Areas of Square and Circle



In the figure above, a circle with center *O* and radius of length 3 is inscribed in a square. What is the area of the shaded region?

A. 3.86

(B.) 7.73

C. 28.27

D. 32.86

E. 36.00

(Students had access to a calculator.)

Example 21	Grade 8
United States	29
	% correct

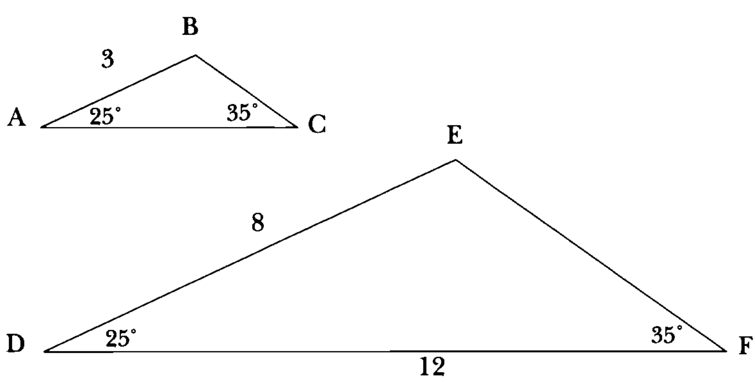
69

Example 22 is a multifaceted TIMSS problem involving the concept of similarity, properties of triangles, and proportional reasoning. It also involves the ability to compute accurately with fractions and decimals as students are asked to find the length of the side AC, given that the triangles are similar (e.g., $\frac{3}{8} \times 12$ or $\frac{8}{12} = \frac{3}{x}$). U.S. eighth graders (28% correct) performed below the international average (38%) and well below their counterparts in Japan (71%).

TIMSS 1995

Example 22: Length of Side of Similar Triangle

Triangles *ABC* and *DEF* are similar triangles.



What is the length of side *AC*?

A. 2

B. 4

(C.) 4.5

D. 5.5

E. 32

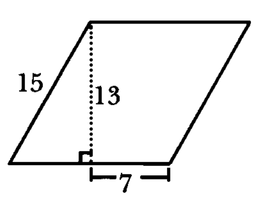
Example 22	Grade 8
United States	28
Japan	71
International Avg.	38

% correct

The NCTM analysts of NAEP's 1992 geometry results observed that even though the Pythagorean theorem is probably the most universally addressed theorem in geometry, there is evidence that many students cannot apply it and probably do not understand it well. [15] Example 23, which involves actually applying the Pythagorean theorem to obtain the area of the parallelogram, was administered to twelfth-grade students as part of NAEP's 1992 assessment. Considering that this test question fits the 1996 NAEP Framework for grade 8, which covers using the Pythagorean theorem to solve problems, it is perhaps surprising that only 8 percent of the twelfth graders provided a correct answer.

NAEP 1992

Example 23: Use Pythagorean Theorem to Find Area of Parallelogram



To the nearest whole number, what is the area of the parallelogram above?

Answer: 188

(Students had access to a calculator.)

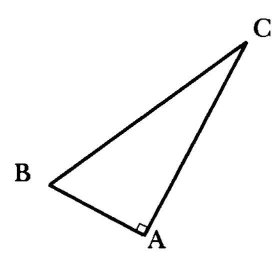
Example 23	**Grade 12**
United States	8
	% correct

71

To give students more experience in recognizing situations involving the Pythagorean theorem, they might be given questions such as that shown in Example 24. Even though this item was not actually included in either NAEP or TIMSS assessments, it addresses the frameworks and is the type of item that eighth-grade students might encounter in the future.

Can the students in your classroom, school, district, or state apply properties of geometric figures to solve problems?

Example 24: Pythagorean Theorem in Problem-solving Context



For a workout Anna bikes along a course in the shape of the right triangle shown above. If the distance from A to B is 5 kilometers and from B to C is 13 kilometers, what is the total distance, in kilometers, that Anna bikes during her workout?

A. 18

B. 20

C. 25

D. 26

(E.) 30

Sample Item for Illustrative Purposes (no results available)

72

AREA FIVE: Geometry and Spatial Sense

Represent Geometric Figures and Properties Algebraically Using Coordinates and Properties of Lines

Some NAEP and TIMSS geometry items were presented analytically. In comparison to synthetic contexts, where students reason about figures in the absence of information about their locations, analytic contexts give students coordinates that place figures at particular locations in two or three dimensions. To solve analytic problems, students use approaches such as slopes and distances between points that draw heavily on the interplay between geometry and algebra. To gain practice in this area, students might be asked to:

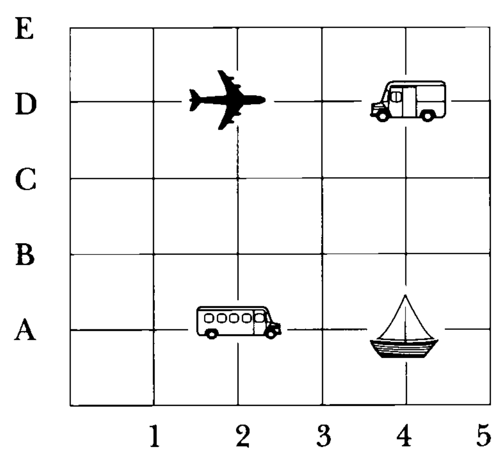
- Reason about the distance between points on a line to determine the relation among the points.
- Use the Pythagorean theorem to determine the hypotenuse of a right triangle on a coordinate plane.
- Represent and interpret geometric figures and properties algebraically using coordinates in a plane.
- Describe the slope of a line in terms of ratio or proportional thinking.
- Develop equations of lines from graphs.

Example 25 was given to fourth graders as part of TIMSS. Identifying the object located at coordinate (2,D) was relatively easy for students in most countries. Fourth graders in the United States and the Netherlands had the best performance, with nearly all students answering correctly. A corresponding eighth-grade item that required knowing which axis to read first, as well as some interpolation, was more difficult for students in all except the Asian countries (see Example 26). For example, compared to 91 percent of the eighth graders in Singapore, 73 percent in the Netherlands and 58 percent in the United States correctly identified the coordinates of point P.

TIMSS 1995

Example 25: Grid on Game Board

This is a game board.



Which object is located at (2,D)?

(A.) The plane 

B. The truck 

C. The bus 

D. The boat 

Example 25	Grade 4
United States	97
Netherlands	97
International Avg.	88
	% correct

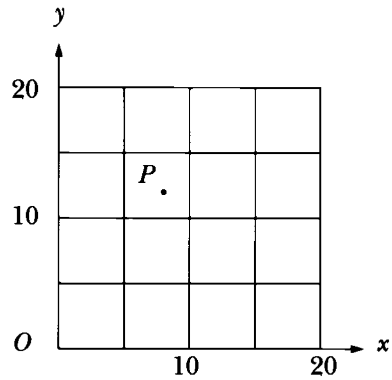
Can the students in your classroom, school, district, or state represent geometric figures using coordinates and properties of lines?

74

TIMSS 1995

Example 26: Coordinates of Point *P*

Which of the following are most likely to be the coordinates of point *P*?



(A.) (8, 12)

B. (8, 8)

C. (12, 8)

D. (12, 12)

Example 26	Grade 8
United States	58
Singapore	91
Netherlands	73
International Avg.	55

% correct

When given its coordinates and asked about another point on a line (Example 27), students showed great variation from country to country. On average across countries, 41 percent of the eighth graders provided correct responses, and the United States performed at the average. The Netherlands was the top-performing country on this item (66%), followed by Singapore (59%).

TIMSS 1995

Example 27: Identify Another Point on a Line

A straight line on a graph passes through the points (3,2) and (4,4). Which of these points also lies on the line?

A. (1,1)

B. (2,4)

(C.) (5,6)

D. (6,3)

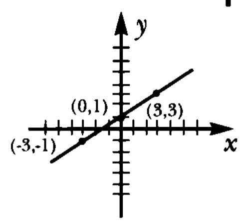
E. (6,5)

Example 27	Grade 8
United States	41
Netherlands	66
Singapore	59
International Avg.	41

% correct

NAEP 1992

Example 28: Find the Slope of a Line



What is the slope of the line shown in the graph above?

A. $\frac{1}{3}$

(B.) $\frac{2}{3}$

C. 1

D. $\frac{3}{3}$

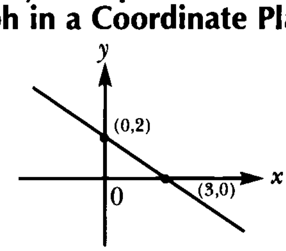
E. 3

Examples 28 and 29 illustrate some questions in this geometry area that would be challenging for many eighth graders, but they assess topics covered in the NAEP and TIMSS frameworks. Example 28, about the slope of a line, was administered to twelfth graders as part of the 1992 NAEP assessment. That only 41 percent answered correctly suggests that students had some difficulty with analytic geometry items. Another 14 percent of the twelfth graders did select the reciprocal of the correct response, suggesting that these students may have found the horizontal and vertical change.

Example 28	Grade 12
United States	41
	% correct

Even though NAEP and TIMSS items involving the equation of a line were not available for presentation here, these types of questions are included in the analytic areas of the frameworks. Example 29 is a sample of a question that relates the equation of a line to its x- and y-intercepts—that is, (3,0) and (0,2). The y-coordinate of the x-intercept is 0. Determining the x- and y-coordinates of a line is a useful way of quickly locating points on a line. Students might also be given an equation of a line and asked to select the corresponding graph from among several choices (see AREA FOUR under Algebra and Functions).

