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ABSTRACT

A 3-year high school program based on functional mathematics (FM) benefits all students, regardless of career goals, by providing a strong background for students entering the workforce and for those moving directly into postsecondary education. A high school curriculum that helps all students master FM would effectively address issues of both equity and competitiveness. All students would have equal opportunity to master the mathematics required for the new world of work. To achieve its goal of preparing students both for work and for further education, FM respects the many parts of the traditional curriculum that are broadly useful, while including topics such as spreadsheets, data analysis, and statistical quality control that are hardly ever part of high school mathematics. In FM, utility is center stage. The elements of FM can be embedded in many different curricula, but any mathematics curriculum designed on functional grounds emphasizes authentic applications from everyday life and work. It requires appropriate content, authentic contexts, engaging tasks, and active learning. FM provides a rich foundation of experience and examples on which students can build subsequent abstractions and generalizations. It addresses many needs that are otherwise often neglected, including the technical and problem-solving needs of the contemporary workforce or the modern demands of active citizenship. (Appendixes contain the elements of FM. Contains 51 references.) (YLB)

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National Center for Research in Vocational Education
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Beyond Eighth Grade: Functional Mathematics for Life and Work

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Contents

Acknowledgments

Beginning in 1996, several industry associations and coalitions published occupational skill standards to document the entry-level expectations of modern high-performance industries. At the same time, the Institute on Education and the Economy (IEE) at Teachers College, Columbia University, organized a series of meetings to stimulate discussion about integrating academic and industry skill standards. The meetings and subsequent publications were undertaken by IEE in its capacity as a site of the National Center for Research in Vocational Education. One such meeting, held in November 1997 at the Arden Conference Center in Harriman, New York, focused on mathematics. *Beyond Eighth Grade: Functional Mathematics for Life and Work* is one outcome of that meeting, a summary of issues and an interpretation of ideas that blend desires of employers with the expectations of academics. We wish to thank Thomas Bailey, Director of IEE, for suggesting and supporting this work, as well as IEE staff members Eric Larsen, Donna Merritt, and Lisa Rothman for their help with editorial and publishing details.

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Executive Summary

Mathematics is the key to many of the most secure and financially rewarding careers in every sector of the economy. The impact of computers and information technology can be seen not just in engineering and science, but in such diverse areas as manufacturing, agriculture, health care, and advertising. To be prepared for careers in virtually any industry, and especially for changing careers during a lifetime, secondary school students need to learn a substantial core of mathematics. However, this core is not like either the abstract pre-engineering mathematics of the academic curriculum or the restricted topics of the discredited "vocational math." New approaches are needed to meet today's challenges.

For most of this century, schools have been organized to prepare some students for college and others for work. In the future, most jobs and all careers will require some form of postsecondary education. Thus, all students, regardless of career goals, can benefit from a mathematics curriculum that prepares them for both work and higher education. A three-year high school program based on "functional" mathematics can do just that: provide a strong background for students entering the workforce as well as for those moving directly into postsecondary education.

A high school curriculum that helps all students master functional mathematics would effectively address issues of both equity and competitiveness. All students would have equal opportunity to master the mathematics required for the new world of work. Moreover, a three-year core of functional mathematics would give students a strong foundation from which to enter the workplace or pursue advanced education.

Most of the elements of functional mathematics are unsurprising. Percentages and ratios; linear and quadratic equations; areas, angles, and volumes; and exponential growth and trigonometric relations are included in any strong high school mathematics program. Although some topics in functional mathematics are uncommon (e.g., index numbers, tolerances, three-dimensional geometry, indirect measurement, financial mathematics), most are taught in any high-quality high school mathematics program. To achieve its goal of preparing students both for work and for further education, functional mathematics respects the many parts of the traditional curriculum that are broadly useful, even while including topics such as spreadsheets, data analysis, and statistical quality control that are hardly ever part of high school mathematics. In functional mathematics, utility is center stage.

The elements of functional mathematics can be embedded in many different curricula. However, any mathematics curriculum designed on functional grounds—whether organized around external contexts or mathematical themes—will emphasize authentic applications from everyday life and work. In such a curriculum, students will gain considerable experience with mathematical tasks that are concrete yet sophisticated, conceptually simple yet cognitively complex. By highlighting the rich mathematics embedded in everyday tasks, this approach (in contrast to traditional "vocational math") can dispel both minimalist views about the mathematics required for work and elitist views of academic mathematics as an area with little to learn from work-based problems.

A curriculum based on functional mathematics requires appropriate content, authentic contexts, engaging tasks, and active learning. By featuring mathematics in common contexts, a functional curriculum can motivate students to link meaning with mathematics. Such contexts invite variations that can propel students to deep understanding and stimulate mathematical habits of mind.

Functional mathematics channels the much-criticized "mile-wide, inch-deep" curricular river into a narrower stream of ideas and procedures that reinforce each other as students progress through school and college and on into careers. It provides a rich foundation of experience and examples on which students can build subsequent abstractions and generalizations. Indeed, to fulfill its goals, a functional curriculum must leave students well prepared not only for work but also for subsequent courses in more abstract mathematics.

Functional mathematics addresses many needs that are otherwise often neglected. Neither traditional college-preparatory mathematics curricula nor the newer standards-inspired curricula were designed specifically to meet either the technical and problem-solving needs of the contemporary workforce or the modern demands of active citizenship. Although each includes much that is of value for employment and citizenship, neither provides the context, motivation, or balance of mathematical topics necessary for employees or citizens in a data-drenched world.

Indeed, for most students, functional mathematics will provide better preparation than current high school curricula. Functional mathematics stresses an in-depth understanding of topics that are most likely to be used by large numbers of people. By employing concrete methods in settings that are both complex and realistic, functional mathematics pushes all students to draw on the full breadth of mathematics. In short, by focusing on what is useful, functional mathematics increases both student interest and student learning.

Beyond Eighth Grade: Functional Mathematics for Life and Work

For most of this century, schools have been organized to prepare some students for college and others for work. But the world of work is changing, much of it requiring increased use of data and computers, measurements and graphs. In the twenty-first century, most jobs and all careers will require some form of postsecondary education. Thus, all students, regardless of career goals, will benefit from a curriculum that prepares them for both work and higher education. Instead of either/or, schools must now focus on both/and.

In this paper, we propose a three-year high school program based on *functional mathematics* that prepares students for life and work. Typically, classroom applications are designed to serve mathematics—to offer context, to illustrate use, to motivate new concepts, or to integrate topics. In functional mathematics, the priorities are reversed. Instead of applications being introduced to help students learn a predefined body of knowledge, the mathematical topics are selected to prepare students to cope with common problems they will face in life and work. Examples of such real-life problems appear throughout this paper to illustrate in a concrete way the nature of functional mathematics.

Functional mathematics, if thoughtfully and rigorously developed, can provide a strong background both for students entering the workforce and for those moving directly into postsecondary education. Indeed, for most students, functional mathematics will provide better preparation than the current high school curricula. Functional mathematics stresses an in-depth understanding of fundamental topics that are most likely to be used by large numbers of people. By employing concrete tools in settings that are both complex and realistic, functional mathematics pushes students to draw on the full breadth of mathematics. In short, focusing on useful mathematics increases total learning.

Given the complete record of a soccer league that ended in a three-way tie, devise a fair means of determining which team should be crowned champion.

Need and Urgency

Despite mathematics' reputation as an ancient subject consisting of indisputable facts, mathematics education has recently become the source of passionate public debate. At stake is nothing less than the fundamental nature of school mathematics: its content (what should be taught), pedagogy (how it should be taught), and assessment (what should be expected). At times, these "math wars" have become so heated that Education Secretary Richard Riley (1998b) has issued a public call for a truce.