Example 29: Identify an Equation of a Line from its Graph in a Coordinate Plane



Which of the following is an equation of line *m* shown in the *xy*-plane above?

A.

B.

C.

(D.)

E.

Sample Item for Illustrative Purposes (no results available)

76

AREA SIX: Geometry and Spatial Sense

Solve Real-World Problems Using Geometric Models and Explain Relationships Involving Geometric Concepts

This area encompasses Subtopics 7 and 8 from the 1996 NAEP Framework as well as various aspects of the TIMSS performance expectations. Since the problem situations in this area require students to develop geometric models and often to explain the relationships shown by the models, the tasks generally require extended responses. Although evaluating student responses can be approached in several ways, one strategy is to identify levels of success (as shown for Example 30).

Within this topic area, students can be asked to:

- Use geometric models and/or diagrams to represent a situation (such as fencing a yard or dividing an area).
- Use geometric models and/or diagrams in solving problems involving probability.
- Extend a geometric pattern, describe its rule, and justify why the rule works.

An important aspect of mathematical power as presented in the NAEP framework is the need for students to use logic, models, and diagrams to make sense of a situation and to communicate their reasoning. However, the following examples from NAEP and TIMSS suggest that many students have yet to recognize that models and diagrams can be effective analytical and communication tools.

Example 30, asking eighth-grade students to diagram a problem situation involving the broadcast areas of two radio stations is from NAEP's 1992 assessment. Students needed to assimilate and translate written information in order to draw a diagram that graphically depicts the location of the radio stations and Highway 7 accurately in terms of given boundary conditions. A graphical approach to this task should enable students to determine the length of the overlapping portion of Highway 6, along which both radio stations can be received. Any satisfactory response must clearly illustrate an overlapping region, whereas, in addition, any extended response must clearly identify the overlap and correctly determine its length to be 75 miles.

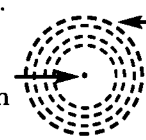
NAEP 1992

Example 30: Radio Stations

This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show all your work.

Radio station KMAT in Math City is 200 miles from radio station KGEO in Geometry City. Highway 7, a straight road, connects the two cities.

KMAT broadcasts can be received up to 150 miles in all directions from the station and KGEO broadcasts can be received up to 125 miles in all directions. Radio waves travel from each radio station through the air, as represented below.

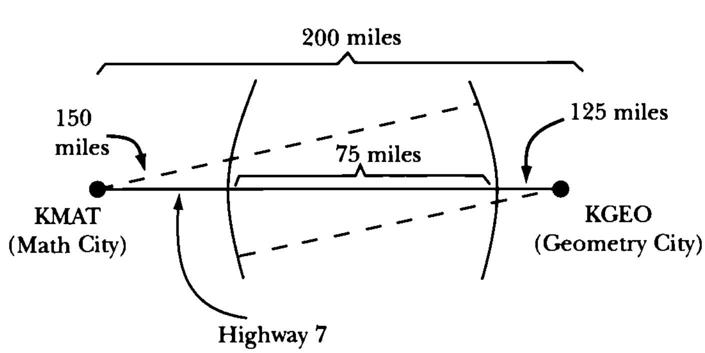


On the next page, draw a diagram that shows the following.

- Highway 7
- The location of the two radio stations
- The part of Highway 7 where both radio stations can be received

Be sure to label the distances along the highway and the length in miles of the part of the highway where both stations can be received.

Example 30
One Possible Solution



Example 30	**Grade 8**
United States	
No Response	15
Incorrect	45
Minimal	22
Partial	13
Satisfactory	4
Extended	1

% of students in response category

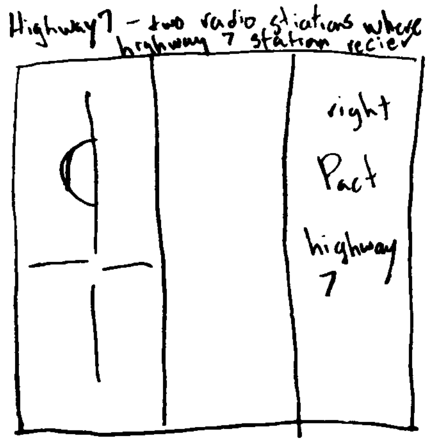
78

Students' responses were evaluated according to the degree of accuracy of their diagrams. Even though a variety of diagrams or explanations could be used to help explain the intersection of the broadcast areas of the two radio stations and no particular approach was preferred, only 5 percent of the eighth graders developed a labeled model that represented the problem. Eighth graders with incorrect responses provided no evidence that they were able to make sense of the problem, often copying a piece of information from the problem or submitting meaningless drawings (or both). Forty-five percent of the eighth graders nationally provided such responses and another 15 percent did not answer the question at all. Although this meant that the majority appeared to be essentially at a loss as to the nature of this task, about one-third did seem to have some understanding of the information presented in relation to the task required. Approximately 22 percent received minimal credit and another 13 percent received partial credit, the difficulty with these responses being an incomplete approach. These sketchy solutions appeared in spite of directions explicitly telling students what to diagram and to be sure to label the distances and the part of the highway where both stations can be received. The following pages present examples for each response category.

NAEP 1992

Example 30
Incorrect Response

The work is completely incorrect, irrelevant, or the response states, "I don't know."



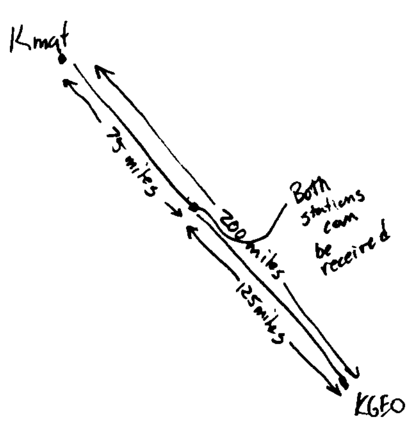
This INCORRECT response does not relate the information given in the problem in a manner that conveys either a meaningful problem-solving approach or an adequate solution.

45% of U.S. students provided an Incorrect response

NAEP 1992

Example 30
Minimal Response

Diagram with only cities, Highway 7, and 200 miles labeled; or a diagram that shows some, but not all, of the given distances; 125, 150, or 200 miles. Minimal responses do not recognize that the common broadcast area is a <u>length</u> along the highway.



This MINIMAL response correctly depicts two pieces of information (radio stations KMAT and KGEO are 200 miles apart and station KGEO can broadcast 125 miles) and shows rudimentary understanding. It does not show the common broadcast area as a length along the highway.

22% of U.S. students provided a Minimal response

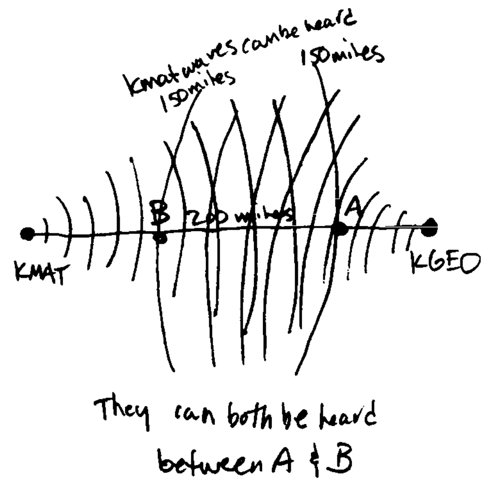
This PARTIAL response indicates considerable understanding of the task relative to the given information. The diagram shows the radio stations to be 200 miles apart and that KMAT can broadcast 150 miles. Additionally, the diagram shows a part of the highway (from A to B) along which both radio stations can be heard. However, the response does not show the broadcast range of station KGEO and does not indicate the length of the common broadcast area.

13% of U.S. students provided a Partial response

NAEP 1992

Example 30
Partial Response

Diagram with cities, Highway 7, and 200 miles labeled <u>and</u> identification of common broadcast area a length along (or not on) the highway. Two or more of the radio wave distances 250, 125, and 75 are insufficiently labeled.

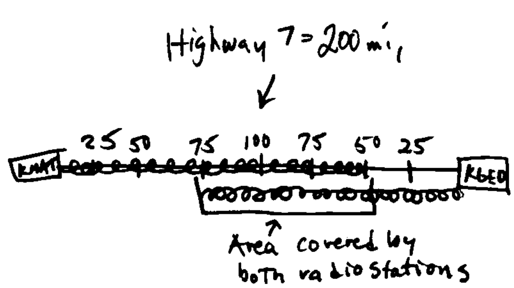


This SATISFACTORY diagram shows a good understanding of the problem. Although the student correctly labeled the common area along Highway 7 where the two stations could be heard, the length in miles of this region was not indicated.

4% of U.S. students provided a Satisfactory response

Example 30
Satisfactory Response

Diagram with only cities, Highway 7, 200 miles and all radio wave distances labeled <u>and</u> identification of common broadcast area on Highway 7 as a length. At the same time, omits or incorrectly computes length of the highway along which both radio stations can be received.

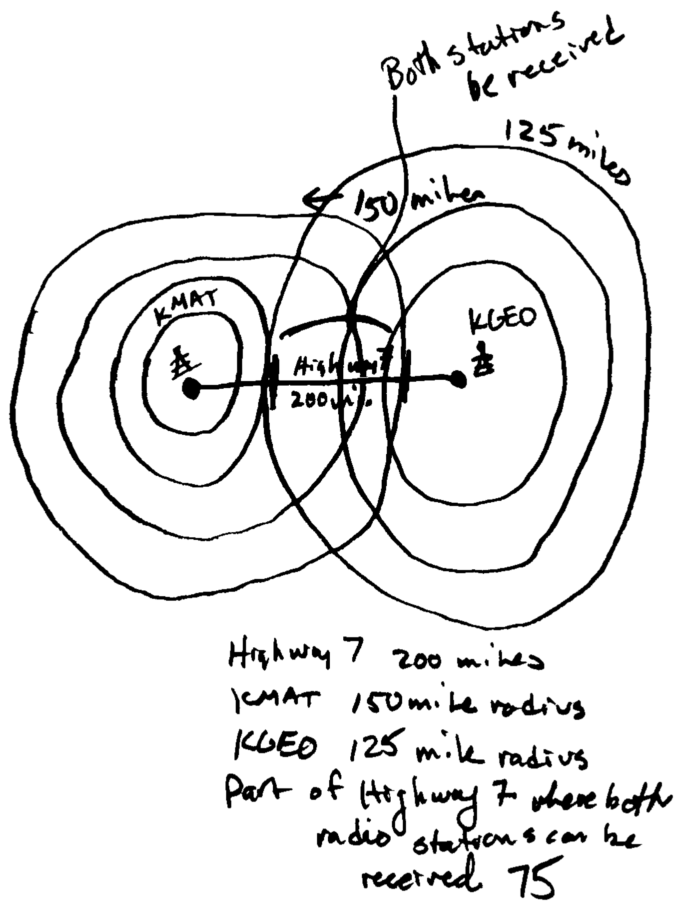


NAEP 1992

NAEP 1992

Example 30
Extended Response

An accurate, well-labeled diagram (as described in the Satisfactory category) clearly indicating that the portion of Highway 7 along which both radio stations can be received is 75 miles in length.



This is a solid EXTENDED response. The diagram is accurate and well labeled. Additionally, below the diagram a statement correctly concludes that the length of the part of Highway 7 along which both radio stations can be heard is 75 miles.

1% of U.S. students provided an Extended response

82

Examples 31 through 33 are three problems that make up a TIMSS performance assessment task. The task involved problem solving in three-dimensional space. Students were supplied with four small plastic balls packed into a square box, some sheets of light cardboard, and an explanation and illustration of a net for the box. With these, and a supply of materials such as a compass, ruler, scissors, adhesive tape, and paper clips, students were to find three other boxes in which the balls could be tightly packed (Example 31), sketch the boxes, draw a net* for each one (Example 32), and then draw one of the nets to the actual size needed to hold the four balls (Example 33). The task was intended to measure the students' sense of spatial relations as evident in their ability to visualize different arrangements of objects in boxes, to translate the three-dimensional models first into a two-dimensional sketch, then into the corresponding net, and finally to scale the net to actual size, working from concrete materials rather than by applying a formula to measurements. Students were provided with the directions and introductory information (shown at right) and then asked to work out the three problems.

Since the emphasis was on using concrete materials and on understanding the relationships between the two-dimensional drawings, the one-dimensional nets, and the three-dimensional models of the boxes, only rough indications were used for sketching the placement of the balls in the boxes. As in the presentation of the item, the sketches of the balls in the students' responses were not required to be accurately rendered in two-dimensional perspective.

* A net is defined here as the two-dimensional pattern that when folded up would yield the three-dimensional object.

Examples 31, 32, 33

PACKAGING

At this station you should have:

- 4 plastic balls packed in a square shaped box
- Blu-tac to stop the balls from rolling around
- Some thin card to make a package for the balls
- A compass
- A 30 cm ruler
- Two pieces of thick card to help measure the balls
- Scissors
- Sellotape
- Paperclips

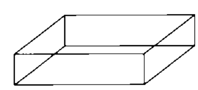
Your task:

Design different boxes which will just hold 4 plastic balls.

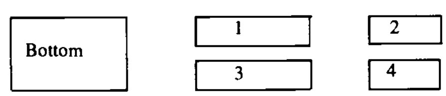
Read this before answering the questions:

The following shows what is meant by the net of a box.

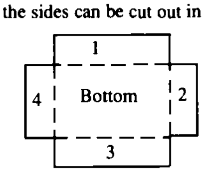
This box has a bottom and 4 sides.



The sides can be cut out separately:

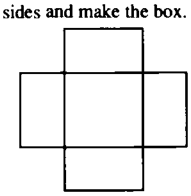


Or the sides can be cut out in one piece and then folded along the dotted lines like this:

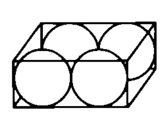


This is a net of a box.

This is the shape of a net of a box like the one that holds the 4 balls. It is not drawn to size but if it were, you could fold up the sides and make the box.



You have been given the box with the four balls just fitting in like this.



Other boxes with different shapes could be made so that the 4 balls would just fit in.

Example 31	**Grade 8**
United States	30
Singapore	82
International Avg.	43

% of students providing fully-correct response

TIMSS 1995

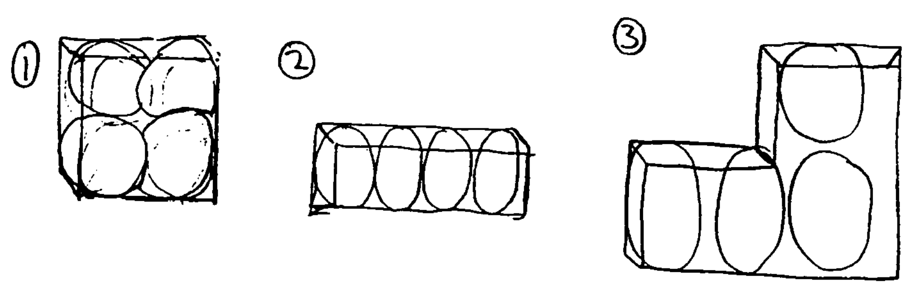
Example 31: Draw Three Boxes

Use the balls to find three other boxes in which the four balls will just fit. Make a drawing of each box with the four balls in it.

Students received full credit for drawings in which each box describes or shows all four balls in "tightly packed" arrangements and at least two of the arrangements were unique. Although there are a number of ways this could have been accomplished, some examples are: a box in which the four balls are arranged in a line in a rectangular shaped box; a box in which the four balls are arranged in an L-shape (three balls in a row and one ball on top of another); a box in which the two rows of two balls are stacked vertically (a square box).

Students receiving partial credit on this question provided only one or two possibilities for boxes that would hold four balls.

Example 31 Possible Response



Eighth-grade performance on Example 31, the task of drawing three boxes in which the four balls would "just fit," varied across the TIMSS countries. In Singapore, 82 percent of students correctly drew three boxes, thus earning full credit. In contrast, only 30 percent of students in the United States correctly drew three boxes; the remaining students drew one or two boxes correctly (22%), or were unable to visualize different shapes that would hold objects and represent them in a drawing (48%).

Example 32: Draw Nets

Now make a drawing of the net for each box.

TIMSS 1995

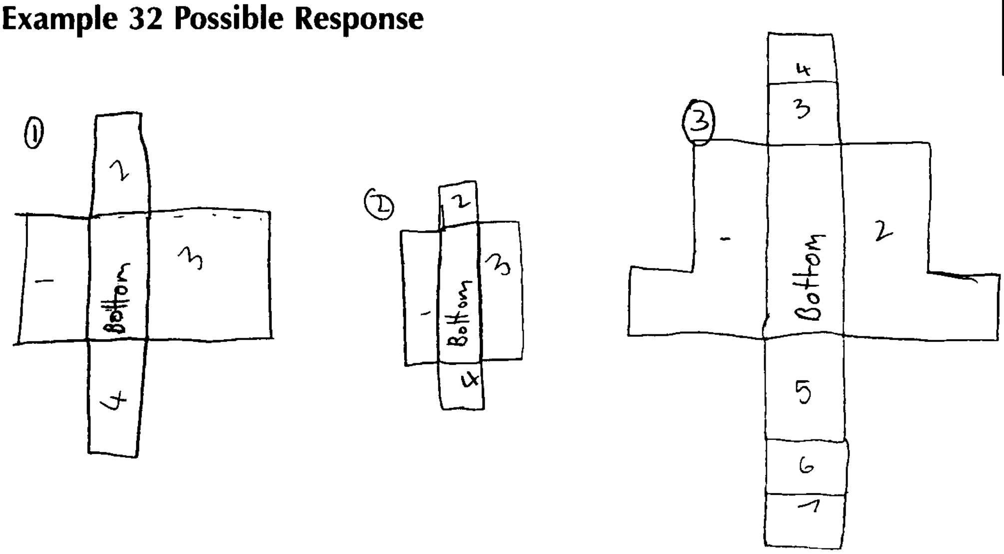
Example 32	Grade 8
United States	17
Switzerland	44
International Avg.	22

% of students providing fully-correct response

In drawing the net of each box the student presented in the previous item (Example 31), the student needed to draw nets consistent with at least two of the ball arrangements, clearly showing the correct shape of the base and side flaps required to constrain the balls in a "tightly packed" arrangement and correct proportions of the base and side flaps.

Students received partial credit if they drew the net correctly for only one box.

Example 32 Possible Response



Example 32 was relatively difficult for students in many of the TIMSS countries, indicating that students are not quite adept at representing the nets of three-dimensional models. Eighth-grade students in Switzerland provided the largest percentage of fully correct responses (44%). Just 17 percent of students in the United States correctly drew the nets of at least two boxes, falling just below the international average of 22 percent.