At the risk of oversimplifying, this debate can be characterized as a clash between "traditionalists" who expect schools to provide the kind of well-focused mathematics curriculum that colleges have historically expected and "reformers" who espouse a broader curriculum that incorporates uses of technology, data analysis, and modern applications of mathematics. The reform approach is championed by the National Council of Teachers of Mathematics, whose standards (NCTM, 1989) advocate a robust eleven-year core curriculum for all students, with supplementary topics for those who are "college-intending." Critics argue, however, that the NCTM standards are diffuse and ambiguous (Cheney, 1997; Raimi & Braden, 1998), that they are based on questionable assumptions about how students learn (Anderson, Reder, & Simon, 1997), and that curricula based on these standards will not provide the kind of rigorous preparation students need to succeed in calculus and other college-level mathematics courses (Wu, 1997).

Largely left out of this debate is a major constituency of mathematics education: employers. In today's information age, economic prosperity—for individuals as well as for the nation—depends on "working smarter, not just working harder." Yet, a majority of America's businesses report deficiencies in the technical and problem-solving skills of their workers and a severe shortage of prospective employees with these requisite skills.

The cost of finding skilled employees has become a serious impediment to growth in many sectors of the U.S. economy (Carnevale, 1998).

Also left out are the voices of democracy and citizenship which were so important in the development of public education in the United States. Now, two centuries later, quantitative literacy is every bit as important as verbal literacy for informed participation in civic affairs. Today's news is not only grounded in quantitative issues (e.g., budgets, profits, inflation, global warming, weather probabilities) but is also presented in mathematical language (e.g., graphs, percentages, charts).

Neither traditional college-preparatory mathematics curricula nor the newer standards-inspired curricula were designed specifically to meet either the technical and problem-solving needs of the modern workforce or the modern demands of active citizenship. Although each includes much that is of value for citizenship and employment, neither provides the context, motivation, or balance of mathematical topics necessary for citizens or prospective employees in a data-drenched world.

The common curricular alternative—vocational or consumer mathematics—is significantly worse. Historically, vocational mathematics has provided only a narrow range of skills limited to middle school topics and devoid of conceptual understanding (National Center for Education Statistics [NCES], 1996). Such programs leave students totally unprepared—not only for modern work and postsecondary education, but even for advanced secondary school mathematics. Although some innovative school-to-career programs are seeking to change this pattern of low expectations, the vast majority of secondary schools in the United States offer students no effective option for mathematics education that meets the expectations of today's high-performance workplace.

A student plans to take out a $10,000 loan at 7% interest with monthly payments of $120, but before she closes the deal, interest rates rise to 7.5%. What will happen if she keeps her monthly payments at $120?

Employment and Education

Mathematics is the key to many of the most secure and financially rewarding careers in every sector of the economy (Business Coalition for Education Reform, 1998). The impact of computers and information technology can be seen not just in engineering and science, but in such diverse areas as manufacturing and agriculture, health care and advertising. To be prepared for careers in virtually any industry, and especially for changing careers during a lifetime, secondary school students need to learn a substantial core of mathematics. However, this core is like neither the abstract pre-engineering mathematics of the academic curriculum nor the restricted topics of the discredited "vocational math." New approaches are needed to meet today's challenges.

A recent survey of 4,500 manufacturing firms revealed that nearly two out of three current employees lack the mathematics skills required for their work, and that half lack the ability to interpret job-related charts, diagrams, and flowcharts (National Association of Manufacturers, 1997). Other reports cite a major shortage of qualified candidates for jobs in the information technology industries (Information Technology Association of America, 1997), as well as for technicians and licensed journeymen in the skilled trades (Mathematical Sciences Education Board, 1995). Even office work has changed, so that technical skills are now at a premium (Carnevale & Rose, 1998).

What current and prospective employees lack is not calculus or advanced algebra, but a plethora of more basic quantitative skills that could be taught in high school but are not (Murnane & Levy, 1996; Packer, 1997). They need statistics and three-dimensional geometry, systems thinking and estimation skills. Even more important, they need the disposition to think through problems that blend quantitative data with verbal, visual, and mechanical information; the capacity to interpret and present technical information; and the ability to deal with situations when something goes wrong (Forman & Steen, 1998).

Business has discovered, and research confirms, that diplomas and degrees do not tell much about students' actual performance capabilities. For example, data from the National Assessment of Educational Progress (NAEP) (1997b) show that twelfth-grade students at the 10th percentile are essentially similar to fourth-grade students at the 80th percentile. Indeed, the level that NAEP considers "advanced," and which is achieved by only 8% of U.S. students, is considered just barely adequate in the context of college expectations (NAEP, 1997a). Enrollment data for postsecondary mathematics courses confirm this discrepancy (Loftsgaarden, Rung, & Watkins, 1997): three out of every four students enrolled in college mathematics courses are studying subjects typically taught in high school or even middle school (see Figure 1). Clearly, covering mathematics in school is no guarantee of mastering it for later use.

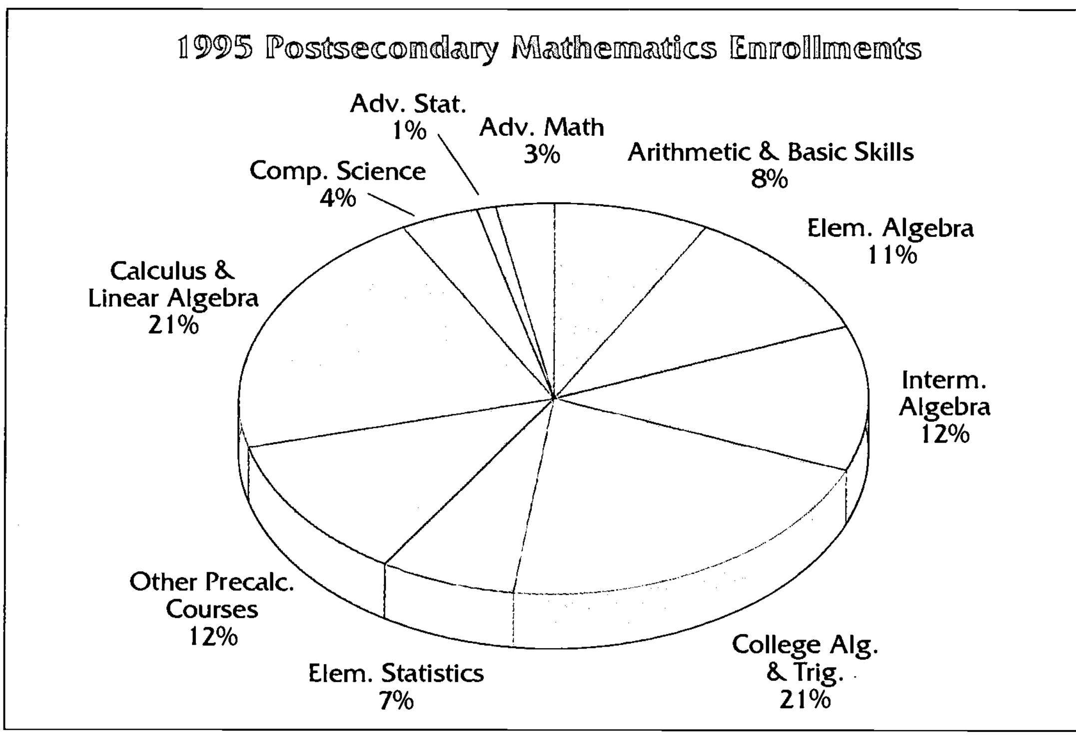

Figure 1

Nearly two-thirds of high school graduates enter postsecondary education primarily in order to obtain further skills and an advanced degree. Unfortunately, fewer than half of those who begin college attain any degree at all within five years. Furthermore, the majority of those who begin a traditional liberal arts program never finish. Although the economy clearly needs employees with advanced technical training (Judy & D'Amico, 1997), these students—the majority—end up with just a list of courses and no degree or job certification (Barton, 1997).