Example 33	Grade 8
United States	4
Romania	51
International Avg.	30

% of students providing fully-correct response

TIMSS 1995

Example 33: Construct Net to Scale

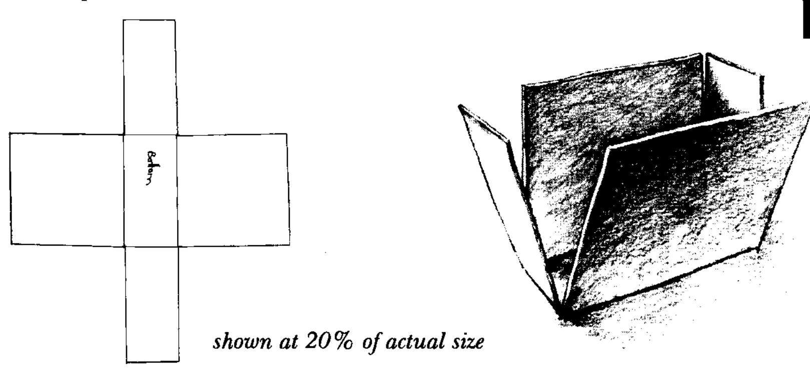
Choose ONE of the boxes you have drawn. Take a piece of plain card. On this card draw the net of the design you have chosen. Draw it to the correct size so that if you made the box it would just hold four balls. Attach the net to this page with a paper clip.

Example 33, students needed to draw the net (actual size) for one of the nets drawn in Example 32. The drawing had to be consistent with the net drawn previously and have dimensions within 4mm of the actual size required to hold the balls.

Students received partial credit if they drew the net of the box, but the dimensions were not consistent with actual sizes or one aspect of the drawing was incorrect (e.g., one or more flaps was missing).

Can the students in your classroom, school, district, or state use geometric models and diagrams to represent situations?

Example 33 Possible Answer



shown at 20% of actual size

Just 4 percent of students in the United States were able to translate one of the nets they drew in Example 33 into a net of dimensions that would actually hold the four balls (scaling the net to actual size). While 51 percent of students in Romania were able to, as were 30 percent of students internationally, the poor performance of U.S. eighth graders suggests that more work is needed in using concrete materials to create geometric shapes.

87

SAMPLE TEST ITEMS: ALGEBRA AND FUNCTIONS

AREA ONE: ALGEBRA AND FUNCTIONS

RECOGNIZE AND EXTEND PATTERNS AND RELATIONSHIPS

The *NCTM Standards* emphasize the importance of recognizing, describing, extending, and creating patterns because such experiences introduce the idea of relationships between variables. Students can progress from understanding a pattern as a predictable change in a single variable to learning how changes in one variable can relate to changes in a second variable. The ability to generalize about relationships between variables prepares students to work more formally with tabular, symbolic, and graphical representations of such relationships.

Beginning in the early grades, students can be asked to work with simple non-numeric patterns. As they progress in their mathematical understanding, the contexts for working with patterns can be more complex, involving tables, equations, and graphical representations. This section illustrates some assessment tasks in which students were asked to do the following:

- Recognize repeating non-numeric and numeric patterns by supplying the "next" element.
- Supply the missing elements in a pattern.
- Complete the next sequence in a pattern or provide elements other than one(s) that came after those given.
- Complete tables and sequences of terms based on a constant or nonconstant rate of increase.
- Explain their reason about whether or not a given number could be in a pattern.
- Translate patterns from one context to another.
- Create an example of a pattern.

- Extend or interpolate with a pattern, given a verbal description of the pattern.
- Determine the specified element in a pattern (e.g., 10th element) and describe how to find it.
- Identify the relationship between two patterns or sequences.
- Determine a specified element in a second pattern based on the relationship between two variables.
- Represent a pattern or relationship between two variables in different ways, including a verbal statement, tabular form, graphical form, or symbolic form.
- Understand and apply the concept of variable.

Example 1 is a TIMSS item based on a non-numeric pattern that relates in some respects to geometry visualization skills. It was relatively easy for fourth-grade students, with 63 percent of the students identifying the correct response, on average, across the 26 participating countries at that grade. In comparison, the item was answered correctly by 59 percent of the fourth graders in the United States and 87 percent of those in Korea.

TIMSS 1995

Example 1: Extend a Pattern

Here is the beginning of a pattern of tiles.

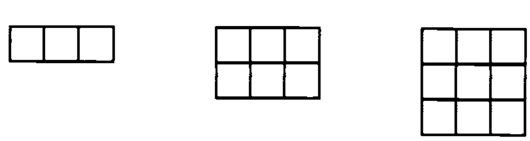


Figure 1 **Figure 2** **Figure 3**

If the pattern continues, how many tiles will be in Figure 6 ?

A. 12

B. 15

(C.) 18

D. 21

Example 1	Grade 4
United States	59
Korea	87
International Avg	63

% correct

89

Example 2, also a TIMSS fourth-grade item, was slightly more difficult for students internationally (57% correct, on average). It required students to explain verbally (subtract 4) or arithmetically (34-4) how to obtain the next number in the given sequence. The U.S. fourth graders (61% correct) performed above the international average on this item but below the achievement of the Dutch students (79% correct).

It should be noted that from a "purist" view, you cannot give a "missing" term without a rule for the nth term. However, the approach to pattern recognition assessed by Examples 1 and 2 is found in many textbooks. Both examples have wording that the pattern continues or that the numbers are part of a pattern.

Example 2	**Grade 4**
United States	61
Netherlands	79
International Avg	57
	% correct

TIMSS 1995

Example 2: Generalization About a Pattern

These numbers are part of a pattern.

50 , 46 , 42 , 38 , 34 , ...

What do you have to do to get the next number?

Answer: Subtract 4 from 34 (30)

· 90

Example 3 is an eighth-grade TIMSS item involving the relationship between two sequences of numbers. Presented in the short-answer format, it requires students to add a constant, although a different constant, to each of the two number sequences to determine the next number that will be seen in both sequences. Eighth graders in the United States performed slightly below the international average, with 42 percent responding correctly. Their counterparts in Korea had the highest achievement with 67 percent correct responses.

Problem situations involving patterns and sequences can involve constant or nonconstant rates of change, the value of which can either be given to students or left for them to determine. In their analysis of the 1992 NAEP algebra results, Blume and Heckman observe that patterns based on nonconstant rates of change can be more difficult for students than patterns based on constant rates of change.[16]

Example 3:
Relationship Between Sequences: Extend

TIMSS 1995

The numbers in the sequence 2, 7, 12, 17, 22, ... increase by fives. The numbers in the sequence 3, 10, 17, 24, 31, ... increase by sevens. The number 17 occurs in both sequences. If the two sequences are continued, what is the next number that will be seen in both sequences?

Answer: 52

Example 3	Grade 8
United States	42
Korea	67
International Avg.	45

% correct

Example 4, also from TIMSS, required eighth-grade students to understand the decreasing rate of the height of the bounce of the rubber ball. This task proved difficult for students, who seemed not to understand that the ball was bouncing both up and down with each decrease. The highest achievement was by eighth graders in Singapore, where 42 percent of the students responded correctly. Twenty-seven percent of the eighth graders in the United States selected the correct answer.

Example 4	Grade 8
United States	27
Singapore	42
International Avg.	34
	% correct

TIMSS 1995

Example 4:
Extend Pattern Given Verbal Description

A rubber ball rebounds to half the height it drops. If the ball is dropped from a rooftop 18 m above the ground, what is the total distance traveled by the time it hits the ground the third time?

A. 31.5 m

B. 40.5 m

(C.) 45 m

D. 63 m

Can the students in your classroom, school, district, or state complete sequences of terms based on constant and nonconstant rates of change?

92

AREA TWO: ALGEBRA AND FUNCTIONS

USE SYMBOLS TO REPRESENT SITUATIONS ALGEBRAICALLY

Students should know how to use symbols to represent situations algebraically, from basic formulas for everyday relationships to more abstract representations of situations using algebraic equations and expressions. The specifications for NAEP's 1996 assessment state that using open sentences and equations as representational tools should be included at grades 4, 8, and 12.

Any number of items can be developed that assess translation from verbal statements to symbolic form. However, students also can be asked to use multiple representations for situations to translate among diagrams, models, and symbolic expressions. Again, initial concepts in this area can be introduced in the primary grades, as students move from understanding how any symbol (such as a box) can represent a number to learning simple, and then, more complex algebraic notation.

Items developed in this area can require students to perform the following:

- Identify a symbolic situation representing a verbal description of a situation involving a variable.
- Translate a verbal description of a situation involving a variable into a symbolic expression representing the situation.
- Translate a statement about geometric figures into algebraic symbols.
- Translate tabular information into an algebraic expression.
- Develop an equation using variables to describe the general setting, given a table of values expressing a relationship between the number of objects, their individual costs, and the total cost.

93

TIMSS 1995

Example 5: Symbolic Representation

Tanya has read the first 78 pages in a book that is 130 pages long. Which number sentence could Tanya use to find the number of pages she must read to finish the book?

A. 130 + 78 = ☐

B. ☐ - 78 = 130

C. 130 ÷ 78 = ☐

(D) 130 - 78 = ☐

Example 5 is a TIMSS multiple-choice item given at grade 4. It demonstrates how the symbol of a box can be used in simple number sentences. In general, fourth graders in most countries did well on this item. Seventy-one percent of the students in the United States performed well above the international average; however, the highest achievement was in Korea (88% correct).

Example 6 was a TIMSS eighth-grade item that students in some countries found more difficult than might have been expected. Even though 82 percent of the students in Singapore answered correctly, the international average was 58 percent correct. Eighth graders in the United States performed well below the international average, with only 46 percent answering correctly. Nearly 40 percent of the U.S. eighth graders selected option C, indicating a basic confusion between the multiplicative and additive nature of the different notations shown.

Example 5	Grade 4
United States	71
Japan	88
International Avg.	62

% correct

Example 6	Grade 8
United States	46
Singapore	82
International Avg.	58

% correct

TIMSS 1995

Example 6: Representation and Notation

If m represents a positive number, which of these is equivalent to $m + m + m + m$?

A. $m + 4$

(B.) $4m$

C. m^4

D. $4(m + 1)$

Examples 7 (from TIMSS) and 8 (from NAEP) required eighth graders to identify how a verbal situation could be translated into an algebraic representation. It is clear from these examples that their understanding is far from complete. Just about 50 percent of the U.S. eighth graders selected the correct linear expression representing Clarissa's hats. This was in comparison to 86 percent of the eighth graders in Singapore. Results were similar on the NAEP item. In the 1996 assessment, 58 percent of the eighth graders selected the correct response.

Example 7: Representation of a Verbal Statement

TIMSS 1995

Juan has 5 fewer hats than Maria, and Clarissa has 3 times as many hats as Juan. If Maria has n hats, which of these represents the number of hats that Clarissa has?

A. $5 - 3n$

B. $3n$

C. $n - 5$

D. $3n - 5$

(E.) $3(n - 5)$

Example 7	Grade 8
United States	49
Singapore	86
International Avg.	47

% correct

Example 8: Representation of a Verbal Statement

NAEP 1996

A plumber charges customers $48 for each hour worked plus an additional $9 for travel. If h represents the number of hours worked, which of the following expressions could be used to calculate the plumber's total charge in dollars?

A. $48 + 9 + h$

B. $48 \times 9 \times h$

C. $48 + (9 \times h)$

D. $(48 \times 9) + h$

(E.) $(48 \times h) + 9$

Example 8	Grade 8
United States	58

% correct

95

AREA THREE: ALGEBRA AND FUNCTIONS

EVALUATE AND PERFORM OPERATIONS ON ALGEBRAIC EXPRESSIONS

This area covers simplifying and evaluating numerical and algebraic expressions, order of operations, exponents (such as cubic expressions), and roots. Essentially, the items in this area tend to be rather straightforward, in the sense that they measure traditional numerical and algebraic operations. Students can be asked to perform the following:

- Evaluate simple numeric and algebraic expressions.
- Evaluate expressions involving more complex order of operations (several decisions).
- Evaluate terms in a sequence.
- Evaluate expressions involving exponents.
- Solve problems involving substitution in expressions and formulas.
- Identify the positive square root of a nonnegative real number.
- Given an expression involving integral values of the variables involved, identify the expression in simplest form.

Example 9 is a TIMSS short-answer item given to eighth graders, requiring them to evaluate the given expression, which involved division. About 50 percent of the students internationally, on average, provided the correct answer of three or an equivalent. Alternative forms such as $^{18}/_{6}$, $^{9}/_{3}$, or $^{6}/_{2}$ also were accepted as correct. Fifty-seven percent of eighth graders in the United States performed above the international average on this item, but well below top-performing Singapore (80% correct).

Example 9	**Grade 8**
United States	57
Singapore	80
International Avg.	53
	% correct

TIMSS 1995

Example 9: Evaluate Expression

If $x = 2$, what is the value of $\frac{7x+4}{5x-4}$?

Answer: 3

Example 10, also from TIMSS, involved the subtraction of two terms of an algebraic expression. Since the fractions had the same denominator, students needed only to subtract the numerators (i.e., x from 2x). A more interesting problem might have been: $\frac{2x}{9} - \frac{x}{3}$ (answer = $\frac{x}{9}$). Still, more than 50 percent of the U.S. eighth graders had difficulty with the question as presented. Even though 42 percent selected the correct answer, about 22 percent selected option B or 2 as the correct answer, indicating they do not understand how to collect like algebraic terms.

Example 11, which involved operations with an algebraic expression using variables, was rather more difficult for the TIMSS eighth graders. Students in the United States performed slightly below the international average, with 37 percent answering correctly. In Hong Kong, however, 65 percent of the eighth graders selected the correct response.

TIMSS 1995

Example 10: Evaluate Expression

Subtract: $\frac{2x}{9} - \frac{x}{3}$

A. $\frac{1}{9}$

B. 2

C. x

(D.) $\frac{x}{9}$

E. $\frac{x}{81}$

Example 10	Grade 8
United States	42
Hong Kong	82
International Avg.	51

% correct

TIMSS 1995

Example 11: Order of Operations Involving Variables

Which one of the following is FALSE when *a*, *b*, and *c* are different real numbers?

A. $(a + b) + c = a + (b + c)$

B. $ab = ba$

C. $a + b = b + a$

D. $(ab)c = a(bc)$

(E.) $a - b = b - a$

Example 11	Grade 8
United States	37
Hong Kong	65
International Avg.	40

% correct

97

Example 12, from NAEP's 1992 mathematics assessment, involved exponents and division. Even though it was given at grade 12, some understanding of exponents is well within the range of the 1996 framework for grade 8. Twelfth-grade students were asked to simplify a quotient given in scientific notation, and about 50 percent of the students selected the correct answer. The most popular distracter was option B, selected by 27 percent of the students. This could indicate that these students knew to subtract the exponents but failed to recognize that in this case they needed to subtract 5 from 3.

Example 12	**Grade 12**
United States	48
	% correct

NAEP 1992

Example 12: Scientific Notation

$$\frac{6 \times 10^{3}}{3 \times 10^{5}} =$$

A. 0.5×10^{2}

B. 2×10^{2}

C. $2 \times 10^{0.6}$

D. 0.5×10^{-2}

(E.) 2×10^{-2}

How would the students in your classroom, school, district, or state perform on these types of items involving symbolic representation and scientific notation?

AREA FOUR: ALGEBRA AND FUNCTIONS

SOLVE LINEAR EQUATIONS AND INEQUALITIES

A substantial portion of the NAEP and TIMSS items dealt with the traditional algebra topics in this area, including solving and graphing equations and inequalities as well as solving and graphing systems of equations. Of the five algebra areas described in this assessment module, this was by far the largest in the TIMSS assessment. Together with the patterns area, this area also was well represented in the 1996 NAEP assessment.

The items requiring students to solve equations tended to fall into four major types: solving simple equations, solving more difficult equations, solving inequalities, and solving systems of equations. This area also includes using number lines and the rectangular coordinate system to represent and analyze relationships and using graphing to solve systems of equations. If one were discussing only the algebra and functions content area, the relationship between equations and their graphs would be treated as a separate topic area. However, since the topic area of graphical representation was introduced in the geometry section, it is simply expanded upon here under algebra.

Keeping in mind the many possibilities for equations, developing items in this topic area can require students to:

- Solve a linear equation with a whole number solution ($3x - 17 = 28$).
- Solve a linear equation with a rational solution ($4x + 15 = 38$).
- Solve a linear equation that requires two or more steps ($15x - 19 = -2x + 8$).
- Write the equation describing the situation illustrated.
- Solve an equation having a negative integral solution.
- Identify the integral pair solution of a system of two equations.
- Identify the equation of a line containing a given pair of points.
- Identify the graph of a line from the equation of the line.

Example 13 from TIMSS illustrates an item requiring students to solve a simple linear equation. Internationally, students did quite well on this item. The correct answer was selected by nearly three-fourths of U.S. eighth graders and nearly all of the students in Singapore. Example 14, written in a short-answer format, represented a more difficult equation for the TIMSS eighth graders to solve, perhaps because of the negative sign that appears in the left side of the equation or the fact that, for many students, the solution process involves two steps. The U.S. students performed below the level of the international average, with only 31 percent of U.S. eighth graders providing the correct solution to this equation. In contrast, 86 percent of the Japanese eighth graders provided the correct solution.

TIMSS 1995

Example 13: Simple Linear Equation

If $3(x + 5) = 30$, then $x =$

A. 2

(B.) 5

C. 10

D. 95

Example 13	Grade 8
United States	73
Singapore	96
International Avg.	73

% correct

TIMSS 1995

Example 14: Two-step Linear Equation

Find x if $10x - 15 = 5x + 20$

Answer 7

Example 14	Grade 8
United States	31
Japan	86
International Avg.	46

% correct

100

Example 15 represents a reasoning item in the context of an algebraic equation that was very difficult for the TIMSS eighth-grade students. The international average was 37 percent correct, and the best performance was 62 percent correct by the Japanese eighth graders. Only about 33 percent of the U.S. students answered correctly. About 33 percent selected option C as the correct answer and another 25 percent selected option D, again indicating little real understanding of algebraic notation by almost 70 percent of eighth-grade students in the United States.

TIMSS 1995

Example 15: Meaning of Equation

Brad wanted to find three consecutive whole numbers that add up to 81. He wrote the equation $(n-1) + n + (n+1) = 81$. What does n stand for?

A. The least of the three whole numbers

(B.) The middle whole number

C. The greatest of the three whole numbers

D. The difference between the least and greatest of the three whole numbers

Example 15	Grade 8
United States	32
Japan	62
International Avg.	37

% correct

101

Example 16 represents an item asking students to solve an inequality. It is difficult to understand why the TIMSS eighth graders had such difficulty with this rather routine problem unless students were simply unfamiliar with how to treat fractions in equations and inequalities. The U.S. students, however, performed above the international average, with 52 percent selecting the correct answer. The NAEP framework encompasses even more difficult inequality problems. None was available for inclusion here, but Example 17 was developed to illustrate this type of item. It involves two unknowns, a minus sign, and a fraction.

Example 16	Grade 8
United States	52
Japan	69
Singapore	69
International Avg.	44

% correct

TIMSS 1995

Example 16: Simple Inequality

$x/2 < 7$ is equivalent to

A. $x < 7/2$

B. $x < 5$

(C.) $x < 14$

D. $x > 5$

E. $x > 14$

Sample Item for Illustrative Purposes (no results available)

Example 17: Complex Inequality

Which of the following is equivalent to $2y - 3x > 4$?