Ever since the publication of *A Nation at Risk* (National Commission on Excellence in Education, 1983), many advocates of educational reform have built their case on international competitiveness: to compete in a global economy that is increasingly technological, U.S. workers need better technical education (Commission on the Skills of the American Workforce, 1990). Yet, data from international comparisons such as the Third International Mathematics and Science Study (TIMSS) show that U.S. students are far from competitive (NCES, 1998). Thus, according to this argument, to remain internationally competitive, we need to radically overhaul mathematics and science education (Riley, 1998a).

In fact, the U.S. economy is thriving despite consistently weak performances by students on both national and international tests. This paradox has led some observers to suggest that the problem with weakness in school mathematics and science education is not so much that it hurts the overall economy, but that it increases economic inequities by providing the means to a good livelihood to only a few, primarily those from upper socioeconomic backgrounds (Barton, 1997; Bracey, 1997). From this perspective, the primary rationale for improving school mathematics is not competitiveness, but equity: in today's data-driven world, there is no justification for approaches to mathematics education that filter out those with greatest need and equip only the best-prepared for productive high-income careers.

A high school curriculum that helps all students master functional mathematics would effectively address issues of both equity and competitiveness. Since all students would study the same curriculum, all would have equal opportunity to master the mathematics required for the new world of work. Moreover, a three-year core of functional mathematics would give all students a strong platform on which to build either technical work experience or advanced education. Either route would lead to productive careers.

A large load of topsoil forms a conical pile. Because of its size, you cannot directly measure either its diameter or its height. Find a strategy for estimating its volume.

Theory and Practice

Historically, education in the United States has vacillated between the liberal and the pragmatic, between Robert Maynard Hutchins and John Dewey. Mathematics reflects a similar tension in the delicate balance of theory and practice, of the pure and the applied (Thurston, 1990). Through most of this century, school mathematics has oscillated back and forth between these poles (Kilpatrick, 1997). Indeed, nearly a century ago, the president of the American Mathematical Society lamented the "grievous" separation of pure from applied mathematics and urged schools to provide a more "practical" mathematics education: "With the momentum of such [education], college students would be ready to proceed rapidly and deeply in any direction in which their personal interests might lead them" (Moore, 1903). Today's effort to make mathematics more functional for all students is just the latest chapter in this long saga.

In recent years, this debate has been expressed in the form of standards, both academic and occupational. Coordinating these standards will involve not only issues of content and pedagogy, but also the balance of school-based vs. work-based learning (Bailey, 1997). Historically, vocational curricula designed to prepare students for work have been burdened by second-class status in comparison with more rigorous academic curricula. Too often, vocational programs became dumping grounds for students who appeared slow or unmotivated—"other people's children." Most programs responded by limiting goals and lowering expectations, thereby offering stunted education to students who were already behind. In contrast, contemporary career-oriented curricula have been designed not primarily as training for low-skill jobs but as motivation for rigorous study, both academic and vocational (Bailey & Merritt, 1997; Hoachlander, 1997). By setting high standards, these programs offer significant responses to the twin challenges of equity and competitiveness.

Mathematics provides a microcosm of the duality between the academic and the vocational. Widely perceived as the epitome of theory and abstraction, mathematics is also valued as a powerful, practical tool (Odom, 1998). In many occupations, quantitative literacy is as important as verbal literacy (Steen, 1997); however, if mathematics education is to serve the world of work, a different type of experience than that found in typical mathematics courses is required (National Research Council, 1998).

Between theory and application lies professional practice—the synthesis of thought and action employed by practitioners in all vocations. Many have argued that practice, properly understood, can be a legitimate and unifying goal of education. Practice is functional knowledge, the kind of know-how that allows people to get things done. According to educator Lee Shulman (1997), practice can provide a context in which theory becomes meaningful, memorable, and internalizable. Peter Denning (1997), a computer scientist, believes that practice—not knowledge or literacy—is what constitutes true expertise. Indeed, practice is what people tend to expect of schools, especially of mathematics education. It is at the heart of functional mathematics.

An infusion of practice into school mathematics can overcome what Shulman (1997) identifies as major deficiencies of theoretical learning: loss of learning ("I forgot it"), illusion of learning ("I thought I understood it"), and uselessness of learning ("I understand it but I can't use it"). Adults who are not professional users of mathematics will recognize these deficiencies from their own experiences. Little of what adults learned in school mathematics is remembered or used, so the accomplishment of "learning" mathematics is often an illusion. In fact, the mathematics many students are force-fed in traditional school environments creates a severe psychological impediment to the practice of mathematics in adult life (Buxton, 1991; Cockroft, 1982). Functional mathematics avoids many of these pitfalls by emphasizing that the goal of mathematics education is not just mathematical theory and word problems, but authentic mathematical practice.

Habitat for Humanity uses volunteer labor to build inexpensive homes, which it sells for the cost of materials. Using information on standard building supplies obtained from a local lumberyard, design a simple home whose building materials can be obtained for $15,000.

High School Mathematics

Traditionally, high school mathematics has served two different purposes—to prepare college-intending students for calculus (and other mathematics-based courses) and to equip other students with necessary skills, mostly arithmetic, so that they can function as employees, homemakers, and citizens. Although most traditionalists—and most parents and grandparents—still support these dual goals, reformers argue for a common curriculum for *all* students which emphasizes problem solving, communication, reasoning, and connections with other disciplines.

Proposed goals for school mathematics can be found in many sources. Some focus directly on K-12, others on the needs of postsecondary education or employers. NCTM (1989) provides a comprehensive set of standards for grade levels K-4, 5-8, and 9-12 that represents the "reform" perspective. In contrast, California recently adopted mathematics standards that represent a more traditional perspective (California Academic Standards Commission, 1997). The American Mathematical Association of Two-Year Colleges (1995) articulated standards for college mathematics before calculus that include expectations for the mathematical foundation that students need to succeed in college. In addition, in the influential report *What Work Requires of Schools* (Secretary's Commission on Achieving Necessary Skills [SCANS], 1991), the U.S. Department of Labor outlined both foundation skills and broad employability competencies for mathematics and other subjects.

These standards differ greatly in both mathematical content and rhetorical style (see Appendix A), although most have overlapping goals. Indeed, to succeed in the real world of teachers and parents, schools and school boards, a mathematics curriculum must

(1) meet society's expectations of what all high school graduates should know and be able to do.

(2) reflect priorities common to state and national guidelines.

(3) increase the number of students who successfully persist in advanced mathematics-based courses, including calculus.

(4) enable students to see and use mathematics in everyday aspects of life and work.

(5) help students understand and use correct mathematical language.

Functional mathematics must also meet these objectives. The first two objectives establish priorities: to focus early and often on what everyone agrees must be learned, leaving to later (or to optional strands) those topics that only some students will find interesting or important. The third objective establishes a standard of quality: to increase the number of students who persist in further mathematics-based courses (including calculus, the traditional hallmark of mathematical success). The fourth objective conveys a commitment to utility—to ensure that students see mathematics as something real in their lives rather than as an alien subject encountered only in school. Finally, the fifth objective stresses command of the language of mathematics, a skill at least as important for success as a command of English.

By meeting these objectives, functional mathematics will satisfy the general public's expectations of school mathematics. In addition, these objectives also enhance functional mathematics' primary goal of preparing students for life and work. Consistent quality and high standards are essential in today's high-performance industries. Persistence in mathematics is not just of academic importance; it is also one of the best predictors of success in careers (Commission on the Skills of the American Workforce, 1990). Moreover, the language of mathematics provides the power to analyze and express complex issues in all aspects of life and work. Fluency in this language is important not only for productive employees but also for careful consumers and critical citizens.