A. $y < \frac{2}{3}x - 2$

B. $y < \frac{3}{2}x + 2$

C. $y > \frac{3}{2}x - 2$

D. $y > \frac{2}{3}x + 2$

(E.) $y > \frac{3}{2}x + 2$

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Example 18 is a 1996 NAEP item involving an ordered pair solution to a linear equation. Even though students needed only to try the various options given to solve the equation, only 42 percent of the eighth graders responded correctly. Here again, however, the NAEP framework calls for much more challenging items, such as those requiring solving systems of linear equations in one and two variables. Example 19, developed especially for this module, illustrates a solution to a pair of linear equations that consists of an ordered pair (x, y) where neither x nor y is an integer. Students cannot easily substitute the values given in the answer choices to determine the solution. For example, if 1 is substituted for x in the first equation, then y = 3. But x = 1 and y = 3 does not satisfy the second equation, since 1 - 2(3) does not equal 7.

NAEP 1996

Example 18: Linear Equation Ordered Pair Solution

Which of the following ordered pairs (x,y) is a solution to the equation $2x - 3y = 6$?

A. (6, 3)

(B.) (3, 0)

C. (3, 2)

D. (2, 3)

E. (0, 3)

Example 19: Simultaneous Equations Non-integer Solution

If $2x + y = 5$ and $x - 2y = 7$, then x equals

A. $-17/15$

B. -1

C. 1

(D.) $17/15$

E. $17/3$

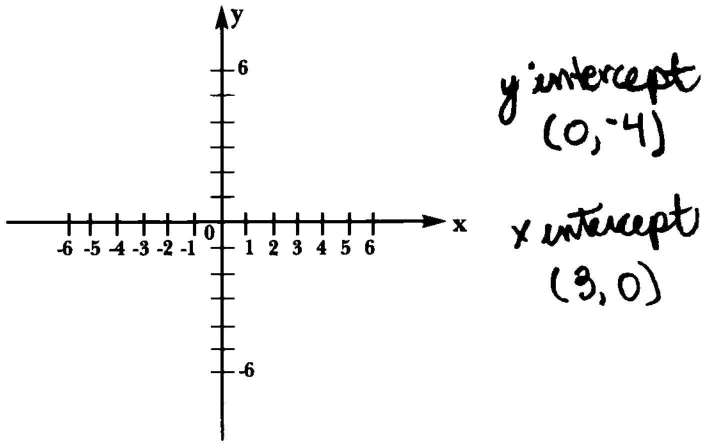
Example 18	Grade 8
United States	42
	% correct

▲ Sample Item for Illustrative Purposes (no results available)

Example 21 provides students with a grid and asks them to sketch the graph of an equation. In the drawing shown, the y-intercept is (0,-4) and the x-intercept is (3,0). This is because some students are likely to determine that x = 0, then y = -4 and that if y = 0, then x = 3. However, students also could determine any other two points, or any one point that satisfies the equation (i.e., 6, 4) and then plot the slope of the line (4/3). Thus, any sketch of the line that correctly intersects the x- and y-axes would be considered a correct solution.

Example 21: Graph an Equation

In the *xy* plane shown below, sketch the graph of $4x - 3y = 12$



Can the students in your classroom, school, district, or state solve linear equations and inequalities?

▲
Sample Item for Illustrative Purposes (no results available)

105

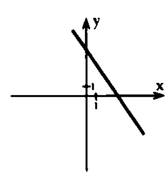
Examples 20 and 21 also were developed especially for this module to illustrate two types of items that assess students' understanding of the relationship between algebraic and graphical representations of equations. In Example 20, students are given an equation and asked to identify a portion of its graph. This item requires students to recognize from the given equation that the y-intercept is negative (when x = 0, y = -5) and E is the only graph where the y-intercept is negative. Also, the slope is positive and, therefore, the line must rise from left to right.

Sample Item for Illustrative Purposes ▶
(no results available)

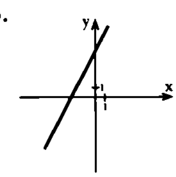
Example 20: Identify Graph of Equation

Which of the following could be a portion of the graph of y = 2x - 5 ?

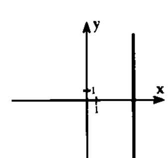
A.



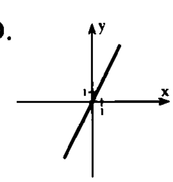
B.



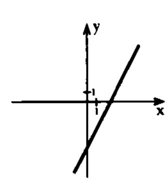
C.



D.



(E.)



AREA FIVE: Algebra and Functions

Solve Real-World Problems Using Algebraic Models and Explain Relationships Involving Algebraic Concepts

The primary reasons for the study of algebra as an important branch of mathematics are (1) to provide students with a knowledge of the fundamental concepts and processes of that branch, (2) to increase and enhance the ability of students to reason and become more adept problem solvers, and (3) to help them communicate and make connections across the various branches of mathematics or from mathematics to other disciplines, such as science. The 1996 NAEP mathematics framework refers to goals two and three above as "mathematical power." The desire to develop mathematical power for all students is a major message of the *NCTM Standards.* The NAEP mathematics assessments have reflected the growing importance of mathematical power in measuring student achievement, namely by including a greater number of constructed-response questions that focus on reasoning, by requiring students to communicate, and by requiring students to connect their learning across NAEP content strands. Although the TIMSS framework does not specifically mention mathematical power per se, the TIMSS assessment does contain items that measure the general concept of mathematical power.

Following are three items that pose problems within a contextual setting that measure algebraic concepts and various aspects of mathematical power: reasoning, connections, and communication.

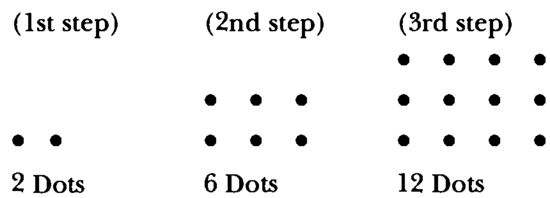
NAEP 1992

Example 22: Marcy's Dot Pattern

Example 22: Marcy's Dot Pattern

This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show all your work.

A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues infinitely.



Marcy has to determine the number of dots in the 20th step, but she does not want to draw all 20 pictures and then count the dots.

Explain or show how she could do this and give the answer that Marcy should get for the number of dots.

(Students had access to a calculator.)

Example 22	Grade 8
United States	
No Response	16
Incorrect	63
Minimal	10
Partial	6
Satisfactory	1
Extended	5

% of students in response category

Example 22 required students to analyze several steps in a pattern of dots in order to conjecture about a general rule for determining the number of dots for any particular step in the pattern. Additionally, students were required to use their rule to find the number of dots at a particular step in an extension of the pattern where it no longer is convenient to draw all of the intermediate dot figures. One approach is to think of the steps in the pattern as consisting of dots in rows and columns and to realize that the number of dots in the n^{th} step can be expressed as $x_n = n(n + 1)$ for $n = 1, 2, 3...$ and thus $x_{20} = 20(20 + 1) = (20)21 = 420$.

Other approaches are possible and students could use arithmetic or algebraic concepts to explain their reasoning. Although a few students did write an algebraic equation to express a rule for the general term in a recursive relationship, it was neither expected nor necessary for students to do so.

Example 22 Possible Solution

The explanation should include one of the following ideas with no false statements:

1) For each successive step, the number of rows and the number of columns is increasing by 1, forming a pattern. For example, the first step shows a pattern of dots that consists of 1 row by 2 columns, the second step shows a pattern of dots that consists of 2 rows by 3 columns, the third step 3 rows by 4 columns, and so on. Continuing in this pattern, the 20th step would have 20 x 21, or 420, dots.

OR

2) Look at successive differences between consecutive steps. The differences 4, 6, 8, 10, ... form a pattern. There are 19 differences forming the pattern 4, 6, 8, 10, ... 38, 40, and this sum is equal to (9 x 44) + 22, or 418. However, 2 must be added for the first step, yielding a response of 420.

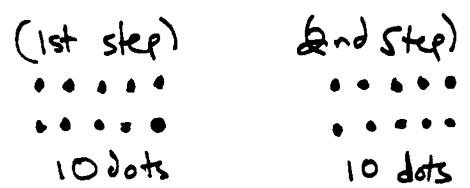
Most of the eighth graders fell short of detecting and communicating a pattern in the dots. Nationally, 16 percent did not respond and 63 percent provided irrelevant or inaccurate information about the dot pattern. Of the students who were able to detect the pattern, most were not able to complete the step of generalizing to a rule that could be used to find the dots in any term. Ten percent were able to demonstrate their understanding of the pattern by working some number of the terms beyond those given and another 6 percent tried to provide a generalization. Only 6 percent of the students provided satisfactory or better responses. The 1 percent of the students providing satisfactory rather than extended responses provided a generalization but failed to apply it to the 20th term. The remaining 5 percent worked on the problem in full.

What teaching strategies could have helped students learn to respond appropriately to question 22?

NAEP 1992

Example 22
Incorrect Response

The response is completely incorrect or irrelevant, or the response states, "I don't know."



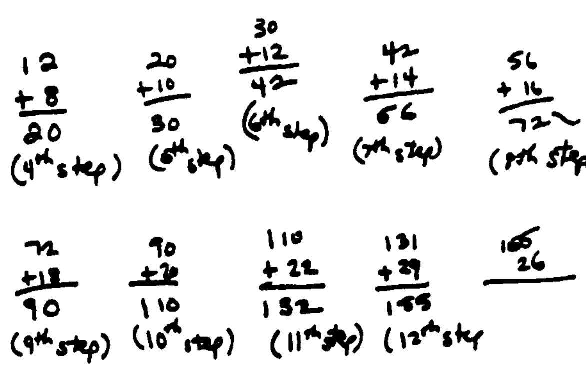
It is difficult to discern an explanation for this INCORRECT response. One possibility is that the student apportioned the total of 20 dots in the three steps shown into two 2 x 5 sets.

63% of U.S. students provided an Incorrect response

NAEP 1992

Example 22
Minimal Response

An attempt to generalize the pattern on a superficial level or to draw all 20 pictures in the pattern (with a clear understanding of the pattern).



This MINIMAL response illustrates a student's attempt to display the first 12 steps in the pattern. There is some understanding of the number of total dots in each entry but no attempt is made to explain the pattern in terms of rows and columns.

10% of U.S. students provided a Minimal response

NAEP 1992

Example 22
Partial Response

The response has communicated a partially correct generalization of the pattern.

When the pattern starts with 2 dots the next step is to add 4 dots to it and the 3rd step is to add 6 dots to it so every time there is a new step you add 2 dots to the last amount you added on to the last step. You would multiply two dots on and on until you reached the 20th step

This PARTIAL response does begin to formulate an explanation of the total number of dots for an entry. However, the last sentence incorrectly uses the term "multiply" in an attempt to discuss the 20th step. At this point, the explanation falters.

6% of U.S. students provided a Partial response

NAEP 1992

Example 22
Satisfactory Response

The response contains a completely correct generalization of the pattern but does not include – or incorrectly states – the number of dots (420) in the 20th step.

Multiply each step by 1# higher such as

1 = 1×2
2 = 2×3
3 = 3×4
4 = 4×5

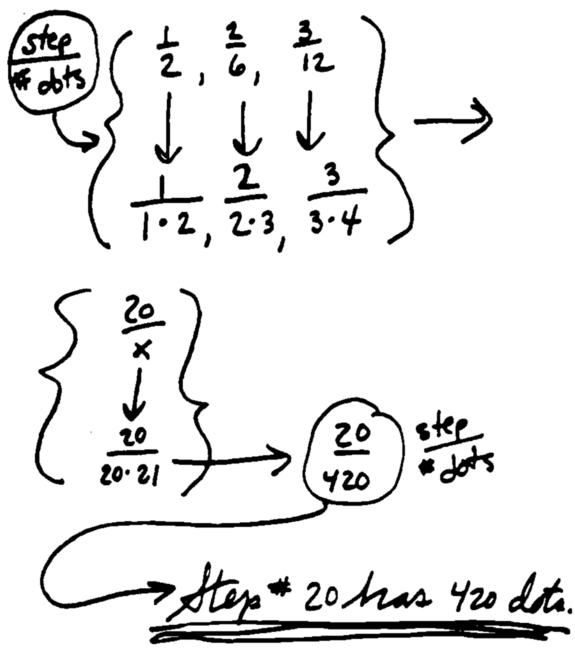
This SATISFACTORY response provides sufficient evidence of how to generate the various steps in the pattern by multiplying the number of rows times the number of columns. However, the student does not determine the number of dots in the 20th step.

1% of U.S. students provided a Satisfactory response

NAEP 1992

Example 22
Extended Response

This response contains a completely correct generalization of the pattern and specifies that there are 420 dots in the 20th step.



This is a strong EXTENDED response. The student clearly related the number of dots in a step to an appropriate multiplication rule. This student then moves directly from step 3 to step 20 and determines the correct number of dots for that step.

5% of U.S. students provided an Extended response

111

Example 23 from TIMSS illustrates an item provided in a contextual setting that could be solved using either arithmetic or algebraic approaches. Because the approach to solving the problem is left to the student, the item provides a measure of whether students can apply their understanding of mathematics to situations that require reasoning. Generally, students who solved this problem successfully either used one of two distinct arithmetic approaches or an algebraic approach, such as setting up an equation or system of equations involving one or two variables. Eighth graders in the United States performed below the international average, with only 25 percent providing a correct solution. In Singapore, the highest performing country on this problem, 71 percent of the students provided a correct solution.

Example 23	**Grade 8**
United States	
Correctness	25
Method	26
Singapore	
Correctness	71
Method	76
International Average	
Correctness	32
Method	34

% of students obtaining full credit

TIMSS 1995

Example 23: Apples in a Box

There are 54 kilograms of apples in two boxes.
The second box of apples weighs 12 kilograms more than the first. How many kilograms of apples are in each box? Show your work.

112

As illustrated by the student responses shown in the following pages, one arithmetic method involved dividing 54 by 2, and then both adding 6 to 27 to obtain 33 kg and subtracting 6 from 27 to obtain 21 kg. In the other frequently used arithmetic approach, students first subtracted 12 from 54 to obtain 42, and then divided by 2 to obtain the 21 kg weight of one box. The weight of the second box, 33 kg, was arrived at by adding 12 to 21.

The illustrative responses also depict algebraic approaches. One method makes use of a linear equation in one variable, n. The use of the variable n as the number of kg of apples in each box, leads to the equation $n + n + 12 = 54$. Correctly solving this equation for $n = 21$ and $n + 12 = 33$ yields the correct answers of 21 kg and 33 kg. The other algebraic method makes use of a system of linear equations that can be solved simultaneously.

The *NCTM Standards* suggest that assessments should yield evidence on students' use of strategies and problem-solving techniques. For this open-ended item, TIMSS not only scored the students' responses for correctness, but kept track of the methods students used. In this way, TIMSS was able to collect information about how many students demonstrated use of an appropriate method (whether it be arithmetic or algebraic), but did not implement it fully by making some type of computational error.

In the United States, the results indicate that most students simply did not know how to approach this problem or could not demonstrate their understanding. Just 26 percent displayed an appropriate method by which a correct solution could have been determined, only 1 percent more than provided a correct solution. In Singapore, 76 percent of the eighth graders displayed a correct method compared to the 71 percent who provided correct responses. About 75 percent of the Singaporean eighth graders communicated their understanding of how to correctly solve the "Apples in a Box" problem compared to only about 25 percent in the United States.

TIMSS 1995

Example 23 Possible Solution A

$$2\overline{)54}\quad = 27$$

$$27 + 6 = 33 \qquad 27 - 6 = 21$$

① 21 ② 33

TIMSS 1995

Example 23 Possible Solution B

$$12 + 21 = 33$$

$$54 - 12 = 42 \qquad 2\overline{)42} = 21$$

21 kl and 33 kl.

114

TIMSS 1995

Example 23 Possible Solution C

Let n be the number of kg of apples in the first box. and n + 12 be the number of kg of apples in the second box.

$$n + n + 12 = 54$$
$$2n = 42$$
$$n = 21$$

There are 21 kg in the first box and 21 + 12 = 33 kg in the second box.

TIMSS 1995

Example 23 Possible Solution D

Let L be the larger number of kg of apples in a box. Let S be the smaller number of kg of apples in a box

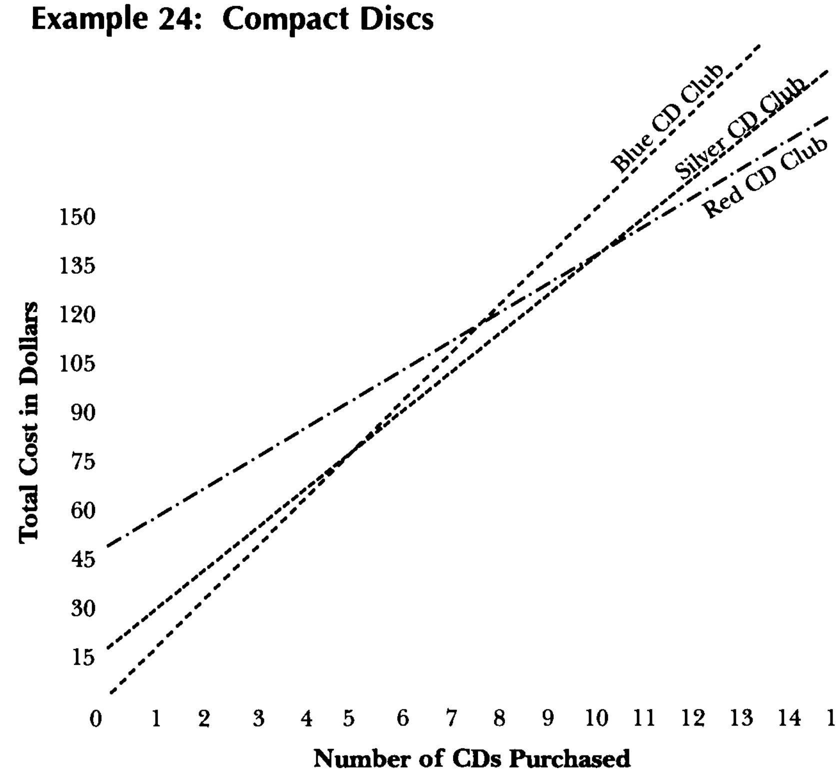
$$L + S = 54$$
$$L - S = 12$$
Add $2L = 66$
$$L = 33$$
$33 - S = 12$ leads to $S = 21$

The answers are 21 kg and 33 kg

Example 24 emphasizes the importance of the connections between equivalent representations by presenting students with a graph that models the costs for purchasing CDs from three CD clubs. Although this item was not administered in NAEP or TIMSS, it represents the type of algebraic reasoning and problem solving that might well be expected of eighth graders taking algebra courses. More specifically, the problem requires students to display their facility in interpreting the graphical properties of lines that model costs.