In functional mathematics, utility is center stage. Other objectives play important but supporting roles. Unfortunately, many mathematicians and mathematics teachers find utility at best a bleak justification (Howe, 1998) for a subject that they chose for its beauty and elegance. For them, the power of mathematics—in Eugene Wigner's famous phrase, its "unreasonable effectiveness"—is not its primary virtue, but merely a consequence of its elegance and internal structure. Thus, mathematicians are wont to stage their subject with theory and abstraction at the center, employing applications, technology, and practice as needed to help promote understanding.

To engage mathematicians and mathematics teachers, functional mathematics needs to be seen in terms of both utility and beauty. For many students, utility can be a path to beauty, while for others, mathematics by itself provides sufficient internal motivation to sustain interest and accomplishment. For any mathematics curriculum to succeed with all students, it must build on the twin foundations of utility and elegance.

What measurements do you need to take in order to tile the floor of a room? How can you use these measurements to determine the number of regular tiles, border tiles, and corner tiles that are needed? What if you decide to lay the main tiles on a 45° angle?

Functional Mathematics

Functional mathematics comprises content, curriculum, context, and pedagogy. By content we simply mean the mathematics students should know and be able to do after finishing the first three years of high school mathematics (see Appendix B). Because mathematics is mathematics—whether traditional, reform, or functional—most of these elements are unsurprising. Although some topics are uncommon (e.g., index numbers, tolerances, three-dimensional geometry, normal curve, quality control charts, standards of proof, financial mathematics, spreadsheets), most are taught in any high-quality high school mathematics program. To achieve its goal of preparing students for both work and further education, functional mathematics respects the many parts of the traditional curriculum that are broadly useful, even while reshaping the boundaries to reflect its distinctive objectives.

The outline of functional mathematics in Appendix B reflects an inventory of mathematical topics selected for their importance in daily life and modern jobs as well as for their value in providing a strong foundation for further education. This outline is organized in predictable strands that cover what is normally subsumed under the umbrella of mathematics: numbers and data, measurement and space, growth and variation, chance and probability, reasoning and inference, variables and equations, modeling and decisions. Real problems cut across all this mathematics, just as these topics cut across the diverse contexts of authentic mathematical practice.

Clearly, many of the elements of functional mathematics are identical to the mathematics found in both traditional and reformed curricula. The core of school mathematics is more or less the same, even if viewed (or taught) from different perspectives. Percentages and ratios; linear and quadratic equations; areas, angles, and volumes; and exponential growth and trigonometric relations must be included in any strong high school mathematics program. The distinctions among traditional, reformed, and functional curricula lie not so much in core content as in contexts, emphases, and pedagogy.

Nonetheless, prospective employees for the new high-performance workplace need expertise in several aspects of mathematics not now emphasized sufficiently in school. On the one hand, students need greater experience recognizing and using some parts of middle school mathematics such as ratio, percentage, and measurement geometry that, although covered in current programs, are not used sufficiently to be learned well. On the other hand, as prospective employees, they need to understand and be able to use mathematical notions such as data analysis, statistical quality control, and indirect measurement that are hardly ever required in high school (Forman & Steen, 1998).

In addition to shifting the balance of topics, functional mathematics provides much greater emphasis on "systems thinking"—on habits of mind that recognize complexities inherent in situations subject to multiple inputs and diverse constraints. Examples of complex systems abound—from managing a small business to scheduling public transportation, from planning a wedding to reforming social security. At all levels from local to national, citizens, policymakers, employees, and managers need to be able to formulate problems in terms of relevant factors and design strategies to determine the influence of those factors on system performance. Although such systems are often so complex that they obscure the underlying mathematics, the skills required to address realistic problems very often include many that are highly mathematical.

A curriculum built on functional mathematics gives students many opportunities to solve realistic problems and build mathematical understanding. Nevertheless, to make this learning valuable for work and further education—as well as to enhance understanding—such a curriculum must also help students become fluent in the language of mathematics. Individuals need to be able to read, understand, and interpret technical material with embedded charts and diagrams (e.g., property tax bills, stock market reports); they need to be able to speak clearly about mathematical ideas (e.g., as a salesman explaining the interest and payoff on an insurance policy); and they need extensive experience writing reports based on mathematical and technical analysis (e.g., a recommendation to a supervisor summarizing the conclusion of a study).

Functional mathematics channels the much-criticized "mile-wide, inch-deep" curricular river into a narrower but deeper stream of ideas and procedures that reinforce each other as students progress through school and college and on into careers. It provides a rich foundation of experience and examples on which students can build subsequent abstractions and generalizations. Indeed, to fulfill its goals, a functional curriculum must leave students well-prepared not only for work but also for subsequent courses in more abstract mathematics.

You are helping your brother-in-law build a garage on gently sloped land next to his house. After leveling the land, you begin staking out the foundation. To check that corners are square, you measure the diagonals and discover that they differ by 3 inches. Is that because the corners may not be perfectly level, or because they are not perfectly square? How can you determine what needs fixing to make sure that you start with a foundation that is both level and square?

Functional Curricula

The elements of functional mathematics can be embedded in many different curricula—the paths students follow through their education. Although some parts of mathematics impose a necessary order on the curriculum (e.g., arithmetic before algebra; linear equations before quadratic), large parts of mathematics can be approached from many different directions. Data analysis can be either a motivation for or an application of graphing and algebra; geometry can either precede or follow algebra, and each can reinforce the other. The order in which elements are listed bears no relationship to the order in which they may be taught through a three-year core curriculum.

The elements of functional mathematics arise from common contexts of life and work—measuring objects, managing money, scheduling time, making choices, and projecting trends. Although it is possible to organize a curriculum around such contexts, without a list of elements such as those in Appendix B to guide instruction, the mathematics itself may remain largely hidden. Alternatively, a functional curriculum can be organized around mathematical themes such as the sections of Appendix B. Indeed, the latter fits better the experience of most mathematics teachers and is more likely to be adaptable to most school settings.

Any mathematics curriculum designed on functional grounds—whether organized around external contexts or mathematical themes—will emphasize authentic applications from everyday life and work. In such a curriculum, students will gain considerable experience with mathematical tasks that are concrete yet sophisticated, conceptually simple yet cognitively complex (Forman & Steen, 1995). A functional curriculum compels a better balance of statistics (numbers), geometry (space), and algebra (symbols)—the three major branches of the mathematical sciences. By highlighting the rich mathematics embedded in everyday tasks, this approach (in contrast to traditional "vocational math") can dispel both minimalist views about the mathematics required for work and elitist views of academic mathematics as an area with little to learn from work-based problems (Bailey & Merritt, 1997; Forman & Steen, 1998).

Because of the history of low standards in traditional vocational programs, many teachers and parents believe that a work-focused curriculum will necessarily lack the rigor of a precollege academic track. Contrary to this belief, the "zero-defect" demands of the high-performance workplace for exacting standards and precise tolerances actually impose a much higher standard of rigor than do academic programs that award students a B for work that is only 80-85% accurate. Moreover, the lengthy and subtle reasoning required to resolve many problems that arise in real contexts provides students with experience in critical thinking that is often lacking in academic courses that rush from topic to topic in order to cover a set curriculum.

Five friends meet for dinner in a restaurant. Some have drinks and others do not; some have dessert and others do not; some order inexpensive entrees, others choose fancier options. When the bill comes they need to decide whether to just add a tip and split it five ways, or whether some perhaps should pay more than others. What is the quickest way to decide how much each should pay?

Teaching Functional Mathematics

Although the public thinks of standards primarily in terms of performance expectations for students, both the mathematics standards

(NCTM, 1989) and the science standards (National Research Council, 1996) place equal emphasis on expectations for teaching, specifically that it be active, student-centered, and contextual:

- *Active instruction* encourages students to explore a variety of strategies; to make hands-on use of concrete materials; to identify missing information needed to solve problems; and to investigate available data.
- *Student-centered instruction* focuses on problems that students see as relevant and interesting; that help students learn to work with others; and that strengthen students' technical communication skills.
- *Contextual instruction* asks students to engage problems first in context, then with mathematical formality; suggests resources that might provide additional information; requires that students verify the reasonableness of solutions in the context of the original problem; and encourages students to see connections of mathematics to work and life.

These expectations for effective teaching are implicitly reinforced in recently published occupational skill standards (National Skill Standards Board, 1998) that outline what entry-level employees are expected to know and be able to do in a variety of trades. Although these standards frequently display performance expectations for basic mathematics as lists of topics, the examples they provide of what workers need to be able to do are always situated in specific contexts and most often require action outcomes (Forman & Steen, in press).

Most students learn mathematics by solving problems. In traditional mathematics courses, exercises came in two flavors: explicit mathematical tasks (e.g., solve, find, calculate) and dreaded "word problems" in which the mathematics is hidden as if in a secret code. Indeed, many students, abetted by their teachers, learn to unlock the secret code by searching for key words (e.g., *less* for *minus, total* for *plus*) rather than by thinking about the meaning of the problem (which may be a good thing, because so many traditional word problems defy common sense).

In a curriculum focused on functional mathematics, tasks are more likely to resemble those found in everyday life or in the workplace than those found in school textbooks. Students need to think about each problem afresh, without the clues provided by a specific textbook chapter.

Rather than just being asked to solve an equation or calculate an answer, students are asked to design, plan, evaluate, recommend, review, define, critique, and explain—all things they will need to do in their future jobs (as well as in college courses). In the process, they will formulate conjectures, model processes, transform data, draw conclusions, check results, and evaluate findings. The challenges students face in a functional curriculum are often nonroutine and open-ended, with solutions taking from minutes to days, and requiring diverse forms of presentation (oral, written, video, or computer). As in real job situations, some work is done alone, and some in teams.

A round chimney 8" in diameter protrudes from a roof that has a pitch of 3:1. Draw a pattern for an aluminum skirt that can be cut out of sheet metal and bent into a cone to seal the chimney against rain.

Mathematics in Context

Students' achievements in school mathematics depend not only on the content of the curriculum and the instructional strategies employed by the teacher but also on the context in which the mathematics is embedded. Traditionally, mathematics has served as its own context: as climbers scale mountains because they are there, so students are expected to solve equations simply because it is in the nature of equations to be solved. From this perspective, mathematics is considered separate from and prior to its applications. Once the mathematics is learned, it supposedly can then be applied to various problems, either artificial or real.

Many of the new curricula developed in response to the NCTM standards or state frameworks give increased priority to applications and mathematical models. In some of these programs, applications are at the center, providing a context for the mathematical tools prescribed by the standards. In others, applications serve more to motivate topics specified in the standards. In virtually all cases, the applications found in current curricula are selected, invented, or simplified to serve the purpose of teaching particular mathematical skills or concepts. In contrast, the mathematical topics in a functional curriculum are determined by the importance of the contexts in which they arise.

For most students, interesting contexts make rigorous learning possible. Realistic problems harbor hidden mathematics that good teachers can illuminate with probing questions. Most authentic mathematical problems require multistep procedures and employ realistic data—which are often incomplete or inconsistent. Problems emerging from authentic contexts stimulate complex thinking, expand students' understanding, and reveal the interconnected logic that unites mathematics.

Devise criteria and procedures for fair addition of a congressional district to a state in a way that will minimize disruption of current districts while creating new districts that are relatively compact (non-gerrymandered) and of nearly equal size.

Employing Computers

It has been clear for many years that technology has changed priorities for mathematics. Much of traditional mathematics (from long division to integration by parts) was created not to enhance understanding but to provide a means of calculating results. This mathematics is now embedded in silicon, so training people to implement these methods with facility and accuracy is no longer as important as it once was. At the same time, technology has increased significantly the importance of certain parts of mathematics (e.g., statistics, number theory, discrete mathematics) that are widely used in information-based industries.

Many mathematics teachers have embraced technology, not so much because it has changed mathematics but because it is a powerful pedagogical tool. Mathematics is the science of patterns (Devlin, 1994; Steen, 1988), and patterns are most easily explored using computers and calculators. Technology enables students to study patterns as they never could before, and in so doing, it offers mathematics what laboratories offer science: a source of evidence, ideas, and conjectures.

The capabilities of computers and graphing calculators to create visual displays of data have also fundamentally changed what it means to understand mathematics. In earlier times, mathematicians struggled to create formal symbolic systems to represent with rigor and precision informal visual images and hand-drawn sketches. However, today's computer graphics are so sophisticated that a great deal of mathematics can be carried out entirely in a graphical mode. In many ways, the medium of computers has become the message of mathematical practice.

Finally, and perhaps most significantly, computers and calculators increase dramatically the number of users of mathematics—many of whom are not well-educated in mathematics. Previously, only those who learned mathematics used it. Today, many people use mathematical tools for routine work with spreadsheets, calculators, and financial systems—tools that are built on mathematics they have never studied. For example, technicians who diagnose and repair electronic equipment employ a full range of elements of functional mathematics—from number systems to logical inferences, from statistical tests to graphical interpretations. Broad competence in the practice of technology-related mathematics can boost graduates up many different career ladders.

This poses a unique challenge for mathematics education: to provide large numbers of citizens with the ability to use mathematics-based tools intelligently without requiring that they prepare for mathematics-based careers. Although mathematicians take for granted that learning without understanding is ephemeral, many others argue that where technology is concerned, it is more important for students to learn how to use hardware and software effectively than to understand all the underlying mathematics. But even those who only use the products of mathematics recognize the value of understanding the underlying principles at a time when things go wrong or unexpected results appear. In a functional curriculum where, for example, algebra emerges from work with spreadsheets, the traditional distinction between understanding and competence becomes less sharp.

— How many school teachers are there in New York City? How many electricians? How many morticians?

— How many words are there in all the books in the school library? How many megabytes of disk storage would be required to store the entire library on a computer?

Avoiding Pitfalls

Those who develop materials and examples for a functional curriculum need to avoid some common pitfalls that plague all attempts at situating mathematics in authentic contexts. On the one hand, there is the temptation to give priority to the mathematics, either by selecting tasks to ensure coverage of mathematical topics without much regard to the tasks' intrinsic importance or by imposing unwarranted structure on a contextually rich problem in the interest of ensuring appropriate mathematical coverage. On the other hand, it is easy to overlook interesting mathematics hidden beneath the surface of many ordinary tasks or to choose problems that fail to help students prepare for advanced study in mathematics. Any curriculum that is to prepare students for subsequent mathematics-dependent courses must recognize the importance of intellectual growth and conceptual continuity in the sequencing of tasks in which mathematical activities are embedded.

Context-rich mathematics curricula often present tasks in the form of worksheets, outlining a series of short-answer steps that lead to a solution. While ostensibly intended to help students organize their thinking and assist teachers in following students' work, these intellectual scaffolds strip tasks of everything that makes them problematic. Indeed, worksheets reveal a didactic posture of traditional teaching (teacher tells, students mimic) that undermines learning and limits understanding. Students will learn and retain much more from the chaotic process of exploring, defending, and arguing their own approaches.

Finally, although a functional mathematics curriculum is motivated largely by examples that seem to lie outside the world of mathematics, it is nonetheless very important for students' future study that instructors bring mathematical closure at appropriate points. Students need to recognize and reflect on what they have learned; to be clear about definitions, concepts, vocabulary, methods, and potential generalizations; and to have sufficient opportunity to reflect on the accomplishments and limitations of mathematics as a tool in helping solve authentic problems.

A patient with an aggressive cancer faces two options for treatment: With Option A, he has a 40% chance of surviving for a year, but if he makes it that long then his chance of surviving a second year is two out of three. With Option B, he has a 50-50 chance of surviving each of the first and second years. Survival rates beyond the second year are similar for each option. Which choice should he make?