Sample Item for Illustrative Purposes (no results available)

Example 24: Compact Discs



The graphs above show the cost for purchasing CDs (Compact Discs) from the Blue, Silver, and Red CD clubs. The questions below are based on these graphs.

a) Which CD club has the greatest membership fee (initial cost)? Give a reason for your answer.

b) Which CD club charges the most for each additional CD purchased? Give a reason for your answer.

c) For what range of numbers of CDs does each CD club have the lowest total cost? Give a reason for your answer.

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In providing their responses, students will need to communicate an understanding of essential algebraic concepts that goes beyond the skill of selecting and applying standard algorithms.

Example 24 Solution

a) The Red CD Club. Reason: The line graph representing the Red CD Club has the greatest y-intercept, 45, which represents the membership fee (initial cost).

b) The Blue CD Club. Reason: The line graph representing the Blue CD Club is steepest or has the greatest slope, 15.

c) The Blue CD Club has the lowest total cost for 1 to 4 CDs. The Silver CD Club has the lowest total cost for 6 to 9 CDs. The Red CD Club has the lowest total cost for 11 or more CDs.

Note: For 5 CDs the Blue and Silver Clubs have the same total cost, $75, and for 10 CDs the Red and Silver Clubs have the same total cost, $135.

Reason: In order for the total cost to be the lowest for a given CD club, the line graph representing the costs for a given CD club must lie below the line graphs representing the costs for the other two CD clubs.

CONCLUSION

Attaining Excellence: TIMSS as a Starting Point to Examine U.S. Mathematics Assessments is a component of *Attaining Excellence: A TIMSS Resource Kit*, which examines U.S. education in an international context. Earlier modules of the resource kit included modules on student achievement, teaching, and curricula. All modules are designed to use national and international studies such as the Third International Mathematics and Science Study (TIMSS), the National Assessment of Educational Progress (NAEP), and findings of similar and related research to help inform our understanding of instructional practice and student achievement in the United States as compared to other countries. Collectively, the modules suggest how the information generated from TIMSS can promote identification of the strengths and weaknesses of education at all levels of the education enterprise in the United States and to make reasonable comparisons with other countries that may be our economic competitors.

In the United States today, mastering mathematics has become more important than ever. Students with a strong grasp of mathematics have an advantage in academics and in the job market. The U.S. results of TIMSS confirm that many of our students enter and leave high school without a solid grounding in mathematics, closing doors very early for further education.

The TIMSS Assessment Module is intended to provide state- and district-level assessment and curriculum specialists with a method of comparing existing assessment frameworks to those found in NAEP and TIMSS. The module also suggests strategies for using materials from NAEP and TIMSS to reflect on local curricula.

Important in achieving these goals is the need for a benchmark with which to compare state and local frameworks. As states and districts continue to develop and implement mathematics standards designed to improve students' skills and to raise student achievement, the need for information about how their efforts compare with those of the nation and of the world increases. NAEP and TIMSS can give educators additional tools with which to benchmark student performance and to examine the curricular expectations associated with that performance. The module provides an overview of the NAEP and TIMSS frameworks, focusing on how the frameworks were developed, describing the major content areas covered in each of the assessments, and tying them to the underlying abilities assessed, such as problem solving and understanding concepts. By viewing state and local systems through the prism of other countries' work, American educators can understand their own practices better and identify possible alternatives for making their unique processes of education more effective.

The module shows how NAEP and TIMSS compare in the areas of geometry and algebra because the U.S. TIMSS results consistently show the need for greater concentration on advanced mathematics topics. This module helps to frame specific questions concerning course options, content rigor, and opportunities to solve geometry and algebra problems as compared to other states and other countries. In proceeding from the global to the specific level, it assists in establishing processes that used to examine the content and abilities contained in frameworks, the relative emphasis of the content and performance areas, the range of assessment item types, and the difficulty of test items not only for geometry and algebra, but all of mathematics, and indeed the entire curriculum.

The TIMSS experience has shown the process of comparison provides a basis for much insightful discussion and consideration about the goals of education and the various ways to achieve those goals. This module enhances that discussion and provides some of the tools needed to achieve educational goals by helping to bridge the gap between assessment and practice.

If you would like to inquire further about NAEP and TIMSS assessments, please see the section of this publication entitled, "Obtaining Publications for Further Study," for information about additional publications and reports.

ENDNOTES

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13 National Assessment Governing Board (1996). *Mathematics Framework for the 1996 National Assessment of Educational Progress*. Washington, DC: Author.

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15 Blume, G.W. and Strutchens, M.E. (1997), "What Do Students Know About Geometry?" in *Results from the Sixth Mathematics Assessment of the National Assessment of Educational Progress*, P.A. Kenney and E.A. Silver (eds.), Reston, VA: National Council of Teachers of Mathematics.

16 Blume, G.W. and Heckman, D.S. (1997), "What Do Students Know About Algebra?" in *Results from the Sixth Mathematics Assessment of the National Assessment of Educational Progress*, P.A. Kenney and E.A. Silver (eds.), Reston, VA: National Council of Teachers of Mathematics.

OBTAINING PUBLICATIONS FOR FURTHER STUDY

OBTAINING TIMSS MATHEMATICS PUBLICATIONS

TIMSS International Reports

Mathematics Achievement in the Primary School Years: IEA's Third International Mathematics and Science Study, $20.00 (prepaid)

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Performance Assessment in IEA's Third International Mathematics and Science Study, $10.00 (prepaid)

➽ To order, contact the TIMSS International Study Center, CSTEEP, Campion Hall 323, Boston College, Chestnut Hill, MA 02167; telephone (617) 552-4521; fax (617) 552-8419; e-mail: timss@bc.edu.

Mathematics Achievement in the Middle School Years: IEA's Third International Mathematics and Science Study, $18 through GPO, stock # 065-000-01023-2

➽ To order, contact the Superintendent of Documents, P.O. Box 371954, Pittsburgh, PA 15250-7954; telephone (202) 512-1800; fax (202) 512-2250.

The TIMSS international reports are also available on the World Wide Web: http//www.csteep.bc.edu/timss.

TIMSS Released Item Sets

TIMSS Mathematics Items, Released Set for Population 1 (Third and Fourth Grades), $20.00 (prepaid)

TIMSS Mathematics Items, Released Set for Population 2 (Seventh and Eighth Grades), $20.00 (prepaid)

TIMSS Released Item Set for the Final Year of Secondary School: Mathematics and Science Literacy, Advanced Mathematics, and Physics, $20.00 (prepaid)

➽ To order, contact the TIMSS International Study Center, CSTEEP, Campion Hall 323, Boston College, Chestnut Hill, MA 02167; telephone (617) 552-4521; fax (617) 552-8419; e-mail: timss@bc.edu.

The TIMSS released item sets are also available on the World Wide Web: http://www.csteep.bc.edu/timss

TIMSS Curriculum Frameworks

TIMSS Monograph Series No. 1: Curriculum Frameworks for Mathematics and Science, 1993

TIMSS Monograph Series No. 2: Mathematics Textbooks: A Comparative Study of Grade 8 Texts, 1995

➻ To order, contact the Pacific Educational Press, Faculty of Education, University of British Columbia, Vancouver, Canada, V6T 1Z4; telephone (604) 822-5385; fax (604) 822-6603.

TIMSS Resource Kit

Attaining Excellence: A TIMSS Resource Kit, $94

➻ To order, contact the Superintendent of Documents, P.O. Box 371954, Pittsburgh, PA 15250-7954; telephone (202) 512-1800; fax (202) 512-2250.

U.S. TIMSS Achievement Reports

Pursuing Excellence: A Study of U.S. Fourth-Grade Mathematics and Science Achievement in International Context, $4.75, GPO Stock # 065-000-01018-6

Pursuing Excellence: A Study of U.S. Eighth-Grade Mathematics and Science Teaching, Learning, Curriculum, and Achievement in International Context, $9.50, GPO Stock # 065-000-00959-5

Pursuing Excellence: A Study of U.S. Twelfth-Grade Mathematics and Science Achievement in International Context, $12.00, GPO Stock # 065-000-01181-6

➻ To order, contact the Superintendent of Documents, P.O. Box 371954, Pittsburgh, PA 15250-7954; telephone (202) 512-1800; fax (202) 512-2250. These reports are also available on the World Wide Web: http://www.nces.ed.gov/timss.

Linking the National Assessment of Educational Progress (NAEP) and the Third International Mathematics and Science Study (TIMSS): A Technical Report

➻ To obtain single copies of the report or ordering information on other U.S. Department of Education products, call toll-free 1-877-4ED-PUBS. This report is also available on the World Wide Web: http://www.nces.ed.gov/pubs98/linking/index.html.

Obtaining NAEP Mathematics Publications

NAEP 1996 Report Card

NAEP 1996 Mathematics Report for the Nation and the States: Findings from the National Assessment of Educational Progress

➻ To obtain single copies of this report or ordering information on other U.S. Department of Education Products, call toll-free 1-877-4ED-PUBS. This report is also available on the World Wide Web: http://www.ed.gov/NCES/naep.

NAEP 1996 Mathematics Framework

Mathematics Framework for the 1996 National Assessment of Educational Progress

➻ To order, contact the National Assessment Governing Board, 800 North Capital Street NW, Suite 825, Washington, DC 20002. This document is also available on the World Wide Web: http://www.nagb.org/abnaep.html.

Implications of NAEP for Policies and Practices

School Policies and Practices Affecting Instruction in Mathematics: Findings from the National Assessment of Educational Progress

➻ This report is available on the World Wide Web: http://www.ed.gov/NCES/naep.

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