Mathematics in Life and Work

The diverse contexts of daily life and work provide many realistic views of functional mathematics—of the mathematical practice underlying routine events of daily life. These contexts offer episodic views, incomplete in scope and less systematic than a list of elements, but more suggestive of the way functional mathematics may be introduced in courses.

Reading Maps. Road maps of cities and states provide crucial information about routes and locations. For those who know how to "read" them, maps also convey scale and direction, helping drivers know which way to turn at intersections, permitting quick estimates of driving time, and revealing compass directions that relate to highway signs at road intersections. Map scales are just ratios—an essential part of school mathematics. Different scales not only convey different detail, but also require different translations to represent distance.

Reading maps is not just a matter of thinking of distances in different scales. In many cases, the geometry of maps represents other features such as temperature or soil content. Most common are weather maps with color-coded regions showing gradations in recorded or predicted temperatures. Similar maps sometimes display recorded or predicted precipitation, barometric pressure, vegetation features, or soil chemistry. Like topographic maps used by hikers, these maps represent some feature of the landscape that changes from place to place. The spacing between regions of similar temperature (or pressure, or elevation) conveys the steepness (or gradient) of change—what mathematicians call the "slope" of a line.

Scale-drawings and blueprints are also widely used to illustrate details of homes, apartments, and office buildings. These drawings represent sizes of rooms, locations of windows and doors, and—if the scale permits—locations of electrical outlets and plumbing fixtures. Architects' rulers with different units representing one foot of real space make it possible to read real distances off scale drawings, taking advantage of the geometrical properties of similar figures. New geographic information systems (GIS) encode spatially oriented data in a form suitable for computer spreadsheets, thereby enabling other factors (e.g., costs, environmental factors) to be logically linked to the geometric structure of a map.

Ensuring Quality. Statistical process control (SPC) and statistical quality control (SQC) are crucial components of high-performance manufacturing, where "zero defect" is the goal. Instead of checking and repairing products after manufacture, firms like Boeing, General Motors, Kodak, Motorola, and Siemens now insist that at every step in the manufacturing process, materials, parts, and final products be manufactured within tight tolerances. Moreover, workers on assembly lines are responsible for ensuring this consistent level of quality.

The two tools that make this possible are based on statistics—the science of collecting and organizing data. The first, statistical process control, occurs during manufacture: assembly line workers chart key indicators of the process—perhaps the temperature of a mixture or the pitch of a grinding tool—on graph paper marked with curves representing the limits determined by the required (or contracted) tolerances. If the process strays outside these limits, or approaches them too often, workers may decide to shut down the assembly line to make adjustments in the manufacturing process rather than risk producing products that do not meet design specifications.

Statistical quality control is like statistical process control, but takes place when components (e.g., computer chips) are completed. By sampling finished products and charting their performance characteristics, workers can identify potential problems before products exceed permitted tolerances—and then take action to prevent the shipping or further manufacture of defective (i.e., out-of-tolerance) products.

Using Spreadsheets. Almost everyone who works with a computer uses a word processor for writing, whether for correspondence or business reports. Almost as popular are "number processors," commonly known as spreadsheets. Originally designed as a tool for accountants, spreadsheets are ubiquitous both in the office and at home—wherever anyone deals with budgets and expenses, taxes and investments. Spreadsheets are used to record business inventories and scientific data, to keep track of medical records and student grades, to organize crop records and airline schedules. Virtually any systematic information can be made more useful by being put in a properly organized spreadsheet.

To a mathematician, a spreadsheet is just algebra playing on a popular stage. The basic operations of a spreadsheet—adding cells together, calculating percentages, projecting growth rates, determining present values—are entered as formulas into the appropriate cells. More complex formulas (e.g., exponential, financial, trigonometric) are available from a pull-down menu. Once the computations are completed, the results can be displayed in graphs of various sorts (lines, bars, pies), often in vivid color.

Figuring out how to translate a task into a spreadsheet design is just like setting up a word problem in algebra: it involves identifying important variables and the relations among them. Preparing a spreadsheet requires equations which are suitably located in the cells. The spreadsheet does the arithmetic, and the designer does the algebra. Then, as in any mathematical exercise, the designer needs to check the results—typically by specifying independent computations to confirm key spots in the spreadsheet. (For example, adding all the entries in a grid can confirm the accuracy of the sum of the row totals, thus catching possible errors in the spreadsheet formulas.) Variables, equations, graphs, word problems—the ingredients of a good algebra course—are just the ticket for mastering spreadsheets.

Building Things. One in every four American workers builds things—automobiles or airplanes, bicycles or buildings, containers or chips. These products are three-dimensional, created by casting and cutting, by folding and fastening, by molding and machining. Designing things to be built (the work of engineers and architects) and building objects as designed (the work of carpenters and machinists) require impressive feats of indirect measurement, three-dimensional geometry, and visual imagination.

In a typical aluminum airplane part, for example, some measurements are specified by the designers, while others must be calculated in order to program the cutting tool that will actually create the part. In three dimensions, things are even more complicated. Planning how to drill holes at specified angles in a block of aluminum whose base is not square and whose sides are tilted in odd directions would tax the skills of most mathematics teachers. But machinists are expected to perform these calculations routinely to determine settings on a "sine plate," a device whose surface can tilt in two different dimensions in order to compensate for odd angles on the part that is to be drilled.

Both designers and builders now use computer-assisted design (CAD) and computer-assisted manufacturing (CAM) to ensure the exacting tolerances required for high-performance manufacturing. To use these tools effectively, workers need to have mastered the basic skills of drawing geometric objects, measuring distances, and calculating angles, distances, areas, and volumes. The basic principles of geometry in three dimensions are the same as those in two dimensions, but the experience of working in three dimensions is startlingly more sophisticated. A good command of geometry and trigonometry is essential for anyone building things in today's manufacturing industries.

Thinking Systemically. Systems surround us—in commerce, science, technology, and society. In complex systems, many factors influence performance, thus making the task of solving problems inherently multidimensional. Indeed, the interaction of different factors is often difficult to predict, sometimes even counterintuitive. Complex systems defy simplistic single answers. Thus, the first step in mathematical analysis is often to prepare an inventory of all possible factors that might need to be considered.

For example, the rise of efficient package delivery services and instantaneous computer communication have enabled many manufacturing companies to operate with minimum inventories, thus saving warehousing costs but risking a shutdown if any part of the network of suppliers fails. Understanding how a system of suppliers, communication, and transportation works requires analysis of capacity, redundancy, single-point failures, and time of delivery—all involving quantitative or logical analyses.

Other system problems arise within the everyday work of a typical small business. For example, the stockroom of a shoe store holds several thousand boxes labeled by manufacturer, style, color, and size and arranged on floor-to-ceiling shelves. Deciding how to arrange these boxes can have a significant impact on the profit margin of the store. Obvious options are by manufacturer, by style, by size, by frequency of demand, or by date of arrival. Clerks need to be able to find and reshelve shoes quickly as they serve customers. But they also need to be able to make room easily for new styles when they arrive, to compare regularly the stockroom inventory with sales and receipt of new shoes, and to locate misshelved shoes. Mathematical thinking helps greatly in exploring the advantages and disadvantages of the many possible systems for arranging the stockroom.

Making Choices. Life is full of choices—to rent an apartment or purchase a home; to lease or buy a car; to pay off credit card debt or use the money instead to increase the down payment on a house. All such choices involve mathematical calculations to compare costs and evaluate risks. For example:

> The rent on your present apartment is $1,200 per month and is likely to increase 5% each year. You have enough saved to put a 25% down payment on a $180,000 townhouse with 50% more space, but those funds are invested in an aggressive mutual fund that has averaged 22% return for the last several years, most of which has been in long-term capital gains (which now have a lower tax rate). Current rates for a 30-year mortgage with 20% down are about 6.75%, with 2 points charged

up front; with a 10% down payment the rate increases to 7.00%. The interest on a mortgage is tax deductible on both state and federal returns; in your income bracket, that will provide a 36% tax savings. You expect to stay at your current job for at least 5-7 years, but then may want to leave the area. What should you do?

This sounds like a problem for a financial planner, and many people make a good living advising people about just such decisions. But anyone who has learned high school mathematics and who knows how to program a spreadsheet can easily work out the financial implications of this situation. Moreover, by doing it on a spreadsheet, it is quite easy to examine "what if" scenarios: What if the interest rate goes up to 7% or 7.25%? What if the stock market goes down to its traditional 10-12% rate of return? What if a job change forces a move after three years?

In contrast to many problems of school mathematics which are routine for anyone who knows the right definitions (e.g., what is cos ($\pi/2$)?) but mystifying otherwise, this common financial dilemma is mathematically simple (it involves only arithmetic and percentages) but logically and conceptually complex. There are many variables, some of which need to be estimated; there are many relationships that interact with each other (e.g., interest rates and tax deductions); and the financial picture changes each year (actually, each month) as payments are made.

The complex sequence of reasoning involved in this analysis is typical of mathematics, which depends on carefully crafted chains of inferences to justify conclusions based on given premises. Students who can confidently reason their way through a lengthy proof or calculation should have no problem being their own financial advisors. And students who learn to deal with long chains of reasoning inherent in realistic dilemmas will be well prepared to use that same logic and careful reasoning if they pursue the study of mathematics in college.

In preparing fertilizer for a garden, a homeowner poured one quart of concentrated liquid fertilizer into a two-gallon can and filled the can with water. Then she discovered that the proper ratio of fertilizer to water should be 1:3. How much more liquid fertilizer should she add to the current mixture to obtain the desired concentration?

Making Mathematics Meaningful

Those who discuss mathematics education frequently describe mathematical knowledge in broad categories such as skills and understanding, concepts and facts, procedures and practices, or insights and knowledge. Whole volumes of educational research are devoted to distinguishing among these different aspects of mathematical knowledge. The standards movement has tended to subsume all these distinctions into two categories of knowledge and performance: what students should know and what students shoud be able to do (Ravitch, 1995; Tucker & Codding, 1998).

The two broad cultures of mathematics education argue with each other largely because they differ in the interpretations they give to these different aspects of mathematical knowledge. Those who favor the traditional curriculum centered on algebra, functions, and Euclidean geometry argue that mastery of facts and basic skills are a prerequisite to understanding and performance. Reformers who favor a broader curriculum take a more constructivist view—that understanding and mastery are an outgrowth of active engagement with contextualized mathematics. Regardless of approach or emphasis, both traditional and reform curricula generally cover a similar set of topics designed to move students along the path from arithmetic to calculus.

Functional mathematics follows much the same path, with variations that reflect its grounding in authentic problems. However, by embedding mathematics in practice, functional mathematics can offer students both theory and know-how. Although in some technical areas, practical "of-the-moment" learning offers little that outlasts the next generation of gadgets, the logical structure that unites mathematics guarantees that all understanding, no matter how specific, has the potential to enhance mastery of other areas. What matters for long-term mastery of mathematics is not so much which particular skills are learned as that the process of learning be, in Shulman's words, "meaningful, memorable, and internalizable." Although topics in functional mathematics may be chosen for proximate utility, their study can provide insight and understandings sufficient for lifelong learning.

A curriculum based on functional mathematics requires appropriate content, authentic contexts, engaging tasks, and active instruction. By featuring mathematics in common contexts, a functional curriculum can motivate students to link meaning with mathematics. The best problem settings offer opportunities for exploration from multiple perspectives, including graphical, numerical, symbolic, verbal, and computational. Technology—from graphing calculators and word processors to spreadsheets and symbolic algebra systems—can enhance understanding from each of these perspectives. Effective contexts provide opportunities for horizontal linkages among diverse areas of life and work as well as vertical integration from elementary ideas to advanced topics. Experience with rich contexts helps students recognize that asking questions is often as important as finding answers. Such contexts invite variations that can stimulate mathematical habits of mind and propel students to deep understanding.

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Appendix A: Excerpts From Different Standards

The diverse ways that different organizations express expectations for mathematics illustrate a variety of approaches to setting standards. The excerpts that follow illustrate this variety in the particular case of algebra, the core of high school mathematics.

From the National Council of Teachers of Mathematics (1989):

In grades 9-12, the mathematics curriculum should include the continued study of algebraic concepts and methods so that all students can

- represent situations that involve variable quantities with expressions, equations, inequalities, and matrices;
- use tables and graphs as tools to interpret expressions, equations, and inequalities;
- operate on expressions and matrices, and solve equations and inequalities;
- appreciate the power of mathematical abstraction and symbolism;

and so that, in addition, college-intending students can

- use matrices to solve linear systems;
- demonstrate technical facility with algebraic transformations, including techniques based on the theory of equations.

From the California Academic Standards Commission (1997):

By the end of Grade 10, all students should be able to:

Solve linear equations and inequalities with rational coefficients; use the slope-intercept equation of a line ($y = mx + b$) to model a linear situation and represent the problem in terms of a graph.

Describe, graph, and solve problems using linear, quadratic, power, exponential, absolute value, polynomial, and rational functions; identify key characteristics of functions (domain, range, intercepts, asymptotes).

Derive and use the quadratic formula to solve any quadratic equation with real coefficients; graph equations of the conic sections (parabola, ellipse, circle, hyperbola), identifying key features such as intercepts and axes.

Describe, extend, and find the nth term of arithmetic, geometric, and other regular series.

And in Grades 11-12, mathematics students should learn about:

Piece-wise defined functions; logarithm function and as inverse of exponential; polar coordinates; parametric equations; recursive formulas, binomial theorem, mathematical induction; trigonometric functions, graphs, identities, key values, and applications; vector decomposition.

From the American Mathematical Association of Two-Year Colleges (1995):

The study of algebra must focus on modeling real phenomena via mathematical relationships. Students should explore the relationship between abstract variables and concrete applications and develop an intuitive sense of mathematical functions. Within this context, students should develop an understanding of the abstract versions of basic number properties and learn how to apply these properties. Students should develop reasonable facility in simplifying the most common and useful types of algebraic expressions, recognizing equivalent expressions and equations, and understanding and applying principles for solving simple equations.

Rote algebraic manipulations and step-by-step algorithms, which have received central attention in traditional algebra courses, are not the main focus. Topics such as specialized factoring techniques and complicated operations with rational and radical expressions should be eliminated. The inclusion of such topics has been justified on the basis that they would be needed later in calculus. This argument lacks validity in view of the reforms taking place in calculus and the mathematics being used in the workplace.

From the Secretary's Commission on Achieving Necessary Skills (1991):

Mathematics. Approaches practical problems by choosing appropriately from a variety of mathematical techniques; uses quantitative data to construct logical explanations for real world situations; expresses mathematical ideas and concepts orally and in writing; and understands the role of chance in the occurrence and prediction of events.

Reasoning. Discovers a rule or principle underlying the relationship between two or more objects and applies it in solving a problem. For example, uses logic to draw conclusions from available information, extracts rules or principles from a set of objects or written text, applies rules and principles to a new situation, or determines which conclusions are correct when given a set of facts and a set of conclusions.

Appendix B: Elements of Functional Mathematics

These elements outline aspects of mathematics that are important for all students in their life and work. They emphasize concrete, realistic topics that arise in common situations in news, sports, finance, work, and leisure. These elements can be taught through many different curricula ranging from traditional to reform, from academic to vocational. Students completing any curriculum that includes these elements would be well-prepared to enter a wide variety of technical and academic programs, including a one-year precalculus course.

Numbers and Data

Mental Estimation. Anticipate total costs, distances, times; estimate unknown quantities (e.g., number of high school students in a state or city) using proportional reasoning; order of magnitude estimates; mental checking of calculator and computer results.

Numbers. Examples of whole numbers (integers), fractions (rational numbers), and irrational numbers (π, $\sqrt{2}$). Number line; mixed numbers; decimals, percentages, scientific notation. Prime numbers, factors; simple number theory; binary numbers and simple binary arithmetic; units and magnitudes; extreme numbers (e.g., national debt, astronomical distances); number sense; scientific notation.

Calculation. Accurate paper-and-pencil methods for simple arithmetic and percentage calculations; calculator use for complex calculations; spreadsheet methods for problems with a lot of data. Strategies for checking reasonableness and accuracy. Significant digits; interval arithmetic; errors; tolerances. Mixed methods (mental, pencil, calculator).

Coding. Number systems (decimal, binary, octal, hex); ASCII code; check digits. Patterns and criteria for credit card, Social Security, telephone, license plate numbers.

Index Numbers. Examples in the news: stock market averages; consumer price index; unemployment rate; SAT scores. Definitions and deficiencies; uses and abuses.

Information Systems. Collecting and organizing data; geographic information systems (GIS) and management information systems (MIS); visual representation of data.

Measurement and Space

Measurement. Direct and indirect means; estimation; use of appropriate instruments (rulers, tapes, micrometers, pacing, electronic gauges); plumb lines and square corners; calculated measurements; accuracy; tolerances; detecting and correcting misalignments.

Measurement Geometry. Measurement formulas for simple plane figures: triangles, circles, quadrilaterals. Calculation of area, angles, lengths by indirect means. Right triangle trigonometry; applications of Pythagorean theorem.

Dimensions. Linear, square, and cubic growth of length, area, volume. Coordinate notation; dimension as factor in multivariable phenomena.

Geometric Relations. Proof of Pythagorean theorem and of other basic theorems. Construction of line and angle bisectors, finding center of circular arc.

Spatial Geometry. Shapes in space; volumes of cylinders and spheres; calculation of angles in three-dimensions (e.g., meeting of roof trusses). Interpreting construction diagrams; nominal vs. true dimensions (e.g., of 2 x 4s); tolerances and perturbations in constructing three-dimensional objects.

Global Positioning: Map projections, latitude and longitude, global positioning systems (GPS); local, regional, and global coordinate systems.

Growth and Variation

Linear Change. Situations in which the rate of change is constant (e.g., uniform motion); contrast with examples where change is nonlinear (e.g., distance vs. time for falling body). Slope as rate of change; slope-intercept equation, with graphical significance of parameters. Difference between rate of change and value of the dependent variable.

Proportion. Situations modeled by similarity and ratio (e.g., height and shadows, construction cost vs. square footage, drug dose vs. body weight); examples where change is disproportional (e.g., height vs. weight). Calculating missing terms. Mental estimation using proportions.

Exponential Growth. Situations such as population growth, radioactivity, and compound interest, where the rate of change is proportional to size; doubling time and half-life as characteristics of exponential phenomena; symbolic representation (2^n, 10^n); ordinary and log-scaled graphs.

Normal Curve. Situations such as distribution of heights, of repeated measurements, and of manufactured goods in which phenomena distribute in a bell-shaped curve. Examples of situations in which they do not (e.g., income, grades, typographical errors, life spans). Parameters and percentages of normal distribution; z-scores, meaning of 1-, 2-, and 3-σ. Area as measure of probability.

Parabolic Patterns. Falling bodies; parabolas; quadratic equations; optimization problems.

Cyclic Functions. Situations such as time of sunrise, sound waves, and biological rhythms that exhibit cyclic behavior. Graphs of sin and cos; relations among graphs; $\sin^2\theta + \cos^2\theta = 1$.

Chance and Probability

Elementary Data Analysis. Measures of central tendency (average, median, mode) and of spread (range, standard deviation, midrange); visual displays of data (pie charts, scatter plots, bar graphs, box and whisker charts). Distributions. Quality control charts. Recognizing and dealing with outliers. "Data = Pattern + Noise."

Probability. Chance and randomness; calculating odds in common situations (dice, coin tosses, lotteries); expected value. Binomial probability, random numbers, hot streaks, binomial approximation of normal distribution; computer simulations; estimating area by Monte Carlo methods. Two-way tables; bias paradoxes.

Risk Analysis. Common examples of risks (e.g., accidents, diseases, causes of death, lotteries). Ways of estimating risk. Confounding factors. Communicating and interpreting risk.

Reasoning and Inference

Statistical Inference. Rationale for random samples; double-blind experiments; surveys and polls; confidence intervals. Causality vs. correlation. Multiple factors; interaction effects; hidden factors. Judging validity of statistical claims in media reports. Making decisions based on data (e.g., research methods, medical procedures).

Scientific Inference. Gathering data; detecting patterns; making conjectures; testing conjectures; drawing inferences.

Mathematical Inference. Logical reasoning and deduction; assumptions and conclusions; axiomatic systems; theorems and proofs; proof by direct deduction, by indirect argument, and by "mathematical induction."

Verification. Levels of convincing argument; persuasion and counterexamples; logical deduction; legal reasoning ("beyond reasonable doubt" vs. "preponderance of evidence"; court decisions interpreting various logical options); informal inference (suspicion, experience, likelihood); classical proofs (e.g., isosceles triangle, infinitude of primes).

Variables and Equations

Algebra. Variables, constants, symbols, parameters; equations vs. expressions. Direct and indirect variation; inverse relations; patterns of change; rates of change. Graphical representations; translation between words and graphs. Symbols and functions.

Equations. Linear and quadratic; absolute value; 2 x 2 systems of linear equations; inequalities; related graphs.

Graphs. Interpretation of graphs; sketching graphs based on relations of variables; connection between graphs and function parameters.

Algorithms. Alternative arithmetic algorithms; flowcharts; loops; constructing algorithms; maximum time vs. average time comparisons.

Modeling and Decisions

Financial Mathematics. Percentages, markups, discounts; simple and compound interest; taxes; investment instruments (stocks, mortgages, bonds); loans, annuities, insurance, personal finance.

Planning. Allocating resources; management information systems; preparing budgets; determining fair division; negotiating differences; scheduling processes, decision trees; PERT charts; systems thinking.

Mathematical Modeling. Abstracting mathematical structures from real-world situations; reasoning within mathematical models; reinterpreting results in terms of original situations; testing interpretations for suitability and accuracy; revision of mathematical structure; repetition of modeling cycle.

Scientific Modeling. Role of mathematics in modeling aspects of science such as acceleration, astronomical geometry, electrical current, genetic coding, harmonic motion, heredity, stoichiometry.

Technological Tools. Familiarity with standard calculator and computer tools: scientific and graphing calculators (including solving equations via graphs); spreadsheets (including presentation of data via charts); statistical packages (including graphical displays of data).

About the Authors

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Forman has served as First Vice-President of the Mathematical Association of America (1992-1994); as Chair of the Association's New York Metropolitan Section (1997-1999); and as President of the New York State Mathematics Association of Two-Year Colleges (1985-1986). She received her doctorate in mathematics education and research from Columbia University.

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Forman and Steen are coauthors of several publications dealing with the mathematical needs of the technical workforce, including "Mathematics for Work" (*Bulletin of the International Commission on Mathematical Instruction,* December 1994); "How School Mathematics Can Prepare Students for Work, Not Just for College" (*Harvard Education Letter,* May/June 1995); "Mathematics for Work and Life" (*Seventy-Five Years of Progress: Prospects for School Mathematics,* NCTM, 1995); and "Applied Academics: Myths and Realities" (*Education Week,* October 19, 1996).

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